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Preprints des Instituts für Mathematik der Universität Potsdam
1 (2012) 13

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Universitätsverlag Potsdam

Bibliografische Information der Deutschen Nationalbibliothek
Die Deutsche Nationalbibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliografie; detaillierte bibliografische Daten sind im Internet über http://dnb.de abrufbar.

Universitätsverlag Potsdam 2012
http://info.ub.uni-potsdam.de/verlag.htm

Am Neuen Palais 10, 14469 Potsdam
Tel.: +49 (0)331 977 2533 / Fax: 2292
E-Mail: verlag@uni-potsdam.de

Die Schriftenreihe **Preprints des Instituts für Mathematik der Universität Potsdam** wird herausgegeben vom Institut für Mathematik der Universität Potsdam.

ISSN (online) 2193-6943

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Titelabbildungen:
1. Karla Fritze | Institutsgebäude auf dem Campus Neues Palais
2. Nicolas Curien, Wendelin Werner | Random hyperbolic triangulation
Published at: http://arxiv.org/abs/1105.5089

Online veröffentlicht auf dem Publikationsserver der Universität Potsdam
URL http://pub.ub.uni-potsdam.de/volltexte/2012/5927/
URN urn:nbn:de:kobv:517-opus-59279
http://nbn-resolving.de/urn:nbn:de:kobv:517-opus-59279

Matrix Methods for Computing Eigenvalues of Sturm-Liouville Problems of Order Four

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Abstract

This paper examines and develops matrix methods to approximate the eigenvalues of a fourth order Sturm-Liouville problem subjected to a kind of fixed boundary conditions, furthermore, it extends the matrix methods for a kind of general boundary conditions. The idea of the methods comes from finite difference and Numerov's method as well as boundary value methods for second order regular Sturm-Liouville problems. Moreover, the determination of the correction term formulas of the matrix methods are investigated in order to obtain better approximations of the problem with fixed boundary conditions since the exact eigenvalues for $q = 0$ are known in this case. Finally, some numerical examples are illustrated.

Keywords: Finite difference method, Numerov's method, Boundary value methods, Fourth order Sturm-Liouville problem, Eigenvalues

1. Introduction

In 1997, Greenberg and Marletta [1] released a software package dealing with the computation of eigenvalues of fourth order Sturm-Liouville problems (SLP). The code is called SLEUTH (Sturm-Liouville Eigenvalues Using THeta matrices). This algorithm can be applied to the fourth order SLP

$$(p(x)y'')'' - (s(x)y')' + q(x)y = \lambda w(x)y, \tag{1}$$

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for $x \in (a,b)$ and is subjected to the general separated self-adjoint boundary conditions

$$A_1 u(a) + A_2 v(a) = 0, B_1 u(b) + B_2 v(b) = 0, \tag{2}$$

where $u = (u_1, u_2)^T, v = (v_1, v_2)^T$, and $u_1 = y, u_2 = y', v_1 = sy' - (py'')', v_2 = py''$, are the quasi-derivatives, A_1, A_2, B_1, B_2 are 2×2 matrices such that $A_1 A_2^T = A_2 A_1^T, B_1 B_2^T = B_2 B_1^T$, and the 2×4 matrices $M_1 = (A_1|A_2), M_2 = (B_1|B_2)$ have rank 2, q and s are in $C^0(a,b)$ and $C^1(a,b)$, respectively. It is known that such problems have an infinite sequence of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ which is bounded from below by a constant τ, i.e. $\tau < \lambda_1 \leq \lambda_2 \leq \dots$ with $\lim_{k\to\infty} \lambda_k = \infty$, see e.g.[2, 3], and each eigenvalue has multiplicity at most 2. There are several more methods which have been applied to approximate the eigensolutions of the fourth order SLP (1) with general separated self-adjoint boundary conditions (2). These methods are shooting method [3], oscillation method [2], an efficient method based on the Adomian decomposition method (ADM) [4], and variational iteration methods [5]. In 2002, Chanane [6] applied Fliess series to compute the eigenvalues of the fourth-order Sturm-Liouville problems

$$y^{(4)} - (s(x)y')' + q(x)y = \lambda y, \tag{3}$$

with self-adjoint boundary condition (2). One year later, Boumenir [7] used the sampling method to compute the eigenvalues of (3) with boundary conditions $y(0) = y'(0) = 0, y(1) = y'(1) = 0$, where s' and q are continuous functions. Chanane [8] introduced a method called the extended sampling method to compute the eigenvalues of fourth-order SLP (3) subjected to the boundary conditions (2) for $x \in [0,1]$.

Everitt [9] considered some of the explicit constructions for the eigenfunctions of the problem (3) with boundary conditions $\sum_{j=0}^{3}(-1)^{j+1}\alpha_i^{4-j}y^{(j)}(a) = 0,\ \sum_{j=0}^{3}(-1)^{j+1}\beta_i^{4-j}y^{(j)}(b) = 0,\ (i = 1,2)$ for $x \in (a,b)$, where $y^{(k)}$ denotes the differential coefficient of order k with respect to x. The existence of positive solutions for the following fourth-order nonlinear singular SLP

$$\frac{1}{p(x)}(p(x)y'''(x))' - \lambda q(x)F(x,y,y'') = 0, x \in (0,1),$$

coupled with boundary conditions

$$\alpha_1 y(0) - \beta_1 y'(0) = 0, \gamma_1 y(1) + \delta_1 y'(1) = 0,$$
$$\alpha_2 y''(0) - \beta_2 \lim_{x\to 0+} p(x)y'''(x) = 0, \gamma_2 y''(1) + \delta_2 \lim_{x\to 0-} p(x)y'''(x) = 0,$$

have been studied in [10, 11]. Thereby, p, q may be singular at $x = 0$ and/or 1, and $F(x, t, y)$ may also have singularity at $t = 0$ and/or $y = 0$.

A regular second order SLP of the form

$$-y'' + q(x)y = \lambda y \tag{4}$$

is an interesting second order SLP. Many methods have been applied to approximate the eigenvalues of (4). However, the two most common matrix methods are finite different method (FDM) and Numerov's method [12]. Correction terms of the methods for second order SLP have been described in [12, 13, 14, 15, 16, 17, 18, 19, 20] to obtain better numerical eigenvalues of the problem.

Recently, a family of boundary value methods (BVMs) obtained as an extension of Numerov's method has been introduced [21] in order to approximate the eigenvalues of the problem of the form (4) subjected to Dirichlet boundary conditions, $y(0) = 0 = y(\pi)$. Later, the boundary value methods have been extended in [22] for the problem with general boundary conditions $a_1y(0) - a_2y'(0) = 0,\ b_1y(\pi) - b_2y'(\pi) = 0$, where $|a_1| + |a_2| \neq 0$ and $|b_1| + |b_2| \neq 0$.

In this paper we extend finite difference method, Numerov's method, and boundary value methods for second order SLP to solve the fourth order SLP of the form

$$y^{(4)} + q(x)y = \lambda y \tag{5}$$

first coupled with fixed boundary conditions

$$y(0) = y''(0) = y(1) = y''(1) = 0, \tag{6}$$

and second the methods are generalized to general boundary conditions

$$\begin{aligned} a_1y(0) - a_2y'(0) &= 0, & (7)\\ b_1y(0) - b_2y''(0) &= 0, & (8)\\ c_1y(1) - c_2y'(1) &= 0, & (9)\\ d_1y(1) - d_2y''(1) &= 0, & (10) \end{aligned}$$

where $a_2, b_2, c_2, d_2 \neq 0$. We have to remark that those boundary conditions fit into the form (2) but in general do not fulfill the conditions $A_1A_2^T = A_2A_1^T$ and $B_1B_2^T = B_2B_1^T$, therefore we assume real discrete eigenvalues.

Since we know the exact eigenvalues for $q = 0$ only for the fixed boundary conditions (6) we give the correction term formulas only for this case but for all investigated matrix methods. Finally, some numerical results are illustrated.

Throughout this paper, fixed boundary conditions mean the boundary conditions of the form (6) and general boundary conditions mean the boundary conditions of the form (7)-(10).

2. Finite Difference Method

In this section we make an extension of finite difference method for second order SLP (4) as described in [12] to get an extended method which can be used to solve the fourth order SLP (5) coupled with both fixed boundary conditions and general boundary conditions.

To use finite difference method to approximate a problem, one must first discretize the problem's domain. This is normally done by dividing the domain into an uniform grid. Here we divide the interval $[0, 1]$ into $N + 1$ equidistance subintervals, i.e. we take an uniformly spaced mesh $0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = 1, i = 0, 1, \ldots, N + 1$ where $x_i = ih$ with $h = \frac{1}{N+1}$. Let us denote $y(x_i)$ by y_i, $q(x_i)$ by q_i, and $y^{(m)}(x_i)$ by $y_i^{(m)}$ for $m = 1, 2, 3, 4$.

The boundary conditions $y(0) = 0$ and $y(1) = 0$ imply that $y_0 = 0$ and $y_{N+1} = 0$. Let us introduce fictitious points $x_{-1} = -h$ and $x_{N+2} = 1 + h$ in order to handle the boundary conditions containing derivatives. Applying centred difference approximations to the boundary conditions $y''(0) = 0$ and $y''(1) = 0$ yields $y_{-1} = -y_1$ and $y_{N+2} = -y_N$, respectively.

Let us write $y^{(4)} + q(x)y = \lambda y$ as $y^{(4)} = (\lambda - q(x))y = f(x, y)$. By replacing centred difference approximation of order four, see e.g. [23], into $y_i^{(4)} = f_i, i = 1, \ldots, N$, one obtains

$$\frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4} = f_i, \quad i = 1, \ldots, N. \tag{11}$$

We now use $f_i = (\lambda_i - q_i)y_i$ to get

$$\frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4} + q_i y_i = \lambda_i y_i, \quad i = 1, \ldots, N. \tag{12}$$

Substituting the boundary conditions into (12), one can write the system of equations to get the matrix form of type (N, N)

$$K\boldsymbol{y} = \lambda \boldsymbol{y} \tag{13}$$

with local truncation error $O(h^2)$ where

$$
\begin{aligned}
K &= \frac{1}{h^4}A+Q, \quad Q=\mathrm{diag}(q_1,\dots,q_N), \quad \boldsymbol{y}=(y_1,\dots,y_N)^T,\\
A &= \begin{pmatrix}
A_{11} & -4 & 1 & & & & \\
A_{21} & 6 & -4 & 1 & & & \\
1 & -4 & 6 & -4 & 1 & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & 1 & -4 & 6 & -4 & 1 \\
 & & & 1 & -4 & 6 & A_{N-1,N} \\
 & & & & 1 & -4 & A_{N,N}
\end{pmatrix}_{N\times N}.
\end{aligned}
$$

and $A_{11}=5, A_{21}=-4, A_{N-1,N}=-4, A_{N,N}=5$.

Using the similar process for general boundary conditions as for fixed boundary conditions by applying difference approximations to the boundary conditions to obtain $y_{-1}, y_0, y_{N+1}, y_{N+2}$, and then substituting these terms into (12) yields the same matrix form (13). But now one has

$$
\begin{aligned}
A_{11} &= 5+\left(\frac{b_1}{b_2}h^2-2\right)\left(\frac{1}{1+\frac{a_1}{a_2}h}\right), & A_{21} &= -4+\frac{1}{1+\frac{a_1}{a_2}h},\\
A_{N-1,N} &= -4+\frac{1}{1-\frac{c_1}{c_2}h}, & A_{N,N} &= 5+\left(\frac{d_1}{d_2}h^2-2\right)\left(\frac{1}{1-\frac{c_1}{c_2}h}\right).
\end{aligned}
$$

3. Modified Numerov's Method

Numerov's method [12] is a fourth order linear multistep method and was developed by Beris Vasilayevich Numerov. It is a matrix method for a mesh of constant stepsize, like finite difference method, and used to solve second order ordinary differential equation $y''=f(x,y)$ where $f=f(x,y)$ is independent of the first derivative.

Numerov's method for second order SLP is modified in this section in order to approximate the eigenvalues of the fourth order SLP (5). We take the same stepsize as for finite difference method.

We recall Taylor's expansion of the following functions

$$
y_{i+2} = y(x_i+2h) = y_i + 2hy_i' + \frac{(2h)^2}{2!}y_i'' + \frac{(2h)^3}{3!}y_i''' + \frac{(2h)^4}{4!}y_i^{(4)} + \dots \quad (14)
$$

$$
y_{i+1} = y(x_i+h) = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' + \frac{h^4}{4!}y_i^{(4)} + \dots \quad (15)
$$

$$y_{i-1} = y(x_i - h) = y_i - hy_i' + \frac{h^2}{2!}y_i'' - \frac{h^3}{3!}y_i''' + \frac{h^4}{4!}y_i^{(4)} + \ldots \tag{16}$$

$$y_{i-2} = y(x_i - 2h) = y_i - 2hy_i' + \frac{(2h)^2}{2!}y_i'' - \frac{(2h)^3}{3!}y_i''' + \frac{(2h)^4}{4!}y_i^{(4)} + \ldots. \tag{17}$$

It follows from equations (14)-(17) that

$$\frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4} = y_i^{(4)} + \frac{h^2}{6}y_i^{(6)} + O(h^4) \tag{18}$$

and $y^{(6)} = f''$. Substituting $y^{(4)}$ and $y^{(6)}$ into (18) and applying centred difference approximations yield

$$\frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4} = f_i + \frac{1}{6}\left(f_{i+1} - 2f_i + f_{i-1}\right) \tag{19}$$

with local truncation error $O(h^4)$. Now using the same idea as in finite difference method and substituting q_0, q_{N+1} respectively by q_1, q_N (since $q_0 \to q_1$ and $q_{N+1} \to q_N$ as $N \to \infty$) yields

$$K\boldsymbol{y} = \lambda B\boldsymbol{y} \tag{20}$$

where $K = (1/h^4)A + BQ,\ B = I + (1/6)T$. In addition, $A, Q, \mathbf{y}$ are defined as in finite difference method, and

$$T = \begin{pmatrix} T_{11} & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & T_{N,N} \end{pmatrix}_{N\times N}$$

with $T_{11} = T_{N,N} = -2$ for fixed boundary conditions (6) and

$$T_{11} = -2 + \frac{1}{1 + \frac{a_1}{a_2}h}, \quad T_{N,N} = -2 + \frac{1}{1 - \frac{c_1}{c_2}h}$$

for general boundary conditions (7)-(10).

4. Boundary Value Methods

A family of boundary value methods obtained as an extension of Numerov's method has been investigated [21] as already mentioned in the Introduction.

In this section we investigate a class of boundary value methods in order to estimate the eigenvalues of the fourth order SLP (5) subjected to fixed boundary conditions (6) and general boundary conditions (7)-(10). The boundary value methods are a linear multistep formula (LMF) coupled with an additional formula for the boundary conditions see [24]. For second order problem, boundary value methods are obtained as an extension of Numerov's method. Here we make an extension of finite different method and modified Numerov's method for fourth order SLP in order to get boundary value methods for the fourth order SLP (5).

Recall that for finite different method and modified Numerov's method, we can write both methods as

$$K\boldsymbol{y} = \lambda B\boldsymbol{y} \tag{21}$$

where K and B are defined as in previous sections. In addition, $B = I$ for finite difference method and $B = I + (1/6)T$ for modified Numerov's method where I is the identity matrix.

Finite different method and modified Numerov's method are special cases of a $(2\nu+2)-$step linear multistep formula (LMF) of the following type

$$\sum_{i=0}^{2\nu} \alpha_i^{(\nu-1,2\nu)} y_{n+i-\nu} = \sum_{i=0}^{2\nu} \beta_i^{(\nu-1,2\nu)} f_{n+i-\nu} \tag{22}$$

where $n = \nu-1, \nu, \ldots, N+1-(\nu-1)$.

Let us define the *main formula* for $\nu \geq 2$ as

$$\frac{y_{n-2} - 4y_{n-1} + 6y_n - 4y_{n+1} + y_{n+2}}{h^4} = \sum_{i=0}^{2\nu} \beta_i^{(\nu-1,2\nu)} f_{n+i-\nu}, \tag{23}$$

where $n = \nu-1, \nu, \ldots, N+1-(\nu-1)$.

We notice that the schemes (11) and (19) are $(2\nu+2)-$step linear multistep formulas of the form (23) with $\nu = 2$ but different coefficients $\beta_i^{(1,4)}$. However, neither finite difference method nor modified Numerov's method have the highest possible order. Therefore we look for boundary value methods that corresponding to $\nu \geq 2$ with highest possible order. One has to try to write the problem to the form (21) with different matrix B, that means the coefficients $\beta_i^{(1,4)}$ must be determined .

As we can see from the main formula, there are only $N-2\nu+4$ equations while there are $N+4$ unknown variables, $y_{-1}, \ldots, y_{N+2}$. If we solve the above system of equations, then the solution is not unique. Suppose that we have fixed boundary conditions $y(0) = y''(0) = 0, y(1) = y''(1) = 0$. The boundary conditions give $y_{-1} = -y_1, y_0 = 0, y_{N+1} = 0, y_{N+2} = -y_N$. Hence we could eliminate 4 unknown variables from the given boundary conditions, i.e. we have only N unknown variables left to determine. Therefore the system needs $2\nu - 4, \nu \geq 2$, extra conditions to get uniqueness of the solution. We now couple the main formula with the *initial formula*

$$\frac{y_{s-2} - 4y_{s-1} + 6y_s - 4y_{s+1} + y_{s+2}}{h^4} = \sum_{i=0}^{2\nu} \beta_i^{(s,2\nu)} f_{i-1}, \tag{24}$$

where $s = 1, 2, \ldots, \nu - 2$, and the *final formula*

$$\frac{y_{m-2} - 4y_{m-1} + 6y_m - 4y_{m+1} + y_{m+2}}{h^4} = \sum_{i=0}^{2\nu} \beta_i^{(s,2\nu)} f_{m-s+i-1}, \tag{25}$$

where $s = \nu, \ldots, 2\nu - 3, m = N + 3 + s - 2\nu$.

Theorem 1. *The multistep formula (23) is at least of order* $2\nu + 1$ *if and only if two following conditions hold :*

1. $\sum_{i=-\nu}^{\nu} \alpha_{i+\nu}^{(s,2\nu)} = 0, \sum_{i=-\nu}^{\nu} i\alpha_{i+\nu}^{(s,2\nu)} = 0, \sum_{i=-\nu}^{\nu} i^2\alpha_{i+\nu}^{(s,2\nu)} = 0, \sum_{i=-\nu}^{\nu} i^3\alpha_{i+\nu}^{(s,2\nu)} = 0,$

2. $\sum_{i=-\nu}^{\nu} i^j\alpha_{i+\nu}^{(s,2\nu)} = j(j-1)(j-2)(j-3) \sum_{i=-\nu}^{\nu} i^{j-4}\beta_{i+\nu}^{(s,2\nu)}, j = 4, \ldots, 2\nu + 4.$

Proof. The proof is similar as in [25] but the Taylor series of the exact solution is expanded at $x = x_\nu$ instead of $x = x_0$ as classically done.

Let us define vector $\boldsymbol{\alpha}^{(s,2\nu)}$ for $s = 1, 2, \ldots, 2\nu - 3$ by

$$\boldsymbol{\alpha}^{(s,2\nu)} = (\alpha_0^{(s,2\nu)}, \ldots, \alpha_{2\nu}^{(s,2\nu)})$$

where nonzero entries are $\alpha_{s-1}^{(s,2\nu)} = 1, \alpha_s^{(s,2\nu)} = -4, \alpha_{s+1}^{(s,2\nu)} = 6, \alpha_{s+2}^{(s,2\nu)} = -4, \alpha_{s+3}^{(s,2\nu)} = 1$.

It is easy to check that the vector $\boldsymbol{\alpha}^{(s,2\nu)}$ satisfies the first condition of Theorem 1. Moreover, the second condition determines the unknown entries of the vector

$$\boldsymbol{\beta}^{(s,2\nu)} = (\beta_0^{(s,2\nu)}, \ldots, \beta_{2\nu}^{(s,2\nu)})$$

by solving the matrix form of type $(2\nu+1, 2\nu+1)$

$$\widetilde{D} V \boldsymbol{\beta}^{(s,2\nu)} = V D^4 \boldsymbol{\alpha}^{(s,2\nu)} \tag{26}$$

where

$$\begin{aligned}
\widetilde{D} &= \operatorname{diag}((4\cdot 3\cdot 2\cdot 1), \ldots, (2\nu+4)\cdot(2\nu+3)\cdot(2\nu+2)\ldots(2\nu+1)),\\
V &= \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ -\nu & 1-\nu & \ldots & \nu-1 & \nu \\ \vdots & \vdots & & \vdots & \vdots \\ (-\nu)^{2\nu} & (1-\nu)^{2\nu} & \ldots & (\nu-1)^{2\nu} & (\nu)^{2\nu} \end{pmatrix},\\
D &= \operatorname{diag}(-\nu, 1-\nu, \ldots, \nu-1, \nu).
\end{aligned}$$

Proposition 1. *The proposed composite scheme (23)-(25) is symmetric, i.e. its coefficient vectors satisfy*

$$\boldsymbol{\alpha}^{(2\nu-s-2,2\nu)} = J\boldsymbol{\alpha}^{(s,2\nu)}, \boldsymbol{\beta}^{(2\nu-s-2,2\nu)} = J\boldsymbol{\beta}^{(s,2\nu)}, s = 1, 2, \ldots, \nu-1. \tag{27}$$

where J is the anti-identity matrix of size $2\nu+1$ in which all the entries are zero except those on the diagonal going from the lower left corner to the upper right corner ($\nearrow$) are one.

Proof. The proof is similar to [21].

The order of accuracy of the composite scheme (23)-(25) means the order of the main formula (23). Since the main formula (23) is a symmetric LMF, i.e. the one corresponding to $s = \nu - 1$ in (27) of Proposition 1, its order of accuracy must be even. From Theorem 1 the main formula is at least of order $2\nu + 1$, therefore the order of accuracy is $p = 2\nu + 2$.

In the case $\nu = 2, 3, 4$, the coefficients of the main formula (23) and of the initial formula (24) are shown in Table 1 and the coefficients of the final formula (25) are obtained from the coefficients of the initial formula by using (27).

The system of equations for the fourth order SLP (5) subjected to the fixed boundary conditions (6) can be written by using boundary value methods as in the form (21) where the matrix B is substituted by the following matrix $B^{(\nu)}$

$$B^{(\nu)} = \left(B_1^{(\nu)}, B_2^{(\nu)}, B_3^{(\nu)}\right)^T$$

	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8
$\nu = 2$									
$\boldsymbol{\beta}^{(1,4)}$	$\frac{-1}{720}$	$\frac{31}{180}$	$\frac{79}{120}$	$\frac{31}{180}$	$\frac{-1}{720}$				
$\nu = 3$									
$\boldsymbol{\beta}^{(1,6)}$	$\frac{-1}{945}$	$\frac{143}{840}$	$\frac{721}{1087}$	$\frac{313}{1890}$	$\frac{1}{280}$	$\frac{-1}{504}$	$\frac{1}{3024}$		
$\boldsymbol{\beta}^{(2,6)}$	$\frac{1}{3024}$	$\frac{-17}{5040}$	$\frac{205}{1157}$	$\frac{436}{669}$	$\frac{205}{1157}$	$\frac{-17}{5040}$	$\frac{1}{3024}$		
$\nu = 4$									
$\boldsymbol{\beta}^{(1,8)}$	$\frac{-24}{52943}$	$\frac{737}{4447}$	$\frac{1366}{2015}$	$\frac{607}{4377}$	$\frac{668}{19453}$	$\frac{-111}{4571}$	$\frac{101}{9784}$	$\frac{-70}{27731}$	$\frac{28}{102117}$
$\boldsymbol{\beta}^{(2,8)}$	$\frac{28}{102117}$	$\frac{-53}{18144}$	$\frac{95}{541}$	$\frac{1093}{1669}$	$\frac{4827}{27865}$	$\frac{-19}{90720}$	$\frac{-61}{48757}$	$\frac{22}{48679}$	$\frac{-13}{230119}$
$\boldsymbol{\beta}^{(3,8)}$	$\frac{-13}{230119}$	$\frac{71}{90720}$	$\frac{-74}{14935}$	$\frac{323}{1791}$	$\frac{971}{1499}$	$\frac{323}{1791}$	$\frac{-74}{14935}$	$\frac{71}{90720}$	$\frac{-13}{230119}$

Table 1: The coefficients of the main and the initial methods with $\nu = 2, 3, 4$.

where

$$B_1^{(\nu)} = \begin{pmatrix} \beta_2^{(1)} - \beta_0^{(1)} & \beta_3^{(1)} & \dots & \beta_\nu^{(1)} & \dots & \beta_{2\nu-1}^{(1)} & \beta_{2\nu}^{(1)} & 0\dots0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ \beta_2^{(\nu-2)} - \beta_0^{(\nu-2)} & \beta_3^{(\nu-2)} & \dots & \beta_\nu^{(\nu-2)} & \dots & \beta_{2\nu-1}^{(\nu-2)} & \beta_{2\nu}^{(\nu-2)} & 0\dots0 \\ \beta_2^{(\nu-1)} - \beta_0^{(\nu-1)} & \beta_3^{(\nu-1)} & \dots & \beta_\nu^{(\nu-1)} & \dots & \beta_1^{(\nu-1)} & \beta_0^{(\nu-1)} & 0\dots0 \end{pmatrix},$$

$$B_2^{(\nu)} = \begin{pmatrix} \beta_1^{(\nu-1)} & \dots & \beta_\nu^{(\nu-1)} & \dots & \beta_1^{(\nu-1)} & \beta_0^{(\nu-1)} & & & & \\ \beta_0^{(\nu-1)} & \beta_1^{(\nu-1)} & \dots & \beta_\nu^{(\nu-1)} & \dots & \beta_1^{(\nu-1)} & \beta_0^{(\nu-1)} & & & \\ \ddots & \ddots & & \ddots & & \ddots & \ddots & & & \\ & \beta_0^{(\nu-1)} & \beta_1^{(\nu-1)} & \dots & \beta_\nu^{(\nu-1)} & \dots & \beta_1^{(\nu-1)} & \beta_0^{(\nu-1)} & & \\ & & \beta_0^{(\nu-1)} & \beta_1^{(\nu-1)} & \dots & \beta_\nu^{(\nu-1)} & \dots & \beta_1^{(\nu-1)} & & \end{pmatrix},$$

$$B_3^{(\nu)} = \begin{pmatrix} 0\dots0 & \beta_{2\nu}^{(\nu-1)} & \beta_{2\nu-1}^{(\nu-1)} & \dots & \beta_\nu^{(\nu-1)} & \dots & \beta_3^{(\nu-1)} & \beta_2^{(\nu-1)} - \beta_0^{(\nu-1)} \\ 0\dots0 & \beta_{2\nu}^{(\nu-2)} & \beta_{2\nu-1}^{(\nu-2)} & \dots & \beta_\nu^{(\nu-2)} & \dots & \beta_3^{(\nu-2)} & \beta_2^{(\nu-2)} - \beta_0^{(\nu-2)} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0\dots0 & \beta_{2\nu}^{(1)} & \beta_{2\nu-1}^{(1)} & \dots & \beta_\nu^{(1)} & \dots & \beta_3^{(1)} & \beta_2^{(1)} - \beta_0^{(1)} \end{pmatrix}.$$

The rest of this section, we consider the general boundary conditions of the form (7)-(10). A standard way to handle a boundary condition containing the derivative of the left endpoint, $x_0 = 0$, and the right endpoint, $x_{N+1} = 1$, is to introduce fictitious points $x_{-1} = -h$ and $x_{N+2} = 1 + h$.

Firstly, let us write our composite scheme (23)-(25) as

$$K\widetilde{\boldsymbol{y}} = \left(\frac{1}{h^4}\widetilde{A} + \widetilde{B}\widetilde{Q}\right)\widetilde{\boldsymbol{y}} = \lambda\widetilde{B}\widetilde{\boldsymbol{y}}$$

where

$$\begin{aligned}
\widetilde{\boldsymbol{y}} &= (y_{-1}, y_0, \ldots, y_{N+1}, y_{N+2})^T \in \mathbb{R}^{N+4} \\
\widetilde{Q} &= \operatorname{diag}(q_{-1}, q_0, Q, q_{N+1}, q_{N+2}) \in \mathbb{R}^{(N+4)\times(N+4)}, \\
Q &= \operatorname{diag}(q_1, \ldots, q_N) \in \mathbb{R}^{N\times N}, \\
\widetilde{A} &= \left(A_0\ A_1\ |\widehat{A}\ |\ JA_1\ JA_0\right) \in \mathbb{R}^{N\times(N+4)}, \quad A_0, A_1 \in \mathbb{R}^N \\
\widehat{A} &= \begin{pmatrix}
6 & -4 & 1 & & & & \\
-4 & 6 & -4 & 1 & & & \\
1 & -4 & 6 & -4 & 1 & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & 1 & -4 & 6 & -4 & 1 \\
 & & & 1 & -4 & 6 & -4 \\
 & & & & 1 & -4 & 6
\end{pmatrix}_{N\times N}
\end{aligned}$$

and $A_0 = (1, 0, \ldots, 0)^T$, $A_1 = (-4, 1, 0, \ldots, 0)^T$. In addition, the matrix $\widetilde{B}$ is defined by

$$\widetilde{B}^{(\nu)} = \left(\boldsymbol{\beta}_0^{(\nu)}\ \boldsymbol{\beta}_1^{(\nu)}\ |\ B^{(\nu)}\ |\ J\boldsymbol{\beta}_1^{(\nu)}\ J\boldsymbol{\beta}_0^{(\nu)}\right) \in \mathbb{R}^{N\times(N+4)}, \quad \boldsymbol{\beta}_0^{(\nu)}, \boldsymbol{\beta}_1^{(\nu)} \in \mathbb{R}^N$$

with $\boldsymbol{\beta}_0^{(\nu)} = \left(\beta_0^{(1)}, \ldots, \beta_0^{(\nu-1)}, 0, \ldots, 0\right)^T$, $\boldsymbol{\beta}_1^{(\nu)} = \left(\beta_1^{(1)}, \ldots, \beta_1^{(\nu-1)}, 0, \ldots, 0\right)^T$, and $B^{(\nu)} = \left(B_1^{(\nu)}, B_2^{(\nu)}, B_3^{(\nu)}\right)^T$ where

$$B_1^{(\nu)} = \begin{pmatrix}
\beta_2^{(1)} & \beta_3^{(1)} & \ldots & \beta_\nu^{(1)} & \ldots & \beta_{2\nu-1}^{(1)} & \beta_{2\nu}^{(1)} & 0\cdots 0 \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
\beta_2^{(\nu-2)} & \beta_3^{(\nu-2)} & \ldots & \beta_\nu^{(\nu-2)} & \ldots & \beta_{2\nu-1}^{(\nu-2)} & \beta_{2\nu}^{(\nu-2)} & 0\cdots 0 \\
\beta_2^{(\nu-1)} & \beta_3^{(\nu-1)} & \ldots & \beta_\nu^{(\nu-1)} & \ldots & \beta_1^{(\nu-1)} & \beta_0^{(\nu-1)} & 0\cdots 0
\end{pmatrix},$$

$$B_2^{(\nu)} = \begin{pmatrix} \beta_1^{(\nu-1)} & \dots & \beta_\nu^{(\nu-1)} & \dots & \beta_1^{(\nu-1)} & \beta_0^{(\nu-1)} & & & \\ \beta_0^{(\nu-1)} & \beta_1^{(\nu-1)} & \dots & \beta_\nu^{(\nu-1)} & \dots & \beta_1^{(\nu-1)} & \beta_0^{(\nu-1)} & & \\ \ddots & \ddots & & \ddots & & \ddots & \ddots & & \\ & \beta_0^{(\nu-1)} & \beta_1^{(\nu-1)} & \dots & \beta_\nu^{(\nu-1)} & \dots & \beta_1^{(\nu-1)} & \beta_0^{(\nu-1)} \\ & & \beta_0^{(\nu-1)} & \beta_1^{(\nu-1)} & \dots & \beta_\nu^{(\nu-1)} & \dots & \beta_1^{(\nu-1)} \end{pmatrix},$$

$$B_3^{(\nu)} = \begin{pmatrix} 0\dots0 & \beta_{2\nu}^{(\nu-1)} & \beta_{2\nu-1}^{(\nu-1)} & \dots\beta_\nu^{(\nu-1)}\dots & \beta_3^{(\nu-1)} & \beta_2^{(\nu-1)} \\ 0\dots0 & \beta_{2\nu}^{(\nu-2)} & \beta_{2\nu-1}^{(\nu-2)} & \dots\beta_\nu^{(\nu-2)}\dots & \beta_3^{(\nu-2)} & \beta_2^{(\nu-2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0\dots0 & \beta_{2\nu}^{(1)} & \beta_{2\nu-1}^{(1)} & \dots\ \beta_\nu^{(1)}\ \dots & \beta_3^{(1)} & \beta_2^{(1)} \end{pmatrix}.$$

Consider the first boundary condition, $a_1y(0) - a_2y'(0) = 0$. We start with the $(2\nu+2)-$step forward differentiation formula (FDF) given by (see [22] for second order SLP)

$$\sum_{i=0}^{2\nu+2} \omega_i y_i = hy'(0) + \tau_L.$$

By ignoring τ_L and substituting $y'(0)$ by $y'(0) = \frac{a_1}{a_2}y(0)$ yield

$$a_2 \sum_{i=0}^{2\nu+2} \omega_i y_i = ha_1y_0.$$

This equation is equivalent to

$$y_0 = \frac{a_2}{ha_1 - a_2\omega_0} \sum_{i=1}^{2\nu+2} \omega_i y_i = \gamma_L \boldsymbol{\omega}^T \boldsymbol{y}, \quad \gamma_L = \frac{a_2}{ha_1 - a_2\omega_0}$$

where $\boldsymbol{y} = (y_1, \dots, y_N)^T$ and $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_{2\nu+2}, 0, \dots, 0)^T \in \mathbb{R}^N$. Now consider the boundary condition $b_1y(0) - b_2y''(0) = 0$. Applying the centred difference formula yields

$$b_1y_0 - b_2\frac{y_1 - 2y_0 + y_{-1}}{h^2} = 0.$$

This equation can be written as

$$y_{-1} = \left(\frac{b_1}{b_2}h^2 + 2\right) y_0 - y_1 = \left(\eta_L\gamma_L\boldsymbol{\omega}^T - \boldsymbol{e}^T\right) \boldsymbol{y}, \quad \eta_L = \frac{b_1}{b_2}h^2 + 2$$

	ω_0	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$\nu=2$	-25/12	4	-3	4/3	-1/4				
$\nu=3$	-49/20	6	-15/2	20/3	-15/4	6/5	-1/6		
$\nu=4$	-761/280	8	-14	56/3	35/2	56/5	14/3	8/7	1/8

Table 2: Coefficients of the $(2\nu+2)$-step FDF with $\nu = 2, 3, 4$

where $\boldsymbol{e} = (1, 0, \ldots, 0)^T \in \mathbb{R}^N$. We obtain the coefficients of the $(2\nu+2)-$step forward differentiation formula (FDF) with $\nu = 2, 3, 4$ as shown in Table 2. Applying the $(2\nu+2)-$step backward differentiation formula (BDF) to the boundary condition $c_1 y(1) - c_2 y'(1) = 0$ gives

$$c_2 \sum_{i=0}^{2\nu+2} \widehat{\omega}_i y_{N-1-2\nu+i} = hc_1 y_{N+1}.$$

Since the coefficients of the FDF and the BDF with the same step number satisfy $\omega_j = -\widehat{\omega}_{2\nu+2-j}$ for $j = 0, 1, \ldots, 2\nu+2$, we have

$$y_{N+1} = -\frac{c_2}{hc_1 + c_2\omega_0}(\boldsymbol{\omega}^T J)\boldsymbol{y} = \gamma_R(\boldsymbol{\omega}^T J)\boldsymbol{y}, \quad \gamma_R = -\frac{c_2}{hc_1 + c_2\omega_0}.$$

Similarly, using the same idea to the boundary condition $d_1 y(1) - d_2 y''(1) = 0$ yields

$$y_{N+2} = \left(\frac{d_1}{d_2}h^2 + 2\right) y_{N+1} - y_N = \left(\eta_R\gamma_R(\boldsymbol{\omega}^T J) - \boldsymbol{e}^T J\right)\boldsymbol{y}, \quad \eta_R = \frac{d_1}{d_2}h^2 + 2.$$

Consider

$$\begin{aligned} K\widetilde{\boldsymbol{y}} &= \left(\frac{1}{h^4}\widetilde{A} + \widetilde{B}\widetilde{Q}\right)\widetilde{\boldsymbol{y}} \\ &= \left(\frac{1}{h^4}\widehat{A} + BQ\right)\boldsymbol{y} + \frac{1}{h^4}\left(A_0 y_{-1} + A_1 y_0 + JA_1 y_{N+1} + JA_0 y_{N+2}\right) \\ &\quad + \boldsymbol{\beta}_0 q_{-1} y_{-1} + \boldsymbol{\beta}_1 q_0 y_0 + J\boldsymbol{\beta}_1 q_{N+1} y_{N+1} + J\boldsymbol{\beta}_0 q_{N+2} y_{N+2} \\ \lambda\widetilde{B}\widetilde{\boldsymbol{y}} &= \lambda\left(B\boldsymbol{y} + \boldsymbol{\beta}_0 y_{-1} + \boldsymbol{\beta}_1 y_0 + J\boldsymbol{\beta}_1 y_{N+1} + J\boldsymbol{\beta}_0 y_{N+2}\right) \end{aligned}$$

Substituting $y_{-1}, y_0, y_{N+1}, y_{N+2}$ into the system $K\widetilde{\boldsymbol{y}} = \lambda\widetilde{B}\widetilde{\boldsymbol{y}}$, one obtains the linear system

$$K\boldsymbol{y} = \lambda M\boldsymbol{y} \tag{28}$$

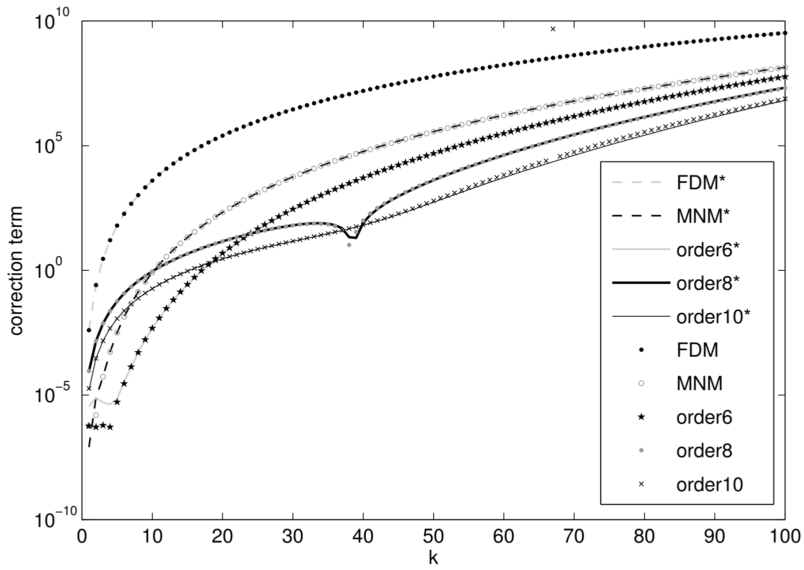


Figure 1: Graph of the correction term $\epsilon_k, k = 1, \ldots, 100$ and the approximated correction term $\widetilde{\epsilon}_k, k = 1, \ldots, 100$. The name of the methods with * denotes the approximated correction terms $\widetilde{\epsilon}$.

where

$$\begin{aligned}
K &= \frac{1}{h^4}\widehat{A} + BQ + \frac{1}{h^4}\left(A_0\left(\eta_L\gamma_L\boldsymbol{\omega}^T - \boldsymbol{e}^T\right) + JA_0\left(\eta_R\gamma_R(\boldsymbol{\omega}^T J) - \boldsymbol{e}^T J\right)\right) \\
&\quad + \frac{1}{h^4}\left(A_1\gamma_L\boldsymbol{\omega}^T + JA_1\gamma_R(\boldsymbol{\omega}^T J)\right) + \boldsymbol{\beta}_0 q_{-1}\left(\eta_L\gamma_L\boldsymbol{\omega}^T - \boldsymbol{e}^T\right) \\
&\quad + \boldsymbol{\beta}_1 q_0\gamma_L\boldsymbol{\omega}^T + J\boldsymbol{\beta}_1 q_{N+1}\gamma_R(\boldsymbol{\omega}^T J) + J\boldsymbol{\beta}_0 q_{N+2}\left(\eta_R\gamma_R(\boldsymbol{\omega}^T J) - \boldsymbol{e}^T J\right), \\
M &= B + \boldsymbol{\beta}_0\left(\eta_L\gamma_L\boldsymbol{\omega}^T - \boldsymbol{e}^T\right) + \boldsymbol{\beta}_1\gamma_L\boldsymbol{\omega}^T + J\boldsymbol{\beta}_0\left(\eta_R\gamma_R(\boldsymbol{\omega}^T J) - \boldsymbol{e}^T J\right) \\
&\quad + J\boldsymbol{\beta}_1\gamma_R(\boldsymbol{\omega}^T J).
\end{aligned}$$

5. Correction Terms

It is stated in [12] that the correction was first introduced by Paine, J.W. in his PhD Thesis (1979). Then it is followed up in various papers [13, 14, 16, 19, 20, 26]. The correction terms can be added to the numerical eigenvalues in order to obtain better eigenvalues of the problem.

In this section the correction terms of finite difference method, modified Numerov's method and boundary value methods are investigated.

The correction terms can be computed by

$$\epsilon_k = \lambda^0_{\text{exa},k} - \lambda^0_{\text{exp},k}$$

where $\lambda^0_{\text{exa},k}$ and $\lambda^0_{\text{exp},k}$ denote the kth exact and expected numerical eigenvalues for $q = 0$, respectively. The expected numerical eigenvalues are obtained by replacing $\boldsymbol{y}$ in the matrix form (13), (20) and (28) by the exact eigenfunctions. Then one has to solve the system of equations. However, we may compute the correction terms by

$$\epsilon_k \approx \widetilde{\epsilon_k} = \lambda^0_{\text{exa},k} - \lambda^0_{\text{num},k}$$

where $\lambda^0_{\text{num},k}$ is the kth numerical eigenvalue for $q = 0$ by using matrix methods. The kth corrected numerical eigenvalue $\lambda^0_{\text{cor},k}$ can be found by $\lambda_{\text{cor},k} = \lambda_{\text{num},k} + \epsilon_k$ where $\lambda_{\text{num},k}$ is the kth numerical eigenvalue by using matrix methods for arbitrary q.

We know that the exact eigenvalues of problem (5) for the case $q = 0$ coupled with fixed boundary conditions [27] are $\lambda_k = (k\pi)^4, k = 1, 2, \ldots$ and the exact eigenfunctions are $y_k = \sqrt{2}\sin k\pi x, k = 1, 2, \ldots$.

5.1. Finite Difference Method

Recall that by using finite difference method, we can write the fourth order SLP (5) with $q(x) = 0$ subjected to the fixed boundary conditions (6) as $(1/h^4)A\boldsymbol{y} = \lambda\boldsymbol{y}$.

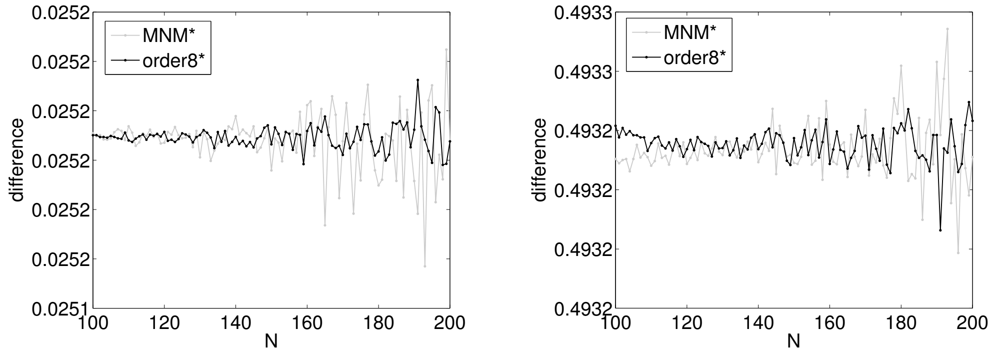
Replacing $\boldsymbol{y}$ by the exact eigenfunctions and solving these equations yields

$$\frac{1}{h^4}(5\sin k\pi h - 4\sin 2k\pi h + \sin 3k\pi h) = \lambda_k \sin k\pi h.$$

Then

$$\lambda^0_{\text{exp},k} = \frac{1}{h^4}\left(\frac{5\sin k\pi h - 4\sin 2k\pi h + \sin 3k\pi h}{\sin k\pi h}\right), \quad k = 1, 2, \ldots$$

and the correction term of finite difference method is $\epsilon_k = (k\pi)^4 - \lambda^0_{\text{exp},k}, \;\; k = 1, 2, \ldots$.



(a) $k = 20$

(b) $k = 40$

Figure 2: Differences in the approximations of $\lambda_k, k = 20, 40$ with respect to the numerical eigenvalues using the code SLEUTH for Example 2.

5.2. Modified Numerov's Method

For modified Numerov's method, we can write the fourth order SLP (5) subjected to the fixed boundary conditions (6) as $(1/h^4)A\boldsymbol{y} = \lambda B\boldsymbol{y}$.

Similar to finite difference method, one obtains

$$\frac{1}{h^4}(5\sin k\pi h - 4\sin 2k\pi h + \sin 3k\pi h) = \lambda_k \left(\frac{4}{6}\sin k\pi h + \frac{1}{6}\sin 2k\pi h\right).$$

Then

$$\lambda^0_{\exp,k} = \frac{6}{h^4}\left(\frac{5\sin k\pi h - 4\sin 2k\pi h + \sin 3k\pi h}{4\sin k\pi h + \sin 2k\pi h}\right), \quad k = 1, 2, \ldots$$

and the correction term with Numerov's method is $\epsilon_k = \lambda^0_{\exp,k}, \;\; k = 1, 2, \ldots$.

5.3. Boundary Value Methods

Now we illustrate the correction terms of boundary value methods. Using the same process as before and taking the $(\nu - 1)$th rows of system of equations one obtains the correction terms as follows.
Case I : $\nu = 2$ (order $p = 6$),

$$\epsilon_k = (k\pi)^4 - \frac{1}{h^4}\left(\frac{5\sin k\pi h - 4\sin 2k\pi h + \sin 3k\pi h}{\sum_{i=1}^{3}\beta^{(1)}_{i+1}\sin ik\pi h - \beta^{(1)}_0\sin k\pi h}\right),$$

Method	λ_1	$\lambda_2(\times 10^3)$	$\lambda_3(\times 10^3)$	$\lambda_4(\times 10^4)$	$\lambda_5(\times 10^4)$
FDM	98.40512350352969	1.559291649027748	7.888245612960320	2.492148721298525	6.081974107416002
MNM	98.40909045039442	1.559545462093629	7.891136323896369	2.493772676808160	6.088167873502381
order6	98.40909713844243	1.559545465093973	7.891136379782363	2.493772731831868	6.088168190139762
order8	98.40918044820025	1.559546913892838	7.891143757429511	2.493775064473476	6.088173887496135
order10	98.40907129437645	1.559545159984350	7.891134875084986	2.493772257536026	6.088167035667270
FDM*	98.40909079949648	1.559545456837260	7.891136372791951	2.493772730528151	6.088168189834414
MNM*	98.40909053149949	1.559545454217382	7.891136377248733	2.493772730341868	6.088168189476284
order6*	98.40910060933985	1.559545472802071	7.891136384826430	2.493772732244701	6.088168190819246
order8*	98.40909149706071	1.559545456584187	7.891136373841368	2.493772730405689	6.088168189596387
order10*	98.40908696311035	1.559545454175871	7.891136371808981	2.493772730319019	6.088168189433142
exact	98.40909103400243	1.559545456544039	7.891136373754197	2.493772730470462	6.088168189625152
Method	$\lambda_{10}(\times 10^5)$	$\lambda_{20}(\times 10^7)$	$\lambda_{50}(\times 10^8)$	$\lambda_{100}(\times 10^9)$	$\lambda_{200}(\times 10^{11})$
FDM	9.701331399863446	1.533348178893714	5.495995567986329	6.427316495699802	0.261126630442447
MNM	9.740910982415146	1.558524402679567	6.084432463310545	9.603450089873486	0.783284218576547
order6	9.740919056373589	1.558545071524078	6.087584188390536	9.682572831969589	0.839218449813391
order8	9.740928214852355	1.558547003781238	6.088027291697892	9.719669364298733	0.900453954058430
order10	9.740917265751835	1.558545267639386	6.088062839506598	9.734336960097340	0.947698991908945
FDM*	9.740919103371439	1.558545556543930	6.088068199625164	9.740909104400242	1.558545456554039
MNM*	9.740919103339907	1.558545556543324	6.088068199625074	9.740909104400244	1.558545456554039
order6*	9.740919103394907	1.558545556545050	6.088068199625179	9.740909104400244	1.558545456554039
order8*	9.740919103393357	1.558545556544351	6.088068199625161	9.740909104400225	1.558545456554037
order10*	9.740919103368853	1.558545556543903	6.088068199625131	9.740909104400244	1.558545456554040
exact	9.740919103400243	1.558545556544039	6.088068199625152	9.740909104400244	1.558545456554039

Table 3: The exact eigenvalues and the numerical eigenvalues λ_k for Example 1 by using FDM, modified Numerov's method (MNM), and BVMs of order $p = 6, 8, 10$ and $N = 200$. * denotes the methods with correction terms .

Case II : $\nu = 3$ (order $p = 8$),

$$\epsilon_k = (k\pi)^4 - \frac{1}{h^4}\left(\frac{-4\sin k\pi h + 6\sin 2k\pi h - 4\sin 3k\pi h + \sin 4k\pi h}{\sum_{i=1}^{5}\beta_{i+1}^{(2)}\sin ik\pi h - \beta_0^{(2)}\sin k\pi h}\right),$$

Case III : $\nu = 4, 5, \ldots$ (order $p = 2\nu + 2$),

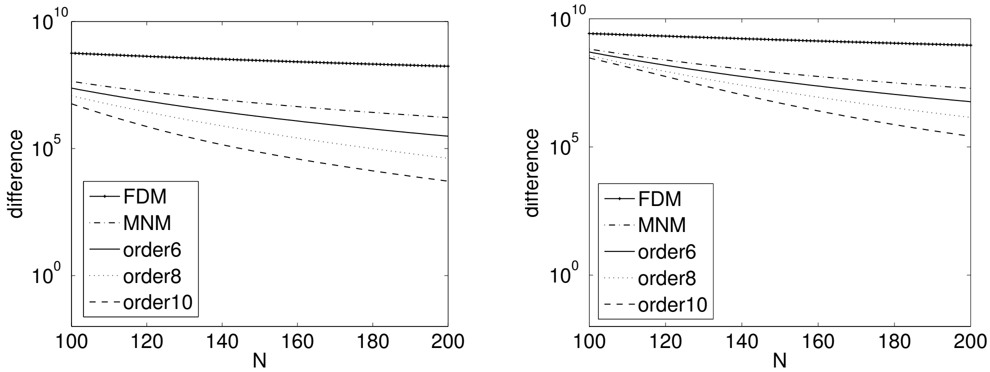
$$\epsilon_k = (k\pi)^4 - \frac{1}{h^4}\left(\frac{\chi_\nu}{\sum_{i=1}^{2\nu-1}\beta_{i+1}^{(\nu-1)}\sin ik\pi h - \beta_0^{(\nu-1)}\sin k\pi h}\right),$$

where $\chi_\nu = \sin(\nu-3)k\pi h - 4\sin(\nu-2)k\pi h + 6\sin(\nu-1)k\pi h - 4\sin(\nu)k\pi h + \sin(\nu+1)k\pi h, \quad k = 1, 2, \ldots.$

Figure 1 shows the graph of the correction terms $\epsilon_k, k = 1, \ldots, 100$ and the approximated correction terms $\widetilde{\epsilon}_k, k = 1, \ldots, 100$ by using matrix methods. The graphs of $\widetilde{\epsilon}$ are only a little bit different from ϵ. This demonstrates that $\widetilde{\epsilon}$ could be used instead of ϵ.

6. Numerical results

In this section, some numerical results are illustrated by using all matrix methods with MATLAB software.



(a) $k = 60$ (b) $k = 80$

Figure 3: Differences in the approximations of $\lambda_k, k = 60, 80$ with respect to the numerical eigenvalues using the code SLEUTH for Example 3.

Example 1. *Let us consider the fourth order SLP with* $q(x) = 1$ *subjected to the fixed boundary conditions.*

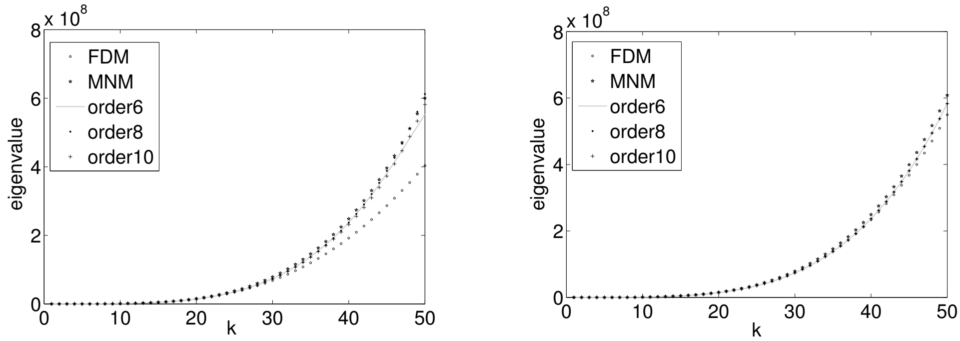
The exact eigenvalues of this problem are given in [27] $\lambda_k = (k\pi)^4 + 1, k = 1, 2, \ldots$. Table 3 shows the exact eigenvalues and the numerical eigenvalues $\lambda_k, k = 1, \ldots, 5, 10, 20, 50, 100, 200$.

Example 2. *Consider the fourth order SLP with* $q(x) = \sin(x) + 2$ *subjected to the fixed boundary conditions.*

Since the exact eigenvalues of this problem could not be found, we illustrate the differences of numerical eigenvalues $\lambda_k, k = 20, 40$ compared with the approximated eigenvalues obtained from the code SLEUTH in Figure 2 .

Example 3. *Let us consider the fourth order SLP with* $q(x) = \exp(x/2)$ *coupled with the fixed boundary conditions.*

Since the exact eigenvalues of this problem could not be found, Figure 3 shows the differences of numerical eigenvalues $\lambda_k, k = 60, 80$ with respect to the approximated eigenvalues by using the code SLEUTH.



(a) Number of stepsize=100 (b) Number of stepsize=200

Figure 4: The first 50 numerical eigenvalues for Example 4.

Example 4. *Consider the fourth order SLP with* $q(x) = \sin(x) + 2$ *coupled with general boundary conditions where* $a_i, b_i, c_i, d_i = 1$ *for* $i = 1, 2$*, i.e. subjected to*

$$y(0) - y'(0) = 0, y(0) - y''(0) = 0,$$
$$y(1) - y'(1) = 0, y(1) - y''(1) = 0.$$

The exact eigenvalues of this problem could not be compute and there is no other method to our best knowledge to approximate them. The numerical eigenvalues $\lambda_k, k = 1, \ldots, 50$ with different number of stepsize are presented in Figure 4.

Example 5. *Consider the fourth order SLP with* $q(x) = x$ *subjected to fixed boundary conditions.*

The numerical eigenvalues $\lambda_k, k = 1, \ldots, 5, 10, 20, 50, 100, 200$ by using different methods are shown in Table 4.

7. Conclusions

We extended finite different method, Numerov's method, and boundary value methods from second order regular Sturm-Liouville problem to a fourth order Sturm-Liouville problem not only for standard boundary conditions but also for a particular kind of general boundary conditions. It is shown that the

Method	λ_1	$\lambda_2(\times 10^3)$	$\lambda_3(\times 10^3)$	$\lambda_4(\times 10^4)$	$\lambda_5(\times 10^4)$
FDM	97.90510331909168	1.558791666146541	7.887745621216823	2.492098721158018	6.081924107325172
MNM	97.90906783358864	1.559045469856773	7.890636324503293	2.493722677603690	6.088117873746771
order6	97.90906830465880	1.559045473344954	7.890636376268214	2.493722730549470	6.088118189204394
order8	97.90915769215033	1.559046929411862	7.890643762503703	2.493725064642844	6.088123887772959
order10	97.90905008131770	1.559045176457534	7.890634878155296	2.493722257693582	6.088117035666337
FDM*	97.90907061505847	1.559045473956054	7.890636381048454	2.493722730387645	6.088118189743584
MNM*	97.90906791469371	1.559045461980526	7.890636377855657	2.493722731137398	6.088118189720674
order6*	97.90907177555621	1.559045481053052	7.890636381312281	2.493722730962303	6.088118189883878
order8*	97.90906874101080	1.559045472103210	7.890636378915559	2.493722730575056	6.088118189873210
order10*	97.90906575005161	1.559045470649055	7.890636374879291	2.493722730476574	6.088118189432209
ADM	97.90906881979826	1.559045472766815	7.890636377436013	2.4937216986103	6.116550334676592
SLEUTH	97.9090688	1.55904547	7.89063638	2.49372273	6.08811819

Method	$\lambda_{10}(\times 10^5)$	$\lambda_{20}(\times 10^7)$	$\lambda_{50}(\times 10^8)$	$\lambda_{100}(\times 10^9)$	$\lambda_{200}(\times 10^{11})$
FDM	9.701326399909467	1.533348128893402	5.495995562986329	6.427316495199812	0.261126630437447
MNM	9.740905982466708	1.558524352679611	6.084432458310583	9.603450089373493	0.783284218571547
order6	9.740914056322698	1.558545021523828	6.087584183390558	9.682572831469570	0.839218449808391
order8	9.740923214854087	1.558546953781296	6.088027286697879	9.719669363798729	0.900453954053431
order10	9.740912265764668	1.558545217639233	6.088062834506621	9.734336959597340	0.947698991903944
FDM*	9.740914103417461	1.558545506543618	6.088068194625164	9.740909103900250	1.558545456549039
MNM*	9.740914103391469	1.558545506543369	6.088068194625112	9.740909103900251	1.558545456549039
order6*	9.740914103344017	1.558545506544800	6.088068194625200	9.740909103900225	1.558545456549039
order8*	9.740914103395089	1.558545506544409	6.088068194625148	9.740909103900221	1.558545456549038
order10*	9.740914103381686	1.558545506543750	6.088068194625155	9.740909103900244	1.558545456549040
SLEUTH	9.7409141	1.55854551	6.08806820	9.74090911	1.55854546

Table 4: The numerical eigenvalues λ_k for Example 5 by using FDM, modified Numerov's method (MNM), boundary value methods of order $p = 6, 8, 10$, the code SLEUTH ($Tol = 10^{-9}$), ADM (using 10 terms and see [4]), and BVMs of order $p = 6, 8, 10$, and $N = 200$. * denotes the methods with correction terms .

orders of accuracy of finite difference method, modified Numerov's method, and boundary value methods are $p = 2$, $p = 4$, and $p = 2\nu + 2, \nu = 2, 3, \ldots$, respectively. The numerical solutions are improved by increasing the number of stepsize N or by adding the correction terms to the numerical solutions. The correction term formulas subjected to fixed boundary conditions $y(0) = y''(0) = y(1) = y''(1) = 0$ have been investigated. Moreover, we have shown by graphs that we can use numerical eigenvalues with $q = 0$ instead of expected eigenvalues with $q = 0$ in order to find the correction terms of each method. Some numerical results have been demonstrated to ensure that the methods work well. In addition, one can use matrix methods not only to compute eigenvalues of the problem, but we can use them also to approximate the eigenfunctions corresponding to the eigenvalues of the problem. Finally, the matrix methods can be used to solve the fourth order SLP subjected to other kinds of fixed boundary conditions.

Acknowledgement We would like to thank Prof. Greenberg and Prof. Marletta for providing the software SLEUTH for downloading.

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