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Point Processes in Statistical Mechanics A Cluster Expansion Approach

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Summary

A point process is a mechanism, which realizes randomly locally finite point measures. One of the main results of this thesis is an existence theorem for a new class of point processes with a so called *signed Lévy pseudo measure* L, which is an extension of the class of infinitely divisible point processes. The construction approach is a combination of the classical point process theory, as developed by Kerstan, Matthes and Mecke, with the method of cluster expansions from statistical mechanics. Here he starting point is a family $(\Theta_m)_m$ of signed Radon measures, which defines on the one hand the Lévy pseudo measure L, and on the other hand locally the point process. The relation between L and the process is the following: this point process solves the integral *cluster equation* determined by L, which we denote by $(\mathbf{\Sigma}_L)$.

We show that the results from the classical theory of infinitely divisible point processes carry over in a natural way to the larger class of point processes with a signed Lévy pseudo measure. In this way we obtain e.g. a criterium for simplicity and a characterization through the cluster equation $(\mathbf{\Sigma}_L)$, interpreted as an integration by parts formula, for such point processes.

Our main result in chapter 3 is a representation theorem for the factorial moment measures of the above point processes. With its help we will identify the permanental respective determinantal point processes, which belong to the classes of Boson respective Fermion processes. As a by-product we obtain a representation of the (reduced) Palm kernels of infinitely divisible point processes.

In chapter 4 we see how the existence theorem enables us to construct (infinitely extended) Gibbs, quantum-Bose and polymer processes. The so called polymer processes, point processes in the space of finite connected subsets of $\mathbb{Z}^d$, seem to be constructed here for the first time.

In the last part of this thesis we prove that the family of cluster equations $(\mathbf{\Sigma}_L)_L$ has certain stability properties with respect to the transformation of its solutions. At first this will be used to show how large the class of solutions of such equations is, and secondly to establish the cluster theorem of Kerstan, Matthes and Mecke in our setting. With its help we are able to enlarge the

class of Pôlya processes to the so called branching Pôlya processes.

The last sections of this work are about thinning and splitting of point processes. One main result is that the classes of Boson and Fermion processes remain closed under thinning. We use the results on thinning to identify a subclass of point processes with a signed Lévy pseudo measure as doubly stochastic Poisson processes. We also pose the following question: Assume you observe a realization of a thinned point process. What is the distribution of deleted points? Surprisingly, the Papangelou kernel of the thinning, besides a constant factor, is given by the intensity measure of this conditional probability, called splitting kernel.

Zusammenfassung

Ein Punktprozess ist ein Mechanismus, der zufällig ein lokalendliches Punktmaß realisiert. Ein Hauptresultat dieser Arbeit ist ein Existenzsatz für eine sehr große Klasse von Punktprozessen mit einem *signierten Lévy Pseudomaß* L. Diese Klasse ist eine Erweiterung der Klasse der unendlich teilbaren Punktprozesse. Die verwendete Methode der Konstruktion ist eine Verbindung der klassischen Punktprozesstheorie, wie sie von Kerstan, Matthes und Mecke ursprünglich entwickelt wurde, mit der sogenannten Methode der Cluster-Entwicklungen aus der statistischen Mechanik. Ausgangspunkt ist eine Familie $(\Theta_m)_m$ von signierten Radonmaßen. Diese definiert einerseits das Lévysche Pseudomaß L; andererseits wird mit deren Hilfe der Prozeß lokal definiert. Der Zusammenhang zwischen L und dem Prozeß ist so, daß der Prozeß die durch L bestimmte Integralgleichung (genannt *Clustergleichung* und notiert $\mathbf{\Sigma}_L$) löst.

Wir zeigen, dass sich die Resultate aus der klassischen Theorie der unendlich teilbaren Punktprozesse auf natürliche Weise auf die neue Klasse der Punktprozesse mit signiertem Lévy Pseudomaß erweitern lassen. So erhalten wir z.B. ein Kriterium für die Einfachheit und eine Charackterisierung durch $\mathbf{\Sigma}_L$ für jene Punktprozesse.

Unser erstes Hauptresultat in Kapitel 3 zur Analyse der konstruierten Prozesse ist ein Darstellungssatz der faktoriellen Momentenmaße. Mit dessen Hilfe werden wir die permanentischen respektive determinantischen Punktprozesse, die in die Klasse der Bosonen respektive Fermionen Prozesse fallen, identifizieren. Als ein Nebenresultat erhalten wir eine Darstellung der (reduzierten) Palm Kerne von unendlich teilbaren Punktprozessen.

Im Kapitel 4 konstruieren wir mit Hilfe unseres Existenzsatzes unendlich ausgedehnte Gibbsche Prozesse sowie Quanten-Bose und Polymer Prozesse. Diese letzte Klasse enthält Punktprozesse auf dem Raum der zusammenhängenden endlichen Teilmengen des $\mathbb{Z}^d$. Unseres Wissens sind sie bisher nicht konstruiert worden.

Im letzten Teil der Arbeit zeigen wir, dass die Familie der Integralgleichungen $(\mathbf{\Sigma}_L)_L$ gewisse Stabilitätseigenschaften gegenüber gewissen Trans-

formationen ihrer Lösungen aufweist. Dies wird erstens verwendet, um zu verdeutlichen, wie groß die Klasse Punktprozeßlösungen einer solchen Gleichung ist. Zweitens wird damit der Ausschauerungssatz von Kerstan, Matthes und Mecke in unserer allgemeineren Situation gezeigt. Mit seiner Hilfe können wir die Klasse der Pôlyaschen Prozesse auf die der von uns genannten Pôlya Verzweigungsprozesse vergrößern. Der letzte Abschnitt der Arbeit beschäftigt sich mit dem Ausdünnen und dem Splitten von Punktprozessen. Wir beweisen, dass die Klassen der Bosonen und Fermionen Prozesse abgeschlossen unter Ausdünnung ist. Die Ergebnisse über das Ausdünnen verwenden wir, um eine Teilklasse der Punktprozesse mit signiertem Lévy Pseudomaß als doppelt stochastische Poissonsche Prozesse zu identifizieren. Wir stellen uns auch die Frage: Angenommen wir beobachten eine Realisierung einer Ausdünnung eines Punktprozesses. Wie sieht die Verteilung der gelöschten Punktkonfiguration aus? Diese bedingte Verteilung nennen wir *splitting Kern*, und ein überraschendes Resultat ist, dass der Papangelou-Kern der Ausdünnung, abgesehen von einem konstanten Faktor, gegeben ist durch das Intensitätsmaß des splitting Kernes.

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Contents

List of notations

$\mathbb{N}$	$\{1, 2, 3, \ldots\}$
X	Polish space
$\mathcal{B}(X)$	Borel σ-field of X
$\mathcal{B}_0(X)$	Relatively compact sets of X
$\mathcal{M}(X)$	locally finite measures
$\mathcal{M}^{\cdot\cdot}(X)$	locally finite, counting measures
$\mathcal{M}^{\cdot\cdot}_f(X)$	finite counting measures
$\mathcal{M}^{\cdot}(X)$	locally finite, simple counting measures
$\mathcal{M}^{\circ}(X)$	locally finite, diffuse measures
$\mathbf{0}$	the zero measure
$\mathcal{K}_+(X)$	continuous, non negative functions on X with compact support
$F_{bc,+}(X)$	bounded, non negative measurable functions with compact support
$F_+(X)$	non negative measurable functions on X
$[n]$	set of the first $n \in \mathbb{N}$ natural numbers
$\pi([n])$	set of partitions of $[n]$
$\lvert\mathcal{J}\rvert$	number of elements in the partition $\mathcal{J} \in \pi([n])$
S_k	set of permutations of the set $[k]$
$\lvert\sigma\rvert$	number of cycles in $\sigma \in S_k$
S_k^{cy}	$\{\sigma \in S_k \,:\, \lvert\sigma\rvert = 1\}$
$\ell(\omega)$	cycle length, equal to k if $\omega \in S_k^{cy}$
$f \otimes g$	tensorproduct of $f, g \in F_+(X)$. $f \otimes g : (x, y) \mapsto f(x)\, g(y)$
$f_{[n]}$	$\prod_{i \in [n]} f_i$ for $f_1, \ldots, f_n \in F_+(X)$

ζ_f	$\zeta_f(\mu)=\mu(f)$ for $\mu\in\mathcal{M}(X)$, $f\in F_+(X)$
$\mathcal{L}_P$	Laplace transform of a random measure P
C_P	Campbell measure of P
C_P^n	n-th order Campbell measure of P
ν_P^n	n-th order moment measure of P
$C_P^{!,n}$	n-th order reduced Campbell measure of P
$\check{\nu}_P^n$	n-th order factorial moment measure of P
$P_{x_1,\ldots,x_n}$	higher order Palm kernel of P
$P^!_{x_1,\ldots,x_n}$	reduced higher order Palm kernel of P
L	Lévy measure, signed cluster pseudo measure, signed Lévy pseudo measure
L^ϵ	negative $\epsilon=-1$ resp. positive $\epsilon=+1$ part of L
$\lvert L\rvert$	L^++L^-, variation of L
$\mathfrak{L}^1(\lvert L\rvert)$	$\{\varphi\in F_+(\mathcal{M}^{\cdot\cdot}(X)):\ \lvert L\rvert(\varphi)<\infty\}$
$\mathcal{W}$	$\{L$ non negative measure on $\mathcal{M}^{\cdot\cdot}(X):\ L(\mathbf{1}-e^{-\zeta_f})<\infty,\ f\in U\}$
$\Im_L$	random KMM process with signed Lévy pseudo measure L
$\{\Pi_a\}_{a\in E}$	cluster field on the Polish space E
$P\Pi_{(\cdot)}$	point process P clustered with respect to $\{\Pi_a\}_{a\in E}$
$\Gamma_q(P)$	q-thinning of P
$S_q(P)$	splitting law of P
$\Upsilon_q^\mu(P)$	splitting kernel of P
$\{\Theta_n\}_{n=1}^\infty$	family of cumulant measures
$\{\varrho_n\}_{n=0}^\infty$	family of Schur measures
$\{\mathcal{C}_m^x\}_{m=0}^\infty$	family of loop measures
$\mathcal{K}_L$	modified Laplace functional with signed Lévy pseudo measure L
Ξ	partition function
π	Papangelou kernel
$\pi^{(m)}$	iterated Papangelou kernel
ϕ	pair potential

ϕ_x	$y \mapsto \phi(x, y)$ for $x, y \in X$
u	Ursell function
$E(\mu)$	energy of $\mu \in \mathcal{M}^{\cdot\cdot}_f(X)$
P_ρ	Poisson process with intensity $\rho \in \mathcal{M}(X)$
$P^\epsilon_{z,\lambda}$	Pólya sum $\epsilon = +1$ and Pólya difference $\epsilon = -1$ process
$\Im^\epsilon_{\lambda,k}$	permanental $\epsilon = +1$ and determinantal $\epsilon = -1$ process with reference measure λ and kernel k
$G_{z,\phi}$	Gibbs process with pair potential ϕ and activity z
$\Im_{\Re}$	point process constructed from a weakly stable and weakly regular pair potential with signed Lévy pseudo measure $\Re$

Chapter 1

Introduction

1.1 The Cluster Expansion Method (*CEM*)

The cluster expansion method (*CEM* for short) is one of the main mathematical methods to study the structure of the collection of Gibbs processes (or states) and to analyze the properties of these processes. This includes also their construction. The development of these topics will be our main task in the sequel; and also to apply the method to several examples of classical and quantum statistical mechanics. We develop the cluster expansion method within the theory of random measures and point processes as it had been worked out by the Jena school of probability (i.e. Matthes, Kerstan, Mecke and Fichtner) in the sixties.

Historical Remarks

We make these remarks by quoting three relevant sources and begin with Glimm and Jaffe [22].

¶ *There are regimes in statistical physics where the observable quantities admit power series expansions. These expansions, in a suitable parameter range, converge uniformly in the volume, and allow a mathematical definition of all thermodynamic quantities in the infinite volume limit. Thus these expansions are a very powerful tool. They are generally known as cluster expansions.*

The cluster expansions originated in the work of Mayer and Montroll [42, 32, 26], and they have been generalized to apply to a great number of situations. Although these various expansions employ fundamentally the same set of ideas, they have generally contained a proliferation of special and ad *hoc features, so that the theory had to be developed separately in each case. More recently, it has been realized that the notion of polymers unified these various expansions. (...)*

They have been applied to a variety of questions, including their original use to establish convergence of the infinite volume limit, structure of the phase diagram, and exponential decay of correlations, in problems of statistical mechanics of particles and statistical mechanics of classical fields (i.e. Euclidean quantum field theory).

We also cite some comments to the historical development of the method from the Saint-Flour lectures of Dobrushin which can be found in [14].

¶ *The cluster expansion method found also important applications in the quantum field theory which, in its so-called Euclidean variant, leads to the theory of distribution-valued Markov fields with continuous argument. (...) Sometimes it seems that the specialists in probabilistic mathematical physics pronounce the words "Now the cluster expansion can be applied" as some kind of magic incarnation. They mean that now we can be sure that all plausible facts can be rigorously proved. (...)*

The literature on the method of cluster expansion and its application is enormous - see the books and review papers [22, 40, 67, 9, 65, 41] and references therein. The history of the method traces back to the deeps of theoretical physics. The introduction of cluster expansions as a way of rigorous mathematical investigation is due to Glimm, Jaffe and Spencer [24]. Often the cluster expansion method is treated as a class of ideas and approaches which have to be additionally specialized and modified for application to any concrete situation. But there is also a tendency to find a unified approach. An essential contribution to it was made by Gruber and Kunz [25] who introduced the so-called polymer model. (...)

Even a more general and so more convenient model was introduced by Kotecký and Preiss [33].

We quote finally from the recent paper of Poghosyan and Ueltschi [58]:

¶ *The method of cluster expansions was introduced in the 1930s in statistical mechanics in oder to study gases of classical interacting systems. Its main achievement, from the point of view of physics, may be the derivation of the van der Waals equation of state for weakly interacting systems. The method was made rigorous by mathematical physicists in the 1960s, [63] and references therein.*

The method splits afterwards. One branch involves continuous systems, with applications to classical systems ([55, 47, 10]), quantum systems [19, *20, 59], or quantum field theory [23, 41, 10, 11]. The other branch involves polymer systems, i.e. discrete systems with additional internal structure [25, 13, 2, 48, 68, 15, 27]. An important step forward was the article of Kotecky and Preiss with its simplified setting and its elegant condition for the convergence of the cluster expansion [33].*

The methods for proving the convergence are diverse. Let us mention the study of Kirkwood-Salzburg equations that involve correlation functions (see[63]) and references therein, the algebraic approach of Ruelle [63], combinatorial approaches using tree identities [55, 10, 4, 11] and inductions for discrete systems [13, 2, 48].

Important and useful surveys were written by Brydges [9], Pfister [56], and Abdessalam and Rivasseau [1].

Thus historically the first to start an axiomatization of the cluster expansion method were Gruber and Kunz [25], and then Glimm and Jaffe [21]. The first systematic mathematical exposition of the *CEM* can be found in the monography of Malyshev and Minlos [40]. As such it is of fundamental importance. Nevertheless, it seems that all authors didn't arrive at a final formulation of this method which is wide enough to describe at the same time discrete as well as continuous classical and quantum systems. For instance the important case of random measures or point processes is indicated only shortly in [40] as an aside. In the latter case the method had been developed mainly by Poghosyan, starting with his thesis [57] with further developments afterwards as cited in [58].

The next essential step was made recently by Poghosyan and Ueltschi [58] who formulated a general setting for the *CEM*. Furthermore, Poghosyan and Ueltschi found sufficient criteria for its convergence, which where already forshaddowed in the thesis of Kuna [36]. Kuna, in his thesis from 1999, followed the method of cluster expansions to study models of classical and quantum statistical mechanics. He worked within the classical frame of Kirkwood-Salzburg equations as designed already by Ruelle [63, 64], but started to combine it with the approach of Nguyen and Zessin [53], which is presented more precisely under (α) in the next section, instead of using the approach of Dobrushin Lanford and Ruelle in terms of conditional expectations (DLR approach). (The same idea has been indicated already by Kutoviy and Rebenko in [37] but without giving detailed mathematical arguments.)

Our aim will be to do some further steps in direction of a general mathematical method of cluster expansions. Valuable elements in this direction (as well as references concerning the mathematical origins of these ideas) were given already in the book of Daley and Vere-Jones [12]. There the authors also start from the combinatorial device (see lemma 5.2.VI. in [12] resp. lemma 3.1.2 below) which relates local cumulants and Schur measures. Here we proceed a bit further and add general conditions for the convergence of these local characteristics. The formulation of these conditions is inspired by the old work of Mecke [44].

Cluster Representations

The *CEM* consists of two parts: First one has to give a description of the process locally in space, e.g. its local description in terms of a local law, expectations of local observables etc., which are *explicitly represented in terms of certain cluster measures.* Then, in a second step, one has to formulate conditions under which these local characteristics converge in the thermodynamic limit.

Our point of view differs from the traditional one in the following respects. For us a cluster is a finite positive measure on some basic phase space. Examples of clusters are finite point configurations in a Euclidean space, finite measures on X or their supports etc.. We shall therefore work within the frame of random measure and point process theory which seems to be a natural general setting for the *CEM.*

The main new features for us are:

(α) We consider abstract *Gibbs processes* P (which are the main objects to study) not in the sense of Dobrushin, Lanford, Ruelle, where they are specified locally, by means of a given potential, by the Gibbs distribution given the outside environment, but as a *Papangelou process* specified by the kernel $\pi(\mu, \mathrm{d}\, x) = \exp(-\beta E(x, \mu))\, \varrho(\mathrm{d}\, x)$ we call Boltzmann kernel. Here $E(x, \mu)$ is the *conditional energy* of the particle in x, given the *environment* μ, defined in terms of some potential. ϱ is some reference measure on the basic state space X.

That P is such a Papangelou process means that it is a solution of an integration by parts formula, called $(\mathbf{\Sigma}'_\pi)$ and to be found in section 2.5, determined by the kernel π. This point of view is equivalent to the DLR approach in case of the Boltzmann kernel π by a theorem of Nguyen, Zessin [53]. We point out that the class of Papangelou processes is much larger than the class of Gibbs states, containing also quantum mechanical processes like the Bose and Fermi gas. And for this reason we choose this point of view.

(β) Accordingly we shall give a formulation of the *CEM* in a more general and abstract form than one can find usually in the literature. It is formulated in terms of Radon measures rather than functions (correlation and cluster functions etc.).

We start the desciption of our version of the *CEM*: We are interested in two families $\{\varrho_k\}_{k=1}^\infty$ and $\{\Theta_m\}_{m=1}^\infty$ of signed Radon measures, which are in duality with respect to one another in the sense that

$$\varrho_k(\otimes_{j=1}^k f_j) = \sum_{\sigma\in S_k} \prod_{\omega\in\sigma} \Theta_{\ell(\omega)}(\otimes_{j\in\omega} f_j), \qquad f_1,\ldots,f_k \in F_{bc,+}(X), \tag{1.1}$$

where S_k is the symmetric group, the product is taken over the cycles $\omega \in \sigma$ in the permutation σ and $\ell(\omega)$ denotes the length of the cycle ω. For the measures $\varrho_k, k \geq 1$, we choose the name *Schur measures*, whereas the $\Theta_m, m \geq 1$, are called the *cumulant measures*. We also call (1.1) a *cluster representation of* $\{\varrho_k\}_{k=1}^\infty$ *in terms of* $\{\Theta_m\}_{m=1}^\infty$. An equivalent dual cluster representation of the cumulant measures $\{\Theta_m\}_{m=1}^\infty$ in terms of the Schur measures $\{\varrho_k\}_{k=1}^\infty$ is

$$\Theta_m(\otimes_{j=1}^m f_j) = \frac{1}{(m-1)!} \sum_{\mathcal{J}\in\pi(\{1,\ldots,m\})} (-1)^{|\mathcal{J}|-1}(|\mathcal{J}|-1)! \prod_{J\in\mathcal{J}} \varrho_{|\mathcal{J}|}(\otimes_{j\in J} f_j). \tag{1.2}$$

Here $\pi(\{1,\ldots,m\})$ denotes the set of partitions of $\{1,\ldots,m\}$.

Remark 1.1.1. *In several cases the terminology used here is not fixed in the literature. For example, our notion of a cumulant measure, which is used by [12], appears also as a semi-invariant in [40] or as Ursell function [67] or even cluster function [73]. Moreover our notion of Schur measure is new and not used in the literature. We propose it here because it is intimately related to the notions of generalized cycle index respectively generalized Schur function (cf. [45]) in representation theory. Another possibility for this notion is to call it immanantal measure because in some special situations (see [51]) it is a measure having a density given by an immanant.*

The Cluster Structure

The cluster structure is also defined in terms of the cumulant measures. Suppose that we are given signed and (without restricting the generality) symmetric Radon measures Θ_m on X^m.

Define then the *signed cluster pseudo measure* on the space $\mathcal{M}_f^{\cdot\cdot}(X)$ of cluster configurations by

$$L(\varphi) = \sum_{m\geq 1} \int_{X^m} \varphi(\delta_{x_1} + \cdots + \delta_{x_{m-1}} + \delta_x)\ \Theta_m(\mathrm{d}\,x\,\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_{m-1}),$$

where φ lies in some appropriate test function space such that $L(\varphi)$ is well defined. The signed cluster pseudo measure produces the clusters of the process we are going to construct. This justifies its name.

In the sequel we will assume that L satisfies a certain integrability condition in order to guarantee the convergence of the method.

The Cluster Expansion of the Local Process

Starting with representation (1.1) of the Schur measures we next define locally the point process belonging to these Schurr measures by means of the method of cluster expansion.

Suppose we are given the cluster measures Θ_m. Define the Schur measures ϱ_k by means of the corresponding cluster representation (1.1) and make the assumption *that they all are positive*, i.e.

$$(\wp) \qquad \varrho_k \geq 0 \quad , k \geq 1.$$

Then define for any bounded Borel set Λ in X the finite point process

$$Q_\Lambda(\varphi) = \frac{1}{\Xi(\Lambda)} \cdot \sum_{k\geq 0} \frac{1}{k!} \cdot \int_{\Lambda^k} \varphi(\delta_{x_1} + \cdots + \delta_{x_k})\, \varrho_k(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_k), \tag{1.3}$$

where $\varphi \in F_+(\mathcal{M}^{\cdot\cdot}_f(X))$ and $\Xi(\Lambda)$ is the normalizing constant, i.e. the partition function in the context of statistical mechanics. That this constant is finite and strictly positive follows for instance from the integrability assumption that the variation $|L|$ of the signed cluster pseudo measure L is of first order, i.e. its first moment measure is locally finite. This will be formulated precisely and assumed later. The process Q_Λ defined locally in Λ is often a Papangelou process, or even a local Gibbs process, but not always.

Following the terminology of [40, 58] we call the representation (1.3) of the local processes Q_Λ its *cluster expansion in terms of the Schur measures* $\{\varrho_k\}_{k=1}^\infty$ *resp. cumulant measures* $\{\Theta_m\}_{m=1}^\infty$.

The Cluster Equation of the Infinitely Extended Process

Under the assumption that $|L|$, the variation of the signed cluster pseudo measure, is of first order we can then construct the limiting process P. In our approach this corresponds to the convergence step of the *CEM.* If the local process Q_Λ is Papangelou or even Gibbs then the question arises whether the limiting process has the same property. We are able to answer this question in several cases.

The connection between L and P is expressed by the fact that P is the unique solution of an equation having the following structure:

$$(\mathbf{\Sigma}_L) \qquad C_P = C_L \star P.$$

Here $\star$ denotes some version of the convolution operation (see (2.5) for a precise definition). This shows that the Campbell measure of the signed cluster pseudo measure L is a divisor of the Campbell measure of P with respect to the convolution operation $\star$. And this reflects the essential nature of the cluster expansion method of P; for this reason we call equation $(\mathbf{\Sigma}_L)$ the *cluster equation of P for L*. P is called the *random KMM process with signed cluster pseudo measure* L or more shorter *KMM process for* L and is denoted by $\Im_L$.

As we saw in the beginning, for the above construction one can even start with a family $\varrho_k, k \geq 1$, of *positive* Radon measures, and then define the cumulant measures $\Theta_m, m \geq 1$, by means of the dual cluster representation (1.2). Applying the above cluster expansion method then leads to some limiting process, solution of the cluster equation $(\mathbf{\Sigma}_L)$.

To summarize: The *cluster expansion method (CEM)* consists in defining first the signed cluster pseudo measure L by means of the given data $\{\Theta_m\}_{m=1}^{\infty}$ respectively $\{\varrho_k\}_{k=1}^{\infty}$; and next, to construct with their help locally the process. (This construction reflects the cluster structure.) The limiting process is obtained, under the condition that $|L|$ is of first order, as the unique solution of the cluster equation belonging to L.

In the classical case of *positive* measures $\Theta_m, m \geq 1$, there is a direct and transparent construction. Here the signed cluster pseudo measure L is an ordinary positive measure on $\mathcal{M}_f^{\cdot\cdot}(X)$.

Suppose that L is of first order. Then one can consider the Poisson process P_L in the (Polish) phase space of clusters $\mathcal{M}_f^{\cdot\cdot}(X)$ with intensity measure L. This process is well defined and realizes configurations of clusters of the form

$$\nu = \delta_{x_1} + \delta_{x_2} + \ldots \quad , x_j \in \mathcal{M}_f^{\cdot\cdot}(X).$$

The point process P, we are interested in, is a process in X given by the image of P_L under the *dissolution mapping*

$$\xi : \nu = \delta_{x_1} + \delta_{x_2} + \ldots \longmapsto x_1 + x_2 + \ldots \quad .$$

It is the unique solution of equation $(\mathbf{\Sigma}_L)$; moreover, P is infinitely divisible; and conversely, any solution of the cluster equation $(\mathbf{\Sigma}_L)$ is infinitely divisible. Here L is called the *Lévy measure* of P. In particular this construction gives the same process P which one obtains by using the new *CEM*-construction above. All this can be found in [31], see also [52].

It is now however possible that $(\mathbf{\Sigma}_L)$ has a unique solution $P = \Im_L$ even in case of a non-trivial L^-. In this case the process is no longer infinitely divisible.

We observe that the classical construction cannot be used if the signed cluster pseudo measure L has a non-trivial negative part L^- because there is no Poisson process with a signed intensity measure. Thus we see, that we need another method in case of signed L. *The analysis of this signed case can be considered as the main topic of this thesis.*

Here one can see the fundamental importance of the *CEM*: As Dobrushin mentioned, it is a *magic incarnation* of an instrument which enables the solution of problems in situations where classical methods do not function.

1.2 Mathematical Foundations and Summary of the Results

In this thesis we use the *CEM* to construct processes either of Bosonic resp. Fermionic type. Specification of the measures Θ_m respectively ϱ_k then leads to permanental, determinantal processes, classical or quantum Gibbs or Polymer processes.

One remark is in order here. The use of the terms *Boson or Fermion* in the literature is vague. We do not clarify here the use of the names Bose resp. Fermi gas or process. It appears today in the literature in an unspecified manner. It would be very useful to have a clear picture of these notions. For us these processes are examples of KMM processes. We take here the following pragmatic point of view: If in the applications below the cumulant measures depend on the parameter $\epsilon \in \{-1, +1\}$, say $\Theta_m(\epsilon)$, then the process belonging to case $\epsilon = -1$ is called *Fermion process* and otherwise *Boson process*.

The mathematical foundations of the *CEM* are taken mainly from Mecke's book [44]. This is an early document of the achievements of the Jena school of probability theory in the sixties. Other relevant sources of the theory appeared later in the monographies of Matthes, Kerstan and Mecke [31] and Kallenberg [28]. Another early unpublished document are the Sorbonne lectures of Krickeberg from 1975/76 which will appear in [35].

In *chapter 2* we present the classical results of the theory of random measures and point processes as one can find them in [44].

The construction of the random KMM process $\Im_L$ for a given signed cluster pseudo measure L by means of the *CEM* is presented in theorem 3.1.3, *chapter 3*. $\Im_L$ is characterized by the fact that its Laplace transform is given by the modified Laplace functional (mlf)

$$\mathcal{K}_L(f) = \exp(-L(\mathbf{1} - \mathrm{e}^{-\zeta_f})) \quad , f \in F_{bc,+}(X); \tag{1.4}$$

and, under more restrictive assumptions on L, as a solution of the cluster equation $(\mathbf{\Sigma}_L)$ in theorem 3.3.1. As an important consequence of this result we see that the local process Q_Λ converges to $\Im_L$ not only weakly as $\Lambda \uparrow X$ but even in the sense of the Campbell measures, i.e. $\mathcal{C}_{Q_\Lambda} \longrightarrow \mathcal{C}_\Im$. This will be used in section 5.2 for the construction of classical Gibbs processes.

In *chapter 4* we start to apply the *CEM* for the construction of special point processes which are relevant for statistical mechanics. In section 4.2 we present Pólya sum and difference processes. These processes have their origin in the work [75, 50, 51]. As has been shown by Bach and Zessin [2] the sum process corresponds to the quantum mechanical Bose-Einstein statistical operator whereas the difference process is related to the statistical operator of Fermi-Dirac. We shall not discuss here the connections to such functional analytic descriptions of these systems.

The cumulant measures of the Pólya pocesses are given by

$$\Theta_{n,\epsilon}(\bigotimes_{j=1}^{n} f_j) = \epsilon^{n-1} z^n \lambda(\prod_{j=1}^{n} f_j),$$

where $z \in (0,1)$ and $\lambda \in \mathcal{M}(X)$ is some locally finite reference measure on X. It is surprising that these point processes, which seem to be as fundamental as the well known Poisson process, have not been considered before within the point process community.

Our next applications in section 4.5 are the construction of permanental and determinantal processes. These processes also arise in physics in connection with Fermions and eigenvalues of random matrices, or in combinatorics in connection with non-intersecting paths and random spanning trees.

Historically the first who studied determinantal processes was Macchi. She was able to construct quantum mechanical Fermions on a rigorous mathematical level in 1972. (See [38], and for the english translation [39].) The subtitle of this seminal work is *Contribution à l'étude théorique des processus ponctuels. Applications à l'Optique Statistique et aux Communications Optiques.*

Roughly speaking a determinantal process is determined by a kernel $K(x,y)$, called *correlation kernel*, in such a way that the correlation functions, as introduced in section 2.2 definition 2.1.3, of the process are given by $\det(K(x_i,x_j))$. Their cumulant measures are

$$\Theta_{n,\epsilon}(\mathrm{d}\,x_1\ldots\mathrm{d}\,x_n) \\ = \epsilon^{n-1}\,k(x_1,x_2)k(x_2,x_3)\ldots k(x_{n-1},x_n)k(x_n,x_1)\,\lambda(\mathrm{d}\,x_1)\ldots\lambda(\mathrm{d}\,x_n), \quad (1.5)$$

where k is the so called *interaction kernel* and λ is some reference measure on X. The precise relation between interaction and correlation kernel is given in theorem 4.5.1.

These processes appear as invariant measures of a model of interacting Brownian particles, the so called Dyson model, in the work of Spohn [70, 71]. Here the correlation kernel K is the *sine-kernel*. Unfortunately the sine-kernel does not satisfy the assumptions of our existence theorem 4.5.1. Soshnikov [69] constructed these processes in full generality and presented many examples. A weak point in our construction of permanental and determinantal processes is that if we let the interaction kernel be translation invariant $k(x,y)=\psi(x-y)$ then the conditions of theorem 4.5.1 require $\psi\in L^1(X,\lambda)$. But there are some interesting examples like the sine kernel, which are in $L^2(X,\lambda)$ but not in $L^1(X,\lambda)$.

Shortly after Shirai and Takahashi [66] gave constructions of these processes including Bosons. Finally in the work of Borodin and Olshansky (see [7] and the literature cited there) there appear many very special determinantal processes in the representation of the infinite symmetric group.

Our construction method, the *CEM*, differs from the approaches of Soshnikov and Shirai/Takahashi mentioned above. Instead using Kolmogoroff's extension theorem we use our abstract version of the *CEM* and no functional analytic methods as needed in [66, 69]. Also our main examples are coming from other sources.

On the other hand the work of Macchi [38] and Fichtner [16] was important for the development of our reasoning here. In the latter it is shown that the so called position distribution of the *ideal Bose gas* is an infinitely divisible point process in $\mathbb{R}^d$. This example appears in section 4.5 as the ideal Bose gas where $X=\mathbb{R}^d, \epsilon=+1$ and the interaction kernel is given by a scaled Gaussian density where $z\in(0,1)$ and $\beta>0$ are some parameters:

$$g_z(x-y)=\frac{z}{(2\pi\beta)^{d/2}}\exp\left(-\frac{\|x-y\|^2}{2\beta}\right),\quad x,y\in\mathbb{R}^d.$$

The *ideal Fermi gas* of Macchi is obtained if $X=\mathbb{R}, \epsilon=-1$ and the correlation kernel is defined by

$$\chi(x-y) = \gamma \exp(-\frac{|x-y|}{\alpha}), \quad x, y \in \mathbb{R},$$

where $\alpha, \gamma > 0$ are chosen such that $4\alpha\gamma < 1$.

Note that in both cases also the process for the opposite sign of ϵ in (1.5) exists.

In *chapter 5* we'll construct by means of our methods first of all Gibbs processes for classical systems, i.e. point processes in $X = \mathbb{R}^d$ with the Lebesgue measure λ, interacting by means of some given nice pair potential. As mentioned already above, this is done within the framework of Papangelou processes and does not use the DLR approach. The construction of these processes has a long history starting with the work of Ruelle as it is documented in [63, 64] and Minlos [46]. An axiomatic approach to the theory of Gibbs states including their existence can by found in the unpublished seminal work of Preston [60]. The Ruelle approach by solving the Kirkwood-Salzburg equations can be found later again in Kuna [36], the literature cited there, and in [37].

For the *CEM* the classical Gibbs process is specified in terms of the so called *Ursell functions* u by means of the cumulant measures

$$\Theta_n(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n) = \frac{1}{(n-1)!} u(x_1, \ldots, x_n)\, \lambda^n(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n). \tag{1.6}$$

The approach of Poghosyan and Ueltschi allows directly to use our construction. But to show that the corresponding limiting KMM process is a Gibbs process is difficult. A similar result can be found already in [36], but derived by DLR methods.

The approach of Poghosyan and Ueltschi allows also to construct so called polymer systems; and surprisingly, even the construction of a quantum Bose gas. Here the underlying phase space X is a path space of Brownian loops with reference measure λ given by a (non-normalized) Brownian loop measure. The cumulant measures have the same structure as in (1.6), but now understood with the measure λ, on the path space of loops and the Ursell function u defined for these loops. The resulting limiting KMM process can be considered as a discrete version of the interacting Bose gas which appears in the classical work of Ginibre [20]. Unfortunately we are not able to treat the analogous interacting Fermi gas, nor Ginibre's versions of quantum gases. *This important open problem remains to be done.*

In the last *chapter 6* we are interested in the problem how random KMM processes behave under the transformations of cluster dissolution and of clustering, in particular under the operations *thinning and splitting.* These methods are coming from quantum optics and have been introduced into point process theory by Fichtner and Freudenberg. (See [16, 17])

The thinning operation is intimately related to the Pólya difference process; to be more precise, in (6.11) we see that thinning of a point process is in fact a doubly stochastic Pólya difference process. Theorem 6.3.4 is a representation of a thinned KMM process in terms of loop measures. The considered examples are Pólya, permanental and determinantal processes.

Next splitting laws are computed for Pólya and Poisson processes. In theorem 6.3.7 the Papangelou kernel of the thinned process is related to the intensity of its splitting kernel.

Finally, a large class of KMM processes is identified as Cox, i.e. doubly stochstic Poisson processes. The Pólya sum process is an example. As a consequence we obtain the surprising result that Pólya difference processes cannot be Cox.

Chapter 2

Point Processes
Basic Concepts and Key Tools

In this introductory chapter we shall present the main concepts and the classical results of the theory of random measures and point processes which we'll use in the sequel. They are mainly taken from the work of Mecke [44]. Theorem 2.3.2 gives the existence of point processes P for a large class of (non-negative) Lévy measures L having as Laplace transform $\mathcal{K}_L$ or solving the cluster equation $(\mathbf{\Sigma}_L)$. Another result, presented in theorem 2.3.3, which is often used, is Mecke's version of Lévy's continuity theorem.

The point of departure of the main topic of this thesis is the problem as it is formulated in the monography of Matthes, Kerstan and Mecke [31]: Given a finite signed measure K on $\mathcal{M}^{\cdot\cdot}(X)$ with $K^+(\mathcal{M}^{\cdot\cdot}(X)) = K^-(\mathcal{M}^{\cdot\cdot}(X))$, does there exists a point process P being the exponents of K, i.e. $P = \exp(K)$? Or equivalently, does there exist a finite signed measure L on $\mathcal{M}^{\cdot\cdot}(X)$ with $L(\{0\}) = 0$ such that $\mathcal{L}_P = \mathcal{K}_L$? We shall pose this question in a more general form.

After presenting the concepts of cluster point processes and the notion of Campbell measure (also versions of higher orders) we develop the notion of a Papangelou process, which will be of fundamental importance for us in the sequel. We present in lemma 2.5.3 Zessin's [75] existence result for finite Papangelou processes.

2.1 Point Processes

Let X be a complete separable metric space. Such a space is called *Polish.* By $\mathcal{B}(X)$ we will denote the σ-field of its Borel sets. Of great importance is also the ring $\mathcal{B}_0(X)$ of relatively compact sets of X. $\mathcal{B}_0(X)$ will sometimes also be denoted by the collection of *bounded Borel sets*. The triple

$$(X, \mathcal{B}(X), \mathcal{B}_0(X))$$

will be called *phase space*. In a second step we now introduce measures on the phase space

(i) The space of *locally finite measures*

$$\mathcal{M}(X) = \{\mu \text{ measure}: \ \mu(B) < \infty \text{ for } B \in \mathcal{B}_0(X)\},$$

(ii) The space of *locally finite, diffuse measures*

$$\mathcal{M}^\circ(X) = \{\mu \in \mathcal{M}(X): \ \mu(\{x\}) = 0 \text{ for all } x \in X\},$$

(iii) The space of *locally finite, counting measures*

$$\mathcal{M}^{\cdot\cdot}(X) = \{\mu \in \mathcal{M}(X): \ \mu(B) \in \mathbb{N} \text{ for } B \in \mathcal{B}_0(X)\},$$

(iv) The space of *locally finite, simple counting measures*

$$\mathcal{M}^{\cdot}(X) = \{\mu \in \mathcal{M}^{\cdot\cdot}(X): \ \mu(\{x\}) \in \{0,1\} \text{ for } x \in X\},$$

(v) The space of *finite counting measures*

$$\mathcal{M}^{\cdot\cdot}_f(X) = \{\mu \in \mathcal{M}^{\cdot\cdot}(X): \ \mu(X) < \infty\}.$$

Let us denote by $\mathcal{K}_+(X)$ the space of continuous functions on X with compact support and consider the following function for $f \in \mathcal{K}_+(X)$

$$\zeta_f : \begin{cases} \mathcal{M}(X) \to \mathbb{R}_+ \\ \qquad \mu \mapsto \mu(f). \end{cases}$$

We now introduce a topology on $\mathcal{M}(X)$ by requiring that we take the coarsest topology which makes all ζ_f, $f \in \mathcal{K}_+(X)$ continuous. This topology will be called the *vague topology*. It can be found in [31] proposition 3.2.1. that $\mathcal{M}(X)$ equipped with the vague topology is Polish. Furthermore lemma 3.2.4. in [31] says that $\mathcal{M}^{\cdot\cdot}(X)$ is vaguely closed in $\mathcal{M}(X)$ and thus Polish. So we can now introduce phase spaces on the level of measures

$$(\mathcal{M}(X), \mathcal{B}(\mathcal{M}(X)), \mathcal{B}_0(\mathcal{M}(X))) \text{ and } (\mathcal{M}^{\cdot\cdot}(X), \mathcal{B}(\mathcal{M}^{\cdot\cdot}(X)), \mathcal{B}_0(\mathcal{M}^{\cdot\cdot}(X))).$$

We remark that the Borel σ-fields $\mathcal{B}(\mathcal{M}(X))$ and $\mathcal{B}(\mathcal{M}^{\cdot\cdot}(X))$ coincide with the smallest σ- fields making the $\zeta_B = \zeta_{\mathbf{1}_B}$ for $B \in \mathcal{B}_0(X)$ measurable on the respective spaces $\mathcal{M}(X)$ and $\mathcal{M}^{\cdot\cdot}(X)$. So in particular if we let $F_{bc,+}(X)$ be the space of bounded non negative functions with compact support in X then all ζ_f, $f \in F_{bc,+}(X)$, are measurable with respect to $\mathcal{B}(\mathcal{M}(X))$ and $\mathcal{B}(\mathcal{M}^{\cdot\cdot}(X))$.

Now a *random measure* is a probability measure on $(\mathcal{M}(X), \mathcal{B}(\mathcal{M}(X)))$ and a *point process* is a probability measure on $(\mathcal{M}^{\cdot\cdot}(X), \mathcal{B}(\mathcal{M}^{\cdot\cdot}(X)))$. Certainly a point process P is a random measure such that $P(\mathcal{M}^{\cdot\cdot}(X)) = 1$. The *Laplace transform* of a random measure P is given by

$$\mathcal{L}_P(f) = \int\limits_{\mathcal{M}(X)} e^{-\zeta_f(\mu)}\, P(\mathrm{d}\,\mu) \text{ for } f \in F_{bc,+}(X).$$

A random measure is characterized by its Laplace transform: We recall the classical result from [44] proposition 3.

Proposition 2.1.1. *Let P_1 and P_2 be two random measures such that*

$$\mathcal{L}_{P_1}(f) = \mathcal{L}_{P_2}(f) \text{ for } f \in F_{bc,+}(X).$$

Then $P_1 = P_2$.

Definition 2.1.2. *The n-th order moment measure of a random measure P is given by*

$$\nu_P^n(f) = \int_{\mathcal{M}(X)} P(\mathrm{d}\,\mu) \int_{X^n} \mu(\mathrm{d}\,x_1)\dots\mu(\mathrm{d}\,x_n)\, f(x_1,\dots,x_n) \text{ for } f \in F_+(X^n).$$

If $\nu_P^n \in \mathcal{M}(X^n)$ then we say that P is of n-th order. Let now P be a point process then its n-th order factorial moment measure is defined by

$$\check{\nu}_P^n(f) = \int_{\mathcal{M}^{\cdot\cdot}(X)} P(\mathrm{d}\,\mu)\, \mu^{-[n]}(f), \tag{2.1}$$

where $f \in F_+(X^n)$ and $\mu^{-[n]}$ is the following symmetric measure on X^n

$$\begin{aligned} &\mu^{-[n]}(\mathrm{d}\,x_1 \dots \mathrm{d}\,x_n) \\ &= \mu(\mathrm{d}\,x_1)(\mu - \delta_{x_1})(\mathrm{d}\,x_2)\dots(\mu - \sum_{j=1}^{n-1} \delta_{x_j})(\mathrm{d}\,x_n) \text{ for } \mu \in \mathcal{M}^{\cdot\cdot}(X) \end{aligned} \tag{2.2}$$

Most of the time the factorial moment measures have densities with respect to some product measure.

Definition 2.1.3. *If P is a point process and $\lambda \in \mathcal{M}(X)$ is some reference measure on X such that*

$$\check{\nu}_P^n(\mathrm{d}\,x_1 \dots \mathrm{d}\,x_n) = \vartheta_n(x_1,\dots,x_n)\, \lambda(\mathrm{d}\,x_1)\dots\lambda(\mathrm{d}\,x_n),$$

then we call $\{\vartheta_n\}_{n=1}^{\infty}$ the family of correlation functions of P.

2.2 The Campbell Measure

Given a non negative measure K on $\mathcal{M}(X)$ its *Campbell measure* is defined by

$$C_K(h) = \iint h(x,\mu)\,\mu(\mathrm{d}\,x)K(\mathrm{d}\,\mu) \text{ for } h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X)).$$

Remark that the marginal $\nu_K^1(f) = C_K(f \otimes \mathbf{1})$ for $f \in F_+(X)$ is the first moment measure of K. The Campbell measure will be an important tool in this work. In the sequel we will also need the *Campbell measures of higher order*. They are defined as follows. Let K be as above then the *n-th order Campbell measure* is given by

$$C_K^n(h) = \int_{\mathcal{M}(X)} K(\mathrm{d}\,\mu) \int_{X^n} \mu(\mathrm{d}\,x_1)\ldots\mu(\mathrm{d}\,x_n)\,h(x_1,\ldots,x_n;\mu)$$

for $h \in F_+(X^n \times \mathcal{M}(X))$. The marginal

$$\nu_K^n(f) = C_K^n(f \otimes \mathbf{1}) \text{ for } f \in F_+(X^n)$$

is the n-th order moment measure of K. For measures K on $\mathcal{M}^{\cdot\cdot}(X)$ we will also need the following modification of the n-th order Campbell measure. The *reduced Campbell measure of n-th order* is given by

$$C_K^{!,n}(h) = \int_{\mathcal{M}^{\cdot\cdot}(X)} K(\mathrm{d}\,\mu) \int_{X^n} \mu^{-[n]}(\mathrm{d}\,x_1\ldots\mathrm{d}\,x_n)\,h(x_1,\ldots,x_n;\mu - \sum_{i=1}^{n}\delta_{x_i}),$$

for $h \in F_+(X^n \times \mathcal{M}^{\cdot\cdot}(X))$. The marginal

$$\check{\nu}_K^n(f) = C_K^{!,n}(f \otimes \mathbf{1}) \text{ for } f \in F_+(X^n)$$

is the n-th order factorial moment measure of K. In section 4.3 we will need the following measure

$$\check{C}_K^n(h) = \int_{\mathcal{M}^{\cdot\cdot}(X)} K(\mathrm{d}\,\mu) \int_{X^n} \mu^{-[n]}(\mathrm{d}\,x_1\ldots\mathrm{d}\,x_n)\,h(x_1,\ldots,x_n;\mu),$$

which remains unnamed. In section 4.6 we will encounter the so called *Palm kernels* $K_{x_1,\ldots,x_n}$ *of n-th order of* K: Assume K is a σ-finite measure on $\mathcal{M}^{\cdot\cdot}(X)$. We certainly have $C_K^n(\cdot \times N) \ll \nu_K^n$ for any fixed $N \in \mathcal{B}(\mathcal{M}^{\cdot\cdot}(X))$. Let us denote the Radon-Nykodim derivative by

$$K_{x_1,\ldots,x_n}(N) = \frac{\mathrm{d}\,C_K^n(\cdot \times N)}{\mathrm{d}\,\nu_K^n}(x_1,\ldots,x_n).$$

The theory of disintegration of measures ([54], chapter V, theorem 8.1) then yields that for $\nu_K^n - a.e.[(x_1, \ldots, x_n)]$ $K_{x_1,\ldots,x_n}$ can be chosen as a point process in X. So in short hand formulation we have

$$C_K^n(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n \,\mathrm{d}\, \mu) = \nu_K^n(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_k)\, K_{x_1,\ldots,x_n}(\mathrm{d}\, \mu). \tag{2.3}$$

The *reduced n-th order Palm kernels* $K^!_{x_1,\ldots,x_n}$ are defined analogously.

The following result says that it suffices to know the Campbell measure for a certain class of test functions. It can be found in: [44] chapter 4, proof of theorem 10.

Lemma 2.2.1. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two non negative measures on $X^n \times \mathcal{M}^{\cdot\cdot}(X)$ such that*

$$\mathcal{C}_1(\bigotimes_{j=1}^n f_j \otimes e^{-\zeta_g}) = \mathcal{C}_2(\bigotimes_{j=1}^n f_j \otimes e^{-\zeta_g}) < \infty \text{ for } f_1, \ldots, f_n, g \in F_{bc,+}(X),$$

then $\mathcal{C}_1 = \mathcal{C}_2$.

Proof. Define two measures on $\mathcal{M}^{\cdot\cdot}(X)$: For $f_1, \ldots, f_n \in F_{bc,+}(X)$ let

$$R_i^{(f_j)_{j=1}^n}(\varphi) = \mathcal{C}_i(\bigotimes_{j=1}^n f_j \otimes \varphi),\ \varphi \in F_+(\mathcal{M}^{\cdot\cdot}(X)),\ i = 1, 2.$$

Then $R_i^{(f_j)_{j=1}^n}$ are finite measures whose Laplace transforms coincide, which implies that $R_1^{(f_j)_{j=1}^n} = R_2^{(f_j)_{j=1}^n}$ due to proposition 2.1.1. So $\mathcal{C}_1$ and $\mathcal{C}_2$ are finite and coincide on

$$\mathcal{G} = \{B_1 \times \ldots \times B_n \times N \,|\, B_1, \ldots, B_n \in \mathcal{B}_0(X) \text{ and } N \in \mathcal{B}(\mathcal{M}^{\cdot\cdot}(X))\}.$$

Since $\sigma(\mathcal{B}_0(X)) = \mathcal{B}(X)$ we certainly have $\sigma(\mathcal{G}) = \mathcal{B}(X)^{\otimes n} \otimes \mathcal{B}(\mathcal{M}^{\cdot\cdot}(X))$. Furthermore $\mathcal{G}$ is stable under intersections and so $\mathcal{C}_1$ and $\mathcal{C}_2$ coincide due to the uniqueness of measure theorem. □

2.3 Infinitely Divisible Point Processes and a Generalization

Definition 2.3.1. *We denote by $\mathcal{W}$ the class of non negative measures L on $\mathcal{M}^{\cdot\cdot}(X)$ such that*

$$L(\{\mathbf{0}\}) = 0 \text{ and } L(\mathbf{1} - e^{-\zeta_f}) < \infty,\ f \in F_{bc,+}(X),$$

where $\mathbf{0}$ denotes the zero measure.

We collect several key results of [44] in the following theorem.

Theorem 2.3.2. *Let $L \in \mathcal{W}$. Then there exists a unique point process P such that its Laplace transform is given by*

$$\mathcal{L}_P(f) = e^{-L(\mathbf{1}-e^{-\zeta_f})} \text{ for } f \in F_{bc,+}(X), \tag{2.4}$$

and P is infinitely divisible. Moreover if L is of first order then (2.4) is equivalent to

$$(\mathbf{\Sigma}_L) \qquad C_P(h) = \int_{\mathcal{M}^{\cdot\cdot}(X)} \int_{\mathcal{M}^{\cdot\cdot}(X)} \int_X h(x, \eta + \mu)\, C_L(\mathrm{d}\, x\, \mathrm{d}\, \eta) P(\mathrm{d}\, \mu)$$

for all $h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X))$. This point process will be denoted $\Im_L$ in the sequel.

The existence of P for given $L \in \mathcal{W}$ can be found in lemma 8 chapter 2 of [44] and the equivalence of the representation of the Laplace transform with $(\mathbf{\Sigma}_L)$ in theorem 11 chapter 4 of [44]. As an aside we remark that also the converse holds true: To every infinitely divisible point process P there belongs some $L \in \mathcal{W}$ such that (2.4) is valid. The measure L is called the *Lévy measure* of the infinitely divisible point process $\Im_L$. The right hand side of $(\mathbf{\Sigma}_L)$ can be seen as a kind of convolution of C_L with P so we will denote it by

$$(C_L \star P)(h) := \int_{\mathcal{M}^{\cdot\cdot}(X)} \int_{\mathcal{M}^{\cdot\cdot}(X)} \int_X h(x, \eta + \mu)\, C_L(\mathrm{d}\, x\, \mathrm{d}\, \eta) P(\mathrm{d}\, \mu) \tag{2.5}$$

Here $\star$ should not be confused with the convolution operator $*$ used below. Furthermore let $\mathcal{K}_L$ be the functional on $F_{bc,+}(X)$ such that

$$\mathcal{K}_L(f) = e^{-L(\mathbf{1}-e^{-\zeta_f})},$$

the so called *modified Laplace functional*. Mecke's proof in [44] relies on the following result, also due to him, which is a version of Lévy's continuity theorem for random measures

Theorem 2.3.3. *Let $(P_m)_m$ be a sequence of laws on $\mathcal{M}(X)$ such that*

$$\mathcal{L}_{P_m}(f) \to \mathcal{K}(f) \text{ as } m \to \infty$$

for $f \in F_{bc,+}(X)$ and the limiting functional $\mathcal{K}$ on $F_{bc,+}(X)$ has the following continuity property:

For any sequence $u_n \in F_{bc,+}(X)$ with $u_n \downarrow 0, \mathcal{K}(u_n) \to 1$ as $n \to \infty$.

Then there exists a law P on $\mathcal{M}(X)$ such that $\mathcal{L}_P = \mathcal{K}$.

A natural question is whether $\mathcal{K}_L$ is still a Laplace transform of a point process if we let L be a finite signed measure on $\mathcal{M}^{\cdot\cdot}(X)$. In the monograph [31] the authors asked whether for a finite signed measure K on $\mathcal{M}^{\cdot\cdot}(X)$ with $K^+(\mathcal{M}^{\cdot\cdot}(X)) = K^-(\mathcal{M}^{\cdot\cdot}(X))$ does there exists a point process P such that

$$P = \exp(K) = \sum_{j=0}^{\infty} \frac{K^{*j}}{j!}?$$

Here $*$ is the usual convolution operator. It can be seen that this question is equivalent to the existence of a finite signed measure L on $\mathcal{M}^{\cdot\cdot}(X)$ with $L(\{0\}) = 0$ such that $\mathcal{L}_P = \mathcal{K}_L$. They showed that this is the case if and only if

$$\Im_{L^+} = \Im_{L^-} * P. \tag{2.6}$$

Here L^+ resp. L^- is the positive resp. negative part in the Hahn-Jordan decomposition of L. This "is quite a complicated question" as the authors of [31] on page 79 remarked. The negative part L^- of the Lévy measure L can be interpreted as to contribute to a *deletion of points* in $\Im_{L^+}$, since according to the convolution equation (2.6) to obtain a realization of $\Im_{L^+}$ we have to take a realization of P and superpose it independently by a realization of $\Im_{L^-}$. As Matthes et al. [31] formulated, P is the convolution quotient of the infinitely divisible point processes $\Im_{L^+}$ and $\Im_{L^-}$.

The main problem studied here: Given any two L^+, $L^- \in \mathcal{W}$, does there exist a point process P such that (2.6) holds? In the sequel we will denote $L = L^+ - L^-$, the so called *signed Lévy pseudo measure* of a point process P if either (2.6) or $\mathcal{L}_P = \mathcal{K}_L$ on $F_{bc,+}(X)$ are valid. But note here: L is not even a signed measure on $\mathcal{M}^{\cdot\cdot}(X)$ since undefined expressions like $\infty - \infty$ can occur, which justifies the name signed Lévy pseudo measure. Remark that L is well defined on the set of $|L|$-integrable functions,

$$\mathfrak{L}^1(|L|) = \{\varphi \in F_+(\mathcal{M}^{\cdot\cdot}(X)) : \ |L|(\varphi) < \infty\}.$$

Here $|L|$ denotes the variation $L^+ + L^-$. In the introduction we also used the term signed cluster pseudo measure. Let us agree on the following convention: Any $L = L^+ - L^-$ with L^+, $L^- \in \mathcal{W}$ will be called a signed cluster pseudo measure. If L is a signed cluster pseudo measure such that a point process P exists and (2.6) holds then we say it is a signed Lévy pseudo measure.

2.4 Cluster Processes and Thinning

Let E and X be two Polish spaces and P be a point process in E. A *cluster field* $\{\Pi_a\}_{a\in E}$ is a measurable mapping from E to the set of point processes in X. Measurability is meant here in the following sense: For any $N \in \mathcal{B}(\mathcal{M}^{\cdot\cdot}(X))$ the mapping $a \mapsto \Pi_a(N)$ is measurable with respect to $\mathcal{B}(E)$. The *cluster process* of a point process P is given by

$$P\Pi_{(\cdot)} = \int P(\mathrm{d}\,\mu) \underset{a\in\mu}{*} \Pi_a.$$

A realization of $P\Pi_{(\cdot)}$ can be described as follows: First P realizes a point measure μ in E. Then, independently for any $a \in \mu$, Π_a realizes a point measure η_a in X. The realization of $P\Pi_{(\cdot)}$ is now given by the *superposition* $\sum_{a\in\mu} \eta_a$. The clustering can be thought of as a one step branching process in that $P\Pi_{(\cdot)}$ describes the distribution of the daughter generation.
There arises now one *major question*: Does the cluster process exist as a point process? In general $\sum_{a\in\mu} \eta_a$ might contain an infinite number of points in a bounded domain and this case has to be excluded.
Let us denote $\Pi_\mu = *_{a\in\mu}\Pi_a$. Then we say that the cluster process exists if

$$\Pi_\mu \text{ is a point process } P - a.s.[\mu].$$

The following result, proposition 4.2.3. in [31], will provide a sufficient condition for existence.

Proposition 2.4.1. *For all point processes P in E and all cluster fields $\{\Pi_a\}_{a\in E}$ the following statements are equivalent:*

a) $P\Pi_{(\cdot)}$ exists and is of first order.

b) $\nu_P^1(\nu_{\Pi_{(\cdot)}}^1(f)) < \infty$ for $f \in F_{bc,+}(X)$.

In this case $\nu_{P\Pi_{(\cdot)}}^1(f) = \nu_P^1(\nu_{\Pi_{(\cdot)}}^1(f))$ for $f \in F_{bc,+}(X)$.

An important example of a cluster field is given by

$$\Phi_x = (1-q)\delta_{\mathbf{0}} + q\delta_{\delta_x} \text{ for } q \in (0,1) \text{ and } x \in X,$$

where $\mathbf{0}$ denotes the zero measure. The corresponding cluster process $P\Phi_{(\cdot)}$ will be denoted by $\Gamma_q(P)$ and is called the *independent q-thinning of P*.

2.5 Papangelou Processes

We recall some facts on these processes from Zessin [75]. Let $\pi(\mu, \mathrm{d}\,x)$ be a kernel from $\mathcal{M}^{\cdot\cdot}(X)$ to $\mathcal{M}(X)$.

Definition 2.5.1. *A point process P is a Papangelou process with kernel π if*

$$(\mathbf{\Sigma}'_\pi) \quad C_P(h) = \iint h(x, \mu + \delta_x)\, \pi(\mu, \mathrm{d}\, x) P(\mathrm{d}\, \mu) \text{ for } h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X)).$$

We also say that π is the Papangelou kernel of the point process P, if P satisfies $(\mathbf{\Sigma}'_\pi)$.

For a detailed discussion on the $(\mathbf{\Sigma}'_\pi)$ condition the interested reader is referred to [28]. Define for $\eta \in \mathcal{M}^{\cdot\cdot}(X)$, $m \geq 1$,

$$\pi^{(m)}(\eta; \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_m) = \pi(\eta, \mathrm{d}\, x_1)\pi(\eta+\delta_{x_1}, \mathrm{d}\, x_2) \ldots \pi(\eta+\delta_{x_1}+\ldots+\delta_{x_{m-1}}, \mathrm{d}\, x_m)$$

the *iterated kernel* $\pi^{(m)}$ from $\mathcal{M}^{\cdot\cdot}(X)$ to $\mathcal{M}(X^m)$.

Definition 2.5.2. *We say that the kernel π satisfies the cocycle condition if for any $\eta \in \mathcal{M}^{\cdot\cdot}(X)$, $\pi^{(2)}(\eta; \mathrm{d}\, x \, \mathrm{d}\, y)$ is a symmetric measure. That is*

$$\pi^{(2)}(\eta; f_1 \otimes f_2) = \pi^{(2)}(\eta; f_2 \otimes f_1) \text{ for } f_1, f_2 \in F_+(X).$$

In particular if π satisfies the cocycle condition then for any $\eta \in \mathcal{M}^{\cdot\cdot}(X)$, $\pi^{(m)}(\eta; \cdot)$ is a symmetric measure.
Let now π be a kernel from $\mathcal{M}^{\cdot\cdot}_f(X)$ to $\mathcal{M}_f(X)$ such that for some $\eta \in \mathcal{M}^{\cdot\cdot}_f(X)$

$$0 < \Xi(\eta) := \sum_{m=0}^{\infty} \frac{1}{m!} \pi^{(m)}(\eta; X^m) < \infty$$

then we say that π *is η-integrable*. Under the condition of η-integrability of π the following finite point process

$$P^\eta_\pi(\varphi) := \frac{1}{\Xi(\eta)} \sum_{m=0}^{\infty} \frac{1}{m!} \int_{X^m} \varphi(\delta_{x_1} + \ldots + \delta_{x_m})\, \pi^{(m)}(\eta; \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_m) \qquad (2.7)$$

for $\varphi \in F_+(\mathcal{M}^{\cdot\cdot}_f(X))$ is well defined. The following result from [75] will later serve as a main lemma for us.

Lemma 2.5.3. *Assume that π is η-integrable for some $\eta \in \mathcal{M}^{\cdot\cdot}_f(X)$ and satisfies the cocycle condition. Then P^η_π, given by (2.7), is a Papangelou process with boundary condition η. That is P^η_π is a solution to*

$$C_P(h) = \iint h(x, \mu + \delta_x)\, \pi(\eta + \mu, \mathrm{d}\, x) P(\mathrm{d}\, \mu) \text{ for } h \in F_+(X \times \mathcal{M}^{\cdot\cdot}_f(X)).$$

Thus in particular we have that $P^{\mathbf{0}}_\pi$ is a Papangelou process with Papangelou kernel π. Here $\mathbf{0}$ denotes the zero measure on X.

2.6 Möbius Inversion Formula

For a more detailed exposition to the subject see [5] or [62].

Theorem 2.6.1 (Möbius inversion formula). *Let X be a finite set endowed with a partial order $\preceq$ (reflexive, transitive, anti-symmetric). Assume $(X, \preceq)$ has a minimal element, denoted by 0. Let f and g be two real valued functions on X. Suppose that for all $x \in X$*

$$g(x) = \sum_{0 \preceq z \preceq x} f(z). \tag{2.8}$$

Then

$$f(x) = \sum_{0 \preceq z \preceq x} \mu(z, x) g(z),\ x \in X.$$

Here μ is the so called Möbius function.

Corollary 2.6.2. *Asume $\{\alpha_J\}_{J \subset \mathbb{N}}$ and $\{\beta_J\}_{J \subset \mathbb{N}}$ are two families of real numbers, indexed by subsets of $\mathbb{N}$, such that there holds for $I \subset \mathbb{N}$*

$$\beta_I = \sum_{\mathcal{J} \in \pi(I)} \prod_{J \in \mathcal{J}} \alpha_J.$$

Then

$$\alpha_{\{1,\ldots,n\}} = \sum_{\mathcal{J} \in \pi(\{1,\ldots,n\})} (-1)^{|\mathcal{J}|-1} (|\mathcal{J}| - 1)! \prod_{J \in \mathcal{J}} \beta_J,\ n \in \mathbb{N},$$

where $\pi(\{1, \ldots, n\})$ denotes the set of partitions of the set $\{1, \ldots, n\}$ and $|\mathcal{J}|$ denotes the number of elements in the partition $\mathcal{J}$.

Proof. Take $X = \pi(\{1, \ldots, n\})$ and let $\mathcal{J} \preceq \mathcal{J}'$ if $\mathcal{J}$ is a refinement of $\mathcal{J}'$. It is easily seen that $(X, \preceq)$ is a partially ordered set with zero. We define two functions on X as follows:

$$g(\mathcal{J}) = \prod_{J \in \mathcal{J}} \beta_J \text{ and } f(\mathcal{J}) = \prod_{J \in \mathcal{J}} \alpha_J.$$

They satisfy the relation (2.8). In [5] Brender and Goldmann have computed the Möbius function of $(X, \preceq)$. In particular they obtain

$$\mu(\mathcal{J}, \{\{1, \ldots, n\}\}) = (-1)^{|\mathcal{J}|-1} (|\mathcal{J}| - 1)!.$$

□

Chapter 3

Point Processes with a Signed Lévy Pseudo Measure

Here we present the basic theorem 3.1.3 which yields the existence of the random KMM process (or shortly) KMM process for L. Specializing L will provide us later with many examples from classical and quantum statistical mechanics. It is formulated in terms of the cluster expansion method. Here we add a comparison with the approach of Malyshev and Minlos [40] to the CEM in case of point processes. Proposition 3.1.6 gives a new criterium for simplicity of a KMM process $\Im_L$.

3.1 Existence

For the moment let us consider L to be a finite signed measure on $\mathcal{M}^{\cdot\cdot}(X)$, that is $L = L^+ - L^-$ with $|L| \in \mathcal{W}$ and $|L|(\mathcal{M}^{\cdot\cdot}(X)) < \infty$. Then the question of existence of $\Im_L$ is equivalent to the problem if L is the logarithm of a point process P as is seen by

Remark 3.1.1. *Let P be a point process. Then the following statements are equivalent:*
(i) There exists some finite signed measure K on $\mathcal{M}^{\cdot\cdot}(X)$ with

$$P = \exp(K) := \sum_{n \geq 0} \frac{1}{n!} K^{*n} \ \textit{with}\ K^0 = \delta_{\mathbf{0}}.$$

(ii) There exists some finite signed measure L on $\mathcal{M}^{\cdot\cdot}(X)$ with $L(\{\mathbf{0}\}) = 0$ such that

$$P = \Im_L.$$

Proof. "$(i) \Rightarrow (ii)$" Let us introduce $Y = \mathcal{M}^{\cdot\cdot}(X) \setminus \{0\}$ and define $L = K(\cdot \cap Y)$ then

$$\mathcal{L}_P(f) = \exp(L + (K(\mathcal{M}^{\cdot\cdot}(X)) - K(Y))\,\delta_0)(e^{-\zeta_f}) = \exp(-L(\mathbf{1} - e^{-\zeta_f})).$$

The last equality follows by observing that $K(\mathcal{M}^{\cdot\cdot}(X)) = 0$, because otherwise we would get a contradiction to $P(\mathcal{M}^{\cdot\cdot}(X)) = 1$.
"$(ii) \Rightarrow (i)$" Define $K = L - L(Y)\delta_0$ then

$$P(e^{-\zeta_f}) = \exp(L(e^{-\zeta_f}) - L(Y)) = \exp(K(e^{-\zeta_f})) = \exp(K)(e^{-\zeta_f}).$$

□

We will restrict our investigation to measures $L^+, L^- \in \mathcal{W}$ which are concentrated on $\mathcal{M}^{\cdot\cdot}_f(X)$ and which can be represented as follows

$$L^\epsilon(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n} \int_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \Theta^\epsilon_n(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_n), \qquad (3.1)$$

for all $\varphi \in F_+(\mathcal{M}^{\cdot\cdot}_f(X))$. Here Θ^ϵ_n is a non negative measure on X^n. Certainly they can always chosen to be symmetric. Only in section 4.5 we will start with a family Θ^ϵ_n, which has no more symmetry properties than invariance under cyclic permutations. At the end of this section we discuss in more detail this problem of symmetry of the Θ^ϵ_n. We also introduce

$$\Theta_n = \Theta^+_n - \Theta^-_n.$$

We shall call $\{\Theta_n\}_{n=1}^\infty$ the family of *cumulant measures*. We will see that under the condition $|L|(\mathbf{1} - e^{-\zeta_f}) < \infty$ for $f \in F_{bc,+}(X)$, $|\Theta_n| = \Theta^+_n + \Theta^-_n$ is a locally finite (or Radon) measure on X^n. We refer to remark 1.1.1 on page 19 for the used terminology. Remark that Θ_n is only a well defined finite signed measure if restricted to the bounded sets of X^n. Θ_n evaluated for unbounded sets might lead to undefined expressions like $\infty - \infty$. Such objects will be called *signed Radon measures* in the sequel.

Let us agree on the following convention. If we say, we are considering a signed cluster pseudo measure of the form

$$L(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n} \int_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \vartheta(x_1, \ldots, x_n)\, \lambda(\mathrm{d}\,x_1) \ldots \lambda(\mathrm{d}\,x_n), \quad (3.2)$$

where $\varphi \in \mathfrak{L}^1(|L|)$, $\vartheta : \sqcup_{n=1}^\infty X^n \mapsto \mathbb{R}$ is a real valued measurable function and λ is some non negative measure on X, then it is always understood that we have made the canonical choice

$$\Theta^\epsilon_n(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_n) = \vartheta_\epsilon(x_1, \ldots, x_n)\, \lambda(\mathrm{d}\,x_1) \ldots \lambda(\mathrm{d}\,x_n),$$

where $\vartheta_\epsilon = \max\{\epsilon\,\vartheta, 0\}$ are the positive $\epsilon = +1$ resp. negative $\epsilon = -1$ part of ϑ. The following combinatorial result is a direct consequence of Ruelle's algebraic approach ([63], chapter 4, eqn. (4.14)). It can also be found in the book [72] of Stanley, corollary 5.1.6, where it is called the exponential formula.

Lemma 3.1.2. *For a sequence h_k of real numbers, such that the series on the below left hand side converges absolutely, the series on the below right hand side converges absolutely and we have*

$$\exp\left[\sum_{k=1}^{\infty}\frac{h_k}{k!}\right] = 1 + \sum_{k=1}^{\infty}\frac{1}{k!}\sum_{\mathcal{J}\in\pi([k])}\prod_{J\in\mathcal{J}} h_{|J|}.$$

Here $\pi([k])$ denotes the set of all partitions of the set $[k] = \{1,\dots,k\}$.

Now we can formulate our main existence result.

Theorem 3.1.3. *Let $L^+, L^- \in \mathcal{W}$ be of the form (3.1). Then $|\Theta_n|$ has to be a Radon measure. Furthermore we assume that the following signed measures*

$$\varrho_k(\otimes_{j=1}^k f_j) = \sum_{\sigma\in S_k}\prod_{\omega\in\sigma}\Theta_{\ell(\omega)}(\underset{i\in\omega}{\otimes} f_i),\ \ f_1,\dots,f_k \in F_{bc,+}(X). \tag{3.3}$$

are actually non negative Radon measures. Here the above product has to be taken over all cycles ω in the permutation σ and $\ell(\omega)$ denotes the cycle length. Then there exists a point process $\Im_L$ such that $\Im_{L^+} = \Im_{L^-} * \Im_L$ or equivalently $\mathcal{L}_{\Im_L} = \mathcal{K}_L$ on $F_{bc,+}(X)$.

Proof. Let $\Lambda \in \mathcal{B}_0(X)$ and $\mathcal{M}^{\cdot\cdot}(\Lambda) = \{\mu \in \mathcal{M}^{\cdot\cdot}(X)|\,\mathrm{supp}(\mu) \subset \Lambda\}$. The method of the proof will be to investigate the restriction

$$L_\Lambda(\varphi) = L(\mathbf{1}_{\mathcal{M}^{\cdot\cdot}(\Lambda)}\varphi) \text{ for } \varphi \in \mathfrak{L}^1(|L|_\Lambda)$$

of L to point measures in Λ. By a combinatorial argument (i.e. lemma 3.1.2) we will then observe that there exist finite point processes Q_Λ in Λ such that L_Λ is the signed Lévy pseudo measure of Q_Λ. As $\Lambda \uparrow X$ we will see that $\mathcal{K}_{L_\Lambda} \to \mathcal{K}_L$. By using theorem 2.3.3 we will obtain the assertion.

Let us first observe that L_Λ is a finite signed measure on $\mathcal{M}^{\cdot\cdot}_f(X)$:

$$|L_\Lambda|(\mathbf{1}) = 2\,|L|(\mathbf{1}_{\mathcal{M}^{\cdot\cdot}(\Lambda)}\frac{1}{2}) \le 2\,|L|(\mathbf{1} - e^{-\zeta_\Lambda}) < \infty.$$

For the first inequality observe that $\zeta_\Lambda \ge 1$ on $\mathcal{M}^{\cdot\cdot}(\Lambda)$, $|L|$-a.e. and $\frac{1}{2} \le 1 - e^{-x}$ for $x \ge 1$. In particular $|L_\Lambda|(\mathbf{1}) < \infty$ implies that Θ_n^ϵ are Radon

measures.
Now set $\Xi(\Lambda) := \exp[L_\Lambda(\mathbf{1})]$, the so called *partition function*. Thus we have

$$\mathcal{K}_{L_\Lambda}(f) = \frac{1}{\Xi(\Lambda)} \exp[\mathcal{L}_{L_\Lambda}(f)],\ f \in F_{bc,+}(X),$$

where

$$\mathcal{L}_{L_\Lambda}(f) = \sum_{n=1}^{\infty} \frac{\Theta_n^+((\mathbf{1}_\Lambda e^{-f})^{\otimes n}) - \Theta_n^-((\mathbf{1}_\Lambda e^{-f})^{\otimes n})}{n}.$$

The above sum converges absolutely due to $|L_\Lambda|(\mathbf{1}) < \infty$. So choose in lemma 3.1.2 $h_n = (n-1)!\,\Theta_n((\mathbf{1}_\Lambda e^{-f})^{\otimes n})$ and combine it with the fact that $|S_n^{cy}| = (n-1)!$, here S_n^{cy} is the set of permutations of the set $[n]$, which consists of one cycle, then

$$\exp[\mathcal{L}_{L_\Lambda}(f)] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{\omega \in \sigma} \Theta_{\ell(\omega)}((\mathbf{1}_\Lambda e^{-f})^{\otimes \ell(\omega)}).$$

Inserting the definition (3.3) of the measures ϱ_n leads to

$$\mathcal{K}_{L_\Lambda}(f) = \frac{1}{\Xi(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} e^{-f(x_1)} \dots e^{-f(x_n)}\, \varrho_n(\mathrm{d}\, x_1 \dots \mathrm{d}\, x_n),\ f \in F_{bc,+}(X).$$

So we have identified $\mathcal{K}_{L_\Lambda}$ as the Laplace functional of the finite point process

$$Q_\Lambda(\varphi) = \frac{1}{\Xi(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \varphi(\delta_{x_1} + \dots + \delta_{x_n})\, \varrho_n(\mathrm{d}\, x_1 \dots \mathrm{d}\, x_n),\ \varphi \in F_+(\mathcal{M}_f^{\cdot\cdot}(X)).$$

Now let us check that the assumptions of Mecke's theorem 2.3.3 are fulfilled. Since

$$|(L - L_\Lambda)(\mathbf{1} - e^{-\zeta_f})| \le |L|((\mathbf{1} - \mathbf{1}_{\mathcal{M}^{\cdot\cdot}(\Lambda)})(\mathbf{1} - e^{-\zeta_f})) \downarrow 0 \text{ as } \Lambda \uparrow X,$$

by dominated convergence, we obtain $\mathcal{L}_{Q_\Lambda} \to \mathcal{K}_L$ as $\Lambda \uparrow X$ on $F_{bc,+}(X)$. Similarly one can establish continuity of $\mathcal{K}_L$ at zero. Let $u_n \in F_{bc,+}(X)$ with $u_n \downarrow 0$ then

$$|L(\mathbf{1} - e^{-\zeta_{u_n}})| \le |L|(\mathbf{1} - e^{-\zeta_{u_n}}) \downarrow 0 \text{ as } \Lambda \uparrow X$$

which is again justified by dominated convergence. So theorem 2.3.3 gives us the existence of a law $\Im_L$ on $\mathcal{M}(X)$ such that $\mathcal{L}_{\Im_L} = \mathcal{K}_L$ on $F_{bc,+}(X)$. Now since $\Im_L$ is the weak limit of the Q_Λ we have $\Im_L(\mathcal{M}^{\cdot\cdot}(X)) = 1$, because the set of point processes is closed with respect to weak convergence (see [28], page 32). □

Furthermore, $\Im_L$ is called here the *random KMM process (or shortly KMM) process for L.* These are the initials of Kerstan, Matthes and Mecke respectively.

Remark that $\Im_L$ does only depend on the difference $\Theta_n = \Theta_n^+ - \Theta_n^-$. If we have found another family $\{\tilde{\Theta}_n^\epsilon\}_{n=1}^\infty$, such that the assumptions of theorem 3.1.3 are satisfied and Θ_n coincides with $\tilde{\Theta}_n$ on bounded sets then $\Im_L = \Im_{\tilde{L}}$ is implied by $Q_\Lambda = \tilde{Q}_\Lambda$ for $\Lambda \in \mathcal{B}_0(X)$.

Remark 3.1.4. *In the proof of theorem 3.1.3 we have constructed finite point processes Q_Λ such that*

$$\Im_{L_\Lambda^+} = \Im_{L_\Lambda^-} * Q_\Lambda.$$

Furthermore we have shown $\Im_{L_\Lambda^\epsilon} \Rightarrow \Im_{L^\epsilon}$ weakly as $\Lambda \uparrow X$, where $\epsilon \in \{+1, -1\}$. Now using a theorem of Matthes et al. ([31], proposition 3.2.9.), which says that if we have three sequences of point processes $(V_n)_n$, $(Q_n)_n$ and $(P_n)_n$ such that $(V_n)_n$ respective $(Q_n)_n$ converge weakly to some point process V respective Q and $V_n = Q_n * P_n$, then also $(P_n)_n$ converges weakly to some point process P and we have $V = Q * P$, we can conclude that there exists a point process P, the weak limit of the Q_Λ such that

$$\Im_{L^+} = \Im_{L^-} * P.$$

Thus this gives an alternative to theorem 2.3.3 for establishing the existence of the thermodynamic limit.

Remark 3.1.5. *Since $1 - e^{-x} \leq x$ we have $|L|(\mathbf{1} - e^{-\zeta_f}) \leq \nu^1_{|L|}(f)$, $f \in F_{bc,+}(X)$. So $|L|$ being of first order is a sufficient condition for $|L| \in \mathcal{W}$. In all upcoming examples this will be the case.*

Let us give a sufficient condition for the simplicity of $\Im_L$.

Proposition 3.1.6. *Assume that for some measurable function $\vartheta : \sqcup_{n=1}^\infty X^n \to \mathbb{R}$ and $\lambda \in \mathcal{M}^\circ(X)$, i.e. λ is a diffuse Radon measure,*

$$L(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n} \int_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \vartheta(x_1, \ldots, x_n)\, \lambda(\mathrm{d}\, x_1) \ldots \lambda(\mathrm{d}\, x_n),$$

for $\varphi \in \mathfrak{L}^1(|L|)$, is a signed Lévy pseudo measure such that L^+ is of first order (we take for L^+, L^- the canonical choice as in (3.2)). Then $\Im_L$ is a simple point process.

Proof. The result is an immediate consequence from [31] proposition 2.2.9. which says that an infinitely divisible point process $\Im_H$ is simple if and only if $H(\mathcal{M}^{\cdot\cdot}(X) \setminus \mathcal{M}^{\cdot}(X)) = 0$ and $H(\{\zeta_{\{x\}} > 0\}) = 0$ for all $x \in X$. Now it is well known that for a diffuse λ the product λ^n is concentrated on $\grave{X}^n = \{(x_1, \ldots, x_n) \in X^n | \, x_i \neq x_j \text{ for } i \neq j\}$ (see [34] theorem 3), which yields $L^+(\mathcal{M}^{\cdot\cdot}(X) \setminus \mathcal{M}^{\cdot}(X)) = 0$. Moreover for $f \in F_{bc,+}(X)$

$$\nu^1_{L^+}(f) = \int_X f(x) \sum_{n=1}^{\infty} \int_{X^{n-1}} \vartheta_+(x, x_2, \ldots, x_n) \lambda(\mathrm{d}\, x_2) \ldots \lambda(\mathrm{d}\, x_n)\, \lambda(\mathrm{d}\, x).$$

So $\nu^1_{L^+}$ is a diffuse Radon measure and we obtain $L^+(\{\zeta_{\{x\}} > 0\}) \leq L^+(\zeta_{\{x\}}) = 0$. Since $\Im_{L^+}$ is simple the equation $\Im_{L^+} = \Im_{L^-} * \Im_L$ forces $\Im_{L^-}$ and $\Im_L$ to be simple. □

Symmetry of Measures

Lemma 3.1.7. *Let ϱ_k be symmetric signed Radon measures on X^k and let α be a real valued function on $\cup_{k \geq 1} \pi([k])$ then*

$$\Theta_k(\otimes_{j=1}^k f_j) = \sum_{\mathcal{J} \in \pi([k])} \alpha(\mathcal{J}) \prod_{J \in \mathcal{J}} \varrho_{|J|}(\otimes_{j \in J} f_j), \quad f_1, \ldots, f_k \in F_{bc,+}(X)$$

are symmetric signed Radon measures.

Proof. For the symmetry it suffices to show that Θ_k remains invariant under transpositions. For the sake of clearness of notation let us take the transposition (1 2). We also abbreviate $G(\mathcal{J}) := \alpha(\mathcal{J}) \prod_{J \in \mathcal{J}} \varrho_{|J|}(\otimes_{j \in J} f_j)$. Let us now decompose the sum over all partitions in two parts. First we sum over all partitions of $[k]$ such that 1 and 2 lie in the same set. The symmetry of the ϱ_j then implies that this sum remains invariant under transposition of 1 and 2. So let us consider the sum where 1 and 2 lie in different sets, this is indicated by the * in the below sum

$$\sum_{\mathcal{J} \in \pi([k])}^{*} G(\mathcal{J}) = \sum_{\mathcal{J} \in \pi(\{3,\ldots,k\})} \sum_{i \neq j} G(\mathcal{J}_{i,j}) + \sum_j (G(\{\{1\}, \mathcal{J}_j^2\}) + G(\{\{2\}, \mathcal{J}_j^1\}) + G(\{\{1\}, \{2\}, \mathcal{J}\}).$$

Here for $\mathcal{J} \in \pi(\{3, ..., k\})$ let us give it some arbitrary numbering $\mathcal{J} = \{J_1, ..., J_{|\mathcal{J}|}\}$ then $\mathcal{J}_{i,j}$ denotes the partition of $[k]$ such that we replace J_i by $J_i \cup \{1\}$ and J_j by $J_j \cup \{2\}$. And $\mathcal{J}_j^1$ denotes the partition where we have added 1 to the j-th set of $\mathcal{J}$, analogously for $\mathcal{J}_j^2$. All three inner sums remain invariant under the transposition of 1 and 2. □

This gives us

$$\text{symmetry of } \{\varrho_k\}_{k\geq 1} \Leftrightarrow \text{symmetry of } \{\Theta_k\}_{k\geq 1}.$$

3.2 The Cluster Expansion Method (*CEM*)

We recall shortly the *CEM*

In theorem 3.1.3 we started with a family of signed Radon measures $\{\Theta_n\}_{n=1}^{\infty}$, the cumulant measures and obtained the family of Schur measures $\{\varrho_k\}_{k=1}^{\infty}$ by means of (3.3). We then say that the family of Schur measures admits a *cluster representation* in terms of the cumulant measures. Certainly (3.3) gives us a duality between the Schur and cumulant measures. If we prescribe a family of non negative symmetric Radon measures $\{\varrho_k\}_{k=1}^{\infty}$ then we obtain the cumulant measures by a Möbius inversion as described in section 2.6

$$\Theta_n(\otimes_{j=1}^n f_j) = \frac{1}{(n-1)!} \sum_{\mathcal{J}\in\pi([n])} (-1)^{|\mathcal{J}|-1} (|\mathcal{J}|-1)! \prod_{J\in\mathcal{J}} \varrho_{|J|}(\otimes_{j\in J} f_j), \quad (3.4)$$

where $f_1,\dots,f_n \in F_{bc,+}(X)$. For the existence of the limiting point process $\Im_L$ according to theorem 3.1.3 it remains to check that the signed cluster pseudo measure L, built on the cumulant measures defined in (3.4) and for some choice of Θ_n^+, Θ_n^-, satisfies $|L|(1-e^{-\zeta_f}) < \infty$, $f \in F_{bc,+}(X)$. We say that (3.4) is the *dual cluster representation* of the cumulant measures in terms of the Schur measures. In the upcoming examples it will always be the case that the Schur measures are of the form $\varrho_k = \psi(x_1,\dots,x_n)\lambda(\mathrm{d}\,x_1)\dots\lambda(\mathrm{d}\,x_k)$ for some symmetric measurable function $\psi : \sqcup_{n=0}^{\infty} X^n \mapsto [0,\infty)$. So with the help of the dual cluster representation we obtain that the corresponding signed cluster pseudo measure has the form (3.2) with

$$\vartheta = \frac{1}{(n-1)!} \sum_{\mathcal{J}\in\pi([n])} (-1)^{|\mathcal{J}|-1} (|\mathcal{J}|-1)! \prod_{J\in\mathcal{J}} \psi((x_j)_{j\in J}). \quad (3.5)$$

The *method of cluster expansion* thus consists in defining first the signed cluster pseudo measure L by means of the given data $\{\Theta_n\}_{n=1}^{\infty}$ respectively $\{\varrho_k\}_{k=1}^{\infty}$; and next, to construct with their help the local processes $\{Q_\Lambda\}_{\Lambda\in\mathcal{B}_0(X)}$. (This construction reflects the cluster structure.) The limiting process is obtained under the condition that $|L|(1-e^{-\zeta_f}) < \infty$, $f \in F_{bc,+}(X)$. If furthermore $|L|$ is of first order it is the unique solution to the cluster equation $(\mathbf{\Sigma}_L)$, see theorem 3.3.1.

Comparison to the Approach of Malyshev and Minlos

Here we give some relations to the work [40] of Malyshev and Minlos. They are in the following setting: Let $\lambda \in \mathcal{M}^\circ(X)$ be a diffuse Radon measure and $\vartheta : \mathcal{M}_f^{\cdot\cdot}(X) \setminus \{\mathbf{0}\} \mapsto \mathbb{R}$ a measurable function. Moreover, let the cumulant measures be of the form

$$\Theta_n(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n) = \frac{1}{(n-1)!}\vartheta(x_1, \ldots, x_n)\, \lambda(\mathrm{d}\, x_1) \ldots \lambda(\mathrm{d}\, x_n), \tag{3.6}$$

where we have used that ϑ can be thought of as a symmteric function on $\sqcup_{n=1}^{\infty} X^n$. If we introduce the following measure Π on $\mathcal{M}_f^{\cdot\cdot}(X)$

$$\Pi(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \lambda(\mathrm{d}\, x_1) \ldots \lambda(\mathrm{d}\, x_n),\ \varphi \in F_+(\mathcal{M}_f^{\cdot\cdot}(X)),$$

the signed cluster pseudo measure can be written as

$$L(\varphi) = \int_{\mathcal{M}_f^{\cdot\cdot}(X)\setminus\{0\}} \varphi(\mu)\, \vartheta(\mu)\, \Pi(\mathrm{d}\, \mu) \text{ for } \varphi \in \mathfrak{L}^1(|L|). \tag{3.7}$$

Malyshev and Minlos impose the following condition on ϑ. For any $\Lambda \in \mathcal{B}_0(X)$ there holds

$$\int_{\mathcal{M}^{\cdot\cdot}(\Lambda)\setminus\{0\}} \Pi(\mathrm{d}\, \mu) \int_{\mathcal{M}_f^{\cdot\cdot}(X)} \Pi(\mathrm{d}\, \eta)\, |\vartheta(\mu + \eta)| < \infty. \tag{3.8}$$

Then they can show that there exists a weak limit of the Q_Λ, the local Gibbs modifications, as $\Lambda \uparrow X$ (see [40], chapter 3, theorem 2). Condition (3.8) can also be formulated in terms of the factorial moment measures of $|L|$. The following representation will be shown in example 4.4.2: For $\Lambda \in \mathcal{B}_0(X)$

$$\check{\nu}^n_{|L|}(\Lambda^n) = \int_{\Lambda^n} \lambda(\mathrm{d}\, y_1) \ldots \lambda(\mathrm{d}\, y_n) \int_{\mathcal{M}_f^{\cdot\cdot}(X)} \Pi(\mathrm{d}\, \eta)\, |\vartheta(\delta_{y_1} + \ldots + \delta_{y_n} + \eta)| \tag{3.9}$$

So (3.8) can be formulated in terms of $|L|$ as

$$\sum_{n=1}^{\infty} \frac{\check{\nu}^n_{|L|}(\Lambda^n)}{n!} < \infty \text{ for } \Lambda \in \mathcal{B}_0(X).$$

We required only that $\nu^1_{|L|}(\Lambda) = \check{\nu}^1_{|L|}(\Lambda) < \infty$, which is a bit weaker.
Let us remark that it is immediate from the proof of theorem 3.1.3 that the

partition function $\Xi(\Lambda)$ can be either expressed as

$$\log(\Xi(\Lambda)) = \int_{\mathcal{M}^{\cdot\cdot}(\Lambda)\setminus\{0\}} \Pi(\mathrm{d}\,\mu)\,\vartheta(\mu),$$
$$\Xi(\Lambda) = \int_{\mathcal{M}^{\cdot\cdot}(\Lambda)} \Pi(\mathrm{d}\,\mu) \sum_{\mathcal{J}\in\pi([|\mu|])} \prod_{j\in\mathcal{J}} \vartheta((x_j)_{j\in J}).$$

Here we have given each $\mu = \delta_{x_1} + ... + \delta_{x_{|\mu|}}$ some arbitrary numbering. Malyshev and Minlos call the second identity a cluster representation of the partition function.
We remark that, due to proposition 3.1.6, all point processes constructed in this section are simple.

3.3 The Cluster Equation

Theorem 3.3.1 shows that the condition that the variation $|L|$ is of first order implies that the KMM process $\Im_L$ has this property too, and furthermore is a solution of the cluster equation $(\mathbf{\Sigma}_L)$. We saw already before that $\Im_L$ is a solution of the equation $\mathcal{L}_{\Im_L} = \mathcal{K}_L$. As an important corollary for later applications we obtain that the local process Q_Λ converges not only weakly to $\Im_L$ as $\Lambda \uparrow X$ but even in the sense that $\mathcal{C}_{Q_\Lambda} \longrightarrow \mathcal{C}_{\Im_L}$.

Theorem 3.3.1. *Let L be a signed Lévy pseudo measure such that $|L|$ is of first order. Then $\Im_L$ is of first order and solves the cluster equation*

$$(\mathbf{\Sigma}_L) \qquad C_P(h) + C_{L^-} \star P(h) = C_{L^+} \star P(h),\;\; h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X)).$$

Here the $\star$ operation is as defined in (2.5).

Proof. By using the representation of the Laplace transform of $\Im_L$ and applying two times $1 - e^{-x} \leq x$ for $x \in \mathbb{R}$, we obtain for $f \in F_{bc,+}(X)$ and $s > 0$

$$\Im_L\left(\frac{\mathbf{1} - e^{-s\zeta_f}}{s}\right) = \frac{1 - e^{-L(\mathbf{1}-e^{-s\zeta_f})}}{s} \leq |L|(\zeta_f) = \nu^1_{|L|}(f).$$

So if we use the lemma of Fatou then we get

$$\nu^1_{\Im_L}(f) = \Im_L(\zeta_f) = \Im_L\left(\liminf_{s\downarrow 0} \frac{\mathbf{1} - e^{-s\zeta_f}}{s}\right) \leq \nu^1_{|L|}(f).$$

Now it is well known (see [44]) that if P is a point process of first order (this allows to interchange integration and differentiation below) then for $f, g \in F_{bc,+}(X)$ we have

$$C_P(f \otimes e^{-\zeta_g}) = -\frac{\mathrm{d}}{\mathrm{d}\, s}\mathcal{L}_P(sf+g)\bigg|_{s=0}.$$

So we have to compute

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}\, s}\mathcal{L}_{\Im_L}(sf+g) &= \mathcal{L}_{\Im_L}(sf+g)\,\frac{\mathrm{d}}{\mathrm{d}\, s}L(\mathbf{1}-e^{-\zeta_{sf+g}}) \\ &= \mathcal{L}_{\Im_L}(sf+g)L(\zeta_f e^{-\zeta_g}) \\ &= C_L(f\otimes e^{-\zeta_g})\mathcal{L}_{\Im_L}(sf+g). \end{aligned}$$

Again the second equality holds since we are allowed to interchange differentiation and integration with respect to L since $|L|$ is of first order. Also we have abbreviated $C_L(f \otimes e^{-\zeta_g}) = C_{L^+}(f \otimes e^{-\zeta_g}) - C_{L^-}(f \otimes e^{-\zeta_g})$. We arrive at $C_{\Im_L}(f \otimes e^{-\zeta_g}) = C_L \star \Im_L(f \otimes e^{-\zeta_g})$, which certainly can be brought into the form $(\mathbf{\Sigma}_L)$. Now lemma 2.2.1 yields that $(\mathbf{\Sigma}_L)$ holds true for all $h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X))$. □

In the following we shall see that besides the weak convergence $Q_\Lambda \Rightarrow \Im_L$ as $\Lambda \uparrow X$, we also have convergence of the Campbell measures for a sufficiently large class of test functions.

Corollary 3.3.2. *Let $f, g \in F_{bc,+}(X)$ and assume that L is a signed Lévy pseudo measure such that $|L|$ is of first order. Then we have for $h = f \otimes e^{-\zeta_g}$*

$$C_{Q_\Lambda}(h) \to C_{\Im_L}(h) \text{ as } \Lambda \uparrow X.$$

Proof. Note that $C_{Q_\Lambda}(h) = C_{L_\Lambda}(h)\mathcal{L}_{Q_\Lambda}(g)$, since Q_Λ is a solution to $(\mathbf{\Sigma}_{L_\Lambda})$. Similarly we have $C_{\Im_L}(h) = C_L(h)\mathcal{L}_{\Im_L}(g)$. Since $\mathcal{L}_{Q_\Lambda}(g) \to \mathcal{L}_{\Im_L}(g)$ is already established, $C_{L_\Lambda}(h) \to C_L(h)$ remains to be seen. But

$$|(C_L - C_{L_\Lambda})(h)| = |(L-L_\Lambda)(\zeta_f e^{-\zeta_g})| \quad \leq |L|((\mathbf{1}-\mathbf{1}_{\mathcal{M}^{\cdot\cdot}(\Lambda)})\zeta_f e^{-\zeta_g}) \downarrow 0,$$

as $\Lambda \uparrow X$, by dominated convergence. □

We remark that the result remains valid if $f \in F_{bc,+}(X)$ is replaced by $f \in F_+(X)$ satisfying $\nu^1_{|L|}(f) < \infty$. The converse to theorem 3.3.1 is valid, too:

Proposition 3.3.3. *Let L be a signed cluster pseudo measure such that $|L|$ is of first order and let P be a solution to $(\mathbf{\Sigma}_L)$. Then L is a signed Lévy pseudo measure with $P = \Im_L$.*

Proof. Assume P solves $(\mathbf{\Sigma}_L)$. We certainly have $C_P(h) \leq C_{|L|} \star P(h)$ for all $h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X))$. This implies $\nu^1_{\Im_L} \leq \nu^1_{|L|}$ on $F_+(X)$. So if $|L|$ is of first order then also P is. Thus for $f \in F_{bc,+}(X)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,s}\mathcal{L}_P(sf) &= \frac{\mathrm{d}}{\mathrm{d}\,s}P(e^{-s\zeta_f}) = -P(\zeta_f e^{-s\zeta_f}) = -C_P(f \otimes e^{-s\zeta_f}) \\ &= -(C_L \star P)(f \otimes e^{-s\zeta_f}) = -C_L(f \otimes e^{-s\zeta_f})\ \mathcal{L}_P(sf).\end{aligned}$$

Therefore $s \mapsto \mathcal{L}_P(sf)$ satisfies a first order differential equation with initial value $\mathcal{L}_P(0) = 1$. An integration yields that $\mathcal{L}_P(f) = \mathcal{K}_L(f)$, so $P = \Im_L$. $\square$

Chapter 4

Palm and Moment Measures

We first present a decomposition of the moment measures of a point process or, more generally, of a measure on $\mathcal{M}^{\cdot\cdot}(X)$ of order k into its factorial moment measures. One of the main lemmata will be in the sequel the representation 4.1.3. It will serve as a tool to check the non-negativity of the Schur measures in concrete situations.

In section 4.2 our construction method is used for the first time to prove the existence of the so called Pólya sum and difference processes.

Furthermore, a construction of permanental and determinantal point processes is given by means of theorem 3.1.3 . Special examples covered by this result are the ideal Bose gas of Fichtner [16] and the 1-dimensional Fermion gas of Macchi [38].

4.1 Decomposition of Moment Measures

For $\mu \in \mathcal{M}^{\cdot\cdot}(X)$ and $\epsilon \in \{-1, +1\}$ let us introduce

$$\mu^{\epsilon[k]} = \mu(\mathrm{d}\, x_1)(\mu + \epsilon\delta_{x_1})(\mathrm{d}\, x_2) \dots (\mu + \epsilon \sum_{j=1}^{k-1} \delta_{x_j})(\mathrm{d}\, x_k). \tag{4.1}$$

Remark that $\mu^{+[k]}$ is also well defined for $\mu \in \mathcal{M}(X)$ and $\mu^{\epsilon[k]}$ are symmetric measures. The $\mu^{\epsilon[k]}$ are given recursively by:

$$\mu^{\epsilon[k]}(\bigotimes_{j=1}^{k} f_j) = \mu(f_k)\mu^{\epsilon[k-1]}(\bigotimes_{j=1}^{k-1} f_j) + \epsilon \sum_{j=1}^{k-1} \mu^{\epsilon[k-1]}(\bigotimes_{i\in[k-1]\setminus\{j\}} f_i \otimes f_{\{j,k\}}), \tag{4.2}$$

for $f_1, \dots, f_k \in F_{bc,+}(X)$, where we denoted $f_J = \prod_{j\in J} f_j$ for $J \subset [k]$.

Theorem 4.1.1. *Let R be a measure on $\mathcal{M}^{\cdot\cdot}(X)$ of k-th order. Then the family $\{\check{\nu}_R^k\}_k$ of factorial moment measures of R is the unique family of symmetric Radon measures such that they decompose the moment measures $\{\nu_R^k\}_k$ in the following way*

$$\nu_R^k(\bigotimes_{j=1}^{k} f_j) = \sum_{\mathcal{I}\in\pi([k])} \check{\nu}_R^{|\mathcal{I}|}(\bigotimes_{I\in\mathcal{I}} f_I) \text{ for } f_1, \ldots, f_k \in F_{bc,+}(X).$$

Proof. Recall the definition of (factorial) moment measures given in (2.1) section 2.1. It will be sufficient to show by induction

$$\sum_{\mathcal{I}\in\pi([k])} \mu^{-[|\mathcal{I}|]}(\bigotimes_{I\in\mathcal{I}} f_I) = \mu(f_k) \sum_{\mathcal{I}\in\pi([k-1])} \mu^{-[|\mathcal{I}|]}(\bigotimes_{I\in\mathcal{I}} f_I) = \mu(f_1)\ldots\mu(f_k).$$

Taking expectations on both sides of the above equation then yields the result.
But we have

$$\sum_{\mathcal{I}\in\pi([k])} \mu^{-[|\mathcal{I}|]}(\bigotimes_{I\in\mathcal{I}} f_I) = \sum_{\mathcal{I}\in\pi([k-1])} \Bigg[\mu^{-[|\mathcal{I}|+1]}(\bigotimes_{I\in\mathcal{I}} f_I \otimes f_k) + \sum_{H\in\mathcal{I}} \mu^{-[|\mathcal{I}|]}(\bigotimes_{I\in\mathcal{I}\setminus\{H\}} f_I \otimes f_{H\cup\{k\}})\Bigg].$$

Now by (4.2) the above expression in square brackets equals $\mu(f_k)\mu^{-[|\mathcal{I}|]}(\bigotimes_{I\in\mathcal{I}} f_I)$.
The uniqueness is clear. For the symmetry we refer to [28] page 109. □

Remark 4.1.2. *Let $\grave{X}^k$ be the space of all k-tuples with distinct components. If R is concentrated on $\mathcal{M}^{\cdot}(X)$ its factorial moment measures coincide with the restriction of its moment measures to the set $\grave{X}^k$, $\check{\nu}_R^k = 1_{\grave{X}^k}\nu_R^k$.*

Lemma 4.1.3. *We have*

$$\mu^{\epsilon[k]}(\bigotimes_{j=1}^{k} f_j) = \sum_{\mathcal{J}\in\pi([k])} \epsilon^{k-|\mathcal{J}|} \prod_{J\in\mathcal{J}} (|J|-1)!\mu(f_J) = \sum_{\sigma\in S_k} \epsilon^{k-|\sigma|} \prod_{\omega\in\sigma} \mu(f_\omega),$$

where $f_1, \ldots, f_k \in F_{bc,+}(X)$.

Proof. As above the proof will be given by induction. By (4.2) and using the inductive hypothesis we have

$$\mu^{\epsilon[k]}(\bigotimes_{j=1}^{k} f_j)$$
$$= \sum_{\mathcal{J}\in\pi([k-1])} \Bigg(\epsilon^{k-1-|\mathcal{J}|} \prod_{J\in\mathcal{J}} (|J|-1)!\mu(f_J)\mu(f_k)$$
$$+\epsilon \sum_{\mathcal{J}\in\pi([k-1])} \sum_{j=1}^{k-1} \epsilon^{k-1-|\mathcal{J}|}(|H_j|-1)!\mu(f_{H_j\cup\{k\}}) \prod_{J\in\mathcal{J}\setminus\{H_j\}} (|J|-1)!\mu(f_J) \Bigg),$$

where $H_j \in \mathcal{J} \in \pi([k-1])$ is such that $j \in H_j$. Observe now that

$$\sum_{j=1}^{k-1}(|H_j|-1)!\mu(f_{H_j\cup\{k\}}) \prod_{J\in\mathcal{J}\setminus\{H_j\}} (|J|-1)!\mu(f_J)$$
$$= \sum_{H\in\mathcal{J}} |H|!\mu(f_{H\cup\{k\}}) \prod_{J\in\mathcal{J}\setminus\{H\}} (|J|-1)!\mu(f_J).$$

Thus we have established the first equality in lemma 4.1.3. By using $S_l^{cy} \simeq S_{l-1}$ we also obtain

$$\sum_{\sigma\in S_k} \epsilon^{k-|\sigma|} \prod_{\omega\in\sigma} \mu(f_\omega) = \sum_{\mathcal{J}\in\pi([k])} \epsilon^{k-|\mathcal{J}|} \sum_{\omega_1\in S^{cy}_{|J_1|}} \cdots \sum_{\omega_{|\mathcal{J}|}\in S^{cy}_{|J_{|\mathcal{J}|}|}} \mu(f_{\omega_1})\dots\mu(f_{\omega_{|\mathcal{J}|}})$$
$$= \sum_{\mathcal{J}\in\pi([k])} \epsilon^{k-|\mathcal{J}|} \prod_{J\in\mathcal{J}} (|J|-1)!\mu(f_J).$$

□

Corollary 4.1.4. *Let R be a measure on $\mathcal{M}^{\cdot\cdot}(X)$ then there holds*

$$\check{\nu}_R^k(\bigotimes_{j=1}^{k} f_j) = \sum_{\mathcal{I}\in\pi([k])} (-1)^{k-|\mathcal{I}|}\, \nu_R^{|\mathcal{I}|}(\bigotimes_{I\in\mathcal{I}} f_I) \prod_{I\in\mathcal{I}}(|\mathcal{I}|-1)!$$
$$= \sum_{\sigma\in S_k} (-1)^{k-|\sigma|}\nu_R^{|\sigma|}(\bigotimes_{\omega\in\sigma} f_\omega).$$

Proof. Taking expectation of the expression in lemma 4.1.3 for $\epsilon = -1$ yields the result. □

4.2 The Pólya Sum Process and the Pólya Difference Process

Let $z \in (0,1)$ and $\lambda \in \mathcal{M}^{\cdot\cdot}(X)$. In case of $\epsilon = +1$ we will also permit $\lambda \in \mathcal{M}(X)$. Now consider the following two families of cumulant measures

$$\Theta_{n,\epsilon}(\overset{n}{\underset{j=1}{\otimes}} f_j) = \epsilon^{n-1} z^n \, \lambda\left(\prod_{j=1}^{n} f_j\right) \text{ for } f_1, \ldots, f_n \in F_{bc,+}(X). \qquad (4.3)$$

Let us check whether the corresponding point process exists, that is whether the conditions of theorem 3.1.3 are satisfied. We have

$$\nu^1_{|L_{\epsilon,z,\lambda}|}(f) = \sum_{n=1}^{\infty} z^n \lambda(f) < \infty \text{ for } f \in F_{bc,+}(X).$$

But if we compare (3.3) and lemma 4.1.3 we see that the Schur measures are given by ($\lambda^{\epsilon[n]}$ is as defined in (4.1))

$$\varrho_{n,\epsilon} = z^n \lambda^{\epsilon[n]},$$

which are non negative.

Theorem 4.2.1. *If $\lambda \in \mathcal{M}^{\cdot\cdot}(X)$ and $0 < z < 1$ then there exists for $\epsilon \in \{-1,+1\}$ exactly one point process $P^{\epsilon}_{z,\lambda}$ which is the KMM process belonging to the signed Lévy pseudo measure $L_{\epsilon,z,\lambda}$ built on the cumulant measures $\{\Theta_{n,\epsilon}\}_{n=1}^{\infty}$. Again in case of $\epsilon = +1$ we also permit $\lambda \in \mathcal{M}(X)$.*

The point process $P^{+}_{z,\lambda}$ corresponding to $\{\Theta_{n,+1}\}_{n=1}^{\infty}$ is the so called *Pólya sum process* and the point process $P^{-}_{z,\lambda}$ corresponding to $\{\Theta_{n,-1}\}_{n=1}^{\infty}$ is the so called *Pólya difference process*. These processes have their origin in the work [75, 50, 51].

4.3 Iterated Cluster Equations

Theorem 4.3.1 will be used to determine factorial moment measures of KMM point processes. The associated corollary 4.3.2 allows the calculation of the moment measures respectively factorial moment measures of the KMM process by means of the corresponding measures of the signed Lévy pseudo measure L.

We will need the iterated cluster equations $(\mathbf{\Sigma}^n_L)$ and $\left(\mathbf{\Sigma}^{!,n}_L\right)$ in section 4.6 to determine the Palm respective reduced Palm kernels of infinitely divisible point processes. Moreover $(\mathbf{\Sigma}^n_L)$ will be needed later to show the Gibbs property of certain point processes in section 5.2.

Theorem 4.3.1. *Assume that $|L|$ is of first order. Let $n \geq 1$ and $f_1, \ldots, f_n, g \in F_+(X)$ such that $\nu^k_{|L|}(\max_{j\in[n]}\{f_j\}^{\otimes k}) < \infty$ for all $1 \leq k \leq n$ then $\nu^n_{\Im_L}(\max_{j\in[n]}\{f_j\}^{\otimes n}) < \infty$ and we have*

$$(\boldsymbol{\Sigma}^n_L) \qquad C^n_{\Im_L}(\bigotimes_{j=1}^n f_j \otimes e^{-\zeta_g}) = \mathcal{L}_{\Im_L}(g) \sum_{\mathcal{J}\in\pi([n])} \prod_{J\in\mathcal{J}} C^{|J|}_L(\bigotimes_{j\in J} f_j \otimes e^{-\zeta_g})$$

$$(\check{\boldsymbol{\Sigma}}^n_L) \qquad \check{C}^n_{\Im_L}(\bigotimes_{j=1}^n f_j \otimes e^{-\zeta_g}) = \mathcal{L}_{\Im_L}(g) \sum_{\mathcal{J}\in\pi([n])} \prod_{J\in\mathcal{J}} \check{C}^{|J|}_L(\bigotimes_{j\in J} f_j \otimes e^{-\zeta_g})$$

$$\left(\boldsymbol{\Sigma}^{!,n}_L\right) \qquad C^{!,n}_{\Im_L}(\bigotimes_{j=1}^n f_j \otimes e^{-\zeta_g}) = \mathcal{L}_{\Im_L}(g) \sum_{\mathcal{J}\in\pi([n])} \prod_{J\in\mathcal{J}} C^{!,|J|}_L(\bigotimes_{j\in J} f_j \otimes e^{-\zeta_g}).$$

Proof. The proof will be given by induction. Let us start by establishing $(\boldsymbol{\Sigma}^n_L)$. The case $n = 1$ has been dealt with in theorem 3.3.1. Denote $f_{\max} = \max_{j\in[n]} f_j$. Now let us assume that $\nu^k_{\Im_L}(f^{\otimes k}_{max}) < \infty$ and $\left(\boldsymbol{\Sigma}^k_L\right)$ holds for $1 \leq k \leq n-1$.

$$\begin{aligned}
\nu^n_{\Im_L}(f^{\otimes n}_{\max}) &= C_{\Im_L}(f_{\max} \otimes \zeta^{n-1}_{f_{\max}}) \\
&\leq C_{|L|} \star \Im_L(f_{\max} \otimes \zeta^{n-1}_{f_{\max}}) \\
&= \sum_{B\subset\{2,\ldots,n\}} \nu^{|B|+1}_{|L|}(f^{\otimes(|B|+1)}_{\max})\, \nu^{|B^c|}_{\Im_L}(f^{\otimes|B^c|}_{\max}) < \infty.
\end{aligned}$$

1. So that

$$C^n_{\Im_L}(\bigotimes_{j=1}^n f_j \otimes e^{-\zeta_g}) = -\frac{\mathrm{d}}{\mathrm{d}\, s} C^{n-1}_{\Im_L}(\bigotimes_{j=1}^{n-1} f_j \otimes e^{-\zeta_{sf_n+g}})\Big|_{s=0}$$

Now $(\boldsymbol{\Sigma}^n_L)$ follows by just using the product rule of differentiation and the fact that

$$C^k_L(\bigotimes_{j=1}^k f_j \otimes e^{-\zeta_g}) = -\frac{\mathrm{d}}{\mathrm{d}\, s} C^{k-1}_L(\bigotimes_{j=1}^{k-1} f_j \otimes e^{-\zeta_{sf_k+g}})\Big|_{s=0} \quad \text{for } 1 \leq k \leq n.$$

Indeed we obtain then

$$\begin{aligned}
C^n_{\Im_L}(\bigotimes_{j=1}^n f_j \otimes e^{-\zeta_g}) &= \mathcal{L}_{\Im_L}(g) C_L(f_n \otimes e^{-\zeta_g}) \sum_{\mathcal{J}\in\pi([n-1])} \prod_{J\in\mathcal{J}} C^{|J|}_L(\bigotimes_{j\in J} f_j \otimes e^{-\zeta_g}) \\
&+ \mathcal{L}_{\Im_L}(g) \sum_{\mathcal{J}\in\pi([n-1])} \sum_{H\in\mathcal{J}} C^{|H|+1}_L(\bigotimes_{j\in H\cup\{n\}} f_j \otimes e^{-\zeta_g}) \prod_{J\in\mathcal{J}\setminus\{H\}} C^{|J|}_L(\bigotimes_{j\in J} f_j \otimes e^{-\zeta_g}).
\end{aligned}$$

2. Now let us show $\left(\check{\boldsymbol{\Sigma}}_L^n\right)$. Due to (4.2) we have

$$\begin{aligned}\check{C}^n_{\Im_L}(\bigotimes_{j=1}^n f_j \otimes e^{-\zeta_g}) &= -\frac{\mathrm{d}}{\mathrm{d}\, s}\check{C}^{n-1}_{\Im_L}(\bigotimes_{j=1}^{n-1} f_j \otimes e^{-\zeta_{sf_n+g}})\Big|_{s=0} \\ &\quad - \sum_{j=1}^{n-1} \check{C}^{n-1}_{\Im_L}(\bigotimes_{i\in[n-1]\setminus\{j\}} f_i \otimes f_{\{j,n\}} \otimes e^{-\zeta_g}) \\ &= T_1 + T_2.\end{aligned}$$

T_1 can be treated as in case of $(\boldsymbol{\Sigma}_L^n)$:

$$\begin{aligned}T_1 =& \mathcal{L}_{\Im_L}(g) C_L(f_n \otimes e^{-\zeta_g}) \sum_{\mathcal{J}\in\pi([n-1])} \prod_{J\in\mathcal{J}} \check{C}_L^{|J|}(\bigotimes_{j\in J} f_j \otimes e^{-\zeta_g}) \\ &+ \mathcal{L}_{\Im_L}(g) \sum_{\mathcal{J}\in\pi([n-1])} \sum_{H\in\mathcal{J}} \int \mu^{-[|H|]}(\bigotimes_{i\in H} f_i)\mu(f_n)e^{-\mu(g)}\, L(\mathrm{d}\,\mu)\times \\ &\times \prod_{J\in\mathcal{J}\setminus\{H\}} \check{C}_L^{|J|}(\bigotimes_{j\in J} f_j \otimes e^{-\zeta_g}) \\ &= T_{11} + T_{12}.\end{aligned}$$

Now as above let us denote by $H_j \in \mathcal{J} \in \pi([n-1])$ the set such that $j \in H_j$. Then we have

$$\begin{aligned}-T_2 &= \sum_{\mathcal{J}\in\pi([n-1])} \sum_{j=1}^{n-1} \check{C}_L^{|H_j|}(\bigotimes_{i\in H_j\setminus\{j\}} f_i \otimes f_{\{j,n\}} \otimes e^{-\zeta_g}) \prod_{J\in\mathcal{J}\setminus\{H_j\}} \check{C}_L^{|J|}(\bigotimes_{i\in J} f_i \otimes e^{-\zeta_g}) \\ &= \sum_{\mathcal{J}\in\pi([n-1])} \sum_{H\in\mathcal{J}} \sum_{j\in H} \check{C}_L^{|H|}(\bigotimes_{i\in H\setminus\{j\}} f_i \otimes f_{\{j,n\}} \otimes e^{-\zeta_g}) \prod_{J\in\mathcal{J}\setminus\{H\}} \check{C}_L^{|J|}(\bigotimes_{i\in J} f_i \otimes e^{-\zeta_g})\end{aligned}$$

But now we have due to (4.2)

$$T_{12} + T_2 = \sum_{\mathcal{J}\in\pi([n-1])} \sum_{H\in\mathcal{J}} \check{C}_L^{|H|+1}(\bigotimes_{i\in H\cup\{n\}} f_i \otimes e^{-\zeta_g}) \prod_{J\in\mathcal{J}\setminus\{H\}} \check{C}_L^{|J|}(\bigotimes_{i\in J} f_i \otimes e^{-\zeta_g}).$$

So we conclude as in case of $(\boldsymbol{\Sigma}_L^n)$.
3. Observe finally that for any measure K on $\mathcal{M}^{\cdot\cdot}(X)$ there holds

$$C_K^{!,n}(\bigotimes_{j=1}^n f_j \otimes e^{-\zeta_g}) = \check{C}_K^n(\bigotimes_{j=1}^n (f_j e^g) \otimes e^{-\zeta_g})$$

and so $\left(\boldsymbol{\Sigma}_L^{!,n}\right)$ straightforwardly follows. □

Corollary 4.3.2. *Let $|L|$ be of arbitrary order then we have for $f_1, \dots, f_n \in F_{bc,+}(X)$*

$$\nu^n_{\Im_L}(\bigotimes_{j=1}^n f_j) = \sum_{\mathcal{J}\in\pi([n])} \prod_{J\in\mathcal{J}} \nu_L^{|J|}(\bigotimes_{j\in J} f_j), \quad \check{\nu}^n_{\Im_L}(\bigotimes_{j=1}^n f_j) = \sum_{\mathcal{J}\in\pi([n])} \prod_{J\in\mathcal{J}} \check{\nu}_L^{|J|}(\bigotimes_{j\in J} f_j).$$

Proof. Since the (factorial) moment measures are the marginals of the Campbell measures of higher order, this is immediately obtained by setting $g = 0$ in theorem 4.3.1. □

4.4 Representation of Factorial Moment Measures

The main result is theorem 4.4.4. It gives, for a certain class of signed Lévy pseudo measures L, a representation of the factorial moment measures of the KMM process $\Im_L$ in terms of certain loop measures.

We shall now assume that the cumulant measures are invariant under cyclic permutations and can be represented as

$$\Theta_n(\mathrm{d}\, x_1 \dots \mathrm{d}\, x_n) = \mathcal{B}^{x_1}_{n-1}(\mathrm{d}\, x_2 \dots \mathrm{d}\, x_n)\lambda(\mathrm{d}\, x_1), \tag{4.4}$$

where $\mathcal{B}^x_n$ is a finite signed measure on X^n, $\mathcal{B}^x_0$ is a real number and $\lambda \in \mathcal{M}(X)$. Let us denote by $\mathcal{B}^{x,\epsilon}_n$ the positive resp. negative part of $\mathcal{B}^x_n$ and by $|\mathcal{B}^x_n|$ the variation of $\mathcal{B}^x_n$. We will assume throughout this section

Assumption 4.4.1. *The variation*

$$|L|(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n} \int_{X^n} \varphi(\delta_{x_1} + \dots + \delta_{x_n})\, |\mathcal{B}^{x_1}_{n-1}|(\mathrm{d}\, x_2 \dots \mathrm{d}\, x_n)\lambda(\mathrm{d}\, x_1),$$

$\varphi \in F_+(\mathcal{M}^{\cdot\cdot}(X))$, of the signed Lévy pseudo measure L has moments of all orders.

Lemma 4.4.2. *If the cumulant measures are invariant under cyclic permutations then we have*

$$C_L(h) = \sum_{n=1}^{\infty} \int_{X^n} h(x_1, \delta_{x_1} + \dots + \delta_{x_n})\, \Theta_n(\mathrm{d}\, x_1 \dots \mathrm{d}\, x_n) \quad , h \in \mathfrak{L}^1(C_{|L|}).$$

Proof. We have

$$C_L(h) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=1}^{n} \int_{X^n} h(x_j, \delta_{x_1} + \ldots + \delta_{x_n})\, \Theta_n(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n).$$

Now let us take a cyclic permutation such that $j \mapsto 1$. Since $\delta_{x_1} + \ldots + \delta_{x_n}$ is unaffected by permutation of the x_i we obtain for any $j \in [n]$,

$$\begin{aligned} &\int_{X^n} h(x_j, \delta_{x_1} + \ldots + \delta_{x_n})\, \Theta_n(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n) \\ &= \int_{X^n} h(x_1, \delta_{x_1} + \ldots + \delta_{x_n})\, \Theta_n(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n). \end{aligned}$$

□

Proposition 4.4.3. *The factorial moment measures of L are given by*

$$\check{\nu}_L^k = \sum_{\omega \in S_k^{cy}} \mathcal{C}_{k-1}^{x_1}(\mathrm{d}\, x_{\omega(1)} \ldots \mathrm{d}\, x_{\omega^{k-1}(1)})\lambda(\mathrm{d}\, x_1),$$

where for $m \geq 1$ and $f_1, \ldots, f_m \in F_{bc,+}(X)$

$$\mathcal{C}_m^x(\bigotimes_{j=1}^{m} f_j) = \sum_{n=m}^{\infty} \int_{X^n} \sum_{1 \leq i_1 < \ldots < i_m \leq n} f_1(x_{i_1}) \ldots f_m(x_{i_m})\, \mathcal{B}_n^x(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n) \quad (4.5)$$

and $\mathcal{C}_0^x = \sum\limits_{n=1}^{\infty} \mathcal{B}_{n-1}^x(X^{n-1})$. We will call $\{\mathcal{C}_m^x\}_{m=0}^{\infty}$ the family of loop measures.

Proof. Let $f_1, \ldots, f_k \in F_{bc,+}(X)$, we have

$$\begin{aligned} \check{\nu}_L^k(\bigotimes_{j=1}^{k} f_j) &= \int L(\mathrm{d}\, \mu)\, \mu^{-[k]}(\bigotimes_{j=1}^{k} f_j) \\ &= \int L(\mathrm{d}\, \mu)\mu(\mathrm{d}\, x)\, (\mu - \delta_x)^{-[k-1]}(\bigotimes_{j=2}^{k} f_j) f_1(x) = C_L(h), \end{aligned}$$

where $h(x, \mu) = f_1(x)(\mu - \delta_x)^{-[k-1]}(\bigotimes_{j=2}^{k} f_j)$. Now observe that

$$h(x_1, \delta_{x_1} + \ldots + \delta_{x_n}) = \mathbf{1}_{[n]}(k)\, f_1(x_1) \sum_{\substack{i_2, \ldots, i_k = 2 \\ i_j \neq i_l, j \neq l}}^{n} f_2(x_{i_2}) \ldots f_k(x_{i_k}).$$

Furthermore

$$\sum_{\substack{i_2,\dots,i_k=2\\ i_j\neq i_l, j\neq l}}^{n} f_2(x_{i_2})\dots f_k(x_{i_k}) = \sum_{\sigma\in S_{\{2,\dots,k\}}} \sum_{2\leq i_2<\dots<i_k\leq n} f_{\sigma(2)}(x_{i_2})\dots f_{\sigma(k)}(x_{i_k}).$$

So applying lemma 4.4.2 yields

$$\begin{aligned}\check{\nu}_L^k(\bigotimes_{j=1}^k f_j) &= \sum_{\sigma\in S_{\{2,\dots,k\}}} \int_X f_1(x_1) \sum_{n=k}^{\infty} \int_{X^{n-1}} \\ &\qquad \sum_{2\leq i_2<\dots<i_k\leq n} f_{\sigma(2)}(x_{i_2})\dots f_{\sigma(k)}(x_{i_k})\, \mathcal{B}_{n-1}^{x_1}(\mathrm{d}\,x_2\dots\mathrm{d}\,x_n)\lambda(\mathrm{d}\,x_1)\\ &= \sum_{\sigma\in S_{\{2,\dots,k\}}} \int_X f_1(x_1)\mathcal{C}_{k-1}^{x_1}(f_{\sigma(2)}\otimes\dots\otimes f_{\sigma(k)})\,\lambda(\mathrm{d}\,x_1).\end{aligned}$$

Now we observe as in [16] that $S_{\{2,\dots,m\}} \simeq S_{[m]}^{cy}$. Moreover for $m\geq 2$

$$\begin{aligned} S_{\{2,\dots,m\}} &\to S_{[m]}^{cy}\\ \sigma &\mapsto (1,\sigma(2),\dots,\sigma(m))\end{aligned}$$

is a bijection with inverse $\omega\mapsto(i\mapsto\omega^{i-1}(1))$ for $i\in\{2,\dots,m\}$. So we finally conclude

$$\nu_L^k(\bigotimes_{j=1}^k f_j) = \sum_{\omega\in S_{[k]}^{cy}} \int_X f_1(x)\mathcal{C}_{k-1}^{x}(f_{\omega^1(1)}\otimes\dots\otimes f_{\omega^{k-1}(1)})\,\lambda(\mathrm{d}\,x).$$

□

Example 4.4.1. *If you recall the cumulant measures (4.3) of the Pólya point processes we have*

$$\mathcal{B}_n^x(\bigotimes_{j=1}^n f_j) = \epsilon^n z^{n+1} f_1(x)\dots f_n(x)$$

and $\mathcal{B}_0^x = z$. *So if we use that*

$$\sum_{n=m}^{\infty}\binom{n}{m} y^n = \frac{y^m}{(1-y)^{m+1}} \text{ for } y\in(-1,1),$$

one obtains

$$\mathcal{C}_m^x(\bigotimes_{j=1}^m f_j) = \epsilon^m \frac{z^{m+1}}{(1-\epsilon z)^{m+1}} f_1(x)\dots f_m(x).$$

Thus the factorial moment measures of the signed Lévy pseudo measure of the Pólya processes are given by

$$\check{\nu}^k_{L_{\epsilon,z,\lambda}}(\bigotimes_{j\in[k]} f_j) = \frac{z^k}{(1-\epsilon z)^k}(k-1)!\,\lambda(f_{[k]}) \text{ for } f_1,\dots,f_k \in F_{bc,+}(X). \quad (4.6)$$

We now have the tools at hand to establish (3.9). That is the representation of the factorial moment measures of the variation $|L|$ of a signed Lévy pseudo measure L of the form (3.7).

Example 4.4.2. *Let* $\vartheta : \mathcal{M}^{\cdot\cdot}_f(X)\setminus\{\mathbf{0}\} \to \mathbb{R}$ *be some measurable function and* $\lambda \in \mathcal{M}^\circ(X)$ *a diffuse Radon measure. Again we can think of* ϑ *as a measurable symmetric function* $\sqcup_{n=0}^\infty X^n \to \mathbb{R}$*. Since the cumulant measures are given by (3.6) we have*

$$\mathcal{B}^x_{n-1} = \frac{1}{(n-1)!}|\vartheta(x,x_2,\dots,x_n)|\,\lambda(\mathrm{d}\,x_2)\dots\lambda(\mathrm{d}\,x_n).$$

So by symmetry of ϑ *we obtain via the definition of the loop measures (4.5)*

$$\mathcal{C}^x_{m-1}(\bigotimes_{j=1}^{m-1} f_j) = \sum_{n=m-1}^\infty \frac{1}{n!}\binom{n}{m-1}\int_{X^n} f_1(x_1)\dots f_{m-1}(x_{m-1})\times$$
$$\times\,|\vartheta(x,x_1,\dots,x_n)|\lambda(\mathrm{d}\,x_1)\dots\lambda(\mathrm{d}\,x_n)$$

for $f_1,\dots,f_{m-1}\in F_+(X)$*. So finally*

$$\check{\nu}^m_{|L|}(\bigotimes_{j=1}^m f_j) = (m-1)!\sum_{n=m-1}^\infty \frac{1}{(m-1)!(n-(m-1))!}\int_X\int_{X^n}$$
$$f_1(x_1)\dots f_{m-1}(x_{m-1})f_m(x)|\vartheta(x,x_1,\dots,x_n)|\lambda(\mathrm{d}\,x_1)\dots\lambda(\mathrm{d}\,x_n)\lambda(\mathrm{d}\,x)$$
$$= \int_{X^m}\lambda(\mathrm{d}\,x_1)\dots\lambda(\mathrm{d}\,x_m)\,f_1(x_1)\dots f_m(x_m)\sum_{n=m-1}^\infty\frac{1}{(n-(m-1))!}$$
$$\int_{X^{n+1-m}}\lambda(\mathrm{d}\,x_{m+1})\dots\lambda(\mathrm{d}\,x_{n+1})\,|\vartheta(x_1,\dots,x_m,x_{m+1},\dots,x_{n+1})|$$
$$= \int_{X^m}\lambda(\mathrm{d}\,x_1)\dots\lambda(\mathrm{d}\,x_m)\,f_1(x_1)\dots f_m(x_m)$$
$$\int_{\mathcal{M}^{\cdot\cdot}_f(X)}\Pi(\mathrm{d}\,\mu)\,|\vartheta(\delta_{x_1}+\ldots+\delta_{x_m}+\mu)|,$$

which gives us (3.9).

If we now combine corollary 4.3.2 and proposition 4.4.3 we get

Theorem 4.4.4. *The factorial moment measures of $\Im_L$, where the signed Lévy pseudo measure L is defined by the family of cumulant measures given in (4.4), can be represented by*

$$\check{\nu}^k_{\Im_L}(\overset{k}{\underset{j=1}{\otimes}} f_j) = \sum_{\sigma\in S_k}\prod_{\omega\in\sigma}\int f_{i_\omega}(x)\mathcal{C}^x_{\ell(\omega)-1}(f_{\omega^1(i_\omega)}\otimes\ldots\otimes f_{\omega^{\ell(\omega)-1}(i_\omega)})\,\lambda(\mathrm{d}\,x),$$

where $f_1,\ldots,f_k \in F_{bc,+}(X)$. The above product has to be taken over all cycles ω in σ, i_ω is any element of the cycle ω and $\ell(\omega)$ denotes its length.

Example 4.4.3. *The factorial moment measures of the Pólya point processes are given by*

$$\check{\nu}^k_{P^\epsilon_{z,\lambda}} = \frac{z^k}{(1-\epsilon z)^k}\lambda^{\epsilon[k]}.$$

Proof. We have

$$\check{\nu}^k_{P^\epsilon_{z,\lambda}}(\overset{k}{\underset{j=1}{\otimes}} f_j) = \frac{z^k}{(1-\epsilon z)^k}\sum_{\sigma\in S_k}\epsilon^{k-|\sigma|}\prod_{\omega\in\sigma}\lambda(f_\omega).$$

But the above sum coincides with $\lambda^{\epsilon[k]}(\overset{k}{\underset{j=1}{\otimes}} f_j)$ due to lemma 4.1.3. □

So apart from a constant the factorial moment measures and Schur measures coincide in case of the Pólya point processes.

4.5 Permanental and Determinantal Processes

Let $\lambda \in \mathcal{M}(X)$ and $k : X\times X \mapsto \mathbb{R}$ be a positive definite kernel, that is for every $x_1,\ldots,x_n \in X$ and $z_1,\ldots,z_n\in\mathbb{R}$ we have

$$\sum_{i,j=1}^n z_i k(x_i,x_j)z_j \geq 0.$$

Consider the following two families of cumulant measures, $\epsilon\in\{+1,-1\}$

$$\begin{aligned}&\Theta_{n,\epsilon}(\mathrm{d}\,x_1\ldots\mathrm{d}\,x_n)\\ &\quad=\epsilon^{n-1}\,k(x_1,x_2)k(x_2,x_3)\ldots k(x_{n-1},x_n)k(x_n,x_1)\,\lambda(\mathrm{d}\,x_1)\ldots\lambda(\mathrm{d}\,x_n).\end{aligned}\tag{4.7}$$

Let us denote by L_ϵ the signed cluster pseudo measure corresponding to the family $\{\Theta_{n,\epsilon}\}_{n=1}^\infty$ of cumulant measures given by formula (4.7). Furthermore let us denote by $k^{(n)}(x,y) = \int k(x,z)k^{(n-1)}(z,y)\,\lambda(\mathrm{d}\,z)$, $k^{(1)} = k$, $n\in\mathbb{N}$ the convoluted kernels of k, in case the integral is well defined. Borrowing the terminology from Georgii and Yoo [18] we will call k the *interaction kernel*.

Theorem 4.5.1. *If the interaction kernel k is positive definite with*

$$\|k\|_\infty := \sup_{x,y\in X} |k(x,y)| < \infty$$

$$\alpha := \sup_{x\in X} \int |k(x,y)|\,\lambda(\mathrm{d}\,y) < 1, \tag{4.8}$$

then there exist point processes $\Im^\epsilon_{\lambda,k}$ such that $\Theta_{n,\epsilon}$, given in (4.7), are their associated cumulant measures. Moreover $\Im^+_{\lambda,k}$ is a permanental point process to the kernel $K_+ = \sum_{m\geq 1} k^{(m)}$ and $\Im^-_{\lambda,k}$ is a determinantal point process to the kernel $K_- = \sum_{m\geq 1}(-1)^{m-1}k^{(m)}$. That is the correlation functions $\vartheta_{n,\epsilon}$ (see definition 2.1.3) of the processes $\Im^\epsilon_{\lambda,k}$ are given by

$$\vartheta_{n,\epsilon}(x_1\dots x_n) = \sum_{\sigma\in S_n} \epsilon^{n-|\sigma|} \prod_{j=1}^{n} K_\epsilon(x_j, x_{\sigma(j)}).$$

We call K_ϵ the correlation kernel of $\Im^\epsilon_{\lambda,k}$.

Proof. A straightforward computation shows that

$$\nu^1_{|L_\epsilon|}(f) = \sum_{n\geq 1} \int f(x)|k|^{(n)}(x,x)\,\lambda(\mathrm{d}\,x) \text{ for } f\in F_{bc,+}(X).$$

Furthermore we have the following estimate $\||k|^{(n)}\|_\infty \leq \|k\|_\infty \alpha^{n-1}$, which yields $\nu^1_{|L_\epsilon|}(f) < \infty$. Let us verify non-negativity of the Schur measures. Recall from theorem 3.1.3 that the Schur measures are expressed in terms of the cumulant measures by

$$\varrho_{k,\epsilon}(\bigotimes_{j=1}^{k} f_j) = \sum_{\sigma\in S_k} \prod_{\omega\in\sigma} \Theta_{\ell(\omega),\epsilon}(\bigotimes_{j\in\omega} f_j).$$

Since $\Theta_{n,\epsilon}$ is only invariant under cyclic permutations and not symmetric we have to set the order of integration in $\Theta_{\ell(\omega),\epsilon}(\bigotimes_{j\in\omega} f_j)$. The convention will be that we integrate the coordinates along the cycle ω. That is

$$\begin{aligned}
\varrho_{k,\epsilon} &= \sum_{\sigma\in S_k} \epsilon^{k-|\sigma|} \prod_{\omega\in\sigma} \prod_{j=1}^{\ell(\omega)} k(x_{\omega^{j-1}(i_\omega)}, x_{\omega^j(i_\omega)})\,\lambda(\mathrm{d}\,x_1)\dots\lambda(\mathrm{d}\,x_k)\\
&= \sum_{\sigma\in S_k} \epsilon^{k-|\sigma|} \prod_{\omega\in\sigma} \prod_{j=1}^{\ell(\omega)} k(x_{\omega^{j-1}(i_\omega)}, x_{\sigma(\omega^{j-1}(i_\omega))})\,\lambda(\mathrm{d}\,x_1)\dots\lambda(\mathrm{d}\,x_k)\\
&= \sum_{\sigma\in S_k} \epsilon^{k-|\sigma|} \prod_{j=1}^{k} k(x_j, x_{\sigma(j)})\,\lambda(\mathrm{d}\,x_1)\dots\lambda(\mathrm{d}\,x_k).
\end{aligned}$$

where as above $i_\omega \in \omega$, $\omega^0(i_\omega) = i_\omega$ and $|\sigma|$ denotes the number of cycles in σ. So we obtain that the Schur measures have densities with respect to the products of λ and are given by

$$\frac{\mathrm{d}\,\varrho_{n,+}}{\mathrm{d}\,\lambda^n} = \mathrm{per}(M) \text{ and } \frac{\mathrm{d}\,\varrho_{n,-}}{\mathrm{d}\,\lambda^n} = \det(M),$$

where M is the matrix $\{k(x_i, x_j)\}_{1\le i,j\le n}$ and $\mathrm{per}(M) = \sum_{\sigma\in S_n} \prod_{j=1}^n M(j, \sigma(j))$ is the so called *permanent of* M. Since M is a positive definite matrix it is well known (see [6]) that $\mathrm{per}(M) \ge \det(M) \ge 0$, which implies non negativity of the Schur measures. So theorem 3.1.3 gives us the existence of point processes $\Im^\epsilon_{\lambda,k}$.

Now let us compute the loop measures of $\Im^\epsilon_{\lambda,k}$. Let $n \ge m \ge 1$ and $1 \le i_1 < \ldots < i_m \le n$ and $f_1, \ldots, f_m \in F_{bc,+}(X)$ then we have

$$\begin{aligned}
&\int\limits_{X^n} f_1(x_{i_1}) \ldots f_m(x_{i_m})\, \mathcal{B}^x_n(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_n) \\
&= \epsilon^n \int\limits_{X^n} f_1(x_{i_1}) \ldots f_m(x_{i_m})\, k(x, x_1)k(x_1, x_2) \ldots k(x_{i_1-1}, x_{i_1})k(x_{i_1}, x_{i_1+1}) \ldots \\
&\qquad k(x_{i_2-1}, x_{i_2})k(x_{i_2}, x_{i_2+1}) \ldots k(x_{i_m-1}, x_{i_m})k(x_{i_m}, x_{i_m+1}) \ldots k(x_n, x)\,\mathrm{d}\,\lambda^n \\
&= \epsilon^m \int\limits_{X^m} f_1(y_1) \ldots f_m(y_m)\, \epsilon^{i_1-1}k^{(i_1)}(x, y_1)\epsilon^{i_2-i_1-1}k^{(i_2-i_1)}(y_1, y_2) \ldots \\
&\qquad \epsilon^{i_m-i_{m-1}-1}k^{(i_m-i_{m-1})}(y_{m-1}, y_m)\epsilon^{n-i_m}k^{(n+1-i_m)}(y_m, x)\,\lambda(\mathrm{d}\,y_1) \ldots \lambda(\mathrm{d}\,y_m)
\end{aligned}$$

Now since we have for $g_j : \mathbb{N} \to \mathbb{R}$, $j \in [m]$

$$\begin{aligned}
&\sum_{n=m}^{\infty} \sum_{1\le i_1<\ldots<i_m\le n} g_1(i_1)g_2(i_2 - i_1) \ldots g_m(i_m - i_{m-1})g_{m+1}(n + 1 - i_m) \\
&= \sum_{j_1,\ldots,j_{m+1}=1}^{\infty} g_1(j_1) \ldots g_{m+1}(j_{m+1})
\end{aligned}$$

we obtain by introducing $K_\epsilon = \sum\limits_{j=1}^{\infty} \epsilon^{j-1}k^{(j)}$

$$\mathcal{C}^x_m(\bigotimes_{j=1}^m f_j) = \epsilon^m \int\limits_{X^m} f_1(x_1) \ldots f_m(x_m)\, K_\epsilon(x, x_1)K_\epsilon(x_1, x_2) \ldots K_\epsilon(x_m, x)\,\mathrm{d}\,\lambda^m \tag{4.9}$$

Now theorem 4.4.4 yields the assertion

$$\check{\nu}^n_{\Im^\epsilon_{\lambda,k}} = \sum_{\sigma\in S_n} \epsilon^{n-|\sigma|} \prod_{j=1}^n K_\epsilon(x_j, x_{\sigma(j)})\,\mathrm{d}\,\lambda^n.$$

□

For another approach to the construction of permanental and determinantal point processes we refer to [66] and [69], who use the Kolmogorov extension theorem for the existence of the thermodynamic limit. Also, differently to [66, 69], we are imposing conditions on the interaction and not on the correlation kernel. If $\lambda \in \mathcal{M}^\circ(X)$ then proposition 3.1.6 shows that $\Im^\epsilon_{\lambda,k}$ is simple.
There are interesting examples, which will be presented below, where the interaction kernel $k(x,y)$ does only depend on $x-y$. In this case theorem 4.5.1 can be formulated as follows:

Corollary 4.5.2. *Let $k(x,y) = \psi(x-y)$, where ψ is a bounded positive definite function such that $\|\psi\|_1 := \lambda(|\psi|) < 1$. Then the corresponding permanental and determinantal processes $\Im^\epsilon_{\lambda,k}$ exist.*

Characteristic functions ψ are positive definite and bounded. So we still need to verify $\|\psi\|_1 < 1$. In the case $X = \mathbb{R}$ Móricz [49] has given sufficient conditions for the Lebesgue integrability of ψ. Let us give two typical examples:

Example 4.5.1. *Let $X = \mathbb{R}^d$ and*

$$g_z(x) = \frac{z}{(2\pi\beta)^{d/2}} \exp\left(-\frac{\|x\|^2}{2\beta}\right), \quad x \in \mathbb{R}^d,$$

be a scaled gaussian density where $z \in (0,1)$ and $\beta > 0$ are some parameters. It is well known that g_z is positive definite and if we let λ the Lebesgue measure on $\mathbb{R}^d$ then $\|g_z\|_1 = z < 1$. Thus there exists a permanental point process $\Im^+_{\lambda,g_z}$ with interaction kernel $g_z(x-y)$. It is called the ideal Bose gas and was studied in [16]. Fichtner has shown in [16] that the correlation functions of $\Im^+_{\lambda,g_z}$ coincide with the reduced density matrices of a locally normal state ω of Boson systems on $\mathbb{R}^d$.
Moreover there exists a determinantal point process $\Im^-_{\lambda,g_z}$ with interaction kernel $g_z(x-y)$. We call here $\Im^-_{\lambda,g_z}$ the corresponding ideal Fermi gas.

Example 4.5.2. *Let $X = \mathbb{R}$ and λ the Lebesgue measure on $\mathbb{R}$*

$$\chi(x) = \gamma\,\exp(-\frac{|x|}{\alpha}), \quad x \in \mathbb{R}, \tag{4.10}$$

where $\alpha, \gamma > 0$ are chosen such that $\|\chi\|_1 = 2\alpha\gamma < 1$. It is well known that χ is positive definite. So corollary 4.5.2 yields that there exist permanental and determinantal point processes with interaction kernel given by $\chi(x-y)$ under the condition $2\alpha\gamma < 1$.

Macchi [38] gave an example of a determinantal point process with correlation kernel given by χ as defined in (4.10). In order to apply the above construction we have to ask which interaction kernel $j(x-y)$ has the corresponding correlation kernel $\chi(x-y)$. En passant, we will also construct the permanental process with correlation kernel $\chi(x-y)$. To incorporate also the Boson case we write the index j_ϵ. If we recall theorem 4.5.1 (and corollary 4.5.2) then we see that j_ϵ should be chosen such that $\|j_\epsilon\|_\infty < \infty$, $\|j_\epsilon\|_1 < 1$ and

$$\chi = \sum_{n=1}^{\infty} \epsilon^{n-1} j_\epsilon^{*n}, \tag{4.11}$$

where j_ϵ^{*n} denotes n-times convolution of j_ϵ with itself. Taking characteristic functions, indicated by the $\hat{}$, of both sides of (4.11) yields

$$\hat{\chi} = \frac{\hat{j}_\epsilon}{1-\epsilon\,\hat{j}_\epsilon} \quad \text{that is} \quad \hat{j}_\epsilon = \frac{\hat{\chi}}{1+\epsilon\,\hat{\chi}}.$$

Now it is well known (see also [18] example 3.11) that $\hat{\chi}(t) = \frac{2\alpha\gamma}{1+(\alpha t)^2}$. Therefore the characteristic function of j_ϵ should be given by $\hat{j}_\epsilon = \frac{2\alpha\gamma}{1-2\alpha\gamma+(\alpha t)^2}$. So

$$j_\epsilon(x) = \gamma_\epsilon \exp(-\frac{|x|}{\alpha_\epsilon}), \quad \alpha_\epsilon = \frac{\alpha}{\sqrt{1+\epsilon\,2\alpha\gamma}},\ \gamma_\epsilon = \frac{\gamma}{\sqrt{1+\epsilon\,2\alpha\gamma}},$$

is the right choice. j_ϵ is bounded and we require $\|j_\epsilon\|_1 = 2\alpha_\epsilon\gamma_\epsilon = \frac{2\alpha\gamma}{1+\epsilon\,2\alpha\gamma} < 1$, that means $(1-\epsilon)2\alpha\gamma < 1$. To summarize we have

Example 4.5.3. *There exists a determinantal point process with correlation kernel $\chi(x-y)$ if $4\alpha\gamma < 1$.*
Furthermore there exists a permanental point process with correlation kernel $\chi(x-y)$ for any choice of $\alpha, \gamma > 0$.

Remark that Macchi [38] and Soshnikov [69] only require $2\alpha\gamma \leq 1$ for the construction of the corresponding determinantal process $\Im^-_{\lambda,j_-}$. Macchi [38] has shown that if restricted to $[0,\infty)$, $\Im^-_{\lambda,j_-}$ is a renewal process. That is if we order its realization $\mu = \delta_{x_1} + \delta_{x_2} + \delta_{x_3} + \ldots$ such that $x_1 \leq x_2 \leq x_3 \leq \ldots$ then the variables $s_1 = x_2 - x_1$, $s_2 = x_3 - x_2, \ldots$ are i.i.d. under $\Im^-_{\lambda,j_-}$. Furthermore Macchi computed the distribution of the s_j explicitly. Later Soshnikov obtained in [69] that if a determinantal point process is a renewal process its correlation kernel has to be given by χ.

4.6 Palm Kernels in the Classical Case

Let us for the moment consider the classical case of non negative L that is L^- is the zero measure. Corollary 4.3.2 yields that for any $\mathcal{J} \in \pi([n])$ the measure $\prod_{J\in\mathcal{J}} \nu_L^{|J|}((\mathrm{d}\, x_j)_{j\in J})$ is absolutely continuous to $\nu_{\Im_L}^n$ and so we can introduce the Radon-Nykodim density

$$\vartheta_{\mathcal{J}}(x_1,...,x_n) = \frac{\prod_{J\in\mathcal{J}} \nu_L^{|J|}((\mathrm{d}\, x_j)_{j\in J})}{\nu_{\Im_L}^k(\mathrm{d}\, x_1 \dots \mathrm{d}\, x_n)}.$$

By Mecke's argument as outlined in lemma 2.2.1 the equations $(\mathbf{\Sigma}_L^n)$ and $\left(\mathbf{\Sigma}_L^{!,n}\right)$ can be extended from test functions $f_1 \otimes \ldots \otimes f_n \otimes e^{-\zeta_g}$, where $f_1, \dots, f_n, g \in F_{bc,+}(X)$ to arbitrary $h \in F_+(X^n \times \mathcal{M}^{\cdot\cdot}(X))$. That is

$$(\mathbf{\Sigma}_L^n) \quad C_{\Im_L}^n(h) = \int \Im_L(\mathrm{d}\, \mu) \sum_{\mathcal{J}\in\pi([n])} \int \prod_{J\in\mathcal{J}} C_L^{|J|}((\mathrm{d}x_j)_{j\in J}, \mathrm{d}\, \eta_J) \\ h(x_1, \dots, x_n; \sum_{J\in\mathcal{J}} \eta_J + \mu).$$

Now if we disintegrate the higher order Campbell measures of L according to (2.3) and using the densities $\vartheta_{\mathcal{J}}$ we have

$$C_{\Im_L}^n(h) = \int \nu_{\Im_L}^n((\mathrm{d}\, x_j)_{j\in[n]}) \int \Im_L(\mathrm{d}\, \mu) \sum_{\mathcal{J}\in\pi([n])} \vartheta_{\mathcal{J}}((x_j)_{j\in[n]}) \times \\ \times \prod_{J\in\mathcal{J}} L_{(x_j)_{j\in J}}(\mathrm{d}\, \eta_J)\, h((x_j)_{j\in[n]}; \sum_{J\in\mathcal{J}} \eta_J + \mu).$$

So the higher order Palm kernels, as defined in section 2.2, of $\Im_L$ can be directly read of as

$$(\Im_L)_{x_1,\dots,x_n} = \Im_L * \sum_{\mathcal{J}\in\pi([n])} \vartheta_{\mathcal{J}}((x_j)_{j\in[n]}) \mathop{*}_{J\in\mathcal{J}} L_{(x_j)_{j\in J}}.$$

This representation is due to Kallenberg [29] section 5. If we take a look at theorem 4.3.1 and corollary 4.3.2 then we see that the reduced higher order Palm kernels of $\Im_L$ have a similar representation

$$(\Im_L)^!_{x_1,\dots,x_n} = \Im_L * \sum_{\mathcal{J}\in\pi([n])} \check{\vartheta}_{\mathcal{J}}((x_j)_{j\in[n]}) \mathop{*}_{J\in\mathcal{J}} L^!_{(x_j)_{j\in J}}. \tag{4.12}$$

Here $\check{\vartheta}_{\mathcal{J}}$ is the derivative of the corresponding factorial moment measures. As Kallenberg remarked in [29] in the non-reduced situation we have

$$\sum_{\mathcal{J}\in\pi([n])} \check{\vartheta}_{\mathcal{J}}(x_1,\ldots,x_n) = 1 \qquad \check{\nu}^n_{\Im_L} - a.e.[(x_1,\ldots,x_n)]. \tag{4.13}$$

This can be established by taking the left hand side of (4.13) as a density for $\check{\nu}^n_{\Im_L}$ and then by observing that this new measure is actually the old one $\check{\nu}^n_{\Im_L}$. In particular (4.13) says that $\mathcal{J}\mapsto\check{\vartheta}_{\mathcal{J}}(x_1,\ldots,x_n)$ is a probability on $\pi([n])$.

Application to Pólya Sum Processes

Let us start by computing the reduced Palm kernels of the Lévy measure L of the Pólya sum process.

Lemma 4.6.1. *Let L be the Lévy measure of the Pólya sum process $P^+_{z,\lambda}$ then we have*

$$L^!_{x_1,\ldots,x_m} = P^+_{z,\delta_{x_1}+\ldots+\delta_{x_m}} \text{ for } x_1,\ldots,x_m\in X.$$

Proof. Remark that for $n\geq m$, $(n\delta_x)^{-[m]}(f_1\otimes\ldots\otimes f_m) = \frac{n!}{(n-m)!}f_1(x)\ldots f_m(x)$ and for $n<m$, $(n\delta_x)^{-[m]}(f_1\otimes\ldots\otimes f_m)=0$. Let now $h\in F_+(X\times\mathcal{M}^{\cdot\cdot}(X))$

$$C^{!,m}_L(h) = \sum_{n=1}^{\infty}\frac{z^n}{n}\int\lambda(\mathrm{d}\,x)(n\delta_x)^{-[m]}(\mathrm{d}\,x_1\ldots\mathrm{d}\,x_m)\,h(x_1,\ldots,x_m;n\delta_x-\sum_{j=1}^{m}\delta_{x_j})$$

$$= \frac{z^m}{(1-z)^m}(m-1)!\int\lambda(\mathrm{d}\,x)\delta_x(\mathrm{d}\,x_1)\ldots\delta_x(\mathrm{d}\,x_m)$$
$$\frac{(1-z)^m}{z^m}\frac{1}{(m-1)!}\sum_{n=m}^{\infty}\frac{z^n}{n}\frac{n!}{(n-m)!}h(x_1,\ldots,x_m;(n-m)\delta_{x_1})$$

$$= \int\check{\nu}^m_L(\mathrm{d}\,x_1\ldots\mathrm{d}\,x_m)\,(1-z)^m\sum_{n=m}^{\infty}z^{n-m}\binom{n-1}{m-1}h(x_1,\ldots,x_m;(n-m)\delta_{x_1}).$$

For the last equation observe that $\check{\nu}^m_L$ has been determined in (4.6). So after an index shift we obtain

$$L^!_{x,\ldots,x}(\varphi) = (1-z)^m\sum_{j=0}^{\infty}z^j\binom{j+m-1}{j}\varphi(j\delta_x) \text{ for } \varphi\in F_+(\mathcal{M}^{\cdot\cdot}(X)),$$

that means $L^!_{x,\dots,x}$ places a negative binomial distributed number of points on x. Since $\check{\nu}^m_L$ is concentrated on the diagonal $D_m = \{(x,\dots,x) \in X^m | \, x \in X\}$ see (4.6), $L^!_{x_1,\dots,x_m}$ can be set arbitrarily for $(x_1,\dots,x_m) \in X^m \setminus D_m$. Now recall that

$$P^+_{z,m\delta_x}(\varphi) = (1-z)^m \sum_{j=0}^{\infty} \frac{z^j}{j!} \int \varphi(\delta_{x_1} + \ldots + \delta_{x_j})\, (m\delta_x)^{+[j]}(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_j)$$

for $\varphi \in F_+(\mathcal{M}^{\cdot\cdot}(X))$. But $(m\delta_x)^{+[j]}(f_1 \otimes \ldots \otimes f_j) = \frac{(j+m-1)!}{(m-1)!} f_1(x) \ldots f_j(x)$, whence $L^!_{x,\dots,x} = P^+_{z,m\delta_x}$ follows. □

Now we are prepared to state the result

Theorem 4.6.2. *Let $z \in (0,1)$ and $\lambda \in \mathcal{M}(X)$ then the reduced Palm kernels of the Pólya sum process are given by*

$$\left(P^+_{z,\lambda}\right)^!_{x_1,\dots,x_m} = P^+_{z,\lambda+\delta_{x_1}+\ldots+\delta_{x_m}} \text{ for } x_1,\dots,x_m \in X.$$

Proof. Recall (4.12) and let us start by computing $\underset{J\in\mathcal{J}}{*} L^!_{(x_j)_{j\in J}}$. The Pólya sum process has the following property: For $\lambda, \rho \in \mathcal{M}(X)$ we have $P^+_{z,\lambda} * P^+_{z,\rho} = P^+_{z,\lambda+\rho}$. Thus

$$\underset{J\in\mathcal{J}}{*} L^!_{(x_j)_{j\in J}} = P^+_{z,\delta_{x_1}+\ldots+\delta_{x_m}}.$$

But if we insert this in (4.12) and use (4.13) we obatin the assertion. □

In [28] Kallenberg remarked in section 12.4. "Links to conditioning" that the reduced Palm distribution $P^!_{x_1,\dots,x_n}$ of a point process P can be interpreted as P conditioned on the event that $\delta_{x_1} + \ldots + \delta_{x_n}$ is part of the realization and then by removing $\delta_{x_1} + \ldots + \delta_{x_n}$ from it.
For Shirai and Takahashi [66] a Boson process is a point process such that a point realization increases the probability of other points occuring in its neighborhood (attractiveness property). In this sense theorem 4.6.2 says that the Pólya sum process can be thought of as a Boson process.

Chapter 5

Gibbs Processes, Polymers and Quantum Gases

In this chapter we will see that a rich class of point processes appearing in statistical mechanics can be constructed via CEM. A main tool is that the abstract cluster expansion developed in the recent paper [58] of Phogosyan and Ueltschi gives sufficient conditions for $|L|$, the variation of the signed cluster pseudo measure L, to be of first order. Polymers, classical and quantum continuous systems are constructed. In particular, classical limiting Gibbs states exist and can be characterized as Gibbs states. Moreover also Ginibre's quantum Bose gas is constructed but not characterized as a Gibbs state.

5.1 Abstract Cluster Expansions for Point Processes

We are in the old setting: X is a Polish space equipped with the Borel σ-field $\mathcal{B}(X)$ and $\mathcal{B}_0(X)$ denotes the ring of bounded sets of X as before. Moreover a Radon measure $\rho \in \mathcal{M}(X)$ and a measurable symmetric function $\phi : X \times X \to \mathbb{R} \cup \{+\infty\}$ is given. ϕ will be called *pair potential* in the sequel. Set

$$\xi(x, y) = e^{-\phi(x,y)} - 1, \; x, y \in X.$$

We let $\xi = -1$ if $\phi = +\infty$. Several assumptions on the potential ϕ have to be imposed:

($\mathcal{A}1$) (**Weak stability**)
There exists $b \in F_+(X)$ such that for $n \geq 1$

$$\sum_{1\leq i<j\leq n} \phi(x_i, x_j) \geq -\sum_{j=1}^{n} b(x_j) \quad \rho^n - a.e.[(x_1, \dots, x_n)].$$

If b can be chosen bounded then weak stability becomes stability in the sense of Ruelle [63] definition 3.2.1.

($\mathcal{A}2$) (**Weak regularity**)
There exists $a \in F_+(X)$ such that

$$\int \rho(\mathrm{d}\, y)\, |1 - e^{-\phi(x,y)}|\, e^{(a+2b)(y)} \leq a(x) \quad \rho - a.e.[x].$$

We remark that for bounded a and b weak regularity implies regularity of ϕ in the sense of Ruelle [63] definition 4.1.2.
The following can replace ($\mathcal{A}2$):

($\mathcal{A}2'$) There exists $a \in F_+(X)$ such that

$$\int \rho(\mathrm{d}\, y)\, |\bar{\phi}(x, y)|\, e^{(a+b)(y)} \leq a(x) \quad \rho - a.e.[x],$$

where

$$\bar{\phi}(x, y) = \begin{cases} \phi(x, y), & \text{if } \phi(x, y) < \infty \\ 1, & \text{if } \phi(x, y) = \infty. \end{cases}$$

($\mathcal{A}3$) (**Integrability of** a**,** b)

$$e^{(a+2b)(x)}\, \rho(\mathrm{d}\, x) \in \mathcal{M}(X).$$

Under condition ($\mathcal{A}2'$) we will always work instead of condition ($\mathcal{A}3$) with

($\mathcal{A}3'$)

$$e^{(a+b)(x)}\, \rho(\mathrm{d}\, x) \in \mathcal{M}(X).$$

For our further analysis in this chapter the so called *Ursell function*

$$u(x_1, ..., x_n) = \sum_{G\in\mathcal{C}_n} \prod_{\{i,j\}\in G} \xi(x_i, x_j) \text{ and } u(x_1) = 1,$$

for $x_1, \dots, x_n \in X$, where we denote by $\mathcal{C}_n$ the set of all undirected connected graphs with n vertices and the product has to be taken over all edges in G, will be of considerable importance. Remark that the Ursell function is symmetric in its arguments. If we introduce the *energy*

$$E(\mu) = \sum_{1\le i<j\le n} \phi(x_i, x_j)$$

of a finite point measure $\mu = \delta_{x_1} + \ldots + \delta_{x_n}$, then Ruelle's algebraic approach [63] chapter four yields that

$$e^{-E(\delta_{x_1}+\ldots+\delta_{x_n})} = \sum_{\mathcal{J}\in\pi([n])} \prod_{J\in\mathcal{J}} u((x_j)_{j\in J}). \tag{5.1}$$

The following theorem will serve as a main lemma for us.

Theorem 5.1.1 (Poghosyan/Ueltschi [58])**.** *Assume conditions* ($\mathcal{A}1$) *and* ($\mathcal{A}2$) *respectively* ($\mathcal{A}1$) *and* ($\mathcal{A}2'$). *Then we have*

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int\limits_{X^{n-1}} \rho(\mathrm{d}\, x_1)\ldots\rho(\mathrm{d}\, x_{n-1})\, |u(x, x_1, \ldots, x_{n-1})| \le e^{a(x)+2b(x)}.$$

Under condition ($\mathcal{A}2'$) *this holds true with* $e^{b(x)}$ *instead of* $e^{2b(x)}$.

Now we can formulate our existence result.

Theorem 5.1.2. *Assume ϕ is a pair potential which satisfies* ($\mathcal{A}1$), ($\mathcal{A}2$) *and* ($\mathcal{A}3$) *respectively* ($\mathcal{A}1$), ($\mathcal{A}2'$) *and* ($\mathcal{A}3'$). *Then there exists a point process* $\mathfrak{I}_{\mathfrak{R}}$ *with signed Lévy pseudo measure*

$$\mathfrak{R}(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n!} \int\limits_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, u(x_1, \ldots, x_n) \rho^n(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n), \tag{5.2}$$

for $\varphi \in \mathfrak{L}^1(|\mathfrak{R}|)$. *That is its cumulant measures are given by*

$$\Theta_n(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n) = \frac{1}{(n-1)!} u(x_1, \ldots, x_n)\, \rho^n(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n).$$

Proof. For theorem 3.1.3 to hold it suffices to show non negativity of the Schur measures ϱ_n and that $|\mathfrak{R}|$ is of first order. Let us first consider the Schur measures

$$\begin{aligned}\varrho_n(\bigotimes_{j=1}^{n} f_j) &= \sum_{\sigma\in S_n} \prod_{\omega\in\sigma} \Theta_{\ell(\omega)}(\bigotimes_{i\in\omega} f_i) = \sum_{\mathcal{J}\in\pi([n])} \prod_{J\in\mathcal{J}} (|J|-1)!\, \Theta_{|J|}(\bigotimes_{j\in J} f_j)\\ &= \int\limits_{X^n} f_1(x_1)\ldots f_n(x_n)\, e^{-E(\delta_{x_1}+\ldots+\delta_{x_n})} \rho^n(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n),\end{aligned}$$

for $f_1, \dots, f_n \in F_{bc,+}(X)$. The last equation is due to (5.1). Now let us give the estimate for the first moment measure of $|\Re|$:

$$\begin{aligned}
&\nu^1_{|\Re|}(f) \\
&= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_X \int_{X^{n-1}} f(x)\, |u(x, x_1, \dots, x_{n-1})| \rho^{n-1}(\mathrm{d}\, x_1 \dots \mathrm{d}\, x_{n-1}) \rho(\mathrm{d}\, x) \\
&\leq \int_X f(x)\, e^{a(x)+2b(x)}\, \rho(\mathrm{d}\, x) < \infty \text{ for } f \in F_{bc,+}(X).
\end{aligned}$$

The first inequality is due to theorem 5.1.1 and the second holds true by $(\mathcal{A}3)$ respectively $(\mathcal{A}3')$ if $(\mathcal{A}2')$ is satisfied. □

5.2 The Classical Case: Gibbs Processes

As in the previous section X denotes any Polish space. We let

$$\rho = z\, \lambda$$

where $\lambda \in \mathcal{M}(X)$ and $z \in (0, \infty)$ is some parameter called the *activity*. The pair potential ϕ will satisfy

$(\mathcal{A}\, i)$ (**Stability**)
In assumption $(\mathcal{A}1)$ we can choose b to be a non negative constant $B \geq 0$.

$(\mathcal{A}\, ii)$ (**Regularity**)
In assumption $(\mathcal{A}2)$ we can also choose a to be constant. That is

$$C_\phi := \sup_{x \in X} \int \lambda(\mathrm{d}\, y)\, |1 - e^{-\phi(x,y)}| \leq z^{-1} e^{-2B}\, a e^{-a} \leq z^{-1} e^{-2B-1},$$

since $a \mapsto a e^{-a}$ attains its maximum at $a = 1$.

Now theorem 5.1.2 yields

Corollary 5.2.1. *Let ϕ be a stable and regular, that is $z \in (0, \frac{e^{-2B-1}}{C_\phi})$, pair potential. Then $\Im_\Re$, which will be denoted $G_{z,\phi}$ in the sequel, exists.*

Remark that stable pair potentials are bounded from below $E(\delta_x + \delta_y) = \phi(x, y) \geq -2B$.
In [63] remark to definition 4.1.2. Ruelle gave the important hint that regularity of a pair potential implies the existence of a set with finite Lebesgue

measure such that the potential is absolutely integrable on its complement. Ruelle is in the setting of translation invariant potentials. So the below remark is a straightforward generalization to arbitrary pair potentials. In the sequel let us denote $\phi_x : y \mapsto \phi(x, y)$.

Remark 5.2.2. *Let $\epsilon > 0$. Since $|1 - e^{-t}| \geq c(\epsilon)$ for $|t| > \epsilon$ for some $c(\epsilon) > 0$ we have*

$$C_\phi \geq \sup_{x\in X} \int_{\{|\phi_x|>\epsilon\}} \lambda(\mathrm{d}\, y)\, |1 - e^{-\phi_x(y)}| \geq c(\epsilon) \sup_{x\in X} \lambda(\{|\phi_x| > \epsilon\}).$$

Furthermore since $|1 - e^{-t}| \geq \tilde{c}(\epsilon)|t|$ for $|t| \leq \epsilon$ for some $\tilde{c}(\epsilon) > 0$ we have

$$C_\phi \geq \sup_{x\in X} \int_{\{|\phi_x|\leq\epsilon\}} \lambda(\mathrm{d}\, y)\, |1 - e^{-\phi_x(y)}| \geq \tilde{c}(\epsilon) \sup_{x\in X} \int_{\{|\phi_x|\leq\epsilon\}} \lambda(\mathrm{d}\, y)\, |\phi_x(y)|.$$

The *conditional energy* at $x \in X$ given the configuration $\mu \in \mathcal{M}_f^{\cdot\cdot}(X)$ is defined as

$$E(x, \mu) = \int \phi(x, y)\, \mu(\mathrm{d}\, y).$$

Later it will be important to consider $E(x, \mu)$ also for infinite μ. Remark that for any $x \in X$ and $\mu \in \mathcal{M}_f^{\cdot\cdot}(X)$

$$E(\mu + \delta_x) = E(\mu) + E(x, \mu). \tag{5.3}$$

Furthermore let us denote by

$$\gamma(\mu, \mathrm{d}\, x) = e^{-E(x,\mu)}\, \rho(\mathrm{d}\, x) \text{ for } \mu \in \mathcal{M}_f^{\cdot\cdot}(X)$$

the so called Boltzmann kernel. Remark that due to (5.3) the iterated kernels of γ evaluated for the zero boundary configuration are given by

$$\gamma^{(k)}(0; \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_k) = e^{-E(\delta_{x_1}+\ldots+\delta_{x_k})}\, \rho(\mathrm{d}\, x_1) \ldots \rho(\mathrm{d}\, x_k). \tag{5.4}$$

The following tree estimate of the Ursell function due to Poghosyan and Ueltschi will be fundamental in the sequel.

Theorem 5.2.3 (Poghosyan, Ueltshi)**.** *Let ϕ be a weakly stable pair potential then*

$$|u(x_1, ..., x_n)| \leq \prod_{j=1}^{n} e^{2b(x_j)} \sum_{G\in\mathcal{T}_n} \prod_{\{i,j\}\in G} |\xi(x_i, x_j)|.$$

Here $\mathcal{T}_n$ denotes the set of trees with n vertices.

Since we are here in the setting of stable ϕ the $b(x_j)$ can be replaced by the stability constant $B \geq 0$ in the above equation. As already remarked in [40] the number $|\mathcal{T}_n|$ is dominated by $c^n n!$ for some constant c, which can be taken to be $c = e$. We obtain that, uniformly in $x \in X$,

$$\int \lambda(\mathrm{d}\, x_2) \dots \lambda(\mathrm{d}\, x_n)\, |u(x, x_2, \dots, x_n)| \leq \left(e^{2B}\right)^n e^n n!\, C_\phi^{n-1}. \tag{5.5}$$

So by using (5.5) we obtain as in theorem 5.1.2 for $f \in F_{bc,+}(X)$

$$\nu^1_{|\Re|}(f) = \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!} \int_{X^n} f(x)|u(x, x_2, \dots, x_n)|\, \lambda(\mathrm{d}\, x)\lambda(\mathrm{d}\, x_2) \dots \lambda(\mathrm{d}\, x_n) < \infty$$

if we let $z \in (0, \frac{e^{-2B-1}}{C_\phi})$. In fact the tree estimate, theorem 5.2.3, is vital in the proof of the main result, theorem 5.1.1, in [58], which provides a sufficient condition for the convergence of the cluster expansion method. The next task is to identify the above constructed point process $G_{z,\phi}$ as a Gibbs point process where the interaction is given by the pair potential ϕ. Remark that $G_{z,\phi}$ is simple due to proposition 3.1.6 if $\lambda \in \mathcal{M}^\circ(X)$.

Theorem 5.2.4. *Let ϕ be a stable and regular pair potential. Then for $z \in (0, \frac{e^{-4B-1}}{C_\phi})$ $G_{z,\phi}$ exists, the Boltzmann kernel γ is $G_{z,\phi}$-a.s. well defined and $G_{z,\phi}$ is a solution to $\left(\mathbf{\Sigma}'_\gamma\right)$. That is $G_{z,\phi}$ is a Papangelou process with the Boltzmann kernel γ as Papangelou kernel. Papangelou processes are introduced in definition 2.5.1.*

Proof. Since existence of $G_{z,\phi}$ was already established it remains to be seen that it satisfies the integration by parts formula $\left(\mathbf{\Sigma}'_\gamma\right)$. Since the Schur measures ϱ_k coincide with $\gamma^{(k)}(0; \cdot)$, see (5.4) the finite point process Q_Λ coincides with the Papangelou point process $P^0_{\gamma_\Lambda}$ for $\Lambda \in \mathcal{B}_0(X)$, where γ_Λ denotes the Boltzmann kernel restricted to Λ. It is straightforward to see that γ satisfies the cocycle condition. So with Zessin's lemma 2.5.3 we conclude that Q_Λ is a solution to $\left(\mathbf{\Sigma}'_{\gamma_\Lambda}\right)$. That is

$$C_{Q_\Lambda}(h) = \int_{\mathcal{M}^{\cdot\cdot}(\Lambda)} \int_\Lambda h(x, \mu + \delta_x)\, \gamma(\mu, \mathrm{d}\, x) Q_\Lambda(\mathrm{d}\, \mu), \tag{5.6}$$

for $h \in F_+(\Lambda \times \mathcal{M}^{\cdot\cdot}(\Lambda))$. In the sequel let h be of the form $f \otimes e^{-\zeta_g}$ for $f, g \in F_{bc,+}(X)$ and Λ such that $\mathrm{supp}(f), \mathrm{supp}(g) \subset \Lambda$. In corollary 3.3.2 convergence of $C_{Q_\Lambda}(h) \to C_{G_{z,\phi}}(h)$ as $\Lambda \uparrow X$ was proved. Remark that $E(x, \mu) = \zeta_{\phi_x}(\mu)$, so the right hand side of (5.6) can be written as

$$\int_X f(x) e^{-g(x)} \mathcal{L}_{Q_\Lambda}(g + \phi_x)\, z\lambda(\mathrm{d}\, x). \tag{5.7}$$

Certainly in order to prove the theorem we would like to have

$$\mathcal{L}_{Q_\Lambda}(g+\phi_x) \to \mathcal{L}_{G_{z,\phi}}(g+\phi_x) \text{ as } \Lambda \uparrow X.$$

We already now that $\mathcal{L}_{Q_\Lambda}(f) \to \mathcal{L}_{G_{z,\phi}}(f)$ for $f \in F_{bc,+}(X)$. But since $g+\phi_x$ can be unbounded, negative and does not need to have bounded support it is not clear whether convergence of the Laplace transforms on $F_{bc,+}(X)$ imply convergence for $g+\phi_x$. This difficulty will be overcome by the use of the modified Laplace functionals $\mathcal{K}_{\Re_\Lambda}$ resp. $\mathcal{K}_{\Re}$ of Q_Λ resp. $G_{z,\phi}$. We will first establish $\mathcal{K}_{\Re_\Lambda}(g+\phi_x) \to \mathcal{K}_{\Re}(g+\phi_x)$ and then show that we actually have $\mathcal{L}_{Q_\Lambda}(g+\phi_x) = \mathcal{K}_{\Re_\Lambda}(g+\phi_x)$ and $\mathcal{L}_{G_{z,\phi}}(g+\phi_x) = \mathcal{K}_{\Re}(g+\phi_x)$. The main technical result will be the following

Lemma 5.2.5. *Choose $\epsilon > 0$ such that $z < \frac{e^{-4B-1-\epsilon}}{C_\phi}$ and let Υ be the function on $\mathcal{M}^{\cdot\cdot}_f(X)$*

$$\Upsilon(\mu) = \begin{cases} 2\, e^{2B\mu(X)}, & \text{for } supp(\mu)\cap O_x \neq \emptyset \\ \mu(|\phi_x|)\, e^{\epsilon\mu(X)}, & \text{for } \mu \in \mathcal{M}^{\cdot\cdot}_f(O_x^c), \end{cases}$$

where $O_x = supp(g) \cup \{|\phi_x| > \epsilon\}$. Then we have $|1-e^{-\zeta_{g+\phi_x}}| \leq \Upsilon$ on $\mathcal{M}^{\cdot\cdot}_f(X)$ and there exists some $\alpha \in \mathbb{R}_+$, independent of $x \in X$, such that $|\Re|(\Upsilon) \leq \alpha$.

Proof. Since $\phi_x(y) \geq -2B$ for all $y \in X$ we certainly have $|1-e^{-\mu(g+\phi_x)}| \leq 1+e^{2B\mu(X)} \leq 2e^{2B\mu(X)}$ for all $\mu \in \mathcal{M}^{\cdot\cdot}_f(X)$. Let now $\mu \in \mathcal{M}^{\cdot\cdot}_f(O_x^c)$ since $O_x^c = \mathrm{supp}(g)^c \cap \{|\phi_x| \leq \epsilon\}$ we have $\mu(g) = 0$ and $|\mu(\phi_x)| \leq \epsilon\mu(X)$. So due to $|1-e^t| \leq |t|e^{|t|}$ we have $|1-e^{-\mu(g+\phi_x)}| \leq \Upsilon(\mu)$. Let us name $\gamma = e^{2B+1}C_\phi$ in the sequel.

$$\begin{aligned}
&|\Re|(\mathbf{1}_{\mathcal{M}^{\cdot\cdot}_f(O_x^c)}\,\Upsilon) \\
= \;& \sum_{n=1}^\infty \frac{z^n}{n!}e^{\epsilon n} \int\limits_{(O_x^c)^n} \sum_{i=1}^n |\phi_x(x_i)||u(x_1,...,x_n)|\,\lambda(\mathrm{d}\,x_1)...\lambda(\mathrm{d}\,x_n) \\
\leq \;& \sum_{n=1}^\infty \frac{z^n e^{\epsilon n}}{(n-1)!} \int\limits_{O_x^c} \lambda(\mathrm{d}\,y)\,|\phi_x(y)| \int\limits_{X^{n-1}} \lambda(\mathrm{d}\,x_1)...\lambda(\mathrm{d}\,x_{n-1})\,|u(y,x_1,...,x_{n-1})| \\
\leq \;& \frac{1}{C_\phi}\sum_{n=1}^\infty n(ze^\epsilon\gamma)^n \int\limits_{O_x^c} \lambda(\mathrm{d}\,y)\,|\phi_x(y)| \\
\leq \;& \frac{1}{C_\phi}\frac{1}{(1-ze^\epsilon\gamma)^2}\lambda\left(|\phi_x| 1_{\{|\phi_x|\leq\epsilon\}}\right) \leq \frac{1}{(1-ze^\epsilon\gamma)^2}\frac{1}{\tilde{c}(\epsilon)} =: \alpha_1.
\end{aligned}$$

The first inequality is due to the symmetry of u and replacing $(n-1)$-times integration over O_x^c by integration over X. For the second inequality we have used the estimate (5.5) and the last inequality is due to remark 5.2.2. Since $X^n \setminus (O_x^c)^n = (O_x \times X^{n-1}) \cup (X \times O_x \times X^{n-2}) \cup ... \cup (X^{n-1} \times O_x)$ we have

$$\begin{aligned}
& |\Re|(\mathbf{1}_{\{\mu \in \mathcal{M}_f^{\cdot\cdot}(X) | \operatorname{supp}(\mu) \cap O_x \neq \emptyset\}} \Upsilon) \\
\leq\ & \sum_{n=1}^{\infty} \frac{z^n}{n!} n 2 \left(e^{2B}\right)^n \int\limits_{O_x} \lambda(\mathrm{d}\, y) \int\limits_{X^{n-1}} \lambda(\mathrm{d}\, x_1) ... \lambda(\mathrm{d}\, x_{n-1}) \, |u(y, x_1, ..., x_{n-1})| \\
\leq\ & \frac{2}{C_\phi} \sum_{n=1}^{\infty} n (z e^{2B} \gamma)^n \, \lambda(O_x) \\
\leq\ & \frac{2}{C_\phi} \frac{1}{(1 - z e^{2B} \gamma)^2} (\lambda(\operatorname{supp}(g)) + \lambda(\{|\phi_x| > \epsilon\})) \\
\leq\ & \frac{2}{C_\phi} \frac{1}{(1 - z e^{2B} \gamma)^2} (\lambda(\operatorname{supp}(g)) + \frac{C_\phi}{c(\epsilon)}) =: \alpha_2.
\end{aligned}$$

For the second inequality we have used the estimate (5.5). The last inequality follows by remark 5.2.2. And so we can choose $\alpha = \alpha_1 + \alpha_2$. □

Corollary 5.2.6. *We have $\mathcal{K}_{\Re_\Lambda}(g + \phi_x) \to \mathcal{K}_{\Re}(g + \phi_x)$ as $\Lambda \uparrow X$ and there exists a constant $\tilde{\alpha}$, independent of Λ and x, such that $\mathcal{K}_{\Re_\Lambda}(g + \phi_x) \leq \tilde{\alpha}$.*

Proof. By the preceding lemma we have $|(\Re - \Re_\Lambda)(\mathbf{1} - e^{-\zeta_{g+\phi_x}})| \leq |\Re|((\mathbf{1} - \mathbf{1}_{\mathcal{M}^{\cdot\cdot}(\Lambda)})\Upsilon) \downarrow 0$ as $\Lambda \uparrow X$ by dominated convergence. Furthermore we certainly have $|\Re_\Lambda(\mathbf{1} - e^{-\zeta_{g+\phi_x}})| \leq |\Re|(\Upsilon) \leq \alpha$, so we can choose $\tilde{\alpha} = e^\alpha$. □

To shorten notation let us denote $\tilde{\phi} = g + \phi_x$ in the sequel.

Lemma 5.2.7. *We have*

$$\mathcal{L}_{G_{z,\phi}}(\tilde{\phi}) = \mathcal{K}_{\Re}(\tilde{\phi}) \text{ and } \mathcal{L}_{Q_\Lambda}(\tilde{\phi}) = \mathcal{K}_{\Re_\Lambda}(\tilde{\phi}).$$

Proof. Let us start by establishing $\mathcal{L}_{G_{z,\phi}}(\tilde{\phi}_+) = \mathcal{K}_{\Re}(\tilde{\phi}_+)$. Let $(f_n)_{n \geq 1}$ be an increasing sequence of functions in $F_{bc,+}(X)$ such that $f_n \uparrow \tilde{\phi}_+$ as $n \to \infty$, i.e. $f_n = \min\{\tilde{\phi}_+, n\} \mathbf{1}_{\Lambda_n}$ with $\Lambda_n \in \mathcal{B}_0(X)$ and $\Lambda_n \uparrow X$. Remark that due to monotone convergence there holds $\lim_{n \to \infty} \zeta_{f_n} = \zeta_{\tilde{\phi}_+}$ on $\mathcal{M}^{\cdot\cdot}(X)$. Thus with $\mathbf{1} - e^{-\zeta_{f_n}} \leq \mathbf{1} - e^{-\zeta_{\tilde{\phi}_+}} \leq \Upsilon$ we conclude by dominated convergence $\Re(\mathbf{1} - e^{-\zeta_{f_n}}) \to \Re(\mathbf{1} - e^{-\zeta_{\tilde{\phi}_+}})$ as $n \to \infty$, which implies the last equality in the following expression.

$$\mathcal{L}_{G_{z,\phi}}(\tilde{\phi}_+) = \lim_{n \to \infty} \mathcal{L}_{G_{z,\phi}}(f_n) = \lim_{n \to \infty} \mathcal{K}_{\Re}(f_n) = \mathcal{K}_{\Re}(\tilde{\phi}_+).$$

The first equality above can also be justified by dominated convergence and the second is due to $f_n \in F_{bc,+}(X)$. Now let us treat the general case.

$$\begin{aligned}\Re(\mathbf{1} - e^{-\zeta_{\tilde{\phi}}}) &= \int \left(1 - e^{-\mu(\tilde{\phi}_+)} \sum_{j=0}^{\infty} \frac{\mu(\tilde{\phi}_-)^j}{j!}\right) \Re(\mathrm{d}\,\mu) \\ &= \int \left(1 - e^{-\mu(\tilde{\phi}_+)}\right) \Re(\mathrm{d}\,\mu) - \int \sum_{j=1}^{\infty} \frac{e^{-\mu(\tilde{\phi}_+)}\mu(\tilde{\phi}_-)^j}{j!} \Re(\mathrm{d}\,\mu)\end{aligned}$$

The second equality above is due to the following: Since $e^{\zeta_{\tilde{\phi}_-}} - 1 \leq \Upsilon$ we have

$$\int \sum_{j=1}^{\infty} \frac{e^{-\mu(\tilde{\phi}_+)}\mu(\tilde{\phi}_-)^j}{j!} |\Re|(\mathrm{d}\,\mu) \leq \int \left(e^{\mu(\tilde{\phi}_-)} - 1\right) |\Re|(\mathrm{d}\,\mu) < \infty. \tag{5.8}$$

According to the monotone convergence theorem we are allowed to exchange in the below equation the sum with the integrals.

$$\int \sum_{j=1}^{\infty} \frac{e^{-\mu(\tilde{\phi}_+)}\mu(\tilde{\phi}_-)^j}{j!} \Re(\mathrm{d}\,\mu) = \sum_{j=1}^{\infty} \frac{1}{j!} C_{\Re}^{j}(\tilde{\phi}_-^{\otimes j} \otimes e^{-\zeta_{\tilde{\phi}_+}}).$$

The estimate (5.8) also shows absolute convergence of the above right hand side. So we have by lemma 3.1.2

$$\exp\left[\sum_{j=1}^{\infty} \frac{C_{\Re}^{j}(\tilde{\phi}_-^{\otimes j} \otimes e^{-\zeta_{\tilde{\phi}_+}})}{j!}\right] = 1 + \sum_{j=1}^{\infty} \frac{1}{j!} \sum_{\mathcal{J} \in \pi([j])} \prod_{J \in \mathcal{J}} C_{\Re}^{|J|}(\tilde{\phi}_-^{\otimes |J|} \otimes e^{-\zeta_{\tilde{\phi}_+}}).$$

Collecting everything together we obtain

$$\begin{aligned}\mathcal{K}_{\Re}(\tilde{\phi}) &= \mathcal{K}_{\Re}(\tilde{\phi}_+) \exp\left[\sum_{j=1}^{\infty} \frac{C_{\Re}^{j}(\tilde{\phi}_-^{\otimes j} \otimes e^{-\zeta_{\tilde{\phi}_+}})}{j!}\right] \\ &= \mathcal{L}_{G_{z,\phi}}(\tilde{\phi}_+) \left(1 + \sum_{j=1}^{\infty} \frac{1}{j!} \sum_{\mathcal{J} \in \pi([j])} \prod_{J \in \mathcal{J}} C_{\Re}^{|J|}(\tilde{\phi}_-^{\otimes |J|} \otimes e^{-\zeta_{\tilde{\phi}_+}})\right) \\ &= \mathcal{L}_{G_{z,\phi}}(\tilde{\phi}_+) + \sum_{j=1}^{\infty} \frac{C_{G_{z,\phi}}^{j}(\tilde{\phi}_-^{\otimes j} \otimes e^{-\zeta_{\tilde{\phi}_+}})}{j!} \\ &= \int e^{-\mu(\tilde{\phi}_+)} \sum_{j=0}^{\infty} \frac{\mu(\tilde{\phi}_-)^j}{j!} G_{z,\phi}(\mathrm{d}\,\mu) = \mathcal{L}_{G_{z,\phi}}(\tilde{\phi}).\end{aligned}$$

The third equation is due to theorem 4.3.1 equation $(\mathbf{\Sigma}_L^n)$, since (5.8) implies $\nu^n_{|\Re|}(\tilde{\phi}_-^{\otimes n}) < \infty$ for $n \geq 1$ and therefore the assumptions of that theorem are satisfied. In particular we have $\nu^1_{G_{z,\phi}}(\tilde{\phi}_-) < \infty$ which implies $\zeta_{\tilde{\phi}_-} < \infty$, $G_{z,\phi}-$ a.s. so the conditional energy $E(x,\mu)$ is $G_{z,\phi}-$ a.s. well defined.

Certainly all arguments are valid if we replace $\Re$ by $\Re_\Lambda$ therefore we also obtain the second assertion. □

We can now finish the proof of the theorem:

With corollary 5.2.6 we obtain $\mathcal{L}_{Q_\Lambda}(g+\phi_x) \to \mathcal{L}_{G_{z,\phi}}(g+\phi_x)$ as $\Lambda \uparrow X$ and $\mathcal{L}_{Q_\Lambda}(g+\phi_x) \leq \tilde{\alpha}$. So we can take the limit $\Lambda \uparrow X$ inside the integral of equation (5.7) and thus obtain that $G_{z,\phi}$ solves $\left(\mathbf{\Sigma}'_\gamma\right)$ for test functions $h = f \otimes e^{-\zeta_g}$, $f, g \in F_{bc,+}(X)$. But again this can be extended to all $h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X))$ by lemma 2.2.1. □

In [53] Nguyen and Zessin have shown that each solution of $\left(\mathbf{\Sigma}'_\gamma\right)$ has the local specifications of a Gibbs point process with parameters z and ϕ in the sense of Dobrushin-Lanford-Ruelle. So we have identified $G_{z,\phi}$ as a Gibbs point process. Since X can be a general Polish space, lattice as well as continuous systems are covered by theorem 5.2.4.

A small drawback is that we could construct $G_{z,\phi}$ for $z \in (0, \frac{e^{-2B-1}}{C_\phi})$ but the Gibbs property could only be shown for $z \in (0, \frac{e^{-4B-1}}{C_\phi})$. If there exists a sharper estimate than the one given by lemma 5.2.5 this difficulty might be overcome.
In [64] Ruelle found that if the conditions of regularity, superstability and lower regularity are imposed on a translation invariant pair potential then there exists a Gibbs point process for *any* activity. Kuna obtained in [36] by the method of cluster expansions combined with an analysis of Kirkwood-Salzburg equations that Ruelle's superstability assumption can be replaced by stability if we choose the activity sufficiently small, that is $z \in (0, \frac{e^{-2B-1}}{C_\phi})$. Like us, Kuna is in the setting of general pair potentials ϕ, which means that ϕ does not have to be translation invariant. Here we have shown that if one chooses the activity even smaller, $z \in (0, \frac{e^{-4B-1}}{C_\phi})$, then also the condition of lower regularity can be dropped.
We have shown that the cluster equation $(\mathbf{\Sigma}_\Re)$ implies $\left(\mathbf{\Sigma}'_\gamma\right)$. A natural question is if also $\left(\mathbf{\Sigma}'_\gamma\right) \Rightarrow (\mathbf{\Sigma}_\Re)$ holds. If this is the case then no phase transition can occur since $(\mathbf{\Sigma}_\Re)$ has a unique solution. In [64] theorem 5.7. Ruelle remarked that for small z there exists a unique solution to the Dobrushin-

Lanford- Ruelle equations for his class of superstable pair potentials. So it is suggestive to ask whether this remains true in our setting.

Remark 5.2.8. *The proof to theorem 5.2.4 gives a general scheme for showing that a point process with a signed Lévy pseudo measure L, such that $|L|$ is of first order, is a Papangelou point process. Assume you have a candidate π for the Papangelou kernel of $\Im_L$ and π satisfies the cocycle condition. Assume also that the Schur measures are of the form $\varrho_k = \pi^{(k)}(0;\cdot)$. Then if for $f, g \in F_{bc,+}(X)$*

$$\lim_{\Lambda \uparrow X} \int\limits_{\mathcal{M}^{\cdot\cdot}(\Lambda)} \pi(\mu, f) e^{-\mu(g)}\, Q_\Lambda(\mathrm{d}\,\mu) = \int\limits_{\mathcal{M}^{\cdot\cdot}(X)} \pi(\mu, f) e^{-\mu(g)}\, \Im_L(\mathrm{d}\,\mu),$$

we can indentify π as the Papangelou kernel of $\Im_L$. Later the same method will be applied to determine the Papangelou kernel of the so called Pólya branching process in theorem 6.2.3.

In the end let us give an open question: Is theorem 5.2.4, for a sufficiently small activity, still valid if we only assume ($\mathcal{A}1$) weak stability and ($\mathcal{A}2$) weak regularity?

5.3 Polymer Systems

Again quoting Dobrushin from his Saint-Flour lectures [14]: ¶*An essential contribution to it* [the cluster expansion method] *was made by Gruber and Kunz [25] who introduced the so-called polymer model.*

Now consider the lattice $\mathbb{Z}^d$ for some $d \in \mathbb{N}$. Our space X is the set of finite connected subsets of the lattice. It is of countably infinite cardinality. We equip X with the discrete topology, that is every subset of X is open (and hence closed). Remark that a set in X is compact if and only if it is finite. By $|x|$ for $x \in X$ we denote the number of lattice sites x consists of. The measure ρ is now given by

$$\rho(\mathrm{d}\,x) = z(x)\ \mathrm{cd}(\mathrm{d}\,x),\ z(x) = e^{-\gamma|x|} \text{ for some } \gamma > 0, \tag{5.9}$$

and cd is the counting measure on X. $z(x)$ is called the *activity.* Now let us introduce a pair potential on X:

$$\phi(x, y) = \begin{cases} \infty, & \text{if } x \cap y \neq \emptyset \\ -\eta\, c(x, y), & \text{if } x \cap y = \emptyset \\ 0, & x \text{ and } y \text{ have no neighbours,} \end{cases} \tag{5.10}$$

where $c(x, y)$ denotes the number of neighbours of x and y and $\eta > 0$ is some parameter. The hard core condition excludes the possibility of overlapping polymers. Furthermore ϕ is of finite range. Poghosyan and Ueltschi have shown in [58] that ϕ satisfies $(\mathcal{A}1)$ with

$$b(x) = \eta d|x|$$

and $(\mathcal{A}2')$ is satisfied with $a(x) = (2d)^{-\frac{3}{2}}|x|$ if the parameter γ in the activity (5.9) is chosen sufficiently large, that is

$$\gamma \geq 2(2d)^{-\frac{3}{2}} + 3d\eta + 2\log(2d). \tag{5.11}$$

Now since every $B \in \mathcal{B}_0(X)$ has finite cardinality as remarked above, condition $(\mathcal{A}3')$ is certainly satisfied because a finite sum of finite expressions is finite. Therefore theorem 5.1.2 yields the existence of the corresponding point process $\mathfrak{I}_{\mathfrak{R}}$ of polymers. To summarize:

Theorem 5.3.1. *Under the condition (5.11) and for the choice of ρ and ϕ as in (5.9) and (5.10) there exists a point process $\mathfrak{I}_{\mathfrak{R}}$ in the space of finite connected subsets of $\mathbb{Z}^d$ such that its signed Lévy pseudo measure $\mathfrak{R}$ is given by formula (5.2).*

A natural question now is whether percolation occurs in this model, that is

$$\mathfrak{I}_{\mathfrak{R}}(\text{There exists an infinite polymer}) > 0.$$

If the attractive part of ϕ is large enough that is for large η this should be the case. Poghosyan and Ueltschi noted:
¶ *For large η one should expect interesting phases with many contacts between the polymers.*

5.4 Quantum Continuous Systems: The Bose Gas

This model has been introduced by Ginibre [19].

Here we let X be the space of winding loops in $\mathbb{R}^d$ of length $m\beta$, $m \in \mathbb{N}$ is free, and $\beta > 0$ is a fixed parameter. That is X is the set of all continuous paths $x : [0, m\beta] \to \mathbb{R}^d$ such that $x(0) = x(m\beta)$. Now the major question is: With what kind of topology should we equip X and what are the relatively compact sets in X? Let us consider the space $\mathcal{C}(\mathbb{R}^d)$ of continuous functions

$x : \mathbb{R}_+ \to \mathbb{R}^d$. X can be thought of as a subset of $\mathcal{C}(\mathbb{R}^d)$ by setting every $x \in X$ constant outside $[0, m\beta]$. It is well known that this space is Polish for the topology of uniform convergence on compacta. The bounded, measurable sets $B \in \mathcal{C}(\mathbb{R}^d)$ for the local uniform topology are characterized by the Ascoli-Arzela theorem by the following two properties:

(α) $\sup_{x \in B} |x(0)| < \infty$

(β) for all $n \geq 1$, $\lim_{r \downarrow 0} \sup_{x \in B} \omega_n(x, r) = 0$, where

$$\omega_n(x,r) = \sup_{t \in \mathbb{R}_+}\{ \sup_{s_1, s_2 \in [t, t+r]} |x(s_1) - x(s_2)| : 0 \leq t \leq t + r \leq n\}, \; r > 0, n \geq 1.$$

Recall that the Borel σ-field in X coincides with the σ-field generated by all coordinate projections $pr_t : x \mapsto x(t)$. All this can be found e.g. in [74]. In particular due to condition (α) we have that for every $B \in \mathcal{B}_0(X)$ there exists $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$ such that $B \subset X_\Lambda = \{x \in X | \, x(0) \in \Lambda\}$, take $\Lambda = \{x(0) : x \in B\}$.
Let us denote by $P_t^{a,b}$ the non-normalized law for the Brownian bridge from a to b in time t. The non-normalization is chosen such that

$$P_t^{a,b}(\mathcal{C}(\mathbb{R}^d)) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{\|a-b\|^2}{2t}}.$$

Now consider the following non negative measure on the loop space X:

$$\rho(g) = \sum_{m=1}^{\infty} \frac{z^m}{m} \int_{\mathbb{R}^d} \int_X g(x) \, e^{-v(x)} \, P_{m\beta}^{a,a}(\mathrm{d}\, x) \, \mathrm{d}\, a, \; g \in F_+(X), \tag{5.12}$$

where $z \in (0, \infty)$ is some parameter and v is the so called *self potential* on X. It is defined as follows: Let the loop $x \in X$ return in time $m\beta$ to its starting point. That is we say x consists of m elementary components $x(0, \beta]$, $x(\beta, 2\beta], \ldots, x((m-1)\beta, m\beta]$. Then we define

$$v(x) := \sum_{0 \leq j_1 < j_2 \leq m-1} \frac{1}{2} \int_0^\beta V(x(s + j_1\beta) - x(s + j_2\beta)) \, \mathrm{d}\, s, \tag{5.13}$$

here $V(a - b)$ is an ordinary pair potential on $\mathbb{R}^d$, which is stable ($\mathcal{A}i$) with stability constant $B \geq 0$. The integral in (5.13) describes the interaction of the j_1+1 with the j_2+1 component. Now we can also introduce a pair interaction on X:

$$\phi(x_1,x_2) := \sum_{j_1=0}^{m_1-1}\sum_{j_2=0}^{m_2-1}\frac{1}{2}\int_0^\beta V(x_1(s+j_1\beta)-x_2(s+j_2\beta))\,\mathrm{d}\,s, \qquad (5.14)$$

here x_i is a loop of length $m_i\beta$. Poghosyan and Ueltschi have shown in [58] that $(\mathcal{A}1)$ holds true with

$$b(x) = m(x)\beta B + v(x),$$

where $m(x)$ denotes the number of elementary components of x. Moreover under the condition $\|V\|_1 < \infty$ they obtained that $(\mathcal{A}2')$ is valid for a of the form

$$a(x) = c\cdot m(x),$$

where $c>0$ is a positive constant satisfying

$$\frac{\beta\|V\|_1}{(4\pi\beta)^{\frac{d}{2}}}\sum_{m=1}^{\infty}\frac{z^m e^{(c+\beta B)m}}{m^{\frac{d}{2}}} \le c. \qquad (5.15)$$

Inequality (5.15) is certainly satisfied for small z. In order to apply theorem 5.1.2 we are left with verifying condition $(\mathcal{A}3')$. So let $B\in\mathcal{B}_0(X)$ then as we have seen in the beginning there has to exist a $\Lambda\in\mathcal{B}_0(\mathbb{R}^d)$ such that $B\subset X_\Lambda$,

$$\begin{aligned}\rho(\mathbf{1}_B e^{a+b}) &\le \rho(\mathbf{1}_{X_\Lambda} e^{a+b})\\ &= \sum_{m=1}^{\infty}\frac{z^m}{m}\int_\Lambda\int_{X_\Lambda} e^{-v(x)+a(x)+b(x)}\,P_{m\beta}^{\tau,\tau}(\mathrm{d}\,x)\,\mathrm{d}\,\tau\\ &= \sum_{m=1}^{\infty}\frac{z^m}{m}\int_\Lambda\int_{X_\Lambda} e^{\beta Bm+cm}\,P_{m\beta}^{\tau,\tau}(\mathrm{d}\,x)\,\mathrm{d}\,\tau\\ &= \lambda(\Lambda)\sum_{m=1}^{\infty}\frac{z^m e^{(c+\beta B)m}}{m}\frac{1}{(2\pi m\beta)^{\frac{d}{2}}}.\end{aligned}$$

Thus we obtain that if z is chosen sufficiently small such that (5.15) holds true then also $\rho(\mathbf{1}_B e^{a+b})<\infty$ for all $B\in\mathcal{B}_0(X)$ is valid. So we have shown the following

Theorem 5.4.1. *Let V be an integrable stable pair potential on $\mathbb{R}^d$. Choose the measure ρ and the pair potential ϕ as in (5.12) and (5.14). Then for sufficiently small z, such that (5.15) holds, there exists a point process $\Im_{\Re}$ in the loop space X with signed Lévy pseudo measure $\Re$ as given by theorem 5.1.2 in formula (5.2).*

Chapter 6

Clustering, Thinning and Splitting. Cox Processes

We continue our study of KMM processes, in particular transformation properties. We want to know what happens under the transformations of cluster dissolution resp. clustering, in particular thinning and splitting.

In proposition 6.1.1 and theorem 6.1.5 we construct starting from solutions P to $(\mathbf{\Sigma}_L)$ new solutions $\tilde{P}$ to $(\mathbf{\Sigma}_{\tilde{L}})$. This then is applied to Pólya sum and difference processes. Two other examples are given. A new class, called branching Pólya processes, is introduced and characterized as Papangelou processes in theorem 6.2.3.

Next we consider the operations of thinning and splitting of point processes. We see in (6.11) that the thinning operation is intimately related to the Pólya difference process. On the other hand we show in corollary 6.3.3 that this operation can be reduced to the computation of factorial moment measures of some modification of the signed Lévy pseudo measure. The main result is theorem 6.3.4 which gives a representation of a q-thinned KMM process in terms of loop measures. As examples we consider Pólya, permanental and determinantal processes.

Then we pose the question: Having observed a realization μ of a q-thinning of a given point process, what is the distribution of that part of the configuration which has been deleted? This conditional distribution is described by the splitting kernel. Such kernels are computed for Pólya and Poisson processes. The main result then is theorem 6.3.7 which relates the Papangelou kernel of the thinning of a given point process to the intensity of its splitting kernel.

Finally point processes with signed Lévy pseudo measures from a certain class are identified as Cox processes. This is based on Mecke's characteriza-

tion of Cox processes in theorem 6.4.1. Examples are Pólya sum processes. An important consequence of this result is that Pólya difference processes cannot be Cox.

6.1 Invariance Properties of the Cluster Equation

Invariance under Cluster Dissolution

Let $Y = \mathcal{M}^{\cdot\cdot}(X) \setminus \{\mathbf{0}\}$. The following so called cluster dissolution mapping has its origin in [43] and was later used by [52].

$$\xi : \begin{cases} \mathcal{M}^{\cdot\cdot}(\mathcal{M}^{\cdot\cdot}(X) \setminus \{\mathbf{0}\}) \to \text{measures on } X \\ \mu \mapsto \int_Y \nu \, \mu(d\nu). \end{cases}$$

Proposition 6.1.1. *Let L be a signed cluster pseudo measure on $\mathcal{M}^{\cdot\cdot}(Y)$. If there exists a point process P in Y such that P solves $(\mathbf{\Sigma}_L)$, then the point process ξP in X solves $(\mathbf{\Sigma}_{\xi L})$.*

Proof. We have

$$C_{\xi P}(h) = \int h(x, \xi(\mu)) \, \nu(\mathrm{d}\, x)\mu(\mathrm{d}\, \nu)P(\mathrm{d}\, \mu) \text{ for } h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X)).$$

Now the right hand side of the above equation is the Campbell measure of P evaluated at the function $\widetilde{h}(\nu, \mu) = \int h(x, \xi(\mu))\nu(dx)$. By using the transformation theorem one obtains $(C_{L^\epsilon} \star P)(\tilde{h}) = (C_{\xi L^\epsilon} \star \xi P)(h)$. So if P is a solution to $(\mathbf{\Sigma}_L)$ then ξP is a solution to $(\mathbf{\Sigma}_{\xi L})$. □

Remark that if $\xi|L|$ is of first order then we have for $B \in \mathcal{B}_0(X)$

$$\xi|L|(\zeta_B) = \int \xi(\mu)(B) \, |L|(\mathrm{d}\, \mu) = \int \mu(\zeta_B) \, |L|(\mathrm{d}\, \mu) = \nu^1_{|L|}(\zeta_B) < \infty.$$

Now since the bounded sets in Y are given by $\{\zeta_B > 0\}$ for some $B \in \mathcal{B}_0(X)$ and we certainly have $\mathbf{1}_{\{\zeta_B > 0\}} \leq \zeta_B$ on Y, in this case also $|L|$ is of first order and proposition 6.1.1 can now be reformulated as follows:

Corollary 6.1.2. *Let L be a signed Lévy pseudo measure on $\mathcal{M}^{\cdot\cdot}(Y)$ such that $\xi|L|$ is of first order. Then ξL is a signed Lévy pseudo measure on $\mathcal{M}^{\cdot\cdot}(X)$ and $\xi \Im_L = \Im_{\xi L}$.*

Proof. By theorem 3.3.1 $\Im_L$ solves $(\boldsymbol{\Sigma}_L)$. Applying now proposition 6.1.1 yields that $\xi\Im_L$ solves $(\boldsymbol{\Sigma}_{\xi L})$. But since $\xi|L|$ is assumed to be of first order we conclude by proposition 3.3.3. □

Example 6.1.1. *Let H_ϵ be a measure on Y of first order, $\epsilon \in \{-1, +1\}$. In case $\epsilon = -1$ we additionally require that $H_{-1} \in \mathcal{M}^{\cdot\cdot}(Y)$. Consider the following signed cluster pseudo measure*

$$L_\epsilon(\varphi) = \sum_{n=1}^{\infty} \epsilon^{n-1} \frac{z^n}{n} \int_Y \varphi(n\delta_\mu)\, H_\epsilon(\mathrm{d}\,\mu) \text{ for } \varphi \in \mathfrak{L}^1(|L_\epsilon|).$$

And let us verify that the conditions of corollary 6.1.2 are satisfied. A straightforward computation yields that

$$\nu^1_{\xi|L_\epsilon|}(f) = \frac{z}{1-z} \nu^1_{H_\epsilon}(f) < \infty \text{ for } f \in F_{bc,+}(X).$$

And again due to $\mathbf{1}_{\{\zeta_B > 0\}} \leq \zeta_B$ on Y for $B \in \mathcal{B}_0(X)$ implies $H_\epsilon \in \mathcal{M}(Y)$, so that $\Im_{L_+} = P^+_{z,H_+}$ is a Pólya sum and $\Im_{L_-} = P^-_{z,H_-}$ a Pólya difference process in Y. Now corollary 6.1.2 gives us that the Pólya cluster dissolution $\xi P^\epsilon_{z,H_\epsilon}$ has a signed Lévy pseudo measure given by

$$\xi L_\epsilon(\varphi) = \sum_{n=1}^{\infty} \epsilon^{n-1} \frac{z^n}{n} \int_Y \varphi(n\mu)\, H_\epsilon(\mathrm{d}\,\mu) \text{ for } \varphi \in \mathfrak{L}^1(\xi|L_\epsilon|).$$

So ξL_ϵ is given by the (non negative measure) H_ϵ except that each realization of H_ϵ gets a weight according to the "pseudo distribution" $n \mapsto \epsilon^{n-1}\frac{z^n}{n}$ on $\mathbb{N}$.

In corollary 6.1.3 below we will see that a large class of point processes admit a signed Lévy pseudo measure. But at first, we need to introduce the so called *simple Fermi process.*

Example 6.1.2 (Simple Fermi process)**.** *Let η be a non negative finite measure on X with total mass $\eta(X) \in (0,1)$ and let the cumulant measures be given by $\Theta_k = (-1)^{k-1}\, \eta^{\otimes k}$. Then*

$$\nu^1_{|L|}(f) = \frac{\eta(f)}{1-\eta(X)} \text{ for } f \in F_{bc,+}(X),$$

and the Schur measures are given by

$$\varrho_k(\bigotimes_{j=1}^{k} f_j) = \sum_{\sigma \in S_k} (-1)^{k-|\sigma|} \prod_{j=1}^{k} \eta(f_j) = \begin{cases} \eta(f_1), & \text{for } k = 1 \\ 0, & \text{for } k \geq 2, \end{cases}$$

for $f_1, \dots, f_k \in F_{bc,+}(X)$*. Since the determinant of the matrix with* 1 *in every entry is zero. Furthermore*

$$\Im_L(\varphi) = \frac{\varphi(0) + \int_X \varphi(\delta_x)\,\eta(\mathrm{d}\,x)}{1+\eta(X)} \text{ for } \varphi \in F_+(\mathcal{M}^{\cdot\cdot}(X)).$$

Remark that its Papangelou kernel is given by

$$\pi(\mathbf{0}, \mathrm{d}\,x) = \eta(\mathrm{d}\,x) \text{ and } \pi(\mu, \mathrm{d}\,x) = \mathbf{0} \text{ for } \mu \neq \mathbf{0}.$$

A similar version of the next result can already be found in [31] page 74 remark 1.9.9., we present an alternative proof.

Corollary 6.1.3. *Let* P *be a point process in* X *with* $P(\{0\}) > \frac{1}{2}$*. Then* P *is a solution to* $(\mathbf{\Sigma}_L)$ *where* L *is given by*

$$L = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} Q^{*n},$$

with $Q = \frac{P|_Y}{P(\{0\})}$ *and* $P|_Y$ *denotes the restriction of* P *to* $Y = \mathcal{M}^{\cdot\cdot}(X) \setminus \{\mathbf{0}\}$*.*

Proof. The condition $P(\{0\}) > \frac{1}{2}$ implies $Q(Y) < 1$. So let R be the simple Fermi process (see example 6.1.2) in Y to Q. And denote by F its signed Lévy pseudo measure , so that

$$F(\varphi) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \int_{Y^n} \varphi(\delta_{\nu_1} + \ldots + \delta_{\nu_n})\, Q(\mathrm{d}\,\nu_1) \ldots Q(\mathrm{d}\,\nu_n),\ \varphi \in \mathfrak{L}^1(|F|).$$

Proposition 6.1.1 tells us that ξR is a solution to $(\mathbf{\Sigma}_{\xi F})$. But $\xi F = L$ and (see example 6.1.2)

$$R(\varphi) = \frac{\varphi(0) + \int_Y \varphi(\delta_\nu)\, Q(\mathrm{d}\,\nu)}{1+Q(Y)} \text{ for } \varphi \in F_+(\mathcal{M}^{\cdot\cdot}(Y)).$$

Which yields combined with $1 + Q(Y) = P(\{0\})^{-1}$

$$(\xi R)(\varphi) = \frac{\varphi(0) + \int_Y \varphi(\nu)\, Q(\mathrm{d}\,\nu)}{1+Q(Y)} = \varphi(0)P(\{0\}) + \int_Y \varphi(\nu)\, P(\mathrm{d}\,\nu) = P(\varphi)$$

for $\varphi \in F_+(\mathcal{M}^{\cdot\cdot}(X))$. □

Invariance under Clustering

For an introduction to the notions concerning clustering we refer to section 2.4.

The next lemma can also be found in [31] proposition 5.2.3. Here we give an alternative proof.

Lemma 6.1.4. *Let* $\{\Pi_a\}_{a\in E}$ *be a cluster field on* E. *Assume that for some* $\mu \in \mathcal{M}^{\cdot\cdot}(E)$ *we have*

$$\int \nu^1_{\Pi_a}(f)\,\mu(\mathrm{d}\,a) < \infty \text{ for } f \in U. \tag{6.1}$$

Then Π_μ *exists and there holds*

$$C_{\Pi_\mu}(h) = \int (C_{\Pi_a} \star \Pi_{\mu-\delta_a})(h)\,\mu(\mathrm{d}\,a) \text{ for } h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X)).$$

Proof. Existence of Π_μ follows by proposition 2.4.1. Now, if $a \in \mu$, then we have

$$\begin{aligned} C_{\Pi_\mu}(h) &= \int h(x, \kappa_1 + \kappa_2)\,(\kappa_1 + \kappa_2)(\mathrm{d}\,x)\Pi_{\mu-\delta_a}(\mathrm{d}\,\kappa_1)\Pi_a(\mathrm{d}\,\kappa_2) \\ &= (C_{\Pi_{\mu-\delta_a}} \star \Pi_a)(h) + (C_{\Pi_a} \star \Pi_{\mu-\delta_a})(h) \end{aligned} \tag{6.2}$$

Now since we have for any measure C on $X \times \mathcal{M}^{\cdot\cdot}(X)$ and point processes $P_1, P_2 \in \mathcal{PM}^{\cdot\cdot}(X)$ that

$$(C \star P_1) \star P_2 = C \star (P_1 * P_2),$$

iterated application of (6.2) yields for any $B \in \mathcal{B}_0(E)$

$$C_{\Pi_\mu}(h) = (C_{\Pi_{\mu_{B^c}}} \star \Pi_{\mu_B})(h) + \int_B (C_{\Pi_a} \star \Pi_{\mu-\delta_a})(h)\,\mu(\mathrm{d}\,a),$$

where μ_B denotes the restriction of μ to B. Let now h be of the form $f \otimes e^{-\zeta_g}$ with $f, g \in F_{bc,+}(X)$. Then

$$(C_{\Pi_{\mu_{B^c}}} \star \Pi_{\mu_B})(h) \le \Pi_{\mu_{B^c}}(\zeta_f) = \int_{B^c} \nu^1_{\Pi_a}(f)\,\mu(\mathrm{d}\,a).$$

Now if we let $B \uparrow E$ the above right hand side tends to zero due to (6.1). So we obtain the assertion for this special class of h. Lemma 2.2.1 now yields that this can be extended to all non negative measurable h. $\square$

The following theorem can be already found in [31] theorem 4.3.3. in the classical case of a non negative Lévy measure L. But in contrast to [31] we will use the cluster equation $(\mathbf{\Sigma}_L)$, which simplifies the proof but requires existence of the first moments, which is not needed in [31].

Theorem 6.1.5. *Let a signed Lévy pseudo measure L and a cluster field $(\Pi_a)_{a\in E}$ be given such that $\nu^1_{|L|}(\nu^1_{\Pi_{(\cdot)}}(f)) < \infty$ for $f \in F_{bc,+}(X)$. Then $\Im_L \Pi_{(\cdot)} = \Im_{L\Pi_{(\cdot)}}$.*

So the clustering of a KMM process with signed Lévy pseudo measure L is given by the KMM process with clustered L.

Proof. Recall that $\nu^1_{\Im_L} \leq \nu^1_{|L|}$ on $F_+(X)$, so condition (6.1) is satisfied $\Im_L - a.s.[\mu]$ and $|L| - a.e.[\mu]$. Now due to lemma 6.1.4 there holds for $W = \Im_L$ or L^ϵ and $h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X))$

$$C_{W\Pi_{(\cdot)}}(h) = \int C_{\Pi_\mu}(h)\, W(\mathrm{d}\,\mu) = \iint (C_{\Pi_a} \star \Pi_{\mu-\delta_a})(h)\, \mu(\mathrm{d}\,a) W(\mathrm{d}\,\mu). \quad (6.3)$$

Let us denote $h_\kappa(x,\eta) = h(x, \eta + \kappa)$. So we have $(C_{L^\epsilon\Pi_{(\cdot)}} \star \Im_L\Pi_{(\cdot)})(h) = \int C_{L^\epsilon\Pi_{(\cdot)}}(h_\kappa)\, (\Im_L\Pi_{(\cdot)})(\mathrm{d}\,\kappa)$. By using (6.3) with $W = L^\epsilon$ we obtain

$$\begin{aligned} C_{L^\epsilon\Pi_{(\cdot)}}(h_\kappa) &= \iint (C_{\Pi_a} \star \Pi_{\mu-\delta_a})(h_\kappa)\, \mu(\mathrm{d}\,a) L^\epsilon(\mathrm{d}\,\mu) \\ &= \iint (C_{\Pi_a} \star (\Pi_{\mu-\delta_a} * \delta_\kappa))(h)\, \mu(\mathrm{d}\,a) L^\epsilon(\mathrm{d}\,\mu) \end{aligned}$$

If we now employ the definition of the cluster process $\Im_L\Pi_{(\cdot)} = \int \Pi_\eta\, \Im_L(\mathrm{d}\,\eta)$ one obtains

$$\begin{aligned} (C_{L^\epsilon\Pi_{(\cdot)}} \star \Im_L\Pi_{(\cdot)})(h) &= \iiint (C_{\Pi_a} \star (\Pi_{\mu-\delta_a} * \Pi_\eta))(h)\, \mu(\mathrm{d}\,a) L^\epsilon(\mathrm{d}\,\mu) \Im_L(\mathrm{d}\,\eta) \\ &= \iiint (C_{\Pi_a} \star \Pi_{\eta+\mu-\delta_a})(h)\, \eta(\mathrm{d}\,a) L^\epsilon(\mathrm{d}\,\mu) \Im_L(\mathrm{d}\,\eta) \\ &= (C_{L^\epsilon} \star \Im_L)(\tilde{h}), \end{aligned}$$

where $\tilde{h}(a,\mu) = (C_{\Pi_a} \star \Pi_{\mu-\delta_a})(h)$. Moreover (6.3) for $W = \Im_L$ yields $C_{\Im_L\Pi_{(\cdot)}}(h) = C_{\Im_L}(\tilde{h})$. Now we are ready to show that $\Im_L\Pi_{(\cdot)}$ solves $\left(\mathbf{\Sigma}_{L\Pi_{(\cdot)}}\right)$, by using that $\Im_L$ solves $(\mathbf{\Sigma}_L)$.

$$\begin{aligned} &C_{\Im_L\Pi_{(\cdot)}}(h) + (C_{L^-\Pi_{(\cdot)}} \star \Im_L\Pi_{(\cdot)})(h) = C_{\Im_L}(\tilde{h}) + (C_{L^-} \star \Im_L)(\tilde{h}) \\ &= (C_{L^+} \star \Im_L)(\tilde{h}) = (C_{L^+\Pi_{(\cdot)}} \star \Im_L\Pi_{(\cdot)})(h). \end{aligned}$$

Now since $|L|\Pi_{(\cdot)}$ was assumed to be of first order we conclude by proposition 3.3.3. □

6.2 The Branching Pólya Processes

Let κ be a probability kernel from X to X. Then a natural cluster field is given by

$$\Pi_x = \int \kappa_x(\mathrm{d}\, y)\delta_{\delta_y}.$$

In this section we want to study the Pólya processes clustered according to $\{\Pi_x\}_{x\in X}$. That is the point processes

$$\mathbf{P}^\epsilon = \int P^\epsilon_{z,\lambda}(\mathrm{d}\,\mu) \underset{x\in\mu}{*} \int \kappa_x(\mathrm{d}\, y)\delta_{\delta_y}.$$

If we denote by L_ϵ the signed Lévy pseudo measure of $P^\epsilon_{z,\lambda}$ then theorem 6.1.5 tells us that if $\nu^1_{|L_\epsilon|}(\nu^1_{\Pi_{(\cdot)}}(f)) = \frac{z}{1-z}\lambda(\kappa(f)) < \infty$, for all $f \in F_{bc,+}(X)$, $\mathbf{P}^\epsilon$ exists and has signed Lévy pseudo measure

$$\mathbf{L}_\epsilon(\varphi) = \sum_{n=1}^{\infty} \frac{\epsilon^{n-1}}{n} z^n \int\limits_{X^{n+1}} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \kappa_x(\mathrm{d}\, x_1) \ldots \kappa_x(\mathrm{d}\, x_n)\, \lambda(\mathrm{d}\, x), \quad (6.4)$$

for $\varphi \in \mathfrak{L}^1(|\mathbf{L}_\epsilon|)$. Remark that we have $\nu^1_{\mathbf{P}^\epsilon}(f) = \frac{z}{1-\epsilon z}\lambda(\kappa(f))$. In the sequel we will denote $(\lambda\circ\kappa)(f) = \lambda(\kappa(f))$ for $f \in F_{bc,+}(X)$. The Pólya sum respective difference processes owe their name to the structure of their Papangelou kernel. In [50] it is shown that the Papangelou kernel of $P^\epsilon_{z,\lambda}$ is given by

$$\pi_\epsilon(\mu, \mathrm{d}\, x) = z(\lambda + \epsilon\mu)(\mathrm{d}\, x).$$

Now the *main question* is: What is the Papangelou kernel $\boldsymbol{\pi}_\epsilon$ of $\mathbf{P}^\epsilon$? A natural Ansatz would be $\boldsymbol{\pi}_\epsilon(\mu, f) = \pi_\epsilon(\mu, \kappa(f))$ for $\mu \in \mathcal{M}^{\cdot\cdot}(X)$ and $f \in F_{bc,+}(X)$. But the next result shows that $\boldsymbol{\pi}_\epsilon$ can only be a Papangelou kernel if κ is a regular conditional probability on X.

Lemma 6.2.1. *Let κ be a probability kernel from X to X.*

(i) *$\boldsymbol{\pi}_\epsilon$ satisfies the cocycle condition that is for all $\mu \in \mathcal{M}^{\cdot\cdot}(X)$*

$$\boldsymbol{\pi}^{(2)}_\epsilon(\mu, f_1 \otimes f_2) = \boldsymbol{\pi}^{(2)}_\epsilon(\mu, f_2 \otimes f_1), \text{ for } f_1, f_2 \in F_+(X).$$

(ii) *κ coincides with the regular conditional probability $\kappa_x(A) = H[A|\mathcal{E}](x)$, where $\mathcal{E}$ is a sub σ-field of $\mathcal{B}(X)$ and the probability H on X is given by*

$$H(f) = \int \lambda(\mathrm{d}\, x)\, \phi(x) \int \kappa_x(\mathrm{d}\, y)\, f(y) \quad \text{for } f \in F_+(X),$$

where $\phi > 0$ is chosen so that $\lambda(\phi) = 1$.

(ii') *For any* $N \in \mathcal{B}(X)$, $\mathbf{P}^\epsilon\{\zeta_N > 0\} = 0$ *implies* $P^\epsilon_{z,\lambda}\{\zeta_N > 0\} = 0$.

(iii) *There exists* $N \in \mathcal{B}(X)$ *such that* $\mathbf{P}^\epsilon\{\zeta_N > 0\} = 0$ *and for all* $\mu \in \mathcal{M}^{\cdot\cdot}(N^c)$

$$\boldsymbol{\pi}^{(2)}_\epsilon(\mu, f_1 \otimes f_2) = \boldsymbol{\pi}^{(2)}_\epsilon(\mu, f_2 \otimes f_1), \text{ for } f_1, f_2 \in F_+(X). \tag{6.5}$$

Then the following implications hold true (i) $\Rightarrow$ (ii) *and* (ii) + (ii') $\Rightarrow$ (iii).

Proof. Let us assume (i) and compute

$$\boldsymbol{\pi}^{(2)}_\epsilon(\mu, f_1 \otimes f_2) = \prod_{i=1}^{2} \pi_\epsilon(\mu, f_i) + \epsilon\pi_\epsilon(\mu, \kappa(f_1\kappa(f_2))).$$

If we evaluate (i) for μ equal to the zero measure we obtain $\lambda(\kappa(f_1\kappa(f_2))) = \lambda(\kappa(f_2\kappa(f_1)))$. Setting now $\mu = \delta_x$ yields

$$\kappa_x(f_1\kappa(f_2)) = \kappa_x(f_2\kappa(f_1)) \text{ for } x \in X. \tag{6.6}$$

Recall Bahadurs [3] characterization of conditional expectation: Let H be a probability on $(X, \mathcal{B})$ and T an operator on $L^2(X, \mathcal{B}, H)$ into itself. Then T is a conditional expectation if and only if:

(1) T is linear (2) $f \geq 0 \Rightarrow Tf \geq 0$ (3) $T^2 = T$
(4) $H(f\,Tg) = H(g\,Tf)$ (5) T is constant preserving,

and the above properties $(1)-(5)$ have to hold H-a.s.. Condition (3) respective (4) mean that T is a idempotent respective self adjoint operator.
Now let H be as in (ii). Since κ is a probability kernel it certainly satisfies $(1), (2), (5)$. (3) holds on $F_+(X)$ due to (6.6) and if we apply H to both sides of (6.6) and apply (3) we obtain (4) for $f, g \in F_+(X)$. It remains to be seen that $\kappa : L^2(H) \to L^2(H)$. First we have to show that for $f \in L^2(H) \subset L^1(H)$, $\kappa(f)$ is well defined, that is

$$\kappa(|f|) < \infty \quad H - a.s. \tag{6.7}$$

But if we apply (4) with $f = \mathbf{1}$ and $g = |f|$ then we obtain $H(\kappa(|f|)) = H(|f|) < \infty$. It remains to be seen that $\kappa(f) \in L^2(H)$. Due to (6.7) we can apply Jensens inequality H-a.s. and obtain $H(\kappa(f)^2) \leq H(\kappa(f^2)) = H(f^2) < \infty$. By decomposition of the respective functions in positive and negative part the properties $(1)-(5)$ extend from $F_+(X)$ to $L^2(H)$.

Let us now discuss the the condition (ii').For any point process $\mathtt{P}$ we have $\mathtt{P}\{\zeta_N > 0\} = 0$ is equivalent to $\nu^1_{\mathtt{P}}(N) = 0$. Now since we have

$$\nu^1_{\mathbf{P}^\epsilon} = \frac{z}{1-\epsilon z}\lambda \circ \kappa \text{ and } \nu^1_{P^\epsilon_{z,\lambda}} = \frac{z}{1-\epsilon z}\lambda$$

condition (ii') is equivalent to $\lambda \ll \lambda \circ \kappa$. Furthermore the probability measure H in (ii) is equivalent to $\lambda \circ \kappa$, which leads to $\lambda \ll H$.

Let us now assume (ii) and (ii'). Remark that a ϕ as in (ii) always exists, since we can choose a partition $\{B_j\}_{j=1}^\infty$ of X such that $\lambda(B_j) > 0$. Now $\phi = \sum_{j\geq 1}(2)^{-j}(\lambda(B_j))^{-1}\mathbf{1}_{B_j}$ has the required properties. Since κ is a regular conditional probability there exists $N \in \mathcal{B}(X)$ with $H(N) = 0$ and

$$\kappa_x(f_1\kappa(f_2)) = \kappa_x(f_2\kappa(f_1)) \text{ for } x \in N^c. \tag{6.8}$$

But due to the above remark we also have $\lambda(N) = 0$ and so we can integrate (6.8) with respect to λ and obtain $\lambda(\kappa(f_1\kappa(f_2))) = \lambda(\kappa(f_2\kappa(f_1)))$, which yields (iii). □

In order not to get confused with null sets we shall in the sequel work under the

Assumption 6.2.2. *κ is a probability kernel such that for all $x \in X$*

$$\kappa_x(f_1\kappa(f_2)) = \kappa_x(f_1)\kappa_x(f_2) \textit{ for } f_1, f_2 \in F_+(X).$$

Theorem 6.2.3. *Let κ be as in assumption 6.2.2 and $\lambda \circ \kappa \in \mathcal{M}(X)$. Then $\mathbf{P}^\epsilon$ exists and its Papangelou kernel is given by $\boldsymbol{\pi}_\epsilon$.*

Proof. The existence of $\mathbf{P}^\epsilon$ under the condition $\lambda(\kappa(f)) < \infty$ for $f \in U$ has already been established in the introduction. It remains to be seen if $\mathbf{P}^\epsilon$ satisfies the integration by parts formula $\left(\boldsymbol{\Sigma}'_{\boldsymbol{\pi}_\epsilon}\right)$. We will follow the same scheme already used in the proof of theorem 5.2.4 and explicitly described in remark 5.2.8. Let us start by computing the Schur measures. Since the cumulant measures can be directly read of from (6.4) we have

$$\begin{aligned}\boldsymbol{\varrho}_{\epsilon,n}(\bigotimes_{j=1}^n f_j) &= z^n \sum_{\sigma\in S_n} \epsilon^{n-|\sigma|} \prod_{\omega\in\sigma} \int \prod_{j\in\omega} \kappa_x(f_j)\, \lambda(\mathrm{d}\, x) = \varrho_{\epsilon,n}(\bigotimes_{j=1}^n \kappa(f_j)) \\ &= \pi_\epsilon^{(n)}(0, \bigotimes_{j=1}^n \kappa(f_j)) \text{ for } f_1,\dots,f_k \in F_{bc,+}(X).\end{aligned}$$

The following lemma will be decisive.

Lemma 6.2.4. *We have for $m \geq 1$ and $\mu \in \mathcal{M}^{\cdot\cdot}(X)$*

$$\boldsymbol{\pi}_\epsilon^{(m)}(\mu, f_1 \otimes \ldots \otimes f_m) = \pi_\epsilon^{(m)}(\mu, \kappa(f_1) \otimes \ldots \otimes \kappa(f_m)).$$

Proof. This will be done by induction.

$$\begin{aligned}
&\boldsymbol{\pi}_\epsilon^{(m)}(\mu, f_1 \otimes \ldots \otimes f_m) \\
&= \int_{X^{m-1}} f_1(x_1) \ldots f_{m-1}(x_{m-1}) \boldsymbol{\pi}_\epsilon(\mu + \delta_{x_1} + \ldots + \delta_{x_{m-1}}, f_m) \\
&\boldsymbol{\pi}_\epsilon^{(m-1)}(\mu, \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_{m-1}) \\
&= \int_{X^{m-1}} \kappa_{x_1}(f_1) \ldots \kappa_{x_{m-1}}(f_{m-1}) \boldsymbol{\pi}_\epsilon(\mu + \delta_{x_1} + \ldots + \delta_{x_{m-1}}, f_m) \\
&\pi_\epsilon^{(m-1)}(\mu, \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_{m-1}) \\
&= \int_{X^{m-1}} \kappa_{x_1}(f_1) \ldots \kappa_{x_m}(f_m)\, \pi_\epsilon^{(m)}(\mu, \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_m)
\end{aligned}$$

The second equation is due to the inductive Hypothesis and the fact that for any $\mu \in \mathcal{M}^{\cdot\cdot}(X)$ and $x \in X$, $\kappa_x(\boldsymbol{\pi}_\epsilon(\mu + \delta_{(\cdot)}, f)\, g) = \boldsymbol{\pi}_\epsilon(\mu + \delta_x, f)\, \kappa_x(g)$ for $f, g \in F_+(X)$ due to assumption 6.2.2. □

So lemma 6.2.4 gives us $\boldsymbol{\varrho}_{\epsilon,n} = \boldsymbol{\pi}_\epsilon^{(n)}(0, \cdot)$. Thus we have identified $\mathbf{Q}_{\epsilon,\Lambda}$ as the finite Papangelou process $P^0_{\boldsymbol{\pi}_\epsilon}$ in $\Lambda \in \mathcal{B}_0(X)$, that is

$$C_{\mathbf{Q}_{\epsilon,\Lambda}}(h) = \iint h(x, \mu + \delta_x)\, \boldsymbol{\pi}_\epsilon(\mu, \mathrm{d}\, x) \mathbf{Q}_{\epsilon,\Lambda}(\mathrm{d}\, \mu) \text{ for } h \in F_+(\Lambda \times \mathcal{M}^{\cdot\cdot}(\Lambda)).$$

Let now h be of the form $f \otimes e^{-\zeta_g}$ for $f, g \in F_{bc,+}(X)$. The method of proof will now be to show that in the limit $\Lambda \uparrow X$ the above integration by parts formula remains valid. In corollary 3.3.2 we have already shown $C_{\mathbf{Q}_{\epsilon,\Lambda}}(h) \to C_{\mathbf{P}_\epsilon}(h)$ as $\Lambda \uparrow X$. Now if we choose Λ such that $\mathrm{supp}(f), \mathrm{supp}(g) \subset \Lambda$ and evaluate the above right hand side we obtain

$$z\lambda(\kappa(fe^{-g})) \mathcal{L}_{\mathbf{Q}_{\epsilon,\Lambda}}(g) + \epsilon z C_{\mathbf{Q}_{\epsilon,\Lambda}}(\kappa(fe^{-g}) \otimes e^{-\zeta_g}).$$

Since $\mathcal{L}_{\mathbf{Q}_{\epsilon,\Lambda}}(g) \to \mathcal{L}_{\mathbf{P}_\epsilon}(g)$ as $\Lambda \uparrow X$ holds,

$$C_{\mathbf{Q}_{\epsilon,\Lambda}}(\kappa(fe^{-g}) \otimes e^{-\zeta_g}) \to C_{\mathbf{P}_\epsilon}(\kappa(fe^{-g}) \otimes e^{-\zeta_g}) \quad \text{as } \Lambda \uparrow X$$

remains to be seen. According to the remark after corollary 3.3.2 the above convergence holds if $\nu^1_{|\mathbf{L}_\epsilon|}(\kappa(fe^{-g})) < \infty$. But

$$\nu^1_{|\mathbf{L}_\epsilon|}(\kappa(fe^{-g})) = \frac{z}{1+z} \lambda(\kappa(\kappa(fe^{-g}))) = \frac{z}{1+z} \lambda(\kappa(fe^{-g})) < \infty.$$

□

Now let us consider some examples of regular conditional probabilities $\kappa_x(A) = H(A|\mathcal{E})(x)$, conditioned on some sub σ-field $\mathcal{E}$.

Example 6.2.1. *(i) Let $\mathcal{E} = \mathcal{B}(X)$, then a regular version of $H(\cdot|\mathcal{E})$ is given by*

$$\kappa_x = \delta_x.$$

(ii) Let $\{B_j\}_{j=1}^N$, with $N \in \mathbb{N} \cup \{+\infty\}$, be a partition of X such that $\lambda(B_j) < \infty$. Then a regular version is given by

$$\kappa_x(f) = \sum_{j=1}^{N} \frac{H(1_{B_j} f)}{H(B_j)} 1_{B_j}(x).$$

(iii) Let $X = \mathbb{R}^d$ and $\mathcal{E} = \{A \in \mathcal{B}(X) | \, A = \sigma(A), \, \sigma \in S_d\}$ the σ- algebra of permutation invariant sets then a regular version is given by

$$\kappa_{(x_1,\ldots,x_d)} = \frac{1}{d!} \sum_{\sigma \in S_d} \delta_{(x_{\sigma(1)},\ldots,x_{\sigma(d)})}.$$

Thus in the first case points stay at their location and the branching process is the process itself. In the second case, X is partitioned into seperate islands and points are not allowed to migrate between them. In the last case a point x is transformed by a permutation of its coordinates. Remark that the examples $(i) - (iii)$ satisfy assumption 6.2.2 as well as $\lambda \circ \kappa \in \mathcal{M}(X)$.

Let $B \in \mathcal{E}$, then for $H - a.s.[x]$ we have $\kappa_x(B) = \mathbf{1}_B(x)$ since κ_x is given by a conditional probability and the child of x is again contained in B. Thus the richer $\mathcal{E}$ is the more restrictions are put on the branching mechanism.

Remark 6.2.5. *Let $\lambda \ll \lambda \circ \kappa$ that is for any $N \in \mathcal{B}(X)$, $\mathbf{P}^\epsilon\{\zeta_N > 0\} = 0$ implies $P^\epsilon_{z,\lambda}\{\zeta_N > 0\} = 0$, which is the same assumption as in (iii) of lemma 6.2.1, then*

$$\mathcal{L}_{\mathbf{P}^\epsilon}(g) = \mathcal{L}_{P^\epsilon_{z,\lambda}}(g) \text{ for } g \in F_{bc,+}(X, \mathcal{E}).$$

Proof. By using the signed Lévy pseudo measure (see (6.4)) of $\mathbf{P}^\epsilon$ and the expansion of the logarithm $\log(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$, $x \in (-1, 1)$ we obtain for $f \in F_{bc,+}(X)$

$$\mathcal{L}_{\mathbf{P}^\epsilon}(f) = \exp\left(-\epsilon \int \log\left(\frac{1 - \epsilon z \kappa_x(e^{-f})}{1 - \epsilon z}\right) \lambda(\mathrm{d}\, x)\right). \tag{6.9}$$

Now we have $\kappa_x(e^{-g}) = e^{-g(x)}$ $H - a.s.[x]$ for $g \in F_{bc,+}(X, \mathcal{E})$ and as discussed in lemma 6.2.1 (ii) there holds $\lambda \ll H$. So $\kappa_x(e^{-g}) = e^{-g(x)}$ $\lambda - a.e.[x]$ holds true, too. □

Under the technical condition $\lambda \ll \lambda \circ \kappa$ we now obtain the following *weak independence of increments* property: For $B_1, \ldots, B_n \in \mathcal{E} \cap \mathcal{B}_0(X)$ the joint distribution of $\zeta_{B_1}, \ldots, \zeta_{B_n}$ coincide under $\mathbf{P}^\epsilon$ and $P^\epsilon_{z,\lambda}$.

6.3 Thinning and Splitting

Heuristically an independent q-thinning, with $q \in (0,1)$, of a point process P is given as follows: Each point x in a realization μ of P will be either deleted with probability $1-q$ or will survive with probability q and this will be done independently for all points in μ. Certainly $\Gamma_q(P)$ always exists and it is of first order if and only if P has this property. One can generalize the independent thinning by letting the survival probability be position dependent $q : X \to (0,1)$. We will work in the setting of constant q but nearly all results carry over to this more general setting.

For an introduction to thinning and the main notations we refer to section 2.4.

Corollary 6.3.1 (to theorem 6.1.5)**.** *If L is a signed Lévy pseudo measure such that $|L|$ is of first order then $\Gamma_q(\Im_L) = \Im_{\Gamma_q(L)}$, where $\Gamma_q(L)^\epsilon = \Gamma_q(L^\epsilon)$.*

So a thinning of a KMM process with signed Lévy pseudo measure L is given by the KMM process with thinned L. In the following we want to establish a connection between the Pólya difference process $P^-_{z,\lambda}$ and the family $(\Phi_x)_{x\in X}$, as introduced in section 2.4.

Lemma 6.3.2. *We have $\Phi_\mu = P^-_{\frac{q}{1-q},\mu}$ for $\mu \in \mathcal{M}^{\cdot\cdot}$ and $q \in [0,1)$.*

Proof. Remember that the finite Pólya difference process is given by

$$P^-_{z,\mu}(\varphi) = (1+z)^{-\mu(X)} \sum_{n\geq 0} \frac{z^n}{n!} \int_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \mu^{-[n]}(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n). \quad (6.10)$$

If we evaluate (6.10) for $\mu = \delta_x$ we obtain $P^-_{z,\delta_x}(\varphi) = \frac{1}{1+z}\varphi(0) + \frac{z}{1+z}\varphi(\delta_x)$. After inserting $z = \frac{q}{1-q}$ we get $\Phi_x = P^-_{\frac{q}{1-q},\delta_x}$ and by $P^-_{z,\mu} = \mathop{*}_{x\in\mu} P^-_{z,\delta_x}$ the assertion follows. □

In particular the independent thinning

$$\Gamma_q(P) = \int P^-_{\frac{q}{1-q},\mu}\, P(\mathrm{d}\, \mu) \quad (6.11)$$

is a doubly stochastic Pólya difference process. The next corollary will help us to identify the thinned process $\Gamma_q(\Im_L)$.

Corollary 6.3.3. *Assume that the signed Lévy pseudo measure L is concentrated on $\mathcal{M}^{\cdot\cdot}_f(X)$ and $|L|$ is of first order. Then $\Gamma_q(\Im_L)$ has a signed Lévy pseudo measure given by*

$$\sum_{n\geq 1} \frac{q^n}{(1-q)^n n!} \int_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \check{\nu}^n_H(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n), \quad \varphi \in \mathfrak{L}^1.$$

Here $\check{\nu}^n_H$ is the n-th factorial moment measure of $H(\mathrm{d}\,\mu) := (1-q)^{\mu(X)} L(\mathrm{d}\,\mu)$. It is a signed Radon measure on X^n and we even have $\check{\nu}^n_{|H|}(\Lambda \times X^{n-1}) < \infty$ for $\Lambda \in \mathcal{B}_0(X)$.

Proof. Corollary 6.3.1 combined with (6.11) imply that the positive respective negative part of the signed Lévy pseudo measure of $\Gamma_q(\Im_L)$ are given by $\int P^-_{\frac{q}{1-q},\mu} L^\epsilon(\mathrm{d}\,\mu)$. If we now use the representation of the finite Pólya difference process (6.10) and the monotone convergence theorem we obtain for any $\varphi \in F_+(\mathcal{M}^{\cdot\cdot}(X))$

$$\int P^-_{\frac{q}{1-q},\mu}(\varphi)\, L^\epsilon(\mathrm{d}\,\mu) = \sum_{n\geq 1} \frac{q^n}{(1-q)^n n!} \int_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \check{\nu}^n_{H^\epsilon}(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n), \tag{6.12}$$

with $H^\epsilon(\mathrm{d}\,\mu) = (1-q)^{\mu(X)} L^\epsilon(\mathrm{d}\,\mu)$, which implies the assertion. Certainly L^ϵ can be replaced by $|L|$ in (6.12). If we then evaluate (6.12) for $\varphi = \zeta_\Lambda$ we obtain the last statement due to the finiteness of the left hand side. □

Now we have reduced the task of identifying the thinned point process to the computation of the factorial moment measures of H. We return to the setting of section 4.4 and consider point processes with signed Lévy pseudo measure of the form

$$L(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n} \int_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \mathcal{B}^x_{n-1}(\mathrm{d}\, x_2 \ldots \mathrm{d}\, x_n)\lambda(\mathrm{d}\, x_1), \tag{6.13}$$

for $\varphi \in \mathfrak{L}^1(|L|)$. Let us denote $\mathcal{B}^x_{n,q} = (1-q)^{n+1}\mathcal{B}^x_n$ and by $\mathcal{C}^x_{n,q}$ the corresponding loop measures. Then we have

Theorem 6.3.4. *Let L be a signed Lévy pseudo measure of the form (6.13) such that $|L|$ is of first order. Then $\Gamma_q(\Im_L)$ has signed Lévy pseudo measure*

$$\sum_{n\geq 1} \frac{q^n}{(1-q)^n n} \int_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \mathcal{C}^{x_1}_{n-1,q}(\mathrm{d}\, x_2 \ldots \mathrm{d}\, x_n)\lambda(\mathrm{d}\, x_1), \quad \varphi \in \mathfrak{L}^1.$$

Proof. The result follows by corollary 6.3.3 and proposition 4.4.3. □

Let us now consider some examples.

Example 6.3.1 (Pólya processes)**.** *Since the loop measures of the Pólya point processes have been computed in example 4.4.1 we obtain*

$$\mathcal{C}^x_{m,q}(\bigotimes_{j=1}^m f_j) = \epsilon^m \frac{(z(1-q))^{m+1}}{(1-\epsilon z(1-q))^{m+1}} f_1(x)\ldots f_m(x).$$

This means the signed Lévy pseudo measure of $\Gamma_q(P^\epsilon_{z,\lambda})$ *is given by*

$$\sum_{n\geq 1} \frac{\epsilon^{n-1}}{n} \frac{(zq)^n}{(1-\epsilon z(1-q))^n} \int \varphi(n\delta_x)\,\lambda(\mathrm{d}\,x), \quad \varphi \in \mathfrak{L}^1.$$

Therefore

$$\Gamma_q(P^\epsilon_{z,\lambda}) = P^\epsilon_{\frac{zq}{1-\epsilon z(1-q)},\lambda}.$$

Example 6.3.2 (Permanental and determinantal processes)**.** *We consider those permanental and determinantal point processes* $\Im^\epsilon_{\lambda,k}$ *as constructed by theorem 4.5.1. The transition from* $\mathcal{B}^x_n$ *to* $\mathcal{B}^x_{n,q}$ *is also straightforward since we just have to replace* k *by* $(1-q)k$*. In (4.9) we computed the loop measures of the permanental and determinantal processes. Thus we obtain by introducing* $K_{\epsilon,q} = \sum_{j=1}^{\infty} \epsilon^{j-1}(1-q)^{j-1}k^{(j)}$

$$\mathcal{C}^x_{m,q}(\bigotimes_{j=1}^m f_j) = \epsilon^m(1-q)^{m+1} \int_{X^m} f_1(x_1)\ldots f_m(x_m)\, K_{\epsilon,q}(x,x_1)K_{\epsilon,q}(x_1,x_2)\ldots$$
$$\ldots K_{\epsilon,q}(x_{m-1},x_m)K_{\epsilon,q}(x_m,x)\,\mathrm{d}\,\lambda^m$$

That is the thinning $\Gamma_q(\Im^\epsilon_{\lambda,k})$ *of* $\Im^\epsilon_{\lambda,k}$ *has signed Lévy pseudo measure*

$$\sum_{n=1}^{\infty} \frac{\epsilon^{n-1}}{n} q^n \int_{X^n} \varphi(\delta_{x_1}+\ldots+\delta_{x_n})K_{\epsilon,q}(x_1,x_2)\ldots K_{\epsilon,q}(x_n,x_1)\,\mathrm{d}\,\lambda^n, \quad \varphi \in \mathfrak{L}^1.$$

So we finally obtain

$$\Gamma_q(\Im^\epsilon_{\lambda,k}) = \Im^\epsilon_{\lambda,qK_{\epsilon,q}}.$$

To summarize we see: The last two examples show that the classes of Pôya sum-, Pólya difference-, permanental- and determinantal processes remain invariant under independent thinning.

Splitting of Point Processes

A natural question is: Having observed a realization μ of a thinning $\Gamma_q(P)$ of a point process P, what is the distribution of deleted point configurations given μ. This is the object of interest in this section and it will be denoted $\Upsilon_q^\mu(P)$, the so called *splitting kernel*.
Let us start by introducing the *splitting law* for any point process P and $q \in (0,1)$

$$S_q(P)(h) = \iint P(\mathrm{d}\,\mu) P^-_{\frac{q}{1-q},\mu}(\mathrm{d}\,\nu)\, h(\nu, \mu - \nu),$$

for any $h \in F_+(\mathcal{M}^{\cdot\cdot}(X) \times \mathcal{M}^{\cdot\cdot}(X))$. A realization of the splitting law can be described as follows: P realizes a point configuration μ, which is q-thinned by the Pólya difference process $P^-_{\frac{q}{1-q},\mu}$. The realization of the splitting law is now the pair of point configurations which survived the thinning and those which have been deleted. The marginal laws of the splitting law are given by

$$S_q(P)(\varphi \otimes \mathbf{1}) = \Gamma_q(P)(\varphi), \quad S_q(P)(\mathbf{1} \otimes \varphi) = \Gamma_{1-q}(P)(\varphi)$$

for any $\varphi \in F_+(\mathcal{M}^{\cdot\cdot}(X))$. Thus for $N \in \mathcal{B}(\mathcal{M}^{\cdot\cdot}(X))$ we have $S_q(P)(\cdot \times N) \ll \Gamma_q(P)$, which enables us to disintegrate the splitting law with respect to the q-thinning $\Gamma_q(P)$ of P and so by the theory of disintegration we obtain the splitting kernel $\Upsilon_q^\nu(P)$, that is

$$S_q(P)(\mathrm{d}\,\nu\,\mathrm{d}\,\eta) = \Gamma_q(P)(\mathrm{d}\,\nu)\Upsilon_q^\nu(P)(\mathrm{d}\,\eta).$$

In the case of finite point processes there is a close connection between the splitting kernel and the reduced Palm kernels. The following proposition can already be found in [30]. Here we give a proof by means of disintegration theory.

Proposition 6.3.5. *Let P be a finite point process then the splitting kernel is given by*

$$\Upsilon_q^\nu(P)(\varphi) = \frac{1}{\int (1-q)^{\mu(X)} P_\nu^!(\mathrm{d}\,\mu)} \int \varphi(\eta)(1-q)^{\eta(X)}\, P_\nu^!(\mathrm{d}\,\eta),$$

for $\varphi \in F_+(\mathcal{M}^{\cdot\cdot}(X))$.

Proof. Recall the representation (6.10) of the finite Pólya difference process.

This gives us for any $h \in F_+(\mathcal{M}^{\cdot\cdot}(X) \times \mathcal{M}^{\cdot\cdot}(X))$

$$
\begin{aligned}
&S_q(P)(h)\\
&= \sum_{n=0}^{\infty} \frac{q^n}{(1-q)^n n!} \int\int P(\mathrm{d}\,\mu)\mu^{-[n]}(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_n)\,(1-q)^{\mu(X)} h(\sum_{i=1}^{n} \delta_{x_i}; \mu - \sum_{i=1}^{n} \delta_{x_i})\\
&= \sum_{n=0}^{\infty} \frac{q^n}{(1-q)^n n!} \int\int C_P^{!,n}(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_n; \mathrm{d}\,\eta)\,(1-q)^{n+\eta(X)} h(\sum_{i=1}^{n} \delta_{x_i}; \eta)\\
&= \sum_{n=0}^{\infty} \frac{q^n}{(1-q)^n n!} \int\int \check{\nu}_P^n(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_n) P^!_{x_1,\ldots,x_n}(\mathrm{d}\,\eta)\,(1-q)^{n+\eta(X)} h(\sum_{i=1}^{n} \delta_{x_i}; \eta)\\
&= \int P(\mathrm{d}\,\mu)(1-q)^{\mu(X)} \sum_{n=0}^{\infty} \frac{q^n}{(1-q)^n n!} \int \mu^{-[n]}(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_n) P^!_{x_1,\ldots,x_n}(\mathrm{d}\,\eta)\\
&(1-q)^{n+\eta(X)-\mu(X)} h(\sum_{i=1}^{n} \delta_{x_i}; \eta)\\
&= \iiint P(\mathrm{d}\,\mu) P^-_{\frac{q}{1-q},\mu}(\mathrm{d}\,\nu) P^!_\nu(\mathrm{d}\,\eta)\,(1-q)^{(\nu+\eta-\mu)(X)} h(\nu, \eta).
\end{aligned}
$$

Let now $\varphi \in F_+(\mathcal{M}^{\cdot\cdot}(X))$ and $\tilde{\varphi}(\nu) = \frac{1}{\int (1-q)^{\mu(X)} P^!_\nu(\mathrm{d}\,\mu)} \varphi(\nu)$. Then we have

$$
\begin{aligned}
&\iint P(\mathrm{d}\,\mu) P^-_{\frac{q}{1-q},\mu}(\mathrm{d}\,\nu)\,(1-q)^{(\nu-\mu)(X)}\,\varphi(\nu)\\
&= \iiint P(\mathrm{d}\,\mu) P^-_{\frac{q}{1-q},\mu}(\mathrm{d}\,\nu) P^!_\nu(\mathrm{d}\,\eta)\,(1-q)^{(\nu+\eta-\mu)(X)}\,\tilde{\varphi}(\nu) = S_q(P)(\tilde{\varphi} \otimes 1) = \Gamma_q[P](\tilde{\varphi})
\end{aligned}
$$

Now by choosing $\varphi(\nu) = \int P^!_\nu(\mathrm{d}\,\eta)(1-q)^{\eta(X)}\,h(\nu, \eta)$ we obtain the assertion.

□

So in case of finite point processes the identification of the splitting kernel boils down to the computation of the reduced Palm kernels. In case of infinitely divisible point processes we obtained the reduced Palm kernels, see equation (4.12), in terms of the reduced Palm kernels of the Lévy measure. In general there does not exist a canonical method to determine the splitting kernel of a point process. One has more or less to guess the right solution. We now want to compute the splitting kernel for some examples. Since we know the Laplace transform of the Pólya difference process

$$\mathcal{L}_{P^-_{\frac{q}{1-q},\mu}}(f) = \exp(\mu(\log(1-q+qe^{-f})))$$

a direct computation shows that

$$S_P^{!,q}(e^{-\zeta_f} \otimes e^{-\zeta_g}) = \mathcal{L}_P(g - \log(1-q+qe^{-(f-g)})). \tag{6.14}$$

Example 6.3.3 (Poisson process). *Let P_λ be the Poisson process with intensity $\lambda \in \mathcal{M}^{\cdot\cdot}(X)$. Applying (6.14) one sees that*

$$S_q(P_\lambda) = P_{q\lambda} \otimes P_{(1-q)\lambda}.$$

Since it is well known that $\Gamma_q(P_\lambda) = P_{q\lambda}$ we conclude that

$$\Upsilon_q^\nu(P_\lambda) = P_{(1-q)\lambda}.$$

Example 6.3.4 (Pólya processes). *We have*

$$\Upsilon_q^\nu(P_{z,\lambda}^\epsilon) = P_{z(1-q),\lambda+\epsilon\nu}^\epsilon. \tag{6.15}$$

Recall that the Laplace transforms of the Pólya processes are given by (6.9), so that

$$\mathcal{L}_{P_{z,\lambda}^\epsilon}(f) = \exp[-\epsilon\lambda(\log(\frac{1-\epsilon z e^{-f}}{1-\epsilon z}))].$$

Applying (6.14) we straightforwardly obtain

$$S_q(P_{z,\lambda}^\epsilon)(e^{-\zeta_f} \otimes e^{-\zeta_g}) = \exp[-\epsilon\lambda(\log(\frac{1-\epsilon z(qe^{-f}+(1-q)e^{-g})}{1-\epsilon z}))].$$

We already know that $\Gamma_q(P_{z,\lambda}^\epsilon) = P_{\frac{qz}{1-\epsilon z(1-q)},\lambda}^\epsilon$, see example 6.3.1. Let us now verify that (6.15) is the right choice for the splitting kernel of the Pólya processes. A direct computation yields

$$\begin{aligned}&\iint P_{\frac{qz}{1-\epsilon z(1-q)},\lambda}^\epsilon(\mathrm{d}\,\nu) P_{z(1-q),\lambda+\epsilon\nu}^\epsilon(\mathrm{d}\,\eta)\, e^{-\nu(f)} e^{-\eta(g)} \\ &= \exp[-\epsilon\lambda(\log(\frac{1-\epsilon z(qe^{-f}+(1-q)e^{-g})}{1-\epsilon z}))] = S_q(P_{z,\lambda}^\epsilon)(e^{-\zeta_f}\otimes e^{-\zeta_g}).\end{aligned}$$

Remark 6.3.6. *Let us compute the intensity measures of the splitting kernels, obtained in the last two examples.*

$$\nu^1_{\Upsilon_q^\mu(P_\lambda)} = (1-q)\lambda, \qquad \nu^1_{\Upsilon_q^\mu(P_{z,\lambda}^\epsilon)} = \frac{z(1-q)}{1-z(1-q)}(\varrho+\epsilon\mu)$$

Theorem 6.3.7. *For every $q \in (0,1)$ $\Gamma_q(P)$ is a Papangelou process and its kernel is given by*

$$\pi(\mu, \mathrm{d}\,x) = \frac{q}{1-q}\nu^1_{\Upsilon_q^\mu(P)}(\mathrm{d}\,x).$$

Proof. So let us start by computing

$$\begin{aligned}
C_{\Gamma_q(P)}(h) &= \int P(\mathrm{d}\,\mu)\, C_{P^-_{\frac{q}{1-q},\mu}}(h) \\
&= \int P(\mathrm{d}\,\mu) P^-_{\frac{q}{1-q},\mu}(\mathrm{d}\,\kappa) \frac{q}{1-q}(\mu-\kappa)(\mathrm{d}\,x)\, h(x,\kappa+\delta_x) \\
&= \int \Gamma_q(P)(\mathrm{d}\,\kappa) \Upsilon^\kappa_q(P)(\mathrm{d}\,\eta) \frac{q}{1-q}\eta(\mathrm{d}\,x)\, h(x,\kappa+\delta_x) \\
&= \int \Gamma_q(P)(\mathrm{d}\,\kappa) \frac{q}{1-q}\nu^1_{\Upsilon^\kappa_q(P)}(\mathrm{d}\,x)\, h(x,\kappa+\delta_x).
\end{aligned}$$

For the second equation we have used the integration by parts formula $(\boldsymbol{\Sigma}')$ for the Pólya difference process and the third equation follows by definition of the splitting kernel. □

Let us end this section with an *open question*. We certainly have, as one would expect, $\Gamma_q(P) \Rightarrow P$ weakly as $q \to 1$. This can be established by convergence of the respective Laplace transforms and by using $-\log(1-q+qe^{-f}) \le fe^f$ due to the estimate $-\log(1-x) \le \frac{x}{1-x}$ for $x \in (0,1)$. Now a natural question is: Which conditions have to be imposed for the existence of a kernel π such that for all $f \in F_{bc,+}(X)$

$$\lim_{q\to 1} \frac{q}{1-q}\nu^1_{\Upsilon^\mu_q(P)}(f) = \pi(\mu,f) \quad P-a.s.[\mu]$$

and when is it the Papangelou kernel of P? We remark that both questions have a positive answer in case of the two examples in remark 6.3.6. Recall that $\frac{q}{1-q} = z$ is the parameter such that the q-thinning is given by the mixture of Pólya difference processes $P^-_{z,\mu}$ with respect to μ see (6.11).

6.4 Cox Point Processes

In this section we want to identify a class of point processes with a signed Lévy pseudo measure as Cox processes. The link between independend thinnings and Cox processes is given by the following fundamental theorem of Mecke in [44] theorem 8.

Theorem 6.4.1 (Characterization of Cox distributions)**.** *Denote by $\mathcal{P}$ the set of point processes and by Δ the set of Cox processes then $\bigcap_{0<q<1}\Gamma_q(\mathcal{P}) = \Delta$. The mapping Γ_q is injective and if $P \in \bigcap_{0<q<1}\Gamma_q(\mathcal{P})$ then $P = \int P_\rho\, R(d\rho)$ with*

$$\mathcal{L}_R(f) = \lim_{q\to 0} \mathcal{L}_{\Gamma_q^{-1}(P)}(qf), \;\; f \in U.$$

So in order to verify that a point process Q is a Cox process we have to check that for each $q \in (0,1)$ there exists a point process P_q such that $Q = \Gamma_q(P_q)$. And for this task we will employ theorem 6.3.4.

Theorem 6.4.2. *Let $\Im_z$ be a point process with signed Lévy pseudo measure*

$$L_z(\varphi) = \sum_{n=1}^{\infty} \frac{z^n}{n} \int\limits_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \mathcal{B}^{x_1}_{n-1}(\mathrm{d}\, x_2 \ldots \mathrm{d}\, x_n)\lambda(\mathrm{d}\, x_1), \quad \varphi \in \mathfrak{L}^1,$$

where $z \in (0,1)$, $\lambda \in \mathcal{M}(X)$, $\{\mathcal{B}^x_m\}_{m\geq 0}$ is a projective family of finite signed measures, that is for $k \leq m$ and $1 \leq i_1 < \ldots < i_k \leq m$ we have for any $f_1, \ldots, f_k \in U$

$$\mathcal{B}^x_k(\overset{k}{\underset{j=1}{\otimes}} f_j) = \int \mathcal{B}^x_m(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_m)\, f_1(x_{i_1}) \ldots f_k(x_{i_k}).$$

Furthermore we assume that $\mathcal{B}^{x_1}_{n-1}(\mathrm{d}\, x_2 \ldots \mathrm{d}\, x_n)\lambda(\mathrm{d}\, x_1)$ is as above invariant under cyclic permutations. Then $\Im_z$ is a Cox process and the directing random measure R_z is the weak limit of the sequence $q\Im_{\frac{z}{z+(1-z)q}}$ as q tends to zero.

Proof. Let us start by computing the q reweighted loop measures of $\Im_z$, which were introduced preceding theorem 6.3.4. Due to the projectivity we have

$$\mathcal{C}^x_{m,q}(\overset{m}{\underset{j=1}{\otimes}} f_j) = \sum_{n=m}^{\infty} \binom{n}{m} (z(1-q))^{n+1}\, \mathcal{B}^x_m(\overset{m}{\underset{j=1}{\otimes}} f_j) = \frac{(z(1-q))^{m+1}}{(1-z(1-q))^{m+1}}\, \mathcal{B}^x_m(\overset{m}{\underset{j=1}{\otimes}} f_j).$$

The second equation follows as in example 4.4.1. Now theorem 6.3.4 shows that $\Im_z$ has signed Lévy pseudo measure

$$\sum_{n=1}^{\infty} \frac{(zq)^n}{(1-z(1-q))^n} \frac{1}{n} \int\limits_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \mathcal{B}^{x_1}_{n-1}(\mathrm{d}\, x_2 \ldots \mathrm{d}\, x_n)\lambda(\mathrm{d}\, x_1),\ \varphi \in \mathfrak{L}^1.$$

A straightforward computation yields that if we choose $\gamma(z,q) = \frac{z}{z+(1-z)q}$ then

$$\frac{\gamma(z,q)q}{1-\gamma(z,q)(1-q)} = z,$$

which implies $\Gamma_q(\Im_{\gamma(z,q)}) = \Im_z$ and we conclude with Mecke's theorem 6.4.1. □

The following example is due to Rafler [61].

Example 6.4.1 (Pólya sum process)**.** *The family*

$$\mathcal{B}_n^x = \delta_x(\mathrm{d}\, x_1) \dots \delta_x(\mathrm{d}\, x_n)$$

certainly has the projectivity property. The corresponding point process $\Im_z$ *is the Pólya sum process* $P^+_{z,\lambda}$ *(see section 4.2). So according to theorem 6.4.2 it is a Cox process. Let us compute its directing measure. The Laplace transform of the scaled Pólya sum process* $qP^+_{\gamma(z,q),\lambda}$ *is given by*

$$\mathcal{L}_{qP^+_{\gamma(z,q),\lambda}}(f) = \exp\left(-\int\limits_X \log(\frac{1-\gamma(z,q)e^{-qf(x)}}{1-\gamma(z,q)})\, \lambda(\mathrm{d}\, x)\right).$$

The integrand in the above term on the right hand side is dominated by some λ*-integrable function as follows. Remember* $\log(x) \leq x - 1$*,* $x \geq 1$ *and* $1 - e^{-x} \leq x$*,* $x \geq 0$ *so that*

$$\begin{aligned}\log(\frac{1-\gamma(z,q)e^{-qf(x)}}{1-\gamma(z,q)}) &\leq \frac{\gamma(z,q)(1-e^{-qf(x)})}{1-\gamma(z,q)} \leq \frac{\gamma(z,q)q}{1-\gamma(z,q)} f(x)\\ &= \frac{z}{1-z} f(x).\end{aligned}$$

Let us now investigate the convergence of the integrand. Using l'Hôpitales rule we get

$$\lim_{q\to 0} \frac{1-z(y,q)e^{-qf(x)}}{1-z(y,q)} = 1 + \frac{y}{1-y} f(x).$$

So the Laplace transform of the directing random measure is given by

$$\mathcal{L}_R(f) = \exp\left(-\int \log(1 + \frac{y}{1-y} f(x))\, \lambda(\mathrm{d}\, x)\right).$$

We see that for $f, g \in F_{bc,+}(X)$ *with supp(f)* $\cap$ *supp(g)* $= \emptyset$ *there holds*

$$\mathcal{L}_R(f+g) = \mathcal{L}_R(f)\mathcal{L}_R(g)$$

Therefore R has independent increments. Let us calculate the Laplace transform of the field variable ζ_B *for some* $B \in \mathcal{B}_0(X)$*. We have*

$$\mathcal{L}_{\zeta_B}(t) = \mathcal{L}_R(t 1_B) = \frac{1}{(1 + \frac{y}{1-y} t)^{\lambda(B)}}, \quad t \geq 0.$$

Thus ζ_B *is Gamma distributed, and so we have identified R as the Poisson-Gamma process.*

Remark 6.4.3. *The Pólya difference process $P^-_{z,\lambda}$ can not be a Cox process, because the field variables ζ_B, $B \in \mathcal{B}_0(X)$ are bounded under $P^-_{z,\lambda}$ due to the fact that for every realization μ of $P^-_{z,\lambda}$ we have $\mu \subset \lambda$. More formally assume that $P^-_{z,\lambda}$ is Cox with directing measure R and let $B \in \mathcal{B}_0(X)$ such that $\lambda(B) \geq 1$. Then we must have $R\{\zeta_B > 0\} > 0$ and*

$$\begin{aligned}
1 &= P^-_{z,\lambda}\{\zeta_B \leq \lambda(B)\} \\
&= \int\limits_{\{\zeta_B=0\}} P_\rho\{\zeta_B \leq \lambda(B)\}\, R(\mathrm{d}\,\rho) + \int\limits_{\{\zeta_B>0\}} P_\rho\{\zeta_B \leq \lambda(B)\}\, R(\mathrm{d}\,\rho) \\
&= R\{\zeta_B = 0\} + \int\limits_{\{\zeta_B>0\}} P_\rho\{\zeta_B \leq \lambda(B)\}\, R(\mathrm{d}\,\rho) < 1,
\end{aligned}$$

contradiction.

Bibliography

[1] Abdessalam, A., Rivasseau, V.: Trees, forests, and jungles: a botanical garden for cluster expansions. Lect. Notes Phys. 446, 7 (1994). eprint: arXiv:hep-th/9409094

[2] Bach,A., Zessin,H.: The particle structure of the quantum mechanical Bose and Fermi gas. Unpublished manuscript (2010).

[3] Bahadur, R.R.: Measurable Subspaces and Subalgebras. American Math. Soc., Vol. 6, 565-570, (1955).

[4] Battle,G., Federbush, P.: A phase cell cluster expansion for Euclidean field theories. Ann. Phys. 142, 95 (1982).

[5] Bender E. A., Goldman J.R.: On the Applications of Möbius Inversion in Combinatorial Analysis. American Math. Monthly, 789-803, (1975)

[6] Bhatia, R.: Matrix Analysis. Graduate Texts in Mathematics 169, Springer (1997).

[7] Borodin,A., Olshansky,G.: Point processes and the infinite symmetric group, part VI: A summary of results. Available online at URL http://arxiv.org/abs/math.RT/9810015 .

[8] Bovier, A.: Statistical Mechanics of Disordered Systems: A mathematical perspective. Series in statistical and probabilistic mathematics (2006).

[9] Brydges, D.C.: A short course on cluster expansions. In: Phenomes critiques, systeme aleatoires, theories des jauge, 129 - 183. Les Houches 1984, Elsevier/North Holland (1986).

[10] Brydges, D.C., Federbush, P.: A new form of the Mayer expansion in classical statistical mechanics. J. Math. Phys. 19, 2064 (1978).

[11] Brydges, D.C., Kennedy, T.: Mayer expansions and the Hamilton-Jacobi equations. J. Statist. Phys. 48, 19 (1987).

[12] Daley,D.J., Vere-Jones,D.: An introduction to the Theory of Point Processes. Volume I: Elementary Theory and Methods. Second Edition, Springer (2003).

[13] Dobrushin, R.L.: Estimates of semi-invariants for the Ising model at low temperatures. In: Topics in Statistical and Theoretical Physics, AMS Translational Series 2,177, 59 - 81, AMS (1966).

[14] Dobrushin, R.L.: Perturbation methods of the theory of Gibbsian fields. In: Ed.: P. Bernard, Ecole d'Ete de Probabilites de Saint-Flour, Springer Lecture Notes in Mathematics 1648, 1 - 66 (1996).

[15] Fernandez,R., Procacci,A.: Cluster expansion for abstract polymer models. New bounds from an old approach. Commun. Math. Phys. 274, 123 (2007).

[16] Fichtner, K: On the position distribution of the ideal Bose gas. Math. Nachr. 151, 59-67 (1991).

[17] Freudenberg, F., Fichtner, K.: Characterization of Classical and Quantum Poisson systems by Thinnings and Splittings. Math. Nachr. 218, 25-47 (2000).

[18] Georgii, H., Yoo, H.: Conditional Intensity and Gibbsianness of Determinantal Point Processes. J. stat. phys., Volume 118, Numbers 1-2, 55-84, (2004).

[19] Ginibre, J.: Reduced density matrices of quantum gases. III. Hard-core potentials, J. Math. Phys. 6, 1432 (1965).

[20] Ginibre, J.: Some applications of functional integration in statistical mechanics. In: Mecanique statistique et theorie quantique des champs. Les Houches 1970, 327 - 427, Gordon and Breach (1971).

[21] Glimm,J., Jaffe,A.: Expansions in Statistical Physics. Commun. Pure and Appl. Math. 38, 613 - 630 (1985).

[22] Glimm,J., Jaffe,A.: Quantum Physics. A Functional Integral Point of View. Springer (1987).

[23] Glimm,J., Jaffe,A., Spencer,T.: The Wightman axioms and particle structure in $\mathcal{P}(\phi)_2$-quantum field model. Ann. Math. 100, 585 (1974).

[24] Glimm,J., Jaffe,A., Spencer,T.: A convergent expansion about the mean field theory. I, Ann. Phys. 101, 610 - 630; II, 631 - 669 (1976).

[25] Gruber,C., Kunz,H.: General properties of Polymer systems. Commun. Math. Phys. 22,133 (1971).

[26] Hill,T.L.: Statistical Mechanics, McGraw-Hill (1956).

[27] Jackson,B., Procacci, A., Sokal,A.D.: Complex zero-free regions at large $|q|$ for multivariate Tutte polynomials (alias Potts-model partition functions) with general complex edge weights. eprint arXiv: 0810.4703

[28] Kallenberg, O.: Random measures. Akademie-Verlag, Berlin, Academic press, London (1983).

[29] Kallenberg, O.: Iterated Palm Conditioning and Some Slivnyak-Type Theorems for Cox and Cluster Processes. J. Theo. Prob. 24, 875-893 (2011)

[30] Karr, A.: Point Processes and their Statistical Inference. Dekker, New York (1986)

[31] Kerstan, J., Matthes, K., Mecke, J.: Infinitely divisible point processes. Wiley (1978).

[32] Kirkwood,J.G., Salsbourg,Z.: Discussions. Faraday Soc. 15, p. 28 (1953).

[33] Kotecky, R., Preiss, D.: Cluster expansion for abstract polymer models. Commun. Math. Phys. 103, 491 (1986).

[34] Krickeberg, K.: Moments of Point-processes. Stochastic Geometry. Harding,E.F. and Kendall,D.G. (ed.), Wiley (1974).

[35] Krickeberg, K.: Random Radon measures. To appear in Walter Warmuth Verlag (2012).

[36] Kuna, T.: Studies in Configuration Space Analysis and Applications. PhD Thesis. Bonner Mathematische Schriften Vol. 324 (1999).

[37] Kutoviy, O., Rebenko, A.: Existence of Gibbs state for a continuous gas with many-body interaction. J. Math. Phys. 45, 1593 (2004).

[38] Macchi,O.: Processus ponctuels et coincidences. These presentee au Centre d'Orsay, Universite de Paris Sud (1972).

[39] Macchi,O.: Point processes and coincidences. In preparation. Walter Warmuth Verlag (2012/13).

[40] Malyshev, V., Minlos, R.: Gibbs Random Fields: Cluster Expansions. Mathematics and Its Applications. Kluwer Academic Publishers (1991).

[41] Malyshev, V.A.: Cluster expansion in lattice models of statistical physics and quantum field theory. Russ. Math. Surveys 35, 1 (1980).

[42] Mayer,J.E., Montroll,E.: Molecular distributions. J. Chem. Phys. 9, 2 - -16 (1941).

[43] Mecke, J.: Stationäre zufällige Maße auf lokalkompakten Abelschen Gruppen, Z. Wahrscheinlichkeitstheorie verw, Gebiete 9,36-58 (1967).

[44] Mecke, J.: Random measures. Walter Warmuth Verlag (2011).

[45] Merris, R.: Manifestations of Polya's Counting Theorem. Lin. Alg. and its Appl. 32, 209-234 (1980).

[46] Minlos,R.A.: Limiting Gibbs distribution. Funct. Anal. Appl. 1,141 - 150 (1967).

[47] Minlos, R.A., Poghosyan, S.: Estimates of Ursell functions, group functions, and their derivatives. Theor. Math. Phys. 31, 408 (1977).

[48] Miracle-Sole, S.: On the convergence of cluster expansions. Physica A 279, 244 (2000).

[49] Móricz, F.: Sufficient conditions for the Lebesgue integrability of Fourier transforms. Analysis Mathematica 36, 121-129 (2010).

[50] Nehring, B., Zessin H.: The Papangelou process. A concept for Gibbs, Fermi and Bose processes . J. Cont. Math. Anal. 46, 326-337 (2011).

[51] Nehring, B., Zessin, H.: On the construction of point processes for classical and quantum gases. Proceedings of the inter. math. conf. 50 years of IPPI, 7 pages, Moscow, 25-29 July 2011, http://www.iitp.ru/upload/content/839/Nehring Zessin.pdf

[52] Nehring, B., Zessin, H.: A representation of the moment measures of the general ideal Bose gas. Math. Nachr. 285, 878-888 (2012).

[53] Nguyen, X., Zessin, H.: Integral and Differential Characterizations of the Gibbs ProcessCambridge Studies in Advanced Mathematics, 62. Math. Nachr. 88, 105-115 (1979).

[54] Parthasarathy, K.: Probability measures on metric spaces. New York, London (1967).

[55] Penrose, O.: Statistical mechanics, Foundations and applications. Benjamin (1967).

[56] Pfister, Ch.E.: Large deviations and phase separation in the two-dimensional Ising model. Helv. Phys. Acta 64, 953 (1991).

[57] Poghosyan,S.: Asymptotic expansion of the logarithm of the partition function. Commun. Math. Phys. 95, 227 - 245 (1984).

[58] Poghosyan, S., Ueltschi, D.: Abstract cluster expansion with applications to statistical mechanical systems. J. Math. Phys. 50, 053509 (2009).

[59] Poghosyan, S., Zessin, H.: Decay of correlations of the Ginibre gas obeying Maxwell-Boltzmann statistics. Markov Processes Relat. Fields 7, 581 (2001).

[60] Preston, Ch.J.: Specifications and their Gibbs states. Bielefeld. http://www.math.uni-bielefeld.de/ preston/rest/gibbs/files/specifications.pdf

[61] Rafler, M.: The Pólya sum process: a Cox representation. J. Cont. Math. Ana. 46, 338-345 (2011)

[62] Rota, G.: On the Foundations of Combinatorial Theory: I. Theory of Möbius Inversion. Z. Wahrscheinlichkeitstheorie 2, 340-368 (1964).

[63] Ruelle, D.: Statistical Mechanics. Rigorous results. Benjamin (1969).

[64] Ruelle, D.: Superstable interaction in classical statistical mechanics. Comm. Math. Phys. 18, 127-159 (1970).

[65] Seiler, E.: Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics. Lecture Notes in Physics 159, Springer (1982).

[66] Shirai, T., Takahashi, Y.: Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point processes . J. Funct. Ana. 205, 414-463 (2003).

[67] Simon, B.: The Statistical Mechanics of Lattice Gases, Vol. 1, Princeton University Press (1993).

[68] Sokal, A.D.: Bounds on the complex zeros of (di)chromatic polynomials of Potts-model partition functions. Combinatorics, Probab. Comput. 10, 41 (2001).

[69] Soshnikov, A.: Determinantal Random Point Fields. Russian Mathematical Surveys 55, 923-975 (2000).

[70] Spohn, H.: Interacting Brownian particles: A study of Dyson's model. In: Hydrodynamic behaviour and interacting particle systems (Minneapolis, Minn., 1986), G. Papanicolaou, ed., IMA Vol. Math. Appl. 9 (1987).

[71] Spohn, H.: Tracer dynamics in Dyson's model of interacting Brownian particles. J.Statist.Phys. 47, 669 - 679 (1987).

[72] Stanley, R.: Enumerative Combinatorics. Vol. 2. Cambridge Studies in Advanced Mathematics, 62. Cambridge University Press (1999).

[73] Tracy, C., Widom, H.: Correlation Functions, Cluster Functions, Spacing Distributions for Random Matrices. J.Statist.Phys. 92, 809-835 (1998).

[74] Whitt, W.: Weak convergence of probability measures on the function space $\mathcal{C}[0,\infty)$. Ann. math. statist. 41, 939-944 (1970).

[75] Zessin, H.: Der Papangelou Prozess. J. Cont. Math. Ana. 44, 36-44 (2009).

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