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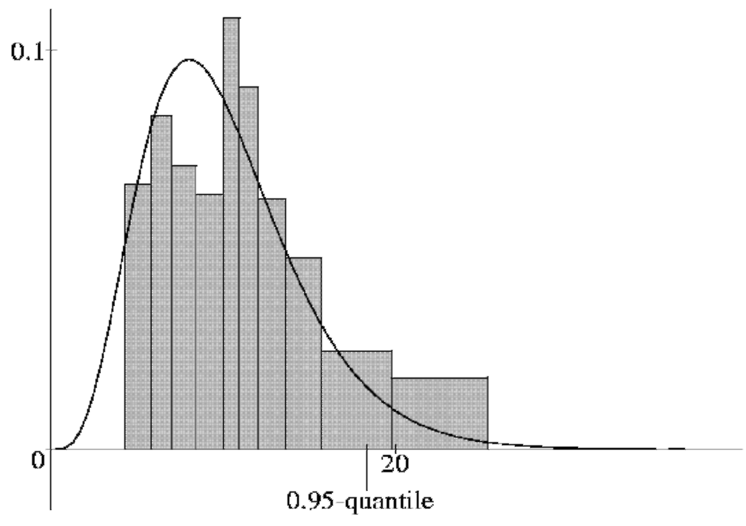

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Complete monotone coupling for Markov processes

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Abstract

We formalize and analyze the notions of monotonicity and complete monotonicity for Markov Chains in continuous-time, taking values in a finite partially ordered set. Similarly to what happens in discrete-time, the two notions are not equivalent. However, we show that there are partially ordered sets for which monotonicity and complete monotonicity coincide in continuous-time but not in discrete-time.

Keywords:

Markov processes, coupling, partial ordering, monotonicity conditions, monotone random dynamical system representation

2000 MSC: 60J05, 60E15, 90C90

Contents

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1 Introduction

The use of Markov chains in simulation has raised a number of questions concerning qualitative and quantitative features of random processes, in particular in connection with mixing properties. Among the features that are useful in the analysis of effectiveness of Markov Chain Monte Carlo (MCMC) algorithms, a relevant role is played by monotonicity. Two notions of monotonicity have been proposed, for Markov chains taking values in a *partially ordered set* S (*poset* from now on). To avoid measurability issues, that are not relevant for our purposes, we shall always assume S to be finite. Moreover, all Markov chains are implicitly assumed to be time-homogeneous.

Definition 1.1. A Markov chain (η_t), $t \in \mathbb{R}^+$ or $t \in \mathbb{Z}^+$, on the poset S, with transition probabilities $P_t(x,y) := P(\eta_t = x|\eta_0 = y)$, is said to be *monotone* if for each pair $y, z \in S$ with $y \leq z$ there exists a Markov chain $(X_t(y,z))$ on $S \times S$ such that

i.) $X_0(y,z) = (y,z)$ a.s.;

ii.) each component $(X_t^i(y,z))$, $i = 1,2$ is a Markov chain on S with transition probabilities $P_t(x,y)$;

iii.) $X_t^1(y,z) \leq X_t^2(y,z)$ for all $t \geq 0$ a.s.

There are various equivalent formulations of monotonicity. For instance, defining the *transition operator* $T_t f(y) := \sum_{x\in S} f(x)P_t(x,y)$, then the chain is monotone if and only if T_t maps increasing functions into increasing functions (see [KKO77] – resp. [KK78] – for this generalization to discrete-time –resp. continuous-time – dynamics of the well-known Strassen's result presented in [Str65]; see also [Lin99]). This characterization can be turned (see Section 2) into a simple algorithm for checking monotonicity (also called attractivity) of Markov chains in terms of the element of the transition matrix (in discrete-time) or in terms of the infinitesimal generator (in continuous-time).

References on the relations between this monotonicity concept and the existence and construction of a monotone coupling for some family of processes in continuous-time, such as diffusions or interacting particle systems, are [Che91, FYMP97, FF97, LS98] and references therein. As consequence of this monotonicity, let us recall the following result [Har77]: a monotone continuous-time Markov chain on a poset is preserving measures with positive correlations if and only if only jumps between comparable states occur.

For various purposes, including simulation, a stronger notion of monotonicity has been introduced.

Definition 1.2. A Markov chain (η_t), $t \in \mathbb{R}^+$ or $t \in \mathbb{Z}^+$, on the poset S, with transition probabilities $P_t(x,y) := P(\eta_t = x|\eta_0 = y)$, is said to be *completely monotone* if there exists a Markov chain $(\xi_t(\cdot))$ on S^S such that

i.) $\xi_0(y) = y$ a.s.;

ii.) for every fixed $z \in S$, the process $(\xi_t(z))$ is a Markov chain with transition probabilities $P_t(x,y)$;

iii.) if $y \leq z$, then for every $t \geq 0$ we have $\xi_t(y) \leq \xi_t(z)$ a.s.

In other words, complete monotonicity means that we can simultaneously couple, in an order preserving way, all processes leaving any possible initial state. This property becomes relevant when one aims at sampling from the stationary measure of a Markov chain using the Propp

& Wilson algorithm (see [PW96]) that we briefly summarize in Subsection 2.4.

We recall that in [FM01] a more general definition of *stochastic monotonicity* and *realizable monotonicity* for a *system of probability measures* is considered:

Definition 1.3. Let $\mathcal{A}$ and $\mathcal{S}$ be two partially ordered sets. A system of probability measures on $\mathcal{S}$ indexed in $\mathcal{A}$, $(P_\alpha : \alpha \in \mathcal{A})$ (which, in what follows, we shall denote by the pair $(\mathcal{A}, \mathcal{S})$) is said to be stochastically monotone if $P_\alpha \preccurlyeq P_\beta$ whenever $\alpha \leq_{\mathcal{A}} \beta$, *i.e.* $P_\alpha f \leq_{\mathbb{R}} P_\beta f$ for every increasing function $f : \mathcal{S} \to \mathbb{R}$ whenever $\alpha \leq_{\mathcal{A}} \beta$.
The system $(P_\alpha : \alpha \in \mathcal{A})$ is said to be realizably monotone if there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a system of $\mathcal{S}$-valued random variables $(X_\alpha : \alpha \in \mathcal{A})$ such that $X_\alpha \leq_{\mathcal{S}} X_\beta$ whenever $\alpha \leq_{\mathcal{A}} \beta$.

With this terminology, a discrete-time Markov Chain on S with transition matrix $P = (P(x, y))_{x,y \in S}$ is monotone if the system of probability measures on S, $(P(x, \cdot) : x \in S)$, *i.e.* with respect to the pair $(\mathcal{S}, \mathcal{S})$, is stochastically monotone. The Chain is completely monotone if $(P(x, \cdot) : x \in S)$ is realizably monotone.

If the transition probabilities, or the infinitesimal generator, are given, no simple rule for checking complete monotonicity is known. Since, obviously, complete monotonicity implies monotonicity, a natural question is to determine for which posets the converse is true. This problem has been completely solved in [FM01] for discrete-time Markov chain (see Theorem 1.1 here). More precisely, it has been solved as a particular case of the more general problem of *equivalence* between stochastic and realizable monotonicity for systems of probability measures w.r.t. $(\mathcal{A}, \mathcal{S})$. In what follows, when such equivalence holds we shall say that *monotonicity equivalence holds.*

In the case of discrete-time Markov Chains the following result holds.

Theorem 1.1. *Every monotone Markov chain in the poset S is also completely monotone if and only if S is* acyclic, *i.e. there is no loop $x_0,\ x_1, \ldots, x_n,\ x_{n+1} = x_0$ such that, for $i = 0, 1, \ldots, n$*

i.) $x_i \neq x_{i+1}$*;*

ii.) either $x_i < x_{i+1}$ *or* $x_i > x_{i+1}$*;*

iii.) $x_i \leq y \leq x_{i+1}$ *or* $x_i \geq y \geq x_{i+1}$ *implies* $y = x_i$ *or* $y = x_{i+1}$*.*

The nontrivial proof of the above statement consists of three steps.

1. For each *minimal* cyclic poset an example is found of a monotone Markov chain which is not completely monotone.

2. Given a general cyclic poset, a monotone but not completely monotone Markov chain is constructed by "lifting" one of the examples in step 1.

3. A proof by induction on the cardinality of the poset shows that, in an acyclic poset, monotone Markov chains are completely monotone.

Note there is no contradiction with the fact that on some cyclic posets, such as product spaces, order preserving coupling may exist for *some* monotone Markovian dynamics. See for instance [Lou05].

Let us now give the following definition:

Definition 1.4. The *Hasse diagram*[1] of a poset is an oriented graph. Its vertices are the elements of the poset. There is an edge from x to y if $x \preceq y$ and $x \preceq z \preceq y$ implies $z = x$ or $z = y$ (it is said that y covers x).

By convention, y is drawn above x in the planar representation of the diagram in order to mean there is an edge from x to y. With this convention of reading the diagram from bottom to the top there is no need to direct any edges. See for example figures 1, 2, 3.

Our aim in this paper is to deal with monotonicity equivalence in continuous-time for *regular* Markov chains, *i.e.* Markov chains possessing an infinitesimal generator (or, equivalently, jumping a.s. finitely many times in any bounded time interval). We have not been able to provide a complete link between discrete and continuous-time. It turns out that if in a poset S monotonicity implies complete monotonicity in discrete-time, then the same holds true in continuous-time (see Subsection 2.3). The converse is not true, however; in the two four-points cyclic posets (the diamond and the bowtie, following the terminology in [FM01], see Figure 1 for their Hasse diagram) equivalence between monotonicity and complete monotonicity holds in continuous but not in discrete-time. There are, however, five-points posets in which equivalence fails in continuous-time as well (see Figure 2).

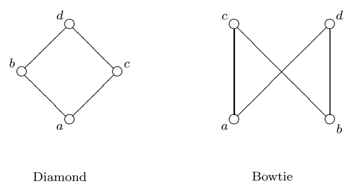

Figure 1: The four-points posets, for which there is no equivalence between the two notions of monotonicity in discrete-time.

Remark, if the poset is a totally ordered one, the two notions of monotonicity coincide using the usual monotone coupling for attractive Markov kernels (see Section 6 in [FM01]).

In this paper we do not achieve the goal of characterizing all posets for which equivalence holds. Via a computer-assisted (but exact) method we find a complete list of five and six point posets for which we state equivalence fails. Moreover we prove that in each poset containing one of the former as sub-poset, equivalence fails as well (this does not follow in a trivial way).

In Section 2 we give some preliminary notions, whose aim is to put the complete monotonicity problem in continuous-time on a firm basis. In Section 3 we perform a systematic investigation of the monotonicity equivalence for five and six points posets, using the software *cdd+* (see [cdd]). Extensions to larger posets are presented in Section 4. Some further considerations and conjectures are contained in Section 5.

[1](or order diagram). According to Birkhoff, the idea of Hasse diagram was introduced and used by Vogt in 1895 before Hasse, 1942.

2 Preliminaries

2.1 Completely monotone coupling

Let $(S, <)$ be a finite poset, and $L = (L_{x,y})_{x,y\in S}$ be the infinitesimal generator of a regular continuous-time Markov chain on S. Assume the chain is completely monotone and let $(\xi_t(\cdot))_{t\geq 0}$ be an order-preserving coupling. Since the original Markov chain is regular, also $(\xi_t(\cdot))_{t\geq 0}$ must be regular: if not, for at least one $z \in S$, $\xi_t(z)$ would be not regular, which is not possible by condition ii. in Definition 1.2. Thus $(\xi_t(\cdot))_{t\geq 0}$ admits an infinitesimal generator $\mathcal{L} = (\mathcal{L}_{f,g})_{f,g\in S^S}$. Let now be $\varphi : S \to \mathbb{R}$, $z \in S$, and define $F_{\varphi,z} : S^S \to \mathbb{R}$ by $F_{\varphi,z}(f) := \varphi(f(z))$. The fact *ii.)* that each component of the chain generated by $\mathcal{L}$ is a Markov chain with generator L, is equivalent to the following statement: for all choices of φ, z, and all $f \in S^S$,

$$\mathcal{L}F_{\varphi,z}(f) = L\varphi(f(z)). \tag{1}$$

By an elementary algebraic manipulation of (1), we can re-express (1) with the following statement: for every $z, x, y \in S$, $x \neq y$ and every $f \in S^S$ such that $f(z) = x$, we have

$$L_{x,y} = \sum_{g\in S^S : g(z)=y} \mathcal{L}_{f,g}. \tag{2}$$

Now let *id* denote the identity on S, and define $\Lambda(f) := \mathcal{L}_{id,f}$. Note that since the Markov chain generated by $\mathcal{L}$ is order preserving, necessarily $\mathcal{L}_{id,f} > 0 \;\Rightarrow\; f \in \mathcal{M}$, where $\mathcal{M}$ is the set of increasing functions from S to S. Note that, by (2), for $x \neq y$,

$$L_{x,y} = \sum_{f\in\mathcal{M} : f(x)=y} \Lambda(f). \tag{3}$$

Identity (3) characterizes the generators of completely monotone Markov chains, in the sense specified by the following proposition.

Proposition 2.1. *A generator L is the generator of a complete monotone Markov chain if and only if there exists $\Lambda : \mathcal{M} \to \mathbb{R}^+$ such that (3) holds.*

Proof. One direction has been proved above. For the converse, suppose (3) holds for some $\Lambda : \mathcal{M} \to \mathbb{R}^+$. For $f, g \in S^S$, define

$$\mathcal{L}_{f,g} := \sum_{h\in\mathcal{M} : g=h\circ f} \Lambda(h),$$

where the sum over the empty set is meant to be zero. It is easily checked that the Markov chain generated by $\mathcal{L}$ is order preserving. Moreover, using (3), a simple computation shows that (2) holds, completing the proof. □

2.2 The cones of monotone and completely monotone generators

Let $S_2 := S \times S \setminus \{(x, x) : x \in S\}$. An infinitesimal generator is a matrix $L = (L_{x,y})_{x,y\in S}$ whose non-diagonal elements are nonnegative, while the terms in the diagonal are given by $L_{x,x} = -\sum_{y\neq x} L_{x,y}$. Thus L may be identified with an element of the cone $(\mathbb{R}^+)^{S_2}$. A subset $\Gamma \subseteq S$ is said to be an *up-set* if

$$x \in \Gamma \text{ and } x \leq y \;\Rightarrow\; y \in \Gamma.$$

The following proposition (see e.g. [Mas87] for the proof) gives a characterization of the generators of monotone Markov chains.

Proposition 2.2. *An element $L \in (\mathbb{R}^+)^{S_2}$ is the generator of a monotone Markov chain if and only if for every up-set Γ the following conditions hold:*

$$\bullet\ x \leq y \notin \Gamma \ \Rightarrow\ \sum_{z \in \Gamma} L_{x,z} \leq \sum_{z \in \Gamma} L_{y,z};$$

$$\bullet\ x \geq y \in \Gamma \ \Rightarrow\ \sum_{z \notin \Gamma} L_{x,z} \leq \sum_{z \notin \Gamma} L_{y,z}.$$

Remark 1. In what follows we shall often call "monotone generator" the infinitesimal generator of a monotone Markov Chain. Given a generator L on S, $x \in S$ and $\Gamma \subset S$, we shall use the symbol $L_{x,\Gamma}$ to denote $\sum_{z \in \Gamma} L_{x,z}$. Moreover, in order to check monotonicity of a generator we shall use the following condition, which is equivalent to the one given in Proposition 2.2:

i) for every up-set Γ, $\quad x \leq y \notin \Gamma \ \Rightarrow\ L_{x,\Gamma} \leq L_{y,\Gamma}$;

ii) for every down-set Γ, $\quad y \geq x \notin \Gamma \ \Rightarrow\ L_{x,\Gamma} \geq L_{y,\Gamma}$

where a down-set is a subset $\Gamma \subset S$ such that $x \in \Gamma$ and $y \leq x \ \Rightarrow\ y \in \Gamma$.

Let $V := \mathbb{R}^{S_2}$ be provided with the natural Euclidean scalar product $\langle \cdot, \cdot \rangle$. For given $x, y \in S$, Γ up-set, let $W^{\Gamma,x,y} \in \mathbb{R}^{S_2}$ be defined by

$$W^{\Gamma,x,y}_{v,z} = \begin{cases} 1 & \text{for } \begin{cases} x \leq y \notin \Gamma,\ v = y,\ z \in \Gamma \\ \text{or } x \geq y \in \Gamma,\ v = y,\ z \notin \Gamma; \end{cases} \\ -1 & \text{for } \begin{cases} x \leq y \notin \Gamma,\ v = x,\ z \in \Gamma \\ \text{or } x \geq y \in \Gamma,\ v = x,\ z \notin \Gamma; \end{cases} \\ 0 & \text{in all other cases.} \end{cases}$$

Proposition 2.2 can be restated as follows: $L \in (\mathbb{R}^+)^{S_2}$ generates a monotone Markov chain if and only if

$$\langle L, W^{\Gamma,x,y} \rangle \geq 0 \ \text{ for every } \Gamma, x, y. \tag{4}$$

In other words, denoting by $\mathcal{G}_{mon}$ the set of monotone generators, the elements of $\mathcal{G}_{mon}$ are characterized by the inequalities

$$\begin{aligned} \langle L, W^{\Gamma,x,y} \rangle &\geq 0 \\ L_{x,y} &\geq 0 \end{aligned}$$

for every Γ, x, y. In other words we are giving $\mathcal{G}_{mon}$ through the *rays* of its *polar cone* (see [Zie95]), *i.e.* the family of vectors $\{W^{\Gamma,x,y}, \delta^{x,y} : (x,y) \in S_2, \ ,\Gamma \text{ up-set in } S\}$, where

$$\delta^{x,y}_{v,z} = \begin{cases} 1 & \text{if } (v,z) = (x,y) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.1 can also be restated to characterize the set of generators of completely monotone Markov chains as a cone in V. For $f \in \mathcal{M}$, let $\mathbb{I}_f \in (\mathbb{R}^+)^{S_2}$ be defined by

$$(\mathbb{I}_f)_{x,y} = \begin{cases} 1 & \text{if } f(x) = y \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Then the set $\mathcal{G}_{c.mon}$ of generators of completely monotone Markov chains is the cone given by linear combination with nonnegative coefficients of the vectors $\mathbb{I}_f$, *i.e.*

$$L = \sum_{f \in \mathcal{M}} \Lambda_f \mathbb{I}_f, \tag{6}$$

with $\Lambda_f \geq 0$. Note that, for each $f \in \mathcal{M}$, Γ up-set, $x, y \in S$, we have

$$\langle \mathbb{I}_f, W^{\Gamma,x,y} \rangle \geq 0,$$

i.e. we recover the inclusion $\mathcal{G}_{c.mon} \subseteq \mathcal{G}_{mon}$. Our aim is to determine for which posets the converse inclusion holds true.

In the next sections, when we want to emphasize the dependence on S, we shall use the notations $\mathcal{G}_{mon}(S)$ and $\mathcal{G}_{c.mon}(S)$.

2.3 Comparisons with the discrete-time case

In this subsection we establish a comparison with the discrete-time case. The proof of proposition 2.4 below relies on analogous representations in terms of cones for discrete-time transition matrix.

Consider a Markov chain with transition matrix P. We recall the following fact.

Proposition 2.3. *$P = (P_{x,y})_{x,y \in S}$ is the transition matrix of a monotone Markov Chain if and only if, for every up-set Γ, the map $x \mapsto \sum_{y \in \Gamma} P_{x,y}$ is increasing.*

Remark 2. Proposition 2.3 above, derives from the following general statement (see [FM01] for more details): a system of probability measures $(P_\alpha : \alpha \in \mathcal{A})$ on $\mathcal{S}$ is stochastically monotone if and only if, for every up-set $\Gamma \subset \mathcal{S}$ the map $\alpha \mapsto P_\alpha(\Gamma)$ is increasing.

The discrete-time version of the argument in Subsection 2.1 shows that P is the transition matrix of a completely monotone Markov chain if and only if there exists a probability Π on $\mathcal{M}$ such that

$$P = \sum_{f \in \mathcal{M}} \Pi_f \mathbb{I}_f, \tag{7}$$

where, with a slight abuse of notation, $\mathbb{I}_f$ given by (5) is now seen as a square matrix, with the diagonal terms too.

Proposition 2.4. *Suppose that in the poset S monotonicity and complete monotonicity are equivalent notions for discrete-time Markov chains. Then the equivalence holds for S-valued continuous-time Markov Chains as well.*

Proof. Let $L = (L_{x,y})_{x,y \in S}$ be the generator of a monotone continuous-time Markov chain. It is easily checked that, for $\epsilon > 0$ sufficiently small, $P := I + \epsilon L$ is the transition matrix of a monotone Markov Chain. Thus, by assumption, (7) holds for a suitable Π. In particular, for each $x, y \in S$ with $x \neq y$,

$$L_{x,y} = \sum_{f \in \mathcal{M}} \frac{\Pi_f}{\epsilon} (\mathbb{I}_f)_{x,y},$$

which is just (6) with $L_f := \frac{\Pi_f}{\epsilon}$. This completes the proof. □

2.4 The Propp & Wilson algorithm revisited

It is well known (see e.g. [Arn98, Bré99, FG, Min04]) that regular finite state Markov processes can be realized as *Random Dynamical Systems with independent increments* (shortly RDSI). To set up notations, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $(\theta_t)_{t \in \mathbb{R}}$ be a one-parameter group (*i.e.*

$\theta_{t+s} = \theta_t \circ \theta_s$, $\theta_0 = id =$ identity map) of $\mathbb{P}$-preserving maps from Ω to Ω, such that the map $(t, \omega) \mapsto \theta_t \omega$ is jointly measurable in t and ω. We still denote by S a finite set, representing the state space.

Definition 2.1. A *Random dynamical system* is a measurable map $\varphi : \mathbb{R}^+ \times \Omega \to S^S$ such that

$$\varphi(0, \omega) \equiv id \tag{8}$$

$$\varphi(t+s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega) \tag{9}$$

for every $s, t \geq 0$ and $\omega \in \Omega$.

Note that, for t fixed, $\varphi(t, \omega)$ can be seen as a S^S-valued random variable.

Definition 2.2. A Random Dynamical System φ is said to have *independent increments* if for each $0 \leq t_0 < t_1 < \cdots < t_n$ the random variables

$$\varphi(t_1 - t_0, \theta_{t_0}\omega),\ \varphi(t_2 - t_1, \theta_{t_1}\omega),\ \ldots,\ \varphi(t_n - t_{n-1}, \theta_{t_{n-1}}\omega)$$

are independent.

In what follows we consider the σ-fields,

$$\mathcal{F}_t := \sigma\{\varphi(s, \omega) :\ 0 \leq s \leq t\}$$

$$\mathcal{F}^- := \sigma\{\varphi(s, \theta_{-t}\omega) :\ 0 \leq s \leq t\}.$$

Proposition 2.5. *(See [Arn98]) For a RDSI, the random process $(\varphi(t, \omega))_{t \geq 0}$ is a S^S-valued, time-homogeneous, $\mathcal{F}_t$-Markov process. Moreover, for any S-valued, $\mathcal{F}^-$-measurable random variable X (in particular any $X = x \in S$ constant), the process $(\varphi(t, \omega)(X))_{t \geq 0}$ is a S-valued, time-homogeneous Markov process, whose transition probabilities do not depend on X.*

The processes $(\varphi(t, \omega)(x))_{t \geq 0}$, with $x \in S$ are called the *one point motions* of φ. When the one point motions are Markov process with infinitesimal generator L, we say that φ *realizes* L. It is nothing else than a complete coupling: copies of the chain starting from every initial conditions are realized on the same probability space.

If we are given a generator L of a Markov chain, it is not difficult to realize it by a RDSI. Let $\mathcal{H}$ be the set whose elements are the locally finite subsets of $\mathbb{R}$. An element $\eta \in \mathcal{H}$ can be identified with the σ-finite point measure $\sum_{t \in \eta} \delta_t$; the topology in $\mathcal{H}$ is the one induced by vague convergence, and the associated Borel σ-field provides a measurable structure on $\mathcal{H}$. Set $\Omega' := \mathcal{H}^{S_2}$. For $\omega = (\omega_{xy})_{(x,y) \in S_2} \in \Omega'$, we define

$$\theta_t \omega := (\theta_t \omega_{xy})_{(x,y) \in S_2},$$

where $\tau \in \theta_t \omega_{xy} \iff \tau - t \in \omega_{xy}$. Consider now a probability $\mathbb{P}$ on Ω' with the following properties:

i.) for $(x, y) \in S_2$, ω_{xy} is, under $\mathbb{P}$, a Poisson process of intensity $L_{x,y}$;

ii.) for $x \in S$ fixed, the point processes $(\omega_{xy})_{y \neq x}$ are independent under $\mathbb{P}$;

iii.) for every I, J *disjoint* intervals in $\mathbb{R}$, the two families of random variables

$$\{|\omega_{xy} \cap I| : (x, y) \in S_2\} \text{ and } \{|\omega_{xy} \cap J| : (x, y) \in S_2\}$$

are independent under $\mathbb{P}$;

iv.) for every $t \in \mathbb{R}$, $\mathbb{P}$ is θ_t-invariant.

It is easy to exhibit one example of a $\mathbb{P}$ satisfying i.-iv.: if $\mathbb{P}_{xy}$ is the law of a Poisson process of intensity $L_{x,y}$, then we can let $\mathbb{P}$ be the product measure

$$\mathbb{P} := \otimes_{(x,y)\in S_2} \mathbb{P}_{xy}. \tag{10}$$

We now construct the map φ *pointwise* in ω. Define

$$\Omega = \{\omega \in \Omega' : \omega_{xy} \cap \omega_{xz} = \emptyset \text{ for every } (x,y),(x,z) \in S_2,\ y \neq z\}.$$

By condition ii. on $\mathbb{P}$, $\mathbb{P}(\Omega) = 1$, and clearly $\theta_t\Omega = \Omega$ for every $t \in \mathbb{R}$. For every $\omega \in \Omega$ the following construction is well posed:

- set $\varphi(0,\omega) = id$ for every $\omega \in \Omega$;
- we run the time in the forward direction. Whenever we meet $t \in \bigcup_{(x,y)\in S_2} \omega_{xy}$ we use the following updating rule:

$$\text{if } \varphi(t^-,\omega)(x) = z \text{ and } t \in \omega_{zy} \text{ then } \varphi(t,\omega)(x) := y.$$

Proposition 2.6. *(see [Min04]). The map φ constructed above is a RDSI, and its one-point motions are Markov chains with generator L.*

Condition i.- iii. leave a lot of freedom on the choice of $\mathbb{P}$. The choice corresponding to (10) is the simplest, but may be quite inefficient when used for simulations.

The following Theorem is just a version of the Propp & Wilson algorithm for perfect simulation ([PW96]).

Theorem 2.1. *Let L be the generator of an irreducible Markov chain on S, π be its stationary distribution, and let φ be a RDSI whose one-point motions are Markov chains with generator L. Define*

$$T := \inf\{t > 0 : \ \varphi(t, \theta_{-t}\omega) = \textit{constant}\}$$

where, by convention, $\inf \emptyset := +\infty$. Assume $T < +\infty$ $\mathbb{P}$-almost surely. Then for each $x \in S$ the random variable $\varphi(T, \theta_{-T}\omega)(x)$ has distribution π.

Proof. Set $X(\omega) := \varphi(T, \theta_{-T}\omega)(x)$ that, by definition of T, is independent of $x \in S$. For $h > 0$

$$\varphi(T+h, \theta_{-T-h}\omega)(x) = \varphi(T, \theta_{-T}\omega)(\varphi(h, \theta_{-T-h}\omega)(x)) = X(\omega).$$

Thus, since we are assuming $T < +\infty$ a.s., we have

$$X(\omega) = \lim_{t\to+\infty} \varphi(t, \theta_{-t}\omega)(x).$$

In particular, this last formula shows that X is $\mathcal{F}^-$-measurable. Denote by ρ the distribution of X, *i.e.* $\rho(x) := P(X = x)$. We have:

$$\begin{aligned}\varphi(h,\omega)(X(\omega)) = \lim_{t\to+\infty} \varphi(h,\omega)(\varphi(t,\theta_{-t}\omega)(x)) &= \lim_{t\to+\infty} \varphi(t+h, \theta_{-t}\omega)(x) \\ &= \lim_{t\to+\infty} \varphi(t+h, \theta_{-t-h}(\theta_h\omega))(x) = X(\theta_h\omega).\end{aligned}$$

By Proposition 2.5, $\varphi(h,\omega)(X(\omega))$ has distribution ρe^{hL}. But, since θ_h is $\mathbb{P}$-preserving, $X(\theta_h\omega)$ has distribution ρ. Thus $\rho = \rho e^{hL}$, *i.e.* ρ is stationary, and therefore $\rho = \pi$.

∎

The condition $\mathbb{P}(T < +\infty) = 1$ depends on the particular choice of the RDSI, and it is not granted by the irreducibility of L. For example, consider $S = \{0, 1\}$ and L given by $L_{0,1} = L_{1,0} = 1$. We can realize this chain by letting ω_{01} be a Poisson process of intensity 1, and $\omega_{10} = \omega_{01}$. Clearly $\varphi(t, \theta_{-t}\omega)(0) \neq \varphi(t, \theta_{-t}\omega)(1)$ for every $t > 0$, so $T \equiv +\infty$. On the other hand, and this holds in general, if we make the choice of $\mathbb{P}$ given by (10), it is not hard to see that $(\varphi(t, \omega))_{t \geq 0}$ is an irreducible Markov chain on S^S. By recurrence, any constant function is hit with probability 1 in finite time, so $T < +\infty$ a.s.

In order to implement the Propp & Wilson algorithm, in principle one needs to run a Markov chain on S^S, which may be computationally unachievable. Some additional structure can make the algorithm much more effective.

Definition 2.3. Let S be a poset. A RDS on S is said to be *monotone* if for every $t \geq 0$ and $\omega \in Q$

$$\varphi(t, \omega) \in \mathcal{M}.$$

Suppose a Markov chain is realized by a monotone RDSI. Then

$$\varphi(t, \theta_{-t}\omega) \text{ is constant} \quad \Longleftrightarrow \quad \varphi(t, \theta_{-t}\omega) \text{ is constant on } A,$$

where A is the set of points in S that are either maximal or minimal. Thus to implement the Propp & Wilson algorithm one needs to run Markov chains starting from every point of A, that may be much smaller than S. Moreover, the following result holds.

Theorem 2.2. *Let S be a* connected *poset (i.e. every two points in S are connected by a path of comparable points), and φ be a monotone RDSI whose one-point motions are irreducible Markov chains. Then $\mathbb{P}(T < +\infty) = 1$.*

Proof. Let x be minimal in S, and $y_1 > x$. By irreducibility, given $s > 0$

$$\mathbb{P}(\varphi(s, \omega)(y_1) = x) > 0.$$

Since $\varphi(s, \omega) \in \mathcal{M}$ and $\mathcal{M}$ is finite, there exists $f_1 \in \mathcal{M}$ with $f_1(y_1) = x$ and

$$\mathbb{P}(\varphi(s, \omega) = f_1) > 0.$$

Note that, necessarily, $f_1(x) = x$, so f_1 is not bijective. Set $S_1 := f_1(S)$. Note that S_1, with the order induced by S, is connected, being the image of a connected posed under an increasing function. Clearly $x \in S_1$ is still minimal in S_1, and $|S_1| < |S|$. Unless $|S_1| = 1$, the same argument can be repeated. Take $y_2 \in S_1$, $y_2 > x$, and $f_2 \in \mathcal{M}$ such that $f_2(y_2) = x$ and $P(\varphi(s, \omega) = f_2) > 0$. Again $f_2(x) = x$, so that $|S_2| := |f(S_1)| < |S_1|$. After a finite number of similar steps, we obtain a finite family $f_1, f_2, \ldots, f_n \in \mathcal{M}$ such that $\mathbb{P}(\varphi(s, \omega) = f_i) > 0$ and

$$f_n \circ f_{n-1} \circ \cdots \circ f_1 \equiv x \text{ is constant.} \tag{11}$$

Now, for $k = 1, 2, \ldots, n$, consider the events

$$\{\varphi(s, \theta_{-ks}\omega) = f_{n-k+1}\}.$$

Since θ_t is $\mathbb{P}$-preserving, all these events have nonzero probability and, by independence of the increments, they are all independent. Thus, by (11)

$$0 < \mathbb{P}\left(\bigcap_{k=1}^{n} \{\varphi(s, \theta_{-ks}\omega) = f_{n-k+1}\}\right) \leq \mathbb{P}(\varphi(ns, \theta_{-ns}\omega) = \text{ const.}).$$

Now, let $t := ns$ and for $N \geq 1$ consider the events $\{\varphi(t, \theta_{-Nt}) = \text{ const.}\}$. Since they are independent and with the same nonzero probability,

$$\mathbb{P}\left(\bigcup_N \{\varphi(t, \theta_{-Nt}) = \text{ const.}\}\right) = 1.$$

Observing that $\varphi(t, \theta_{-Nt}) = \text{ const.}$ implies $\varphi(Nt, \theta_{-Nt}) = \text{ const.}$, we obtain

$$\mathbb{P}\left(\bigcup_N \{\varphi(Nt, \theta_{-Nt}) = \text{ const.}\}\right) = 1.$$

from which $\mathbb{P}(T < +\infty) = 1$ follows.

□

We conclude this section by remarking that a Markov chain with generator L can be realized by a monotone RDSI if and only if it is completely monotone. Indeed, if such RDSI exists, then $(\varphi(t, \cdot))_{t \geq 0}$ is a Markov chain on S^S for which the conditions in Definition 1.2 are satisfied. Conversely, once we have the representation in (3), a RDSI with the desired properties is obtained as follows. For $f \in \mathcal{M}$, let $\mathbb{P}_f$ be the law of a Poisson process on $\mathbb{R}$ with intensity $\Lambda(f)$, and, on the appropriate product space whose elements are denoted by $\omega = (\omega_f)_{f \in \mathcal{M}}$, we define $\mathbb{P} := \otimes_{f \in \mathcal{M}} \mathbb{P}_f$. The map φ is constructed pointwise in ω via the updating rule: if $t \in \omega_f$ and $\varphi(t^-, \omega) = g$ then $\varphi(t, \omega) = f \circ g$.

3 Extremal generators of monotone Markov chains: the monotonicity equivalence for "small" posets

As seen in subsection 2.2, equivalence between complete monotonicity and monotonicity of any Markov Chain on a poset S is equivalent to

$$\mathcal{G}_{c.mon} = \mathcal{G}_{mon}. \tag{12}$$

In this section we study monotonicity equivalence for posets with small cardinality.
First note that the cases $\sharp S = 2$, $\sharp S = 3$ are obvious: in these cases S is acyclic. According to Theorem 1.1, there is equivalence for discrete-time Markov chains and using the result of Proposition 2.4 the equivalence holds for continuous-time Markov chains as well.

In order to further investigate the equality (12) we developed computer computations. The cone $\mathcal{G}_{mon}$ is defined as intersection of half spaces in (4) (so called *H-representation*). The cone $\mathcal{G}_{c.mon}$ is defined by its extremal rays in (6) (so called *V-representation*). The software *cdd+* (see [cdd]) is able to compute exactly one representation given the other one. This is a *C++* implementation for convex polyhedron of the Double Description Method (see for instance [FP96]). Finding the extremal rays of the cone $\mathcal{G}_{mon}$ and the (minimal) set of inequalities defining the cone $\mathcal{G}_{c.mon}$, the inclusion $\mathcal{G}_{mon} \subseteq \mathcal{G}_{c.mon}$ can be easily checked.

We operated by first using the software *GAP* (see [GAP]) in order to

i.) find the up-sets Γ related to the poset S, the vectors $W^{\Gamma,x,y} \in \mathbb{R}^{S_2}$ and then identify the H-representation of $\mathcal{G}_{mon}$;

ii.) compute all the increasing functions $f \in \mathcal{M}$, identify the vectors $\mathbb{I}_f \in (\mathbb{R}^+)^{S_2}$ and then find the V-representation of $\mathcal{G}_{c.mon}$.

We then use the software *cdd+* to produce the other representations of the cones, and the software *Scilab* (see [Sci]) to test if $\mathcal{G}_{mon} \subseteq \mathcal{G}_{c.mon}$.

The difficulty in applying this method to posets with high cardinality is mainly due to the combinatorial complexity of the step (ii) and to the computational time needed to *cdd+* to obtain the dual representation of the cone. Rather than to $\sharp S$, this time is related to the number of facets of the cones, which comes from the partial order structure. It should also be remarked that a systematic analysis, made by generating all posets with a given cardinality, is not doable for "moderate" cardinality. For instance, the number of different unlabeled posets structure – up to an order preserving isomorphism, not necessarily connected – for a given set of cardinality 16 is $\sim 4,48.10^{15}$. It was stated in 2002, see [BM02]. For a set of cardinality 17, the number of unlabeled posets is till now unknown. For a set of cardinality 4, resp. 5, 6, 7, the number of posets is respectively 16, 63, 318, 2045. See [pos] for the list.

Nevertheless, we were able to completely study the cases when $\sharp S \leq 6$. For $\sharp S > 6$, the result of Proposition 4.1 in the next section gives the answer for some posets.

For $\sharp S = 4$ the two relevant poset-structure are the diamond and the bowtie. Their Hasse-Diagram are given by the Figure 1. For those two posets, the algorithm above ensures that $\mathcal{G}_{mon} = \mathcal{G}_{c.mon}$ holds. Note that this result is known to be *false* in discrete-time, see for instance examples 1.1 and 4.5 in [FM01].

Then, we studied all five-points posets which are not linearly totally ordered. For some of these posets (see Figure 2 below), we found extremal rays $L = (L_{x,y})_{(x,y)\in S_2}$ of $\mathcal{G}_{mon}$ which are not in $\mathcal{G}_{c.mon}$. One example for each poset will be given below.

In what follows, a *symmetry* of a poset S is a bijective map from S to S which is either order preserving or order reversing.

Proposition 3.1. *The only posets S with $\sharp S \leq 5$ such that (12) does not hold are, up to symmetries, those whose Hasse-Diagrams are presented in Figure 2.*

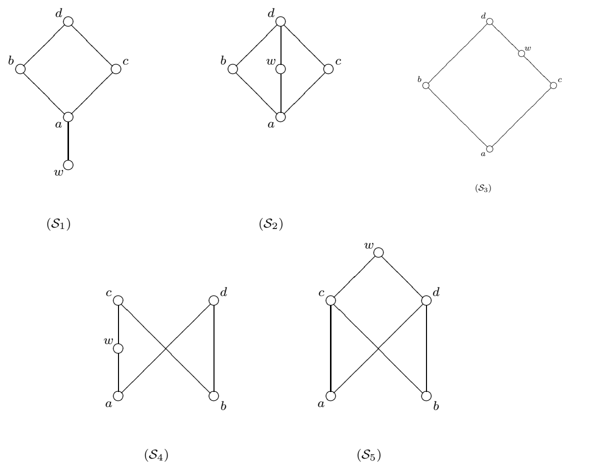

Figure 2: The five-points posets, for which there is no equivalence between the two notions of monotonicity in continuous-time.

As mentioned above, Proposition 3.1 has been obtained by exact computer-aided computations. For completeness, for each one of the posets in Figure 2, we give explicitly a generator in $\mathcal{G}_{mon} \setminus \mathcal{G}_{c.mon}$. We recall that, by Proposition 2.1, a generator L is completely monotone if and only if, for some $\Lambda : \mathcal{M} \to \mathbb{R}^+$, we have $x, y \in S, \quad x \neq y \Rightarrow L_{x,y} = \sum_{f \in \mathcal{M}: f(x)=y} \Lambda(f)$. In particular $L_{x,y} = 0 \Leftrightarrow \Lambda(f) = 0$ for all $f \in \mathcal{M}$ such that $f(x) = y$. Given a function Λ as above, we shall use the abbreviate notation $\mathcal{M}_{x \mapsto y}$ for the set $\{f \in \mathcal{M} : \Lambda(f) > 0 \text{ and } f(x) = y\}$.

Example 1. For $S = S_1$ there is only one extremal ray L of $\mathcal{G}_{mon}$ which is not in $\mathcal{G}_{c.mon}$. It is given by $L_{d,c} = L_{d,b} = L_{b,w} = L_{c,w} = L_{a,w} = 1$, and $L_{x,y} = 0$ for each other pair $x, y \in S_1$ with $x \neq y$. This generator is clearly monotone (the conditions of Proposition 2.2 are easily verified), but it is not completely monotone: indeed, if Proposition 2.1 holds, we have $L_{d,b} = \sum_{f \in \mathcal{M}_{d \mapsto b}} \Lambda(f) = 1$ and $L_{d,c} = \sum_{f \in \mathcal{M}_{d \mapsto c}} \Lambda(f) = 1$. Note that $\mathcal{M}_{d \mapsto b} \cap \mathcal{M}_{d \mapsto c} = \emptyset$. Moreover, for each $f \in \mathcal{M}_{d \mapsto b}$, since $c < d$ and $\Lambda(f) > 0$, by monotonicity of f and the fact that $L_{c,a} = L_{c,b} = 0$, we have necessarily $f(c) = w$ and then $f(a) = w$ to maintain the ordering, *i.e.*, $f \in \mathcal{M}_{a \mapsto w}$. Analogously, $\mathcal{M}_{d \mapsto c} \subset \mathcal{M}_{a \mapsto w}$, then $1 = L_{a,w} = \sum_{f \in \mathcal{M}_{a \mapsto w}} \Lambda(f) \geq \sum_{f \in \mathcal{M}_{d \mapsto b} \sqcup \mathcal{M}_{d \mapsto c}} \Lambda(f) = 2$, so we obtain a contradiction.

Example 2. For $S = S_2$ a generator $L \in \mathcal{G}_{mon} \setminus \mathcal{G}_{c.mon}$ is given by $L_{a,c} = L_{w,c} = L_{d,c} = L_{b,d} = L_{b,a} = 1$ and $L_{x,y} = 0$ for each other pair $x, y \in S_2$ with $x \neq y$. According to Proposition 2.2 it is monotone. Assume it is completely monotone. If f is an increasing function on S with $f(d) = f(a) = c$, since $a < b < d$ we have $f(b) = c$. But $L_b = 0$, then necessarily $\Lambda(f) = 0$. This means that $\mathcal{M}_{d \mapsto c} \cap \mathcal{M}_{a \mapsto c} = \emptyset$. Moreover, $(\mathcal{M}_{d \mapsto c} \sqcup \mathcal{M}_{a \mapsto c}) \subset \mathcal{M}_{w \mapsto c}$ which gives the contradiction $1 = L_{w,c} \geq L_{d,c} + L_{a,c} = 2$.

Example 3. For $S = S_3$, consider the monotone generator given by $L_{a,w} = L_{b,w} = L_{c,d} = L_{w,d} = L_{d,w} = 1$ and $L_{x,y} = 0$ for each other pair $x, y \in S_3$ with $x \neq y$ and suppose it is completely monotone. If $f \in \mathcal{M}$ and $f(d) = f(a) = w$, by monotonicity we have $f(c) = w$ and then, since $L_{c,w} = 0$, we have $\Lambda(f) = 0$. Then $\mathcal{M}_{d \mapsto w} \cap \mathcal{M}_{a \mapsto w} = \emptyset$ and by $(\mathcal{M}_{d \mapsto w} \sqcup \mathcal{M}_{a \mapsto w}) \subset \mathcal{M}_{b \mapsto w}$, it follows that $L_{b,w} \geq 2$, which gives a contradiction.

Example 4. For $S = S_4$ we take the monotone generator given by $L_{a,b} = L_{w,b} = L_{d,b} = L_{b,d} = L_{c,d} = 1$ and $L_{x,y} = 0$ for each other pair $x, y \in S_4$ with $x \neq y$. It is clear that, if L was completely monotone, we should have $\mathcal{M}_{d \mapsto b} \cap \mathcal{M}_{b \mapsto d} = \emptyset$ and the inclusions $\mathcal{M}_{d \mapsto b} \subset \mathcal{M}_{a \mapsto b} \subset \mathcal{M}_{w \mapsto b}$ and $\mathcal{M}_{b \mapsto d} \subset \mathcal{M}_{c \mapsto d} \subset \mathcal{M}_{w \mapsto b}$: but it is not possible, since in that case we should have $L_{w,b} \geq 2$.

Example 5. For $S = S_5$ consider the monotone generator given by $L_{c,a} = L_{d,a} = L_{b,a} = L_{w,c} = L_{w,d} = 1$ and $L_{x,y} = 0$ for each other pair $x, y \in S_5$ with $x \neq y$. If $L \in \mathcal{G}_{c.mon}$, we have $\mathcal{M}_{w \mapsto d} \cap \mathcal{M}_{w \mapsto c} = \emptyset$, $\mathcal{M}_{w \mapsto d} \subset \mathcal{M}_{c \mapsto a} \subset \mathcal{M}_{b \mapsto a}$ and $\mathcal{M}_{w \mapsto c} \subset \mathcal{M}_{d \mapsto a} \subset \mathcal{M}_{b \mapsto a}$, then we obtain the contradiction $1 = L_{b,a} \geq 2$.

For the sake of completeness, for the posets considered in examples 1,..., 5 we give the number of extremal rays generating the cone $\mathcal{G}_{c.mon}$, resp. $\mathcal{G}_{mon}$ in $\mathbb{R}^{20}$, see Table 1.

We now recall the following definition:

Definition 3.1. A subset S' of S is said to be an *induced sub-poset* if for all $x, y \in S'$, $x \leq y$ in S' is equivalent to $x \leq y$ in S.

For $\sharp S = 6$ we shall see in the next section that, if S has one of the 5-points posets above as an induced sub-poset, then there is no equivalence between monotonicity and complete monotonicity. However, there are 6-points posets for which there is no equivalence and such that we have equivalence for each one of their sub-posets.

Poset	$\mathcal{G}_{c.mon}$	$\mathcal{G}_{mon}$
S_1	40	41
S_2	41	47
S_3	40	42
S_4	46	50
S_5	49	53

Table 1: $\sharp S = 5$: minimal number of extremal rays generating the cone $\mathcal{G}_{c.mon}$, resp. $\mathcal{G}_{mon}$, in $\mathbb{R}^{20}$

Proposition 3.2. *The only posets S with $\sharp S = 6$ such that (12) does not hold are, up to symmetries,*

- *those having one of the posets in Proposition 3.1 as subposet;*
- *those whose Hasse-Diagrams are presented in Figure 3.*

Following the terminology of [FM01], the first poset in Figure 3 is a *double diamond*, while the other three are *3-crowns*.

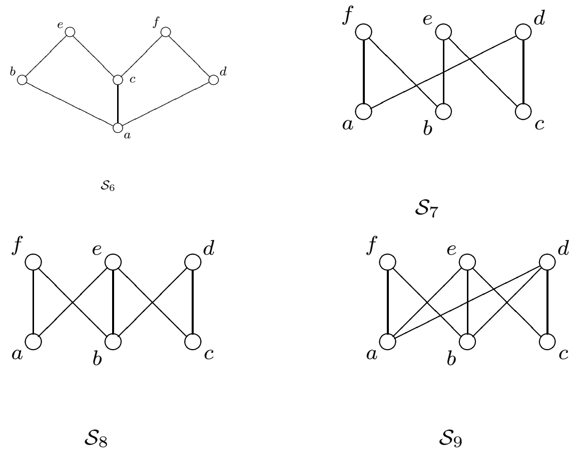

Figure 3: The double diamond S_6 and 3-crowns: S_7, S_8, S_9.

In the next examples we shall see generators of $\mathcal{G}_{mon} \setminus \mathcal{G}_{c.mon}$ for posets S_6, S_7, S_8 and S_9. For the posets considered in examples 6,... 9 we give the number of extremal rays generating the cone $\mathcal{G}_{c.mon}$, resp. $\mathcal{G}_{mon}$ in $\mathbb{R}^{30}$ in Table 2.

Poset	$\mathcal{G}_{c.mon}$	$\mathcal{G}_{mon}$
S_6	126	421
S_7	684	914
S_8	134	312
S_9	84	132

Table 2: $\sharp S = 6$: number of extremal rays generating the cone $\mathcal{G}_{c.mon}$, resp. $\mathcal{G}_{mon}$ in $\mathbb{R}^{30}$

Example 6. Consider the monotone generator L on S_6 defined as follows: $L_{a,c} = L_{d,c} = L_{c,b} = L_{b,e} = L_{f,e} = 1$, and $L_{x,y} = 0$ for each other pair $x, y \in S_6$ with $x \neq y$. If L was completely monotone we should have $\mathcal{M}_{a\mapsto c} \cap \mathcal{M}_{c\mapsto b} = \emptyset$, $\mathcal{M}_{a\mapsto c} \subset \mathcal{M}_{d\mapsto c}$ and $\mathcal{M}_{c\mapsto b} \subset \mathcal{M}_{f\mapsto e} \subset \mathcal{M}_{d\mapsto c}$; but this would give the contradiction $1 = L_{d,c} \geq 2$.

Example 7. For $S = S_7$ we take $L \in \mathcal{G}_{mon}$ with $L_{a,c} = L_{b,c} = L_{f,c} = L_{d,c} = 1$, $L_{c,d} = L_{e,d} = 1$ and $L_{x,y} = 0$ for each other pair x, y with $x \neq y$. Suppose L is completely monotone; then $\mathcal{M}_{d\mapsto c} \cap \mathcal{M}_{c\mapsto d} = \emptyset$, $\mathcal{M}_{d\mapsto c} \subset \mathcal{M}_{a\mapsto c}$ and $\mathcal{M}_{c\mapsto d} \subset \mathcal{M}_{e\mapsto d} \subset \mathcal{M}_{b\mapsto c} \subset \mathcal{M}_{f\mapsto c} \subset \mathcal{M}_{a\mapsto c}$. But in this case we have $1 = L_{a,c} \geq L_{d,c} + L_{c,d} = 2$, then $L \notin \mathcal{G}_{c.mon}$.

Example 8. For $S = S_8$ we consider the same generator of Example 7; L is clearly monotone, but complete monotonicity of L would imply $\mathcal{M}_{d\mapsto c} \cap \mathcal{M}_{c\mapsto d} = \emptyset$, $\mathcal{M}_{d\mapsto c} \subset \mathcal{M}_{b\mapsto c} \subset \mathcal{M}_{f\mapsto c} \subset \mathcal{M}_{a\mapsto c}$ and $\mathcal{M}_{c\mapsto d} \subset \mathcal{M}_{e\mapsto d} \subset \mathcal{M}_{a\mapsto c}$, then $L_{ac} \geq 2$, which is not the case.

Example 9. For $S = S_9$ let L be defined by $L_{a,c} = L_{b,c} = L_{e,c} = 1$, $L_{b,e} = L_{f,e} = L_{d,e} = 1$, $L_{a,d} = L_{f,d} = L_{e,d} = 1$ and $L_{x,y} = 0$ for each other pair x, y with $x \neq y$. L is a monotone generator. Suppose $L \in \mathcal{G}_{c.mon}$. Then, inequalities $a < e$ and $a < f$ imply respectively $\mathcal{M}_{a\mapsto d} \cap \mathcal{M}_{e\mapsto c} = \emptyset$ and $\mathcal{M}_{a\mapsto d} \subset \mathcal{M}_{f\mapsto d}$. Note that we have also $\mathcal{M}_{e\mapsto c} \subset \mathcal{M}_{f\mapsto d}$: indeed $\mathcal{M}_{e\mapsto c} \subset (\mathcal{M}_{f\mapsto e} \cup \mathcal{M}_{f\mapsto d})$ and, since $\mathcal{M}_{b\mapsto e} \subset \mathcal{M}_{f\mapsto e}$ and $L_{b,e} = L_{f,e} = 1$ we have necessarily $\mathcal{M}_{b\mapsto e} = \mathcal{M}_{f\mapsto e}$ and so $\mathcal{M}_{e\mapsto c} \cap \mathcal{M}_{b\mapsto e} = \mathcal{M}_{e\mapsto c} \cap \mathcal{M}_{f\mapsto e} = \emptyset$. Therefore we obtain the contradiction $L_{f,d} \geq 2$.

Remark 3. We recall that, in discrete-time, equivalence does not hold if the graph corresponding to the Hasse diagram of the poset has a subgraph which is a cycle (in the graph-theoretic sense). So, a look at the figures above could suggest that in continuous-time, a sufficient condition for the failure of (12) is the presence of *two* cycles in the Hasse diagram. The poset in Figure 4 (the complete 3-crown) gives a counterexample: it has more than two cycles, but for this set we have *equivalence* between the two concepts of monotonicity. In fact, more generally, if we have a poset $S = \{a_1, \dots, a_n, b_1, \dots, b_m\}$ with $a_i < b_j$ for $i = 1, \dots, n$, $j = 1, \dots, m$, we can show that every monotone generator on S is completely monotone. We use an argument analogous to the one used in section 5 of [FM01]. Let L be a monotone generator on S and consider the poset $\tilde{S} = \{a_1, \dots, a_n, c, b_1, \dots, b_m\}$ with $a_i < c < b_j$ for $i = 1, \dots, n$, $j = 1, \dots, m$. This poset admits S as induced sub-poset and it is acyclic. Now, we take a generator $\tilde{L}$ on $\tilde{S}$ defined as follows: $\tilde{L}_{x,y} = L_{x,y}$ if $x, y \in S$, $\tilde{L}_{x,c} = 0$ for each $x \in S$, $\tilde{L}_{c,a_i} = \sum_{h=1}^{m} L_{b_h,a_i}$ for $i = 1, \dots, n$, $\tilde{L}_{c,b_j} = \sum_{k=1}^{n} L_{a_k,b_j}$ for $j = 1, \dots, m$. It is not hard to check that this generator is monotone, and its restriction to S is given by L. Since $\tilde{S}$ is acyclic, then $\tilde{L}$ is completely monotone. But a monotone realization of $\tilde{L}$ gives a monotone realization of L too. Therefore, L is completely monotone.

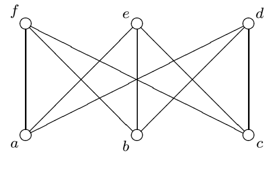

Figure 4: Complete 3-crown

4 Extensions to larger posets

In this section we prove some sufficient conditions on a poset S such that $\mathcal{G}_{mon}(S) \neq \mathcal{G}_{c.mon}(S)$. The general argument we use is analogous to the one used in the discrete-time case (see [FM01]): we take a generator L in $\mathcal{G}_{mon}(S) \setminus \mathcal{G}_{c.mon}(S)$ for a "small" poset S and we define a "monotone extension of L to a larger poset S'", *i.e.*, a generator $L' \in \mathcal{G}_{mon}(S')$, where S' is a poset having S as induced sub-poset, such that $L'_{xy} = L_{xy}$ for all $x, y \in S$. If this construction is possible, L' is not completely monotone too. This is a consequence of the following Lemma

Lemma 4.1. *Let S be an induced sub-poset of a given poset S' and let L be a monotone generator on S which has a monotone extension L' to S'. Then $L' \in \mathcal{G}_{c.mon}(S') \Rightarrow L \in \mathcal{G}_{c.mon}(S)$.*

Proof. We denote by $\mathcal{M}'$ and $\mathcal{M}$ the sets of increasing functions on S' and S respectively. Assume $L' \in \mathcal{G}_{c.mon}(S')$. Then, by Proposition 2.1 there exists $\bar{\Lambda} : \mathcal{M}' \to \mathbb{R}^+$ such that (3) holds for L'. Let us define $\Lambda : \mathcal{M} \to \mathbb{R}^+$ by

$$\Lambda(f) = \sum_{f' \in \mathcal{M}', f'|_S = f} \bar{\Lambda}(f')$$

for each $f \in \mathcal{M}$. Now, since $L'_{x,y} = L_{x,y}$ for all $x, y \in S$, necessarily $L'_{x,z} = 0$ for all $x \in S, z \in S' \setminus S$. Then, by condition (3), for every $f' \in \mathcal{M}'$ with $\bar{\Lambda}(f') > 0$ we have $f'(S) \subset S$; moreover, S is an *induced* sub-poset of S', then we have also $f' |_S \in \mathcal{M}$. Therefore, for $x, y \in S$, $x \neq y$

$$\begin{aligned}
\sum_{f \in \mathcal{M}: f(x) = y} \Lambda(f) &= \sum_{f \in \mathcal{M}: f(x) = y} \Big(\sum_{f' \in \mathcal{M}', f'|_S = f} \bar{\Lambda}(f') \Big) \\
&= \sum_{\substack{f' \in \mathcal{M}', f'(x) = y \\ f'|_S \in \mathcal{M}}} \bar{\Lambda}(f') \\
&= \sum_{f' \in \mathcal{M}'_{x \mapsto y}} \bar{\Lambda}(f') = L'_{x,y} = L_{x,y},
\end{aligned}$$

then the proof is complete using for L Proposition 2.1. □

As the Example below shows, Lemma 4.1 is false if S is a (not necessarily induced) sub-poset of S', *i.e.* a subset of S' such that, for all $x, y \in S$, $x \leq y$ in S implies $x \leq y$ in S'.

Example 10. let S be the poset S_8 of Figure 3: it is a (not induced) sub-poset of the complete crown of Figure 3, which we denote by S'. Now let us consider the generator L on S defined by $L_{f,e} = L_{b,e} = L_{d,e} = 1$, $L_{e,d} = L_{a,c} = L_{c,a} = 1$ and $L_{x,y} = 0$ for each other pair $x, y \in S$ with $x \neq y$. It is easy to check that L is monotone as a generator both on S and on S'. Moreover, $L \notin \mathcal{G}_{c.mon}(S)$. Indeed, if L was completely monotone and $\mathcal{M}$ denotes the set of increasing functions on S we should have $\mathcal{M}_{e \mapsto d} \subset \mathcal{M}_{a \mapsto c} \subset \mathcal{M}_{f \mapsto e}$, $\mathcal{M}_{b \mapsto e} \subset \mathcal{M}_{f \mapsto e}$ and $\mathcal{M}_{b \mapsto e} \cap \mathcal{M}_{e \mapsto d} = \emptyset$ which implies the contradiction $L_{f,e} \geq 2$. On the other hand, as a generator on the complete crown, L is a monotone extension of itself and, by Remark 3, $L \in \mathcal{G}_{c.mon}(S')$.

It must be stressed that the method of monotone extension of generators to larger posets does not always work. First of all, note that, if $L \in \mathcal{G}_{mon}(S) \setminus \mathcal{G}_{c.mon}(S)$ and there is an *acyclic* poset S' which has S as an induced sub-poset, it is impossible to construct a monotone extension of L to S': indeed, by Theorem 1.1 and Proposition 2.4 such an extension would be a generator of $\mathcal{G}_{c.mon}(S')$ and so, by Lemma 4.1 we should have also $L \in \mathcal{G}_{c.mon}(S)$.
As an example, consider the poset S_9 of Figure 3 and the generator L of Example 9. Consider

the poset S'_9 (see Figure 5) obtained by adding to S_9 the points w, w_1, w_2 in such a way that $a < w_1 < w < w_2 < d$, $b < w_1 < f$ and $c < w_2 < e$: we obtain a 9-points poset which is *acyclic*. Therefore, it is impossible to obtain a monotone extension of L to S'_9.

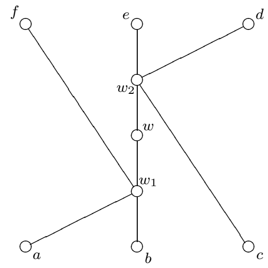

Figure 5: An extension from poset S_9: acyclic posetS'_9

Moreover, the example below shows that, even when S is not "enlargeable" to an acyclic poset (*i.e.*, every poset having S as an induced sub-poset is *non-acyclic*), there are generators which cannot be extended to every "larger" poset.

Example 11. Let L be the generator on poset S_8 (Figure 3) defined in Example 10. Note that (see Proposition 5.11 of [FM01]) S_8 is not enlargeable to any acyclic poset. However, consider a poset S'_8 obtained from S_8 by adding a point z with $z > f$, $z > e$ and $z \not> d$. Suppose L' is a "monotone extension" of L. Since $\{d\}$ is a up-set, $L_{e,d} = 1$ and $z > e$, it must be $L'_{z,d} \geq 1$. Moreover $\Gamma := \{d, b, c\}$ is a down-set; since $L'_{z,\Gamma} \geq 1$ and $z > f$, it should be $L_{f,\Gamma} \geq 1$, which is false, being $L_{f,\Gamma} = 0$. Thus there can be no such extension.
However, as we shall see in subsection 4.2, the method of monotone extension works for S_8, in the sense that for any poset S' having S_8 as sub-poset, *there exists* a generator L in S_8 which is monotone but not completely monotone, that can be extended to a monotone generator in S'. But this generator L has to be chosen appropriately.

4.1 From 5-points posets to larger posets

In this subsection we show that monotonicity equivalence does not hold for any poset S' having one of the 5-points posets of Figure 2 as an induced sub-poset. Note that, in this case, S' is *non-acyclic*: this is an immediate consequence of Proposition 2.7 of [FM01].

Proposition 4.1. *If a poset S' admits as induced sub-poset a poset S, whose Hasse-Diagram is one of those in Figure 2 (up to symmetries), then monotonicity equivalence fails in S' as well.*

Proof. For each $i = 1, \ldots, 5$, let S'_i be a poset which has S_i as an induced sub-poset.
According to Lemma 4.1, it is enough to show that, if we choose a monotone generator L on S_i which is not completely monotone, then we can define a stochastically monotone generator L' on S'_i such that $L'_{x,y} = L_{x,y}$ for all $x, y \in S_i$.
In each case considered below we shall pose $\bar{S} = S'_i \setminus S_i$ and define L' in such a way that the only new transitions allowed are the ones from elements of $\bar{S}$ to elements of S_i. In other words, for each $i = 1, \ldots, 5$, we shall pose $L'_{xy} = L_{xy}$ for all $x, y \in S_i$ and $L'_{xy} = 0$ for all $x \in S'$ and $y \in \bar{S}$. Note that, if Γ' is an up-set in S'_i, then $\Gamma = \Gamma' \cap S_i$ is an up-set in S_i and by the construction of L' it follows that, for each $x \notin \Gamma'$, $L'_{x,\Gamma'} = L'_{x,\Gamma}$. The same property holds for down-sets.
Then (see Remark 1), in order to verify that L' is monotone, it will be sufficient to check that,

for all $x, y \in S'_i$, with $x < y$, if Γ is an up-set in S_i (resp. a down-set) and $x, y \notin \Gamma$, we have $L'_{x,\Gamma} \le L'_{y,\Gamma}$ (resp. $L'_{x,\Gamma} \ge L'_{y,\Gamma}$).

Case I. Let us consider S_1 and the generator L given in Example 1.
We have to define only transition rates from elements of $\bar{S}$ to elements of S_1. Consider the partition of S'_1 given by the sets $A = \{z \in S'_1 : z \le b\} \cup \{z \in S'_1 : z \le c\}$ and $B = S'_1 \setminus A$. Then, if $z \in \bar{S} \cap A$ we pose $L'_{zw} = 1$ and $L'_{zy} = 0$ for each other $y \in S_1$; if $z \in \bar{S} \cap B$, we pose $L'_{zb} = L'_{zc} = 1$ and $L'_{zy} = 0$ for each other $y \in S_1$.
L' is a monotone generator. Indeed, let us take $x, y \in S'_1$ with $x < y$. There are only three possibilities: $x, y \in B$, $x, y \in A$ or $x \in A$ and $y \in B$. If $x, y \in B$, then $L'_{xz} = L'_{yz}$ for each $z \neq x, y$ and there is nothing to verify. Now we show that if Γ is an up-set in S_1 such that $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$ then $L'_{x,\Gamma} \le L'_{y,\Gamma}$. Suppose that $x, y \in A$; since, for $x \in A$ we have $L'_{x,\Gamma} \neq 0$ if and only if $\Gamma = S_1$, then necessarily $x, y \in \bar{S} \cap A$ and $L'_{x,S_1} = L'_{y,S_1} = 1$. For the same reason, if $x \in A$ and $y \in B$, then $\Gamma = S_1$ and $1 = L'_{x,S_1} < L'_{y,S_1} = 2$.

Analogously, consider a down-set Γ in S_1 with $x, y \notin \Gamma$ and $L'_{y,\Gamma} \neq 0$. If $x, y \in A$, since each down-set in S_1 contains w, we have $x, y \neq w$ and so $L'_{x,\Gamma} = L'_{y,\Gamma}$. If $x \in A$ and $y \in B$, then $\Gamma = \{b, a, w\}$ or $\{c, a, w\}$. Indeed, $y \in B$ and $L'_{y,\Gamma} \neq 0$, implies that $\Gamma \cap \{b, c\} \neq \emptyset$. But x is smaller or equal than at least one element of the set $\{b, c\}$ and $x \notin \Gamma$, so we cannot have $\{b, c\} \subset \Gamma$. Therefore $L'_{x,\Gamma} = 1 = L'_{y,\Gamma}$.

Case II. Consider the poset S_2 with the generator L given in Example 2. Now, we take the partition of S'_2 given by the sets $A = \{z \in S'_2 : z \le b\}$, $B = \{z \in S'_2 : z > b\}$, $C = S'_2 \setminus (A \cup B)$ and we pose, for each $z \in \bar{S} \cap C$, $L'_{z,c} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_2$. If $z \in \bar{S} \cap A$ (respectively, if $z \in \bar{S} \cap B$), we pose $L'_{z,a} = 1$ (resp. $L'_{z,d} = 1$) and $L'_{z,y} = 0$ for each other $y \in S_2$. Let us take $x, y \in S'_2$ with $x < y$ and let Γ be an up-set in S_2 such that $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$. Note that we have always $d \in \Gamma$. Suppose first that both x, y belong to one of the sets A, B, C. If $x, y \in A$, then $x < y \Rightarrow x \neq b$, therefore $L'_{x,\Gamma} \neq 0 \Rightarrow \Gamma = S_2$ and $L'_{x,\Gamma} = L'_{y,\Gamma} = 1$. If $x, y \in B$, since $d \in \Gamma$, we have $x, y \neq d$, and so $L'_{x,\Gamma} = L'_{y,\Gamma}$. If $x, y \in C$, since $L'_{x,\Gamma} \neq 0$ we have $c \in \Gamma$ and $x, y \neq c$, then $L'_{x,\Gamma} = L'_{y,\Gamma}$. Now, suppose that x and y are not in the same subset of the given partition. This means that $x \notin B$. If $x \in A$ and $x \neq b$, then $L'_{x,\Gamma} \neq 0 \Rightarrow \Gamma = S_2$ and for $y \in (B \cup C) \setminus S_2$, we have $L'_{x,S_2} = L'_{y,S_2} = 1$. If $x = b$, necessarily we have $y \in B$ and $a \notin \Gamma$, then $L'_{x,\Gamma} = 1 = L'_{y,\Gamma}$. Finally, suppose that $x \in C$: then $L'_{x,\Gamma} \le 1$ and, since $x < y$ and $d \in \Gamma$, we have $y \in B$ and $L'_{y,\Gamma} = 1$, therefore $L'_{x,\Gamma} \le L'_{y,\Gamma}$. In order to check monotonicity of L', we should consider also down-sets of S_2, but in this case the argument for down-sets is perfectly symmetric to the one given above for up-sets.

Case III. For the poset S_3 we consider the generator L of Example 3 and take the partition of S'_3 given by $A = \{z \in S'_3 : z \le b\} \cup \{z \in S'_3 : z < c\}$ and $B = S'_3 \setminus A$. Then, if $z \in \bar{S} \cap A$ we pose $L'_{z,w} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_3$; if $z \in \bar{S} \cap B$, we pose $L'_{z,w} = L'_{z,d} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_3$.

In order to verify monotonicity of L', take $x, y \in S'_3$ with $x < y$. Note that $y \in A \Rightarrow x \in A$, therefore we can have $x, y \in A$, $x, y \in B$ or $x \in A, y \in B$. If $x, y \in A$, then $L'_{x,z} = L'_{y,z}$ for each $z \neq x, y$ and there is nothing to check. Suppose that $x \in A, y \in B$. Let us take an up-set Γ in S_3 with $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$; then, $\{w, d\} \subset \Gamma$ and $L'_{x,\Gamma} = 1 \le L'_{y,\Gamma}$. If we take a down-set Γ in S_3 with $x, y \notin \Gamma$ and $L'_{y,\Gamma} \neq 0$, since $x \in A \Rightarrow x < d$, we have $d \notin \Gamma$, therefore $L'_{y,\Gamma} = 1 = L'_{x,\Gamma}$. Now, suppose that $x, y \in B$. Let us take an up-set Γ with $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$. Then, $x, y \neq d$ and, if $w \notin \Gamma$, $L'_{x,\Gamma} = L'_{y,\Gamma}$. If $\{w, d\} \subset \Gamma$ we have $x \neq w, d$ and (since $y = c \Rightarrow x \in A$) $y \neq w, d, c$, therefore $L'_{y,\Gamma} = 2 \ge L'_{x,\Gamma}$. If Γ is a down-set with $x, y \notin \Gamma$ and $L'_{y,\Gamma} \neq 0$, then $\Gamma = S_3$ or

$\{w, a, c\} \subset \Gamma$. If $\Gamma = S_3$, then $L'_{x,\Gamma} = L'_{y,\Gamma} = 2$. If $\{w, a, c\} \subset \Gamma$ we have $x, y \neq w, a, c$ and so $L'_{x,\Gamma} = L'_{y,\Gamma} = 1$.

Case IV. For S_4 take the generator L given in Example 4. Then we consider the partition of S'_4 given by $A = \{z \in S'_4 : b \leq z \leq d\}$, $B = \{z \in S'_4 : z \geq b\} \setminus A$ and $C = S'_4 \setminus (A \cup B)$. Then, if $z \in \bar{S} \cap A$ we pose $L'_{z,b} = L'_{z,d} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_4$; if $z \in \bar{S} \cap B$, we pose $L'_{z,d} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_4$; if $z \in C$ we pose $L'_{z,b} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_4$.
Now we take $x, y \in S'_4$ with $x < y$. Suppose that $x \in A$; then $y > x \Rightarrow y \in A \cup B$. If Γ is an up-set in S_4 with $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$, then $x, y \neq d$ and $x \notin \Gamma \Rightarrow b \notin \Gamma$, therefore $L'_{x,\Gamma} = 1 = L'_{y,\Gamma}$. If Γ is a down-set in S_4 with $x, y \notin \Gamma$ and $L'_{y,\Gamma} \neq 0$, note that, $d \in \Gamma \Rightarrow x \in \Gamma$, therefore $\Gamma \cap \{b, d\} = \{b\}$ and so $L'_{x,\Gamma} = 1 = L'_{y,\Gamma}$. Suppose now that $x \in B$; since $y > x$ we have also $y \in B$ and there is nothing to verify. Finally, for $x \in C$, if Γ is an up-set with $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$ then $b \in \Gamma$, and so $\{b, d\} \subset \Gamma$, which implies that $y \not\geq b$, *i.e.* $y \in C$ and so $L'_{x,\Gamma} = L'_{y,\Gamma}$. If Γ is a down-set with $x, y \notin \Gamma$ and $L'_{y,\Gamma} \neq 0$, then $b \in \Gamma$, $y \in A \cup C$. Moreover, if $y \in A$ we have $d \notin \Gamma$, therefore $L'_{x,\Gamma} = 1 = L'_{y,\Gamma}$.

Case V. Consider the poset S_5 and the generator L of Example 5. We take the partition of S'_5 given by $A = \{z \in S'_5 : z \leq c\} \cup \{z \in S'_5 : z \leq d\}$, $B = S'_5 \setminus A$. Then, if $z \in \bar{S} \cap A$ we pose $L'_{z,a} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_5$; if $z \in \bar{S} \cap B$, we pose $L'_{z,,c} = L'_{z,d} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_5$.
Now we consider $x, y \in S'_5$ with $x < y$. If both x, y belong to A or B, then $L'_{x,z} = L'_{y,z}$ for each $z \neq x, y$, therefore there is nothing to verify. If x, y are not in the same set of the partition, since $y \in A \Rightarrow x \in A$, we can have only $x \in A$ and $y \in B$. Suppose that Γ is an up-set in S_5 with $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$. Then $a \in \Gamma$ and so $c, d \in \Gamma$, which implies that $L'_{y,\Gamma} = 2 \geq L'_{x,\Gamma}$. If Γ is a down-set in S_5 with $x, y \notin \Gamma$ and $L'_{y,\Gamma} \neq 0$, we have $\{c, d\} \cap \Gamma \neq \emptyset$. Moreover, since $x \leq c$ or $x \leq d$ and $x \notin \Gamma$, we have $\Gamma = \{c, a, b\}$ or $\Gamma = \{d, a, b\}$ and in both cases we have $L'_{x,\Gamma} = 1 = L'_{y,\Gamma}$. □

4.2 From 6-points posets to larger posets

As we saw in the preceding subsection, for a poset of cardinality 6 having one of the 5-points posets of Figure 2 as an induced sub-poset, there is not equivalence between monotonicity and complete monotonicity. Moreover, these are the only 6-points posets for which equivalence fails, together with the posets in Figure 2. Now we apply the method of monotone extension of generator to posets S_6, S_7, S_8 of Figure 3.

Proposition 4.2. *If a poset S' admits as induced sub-poset a poset S, whose Hasse-Diagram is one of the posets S_6, S_7, S_8 of Figure 3 then monotonicity equivalence fails in S as well.*

Proof. Case I. As we did in the preceding section, we call S' a poset which has S_6 (the double diamond of Figure 2) as an induced sub-poset, $\bar{S} = S' \setminus S_6$, and we take the monotone generator L on S_6 defined in Example 6. This generator is not completely monotone.
Now, we want to define L' on S' as a monotone extension of L. We consider the partition of S' given by the sets $A = \{z \in S' : z > b\} \cup \{z \in S' : z > c\} \cup \{z \in S' : z > d\}$ and $B = S' \setminus A$. Then, if $z \in \bar{S} \cap A$ we pose $L'_{z,e} = 1$ and $L'_{z,y} = 0$ for each other $y \in S'$; if $z \in \bar{S} \cap B$, we pose $L'_{z,a} = L'_{z,c} = 1$ and $L'_{z,y} = 0$ for each other $y \in S'$.
L' is a monotone generator. Indeed, let us suppose that $x, y \in S'$ with $x < y$ and at least one of them does not belong to S_6. We have only three possibilities: $x, y \in A$, $x, y \in B$ or $x \in B$ and $y \in A$. If $x, y \in A$, and $x, y \neq e$, there is nothing to verify, since x and y make the same transitions with the same rate. On the other hand, if $x = e$ or $y = e$, then, for every up-set

(down-set) Γ with $x, y \notin \Gamma$ we have $e \notin \Gamma$ and so $L'_{x,\Gamma} = L'_{y,\Gamma} = 0$.
If $x, y \in B$ we have to consider only the cases in which x and y make different transitions, *i.e.* when $y \in \{a, b, c, d\}$ and $x \neq a$ or when $y \notin \{a, b, c, d\}$ and $x = a$.
In the first case, if we take an up-set Γ with $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$, then, since $y \geq a$ we have $a \notin \Gamma$ and $\{c, e\} \subset \Gamma$. This implies $L'_{x,\Gamma} = L'_{y,\Gamma} = 1$; if we take a down-set Γ with $x, y \notin \Gamma$, then $a \in \Gamma$ and $L'_{x,\Gamma} \geq 1 \geq L'_{y\Gamma}$. In the second case we have $x = a$, then, for any up-set Γ with $y, a \notin \Gamma$ we have $L'_{a,\Gamma} \neq 0 \Rightarrow c \in \Gamma \Rightarrow L'_{a,\Gamma} = L'_{y,\Gamma} = 1$; on the other hand, for each down-set Γ in S' with $L'_{y,\Gamma} \neq 0$ we have $x = a \in \Gamma$ and there is nothing to verify.
Now, suppose $x \in B$ and $y \in A$. If Γ is an up-set with $L'_{x,\Gamma} \neq 0$ and $x, y \notin \Gamma$, since $y > a$, we have $a \notin \Gamma$, so $L'_{x,\Gamma} \leq 1$ and $L'_{x,\Gamma} = 1$ if and only if $\{c, e\} \subset \Gamma$ which implies $L'_{y,\Gamma} = 1$; if Γ is a down-set with $L'_{y,\Gamma} \neq 0$, we have necessarily $e \in \Gamma$, which implies $a, c \in \Gamma$, then $2 = L'_{x,\Gamma} \geq L'_{y,\Gamma}$.

Cases II, III. Now, consider the monotone generator L on S_7 given in Example 7. Note that L has the same property also as a generator on the poset S_8. If in the proof which follows we consider S_8 instead of S_7, we obtain the same result.
Let S'_7 a poset which has S_7 as induced sub-poset and $\bar{S} = S'_7 \setminus S_7$. We take the partition of S'_7 given by $A = \{z \in S'_7 : c \leq z \leq d\}$, $B = \{z \in S'_7 : z \geq c\} \setminus A$, $C = S'_7 \setminus (A \cup B)$ and we define a monotone extension L' of L as follows: if $z \in \bar{S} \cap A$ we pose $L'_{z,c} = L'_{z,d} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_7$; if $z \in \bar{S} \cap B$, we pose $L'_{z,d} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_7$; if $z \in C$ we pose $L'_{z,c} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_7$.
Now we take $x, y \in S'_7$ with $x < y$. Suppose that $x \in A$; then $y > x \Rightarrow y \in A \cup B$. If Γ is an up-set in S_7 with $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$, then $x, y \neq d$ and $x \notin \Gamma \Rightarrow c \notin \Gamma$, therefore $L'_{x,\Gamma} = 1 = L'_{y,\Gamma}$. If Γ is a down-set in S_7 with $x, y \notin \Gamma$ and $L'_{y,\Gamma} \neq 0$, note that, $d \in \Gamma \Rightarrow x \in \Gamma$, therefore $\Gamma \cap \{c, d\} = \{c\}$ and so $L'_{x,\Gamma} = 1 = L'_{y,\Gamma}$. Suppose that $x \in B$; since $y > x$ we have also $y \in B$ and there is nothing to verify. Finally, for $x \in C$, if Γ is an up-set with $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$ then $c \in \Gamma$, and so $\{c, d\} \subset \Gamma$, which implies that $y \ngeq c$, *i.e.* $y \in C$ and so $L'_{x,\Gamma} = L'_{y,\Gamma}$. If Γ is a down-set with $x, y \notin \Gamma$ and $L'_{y,\Gamma} \neq 0$, then $c \in \Gamma$ and $L'_{x,\Gamma} = 1$. On the other hand, if $y \in A$, then necessarily $d \notin \Gamma$, therefore in any case we have $L'_{x,\Gamma} = 1 \geq L'_{y,\Gamma}$. ☐

Remark 4. The procedure used in Case II can be applied also to show that monotonicity equivalence fails for every poset which has a *k-crown* (see Figure 6) with $k \geq 3$ as induced sub-poset.

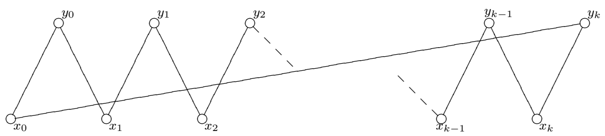

Figure 6: k-crown

Let S be a k-crown and $\{x_0, \ldots, x_k\}$, resp. $\{y_0, \ldots, y_k\}$, be the sets of its minimal, resp. maximal, elements (with $x_k < y_k$, $x_k < y_{k-1}$, $x_{k-1} < y_{k-1}, x_{k-1} < y_{k-2} \ldots x_0 < y_0, x_0 < y_k$). The generator defined by $L_{x_k,y_k} = L_{y_{k-1},y_k} = 1$, $L_{x_i,x_k} = 1$ for $i = 0, \ldots, k-1$, $L_{y_k,x_k} = 1$, $L_{y_i,x_k} = 1$ for $i = 0, \ldots k-2$ and $L_{x,y} = 0$ for each other pair $x, y \in S$ with $x \neq y$, is monotone. Suppose L is completely monotone. Then $\mathcal{M}_{x_k \mapsto y_k} \subset \mathcal{M}_{y_{k-1} \mapsto y_k} \subset \mathcal{M}_{x_{k-1} \mapsto x_k}$ and $\mathcal{M}_{y_k \mapsto x_k} \subset \mathcal{M}_{x_0 \mapsto x_k} \subset \mathcal{M}_{y_0 \mapsto x_k} \subset \ldots \subset \mathcal{M}_{x_{k-1} \mapsto x_k}$. Therefore, since $\mathcal{M}_{x_k \mapsto y_k} \cap \mathcal{M}_{y_k \mapsto x_k} = \emptyset$ we obtain $L_{x_{k-1},x_k} \geq 2$, which gives a contradiction.
Now, as we did for the 3-crown, we consider a poset S' which has a k-crown as induced sub-poset, we pose $\bar{S} = S' \setminus S$ and, in order to construct a monotone extension of the generator L given above, we take the partition of S' given by $A = \{z \in S' : x_k \leq z \leq y_k\}$, $B = \{z \in S' : z \geq x_k\} \setminus A$,

$C = S' \setminus (A \cup B)$. For $z \in \bar{S} \cap A$ we pose $L'_{z,x_k} = L'_{z,y_k} = 1$ and $L'_{z,w} = 0$ for each other $w \in S$; if $z \in \bar{S} \cap B$, we pose $L'_{z,y_k} = 1$ and $L'_{z,w} = 0$ for each other $w \in S$; if $z \in C$ we pose $L'_{z,x_k} = 1$ and $L'_{z,w} = 0$ for each other $w \in S$. The same arguments used above for the 3-crown show that L' is monotone.

5 Conclusions

In this paper we have obtained partial results concerning the relations between monotonicity and complete monotonicity for continuous-time Markov chains on partially ordered sets. We have provided sufficient conditions on the poset for the monotonicity equivalence to hold or to fail, and given a complete classifications for posets of cardinality ≤ 6. Unlike what Fill & Machida have obtained in the discrete-time case, we have not been able to find a characterization of posets for which monotonicity equivalence holds, in terms of their Hasse diagram. We remark, as the example in Figure 5 shows, that there are posets with an acyclic extension for which monotonicity equivalence fails. Therefore, in general, non-equivalence is not preserved by extending the poset.

For posets with no acyclic extensions, we believe the following fact holds true.

Conjecture. Let S be a connected poset having no acyclic extension. Then monotonicity equivalence holds if and only if the following conditions hold:

i) the Hasse diagram of S has a unique cycle, which is a diamond;

ii) S has no Y-shaped subposet (see Figure 7) having at most one point in common with the cycle in point i) and there is no induced subposet of the types from Figure 8 (up to symmetries).

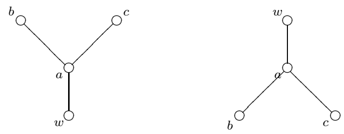

Figure 7: Y shapes

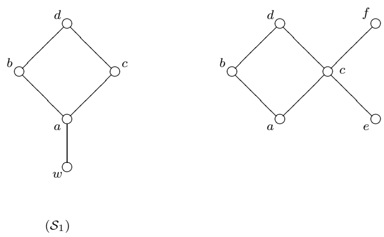

Figure 8: Forbidden posets

The necessity of conditions i) and ii) should actually be not too hard to prove, although many different cases have to be considered. We have tried harder to prove sufficiency of i), ii) by induction on the cardinality of the poset, but, unfortunately, we have not succeeded.

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