A Short Introduction to Boutet de Monvel's Calculus

Elmar Schrohe

Abstract. This paper provides an introduction to Boutet de Monvel's calculus on the half-space $\mathbb{R}^n_+$ in the framework of a pseudodifferential calculus with operator-valued symbols.

Introduction

The development of the theory of pseudodifferential operators has greatly advanced our understanding of partial differential equations, and the pseudodifferential calculus has become an indispensable tool in contemporary analysis, in particular on compact manifolds without boundary.

Boundary value problems cannot be treated directly by pseudodifferential methods. Already in the sixites, however, first essential steps were taken to provide a similar framework allowing the construction of parametrices to elliptic elements, cf. e.g. Vishik and Eskin [**19**].

It was Boutet de Monvel [**1**] who also brought in the operator-algebraic aspect with his calculus established in 1971. As he points out, he constructs a relatively small 'algebra', containing the elliptic differential boundary value problems as well as their parametrices. He considers matrices of operators

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(X, E_1) \\ \oplus \\ C^\infty(\partial X, F_1) \end{matrix} \to \begin{matrix} C^\infty(X, E_2) \\ \oplus \\ C^\infty(\partial X, F_2) \end{matrix} . \tag{0.1}$$

Here X is a manifold with boundary, E_1, E_2 are vector bundles over X, and F_1, F_2 are vector bundles over ∂X; each of them might be zero. P is a pseudodifferential operator on the double of X; the subscript $+$ indicates that the action of P_+ is defined by extending the function by zero to the full manifold, applying P, and then restricting the result to X. S is a usual pseudodifferential operator on the boundary. K and T are generalizations of the potential and trace operators known from the theory of boundary value problems. The entry G, a so-called singular Green operator, is an operator which is smoothing in the interior while it acts like a pseudodifferential operator in directions tangential to the boundary. As an example we may take the difference of two solution operators to (invertible) classical boundary value problems with the same differential part in the interior but different boundary conditions.

Given an arbitrary pseudodifferential operator P, it is in general not true that P_+ maps functions which are smooth up to the boundary to functions with the same property. The mapping property above will therefore not hold if we admit *all* pseudodifferential operators. The crucial requirement here is that the symbol of P has the transmission property, which will be discussed in detail, below. On one hand, this restricts the class of boundary value problems in the calculus, on the other hand, however, it ensures that solutions to elliptic equations with smooth data are smooth; it therefore helps to avoid problems with singularities of solutions at the boundary.

It also is a central point that these operator matrices form an algebra in the following sense: Given another element of the calculus, say,

$$A' = \begin{pmatrix} P'_+ + G' & K' \\ T' & S' \end{pmatrix} : \begin{matrix} C^\infty(X, E_2) \\ \oplus \\ C^\infty(\partial X, F_2) \end{matrix} \to \begin{matrix} C^\infty(X, E_3) \\ \oplus \\ C^\infty(\partial X, F_3) \end{matrix} ,$$

1991 *Mathematics Subject Classification.* 35S15, 58J40.

the composition $A'A$ is again an operator matrix of the type described above. This is far from being obvious. Consider, for example, one of the terms arising in the matrix composition, namely the product P'_+P_+. Except for special cases, it will not coincide with $(P'P)_+$; in fact, the difference $L(P',P) = P'_+P_+ - (P'P)_+$ turns out to be a singular Green operator.

The presentation in Boutet de Monvel's original paper is rather concise. More detailed accounts were given in the books by Rempel and Schulze [**7**] and Grubb [**3**]. The present introduction focuses on a special aspect: The operators in Boutet de Monvel's calculus may be regarded as operator-valued pseudodifferential operators as they were introduced by Schulze, cf. e.g. [**17**]. This point of view, going back to an idea of Schulze, was first sketched in the joint paper [**11**]. It shows the pseudodifferential spirit of Boutet de Monvel's construction more clearly than the older descriptions and makes it easier to understand.

Moreover, this concept has been applied successfully to the analysis of boundary value problems on singular manifolds, because, in this operator-valued set-up, Boutet de Monvel's calculus can be combined very well with pseudodifferential calculi for cone and edge singularities, cf. [**11**], [**12**], [**13**], [**14**], [**15**].

I am giving here an essentially complete and self-contained introduction to the calculus on $\mathbb{R}^n_+$ in terms of operator-valued symbol classes satisfying uniform estimates. I have not included a section on coordinate invariance and the construction of the manifold. For one thing, this allowed me to keep the exposition short, moreover, these constructions are rather standard, and there is no new aspect to be developed. The material for this article is taken from [**8**], where more details can be found.

1. Symbol Spaces

1.1 General notation. In the sequel, $H^s(\mathbb{R}^q), s \in \mathbb{R}$, will denote the usual Sobolev space on $\mathbb{R}^q$. For $s = (s_1, s_2) \in \mathbb{R}^2$ we will write

$$H^s(\mathbb{R}^q) = \{\langle x\rangle^{-s_2}u : u \in H^{s_1}(\mathbb{R}^q)\};$$

here $\langle x\rangle = (1+|x|^2)^{1/2}$.

$\mathcal{S}(\mathbb{R}^q)$ is the Schwartz space of all rapidly decreasing functions on $\mathbb{R}^q$. For a Fréchet space E, $\mathcal{S}(\mathbb{R}^q, E)$ is the vector-valued analog. The dual spaces are $\mathcal{S}'(\mathbb{R}^q)$ and $\mathcal{S}'(\mathbb{R}^q, E) = \mathcal{L}(\mathcal{S}(\mathbb{R}^q), E)$, respectively. The Fourier transform

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^q, E) \to \mathcal{S}(\mathbb{R}^q, E)$$

will in general be indicated by a hat: For $u \in \mathcal{S}(\mathbb{R}^q, E)$

$$\hat{u}(\xi) = (\mathcal{F}u)(\xi) = (2\pi)^{-n/2}\int e^{-ix\xi}u(x)\,dx, \qquad \xi \in \mathbb{R}^q.$$

Given a distribution u on $\mathbb{R}^q$, we write r^+u for its restriction to $\mathbb{R}^q_+$. We define

$$\begin{aligned} H^s(\mathbb{R}^q_+) &= \{r^+u : u \in H^s(\mathbb{R}^q)\}, \qquad s \in \mathbb{R}^2;\\ \mathcal{S}(\mathbb{R}^q_+) &= \{r^+u : u \in \mathcal{S}(\mathbb{R}^q)\}. \end{aligned}$$

$H^s_0(\overline{\mathbb{R}}^q_+)$ denotes the space of all u in $H^s(\mathbb{R}^q)$ which are supported in $\overline{\mathbb{R}}^q_+$. We note that

$$\mathcal{S}(\mathbb{R}^q_+) = \text{proj}-\lim{}_{s_1,s_2\to\infty}H^s(\mathbb{R}^q_+), \quad \mathcal{S}'(\mathbb{R}^q_+) = \text{ind}-\lim{}_{s_1,s_2\to-\infty}H^s_0(\overline{\mathbb{R}}^q_+).$$

1.2 Group actions. A strongly continuous group action on a Banach space E is a family $\kappa = \{\kappa_\lambda : \lambda \in \mathbb{R}_+\}$ of isomorphisms in $\mathcal{L}(E)$ such that $\kappa_\lambda\kappa_\mu = \kappa_{\lambda\mu}$ and the mapping $\lambda \mapsto \kappa_\lambda e$ is continuous for every $e \in E$. For all the above Sobolev spaces on $\mathbb{R}^q$ and $\mathbb{R}^q_+$ we shall use the group action defined on functions by

$$(\kappa_\lambda u)(x) = \lambda^{q/2}u(\lambda x). \tag{1.1}$$

It extends to distributions by $(\kappa_\lambda u)(\varphi) = u(\kappa_{\lambda^{-1}}\varphi), \varphi \in C^\infty_0$. On $E = \mathbb{C}^l, l \in \mathbb{N}$, we use the trivial group action $\kappa_\lambda \equiv \text{id}$. Sums of spaces of the above kind will be endowed with the sum of the group actions.

Lemma 1.3. *There are constants c and M such that*

$$\|\kappa_\lambda\|_{\mathcal{L}(E)} \le c\max\{\lambda, \lambda^{-1}\}^M.$$

Proof. This follows easily from the well-known statement for additive semigroups.

1.4 Operator-valued symbols. Let E, F be Banach spaces with strongly continuous group actions κ and $\tilde\kappa$, respectively. Let $a \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E,F))$ and $\mu \in \mathbb{R}$. We shall write $a \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, F)$ provided that, for all multi-indices α, β, γ, there is a constant $C = C(\alpha, \beta, \gamma)$ with

$$\|\tilde\kappa_{\langle\eta\rangle^{-1}} D_\eta^\alpha D_y^\beta D_{\tilde y}^\gamma a(y, \tilde y, \eta) \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,F)} \le C\,\langle\eta\rangle^{\mu-|\alpha|}.$$

If a is independent of y or $\tilde y$ we shall write $a \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q; E, F)$.

For $E = F = \mathbb{C}$ we recover the definition of the symbol class $S^\mu_{1,0}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q)$.

1.5 Example: Trace operators. Let $\gamma_j : \mathcal{S}(\mathbb{R}_+) \to \mathbb{C}$ be defined by

$$\gamma_j f = \lim_{t\to 0^+} \partial^j_{x_n} f(t) = f^{(j)}(0).$$

The trace theorem for Sobolev spaces ensures that γ_j extends to an element of $\mathcal{L}(H^\sigma(\mathbb{R}_+), \mathbb{C})$ provided that $\sigma_1 > j + \frac12$. We can also view γ_j as an operator-valued symbol independent of the variables y and η; we then have

$$\gamma_j \in S^{j+\frac12}(\mathbb{R}^q \times \mathbb{R}^q; H^\sigma(\mathbb{R}_+), \mathbb{C}) :$$

Recalling that the group action on $H^\sigma(\mathbb{R}_+)$ is given by (1.1) while on $\mathbb{C}$ we choose the identity, we only have to check that

$$\|\gamma_j \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H^\sigma(\mathbb{R}_+),\mathbb{C})} = O(\langle\eta\rangle^{j+\frac12}).$$

This is immediate, since

$$\partial_t^j[\langle\eta\rangle^{\frac12} f(\langle\eta\rangle t)] = \langle\eta\rangle^{j+\frac12}(\partial_t^j f)(\langle\eta\rangle t).$$

1.6 Example: Action in the normal direction. Let $a \in S^\mu_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $\mu \in \mathbb{R}$. For fixed (x', ξ'), the function $a(x', \cdot, \xi', \cdot)$ is an element of $S^\mu_{1,0}(\mathbb{R} \times \mathbb{R})$. For every $s \in \mathbb{R}^2$ it therefore induces a bounded linear operator

$$\mathrm{op}_{x_n} a = \mathrm{op}_{x_n} a(x', x_n, \xi', \xi_n) : H^\sigma(\mathbb{R}) \to H^{\sigma-(\mu,0)}(\mathbb{R}).$$

The subscript x_n indicates that the action is with respect to the variable x_n and the covariable ξ_n only. We then have

$$\kappa_{\langle\xi'\rangle^{-1}}[\mathrm{op}_{x_n} a]\kappa_{\langle\xi'\rangle} = \mathrm{op}_{x_n} a\,(x', x_n/\langle\xi'\rangle, \xi', \langle\xi'\rangle\,\xi_n) : \tag{1.2}$$

For $u \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}&\kappa_{\langle\xi'\rangle^{-1}}[\mathrm{op}_{x_n} a](\kappa_{\langle\xi'\rangle} u)(x_n)\\ &= \int e^{ix_n\xi_n/\langle\xi'\rangle}\,\langle\xi'\rangle^{-1}\,a(x', x_n/\langle\xi'\rangle, \xi', \xi_n)\hat u(\xi_n/\langle\xi'\rangle) đ\xi_n;\end{aligned}$$

the substitution $\eta_n = \xi_n/\langle\xi'\rangle$ then yields the assertion. The theorem, below, shows that $\mathrm{op}_{x_n} a$ is an operator-valued symbol in the sense of 1.4:

Theorem 1.7. *For $a \in S^\mu_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and $\sigma \in \mathbb{R}^2$ we have*

$$\mathrm{op}_{x_n} a \in S^\mu(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H^\sigma(\mathbb{R}), H^{\sigma-(\mu,0)}(\mathbb{R})).$$

Proof. Given multi-indices α, β, we know from 1.6 that we have to estimate

$$\begin{aligned}&\sup_{x',\xi'} \|\,\langle\xi'\rangle^{|\alpha|}\,\kappa_{\langle\xi'\rangle^{-1}}[\mathrm{op}_{x_n}(D^\alpha_{\xi'} D^\beta_{x'} a)]\kappa_{\langle\xi'\rangle}\|_{\mathcal{L}(H^\sigma(\mathbb{R}),H^{\sigma-(\mu,0)}(\mathbb{R}))}\\ &= \sup_{x',\xi'} \|\,\langle\xi'\rangle^{|\alpha|}\,\mathrm{op}_{x_n}(D^\alpha_{\xi'} D^\beta_{x'} a)\,(x', x_n/\langle\xi'\rangle, \xi', \xi_n\langle\xi'\rangle)\,\|_{\mathcal{L}(H^\sigma(\mathbb{R}),H^{\sigma-(\mu,0)}(\mathbb{R}))}.\end{aligned}$$

Since $D^\alpha_{\xi'} D^\beta_{x'} a$ is of order $\mu - |\alpha|$ we may assume that $\alpha = \beta = 0$. Now

$$\mathrm{op}_{x_n} a(x', x_n/\langle\xi'\rangle, \xi', \xi_n\langle\xi'\rangle) : H^\sigma(\mathbb{R}) \to H^{\sigma-(\mu,0)}(\mathbb{R})$$

is continuous, and a bound for its norm is given by the suprema

$$\sup\left\{|D^\alpha_{\xi_n} D^\beta_{x_n}\{a(x', x_n/\langle\xi'\rangle, \xi', \xi_n\langle\xi'\rangle)\}|\,\langle\xi_n\rangle^{-\mu} : x_n, \xi_n \in \mathbb{R}\right\}$$

for a finite number of derivatives. Since each of them is $O(\langle\xi'\rangle^\mu)$ the proof is complete. ◁

As a consequence we easily obtain order-reducing operators for the full space situation.

COROLLARY 1.8. *For real* μ *the symbol* $r^\mu \in S^\mu_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ *given by* $r^\mu(\xi) = \langle\xi\rangle^\mu$ *induces the operator-valued symbol*

$$\mathrm{op}_{x_n} r^\mu \in S^\mu(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H^\sigma(\mathbb{R}), H^{\sigma-(\mu,0)}(\mathbb{R})), \quad \sigma \in \mathbb{R}^2.$$

REMARK 1.9. Definition 1.4 extends to projective and inductive limits. Let $\tilde E$ and $\tilde F$ be Banach spaces with group actions. If $F_1 \hookleftarrow F_2 \hookleftarrow \dots$ and $E_1 \hookrightarrow E_2 \hookrightarrow \dots$ are sequences of Banach spaces with the same group action and $F = \mathrm{proj}-\lim_k F_k, E = \mathrm{ind}-\lim_k E_k$, then we let

$$\begin{aligned} S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; \tilde E, F) &= \mathrm{proj}-\lim_k S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; \tilde E, F_k) \\ S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, \tilde F) &= \mathrm{proj}-\lim_k S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E_k, \tilde F) \\ S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, F) &= \mathrm{proj}-\lim_{k,l} S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E_k, F_l) \end{aligned}$$

In particular, it makes sense to speak of $S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; \mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+))$, $S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; \mathcal{S}'(\mathbb{R}_+), \mathbb{C})$, and $S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; \mathbb{C}, \mathcal{S}(\mathbb{R}_+))$.

We write $S^{-\infty}(\dots) = \bigcap_\mu S^\mu(\dots)$.

THEOREM 1.10. *Given* $a_j \in S^{\mu-j}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, F), j = 0, 1, 2, \dots,$ *there is a symbol* $a \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, F)$ *such that* $a \sim \sum a_j$. *As usual, the equivalence relation* $\sim$ *is defined by the fact that for every* J

$$a - \sum_{j \le J} a_j \in S^{\mu-J-1}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, F).$$

Moreover, a *is unique modulo* $S^{-\infty}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, F)$.

The *proof* follows the standard argument. ◁

DEFINITION 1.11. A symbol $a \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, F)$ is said to be *classical,* if it has an asymptotic expansion $a \sim \sum_{j=0}^\infty a_j$ with $a_j \in S^{\mu-j}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, F)$ satisfying the homogeneity relation

$$a_j(y, \tilde y, \lambda\eta) = \lambda^{\mu-j} \tilde\kappa_\lambda\, a_j(y, \tilde y, \eta)\, \kappa_{\lambda^{-1}} \tag{1.3}$$

for all $\lambda \ge 1, |\eta| \ge R$ with a suitable constant R. We write $a \in S^\mu_{cl}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, F)$. For $E = \mathbb{C}^k,\ F = \mathbb{C}^l$ we recover the standard notion.

The symbols γ_j in 1.5 for example are homogeneous of degree $j + 1/2$ in the sense of (1.3). The lemma, below, is straightforward to prove.

LEMMA 1.12. *Let* $a \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, F), b \in S^{\tilde\mu}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; F, G)$.

(a) $D^\alpha_\eta D^\beta_y D^\gamma_{\tilde y} a \in S^{\mu-|\alpha|}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, F)$ *for all multi-indices* α, β, γ.

(b) *The pointwise composition* $(ba)(y, \tilde y, \eta)$ *yields an element* $ba \in S^{\mu+\tilde\mu}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, G)$.

DEFINITION 1.13. For $a \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, F)$, the pseudodifferential operator

$$\mathrm{op}\, a : \mathcal{S}(\mathbb{R}^q, E) \to \mathcal{S}(\mathbb{R}^q, F)$$

is defined by

$$[\mathrm{op}\, a]u(y) = \iint e^{i(y-\tilde y)\eta} a(y, \tilde y, \eta) u(\tilde y)\, d\tilde y\, đ\eta; \qquad y \in \mathbb{R}^q.$$

Here $đ\eta = (2\pi)^{-q} d\eta$. If a is independent of $\tilde y$, this reduces to $[\mathrm{op}\, a]u(y) = (2\pi)^{-q/2} \int e^{iy\eta} a(y, \eta) \hat u(\eta)\, d\eta$; in this case we call a a *left* symbol for $\mathrm{op}\, a$. If a is independent of y, then

$$[\mathrm{op}\, a]u(y) = \iint e^{i(y-\tilde y)\eta} a(\tilde y, \eta) u(\tilde y)\, d\tilde y\, đ\eta,$$

and a is called a *right* symbol.

THEOREM 1.14. *Let* $a \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, F)$. *Then there is a (unique) left symbol* a_L *and a (unique) right symbol* a_R *for* $\mathrm{op}\, a$.

Given $a \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q; E, F)$ *and* $b \in S^{\tilde\mu}(\mathbb{R}^q \times \mathbb{R}^q; F, G)$ *there is a (left) symbol* $c \in S^{\mu+\tilde\mu}(\mathbb{R}^q \times \mathbb{R}^q; E, G)$ *such that*

$$\mathrm{op}\, b \circ \mathrm{op}\, a = \mathrm{op}\, c.$$

We have the asymptotic expansion $c \sim \sum_\alpha \frac{1}{\alpha!} D^\alpha_y b(y, \eta) \partial^\alpha_y a(y, \eta)$.

Proof. The proof of the first part is obtained by following the construction in Kumano-go, [**6**]. For the second part we choose a right symbol a_R for $\operatorname{op} a$. Then

$$\operatorname{op} b \circ \operatorname{op} a = \operatorname{op} b \circ \operatorname{op} a_R = \operatorname{op} \tilde{c}$$

with $\tilde{c}(y, \tilde{y}, \eta) = b(y, \eta) a_R(\tilde{y}, \eta)$. Switching to the left symbol c of $\operatorname{op} \tilde{c}$ we obtain the assertion. ◁

1.15 Duality. Let (E_-, E_0, E_+) be a triple of Hilbert spaces. We assume that all are embedded in a common vector space V and that $E_0 \cap E_+ \cap E_-$ is dense in $E_\pm$ as well as in E_0. Moreover we assume that there is a continuous, non-degenerate sesquilinear form $(\cdot, \cdot)_E : E_+ \times E_- \to \mathbb{C}$ which coincides with the inner product $(\cdot, \cdot)_E$ of E_0 on $(E_+ \cap E_0) \times (E_- \cap E_0)$. We ask that, via $(\cdot, \cdot)_E$, we may identify E_+ with the dual of E_- and vice versa, and that

$$\|e\|'_{E_-} = \sup_{\|f\|_{E_+} = 1} |(f, e)_E|, \quad \|f\|'_{E_+} = \sup_{\|e\|_{E_-} = 1} |(f, e)_E|$$

furnish equivalent norms on E_- and E_+, respectively. Suppose there is a group action κ on V which has strongly continuous restrictions to each of the spaces, unitary on E_0, i.e., $(\kappa_\lambda e, f)_E = (e, \kappa_{\lambda^{-1}} f)_E$ for $e, f \in E_0$. Then

$$(\kappa_\lambda e, f)_E = (e, \kappa_{\lambda^{-1}} f)_E, \quad e \in E_+, f \in E_-,$$

since the identity holds on the dense set $(E_+ \cap E_0) \times (E_- \cap E_0)$. In other words, the action κ on E_+ is dual to the action κ on E_- and vice versa.

Typical examples for the above situation are given by the triples

$$(H^{-\sigma}(\mathbb{R}), L^2(\mathbb{R}), H^\sigma(\mathbb{R})) \quad \text{and} \quad (H_0^{-\sigma}(\overline{\mathbb{R}}_+), L^2(\mathbb{R}_+), H^\sigma(\mathbb{R}_+)), \quad \sigma \in \mathbb{R}^2.$$

Let (F_-, F_0, F_+) be an analogous triple of Hilbert spaces with group action $\tilde{\kappa}$, and let $a \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E_-, F_-)$. We define a^* by $a^*(y, \tilde{y}, \eta) = a(\tilde{y}, y, \eta)^* \in \mathcal{L}(F_+, E_+)$, where the last asterisk denotes the adjoint operator with respect to the sesquilinear forms $(\cdot, \cdot)_E$ and $(\cdot, \cdot)_F$:

$$(a(\tilde{y}, y, \eta)^* f, e)_E = (f, a(\tilde{y}, y, \eta) e)_F, \quad e \in E_-, f \in F_+.$$

It is not difficult to check that $a^* \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; F_+, E_+)$.

Moreover, we may introduce a continuous non-degenerate sesquilinear form

$$(\cdot, \cdot)_{\mathcal{S}_E} : \mathcal{S}(\mathbb{R}^q, E_+) \times \mathcal{S}(\mathbb{R}^q, E_-) \to \mathbb{C}$$

by $(u, v)_{\mathcal{S}_E} = \int (u(y), v(y))_E dy$. Analogously we define $(\cdot, \cdot)_{\mathcal{S}_F}$.

The symbol a^* induces a continuous mapping $\operatorname{op} a^* : \mathcal{S}(\mathbb{R}^q, F_+) \to \mathcal{S}(\mathbb{R}^q, E_+)$. This is the unique operator satisfying

$$([\operatorname{op} a^*] u, v)_{\mathcal{S}_E} = (u, [\operatorname{op} a] v)_{\mathcal{S}_F}.$$

1.16 Wedge Sobolev spaces. Let E be a Banach space with a group action κ. The wedge Sobolev space $\mathcal{W}^s(\mathbb{R}^q, E)$, $s \in \mathbb{R}$, is the completion of $\mathcal{S}(\mathbb{R}^q, E)$ with respect to the norm

$$\|u\|^2_{\mathcal{W}^s(\mathbb{R}^q, E)} = \int \langle \eta \rangle^{2s} \|\kappa_{\langle \eta \rangle^{-1}} \hat{u}(\eta)\|_E^2 \, d\eta.$$

$\mathcal{W}^s(\mathbb{R}^q, E)$ is a subset of $\mathcal{S}'(\mathbb{R}^q, E)$ and a Hilbert space with the natural inner product. For $\sigma \in \mathbb{R}^2$, $s \in \mathbb{R}$, the dual space of $\mathcal{W}^s(\mathbb{R}^q, H^\sigma(\mathbb{R}_+))$ is $\mathcal{W}^{-s}(\mathbb{R}^q, H_0^{-\sigma}(\overline{\mathbb{R}}_+))$ and vice versa. The dual of $\mathcal{W}^s(\mathbb{R}^q, H^\sigma(\mathbb{R}))$ is $\mathcal{W}^{-s}(\mathbb{R}^q, H^{-\sigma}(\mathbb{R}))$.

For $s \in \mathbb{R}^2$, we can define

$$\mathcal{W}^s(\mathbb{R}^q, E) = \{\langle y \rangle^{-s_2} u : u \in \mathcal{W}^{s_1}(\mathbb{R}^q, E)\}.$$

We then obtain in particular, cf. [**9**, Lemma 1.8, Corollary 1.10]:

$$\operatorname{proj} - \lim_{s_1, s_2, \sigma_1, \sigma_2 \to \infty} \mathcal{W}^s(\mathbb{R}^q, H^\sigma(\mathbb{R}_+)) = \mathcal{S}(\mathbb{R}_+^{q+1});$$

$$\operatorname{ind} - \lim_{s_1, s_2, \sigma_1, \sigma_2 \to -\infty} \mathcal{W}^s(\mathbb{R}^q, H_0^\sigma(\overline{\mathbb{R}}_+)) = \mathcal{S}'(\mathbb{R}_+^{q+1}).$$

Lemma 1.17. *For $s \in \mathbb{R}$ we have*

(a) $\mathcal{W}^s(\mathbb{R}^q, \mathbb{C}) = H^s(\mathbb{R}^q)$.
(b) $\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^k)) = H^s(\mathbb{R}^{q+k})$.
(c) $\mathcal{W}^s(\mathbb{R}^q, H_0^s(\overline{\mathbb{R}}_+)) = H_0^s(\overline{\mathbb{R}}_+^{q+1})$.

(d) $\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}_+)) = H^s(\mathbb{R}^{q+1}_+)$.

Proof. (a) is obvious. For (b) note that the behavior of the Fourier transform under dilations yields

$$\begin{aligned}\|u\|^2_{\mathcal{W}^s(\mathbb{R}^q,H^s(\mathbb{R}^k))} &= \int \langle\eta\rangle^{2s}\|\langle\eta\rangle^{k/2}\langle\tau\rangle^s(\mathcal{F}_{t\to\tau}\mathcal{F}_{y\to\eta}u)(\eta,\langle\eta\rangle\tau)\|^2_{L^2(\mathbb{R}^k_\tau)}\,d\eta\\ &= \int\langle\eta\rangle^{2s+k}\int\langle\tau\rangle^{2s}|(\mathcal{F}_{t\to\tau}\mathcal{F}_{y\to\eta}u)(\eta,\langle\eta\rangle\tau)|^2\,d\tau\,d\eta\\ &= \iint\langle\eta\rangle^{2s}\langle\langle\eta\rangle^{-1}\tau\rangle^{2s}|(\mathcal{F}_{t\to\tau}\mathcal{F}_{y\to\eta}u)(\eta,\tau)|^2\,d\tau\,d\eta.\end{aligned}$$

The statement follows, since $\langle\eta\rangle^{2s}\langle\langle\eta\rangle^{-1}\tau\rangle^{2s} = \langle\eta,\tau\rangle^{2s}$.

(c) follows from (b): $H^s_0(\overline{\mathbb{R}}^{q+1}_+)$ is the closure of $C^\infty_0(\mathbb{R}^{q+1}_+)$ in the topology of $H^s(\mathbb{R}^{q+1})$. Since $C^\infty_0(\mathbb{R}^{q+1}_+)$ is dense in $\mathcal{S}(\mathbb{R}^n, H^s_0(\overline{\mathbb{R}}_+))$ and the norms of $\mathcal{W}^s(\mathbb{R}^q, H^s_0(\overline{\mathbb{R}}_+))$, $\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}))$, and $H^s(\mathbb{R}^{q+1})$ coincide on $C^\infty_0(\mathbb{R}^{n+1}_+)$, we obtain the assertion.

(d) results from duality. ◁

The following theorem was proven by J. Seiler [**18**, Theorem 3.14].

Theorem 1.18. *Let E, F be Hilbert spaces and $a \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q; E, F)$. Then*

$$\operatorname{op} a : \mathcal{W}^s(\mathbb{R}^q, E) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, F)$$

is bounded for every $s \in \mathbb{R}$.

Corollary 1.19. *Under the assumptions of Theorem* 1.18*,*

$$\operatorname{op} a : \mathcal{W}^s(\mathbb{R}^q, E) \to \mathcal{W}^{s-(\mu,0)}(\mathbb{R}^q, F)$$

is even bounded for every $s \in \mathbb{R}^2$.

Proof. By interpolation it is sufficient to treat the case $s_2 \in 2\mathbb{Z}$. The boundedness of $\operatorname{op} a : \mathcal{W}^s(\mathbb{R}^q, E) \to \mathcal{W}^{s-(\mu,0)}(\mathbb{R}^q, F)$ is equivalent to the boundedness of

$$\langle y\rangle^{s_2} \operatorname{op} a\, \langle y\rangle^{-s_2} : \mathcal{W}^{s_1}(\mathbb{R}^q, E) \to \mathcal{W}^{s_1-\mu}(\mathbb{R}^q, F).$$

Either $\langle y\rangle^{-s_2}$ or $\langle y\rangle^{-s_2}$ is a polynomial; without loss of generality let $s_2 \geq 0$. Using the identity $[\operatorname{op} a, y_j] = \operatorname{op} D_{y_j} a$ we may shift the powers of y from the left to the right hand side. According to 1.18, both $\operatorname{op} D^\alpha_y a$ and $y^\beta\,\langle y\rangle^{-s_2}$, $|\beta| \leq s_2$, are bounded operators on the respective spaces, and the assertion follows. ◁

2. The Transmission Property

It has been pointed out in the introduction that one will have to impose a condition on the pseudodifferential entry P in (0.1) in order to ensure the stated mapping property. Following Boutet de Monvel [**1**], one asks that the symbol of P has the transmission property.

Hörmander introduces a similar notion: the *transmission condition*. An operator on the half-space $\mathbb{R}^n_+$ has it, if it maps functions which are smooth up to the boundary to functions with the same property [**5**, Definition 18.2.13]. The difference between both notions is that the transmission *property* assures the transmission *condition* also for the opposite half-space $\mathbb{R}^n_-$. This is needed in order to obtain a calculus which is closed under taking adjoints.

For classical pseudodifferential symbols of integer order the transmission condition can be expressed via homogeneity properties of the terms in the asymptotic expansion, cf. [**5**, Theorem 18.2.18]; in fact, this condition also ensures the transmission property, cf. (2.2), below. For non-classical symbols, this is no longer true, for details see the analysis by Grubb and Hörmander [**4**]. In [**10**] it was shown that the transmission property can be characterized via commutator estimates on wedge Sobolev spaces.

Definition 2.1. Given a function f on $\mathbb{R}^n_+$ we denote by e^+f its extension (by zero) to a function on $\mathbb{R}^n$.

Extension by zero also makes sense for distributions u in $H^s(\mathbb{R}^n_+)$, $s_1 > -\frac{1}{2}$. For $-\frac{1}{2} < s_1 < \frac{1}{2}$, e^+u then is an element of $H^s(\mathbb{R}^n)$; for $s_1 \geq 1/2$, we clearly have $e^+u \in H^\sigma(\mathbb{R}^n_+)$ for all σ with $\sigma_1 < 1/2$ and $\sigma_2 = s_2$.

We let

$$\begin{aligned} H^+ &= \{(e^+u)^\wedge : u \in \mathcal{S}(\mathbb{R}_+)\} \\ H_0^- &= \{(e^-u)^\wedge : u \in \mathcal{S}(\mathbb{R}_-)\}. \end{aligned}$$

H^+ and H_0^- are spaces of smooth functions on $\mathbb{R}$, decaying to first order near infinity. By H' we denote the space of all polynomials. We let

$$H = H^+ \oplus H_0^- \oplus H', \qquad H^- = H_0^- \oplus H', \quad H_0 = H^+ \oplus H_0^-.$$

Parts (b), (c) and (d) of the following proposition provide Paley-Wiener type characterizations of these spaces.

PROPOSITION 2.2. (a) *H^+, H_0^-, H^-, H_0, and H are algebras.*

(b) *A function $h \in C^\infty(\mathbb{R})$ belongs to H^+ if and only if it has an analytic extension to the lower half-plane $\{\operatorname{Im}\zeta < 0\}$, continuous in $\{\operatorname{Im} \le 0\}$, together with an asymptotic expansion*

$$h(\zeta) \sim \sum_{k=1}^{\infty} a_k \zeta^{-k}, \tag{2.1}$$

$|\zeta| \to \infty$ in $\{\operatorname{Im}\zeta \le 0\}$, *which can be differentiated formally.*

(c) *An analogous statement holds if we replace H^+ by H_0^- and $\{\operatorname{Im}\zeta < 0\}$ by $\{\operatorname{Im}\zeta > 0\}$.*

(d) *$h \in C^\infty(\mathbb{R})$ belongs to H_0 if and only if it has an expansion*

$$h \sim \sum_{k=1}^{\infty} a_k \zeta^{-k}$$

for $|\zeta| \to \infty$ in $\mathbb{R}$, which can be differentiated formally.

DEFINITION 2.3. A pseudodifferential symbol $p = p(x, y, \xi) \in S^\mu_{1,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ has the transmission property (at $\{x_n = y_n = 0\}$) provided that, for all k, l

$$\partial^l_{y_n} \partial^k_{x_n} p(x', 0, y', 0, \xi', \langle\xi'\rangle\xi_n) \in S^\mu_{1,0}(\mathbb{R}^{n-1}_{x'} \times \mathbb{R}^{n-1}_{y'} \times \mathbb{R}^{n-1}_{\xi'}) \hat{\otimes}_\pi H_{\xi_n}.$$

The subscripts x', y', ξ', and ξ_n are used only for the moment in order to indicate the variable for which we have the corresponding property.

We write $p \in S^\mu_{tr}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$. If $p = p(x, \xi)$ is a classical symbol of order $\mu \in \mathbb{Z}$ with an asymptotic expansion

$$p \sim \sum_{j=0}^{\infty} p_{\mu-j},$$

where $p_{\mu-j} = p_{\mu-j}(x, \xi) \in S^{\mu-j}$ is positively homogeneous of degree $\mu - j$ in ξ for $|\xi| \ge 1$, i. e.,

$$p_l(x, t\xi) = t^l p_l(x, \xi), \qquad t \ge 1, |\xi| \ge 1.$$

Let us sketch the argument, why p has the transmission property if and only if for all α, k, l

$$\partial^k_{x_n} \partial^\alpha_{\xi'} p_l(x', 0, 0, +1) = (-1)^{l-|\alpha|} p_l(x', 0, 0, -1): \tag{2.2}$$

A Taylor expansion gives

$$\begin{aligned} &\partial^k_{x_n} p_l(x', 0, \xi', \langle\xi'\rangle\nu) = \langle\xi'\rangle^l \partial^k_{x_n} p_l(x', 0, \xi'/\langle\xi'\rangle, \nu) \\ &= \langle\xi'\rangle^l \Big[\sum_{|\alpha| \le N} \frac{1}{\alpha!} \partial^k_{x_n} \partial^\alpha_{\xi'} p_l(x', 0, 0, \nu)(\xi'/\langle\xi'\rangle)^\alpha + r_N(x', \xi', \nu) \Big]. \end{aligned}$$

Here, $r_N(x', \xi', \nu)$ can be estimated in terms of

$$\sup\{|\partial^\beta_{\xi'} p_l(x', 0, \sigma\xi'/\langle\xi'\rangle, \nu)| : |\beta| = N+1, 0 \le \sigma \le 1\},$$

which is $O(\langle\nu\rangle^{l-N-1})$, uniformly in x', ξ'. Moreover,

$$\partial^k_{x_n} \partial^\alpha_{\xi'} p_l(x', 0, 0, \nu) = \partial^k_{x_n} \partial^\alpha_{\xi'} p_l(x', 0, 0, \pm 1)|\nu|^{l-\alpha},$$

so that, eventually, Proposition 2.2(d) gives the assertion.

The next lemma is obvious:

LEMMA 2.4. *Regularizing symbols always have the transmission property, and so do symbols which vanish to infinite order at $\{x_n = 0\}$. Moreover, all symbols which are polynomial in ξ have it according to* (2.2).

EXAMPLE 2.5. The symbol $\langle\xi\rangle$ does *not* have the transmission property. Indeed, since $\langle(\xi', \langle\xi'\rangle\xi_n)\rangle = \langle\xi'\rangle\langle\xi_n\rangle$ this would require that $\nu \mapsto \langle\nu\rangle \in H$. Writing

$$\langle\nu\rangle = |\nu|\left(1+\frac{1}{\nu^2}\right)^{\frac{1}{2}} = |\nu|\sum_{j=0}^{\infty}\binom{1/2}{j}\nu^{-2j} = \sum_{j=0}^{\infty}\binom{1/2}{j}(\operatorname{sgn}\nu)\nu^{-2j+1}$$

we see that (2.2) is violated.

There are symbols with the transmision property of arbitrary order:

EXAMPLE 2.6. Let $q \in S^{\mu}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1})$ and $\varphi \in \mathcal{S}(\mathbb{R})$. Then

$$p(x,\xi) := q(x,\xi')\varphi(\xi_n/\langle\xi'\rangle) \in S^{\mu}_{tr}(\mathbb{R}^n \times \mathbb{R}^n).$$

Indeed $p(x',0,\xi',\langle\xi'\rangle\xi_n) = q(x',0,\xi')\varphi(\xi_n) \in S^{\mu}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \otimes H$. It is straightforward to check the symbol estimates for p.

The following results prove the stability of the transmission property under the usual pseudodifferential constructions.

PROPOSITION 2.7. (a) *If p and q have the transmission property then also all derivatives $D^{\alpha}_{\xi}D^{\beta}_{x}D^{\gamma}_{y}p$, the product pq, and the left and right symbols p_L and p_R, respectively.*

(b) *If p_j are symbols of order $\mu - j$ with the transmission property and $p \sim \sum p_j$, then p has the transmission property.*

(c) *If p is elliptic with the transmission property, then every parametrix has the transmission property.*

The proofs are straightforward.

The following lemma together with Theorem 2.10, below, shows that there are also order-reducing symbols for the half-space situation. The construction goes back to Grubb.

LEMMA 2.8. *Choose $\chi \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp}\mathcal{F}^{-1}\chi \subseteq \mathbb{R}_-$ and $\chi(0) = 1$. For $\mu \in \mathbb{Z}$ and $a \in \mathbb{R}$ with $a \gg \|\chi'\|_{\sup}$ (the sup-norm of the first derivative of χ), let*

$$r^{\mu}_{-}(\xi) = \left(\chi\left(\frac{\xi_n}{a\langle\xi'\rangle}\right)\langle\xi'\rangle - i\xi_n\right)^{\mu}, \qquad \xi \in \mathbb{R}^n. \tag{2.3}$$

Then r^{μ}_{-} is an elliptic element in $S^{\mu}_{tr}(\mathbb{R}^n \times \mathbb{R}^n)$. The same is true for $r^{\mu}_{+} = \overline{r}^{\mu}_{-}$.

Proof. The above definition makes sense, since

$$\frac{\chi\left(\frac{\xi_n}{a\langle\xi'\rangle}\right)\langle\xi'\rangle - i\xi_n}{\langle\xi'\rangle - i\xi_n} = 1 + \langle\xi'\rangle\frac{\chi\left(\frac{\xi_n}{a\langle\xi'\rangle}\right) - \chi(0)}{\langle\xi'\rangle - i\xi_n} = 1 + r,$$

where $|r| \le \|\chi'\|_{\sup}|\xi_n|\langle\xi\rangle^{-1}a^{-1}$ is small. An application of Example 2.6 shows that r^{μ}_{-} is an elliptic symbol of order μ. Moreover,

$$r^{\mu}_{-}(\xi', \langle\xi'\rangle\xi_n) = \langle\xi'\rangle^{\mu}(\chi(\xi_n/a) - i\xi_n)^{\mu}.$$

Using Proposition 2.3 it is not difficult to check that $\chi(\xi_n/a) - i\xi_n$ as well as its integer powers belong to H^-. Hence r^{μ}_{-} has the transmission property.

DEFINITION 2.9. Given $s \in \mathbb{R}^2, s_1 > -\frac{1}{2}$, and a symbol p we define the operator

$$\operatorname{op}^+p : H^s(\mathbb{R}^n_+) \to \mathcal{D}'(\mathbb{R}^n_+) \tag{2.4}$$

$$[\operatorname{op}^+p]u = r^+(\operatorname{op}p(e^+u)).$$

For $P = \operatorname{op}p$ we write $P_+ = \operatorname{op}^+p$.

It is obvious that we may replace $\mathcal{D}'(\mathbb{R}^n_+)$ on the right hand side of (2.4) by $H^{\sigma}(\mathbb{R}^n_+)$, where $\sigma_1 = \min\{s_1, 0\} - \mu$, $\sigma_2 = s_2$, and μ is the order of p.

THEOREM 2.10. *Let $\sigma \in \mathbb{R}^2$. For the symbols $r^\mu_\pm$ we have:*

(a) $\operatorname{op}_{x_n} r^\mu_\pm \in S^\mu(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; H^\sigma(\mathbb{R}), H^{\sigma-(\mu,0)}(\mathbb{R}))$.

(b) $\operatorname{op}^+_{x_n} r^\mu_- \in S^\mu(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; H^\sigma(\mathbb{R}_+), H^{\sigma-(\mu,0)}(\mathbb{R}_+))$, $\sigma_1 > -1/2$.

(c) *Let $e_\sigma : H^\sigma(\mathbb{R}_+) \to H^\sigma(\mathbb{R})$ be an arbitrary extension operator. Then $r^+[\operatorname{op} r^\mu_-]e_\sigma \in S^\mu(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; H^\sigma(\mathbb{R}_+), H^{\sigma-(\mu,0)}(\mathbb{R}_+))$. The operator is independent of the particular choice of e_σ.*

(d) $\operatorname{op}_{x_n} r^\mu_+ \in S^\mu(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; H^\sigma_0(\overline{\mathbb{R}}_+), H^{\sigma-(\mu,0)}_0(\overline{\mathbb{R}}_+))$. *Note that we use neither restriction nor extension, for $H^\sigma_0(\overline{\mathbb{R}}_+) \hookrightarrow H^\sigma(\mathbb{R})$.*

(e) *For $\mu' \in \mathbb{Z}$ we have*

$$[\operatorname{op}_{x_n} r^\mu_+][\operatorname{op}_{x_n} r^{\mu'}_+] = \operatorname{op}_{x_n} r^{\mu+\mu'}_+$$

on each $H^\sigma_0(\overline{\mathbb{R}}_+)$. In particular, as operators $H^\sigma_0(\overline{\mathbb{R}}_+) \to H^{\sigma-(\mu,0)}_0(\overline{\mathbb{R}}_+)$,

$$[\operatorname{op}_{x_n} r^\mu_+]^{-1} = \operatorname{op}_{x_n} r^{-\mu}_+.$$

(f) *For $\mu' \in \mathbb{Z}$ we have*

$$[\operatorname{op}^+_{x_n} r^\mu_-][\operatorname{op}^+_{x_n} r^{\mu'}_-] = \operatorname{op}^+_{x_n} r^{\mu+\mu'}_-$$

on each $H^\sigma(\mathbb{R}_+)$. Here we tacitly assume that the zero extension e^+ is replaced by an arbitrary extension operator if $\sigma_1 \le -1/2$ or $\sigma_1 - \mu' \le -1/2$. In particular,

$$[\operatorname{op}^+_{x_n} r^\mu_-]^{-1} = \operatorname{op}^+_{x_n} r^{-\mu}_-.$$

Proof. (a) follows from Theorem 1.7. For (b) and (c) one makes the following observation: If $e : H^\sigma(\mathbb{R}_+) \to H^\sigma(\mathbb{R})$ is an arbitrary extension operator, then $r^+\operatorname{op} r^\mu_-(eu)$ is independent of the choice of e. Since we can always find a continuous operator $e^s : H^s(\mathbb{R}_+) \to H^s(\mathbb{R})$ satisfying $\tilde\kappa_{\lambda^{-1}} e^s \kappa_\lambda = e^s$ for the standard group actions κ and $\tilde\kappa$ on $H^s(\mathbb{R}_+)$ and $H^s(\mathbb{R})$, respectively, (a) gives the assertion.

(d) For $u \in H^\sigma_0(\overline{\mathbb{R}}_+), [\operatorname{op}_{x_n} r^\mu_+](u)$ has support in $\overline{\mathbb{R}}_+$. Hence the assertion follows from (a). (e) is trivial. (f) is first shown for large σ_1; it then extends to the general case. ◁

The following lemma illustrates the effect of the transmission property:

LEMMA 2.11. *Let $p \in S^\mu_{tr}(\mathbb{R}^n\times\mathbb{R}^n)$. For $l \in \mathbb{N}$, the definition*

$$k_l(x',\xi') = r^+[\operatorname{op}_{x_n} a](\delta^{(l)}_0) \tag{2.5}$$

yields an element $k_l \in S^{\mu+l+1/2}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; \mathbb{C}, \mathcal{S}(\mathbb{R}_+))$. Here $\delta^{(l)}_0$ is the l-th derivative of Dirac's distribution at 0, and we consider k_l as the operator that associates to a complex number c the function $c[\operatorname{op}_{x_n} a]\delta^{(l)}_0$ on $\mathbb{R}_+$.

Note that $k_l = 0$ if p is a polynomial.

Proof. Take a right symbol $p_R = p_R(x', y_n, \xi)$ for $\operatorname{op}_{x_n} p$. Fix a function $\omega \in C^\infty_0(\mathbb{R})$ with $\omega(t) \equiv 1$ near $t = 0$ and write

$$\begin{aligned} & p_R(x', y_n, \xi) \\ = \ & \sum_{j=0}^{l} \frac{y_n^j}{j!}\omega(y_n)\partial^j_{y_n} p_R(x', 0, \xi) + y_n^{l+1} p_{Rl}(x', y_n, \xi) + (1-\omega(y_n))p_R(x', y_n, \xi) \end{aligned}$$

with suitable p_{Rl}. The operators associated with the second and the third summand vanish, so we can focus on the first one. By Proposition 2.7, p_R has the transmission property. Hence $\partial^j_{y_n} p_R(x', 0, \xi', \langle\xi'\rangle\xi_n) \in S^\mu_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\hat\otimes_\pi H$, and we can write

$$\partial^j_{y_n} p_R(x', 0, \xi', \xi_n) = \sum_{k=0}^{\mu} s_{jk}(x',\xi')\xi_n^k + \sum_{k=0}^{\infty} \lambda_{jk} b_{jk}(x',\xi') h_{jk}(\xi_n/\langle\xi'\rangle)$$

with $s_{jk} \in S^{\mu-k}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}), \{\lambda_{jk}\}_k \in l^1$, and null sequences $b_{jk} \in S^\mu_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}), h_{jk} \in H_0$. The polynomial part gives no contribution to (2.5). Hence it is sufficient to consider a single term $b(x',\xi')h(\xi_n/\langle\xi'\rangle)$ under the summation, to show that its contribution to (2.5) is an element of

$S^\mu(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathbb{C},\mathcal{S}(\mathbb{R}_+))$, and to check that the semi-norms for this element depend continuously on those for b and h. Since b is of order μ and since $y_n^j\delta_0^{(l)}=\binom{l}{j}\delta_0^{(l-j)}$, it suffices that, for all $\sigma\in\mathbb{R}^2$,

$$r^+\kappa_{\langle\xi'\rangle^{-1}}[\mathrm{op}_{x_n}D^\alpha_{\xi'}h(\xi_n/\langle\xi'\rangle)]\delta_0^{(l)}:\mathbb{C}\to H^\sigma(\mathbb{R}_+)$$

has norm $O(\langle\xi'\rangle^{-|\alpha|+l+1/2})$. Now $D^\alpha_{\xi'}h(\xi_n/\langle\xi'\rangle)$ is a linear combination of terms of the form

$$(\xi_n/\langle\xi'\rangle)^k h^{(k')}(\xi_n/\langle\xi'\rangle)s(\xi'),$$

where $s\in S^{-|\alpha|}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, and $0\le k\le k'\le|\alpha|$. The function $\nu^kh^{(k')}$ is an element in H_0, so we may focus on the case $\alpha=0$. We observe that

$$\begin{aligned}\kappa_{\langle\xi'\rangle^{-1}}\mathrm{op}_{x_n}h(\xi_n/\langle\xi'\rangle)\delta_0^{(l)} &= c_l\,\langle\xi'\rangle^{-1/2}\,\mathcal{F}^{-1}_{\xi_n\to x_n}[h(\xi_n/\langle\xi'\rangle)\xi_n^l](x_n/\langle\xi'\rangle)\\ &= c_l\,\langle\xi'\rangle^{1/2+l}\,\mathcal{F}^{-1}[h(\nu)\nu^l](x_n),\end{aligned}$$

with $c_l=(2\pi)^{-1/2}i^l$. Since $r^+\mathcal{F}^{-1}(h\nu^l)$ is a function in $\mathcal{S}(\mathbb{R}_+)$, this gives the desired result. ◁

Theorem 2.12. *Let $a\in S^\mu_{tr}(\mathbb{R}^n\times\mathbb{R}^n)$, $\mu\in\mathbb{R}$, and $\sigma\in\mathbb{R}^2$, $\sigma_1>-1/2$. Then*

$$\mathrm{op}^+_{x_n}a\in S^\mu(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};H^\sigma(\mathbb{R}_+),H^{\sigma-(\mu,0)}(\mathbb{R}_+)).$$

Proof. For $-1/2<\sigma_1<1/2$ the assertion is immediate from Theorem 1.7 since then the extension operator $e^+:H^\sigma(\mathbb{R}_+)\to H^\sigma(\mathbb{R})$ is continuous. Using interpolation we may assume that $\sigma_1\in\mathbb{N}$. We proceed by induction in σ_1. Recall that the norm of a function v in $H^{\sigma+(1-\mu,0)}(\mathbb{R}_+)$ can be estimated by its norm in $H^{\sigma-(\mu,0)}(\mathbb{R}_+)$ and the norm of $\partial_{x_n}v$ in $H^{\sigma-(\mu,0)}(\mathbb{R}_+)$. We note that $\partial_{x_n}e^+u=e^+\partial_{x_n}u+u(0)\delta_0$ for $u\in H^{\sigma+(1,0)}(\mathbb{R})$. Hence

$$\partial_{x_n}[\mathrm{op}^+_{x_n}a]u=[\mathrm{op}^+_{x_n}\partial_{x_n}a]u+[\mathrm{op}^+_{x_n}a]e^+\partial_{x_n}u+u(0)r^+[\mathrm{op}_{x_n}a]\delta_0.$$

Since $\partial_{x_n}\kappa_{\langle\xi'\rangle^{-1}}=\langle\xi'\rangle^{-1}\kappa_{\langle\xi'\rangle^{-1}}\partial_{x_n}$, we get

$$\begin{aligned}&\|\partial_{x_n}\kappa_{\langle\xi'\rangle^{-1}}[\mathrm{op}^+_{x_n}a]\kappa_{\langle\xi'\rangle}\|_{\mathcal{L}(H^{\sigma+(1,0)}(\mathbb{R}_+),H^{\sigma-(\mu,0)}(\mathbb{R}_+))}\\ \le\ &\langle\xi'\rangle^{-1}\|\kappa_{\langle\xi'\rangle^{-1}}[\mathrm{op}^+_{x_n}\partial_{x_n}a]\kappa_{\langle\xi'\rangle}\|_{\mathcal{L}(H^{\sigma+(1,0)}(\mathbb{R}_+),H^{\sigma-(\mu,0)}(\mathbb{R}_+))}\\ &+\|\kappa_{\langle\xi'\rangle^{-1}}[\mathrm{op}^+_{x_n}a]\kappa_{\langle\xi'\rangle}\|_{\mathcal{L}(H^{\sigma}(\mathbb{R}_+),H^{\sigma-(\mu,0)}(\mathbb{R}_+))}\\ &+\langle\xi'\rangle^{-1}\|\kappa_{\langle\xi'\rangle^{-1}}r^+[\mathrm{op}_{x_n}a]\delta_0\|_{\mathcal{L}(\mathbb{C},H^{\sigma-(\mu,0)}(\mathbb{R}_+))}\|\gamma_0\kappa_{\langle\xi'\rangle}\|_{\mathcal{L}(H^{\sigma+(1,0)}(\mathbb{R}_+),\mathbb{C})}.\end{aligned}$$

By induction, all are $O(\langle\xi'\rangle^\mu)$; for the last term apply Lemma 2.11 and Example 1.5. ◁

3. Symbol Classes for Boutet de Monvel's Calculus

3.1 The operator ∂_+. In the subsequent text we shall denote by ∂_+ the usual derivative considered as a differential operator on distributions over $\mathbb{R}_+$. We choose this notation in order to distinguish ∂_+ from ∂_t which also acts on distributions on the full line. For $\sigma\in\mathbb{R}^2$, $\sigma_1>-1/2$,

$$\partial_+=r^+\partial_te^+:H^\sigma(\mathbb{R}_+)\to H^{\sigma-(1,0)}(\mathbb{R}_+);$$

on the other hand, ∂_+ acts on *all* spaces $H^\sigma(\mathbb{R}_+)$ and defines an (x',ξ')-independent element of $S^1(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};H^\sigma(\mathbb{R}_+),H^{\sigma-(1,0)}(\mathbb{R}_+))$ for every $\sigma\in\mathbb{R}^2$.

3.2 Boundary symbols on $\mathbb{R}^n_+$.

(a) A *potential symbol* of order m is an element of

$$S^m(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathbb{C},\mathcal{S}(\mathbb{R}_+))=\bigcap_{\sigma\in\mathbb{R}^2}S^m(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathbb{C},H^\sigma(\mathbb{R}_+)).\tag{3.1}$$

This is a Fréchet space with the topology of the projective limit. The group action is the identity on $\mathbb{C}$ and given by (1.1) on $H^\sigma(\mathbb{R}_+)$.

(b) A *trace symbol* of order m and type zero is an element of

$$S^m(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}'(\mathbb{R}_+),\mathbb{C}) = \bigcap_{\sigma\in\mathbb{R}^2} S^m(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};H_0^\sigma(\overline{\mathbb{R}}_+),\mathbb{C}). \tag{3.2}$$

Again this is a Fréchet space with the projective limit topology.

For $\sigma_1 > -1/2$ the weighted Sobolev space $H^\sigma(\mathbb{R}_+)$ is embedded in the space $H_0^\tau(\overline{\mathbb{R}}_+)$, $\tau = (\min\{\sigma_1, 0\}, \sigma_2)$, using extension by zero. A trace symbol t of order m and type 0 therefore defines an element of $S^m(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};H^\sigma(\mathbb{R}_+),\mathbb{C})$ whenever $\sigma_1 > -1/2$.

A *trace symbol* of order m and *type* $d\in\mathbb{N}_0$ is a sum of operator-valued symbols $\sum_{j=0}^d t_j\partial_+^j$, where each t_j is a trace symbol of order $m-j$ and type zero and the summation is in $S^m(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};H^\sigma(\mathbb{R}_+),\mathbb{C})$, $\sigma_1 > d-1/2$. We endow the space of trace symbols of order m and type d with the topology of the non-direct sum of Fréchet spaces, see 3.3, below. Instead of 'type', the notion 'class' is also used.

(c) A *singular Green symbol* of order m and type zero is an element of

$$\begin{aligned} S^m(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}'(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+)) \\ = \bigcap_{\sigma,\tau\in\mathbb{R}^2} S^m(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};H_0^\sigma(\overline{\mathbb{R}}_+),H^\tau(\mathbb{R}_+)), \end{aligned} \tag{3.3}$$

endowed with the Fréchet topology of the projective limit.

A singular Green symbol of order m and type zero furnishes an element of $S^m(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};H^\sigma(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+))$ provided $\sigma_1 > -1/2$. We define the *singular Green symbols* of order m and type d as the sums $\sum_{j=0}^d g_j\partial_+^j$, where each g_j is a singular Green symbol of order $m-j$ and type zero, and the summation is in $S^m(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};H^\sigma(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+))$, $\sigma_1 > d-1/2$. The resulting space carries the Fréchet topology of the non-direct sum.

(d) A *boundary symbol* in Boutet de Monvel's calculus of order m and type d is an operator-valued symbol of the form

$$a = \begin{pmatrix} \mathrm{op}^+_{x_n} p + g & k \\ t & s \end{pmatrix}, \tag{3.4}$$

where $p\in S^m_{tr}(\mathbb{R}^n\times\mathbb{R}^n)$, g is a singular Green symbol of order m and type d, k is a potential symbol of order m, t is a trace symbol of order m and type d, and $s\in S^m_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. We saw in 2.12 that

$$\mathrm{op}^+_{x_n} p \in S^m(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};H^\sigma(\mathbb{R}_+),H^{\sigma-(m,0)}(\mathbb{R}_+)),\quad \sigma_1 > -1/2.$$

Also $s\in S^m_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) = S^m(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathbb{C},\mathbb{C})$ is an operator-valued symbol. A boundary symbol of order m and type d can therefore be considered an element of

$$S^m(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};H^\sigma(\mathbb{R}_+)\oplus\mathbb{C},H^{\sigma-(m,0)}(\mathbb{R}_+)\oplus\mathbb{C}),\quad \sigma_1 > d-1/2. \tag{3.5}$$

We endow the space of boundary symbols of order m and type d with the (Fréchet) topology of the non-direct sum of the Fréchet spaces involved.

(e) We obtain the notions of *regularizing potential, trace, singular Green, and boundary symbols* by taking the intersection of the corresponding spaces over all m.

(f) The definitions in (a), (b), and (c) extend easily to double symbols.

We obtain classical symbol classes by taking $S^m_{cl}(\dots)$.

(g) Since we eventually want to treat operators acting on sections of vector bundles over compact manifolds, we shall have to replace the spaces $\mathbb{C},\mathcal{S}(\mathbb{R}_+),H^\sigma(\mathbb{R}_+),H_0^\sigma(\overline{\mathbb{R}}_+),\mathcal{S}'(\mathbb{R}_+)$ in general by N-fold cartesian products for suitable N: $\mathbb{C}^{N_1},\mathcal{S}(\mathbb{R}_+)^{N_2}$, etc.. In order to avoid superfluous notation, we shall not write the N_j unless clarity demands it.

DEFINITION 3.3. Let E,F be Fréchet spaces and suppose both are continuously embedded in the same Hausdorff vector space. The exterior direct sum $E\oplus F$ is Fréchet and has the closed subspace $\mathcal{N} = \{(a,-a) : a\in E\cap F\}$. The non-direct sum of E and F then is the Fréchet space $E+F := E\oplus F/\mathcal{N}$.

DEFINITION 3.4. We call the symbol a in (3.4) a *generalized singular Green symbol* of order m and type d, if $p=0$. For $d=0$, we obtain an element of $S^m(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}'(\mathbb{R}_+)^{N_1}\oplus\mathbb{C}^{N_2},\mathcal{S}(\mathbb{R}_+)^{N_3}\oplus\mathbb{C}^{N_4})$ with suitable $N_1,\ldots,N_4\in\mathbb{N}_0$.

The following proposition lays the foundation for Theorem 3.7, below.

PROPOSITION 3.5. *Let $u\in L^2(\mathbb{R}\times\mathbb{R})$, and suppose that, for $j=1,2$, and all $k,l\in\mathbb{N}_0$, the distribution $x_j^kD_{x_j}^lu(x_1,x_2)$ is an element of $L^2(\mathbb{R}\times\mathbb{R})$. Then $u\in\mathcal{S}(\mathbb{R}\times\mathbb{R})$.*

Proof. For each $l\in 2\mathbb{N}$ we have $\operatorname{op}(1+\xi_1^l+\xi_2^l)u\in L^2(\mathbb{R}^2)$. Since $1+\xi_1^l+\xi_2^l$ is an elliptic symbol of order l, we conclude that $u\in H^{(l,0)}(\mathbb{R}^2)$ for all l. On the other hand, the assumption with $l=0$ implies that $u\in H^{(0,k)}(\mathbb{R}^2)$ for all $k\in\mathbb{N}$. So it remains to show that

$$\bigcap_{s\geq 0}[H^{(s,0)}(\mathbb{R}^2)\cap H^{(0,s)}(\mathbb{R}^2)]=\mathcal{S}(\mathbb{R}^2).$$

Denote by F the Fréchet space on the left hand side. A system $\{p_k:k\in\mathbb{N}\}$ of semi-norms for F is given by

$$p_k(u)=\|\langle x\rangle^{2k}u\|_{L^2}+\|\langle D\rangle^{2k}u\|_{L^2}.$$

Clearly, $\mathcal{S}(\mathbb{R}^2)$ is a subset of F. It even is dense: Fix $\varphi\in C_0^\infty(\mathbb{R}^2)$ with $\varphi(0)=1$; let $\varphi_\varepsilon=\varphi(\varepsilon\cdot)$. Given $u\in F$ we have $\varphi_\varepsilon u\in\mathcal{S}(\mathbb{R}^2)$; by dominated convergence, $\varphi_\varepsilon u\to u$ in F as $\varepsilon\to 0$.

For $u\in\mathcal{S}(\mathbb{R}^2)$, integration by parts and and Cauchy-Schwarz' inequality imply that

$$\|x^\alpha D_x^\beta u\|_{L^2}^2=\int\left(x^\alpha D_x^\beta u\right)\left(x^\alpha D_x^\beta\overline{u}\right)dx\leq Cp_k(u)^2$$

provided that $k\geq\max\{|\alpha|,|\beta|\}$. Here C is a universal constant depending only on α and β. Hence $x^\alpha D^\beta$ extends to a continuous operator on F, and $F\subseteq\mathcal{S}(\mathbb{R}^2)$. ◁

DEFINITION 3.6. We shall write $\mathbb{R}^2_{++}=\mathbb{R}_+\times\mathbb{R}_+$.

The importance of the following theorem lies in the fact that it shows the equivalence of the operator-valued approach and the standard definition: The estimates in 3.7(i) are those required in the usual presentation of singular Green operators, cf. e.g. Grubb [**3**, (2.3.28)].

3.7 THEOREM: SINGULAR GREEN SYMBOL KERNELS. Let $m\in\mathbb{R}$ and let $\{g(y,\eta):y,\eta\in\mathbb{R}^q\times\mathbb{R}^q\}$ be a family of operators $L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+)$. Then the following are equivalent

(i) Each $g(y,\eta)$ is an integral operator with a kernel $\tilde g(y,\eta;u,v)$ satisfying the following estimates: For all $k,k',l,l'\in\mathbb{N}_0$, $\alpha,\beta\in\mathbb{N}_0^q$, there is a constant c depending on k,k',l,l',α,β, with

$$\|u^kD_u^{k'}v^lD_v^{l'}D_\eta^\alpha D_y^\beta\tilde g(y,\eta;u,v)\|_{L^2(\mathbb{R}_{++})}\leq c\,\langle\eta\rangle^{m-|\alpha|-k+k'-l+l'}.$$

(ii) $g\in S^m(\mathbb{R}^q\times\mathbb{R}^q;\mathcal{S}'(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+))$.

(iii) $g\in S^m(\mathbb{R}^q\times\mathbb{R}^q;L^2(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+))$ and $g^*\in S^m(\mathbb{R}^q\times\mathbb{R}^q;L^2(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+))$, where $g^*=\{g(y,\eta)^*:y,\eta\in\mathbb{R}^q\}$ is the family of point-wise adjoints.

Proof. (i) $\Rightarrow$ (ii). It is easy to check that $\kappa_{\langle\eta\rangle^{-1}}D_\eta^\alpha D_y^\beta g(y,\eta)\kappa_{\langle\eta\rangle}$ is the integral operator with the symbol kernel

$$h_{\alpha,\beta}(y,\eta;u,v)=(D_\eta^\alpha D_y^\beta\tilde g)(y,\eta;\langle\eta\rangle^{-1}u,\langle\eta\rangle^{-1}v)\,\langle\eta\rangle^{-1}.$$

The estimates for $\tilde g$ imply that $h_{\alpha,\beta}(y,\eta;\cdot,\cdot)$ is a function in $\mathcal{S}(\mathbb{R}^2_{++})$, and all its semi-norms are $O(\langle\eta\rangle^{m-|\alpha|})$. In particular, it induces an operator from $\mathcal{S}'(\mathbb{R}_+)$ to $\mathcal{S}(\mathbb{R}_+)$, and we get (ii).

(ii) $\Rightarrow$ (iii). Trivially, $g\in S^m(\mathbb{R}^q\times\mathbb{R}^q;L^2(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+))$. Moreover, $g\in S^m(\mathbb{R}^q\times\mathbb{R}^q;H_0^\sigma(\overline{\mathbb{R}}_+),L^2(\mathbb{R}_+))$ for all $\sigma\in\mathbb{R}^2$. Hence 1.15 shows the asserted property of g^*.

(iii) $\Rightarrow$ (i). The operator $\kappa_{\langle\eta\rangle^{-1}}g(y,\eta)\kappa_{\langle\eta\rangle}:L^2(\mathbb{R}_+)\to\mathcal{S}(\mathbb{R}_+)$ is continuous. In particular, it is a Hilbert-Schmidt operator on $L^2(\mathbb{R}_+)$ and thus has an integral kernel $h_1(y,\eta;\cdot,\cdot)\in L^2(\mathbb{R}^2_{++})$, and

$$\|h_1(y,\eta;\cdot,\cdot)\|_{L^2(\mathbb{R}^2_{++}))}\;=\;\|\kappa_{\langle\eta\rangle^{-1}}g(y,\eta)\kappa_{\langle\eta\rangle}\|_{HS(L^2(\mathbb{R}_+))}.$$

The last norm is bounded by the norm in $\mathcal{L}(L^2(\mathbb{R}_+), H^{(1,1)}(\mathbb{R}_+))$. By a direct calculation, the operator $g(y,\eta)$ then has the integral kernel

$$\tilde{g}_1(y,\eta;u,v) = h_1(y,\eta;\langle\eta\rangle u, \langle\eta\rangle v)\langle\eta\rangle . \tag{3.6}$$

Correspondingly, the operator $\kappa_{\langle\eta\rangle^{-1}} g^*(y,\eta)\kappa_{\langle\eta\rangle}$ has the kernel $h_2(y,\eta;u,v)$, and

$$h_1(y,\eta;u,v) = \overline{h_2}(y,\eta;v,u). \tag{3.7}$$

The mapping $u^k D_u^{k'} \kappa_{\langle\eta\rangle^{-1}} D_\eta^\alpha D_y^\beta g(y,\eta)\kappa_{\langle\eta\rangle}\colon L^2(\mathbb{R}_+) \to \mathcal{S}(\mathbb{R}_+)$ also is continuous. Therefore,

$$\|u^k D_u^{k'} D_\eta^\alpha D_y^\beta h_1(y,\eta;u,v)\|_{L^2(\mathbb{R}^2_{++})} = O(\langle\eta\rangle^{m-|\alpha|}).$$

Using relation (3.7) we also have

$$\|v^l D_v^{l'} D_\eta^\alpha D_y^\beta h_1(y,\eta;u,v)\|_{L^2(\mathbb{R}^2_{++})} = O(\langle\eta\rangle^{m-|\alpha|}).$$

Together with Proposition 3.5, these estimates show that

$$\|u^k D_u^{k'} v^l D_v^{l'} D_\eta^\alpha D_y^\beta h_1(y,\eta;u,v)\|_{L^2(\mathbb{R}^2_{++})} = O(\langle\eta\rangle^{m-|\alpha|}).$$

Combining this with (3.6), we obtain (i). ◁

LEMMA 3.8. *We may replace the estimates in Proposition* 3.7(i) *by*

$$\sup_{u,v>0} |u^k D_u^{k'} v^l D_v^{l'} D_\eta^\alpha D_y^\beta \tilde{g}(y,\eta;u,v)| \le c\,\langle\eta\rangle^{1+m-|\alpha|-k+k'-l+l'} .$$

This is a consequence of the following estimates for functions in $\mathcal{S}(\mathbb{R}_+)$*:*

$$\sup_{t>0}|f(t)|^2 \;\le\; 2\|f\|_{L^2(\mathbb{R}_+)}\|D_t f\|_{L^2(\mathbb{R}_+)}, \quad \text{and} \tag{3.8}$$

$$\|f\|_{L^2(\mathbb{R}_+)} \;\le\; \langle\eta\rangle^{-1/2}\sup_{t>0}(1+\langle\eta\rangle t)\,|f(t)|. \tag{3.9}$$

Proof. (3.8) follows from the identity $|f(t)|^2 = -\int_t^\infty \partial_s\{f(s)\overline{f}(s)\}ds$. For (3.9) we note that

$$\|f\|^2_{L^2(\mathbb{R}_+)} \le \sup_{t>0}(1+\langle\eta\rangle t)^2|f(t)|^2\int_0^\infty (1+\langle\eta\rangle s)^{-2}ds.$$

◁

3.9 THEOREM: POTENTIAL AND TRACE SYMBOL KERNELS. (a) Let $m\in\mathbb{R}$ and let $k=\{k(y,\eta) : y,\eta\in\mathbb{R}^q\}$ be a family of operators in $\mathcal{L}(\mathbb{C}, L(\mathbb{R}_+))$. Then the following are equivalent:

(i) The operators $\{k(y,\eta) : y,\eta\in\mathbb{R}^q\}$ act on $\mathbb{C}$ by multiplication by functions $\tilde{k}(y,\eta;\cdot)$ satisfying the following estimates: For all $l,l'\in\mathbb{N}_0$ and all multi-indices $\alpha,\beta\in\mathbb{N}_0^{n-1}$, there is a constant c, depending on l,l',α,β, such that

$$\|u^l D_u^{l'} D_\eta^\alpha D_y^\beta \tilde{k}(y,\eta;u)\|_{L^2(\mathbb{R}_+)} \le c\,\langle\eta\rangle^{m-|\alpha|-l+l'} .$$

(ii) $k\in S^m(\mathbb{R}^q\times\mathbb{R}^q;\mathbb{C},\mathcal{S}(\mathbb{R}_+))$, i.e. k is a potential symbol.

(iii) The family k^* of point-wise adjoints $k^*=\{k(y,\eta)^* : y,\eta\in\mathbb{R}^q\}$ is an element of $S^m(\mathbb{R}^q\times\mathbb{R}^q;\mathcal{S}'(\mathbb{R}_+),\mathbb{C})$.

(iv) We may replace the estimates in (a) by

$$\sup_u |u^l D_u^{l'} D_\eta^\alpha D_y^\beta \tilde{k}(y,\eta;u)| \le c\,\langle\eta\rangle^{1/2+m-|\alpha|-l+l'} .$$

(b) A trace symbol $t\in S^m(\mathbb{R}^q\times\mathbb{R}^q;\mathcal{S}'(\mathbb{R}_+),\mathbb{C}))$ of order m and type zero is given via

$$t(y,\eta)f = \int_0^\infty \tilde{t}(y,\eta;u)f(u)du,\;\; f\in\mathcal{S}(\mathbb{R}_+),$$

where $\tilde{t}$ satisfies the estimates in (a.i) or (a.iv).

In particular, potential and trace symbols are dual to each other.

Proof. The equivalence of (i) and (ii) can be shown just as in Proposition 3.6. Furthermore, (ii) and (iii) are equivalent, since $H_0^{-\sigma}(\overline{\mathbb{R}}_+) = H^\sigma(\mathbb{R}_+)'$, cf. 1.15. Finally we may use sup-norm estimates by the same arguments as in Lemma 3.8. ◁

EXAMPLE 3.10. In 1.5 we saw that $\gamma_j \in S^{j+1/2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H^\sigma(\mathbb{R}_+), \mathbb{C})$ whenever $\sigma_1 > j + 1/2$. Let us show that γ_j is a trace symbol of order $j + 1/2$ and type $j + 1$. Indeed, we can write

$$\gamma_0 f = \int_0^\infty \langle \xi' \rangle \, e^{-y_n \langle \xi' \rangle} f(y_n) dy_n - \int_0^\infty e^{-y_n \langle \xi' \rangle} \partial_{y_n} f(y_n) dy_n, \tag{3.10}$$

$f \in \mathcal{S}(\mathbb{R}_+)$. Hence $\gamma_0 = t_0 + t_1 \partial_+$, where

$$t_0 f = \langle \xi' \rangle \int_0^\infty e^{-y_n \langle \xi' \rangle} f(y_n) dy_n \quad \text{and} \quad t_1 f = -\int_0^\infty e^{-y_n \langle \xi' \rangle} f(y_n) dy_n.$$

In particular,

$$t_0 \kappa_{\langle \xi' \rangle} f = \langle \xi' \rangle^{1/2} \int_0^\infty e^{-t} f(t) dt = \langle \xi' \rangle^{1/2} \left\langle e^{-\cdot}, f \right\rangle_{\mathcal{S}(\mathbb{R}_+), \mathcal{S}'(\mathbb{R}_+)},$$

so $t_0 \in S^{1/2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}'(\mathbb{R}_+), \mathbb{C})$. In the same way $t_1 \in S^{-1/2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}'(\mathbb{R}_+), \mathbb{C})$. Applying (3.10) to $\partial^j_{x_n} f$ we get the desired result by iteration.

REMARK 3.11. It is obvious that there are many different ways to write $\gamma_j = \sum_{l=0}^{j+1} t_l \partial_+^l$ with trace symbols t_j of order $l + 1/2 - j$ and type zero.

4. The Analysis of Compositions

We may compose two boundary symbols of orders $m_1, m_2 \in \mathbb{Z}$ and types $d_1, d_2 \in \mathbb{N}_0$, say

$$a_1 = \begin{pmatrix} \operatorname{op}^+_{x_n} p_1 + g_1 & k_1 \\ t_1 & s_1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} \operatorname{op}^+_{x_n} p_2 + g_2 & k_2 \\ t_2 & s_2 \end{pmatrix},$$

provided the dimensions of the matrices are compatible. According to Lemma 1.12(b)

$$a_1 a_2 \in S^{m_1+m_2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H^\sigma(\mathbb{R}_+)^{N_1} \oplus \mathbb{C}^{N_2}, H^{\sigma-(m_1+m_2,0)}(\mathbb{R}_+)^{N_3} \oplus \mathbb{C}^{N_4})$$

for suitable $N_1, \ldots, N_4$ supposing $\sigma_1 > d_2 - 1/2$ and $\sigma_1 - m_2 > d_1 - 1/2$. We can compute the composition

$$a_1 a_2 = \begin{pmatrix} p_1^+ + g_1 & k_1 \\ t_1 & s_1 \end{pmatrix} \begin{pmatrix} p_2^+ + g_2 & k_2 \\ t_2 & s_2 \end{pmatrix}$$

$$= \begin{pmatrix} (p_1 \#_n p_2)^+ + l(p_1, p_2) + p_1^+ g_2 + g_1 p_2^+ + g_1 g_2 + k_1 t_2 & p_1^+ k_2 + g_1 k_2 + k_1 s_2 \\ t_1 p_2^+ + t_1 g_2 + s_1 t_2 & t_1 k_2 + s_1 s_2 \end{pmatrix}.$$

We have written p_j^+ instead of $\operatorname{op}^+_{x_n} p_j, j = 1, 2$, in order to save space, and $(p_1 \#_n p_2)^+$ instead of $\operatorname{op}^+_{x_n}(p_1 \#_n p_2)$; the notation $\#_n$ indicates composition with respect to x_n.

We will show:

(i) $l(p_1, p_2) = p_1^+ p_2^+ - (p_1 \#_n p_2)^+$ is a singular Green symbol of type $(m_2)_+ = \max\{m_2, 0\}$;
(ii) $p_1^+ g_2$ is a singular Green symbol of type d_2;
(iii) $g_1 p_2^+$ is a singular Green symbol of type $(m_2 + d_1)_+ = \max\{m_2 + d_1, 0\}$;
(iv) $g_1 g_2$ is a singular Green symbol of type d_2;
(v) $k_1 t_2$ is a singular Green symbol of type d_2;
(vi) $p_1^+ k_2$ is a potential symbol;
(vii) $g_1 k_2$ is a potential symbol;
(viii) $k_1 s_2$ is a potential symbol;
(ix) $t_1 p_2^+$ is a trace symbol of type $(m_2 + d_1)_+$;
(x) $t_1 g_2$ is a singular Green symbol of type d_2;
(xi) $s_1 t_2$ is a trace symbol of type d_2;
(xii) $t_1 k_2$ is a pseudodifferential symbol;
(xiii) $s_1 s_2$ is a pseudodifferential symbol.

In all cases the order of the respective symbols is $m_1 + m_2$. When referring to the symbols in (xii) and (xiii) as 'pseudodifferential' we stress that the Banach spaces $E, \tilde{E}$ in the sense of Definition 1.4 are simply $\mathbb{C}^{N_2}$ and $\mathbb{C}^{N_4}$, respectively.

We therefore obtain the following result:

THEOREM 4.1. *The point-wise composition a_1a_2 of two boundary symbols a_1 and a_2 of orders m_1 and m_2 and types d_1 and d_2, respectively, is a boundary symbol of order m_1+m_2 and type* $\max\{m_2+d_1, d_2\}$. *Its pseudodifferential part is $p_1\#_n p_2$.*

The proof is rather long, and we shall break it up into a sequence of partial results. Let us first deal with the easy cases, namely (ii), (iv), (v), (vi), (vii), (viii), (x), (xi), (xii), and (xiii). We may assume that $N_1, \dots, N_4 = 1$. Write $g_1 = \sum_{j=0}^{d_1} g_{1j}\partial_+^j$ and $g_2 = \sum_{j=0}^{d_2} g_{2j}\partial_+^j$. For (ii) note that

$$p_1^+ g_2 = \sum_{j=0}^{d_2} (p_1^+ g_{2j})\partial_+^j,$$

where, according to Theorem 2.12 and Lemma 1.12, $p_1^+ g_{2j} \in S^{m_1+m_2-j}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+))$. For (iv) we observe that

$$\partial_+^j \in S^j(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, H^\sigma(\mathbb{R}_+), H^{\sigma-(j,0)}(\mathbb{R}_+)) \tag{4.1}$$

for all $\sigma \in \mathbb{R}^2$. Then Lemma 1.12 yields the result. The proof of (v), (vi), (vii), (x), and (xii) is analogous. For (viii), (xi), and (xiii) we recall additionally that $s_2 \in S_{1,0}^{m_2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) = S^{m_2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathbb{C}, \mathbb{C})$.

Compositions (iii) and (ix) are slightly more delicate. Consider (iii), for example: In order to show that $g_1p_2^+$ is a singular Green operator we first note that

$$\partial_+ p_2^+ = r^+\partial_{x_n}[\mathrm{op}_{x_n} p_2]e^+ = \mathrm{op}_{x_n}^+[ip_2(x,\xi)\xi_n + \partial_{x_n}p_2(x,\xi)].$$

By iteration, $\partial_+^j p_2^+ = \mathrm{op}_{x_n}^+ q_j$ for a suitable $q_j \in S_{tr}^{m_2+j}(\mathbb{R}^n \times \mathbb{R}^n)$. So it is no restriction to assume that $d_1 = 0$.

Next we shall establish a central point in 4.4, below: For fixed N we may split p_2 in a 'differential' part and one that acts on the H_0^σ-spaces:

$$\mathrm{op}_{x_n} p_2 = \sum_{j=0}^{N}\sum_{k=0}^{m_{2+}} x_n^j s_{jk}\partial_{x_n}^k + a$$

with $s_{jk} \in S^{m_2-k}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, $a \in S^{m_2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H^\sigma(\mathbb{R}), H^{\sigma-(m_{2+},0)}(\mathbb{R}))$ for each $\sigma \in \mathbb{R}^2$, and

$$\chi_+ a \in S^{m_2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H_0^\sigma(\overline{\mathbb{R}}_+), H_0^{\sigma-(m_{2+},0)}(\overline{\mathbb{R}}_+)), \quad -N \le \sigma_1 \le 0.$$

Here χ_+ denotes multiplication by the characteristic function of $\mathbb{R}_+$. This makes sense: For u in $C_0^\infty(\mathbb{R}_+)$ and x', ξ' fixed, $a(x',\xi')u$ is smooth on $\mathbb{R}$. Multiplication by χ_+ furnishes an element of $H_0^0(\overline{\mathbb{R}}_+)$ which coincides with the one given via restriction to $\mathbb{R}_+$ followed by embedding into $H_0^0(\overline{\mathbb{R}}_+)$. As we shall show, $\chi_+a(x',\xi')$ extends to an element of $\mathcal{L}(H_0^\sigma(\overline{\mathbb{R}}_+), H_0^{\sigma-(m_{2+},0)}(\overline{\mathbb{R}}_+))$ and satisfies the required symbol estimates.

After the above reduction, g_1 is a singular Green symbol of order m_1 and type 0, and

$$g_1p_2^+ = \sum_{j=0}^{N}\sum_{k=0}^{m_{2+}} (g_1x_n^j s_{jk})\partial_+^k + g_1\chi_+a.$$

Clearly, s_{jk} induces an element of $S^{m_2-k}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H_0^\sigma(\overline{\mathbb{R}}_+), H_0^\sigma(\overline{\mathbb{R}}_+))$ and

$$x_n^j \in S^{-j}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H_0^\sigma(\overline{\mathbb{R}}_+), H_0^{\sigma-(0,j)}(\overline{\mathbb{R}}_+)),$$

$\sigma \in \mathbb{R}^2$. The summation therefore yields a singular Green symbol of order m_1+m_2, while $g_1\chi_+a \in S^{m_1+m_2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H_0^\sigma(\overline{\mathbb{R}}_+), \mathcal{S}(\mathbb{R}_+))$ for $\sigma_1 \ge -N$. By Lemma 4.2, below, $g_1p_2^+$ therefore is a singular Green symbol of order m_1+m_2. The type now is m_{2+} since we assumed that $d_1 = 0$; in the general case it is $(m_2+d_1)_+$ by the above consideration.

Also composition (ix) follows from the above representation for $\mathrm{op}_{x_n} p_2$. Statement (i) presents additional complications. We shall deal with them below.

First, however, let us state a lemma which was employed in the proof, above. It is easily established.

Lemma 4.2. *Let $d \in \mathbb{N}_0$ and $g \in S^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H^\sigma(\mathbb{R}_+), L^2(\mathbb{R}_+))$, $\sigma_1 > d - 1/2$. Suppose that, for each $N \in \mathbb{N}$, we find symbols*

$$g_{jN} \in S^{m-j}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H_0^{-(N,N)}(\overline{\mathbb{R}}_+), H^{(N,N)}(\mathbb{R}_+))$$

with $g = \sum_{j=0}^d g_{jN}\partial_+^j$. Then g is a singular Green symbol of order m and type d.

We now turn to composition (i).

Lemma 4.3. *For $h \in H_0$ and all $\sigma \in \mathbb{R}^2$ with $\sigma_1 \leq 0$,*

$$\chi_+ \mathrm{op}_{x_n} h(\xi_n/\langle\xi'\rangle) \in S^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H_0^\sigma(\overline{\mathbb{R}}_+), H_0^\sigma(\overline{\mathbb{R}}_+)).$$

Proof. The crucial point here is the boundedness of $\chi_+\mathrm{op}_{x_n} h(\xi_n) : H_0^{-N}(\overline{\mathbb{R}}_+) \to H_0^{-N}(\overline{\mathbb{R}}_+)$, $N \in \mathbb{N}$. We let $\lambda = \mathrm{op}\,(1+i\xi_n)^N$. Clearly, $\lambda : H_0^0(\overline{\mathbb{R}}_+) \to H_0^{-N}(\overline{\mathbb{R}}_+)$ is an isomorphism with inverse $\lambda^{-1} = \mathrm{op}\,(1+i\xi_n)^{-N}$, so it is sufficient to show that

$$\lambda^{-1}[\chi_+\mathrm{op}\, h]\lambda : C_0^\infty(\mathbb{R}_+) \to H_0^{-N}(\overline{\mathbb{R}}_+)$$

extends to a bounded operator on $H_0^0(\overline{\mathbb{R}}_+)$.

Now $[\mathrm{op}\, h]\lambda = \mathrm{op}_{x_n}(h(1+i\xi_n)^N) = \mathrm{op}\,(h_0 + p_0)$ for some $h_0 \in H_0$ and a polynomial p_0 of degree $\leq N$. The operator $\lambda^{-1}\chi_+\mathrm{op}(h_0 + p_0)$ is easily seen to have a bounded extension to $H_0^0(\mathbb{R}_+)$. ◁

Here is the decomposition result used for (iii) and (ix):

4.4 Decomposing $\mathrm{op}_{x_n}^+ p$. Let $p \in S_{tr}^m(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{Z}$. Fix $N \in \mathbb{N}$ and a function $\omega \in C_0^\infty(\mathbb{R})$ with $\omega(t) \equiv 1$ for t close to zero. Write

$$p(x,\xi) = \sum_{j=0}^{N-1} \frac{x_n^j}{j!}\omega(x_n)\partial_{x_n}^j p(x',0,\xi) + x_n^N\omega(x_n)p_N(x,\xi) + (1-\omega(x_n))p(x,\xi);$$

with the Taylor remainder p_N. As a consequence of the transmission property, $\partial_{x_n}^j p(x',0,\langle\xi'\rangle\,\xi_n)$ is an element of $S^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})\hat{\otimes}_\pi H$. Now $H = H' \oplus H_0$, so we have $\partial_{x_n}^j p(x',0,\xi) = \sum_{k=0}^m s_{kj}(x',\xi')\xi_n^k + q_j(x',\xi)$, where the s_{kj} are elements of $S^{m-k}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, and q_j has a representation

$$q_j(x',\xi) = \sum_{k=0}^\infty \lambda_{kj}c_{kj}(x',\xi')h_{kj}(\xi_n/\langle\xi'\rangle) \tag{4.2}$$

with a sequence $\{\lambda_{kj}\} \in l^1$, and null sequences $\{c_{kj}\} \subseteq S_{1,0}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ and $\{h_{kj}\} \subseteq H_0$. We therefore obtain

$$\begin{aligned} \mathrm{op}_{x_n} p &= \sum_{j=0}^{N-1}\sum_{k=0}^m \frac{x_n^j}{j!}\omega(x_n)s_{kj}\partial_{x_n}^k + a, \quad \text{where} \\ a &= \mathrm{op}_{x_n}\left(\sum_{j=0}^{N-1}\frac{x_n^j}{j!}\omega(x_n)q_j + x_n^N\omega(x_n)p_N(x,\xi) + (1-\omega(x_n))p(x,\xi)\right). \end{aligned} \tag{4.3}$$

Obviously, a is an element of $S^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H^\sigma(\mathbb{R}), H^{\sigma-(m_+,0)}(\mathbb{R}))$ for all $\sigma \in \mathbb{R}^2$. Moreover, we deduce that $\chi_+ a \in S^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H_0^\sigma(\overline{\mathbb{R}}_+), H_0^{\sigma-(m_+,0)}(\overline{\mathbb{R}}_+))$ provided that $-N \leq \sigma_1 \leq 0$. In fact, this is rather straightforward for the last two terms on the right hand side of (4.3); for the ones under the summation we employ the representation (4.2) which allows us to focus on a single term. Then we apply Lemma 4.3.

As explained above, this decomposition of $\mathrm{op}_{x_n} p$ furnishes the statement regarding compositions (iii) and (ix).

4.5 The analysis of the leftover term. The leftover term $l(p,q)$ arises from the composition of the boundary symbols associated with two pseudodifferential operators p and q of orders m_1 and m_2:

$$l(p,q) = [\mathrm{op}_{x_n}^+ p][\mathrm{op}_{x_n}^+ q] - r^+[\mathrm{op}_{x_n} p][\mathrm{op}_{x_n} q]e^+ = r^+[\mathrm{op}_{x_n} p](e^+r^+ - 1)[\mathrm{op}_{x_n} q]e^+.$$

We shall show that this is a singular Green symbol of order m_1+m_2 and type m_{2+}. As in 4.4 we find $s_j \in S^{m_1-j}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}), \tilde{s}_k \in S^{m_2-k}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1})$ such that

$$\mathrm{op}_{x_n} p = \sum_{j=0}^{m_1} s_j \partial^j_{x_n} + a; \qquad \mathrm{op}_{x_n} q = \sum_{k=0}^{m_2} \tilde{s}_k \partial^k_{x_n} + b$$

(with the corresponding notation). In view of the identity

$$\partial_{x_n} e^+ f = e^+ \partial_{x_n} f + f(0)\delta_0 \tag{4.4}$$

valid for $f \in \mathcal{S}(\mathbb{R}_+)$, we conclude that

$$l\Big(\sum_{j=0}^{m_1} s_j \xi_n^j, q\Big) = 0. \tag{4.5}$$

Therefore $l(p,q) = r^+a(e^+r^+-1)\big(\sum_{k=0}^{m_2} \tilde{s}_k \partial^k_{x_n} e^+ + be^+\big)$. Iterating (4.4) we see that

$$\partial^k_{x_n} e^+ f = e^+ \partial^k_{x_n} f + \sum_{l=0}^{k-1} f^{(l)}(0)\delta_0^{(k-l-1)}.$$

Hence

$$(e^+r^+-1)\sum_{k=0}^{m_2} \tilde{s}_k \partial^k_{x_n} e^+ f = -\sum_{k=0}^{m_2}\sum_{l=0}^{k-1} \tilde{s}_k(x',0,\xi')\gamma_l(f)\delta_0^{(k-l-1)}.$$

We know from Example 3.10 that γ_l is a trace symbol of order $1/2+l$ and type $l+1$. So we can write

$$l(p,q) = \sum_{l=0}^{m_2-1} k_l\gamma_l + r^+a(e^+r^+-1)be^+, \tag{4.6}$$

where

$$k_l = -\sum_{j=l+1}^{m_2} r^+a\tilde{s}_j(x',0,\xi')\delta_0^{(j-l-1)}.$$

Using (4.5) we may replace a by p. Since $\tilde{s}_j(x',0,\xi') \in S^{m_2-j}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ it follows from Lemma 2.11 that k_l is a potential symbol of order $m_1+m_2-l-1/2$, hence $k_l\gamma_l$ is a singular Green symbol of order m_1+m_2 and type m_{2+} as asserted.

Now let us consider the second summand in (4.6). On $C_0^\infty(\mathbb{R}_+)$ we have

$$r^+a(e^+r^+-1)be^+ = -r^+ae^-r^-be^+ = -(r^+ae^-J)(Jr^-be^+)$$

where r^- denotes restriction to $\mathbb{R}_-$, e^- extension from $\mathbb{R}_-$ to $\mathbb{R}$ by zero, and J is the reflection operator

$$Ju(x_n) = u(-x_n).$$

Note that, for $u \in C_0^\infty(\mathbb{R}_+) \subseteq \mathcal{S}(\mathbb{R})$, be^+ is a function in $\mathcal{S}(\mathbb{R})$, so that there are no problems with the compositions.

We shall show that r^+ae^-J and Jr^-be^+ are type zero singular Green operators of orders m_1 and m_2 respectively. Let us first analyze r^+ae^-J. From (4.3) we know that

$$a = \mathrm{op}_{x_n}\Big(\sum_{j=0}^{N-1}\frac{x_n^j}{j!}\omega q_j + x_n^N\omega p_N + (1-\omega)p\Big); \tag{4.7}$$

here q_j is the projection onto H_0 of $\partial^j_{x_n}p(x',0,\xi',\cdot)$, and $x_n^Np_N$ is the Taylor remainder; N as well as the cut-off function ω are fixed.

In the argument, below, we shall only need the fact that $p_N \in S^{m_1}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$, which is obvious from Taylor's formula. Since $(1-\omega)p$ vanishes to arbitrary order at $x_n=0$ we find an $r \in S^{m_1}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$ with $x_n^Np_N + (1-\omega)p = x_n^Nr$.

Step 1. The operator of multiplication by $x_n^j\omega$ is an element of $S^{-j}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; H^\sigma(\mathbb{R}_+), H^\sigma(\mathbb{R}_+))$ for all $\sigma \in \mathbb{R}^2$: This follows from the identity

$$\kappa_{\langle\xi'\rangle^{-1}} x_n^j\omega(x_n)\kappa_{\langle\xi'\rangle} = \langle\xi'\rangle^{-j}\, x_n^j\omega(x_n/\langle\xi'\rangle)$$

together with the observation that the family $\{\omega(\cdot/\langle\xi'\rangle) : \xi' \in \mathbb{R}^{n-1}\}$ is uniformly bounded on $H^\sigma(\mathbb{R}_+)$.

Step 2. $r^+\mathrm{op}_{x_n} q_j e^- J \in S^{m_1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+))$: Write q_j in the form (4.2). Then it is clearly sufficient to show that

$$r^+\mathrm{op}_{x_n} h(\xi_n/\langle\xi'\rangle)e^- J \in S^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+)).$$

We note that $\kappa_{\langle\xi'\rangle^{-1}}\mathrm{op}_{x_n} h(\xi_n/\langle\xi'\rangle)\kappa_{\langle\xi'\rangle} = \mathrm{op}_{x_n} h(\xi_n)$, thus it is enough to prove the continuity of $r^+[\mathrm{op}_{x_n} h]e^- J : \mathcal{S}'(\mathbb{R}_+) \to \mathcal{S}(\mathbb{R}_+)$; derivatives can be treated in the same way. The operator $\mathrm{op}_{x_n} h$ has the integral kernel

$$k(x_n, y_n) = \int e^{i(x_n-y_n)\xi_n} h(\xi_n) d\xi_n,$$

so $r^+[\mathrm{op}_{x_n} h]e^- J$ is given via $k(x_n, -y_n) = (\mathcal{F}^{-1}h)(x_n + y_n)$ on $\mathbb{R}^2_{++}$. Since $(\mathcal{F}^{-1}h)|_{\mathbb{R}_+} \in \mathcal{S}(\mathbb{R}_+)$ we obtain the assertion.

Steps 1 and 2 imply that the terms under the summation in (4.7) are singular Green operators of order $m_1 - j$ and type zero.

Step 3. Fix $K \in \mathbb{N}$ and $r \in S^{m_1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Then $r^+[\mathrm{op}_{x_n} x_n^N r]e^- J$ defines an element of $S^{-K}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H_0^{-(K,K)}(\overline{\mathbb{R}}_+), H^{(K,K)}(\mathbb{R}_+))$ provided N is sufficiently large: Let us show that the norm of

$$\begin{aligned} &\kappa_{\langle\xi'\rangle^{-1}} r^+\mathrm{op}_{x_n}(x_n^N D^\alpha_{\xi'} D^\beta_{x'} r)e^- J\kappa_{\langle\xi'\rangle} \\ &= \langle\xi'\rangle^{-N} r^+\mathrm{op}_{x_n} x_n^N (D^\alpha_{\xi'} D^\beta_{x'} r)(x', x_n/\langle\xi'\rangle, \xi', \xi_n\langle\xi'\rangle)e^- J \end{aligned} \tag{4.8}$$

in $\mathcal{L}(H_0^{-(K,K)}(\overline{\mathbb{R}}_+), H^{(K,K)}(\mathbb{R}_+))$ is $O(\langle\xi'\rangle^{-K-|\alpha|})$. It is clearly no restriction to assume $\alpha = \beta = 0$. The operator in (4.8) then has the integral kernel

$$k(x_n, y_n) = \langle\xi'\rangle^{-N} \int e^{i(x_n+y_n)\xi_n} x_n^N r(x', x_n/\langle\xi'\rangle, \xi', \xi_n\langle\xi'\rangle) d\xi_n$$

on $\mathbb{R}^2_{++}$. This makes sense as an oscillatory integral: We may choose $l > (m_1 + 1)/2$ and regularize it as

$$\int e^{i(x_n+y_n)\xi_n} x_n^N (x_n + y_n)^{-2l} \langle\xi'\rangle^{2l} (\Delta^l_{\xi_n} r)(x', x_n/\langle\xi'\rangle, \xi', \xi_n\langle\xi'\rangle) d\xi_n. \tag{4.9}$$

The integrand then is $O(x_n^N(x_n + y_n)^{-2l}\langle\xi'\rangle^{2l}\langle\xi', \xi_n\langle\xi'\rangle\rangle^{m_1-2l})$. In view of the identity $\langle\xi', \xi_n\langle\xi'\rangle\rangle = \langle\xi'\rangle\langle\xi_n\rangle$ we see that

$$k(x_n, y_n) = O(x_n^N(x_n + y_n)^{-2l}\langle\xi'\rangle^{m_1-N}), \quad l > (m_1 + 1)/2.$$

For $x_n, y_n \le 1$ we choose $l = N/2$ (N is large and may be assumed to be even) and conclude that $k(x_n, y_n) = O(\langle\xi'\rangle^{m_1-N})$. Otherwise we let $2l > N + 2$; then $k(x_n, y_n)$ is $O(\langle x_n + y_n\rangle^{-2}\langle\xi'\rangle^{m_1-N})$. Consequently,

$$\begin{aligned} \sup_{y_n\ge 0} \int_0^\infty |k(x_n, y_n)| dx_n &= O(\langle\xi'\rangle^{m_1-N}), \text{ and} \\ \sup_{x_n\ge 0} \int_0^\infty |k(x_n, y_n)| dy_n &= O(\langle\xi'\rangle^{m_1-N}). \end{aligned}$$

By Schur's Lemma the norm in $\mathcal{L}(L^2(\mathbb{R}_+))$ of the operator family in (4.8) is $O(\langle\xi'\rangle^{m_1-N})$.

We next recall a general fact: We can estimate the norm of an operator $T \in \mathcal{L}(H_0^{-(K,K)}(\overline{\mathbb{R}}_+), H^{(K,K)}(\mathbb{R}_+))$ in terms of

$$\{\|x_n^m \partial_{x_n}^{m'} T x_n^l \partial_{x_n}^{l'}\|_{\mathcal{L}(L^2(\mathbb{R}_+))} : m, m', l, l' \le K\},$$

and these operators have kernels $(-1)^l x_n^m \partial_{x_n}^{m'} y_n^l \partial_{y_n}^{l'} k(x_n, y_n)$ if k is the kernel for T. We plug this into our regularized expression (4.9) for k. We shall then have to ask $l > (m_1 + 2K + 1)/2$ to make the integral converge. For $N > m_1 + 2K + 1$ we can apply Schur's lemma as before and we obtain the assertion.

Steps 1, 2, and 3 show that $r^+ae^- J$ is a singular Green operator of order m_1 and type zero. In virtually the same way we can treat $Jr^- be^+$ and prove that it is a singular Green symbol of order m_2 and type zero. Altogether we then know that $l(p, q)$ is a singular Green symbol of order $m_1 + m_2$ and type m_{2+}.

REMARK 4.6. If a_1 and a_2 are classical, then so is the composition a_1a_2.

5. Operators on the Half-Space

We shall now introduce Boutet de Monvel's algebra.

5.1 OPERATORS IN BOUTET DE MONVEL'S CALCULUS. Given a boundary symbol a of order $m \in \mathbb{Z}$ and type $d \in \mathbb{N}_0$ as in (3.4) we call $A = \operatorname{op} a$ an *operator of order m and type d in Boutet de Monvel's calculus* and write $A \in \mathcal{B}^{m,d}(\overline{\mathbb{R}}^n_+)$.

The operator A therefore is a 2×2 matrix of operators

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix}. \tag{5.1}$$

We call $P_+ = \operatorname{op}^+ p = \operatorname{op}_{x'} \operatorname{op}^+_{x_n} p$ the pseudodifferential part of A. The operator $G = \operatorname{op} g$ is a so-called singular Green operator, $T = \operatorname{op} t$ a trace operator, $K = \operatorname{op} k$ a potential (or Poisson) operator; $S = \operatorname{op} s$ is the pseudodifferential part on the boundary.

The classical elements in Boutet de Monvel's calculus are the operators $\operatorname{op} a$, where a is classical in the sense of 1.11. The notation is $\mathcal{B}^{m,d}_{cl}(\overline{\mathbb{R}}^n_+)$.

We call A a generalized singular Green operator if a is a generalized singular Green symbol as in 3.4.

The intersection $\bigcap_{m\in\mathbb{Z}} \mathcal{B}^{m,d}(\overline{\mathbb{R}}^n_+)$ is the space of *regularizing operators* of type d.

We endow these spaces with the topology inherited from the topology on the associated boundary symbols, see 3.2.

REMARK 5.2. The sum in the upper left corner of (5.1) is not direct: Let, for example, P be a regularizing pseudodifferential operator which has an integral kernel in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Then it is easy to see from Theorem 3.7 that P_+ coincides with a regularzing singular Green operator.

On the other hand, Theorem 5.9, below, shows that singular Green operators are regularizing when localized to the interior. If we additionally assume P to be *classical*, we conclude that P_+ can only coincide with a singular Green operator, if it is regularizing.

Combining (3.5) with Corollary 1.19 gives the following result.

THEOREM 5.3. *An operator $A \in \mathcal{B}^{m,d}(\overline{\mathbb{R}}^n_+)$ induces continuous mappings*

$$A: \begin{array}{c} \mathcal{W}^s(\mathbb{R}^{n-1}, H^\sigma(\mathbb{R}_+))^{M_1} \\ \oplus \\ \mathcal{W}^s(\mathbb{R}^{n-1}, \mathbb{C})^{M_2} \end{array} \to \begin{array}{c} \mathcal{W}^{s-(m,0)}(\mathbb{R}^{n-1}, H^{\sigma-(m,0)}(\mathbb{R}_+))^{N_1} \\ \oplus \\ \mathcal{W}^{s-(m,0)}(\mathbb{R}^{n-1}, \mathbb{C})^{N_2} \end{array}$$

for all $\sigma \in \mathbb{R}^2$ with $\sigma_1 > d - 1/2$ and all $s \in \mathbb{R}^2$.

For *unweighted* Sobolev spaces the following statement is an immediate corollary of the above theorem and 1.17(b). The statement for weighted spaces can be obtained by the commutator technique in the proof of 1.19.

THEOREM 5.4. *Let $\sigma \in \mathbb{R}^2$, and $A \in \mathcal{B}^{m,d}(\overline{\mathbb{R}}^n_+)$. Then*

$$A: \begin{array}{c} H^\sigma(\mathbb{R}^n_+)^{M_1} \\ \oplus \\ H^\sigma(\mathbb{R}^{n-1})^{M_2} \end{array} \to \begin{array}{c} H^{\sigma-(m,0)}(\mathbb{R}^n_+)^{N_1} \\ \oplus \\ H^{\sigma-(m,0)}(\mathbb{R}^{n-1})^{N_2} \end{array}$$

is bounded for $\sigma_1 > d - 1/2$.

COROLLARY 5.5. *The operators in 5.4 have continuous restrictions*

$$A : \mathcal{S}(\mathbb{R}^n_+)^{M_1} \oplus \mathcal{S}(\mathbb{R}^{n-1})^{M_2} \to \mathcal{S}(\mathbb{R}^n_+)^{N_1} \oplus \mathcal{S}(\mathbb{R}^{n-1})^{N_2}.$$

5.6 THEOREM: COMPOSITIONS. *Consider two operators*

$$A_1 : \mathcal{S}(\mathbb{R}^n_+)^{M_1} \oplus \mathcal{S}(\mathbb{R}^{n-1})^{M_2} \to \mathcal{S}(\mathbb{R}^n_+)^{N_1} \oplus \mathcal{S}(\mathbb{R}^{n-1})^{N_2}$$

and

$$A_2 : \mathcal{S}(\mathbb{R}^n_+)^{L_1} \oplus \mathcal{S}(\mathbb{R}^{n-1})^{L_2} \to \mathcal{S}(\mathbb{R}^n_+)^{M_1} \oplus \mathcal{S}(\mathbb{R}^{n-1})^{M_2}$$

with $A_l \in \mathcal{B}^{m_l,d_l}(\overline{\mathbb{R}}^n_+)$ for $l = 1, 2$.

(a) *The composition* A_1A_2 *is an element of* $\mathcal{B}^{m_1+m_2,d}(\overline{\mathbb{R}}^n_+)$, $d=\max\{m_2+d_1,d_2\}$.
(b) *The composition is a regularizing operator whenever one of the factors is, and it is a generalized singular Green operator whenever this is the case for* A_1 *or* A_2. *In particular,* $\mathcal{B}^{0,0}(\overline{\mathbb{R}}^n_+)$ *is an algebra, and* $\mathcal{B}^{-\infty,0}(\overline{\mathbb{R}}^n_+)$ *as well as the generalized singular Green operators are ideals.*
(c) A_1A_2 *is a classical operator if both* A_1 *and* A_2 *are classical.*

Proof. (a) Write $A_l = \mathrm{op}\, a_l$ with

$$a_l = \begin{pmatrix} \mathrm{op}^+_{x_n} p_l + \sum_{j=0}^{d_l} g_{lj}\partial_+^j & k_l \\ \sum_{j=0}^{d_l} t_{lj}\partial_+^j & s_l \end{pmatrix}.$$

Next choose left symbols for $p_1, g_{1j}, t_{1j}, k_1, s_1$, and choose right symbols for p_2, g_{2j}, t_{2j}, k_2, s_2 with respect to the x'-action.

$$\mathrm{op}\, b_L(y,\eta)\mathrm{op}\, b_R(y',\eta) = \mathrm{op}\,(b_L(y,\eta)b_R(\tilde y,\eta)).$$

So we get the desired result from Theorems 1.14 and 4.1. ◁

5.7 Theorem: Adjoints. *Let* $A\in\mathcal{B}^{m,0}(\overline{\mathbb{R}}^n_+), m\in\mathbb{Z}, m\le 0$. *Then the formal adjoint* A^* *of*

$$A:\mathcal{S}(\mathbb{R}^n_+)^{M_1}\oplus\mathcal{S}(\mathbb{R}^{n-1})^{M_2}\to\mathcal{S}(\mathbb{R}^n_+)^{N_1}\oplus\mathcal{S}(\mathbb{R}^{n-1})^{N_2}$$

with respect to the inner products in $L^2(\mathbb{R}^n_+)^{M_1}\oplus L^2(\mathbb{R}^{n-1})^{M_2}$ *and* $L^2(\mathbb{R}^n_+)^{N_1}\oplus L^2(\mathbb{R}^{n-1})^{N_2}$, *respectively, is an element of* $\mathcal{B}^{m,0}(\overline{\mathbb{R}}^n_+)$. *If* $A=\mathrm{op}\, a$ *with* a *as in* (3.4), *then*

$$A^* = \begin{pmatrix} \mathrm{op}^+_{x_n} p^* + g^* & k^* \\ t^* & s^* \end{pmatrix}.$$

Here, $g^*(x',y',\xi') = g(y',x',\xi')^*$, $t^*(x',y',\xi') = k(y',x',\xi')^*$, $k^*(x',y',\xi') = t(y',x',\xi')^*$, *and* $s^*(x',y',\xi') = s(y',x',\xi')$.

Note that the assertion is no longer true if d or m are positive.

Proof. Let $(\mathrm{op}\, p)^* = \mathrm{op}\, p^*$ be the formal adjoint of $\mathrm{op}\, p$ with respect to the inner product in $L^2(\mathbb{R}^n)$. Then the formal adjoint $(\mathrm{op}^+ p)^*$ of $\mathrm{op}^+ p$ with respect to the inner product in $L^2(\mathbb{R}^n_+)$ is given by

$$(\mathrm{op}^+ p)^* = \mathrm{op}^+ p^*.$$

Indeed, $\mathrm{op}^+ p^*$ is bounded on $L^2(\mathbb{R}_+)$, since $m\le 0$, and

$$\begin{aligned}(r^+[\mathrm{op}\, p^*]e^+u, v) &= ([\mathrm{op}\, p^*]e^+u, e^+v) = (e^+u, [\mathrm{op}\, p]e^+v)\\ &= (u, r^+[\mathrm{op}\, p]e^+v).\end{aligned}$$

For the symbol g we first apply 3.7 to see that the pointwise adjoint $g(x',y',\xi')^*$ defines a singular Green symbol. Next we deduce from 1.15 that the formal adjoint of $\mathrm{op}\, g$ is $\mathrm{op}\, g^*$ with $g^*(x',y',\xi') = g(y',x',\xi')$. Similarly we see from Theorem 3.9 that the pointwise adjoint of a trace symbol of type zero is a potential symbol and vice versa; we then employ 1.15.

For s the assertion follows from the standard pseudodifferential calculus or, alternatively, also from 1.15. ◁

5.8 Theorem: Asymptotic expansions. *Let* $N\in\mathbb{N}$. *Adopting the notation of* 5.6 *and* 5.7 *we can find boundary symbols* c_N *and* d_N *such that*

$$A_1A_2 - \sum_{|\alpha|=0}^{N-1}\frac{1}{\alpha!}\mathrm{op}\,(\partial^\alpha_{\xi'}a_1(x',\xi')D^\alpha_{x'}a_2(x',\xi')) = \mathrm{op}\, c_N$$

and

$$A^* - \sum_{|\alpha|=0}^{N-1}\frac{1}{\alpha!}\mathrm{op}\,(\partial^\alpha_{\xi'}D^\alpha_{x'}a(x',\xi')^*) = \mathrm{op}\, d_N.$$

The symbols c_N *are of order* m_1+m_2-N *and type* $\max\{m_2+d_1,d_2\}$, *while* d_N *is of order* $m-N$ *and type zero.*

Proof. This follows from Theorem 1.14. ◁

THEOREM 5.9. *Let G be a singular Green operator of order m and type d, T a trace operator of order m and type d, and let K be a potential operator of order m. Furthermore let $\varepsilon > 0$ and $\varphi \in C_b^\infty(\overline{\mathbb{R}}^n_+)$ vanish for $x_n < \varepsilon$. Denote, just for the moment, by M_φ the operator of multiplication by φ. Then*

(a) *GM_φ is a regularizing singular Green operator of type zero.*
(b) *$M_\varphi G$ is a regularizing singular Green operator of type d.*
(c) *TM_φ is a regularizing trace operator of type zero.*
(d) *$M_\varphi K$ is a regularizing potential Green operator.*

Proof. (a) It is no loss of generality to assume the symbols to be scalar. Let $\varphi_N(x', x_n) = \varphi(x', x_n)x_n^{-N}$. Then $\varphi_N \in C_b^\infty(\mathbb{R}^n_+)$, and

$$GM_\varphi = GM_{\varphi_N} M_{x_n^N}.$$

Since $M_{x_n^N}$ is an element of $S^{-N}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H_0^\sigma(\overline{\mathbb{R}}_+), H_0^{\sigma-(0,N)}(\overline{\mathbb{R}}_+))$ for each $\sigma \in \mathbb{R}^2$, the assertion follows easily.

The arguments for (b), (c), and (d) are analogous. ◁

We now introduce ellipticity and parametrices. As usual, *ellipticity* is the property of the symbol (and the associated operator) that enables us to find an inverse up to an error of order -1, while a *parametrix* is an inverse modulo regularizing elements. Both notions are easily seen to be equivalent as a consequence of the symbolic calculus.

We shall finally state a simpler condition for ellipticity, namely the ellipticity of the interior pseudodifferential symbol together with the uniform invertibility of the boundary symbol with the pseudodifferential symbol p replaced by the x_n-independent symbol p_0, where $p_0(x', \xi) = p(x', 0, \xi)$.

DEFINITION 5.10. Let $A \in \mathcal{B}^{m,d}(\overline{\mathbb{R}}^n_+)$ with $m \in \mathbb{Z}$, $d \leq m_+$, and $A = \operatorname{op} a$.

(a) We call A *elliptic*, if there exist boundary symbols b_l, b_r of order $-m$ and type $(-m)_+$ such that both boundary symbols $b_l a - I$ and $ab_r - I$ have order -1. Also the boundary symbol a will then be called elliptic.

  We have $b_l = b_l - b_l ab_r + b_l ab_r - b_r + b_r$, so b_l and b_r differ by a symbol of order $m-1$ only, and we may choose $b_l = b_r$. Note that also b_l and b_r are elliptic. The composition rules imply that the type of $b_l a - I$ is m_+, while that of $ab_r - I$ is $(-m)_+$.

(b) We shall say that the operator $B \in \mathcal{B}^{-m,(-m)_+}(\overline{\mathbb{R}}^n_+)$ is a *parametrix* for A if $BA - I$ and $AB - I$ both are regularizing. Their types are m_+ and $(-m)_+$, respectively.

THEOREM 5.11. *Let $A \in \mathcal{B}^{m,d}(\overline{\mathbb{R}}^n_+), d \leq m_+$. Then A has a parametrix if and only if it is elliptic.*

Proof. If $A = \operatorname{op} a$ has a parametrix $B = \operatorname{op} b$, then 5.8 implies that $ba - I$ and $ab - I$ both are boundary symbols of order -1.

Conversely suppose we find b_l, b_r such that $b_l a - I$ and $ab_r - I$ have order -1. Let $B_l = \operatorname{op} b_l$ and $B_r = \operatorname{op} b_r$. We find boundary symbols r_l and r_r of order -1 and type m_+ and $(-m)_+$ such that

$$B_l A - I = \operatorname{op} r_l, \quad AB_r - I = \operatorname{op} r_r.$$

Indeed, this is immediate from the assumption combined with 5.8. Next choose

$$\tilde{b}_l \sim \sum_{j=0}^\infty (-r_l)^{\#j} \# b_l \quad \text{and} \quad \tilde{b}_r \sim b_r \# \sum_{j=0}^\infty (-r_r)^{\#j}.$$

In this notation, the '#' indicates that we pick a boundary symbol for the corresponding composition, e.g., $\operatorname{op}[(-r_l)^{\#j}] = [\operatorname{op}(-r)]^j$. The type of $r_l^{\#j}$ is m_+ for all j while that of $r_r^{\#j}$ is $(-m)_+$. Finally we carry out the asymptotic summation modulo regularizing boundary symbols. We let $\tilde{B}_l = \operatorname{op} \tilde{b}_l, \tilde{B}_r = \operatorname{op} \tilde{b}_r$, and conclude that

$$\tilde{B}_l A - I = R_l \in \mathcal{B}^{-\infty, m_+}(\overline{\mathbb{R}}^n_+) \quad \text{and} \quad A\tilde{B}_r - I = R_r \in \mathcal{B}^{-\infty,(-m)_+}(\overline{\mathbb{R}}^n_+).$$

Hence

$$\tilde{B}_l \equiv \tilde{B}_l(A\tilde{B}_r - R_r) = \tilde{B}_r + R_l\tilde{B}_r - \tilde{B}_l R_r \equiv \tilde{B}_r$$

modulo $\mathcal{B}^{-\infty,(-m)_+}(\overline{\mathbb{R}}^n_+)$. So both $\tilde{B}_l$ and $\tilde{B}_r$ furnish a parametrix. ◁

With a little more work, we find the following simpler ellipticity criterion which we shall not prove here:

THEOREM 5.12. *Let*

$$a = \begin{pmatrix} \mathrm{op}^+_{x_n} p + g & k \\ t & s \end{pmatrix}$$

be a boundary symbol of order m and type $d \le m_+$. Let $p_0(x',\xi) = p(x',0,\xi)$ and

$$a^0 = \begin{pmatrix} \mathrm{op}^+_{x_n} p_0 + g & k \\ t & s \end{pmatrix}.$$

Then a is elliptic if and only if

1. *$p \in S^m_{tr}(\mathbb{R}^n \times \mathbb{R}^n)$ is an elliptic $N \times N$ matrix-valued symbol on $\overline{\mathbb{R}}^n_+$;*
2. *$a^0(x',\xi') : H^{(m,0)}(\mathbb{R}_+)^N \oplus \mathbb{C}^{M'} \to L^2(\mathbb{R}_+)^N \oplus \mathbb{C}^M$ is an isomorphism for all x' and ξ' with $|\xi'| \ge 1$, satisfying $a^0(x',\xi')^{-1} = O(\langle \xi' \rangle^{-m})$.*

By referring to p as 'elliptic of order m on $\overline{\mathbb{R}}^n_+$' we mean that there is a $q \in S^{-m}_{tr}(\mathbb{R}^n \times \mathbb{R}^n)$ such that $pq - I$ and $qp - I$ coincide with a symbol in $S^{-1}_{tr}(\mathbb{R}^n \times \mathbb{R}^n)$ on $\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n$.

The key ingredient in the proof is the fact that, according to Theorem 5.9, the non-pseudodifferential entries of a are regularizing outside a neighborhood of the boundary.

5.13 OUTLOOK. It remains to establish the invariance of the symbol classes under suitable changes of coordinates in order to introduce Boutet de Monvel's calculus on manifolds with boundary. I shall omit this part, since there are no new aspects to be developed, and all techniques are fairly standard.

The results on compositions, adjoints, and mapping properties carry over to the case of operators acting on sections of vector bundles. In particular, if A is an operator of order m and type d then

$$A : \begin{matrix} H^s(X,E_1) \\ \oplus \\ H^s(\partial X, F_1) \end{matrix} \to \begin{matrix} H^{s-m}(X,E_2) \\ \oplus \\ H^{s-m}(\partial X, F_2) \end{matrix} \tag{5.2}$$

is bounded for each $s > -1/2$, so that we have the mapping property (0.1). In view of the well-known imbedding properties of Soboloev spaces on compact manifolds we see immediately that ellipticity implies the Fredholm property of A in (5.2).

A few more features which might be of interest can be found in the paper [**16**] which also deals with weighted symbols on certain non-compact manifolds. In case $E_1 = E_2 = E$ and $F_1 = F_2 = F$, for example, the algebra $\mathcal{B}^{0,0}(X)$ of all operators of order and type zero is a Fréchet sub-algebra of the Banach algebra of all bounded operators on the Hilbert space $L^2(X,E) \oplus L^2(\partial X, F)$. It is closed under holomorphic functional calculus in several complex variables, hence a pre-C^*-algebra and moreover a Ψ^*-algebra in the sense of Gramsch [**2**].

Using the fact that there exist order-reducing operators in the calculus, cf. Theorem 2.10, one can show that the calculus is closed under inversion: If the operator A in (5.2) is bijective, then its inverse is again an element of the calculus. Moreover, ellipticity is not only sufficient but also necessary for the Fredholm property of $A : H^m(X,E_1) \oplus H^m(\partial X, F_1) \to H^0(X,E_2) \oplus H^0(\partial X, F_2)$.

References

[1] L. Boutet de Monvel. Boundary problems for pseudo-differential operators. *Acta Math.*, 126:11 – 51, 1971.

[2] B. Gramsch. Relative Inversion in der Störungstheorie von Operatoren und Ψ-Algebren. *Math. Annalen* 269:27 - 71, 1984.

[3] G. Grubb. *Functional Calculus for Boundary Value Problems.* Number 65 in Progress in Mathematics. Birkhäuser, Boston, Basel, second edition, 1996.

[4] G. Grubb and L. Hörmander. The transmission property. *Math. Scand.*, 67:273 – 289, 1990.

[5] L. Hörmander. *The Analysis of Linear Partial Differential Operators*, volume I - IV. Springer-Verlag, Berlin, New York, Tokyo, 1983 - 1985.

[6] H. Kumano-go. *Pseudo-Differential Operators.* The MIT Press, Cambridge, MA, and London, 1981.

[7] S. Rempel and B.-W. Schulze. *Index Theory of Elliptic Boundary Problems.* Akademie-Verlag, Berlin, 1982.

[8] E. Schrohe. *Fréchet algebras of pseudodifferential operators and boundary value problems.* Birkhäuser Verlag. In preparation.

[9] E. Schrohe. A characterization of the singular Green operators in Boutet de Monvel's calculus via wedge Sobolev spaces. *Comm. Partial Differential Equations*, 19:677–699, 1994.

[10] E. Schrohe. A characterization of the uniform transmission property for pseudodifferential operators. In M. Demuth et al., editors, *Pseudodifferential Operators and Mathematical Physics, Advances in Partial Differential Equations* **1**, pages 210 – 234. Akademie Verlag, Berlin, 1994.

[11] E. Schrohe and B.-W. Schulze. Boundary value problems in Boutet de Monvel's calculus for manifolds with conical singularities I. In M. Demuth et al., editors, *Pseudodifferential Operators and Mathematical Physics, Advances in Partial Differential Equations* **1**, pages 97 – 209. Akademie Verlag, Berlin, 1994.

[12] E. Schrohe and B.-W. Schulze. Boundary value problems in Boutet de Monvel's calculus for manifolds with conical singularities II. In M. Demuth et al., editors, *Boundary Value Problems, Schrödinger Operators, Deformation Quantization, Advances in Partial Differential Equations* **2**, pages 70 – 205. Akademie Verlag, Berlin, 1995.

[13] E. Schrohe and B.-W. Schulze. Mellin operators in a pseudodifferential calculus for boundary value problems on manifolds with edges. *Operator Theory: Advances Applications* 102:255–285, 1998.

[14] E. Schrohe and B.-W. Schulze. A symbol algebra for pseudodifferential boundary value problems for manifolds with edges. In M. Demuth et al., editors, *Differential Equations, Asymptotic Analysis, and Mathematical Physics*, volume 100 of *Math. Research*, pages 292–324, Berlin, 1997. Akademie Verlag.

[15] E. Schrohe and B.-W. Schulze. Mellin and Green symbols for boundary value problems on manifolds with edges. *Integral Equations Operator Theory*, 34:339–363, 1999.

[16] E. Schrohe. Frèchet algebra techniques for boundary value problems on noncompact manifolds: Fredholm criteria and functional calculus via spectral invariance. *Math. Nachr.*, 199:145 – 185, 1999.

[17] B.-W. Schulze. *Pseudodifferential Operators on Manifolds with Singularities.* North-Holland, Amsterdam, 1991.

[18] J. Seiler. Continuity of edge and corner pseudo-differential operators. *Math. Nachr.*, 205:163–182, 1999.

[19] M.I. Višik and G.I. Eskin. Normally solvable problems for elliptic systems in equations of convolution. *Mat. Sb.*, 3:303–332, 1967.

Institut für Mathematik, Universität Potsdam, 14415 Potsdam, Germany

E-mail address: schrohe@math.uni-potsdam.de