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To the shining memory of my friend
Tommy Tóth

Preface

The research for the present work began in the spring of 2011. As a diploma student of the University of Potsdam, I was introduced to the concept of quasicomplexes, i.e. the sequences with "small" curvature, by my supervisor N. Tarkhanov. The topic of my diploma thesis was to introduce the Euler characteristic for elliptic sequences of smoothing curvature. Afterwards, the results were published in [Wal12] where, in particular, the question was posed how the Lefschetz number can be defined in the context of quasicomplexes. It became the central point of my PhD studies to understand and to solve this problem. The basic ideas for the solution were given in the joint paper [TW12]. Then, in the paper [Wal14] this approach was developed and the results were used to generalise the Lefschetz fixed point formula of Atiyah and Bott [AB67]. In this thesis, the results of the three cited papers are summarised, which results in a unified presentation. Furthermore, we present several unpublished generalisations and examples.

In addition to the aforementioned results, this work gives a summary of the theory of Fredholm quasicomplexes, which has been developed since the 1980s and provides a natural extension of the well known theory of Fredholm complexes. During the compilation of the results, I tried to find an approach as direct as possible to avoid nonnecessary redundancies in the development, which would be caused by simply connecting the cited papers. I clearly separate my own contribution from the results of others, by giving exact references and integration into the historical context. Particularly, no proofs of basic theorems are listed. Moreover, all important facts for the development of the theory are summarised in a thorough basic chapter, containing all necessary aspects for performing this research. Hopefully, this will allow us to introduce the present theory to a large number of interested mathematicians, while giving the advanced reader the opportunity to reach the relevant results quickly without having to filter out of a swamp of familiar material.

I would greatly like to thank to my supervisor for the detailed and patient mentoring during all this years. He has deeply influenced not only my view on mathematics but also on other aspects of my life. My research was also encouraged by my colleagues of the University of Kassel, where I have been working for the last two years. Finally, many thanks to my family who always supported me and my studies.

Kassel
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Introduction

The concept of (cochain) complexes arises in different fields of modern mathematics. In mathematical analysis one normally investigates sequences (also called cochain complexes) of the form

$$\{V^{\cdot}, D\}\colon\ \ 0 \to V^0 \xrightarrow{D^0} V^1 \xrightarrow{D^1} \dots \xrightarrow{D^{N-1}} V^N \to 0\,,$$

where $V^0, \dots, V^N$ are topological vector spaces (for instance function spaces on a manifold) and $D^0, \dots, D^{N-1}$ are continuous linear mappings. Such a sequence is called a complex, if $D^{i+1}D^i = 0$ is satisfied for all $i = 0, 1, \dots, N$. In this case the cohomology H^i can be defined. Those form vector spaces as well, and it turns out that it is possible to draw conclusions from the investigation of the cohomologies for the whole object and for the solvability of equations with the operators involved.

In general a complex of Hilbert spaces is called Fredholm, if it has finite dimensional cohomology. This is satisfied if and only if there is a sequence of operators $P^i\colon V^i \to V^{i-1}$, such that

$$D^{i-1}P^i + P^{i+1}D^i = \mathrm{Id}_{V^i}$$

holds modulo compact operators. Such a $P = (P^1, \dots, P^N)$ is called parametrix of the complex. For any Fredholm complex the Euler characteristic is defined to be

$$\chi := \sum_{i=0}^{N} (-1)^i \dim H^i,$$

which generalises the index of Fredholm operators. This characteristic number is a special case of a more abstract number defined for each endomorphim E of a Fredholm complex $\{V^{\cdot}, D\}$. Such an endomorphism is a family of linear selfmappings $E^i\colon V^i \to V^i$ fulfilling $E^{i+1}D^i = D^iE^i$. Then the induced mappings $HE^i : H^i \to H^i$ are endomorphisms of the finite-dimensional spaces H^i, and so the alternating sum

$$\mathscr{L} := \sum_i (-1)^i \operatorname{tr} HE^i\,,$$

the Lefschetz number of the endomorphism, is well defined.

To each complex the differential D is associated by $Dv = D^iv$ for $v \in V^i$. One may ask what happens with complexes under "small" perturbations of their differentials. Note that it depends on the structure of the underlying spaces whether or not an operator is "small." This leads to a magical mix of perturbation and regularisation theory. In the general setting of Hilbert spaces compact operators are "small." Thus, we may perturb the differential D by a compact operator K. Formally we find

$$(D+K)(D+K) = D^2 + DK + KD + K^2 = DK + KD + K^2.$$

Hence, the product is a compact operator, too. This leads us to the theory of quasicomplexes introduced by Putinar in [Put82]. More precisely, a quasicomplex with Hilbert spaces is a sequence

$$\{V^\cdot, A\}\colon\ 0 \to V^0 \xrightarrow{A^0} V^1 \xrightarrow{A^1} \dots \xrightarrow{A^{N-1}} V^N \to 0$$

where the composions $A^{i+1}A^i$ are compact operators. Note that such objects were called "essential complexes" by Putinar. In [Tar07] the name "quasicomplexes" was suggested, which we will use in this work. For quasicomplexes the cohomology is no longer defined, since the image of A^{i-1} fails in general to lie in the null-space of A^i. However, in order to define Fredholm quasicomplexes we may use the parametrix, where the parametrix of a quasicomplex is defined in the same way as that of a complex.

Obviously, we obtain Fredholm quasicomplexes by perturbing the differential of a Fredholm complex by compact operators. The inverse theorem is also true, i.e. each Fredholm quasicomplex can be obtained by compact perturbations of a Fredholm complex. The proof is much more difficult and it was first shown in [EP96] for more general parametrised quasicomplexes of Banach spaces. In order to define the Euler characteristic of a Fredholm quasicomplex, a proof which uses special Hilbert space methods was given in [Tar07].

An open problem was to introduce reasonably the Lefschetz number in the context of quasicomplexes, see [Wal12]. To highlight the problem, we consider a Fredholm quasicomplex $\{V^\cdot, A\}$ of Hilbert spaces and an endomorphism E, i.e. $E^{i+1}A^i = A^iE^i$. The idea is now to use any reduced complex $\{V^\cdot, D\}$ for $\{V^\cdot, A\}$ to define the Lefschetz number. Since $D^i = A^i + C^i$ implies $E^{i+1}D^i = D^iE^i$ modulo $\mathcal{K}(V^i, V^{i+1})$, the sequence E fails to determine an endomorphism of $\{V^\cdot, D\}$. So nothing changes if we deal with quasiendomorphisms of $\{V^\cdot, A\}$ from the very beginning, i.e. with those sequences which satisfy $E^{i+1}A^i = A^iE^i$ modulo compact operators from V^i to V^{i+1}. However, these latter don not act naturally on the cohomology of reduced complexes. Moreover, it turns out that in the general case of compact curvature no Lefschetz number is available. In particular, for quasicomplexes the Euler characteristic is in general no special Lefschetz number. Nevertheless, it was shown in [TW12] that it makes sense to introduce the Lefschetz number for Fredholm quasicomplexes with trace class curvature and endomorphisms modulo trace class operators by

$$\mathscr{L} := \sum_{i=0}^{N} (-1)^i \operatorname{tr}\left(E^i - A^{i-1}E^{i-1}P^i - E^iP^{i+1}A^i\right),$$

where P is an arbitrary "regularising" parametrix of $\{V^{\cdot}, A\}$. As shown in [Wal14], this definition does not depend on the particular choice of parametrix and the standard properties of Lefschetz number hold true in this more general context. Those results were obtained at the same time independently by J. Eschmeier in the context of Banach spaces, see [Esc13]. Furthermore, the cited paper contains an interesting example concerning Toeplitz operators.

As is well known, many geometric problems lead to elliptic complexes of differential and pseudodifferential operators, which are studied since the 1930s by W. Hodge, G. de Rham, A. Grothendieck, D. C. Spencer, M. F. Atiyah, I. Singer, and others. Namely, let F^i be smooth vector bundles over a compact closed manifold X, for $i = 0, .., N$. We consider complexes of classical pseudodifferential operators A^i on spaces of smooth sections

$$\mathcal{E}(X, F^{\cdot}) \colon 0 \to \mathcal{E}(X, F^0) \xrightarrow{A^0} \mathcal{E}(X, F^1) \xrightarrow{A^1} \ldots \xrightarrow{A^{N-1}} \mathcal{E}(X, F^N) \to 0\,.$$

To such a complex we assign a complex of principal symbols defined on the cotangent bundle of the manifold. $\mathcal{E}(X, F^{\cdot})$ is called elliptic, if the associated symbol complex is exact away from the zero section. The standard example is the familiar de Rham complex.

It is well known that an elliptic complex has finite dimensional cohomology. In order to show this, one uses the fact that any elliptic complex extends to a complex of Sobolev spaces. This is a complex of Hilbert spaces and the Fredholm theory can be applied. Even the theory of elliptic complexes could be extended in the sense of quasicomplexes. In general, it suffices to claim that the symbol sequence is a complex. A natural example is the connection quasicomplex

$$\Omega^{\cdot}(X, F): \; 0 \to C^{\infty}(X, F) \overset{\partial^0}{\to} \Omega^1(X, F) \overset{\partial^1}{\to} \ldots \overset{\partial^{n-1}}{\to} \Omega^n(X, F) \to 0,$$

where ∂ is a connection of a vector bundle F over X, cf. [Wel80]. This is an analogue of the de Rham complex in the case of sections instead of differential forms. Note that $(\partial)^2$ is called the curvature, what inspired us to use this name for the composition of operators in a sequence in general.

Elliptic quasicomplexes on compact manifolds without boundary have been studied in [Wal12] where a generalisation of the Atiyah-Singer index formula was proved. The more difficult case of quasicomplexes on compact manifolds with boundary was studied before in [KTT07]. Another application of the theory can be found in [Wal14], where a generalisation of the Atiyah-Bott-Lefschetz fixed point formula for geometric quasiendomorphisms of elliptic quasicomplexes was proved.

It should be noted that elliptic quasicomplexes of differential operators (of the same order) were already studied by Gilkey in [Gil94]. In particular, for some cases the index of the corresponding block operator $(A + A^*)_e$ was computed, what is in fact the Euler characteristic of the quasicomplex. During a talk at a conference in Potsdam some participants suggested to use this index of the block operator as a definition of the Euler characteristic. However, on the one hand, this seems not to be a good definition because no cohomology is used, and on the other hand, this does not work very well in the abstract Hilbert space setting because the Fredholm property of the block operator follows from the fact that a reduced complex can be found.

To some extent the theory of elliptic complexes is a highlight of mathematics in the 20 th century. Consequently, to formulate and understand the central concepts used in this field, we have to do some preliminary work. Namely, many elements of diverse mathematical disciplines, such as functional analysis, differential geometry, partial differential equation, homological algebra and topology have to be combined. All essential basics are summarised in the first chapter of this thesis. This contains classical elements of index theory, such as Fredholm operators, elliptic pseudodifferential operators and characteristic classes, which can be found for instance [Wel80]. Moreover we study the de Rham complex [War83] and introduce Sobolev spaces of arbitrary order as well as the concept of operator ideals from [Pie78].

In the second chapter, the abstract theory of (Fredholm) quasicomplexes of Hilbert spaces will be developed. From the very beginning we will consider quasicomplexes with curvature in an ideal class, as it was suggested in [TW12]. We introduce the Euler characteristic, the cone of a quasiendomorphism and the Lefschetz number. In particular, we generalise Euler's identity, which will allow us to develop the Lefschetz theory on nonseparable Hilbert spaces.

Finally, in the third chapter the abstract theory will be applied to elliptic quasicomplexes with pseudodifferential operators of arbitrary order. We will show that the Atiyah-Singer index formula holds true for those objects and, as an example, we will compute the Euler characteristic of the connection quasicomplex. In addition to this we introduce geometric quasiendomorphisms and prove a generalisation of the Lefschetz fixed point theorem of [AB67].

This is a work in pure mathematics. Namely, we give some abstract definitions and prove theorems, where we focus on mathematical correctness. Any examples and applications are of inner mathematical nature. Possible physical applications are far from being studied. In general, abstract mathematical material like that done in this work will be applied at the earliest 50 years after it has been developed.

Chapter 1

Basics

The topics explained in this chapter contain some classical elements of modern analysis and can be only sketched. Accordingly, mere basic definitions and theorems are summarised and no proofs are given.

We use the following standard notation:

- $\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.
- $\imath$ denotes the imaginary unit in $\mathbb{C}$, i.e. $\imath^2 = -1$.
- $M \subset N$ means that $x \in N$ for any $x \in M$.
- $\operatorname{im} f$ denotes the image and $\ker f$ the null-space of a (linear) mapping f.
- Id_M denotes the identical mapping on a set M.

1.1 Elements of functional analysis

In this section we summarise some basic facts and results of functional analysis. For this we assume that the reader has elementary knowledge of linear algebra and abstract algebraic structures, as well as analysis. This includes in particular metric spaces and measure and integration theory.

1.1.1 Topological vector spaces

Let X be a nonempty set and τ a family of subsets of X. Then τ is called topology on X, if

i) $\emptyset, X \in \tau$;

ii) $U \cap V \in \tau$, if $U, V \in \tau$;

iii) $\bigcup_{i \in I} U_i \in \tau$, if $U_i \in \tau$ for all $i \in I$.

In this case X, or more precisely the pair $\{X, \tau\}$, is called a topological space. If M is a nonempty subset of X, then by the relative topology on M is meant the family $\tau_M := \{U \cap M | U \in \tau\}$.

In a metric space the family of open subsets forms a topology. Motivated by this example the elements of an arbitrary topology are called open sets. Analogously to metric spaces, a subset $M \subset X$ is said to be closed if $X \setminus M$ is open. The intersection of all closed subsets of X containing M is called the closure of M and denoted by $\overline{M}$. Furthermore, a subset is said to be dense in X, if its closure coincides with X. A topological space X is called separable if there is a countable subset which is dense in X.

The concept of topological vector spaces arose in the 20s of the past century, cf. [CR96]. A central point in this theory is perhaps the idea of abstract compactness. Namely, a topological X space is called compact if every open covering $\{U_j\}_{j\in J}$ of X possesses a finite subcovering, i.e. there are $U_{j_1}, \dots, U_{j_N}$ with $j_1, \dots, j_N \in J$, such that $X = U_{j_1} \cup \ldots \cup U_{j_N}$. A subset M of X is called compact if it is compact with respect to the relative topology on M, and relatively compact if its closure in X is compact. The relative compactness is usually denoted by $M \Subset X$.

It should be noted that each topological space can be compactified by adding a symbolic point ∞. For this purpose, we set $X^+ := X \cup \{\infty\}$ and call $U \subset X^+$ open if U is an open subset of X or if $X^+ \setminus U$ is closed and compact in X. When endowed with this topology X^+ becomes compact. This construction goes back to P. Alexandrov.

Let X and Y be topological spaces with topologies τ_X and τ_Y. A mapping $f\colon X \to Y$ is said to be continuous if $f^{-1}(V) \in \tau_X$ for all $V \in \tau_Y$. The image of a compact subset of X by a continuous mapping f proves to be a compact subset of Y.

Write $\tau_{X\times Y}$ for the family of all subsets O of $X \times Y$ with the property that for each $(x, y) \in O$ there are open sets $U_x \supset x$ and $V_y \supset y$ in X and Y, respectively, such that $U_x \times U_y \subset U$. Then $\tau_{X\times Y}$ defines a topology on the Cartesian product $X \times Y$.

A vector space V over a field $\mathbb{K}$ is called a topological vector space if V is given a topology which is compatible with the vector structure in V, i.e. if the mappings

$$\begin{array}{rclcl} + & : & V \times V \to V, & (v_1, v_2) & \mapsto & v_1 + v_2\,, \\ \cdot & : & \mathbb{K} \times V \to V, & (\lambda, v) & \mapsto & \lambda \cdot v \end{array}$$

are continuous. If W is another topological vector space over the same field $\mathbb{K}$, then the vector space consisting of all continuous linear maps acting from V to W is denoted by $\mathcal{L}(V, W)$.

A mapping $f \in \mathcal{L}(V, W)$ is called a topological isomorphisms if f is bijective and its inverse is continuous. We set $V' := \mathcal{L}(V, \mathbb{K})$ and often write $f(v)$ as $\langle f, v\rangle$ for $f \in V'$ and $v \in V$.

Given any $A, B \in \mathcal{L}(V) := \mathcal{L}(V, V)$, we denote by

$$[A, B] := AB - BA$$

the commutator of A and B. This operation gives a Poisson algebra structure to $\mathcal{L}(V)$.

To endow the space $\mathcal{L}(V,W)$ with a meaningful topology, we have to specify the structur of V and W. We restrict ourselves to the case of normed spaces. If $\|\cdot\|$ is a norm on V, then on setting $d(u,v) := \|u-v\|$ for $u,v \in V$ we get a metric on V. If we endow the vector space with the topology induces by this metric, it becomes a topological vector space. If W is another normed space over the same field, then $\mathcal{L}(V,W)$ is a normed space under the operator norm $\|A\| := \sup_{\|v\|\leq 1} \|Av\|$ for $A \in \mathcal{L}(V,W)$.

Example 1.1.1 Let U be an open set in $\mathbb{R}^n$. We consider the space $C^\infty(U,\mathbb{K}^m)$ consisting of all smooth functions on U with values in $\mathbb{K}^m$. Choose an enlarging sequence of compact sets $\{K_j\}_{j\in\mathbb{N}}$ in U converging to U, i.e. $K_{j+1} \subset K_j$ and the union of K_j is all of U. Set

$$p_j(u) := \sup_{\substack{x\in K_j \\ |\alpha|\leq j}} |\partial^\alpha u(x)|$$

for $u \in C^\infty(U,\mathbb{K}^m)$. Here, $\alpha = (\alpha_1,\ldots,\alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\partial^\alpha u = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n} u$. On setting

$$d(u,v) = \sum_{j=0}^{\infty} 2^{-j} \frac{p_j(u-v)}{1+p_j(u-v)}$$

for $u,v \in C^\infty(U,\mathbb{K}^m)$ we obtain a metric on $C^\infty(U,\mathbb{K}^m)$. One can show that any bounded subset of $C^\infty(U,\mathbb{K}^m)$ is relatively compact (this is often referred to as the Heine-Borel property).

On the contrary, a ball in a normed space is relatively compact if and only if the space is finite dimensional.

Theorem 1.1.2 *A normed space V is finite dimensional if and only if the identity mapping* Id_V *is compact.*

The complete normed vector spaces are called Banach spaces. Note that a finite-dimensional normed space is automatically a Banach space. If W is a Banach space, then the space $\mathcal{L}(V,W)$ is a Banach space, too. In this case any continuous mapping $A \in \mathcal{L}(U,W)$, whose domain is a dense subspace U of V, can be uniquely extended to a continuous mapping $\overline{A} \in \mathcal{L}(V,W)$ of all V by

$$\overline{A}v = \lim_{k\to\infty} Av_k$$

for $v \in V$, where $\{v_k\}$ is any sequence in U converging to v.

If a vector space V is endowed with a scalar product $(\cdot,\cdot)_V$, the scalar product induces a norm on V through $\|v\|_V := \sqrt{(v,v)_V}$ for $v \in V$. Such a space is called unitary. Another designation is a pre-Hilbert space or Hilbert space, if it is complete. These spaces have a very rich structure, which can not be explained here in all aspects. For details on the geometry of Hilbert spaces we refer the reader to textbooks in functional analysis, such as [DS88] or [Hal74]. We only mention that if U is an subspace of a Hilbert space V, then the orthogonal decomposition $V = \overline{U} \oplus U^\perp$ holds, where the orthogonal complement $U^\perp$ is the subspace of V consisting of all $v \in V$ satisfying $(v, u)_V = 0$ for each $u \in U$. Note that the space $U^\perp$ is always closed.

Theorem 1.1.3 *Let U be a closed subspace of a Hilbert space V. Then there exists an orthogonal projection P_U of V onto U, i.e. a linear selfmapping of V satisfying $P_U v = v$, if $v \in U$, and $P_U v = 0$, if $v \in U^\perp$.*

Example 1.1.4 Let X be a compact topological space and V a Banach space. The space of all continuous functions on X with values in V is denoted by $C(X, V)$. This space becomes a Banach space, if we endow it with the supremum norm

$$\|u\|_{\sup} := \sup_{x \in X} \|u(x)\|_V$$

for $u \in C(X, V)$.

Examples 1.1.1 and 1.1.4 present the so-called function spaces, i.e. the vectors in this spaces are functions with values in a Banach space and both addition and multiplication by scalars are defined pointwise. The classical function spaces of integration theory are the Lebesque spaces $L^p(X, \mu)$, where $p \in [1, \infty]$ and $\{X, \mathcal{A}, \mu\}$ is a measure space, i.e. X is a topological space, $\mathcal{A}$ a σ-algebra of subsets of X and μ a σ-additive measure on $\mathcal{A}$. As usual these are separable Banach spaces and even Hilbert spaces, if $p = 2$, with scalar product

$$(u, v)_{L^2(X,\mu)} := \int_X u(x)\overline{v(x)}\, d\mu$$

for $u, v \in L^2(X, \mu)$. In the sequel we set $C^\infty(U) := C^\infty(U, \mathbb{C})$ and write simply $L^p(X) := L^p(X, \mu)$, if the measure μ is evident from the context (a surface area measure as a rule).

The standard Hilbert spaces are separable, still, since later we will investigate sequences of arbitrary, not necessarily separable, Hilbert spaces, we should give an example of such a space.

Example 1.1.5 Consider the functions of $x \in \mathbb{R}$ of the form $u_\lambda(x) = e^{\imath\lambda x}$, where $\lambda \in \mathbb{R}$. Let V be the vector space of all finite linear combinations of such functions. We define a scalar product on V by

$$(u, v) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u(x)\overline{v(x)}\, dx$$

for $u, v \in V$ and denote the completion of V with respect to the corresponding norm by $\mathrm{AP}^2(\mathbb{R})$. The elements of the Hilbert space $\mathrm{AP}^2(\mathbb{R})$ are called almost periodic functions on $\mathbb{R}$.

It is worth pointing out that the space $\mathrm{AP}^2(\mathbb{R})$ does not contain any function $u \in L^2(\mathbb{R})$ except for the zero one.

1.1.2 Tempered distributions

For $u \in L^1(\mathbb{R}^n)$ the Fourier transform $\hat{u}$ is defined as a function on $\mathbb{R}^n$ by

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-\imath\langle \xi, x\rangle} u(x)\, dx$$

for any $\xi \in \mathbb{R}^n$, where $\langle \xi, x\rangle = \xi_1 x_1 + \ldots + \xi_n x_n$ stands for the standard pairing $(\mathbb{R}^n)' \times \mathbb{R}^n \to \mathbb{R}$ of ξ and x.

The Fourier transform exhibits a very interesting behavior, if we apply it to differentiable functions. To this end, we set

$$\langle x\rangle^s := (\sqrt{1+|x|^2})^s$$

for $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$ and call a function $u \in C^\infty(\mathbb{R}^n)$ rapidly decreasing , if

$$p_{j,k}(u) := \sup_{\substack{x\in\mathbb{R}^n \\ |\alpha|\le j}} \langle x\rangle^k |\partial^\alpha u(x)| < \infty$$

holds for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $\mathbb{N}_0^n$ and each $k \in \mathbb{N}_0$. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the topological vector space consisting of all rapidly decreasing functions. The topology on $\mathcal{S}(\mathbb{R}^n)$ is induced by the metric

$$d(u,v) = \sum_{j,k\in\mathbb{N}_0} 2^{-(j+k)} \frac{p_{j,k}(u-v)}{1+p_{j,k}(u-v)}.$$

It is well known that the Fourier transform provides a topological isomorphism $\mathscr{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ with the inverse

$$\mathscr{F}^{-1}(v)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\imath\langle x,\xi\rangle} v(\xi)\, d\xi$$

for $x \in \mathbb{R}^n$ and $v \in \mathcal{S}(\mathbb{R}^n)$. On setting

$$D_j = \frac{1}{\imath}\frac{\partial}{\partial x_j}$$

and $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ we obtain

$$\begin{aligned} \mathscr{F}(D^\alpha u)(\xi) &= \xi^\alpha \mathscr{F}u(\xi), \\ \mathscr{F}(x^\beta u)(\xi) &= (-1)^{|\beta|} D^\beta \mathscr{F}u(\xi) \end{aligned}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$ and multi-indices α, β.

Example 1.1.6 Consider the Gauß function $u\colon \mathbb{R} \to \mathbb{R}$ with $u(x) = e^{-x^2}$. Then $u \in \mathcal{S}(\mathbb{R})$ holds and the Fourier transform is given by $\hat{u}(\xi) = \sqrt{\pi} e^{-\xi^2/4}$.

Let U be open in $\mathbb{R}^n$ and V be a vector space. For $f : U \to V$ the closure of the subset of U where f does not vanish is called the support of f and denoted by $\operatorname{supp} f$. As usual, we write $\mathcal{D}(U)$ for the vector space of all smooth functions on U with compact support.

The following important lemma is known as the stationary phase method, cf. for instance [Dui95].

Lemma 1.1.7 *Assume that* φ *is a* C^4 *function on* $\mathbb{R}^n$ *with real values and* $x_0 \in \mathbb{R}^n$ *any simple stationary point of* φ*, i.e.* $\varphi'(x_0) = 0$ *and* $\det \varphi''(x_0) \neq 0$*. Then, for any* C^2 *function* f *with compact support on* $\mathbb{R}^n$*, such that* $\varphi' \neq 0$ *on* $\operatorname{supp} f \setminus \{x_0\}$*, we have*

$$\frac{1}{(2\pi\hbar)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{\imath}{\hbar}\varphi(x)} f(x)\, dx = \frac{e^{\frac{\imath}{\hbar}\varphi(x_0)}}{\sqrt{|\det \varphi''(x_0)|}}\, e^{\imath \frac{\pi}{4} \operatorname{sgn} \varphi''(x_0)} \left(f(x_0) + O(\hbar) \right)$$

as $\hbar \to 0$*, where* $\operatorname{sgn} \varphi''(x_0)$ *is the signature of* $\varphi''(x_0)$*, i.e. the number of positive eigenvalues minus the number of negative eigenvalues of the matrix.*

The space $\mathcal{D}(U)$ is equipped with a locally convex topology, such that a sequence $\{\varphi_k\}$ converges to $\varphi \in \mathcal{D}(U)$ if and only if there is a compact set $K \subset U$ with the property that $\operatorname{supp} \varphi_k \subset K$ for all k and $\partial^\alpha(\varphi_k - \varphi)$ converges to zero uniformly on K for all multi-indices α. The elements of the dual space $\mathcal{D}'(U)$ are called distributions on U.

Example 1.1.8 The Cauchy principal value integral

$$\varphi \mapsto \text{p.v} \int_{\mathbb{R}} \frac{\varphi(x)}{x} dx := \lim_{\varepsilon \searrow 0} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \frac{\varphi(x)}{x} dx$$

is a distribution on the real axis.

Consider $f \in L^1_{\text{loc}}(U)$, i.e. $f : U \to \mathbb{C}$ is measurable and the integral of $|f|$ over each compact set $K \subset U$ is finite (such a f is called locally integrable on U). As usual, f is thought of as a representative of an equivalence class of functions coinciding with f almost everywhere. Then f provides a linear form $f : \mathcal{D}(U) \to \mathbb{C}$ by

$$\langle f, \varphi \rangle := \int_U f(x) \varphi(x) dx$$

for $\varphi \in \mathcal{D}(U)$. On using Lebesque's theorem of dominated convergence we see that $f \in \mathcal{D}'(U)$. Moreover, the fundamental lemma of the calculus of variations states that $\langle f, \varphi \rangle = 0$ holds for each $\varphi \in \mathcal{D}(U)$ if and only if $\varphi = 0$. Hence, we can specify the locally integrable functions within distributions and regard distributions as "generalised functions." The distributions of this type are called regular. Given any $f, g \in C^\infty(U)$, one verifies readily

$$\begin{aligned} \langle \partial^\alpha f, \varphi \rangle &= (-1)^{|\alpha|} \langle f, \partial^\alpha \varphi \rangle, \\ \langle g f, \varphi \rangle &= \langle f, g \varphi \rangle \end{aligned}$$

for all test functions φ. On using these equalities one defines the generalised derivatives and multiplication with a smooth function for an arbitrary distribution $f \in \mathcal{D}'(U)$.

Example 1.1.9 Pick $x_0 \in \mathbb{R}$. Consider the functional δ_{x_0} on $\mathcal{D}(\mathbb{R})$ given by $\langle \delta_{x_0}, \varphi \rangle := \varphi(x_0)$ for $\varphi \in \mathcal{D}(\mathbb{R})$. It is easy to see that this functional is continuous. The distribution $\delta_{x_0} \in \mathcal{D}'(\mathbb{R})$ called the delta function supported at x_0. If

$x_0 = 0$, one writes δ_{x_0} simply δ. The Heavyside function $\theta(x) = (1/2)(\mathrm{sgn}(x)+1)$ is locally integrable on $\mathbb{R}$. Since

$$\langle \theta, \varphi' \rangle = \int_0^\infty \varphi'(x)\, dx = -\varphi(0)$$

is fulfilled for all $\varphi \in \mathcal{D}(\mathbb{R})$, we conclude that $\theta' = \delta$.

Let U and V be open sets in $\mathbb{R}^n$. Distributions $K \in \mathcal{D}'(U \times V)$ are said to be kernels or, more precisely, Schwartz kernels on $U \times V$. Each kernel gives rise to a continuous linear operator $T : \mathcal{D}(V) \to \mathcal{D}'(U)$ by the formula

$$\langle T\psi, \varphi \rangle_U = \langle K, \varphi \otimes \psi \rangle_{U \times V}$$

for $\varphi \in \mathcal{D}(U)$ and $\psi \in \mathcal{D}(V)$. L. Schwartz proved that this correspondence leads actually to a topological isomorphism

$$\mathcal{D}'(U \times V) \overset{\mathrm{top}}{\cong} \mathcal{L}(\mathcal{D}(V), \mathcal{D}'(U)),$$

the space $\mathcal{L}(\mathcal{D}(V), \mathcal{D}'(U))$ is endowed by the topology of uniform convergence on bounded subsets of $\mathcal{D}(V)$. This result admits also a global formulation for sections of vector bundles.

The elements of the dual space $\mathcal{S}'(\mathbb{R}^n) := (\mathcal{S}(\mathbb{R}^n))'$ are called tempered distributions on $\mathbb{R}^n$. If a sequence converges in $\mathcal{D}(\mathbb{R}^n)$, then it converges in $\mathcal{S}(\mathbb{R}^n)$, too. Thus, if f is a continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$, then its restriction to $\mathcal{D}(\mathbb{R}^n)$ is continuous. Since $\mathcal{D}(\mathbb{R}^n)$ proves to be dense in $\mathcal{S}(\mathbb{R}^n)$, it follows that $\mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$, i.e. tempered distributions are specified within distributions on $\mathbb{R}^n$.

Given any $f \in \mathcal{S}'(\mathbb{R}^n)$, the Fourier transform is defined by

$$\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The Fourier transform is known to be a topological isomorphism of $\mathcal{S}'(\mathbb{R}^n)$ onto $\mathcal{S}'(\mathbb{R}^n)$ itself.

Example 1.1.10 Since

$$\langle \hat{\delta}, \varphi \rangle = \langle \delta, \hat{\varphi} \rangle = \hat{\varphi}(0) = \int_{\mathbb{R}^n} \varphi(x)\, dx$$

holds for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the Fourier transform $\hat{\delta}$ just amounts to the constant function 1.

Denote by $\langle \xi \rangle$ the function on $\mathbb{R}^n$ given by $\xi \mapsto (1 + |\xi|^2)^{1/2}$. For any $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ is the vector space of all temperated distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ with the property that $\langle \xi \rangle^s \hat{f} \in L^2(\mathbb{R}^n)$. This is a Hilbert space with the scalar product

$$(f, g)_{H^s(\mathbb{R}^n)} = (\langle \xi \rangle^s \hat{f}, \langle \xi \rangle^s \hat{g})_{L^2(\mathbb{R}^n)}$$

for $f, g \in H^s(\mathbb{R}^n)$.

1.1.3 Ideals of compact operators

Let V and W be Banach spaces over the same vector field $\mathbb{K}$. An operator in $\mathcal{L}(V, W)$ is said to be compact if it maps a ball in V onto a relativly compact subset of W. The vector space of all compact linear mappings acting from V to W is denoted by $\mathcal{K}(V, W)$. This is a closed subspace of the Banach space $\mathcal{L}(V, W)$, and thus a Banach space under the induced norm.

The composition of a compact operator and a bounded operator is always compact. In particular, $\mathcal{K}(V) := \mathcal{K}(V, V)$ is an ideal in $\mathcal{L}(V)$.

More generally, a class of operators $\mathcal{I} \subset \mathcal{L}$ is said to be an operator ideal, if $\mathcal{I}$ is a vector subspace of $\mathcal{L}$ and both BA and AB belong to $\mathcal{I}$ for all $A \in \mathcal{L}$ and $B \in \mathcal{I}$. The zero operators form the trivial ideal classes which will be denoted by $\mathcal{O}$. Another ideal classes are given by the operators of finite rank $\mathcal{F}$. It should be noted that if V is a Banach space then the only closed ideals in $\mathcal{L}(V)$ are $\mathcal{O}(V)$ and $\mathcal{K}(V)$.

Let us recall some essential properties of compact operators on Hilbert spaces. To this end, let V, W be complex Hilbert spaces. We start with the observation that each operator $A \in \mathcal{L}(V, W)$ possesses an adjoint operator $A^* \in \mathcal{L}(W, V)$ which is uniquely defined by

$$(Av, w)_W = (v, A^*w)_V$$

for all $v \in V$ and $w \in W$. Then, the orthogonal complement of the range of A is easily seen to be the kernel of A^*, i.e. the orthogonal sum decomposition $W = \ker A^* \oplus \overline{\operatorname{im} A}$ holds. Note that an operator A is compact if and only if so is A^*.

If $A \in \mathcal{L}(V)$ satisfies $A^* = A$, then A is said to be selfadjoint, and A is called normal if $[A, A^*] = 0$. The eigenvectors of a normal operator corresponding to different eigenvalues are orthogonal. Suppose $A \in \mathcal{K}(V)$ is normal. Then there are an orthonormal system $e_1, e_2, \ldots$ in V and a sequence $\lambda_1, \lambda_2, \ldots$ in $\mathbb{C} \setminus \{0\}$ converging to zero, such that

$$Av = \sum_{k=1}^{\infty} \lambda_k \, (v, e_k) \, e_k$$

for all $v \in V$. In the case $\mathbb{K} = \mathbb{C}$, the numbers λ_k are actually the eigenvalues of A, counted with their multiplicity. This spectral theorem is a generalisation of the principal axis theorem from linear algebra. There is a more general representation theorem for arbitrary compact operators.

Theorem 1.1.11 *Let V and W be Hilbert spaces and $A \in \mathcal{K}(V, W)$. Then, there are orthonormal systems $e_1, e_2, \ldots$ and $f_1, f_2, \ldots$ in V and W, respectively, and a decreasing sequence of nonnegative numbers $s_1 \geq s_2 \geq \ldots$ converging to zero, such that*

$$Av = \sum_{k=1}^{\infty} s_k \, (v, e_k) \, f_k$$

for all $v \in V$.

The so-called singular numbers $s_k = s_k(A)$ in the theorem above are the eigenvalues of the operator $(A^*A)^{1/2}$ in V counted with their geometric multiplicativity. Note that A^*A is compact and selfadjoint. For any $p \geq 1$, we denote by $\mathfrak{S}_p(V, W)$ the set of all $A \in \mathcal{K}(V, W)$ with the property that

$$\Big(\sum_{k=1}^{\infty} |s_k(A)|^p\Big)^{1/p} < \infty.$$

The so-called Schatten classes $\mathfrak{S}_p$ are ideals of compact operators. The classes $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are of particular interest. As before we set $\mathfrak{S}_p(V) := \mathfrak{S}_p(V, V)$. Then $\mathfrak{S}_1(V)$ is the set of trace class operators and $\mathfrak{S}_2(V)$ is the set of Hilbert-Schmidt operators on V. For a trace class operator $A \in \mathfrak{S}_1(V)$ the trace is defined by

$$\operatorname{tr} A := \sum_{k=1}^{\infty} s_k(A)(f_k, e_k).$$

Example 1.1.12 As usual, we denote by $\ell^\infty(\mathbb{K})$ the vector space of all bounded sequences $x = (x_1, x_2, \ldots)$ in $\mathbb{K}$. On setting $\|x\|_\infty := \sup |x_k|$ we get a norm which makes $\ell^\infty(\mathbb{K})$ a Banach space. This space is nonseparable. For $p \in [1, \infty)$, we write $\ell^p(\mathbb{K})$ for the space of all $x \in \ell^\infty(\mathbb{K})$ satisfying

$$\|x\|_p := \Big(\sum_{k=1}^{\infty} |x_k|^p\Big)^{1/p} < \infty.$$

The space $\ell^p(\mathbb{K})$ is a separable Banach space and even a Hilbert space, if $p = 2$. Classical linear operators on these spaces are multiplication operators of the form $A(x_1, x_2, \ldots) = (a_1x_1, a_2x_2, \ldots)$, where $a = (a_1, a_2, \ldots) \in \ell^\infty(\mathbb{K})$. Such an operator is compact if and only if a converges to 0. Moreover, for $A \in \mathfrak{S}_p(\ell^2(\mathbb{K}))$ it is necessary and sufficient that $a \in \ell^p(\mathbb{K})$. For $p = 1$, the trace of A is given by the formula

$$\operatorname{tr} A = \sum_{k=1}^{\infty} a_k.$$

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be Lebesgue measurable sets. For a kernel function $K \in L^2(X \times Y)$, the associated integral operator $A\colon L^2(Y) \to L^2(X)$ given by

$$Au(x) := \int_Y K(x, y)u(y)\, dy$$

is compact. The Hilbert-Schmidt operators on $L^2[a, b]$ are the integral operators with kernels $K \in L^2([a, b]\times[a, b])$. For those K which are smooth on $[a, b] \times [a, b]$, the induced integral operator is of trace class and the trace is given by the integral of $K(x, x)$ over $[a, b]$. This formula can be also used if K is merely continuous. However, it should be noted that the mere continuity of the kernel does not imply the trace class property.

Example 1.1.13 Let $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ be the unit disk in $\mathbb{C}$ and $\mathbb{S}$ its boundary. For any $p \geq 1$, the Hardy space $H^p(\mathbb{S})$ is defined to consist of all L^p-functions on $\mathbb{S}$ which are weak limit values of holomorphic function in $\mathbb{D}$. This is a closed subspace of $L^p(\mathbb{S})$ and, in particular, a Hilbert space, if $p = 2$. By Theorem 1.1.3, there is an orthogonal projection $P \colon L^2(\mathbb{S}) \to H^2(\mathbb{S})$. For $f \in L^\infty(\mathbb{S})$, the Toeplitz operator $T_f \in \mathcal{L}(H^2(\mathbb{S}))$ is given by $T_f := PM_fP$, where $M_f \in \mathcal{L}(L^2(\mathbb{S}))$ is the operator of multiplication with f. Assume that f and g are smooth functions on $\mathbb{S}$. Then the difference $T_{fg} - T_f T_g$ proves to be of trace class operator on $H^2(\mathbb{S})$ and the trace formula

$$\operatorname{tr}[T_f, T_g] = \frac{1}{2\pi\imath} \int_{\mathbb{S}} f\, d_z g$$

holds.

If V is separable, the trace of an operator $A \in \mathfrak{S}_1(V)$ can be evaluated using the formula

$$\operatorname{tr} A = \sum_{k=1}^{\infty} (Ae_k, e_k),$$

where $\{e_k\}_{k \in \mathbb{N}}$ is an arbitrary orthonormal basis of V, see [Fed96, p. 98]. Furthermore, the trace of A can be expressed in terms of the eigenvalues. Indeed, for any $A \in \mathcal{K}(V)$, there is an orthonormal system $e_1, e_2, \ldots$, such that $\lambda_k(A) = (Ae_k, e_k)$ are the eigenvalues of A. It follows that

$$\operatorname{tr} A = \sum_{k=1}^{\infty} \lambda_k(A).$$

This property of the trace is known as Lidskij's theorem. It has a very important consequence. Namely, by A. Pietsch's principle of related operators, if $A \in \mathcal{L}(V, W)$ and $B \in \mathcal{L}(W, V)$ are operators on Hilbert spaces, such that $BA \in \mathcal{K}(V)$, then $AB \in \mathcal{L}(W)$ admits the same eigenvalues as $BA \in \mathcal{L}(V)$. Applying Lidskij's theorem yields

Theorem 1.1.14 *Let V, W be Hilbert spaces and $A : V \to W$, $B : W \to V$ linear operators (may be unbounded), such that $BA \in \mathfrak{S}_1(V)$ and $AB \in \mathfrak{S}_1(W)$. Then* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ *holds.*

1.1.4 Fredholm operators

Suppose that V and W are Banach spaces. An operator $A \in \mathcal{L}(V, W)$ is called Fredholm, if both $\ker A$ and $\operatorname{coker} A := W/\operatorname{im} A$ are of finite dimension. In this case, the integer

$$\operatorname{ind} A := \dim \ker A - \dim \operatorname{coker} A$$

is called the (Fredholm) index of A. Note that the image of a Fredholm operator is always closed. This was first shown by T. Kato.

Topological isomorphisms between Banach spaces are Fredholm operators with index 0. Roughly speaking, Fredholm operators behave like linear operators in finite-dimensional vector spaces.

Example 1.1.15 Choose $m, n \in \mathbb{N}$ and $p \in [1, \infty]$. Let $A\colon \ell^p(\mathbb{K}) \to \ell^p(\mathbb{K})$ be given by

$$(x_1, x_2, \ldots) \mapsto (\underbrace{0, \ldots, 0}_{m \text{ times}}, x_{n+1}, x_{n+2}, \ldots).$$

Then A is a bounded linear operator with

$$\begin{aligned} \ker A &= \{(x_1, x_2, \ldots) \in \ell^p(\mathbb{K}) : x_k = 0 \text{ for } k > n\}, \\ \operatorname{coker} A &\cong \{(x_1, x_2, \ldots) \in \ell^p(\mathbb{K}) : x_k = 0 \text{ for } k > m\}. \end{aligned}$$

Hence, A is Fredholm with $\operatorname{ind} A = n - m$.

The index of an operator provides statements on the solvability of equations. For instance, if the index is positive, then there are nontrivial solutions of the equation $Av = 0$.

Example 1.1.16 Consider a linear ordinary differential equation of order m with smooth variable coefficients in an interval (a, b),

$$u^{(m)}(x) + a_{m-1}(x)u^{(m-1)}(x) + \ldots + a_0(x)u(x) = f(x).$$

It is well known from the general theory that the corresponding homogeneous equation possesses precisely m linearly independent solutions and the inhomogeneous equation is solvable for each right-hand side $f \in C[a, b]$. Write $A : C^m[a, b] \to C[a, b]$ for the associated bounded linear operator in Banach spaces. We conclude readily that the kernel of A is m-dimensional and the cokernel of A trivial. Thus, A is Fredholm with index m.

Example 1.1.17 Let V and W be Hilbert spaces, and suppose $A \in \mathcal{L}(V, W)$ is Fredholm. Since the image of A is closed, we get

$$\begin{aligned} \ker A &\cong \operatorname{coker} A^*, \\ \operatorname{coker} A &\cong \ker A^*. \end{aligned}$$

Hence, A^* is Fredholm and $\operatorname{ind} A^* = -\operatorname{ind} A$. In particular, any selfadjoint operator has index zero.

In the more general context of Banach space there is a statement similar to that of Example 1.1.17. More precisely, each operator $A \in \mathcal{L}(V, W)$ possesses a transposed operator $A^T \in \mathcal{L}(W^*, V^*)$ given by

$$\langle A^T w, v \rangle := \langle w, Av \rangle$$

for $v \in V$ and $w \in W^*$.

Theorem 1.1.18 *Suppose that $A \in \mathcal{L}(V, W)$ and $B \in \mathcal{L}(W, X)$ are Fredholm and $K \in \mathcal{K}(V, W)$. Then*

i) A^T *is Fredholm and* $\operatorname{ind} A^T = -\operatorname{ind} A$*;*

ii) $A + K$ *is Fredholm and* $\operatorname{ind}(A + K) = \operatorname{ind} A$*;*

iii) BA *is Fredholm and* $\operatorname{ind} BA = \operatorname{ind} A + \operatorname{ind} B$*.*

Assume that $A \in \mathcal{L}(V, W)$ is a Fredholm operator with index 0. Then, either $\dim\ker A = 0$ and thus $\operatorname{im} A = W$, or $\dim\ker A = n \in \mathbb{N}$ and the equation $Av = w$ is solvable if and only if $\langle f, w\rangle = 0$ holds for all $f \in \ker A^T$. Since $\dim\ker A^T = n$, we obtain n solvability conditions for w. For A of the form $A = \lambda \operatorname{Id}_V - K$ with $K \in \mathcal{K}(V)$ and $\lambda \in \mathbb{C} \setminus \{0\}$, we arrive at the Fredholm alternative in its classical form.

Example 1.1.19 For $K \in L^1(\mathbb{R})$, the convolution operator $C\colon L^2[0,\infty) \to L^2[0,\infty)$ defined by

$$Cu(x) = \int_0^\infty K(x-y)u(y)\,dy$$

is compact. The Fredholm operators of the form $A := \operatorname{Id} - C$ are called Wiener-Hopf operators.

A linear map $P \in \mathcal{L}(W, V)$ is called a parametrix for $A \in \mathcal{L}(V, W)$ if

$$\begin{array}{rcl} \operatorname{Id}_V - PA & \in & \mathcal{K}(V), \\ \operatorname{Id}_W - AP & \in & \mathcal{K}(W) \end{array}$$

is satisfied. In other words, by a parametrix of A is meant an inverse modulo compact operators. This property can be described by using a familiar construction with quotient spaces which goes back as far as [Cal41]. Given a Banach space Σ, we set

$$\begin{array}{rcl} \phi_\Sigma(V) & := & \mathcal{L}(\Sigma, V)/\mathcal{K}(\Sigma, V), \\ \phi_\Sigma(W) & := & \mathcal{L}(\Sigma, W)/\mathcal{K}(\Sigma, W). \end{array}$$

Furthermore, for $A \in \mathcal{L}(V, W)$, we introduce a map $\phi_\Sigma(A)\colon \phi_\Sigma(V) \to \phi_\Sigma(W)$ by

$$\phi_\Sigma(A)[O] := [A \circ O]$$

for $O \in \mathcal{L}(\Sigma, V)$. This defines a functor ϕ_Σ from the category of Banach spaces to the category of 'Banach algebras', such that

i) $\phi_\Sigma(A) = 0$, if A is compact;

ii) $\phi_\Sigma(BA) = \phi_\Sigma(B)\,\phi_\Sigma(A)$ for all $A \in \mathcal{L}(V, W)$ and $B \in \mathcal{L}(W, Z)$;

iii) $\phi_\Sigma(\operatorname{Id}_V) = \operatorname{Id}_{\phi_\Sigma(V)}$.

If $\Sigma = V$, then the quotient space $\phi_\Sigma(V) = \mathcal{L}(V)/\mathcal{K}(V)$ is a Banach algebra, indeed.

Let $A \in \mathcal{L}(V, W)$. The operator $\phi_\Sigma(A)$ proves to be invertible for each Banach space Σ if and only if there is an operator $P \in \mathcal{L}(W, V)$ with the property that

$$\begin{array}{rcl} \phi_\Sigma(P)\,\phi_\Sigma(A) & = & \operatorname{Id}_{\phi_\Sigma(V)}, \\ \phi_\Sigma(A)\,\phi_\Sigma(P) & = & \operatorname{Id}_{\phi_\Sigma(W)} \end{array}$$

for all Banach spaces Σ.

Theorem 1.1.20 *Let V and W be Banach spaces and $A \in \mathcal{L}(V,W)$. The following are equivalent:*

i) A is Fredholm.

ii) A possesses a parametrix.

iii) $\phi_\Sigma(A)$ is invertible for each Banach space Σ.

The equivalences above are known as theorem of Th. Atkinson.

Example 1.1.21 Suppose $f \in C(\mathbb{S})$ is a nonvanishing complex-valued function on the unit circle. Then the Toeplitz operator T_f is Fredholm and the Gokhberg-Krein index formula gives

$$\operatorname{ind} T_f = -\deg(f,0),$$

where

$$\deg(f,0) = \frac{1}{2\pi\imath}\int_{\mathbb{S}} \frac{d_z f}{f}$$

is the winding number of f. Note that a parametrix of T_f is given by $T_{1/f}$.

If A is the Wiener-Hopf operator with kernel $K = \mathscr{F}^{-1}(f \circ C - 1)$, where

$$C(\xi) = \frac{\xi - \imath}{\xi + \imath}$$

is the Caley transform, then $\operatorname{ind} A = \operatorname{ind} T_f$ holds.

1.2 Analysis on manifolds

In this section we describe some concepts and results of differential geometry. We assume the reader is familiar with the concept of smooth manifolds.

1.2.1 Vector bundles

Let F and X be smooth manifolds. A smooth map $\pi : F \to X$ is called a smooth $\mathbb{K}$-vector bundle of rank k , if

i) $F_p := \pi^{-1}(p)$ is a $\mathbb{K}$-vector space of dimension k for all $p \in X$ (F_p is called the fibre of F at p).

ii) For any $p \in X$ there is an open set U containing p and a diffeomorphism $t : \pi^{-1}(U) \to U \times \mathbb{K}^k$, such that $t(F_p) \subset \{p\} \times \mathbb{K}^k$ and $F_p \xrightarrow{t} \{p\} \times \mathbb{K}^k \xrightarrow{\text{proj}} \mathbb{K}^k$ is an isomorphism.

The pair $\{U, t\}$ is said to be a local trivialisation of the bundle F close to the point p.

The manifold F itself is sometimes also called smooth vector bundle over X and a map $s \in C^{(\infty)}(X,F)$ is called a (smooth) section if $s(p) \in F_p$ holds for all

$p \in X$. The topological vector space of smooth sections will be denoted by $\mathcal{E}(X, F)$. The (vector) subspace of $\mathcal{E}(X, F)$ consisting of all sections with compact support is denoted by $\mathcal{D}(X, F)$. However, it is given another topology analogous to that described in Section 1.1.2, which makes it complete. Obviously, if X is compact, we obtain $\mathcal{D}(X, F) = \mathcal{E}(X, F)$.

Example 1.2.1 Let X be a smooth manifold. The map $\pi : X \times \mathbb{K}^n \to X$ projecting the Cartesian product onto the first factor is obviously a smooth vector bundle. It is called trivial.

Let X be a smooth manifold and TX the disjunct union of tangent spaces T_pX over all $p \in X$. Define $\pi : TX \to X$ by $\pi(p, v) = p$ for $p \in X$ and $v \in T_pX$. This is a smooth vector bundle over X called the tangent bundle. On passing to the dual spaces in fibres we obtain the so-called cotangent bundle T^*X, which is also a smooth vector bundle over X.

Example 1.2.2 Let F be a smooth vector bundle over X. Suppose that Y is another manifold and f a smooth mapping of Y to X. The vector bundle over Y whose fibre over $y \in Y$ is $F_{f(y)}$ is called the induced bundle (or "pull-back") of F by f and denoted by f^*F. The vector bundle structure is straightforward. For instance, if $\pi : T^*X \to X$ is the cotangent bundle, then π^*F denotes the induced bundle over T^*X.

In order to construct more bundles we turn ourselves to classical operations with vector spaces. To this end, suppose that V and W are vector spaces over the same field. Their direct sum $V \oplus W$ is known to be a vector space of dimension $\dim V + \dim W$. We get another vector space if we endow $V \oplus W$ with the equivalence relation "$(v_1, w_1) \sim (v_2, w_2)$ if and only if $v_1 - v_2 = 0$ or $w_1 - w_2 = 0$." The quotient space obtained in this way is called the tensor product of V and W and it is denoted by $V \otimes W$. The dimension of $V \otimes W$ just amounts to the product of the dimensions of V and W.

Let V be a vector space of dimension n and $k \in \mathbb{N}$. We denote by $\mathcal{S}^i V^*$ the vector space of all symmetric k-linear forms on V. Its dimension is the binomial coefficient C^k_{n+k-1}. The space of all alternating k-linear forms on V is denoted by $\Lambda^k V^*$, its dimension is C^k_n. The exterior product of linear forms $f_1, \ldots, f_k$ on V is the element of $\Lambda^k V^*$ given by

$$(f_1 \wedge \ldots \wedge f_k)\,(v_1, \ldots, v_k) := \det\left(\langle f_i, v_j \rangle\right)_{\substack{i=1,\ldots,k \\ j=1,\ldots,k}}$$

for $v_1, \ldots, v_k \in V$.

Example 1.2.3 The classical operations with vector spaces extend naturally to operations with vector bundles. In particular, given any vector bundles E and F over a manifold X, the direct sum $E \oplus F$, the tensor product $E \otimes F$, the dual bundle F^*, the symmetric product $\mathrm{S}^k F^*$ and the exterior product $\Lambda^k F^*$ are well defined. If E and F are vector bundles over different manifolds X and Y, then the exterior tensor product $E \boxtimes F$ of E and F is the vector bundle over $X \times Y$ whose fibre over $(x, y) \in X \times Y$ is $E_x \otimes F_y$.

Suppose that $\pi_E \colon E \to X$ and $\pi_F \colon F \to X$ are smooth vector bundles over the same manifold. A map $h \in C^\infty(E, F)$ is called a smooth bundle homomorphism if h maps the fibre E_p into the fibre F_p for all $p \in X$ and the restriction $h \colon E_p \to F_p$ is a linear map of vector spaces for any $p \in X$. In other words, by a bundle homomorphism is meant a family of linear mappings in fibres smoothly depending on the point of the base. The bundle homomorpisms constitute a vector bundle over X denoted by $\mathrm{Hom}(E, F)$.

Example 1.2.4 Let X be a smooth manifold and X_1, X_2 be open subsets of X whose union is X. Suppose F_1 and F_2 are smooth bundles over X_1 and X_2, respectively, such that there is a bundle isomorphism $h : F_1 \restriction_Y \to F_2 \restriction_Y$, where $Y = X_1 \cap X_2$. In particular, F_1 and F_2 are of the same rank. We obtain a vector bundle $F_1 \cup_h F_2$ over X by identifying F_1 and F_2 over Y under h. To this end, consider the bundle $F_1 \restriction_Y \oplus F_2 \restriction_Y$ and endow it with the equivalence relation "$(f_1, f_2) \sim 0$ if and only if $f_2 = h(f_1)$" in the fibres.

A Riemannian metric on a manifold X is a smooth section g_X of the bundle $\mathcal{S}^2 T^* X$, such that $g_X(p)$ is a scalar product on $T_p X$ for any $p \in X$. A Riemannian metric allows us to measure length and angles on X, and so to integrate functions over the manifold. In order to integrate sections of vector bundles, we introduce the concept of Hermitian metric in a bundle F. This is a family of scalar products $\langle \cdot, \cdot \rangle_x$ in the fibres of F, which is smooth in $x \in X$ in the sense that, for any open set $U \subset X$ and sections $f, g \in \mathcal{E}(U, F)$, the function $x \mapsto \langle f(x), g(x) \rangle_x$ is smooth in U. This enables us to define the spaces $L^p(X, F)$. Any smooth vector bundle admits a Hermitian metric, cf. [Wel80].

1.2.2 Complexes and cohomology

By a sequence $\{V^\cdot, D\}$ of topological vector spaces of length $N \in \mathbb{N}$ is meant any object of the form

$$\{V^\cdot, D\} \colon\ 0 \longrightarrow V^0 \stackrel{D^0}{\longrightarrow} V^1 \stackrel{D^1}{\longrightarrow} \ldots \stackrel{D^{N-1}}{\longrightarrow} V^N \longrightarrow 0 \qquad (1.2.1)$$

where $V^0, V^1, \ldots, V^N$ are $\mathbb{K}$-vector spaces and $D^0, D^1, \ldots, D^{N-1}$ continous linear mappings. For simplicity we set $V^i = 0$ for $i \in \mathbb{Z} \setminus \{0, 1, \ldots, N\}$ as well as $D^i = 0$ for $i \in \mathbb{Z} \setminus \{0, 1, \ldots, N-1\}$.

The sequence (1.2.1) is called a (cochain) complex , if

$$D^{i+1} D^i = 0$$

for all $i = 0, 1, \ldots, N-1$. For each complex, the differential D is associated by $Dv = D^i v$ for $v \in V^i$. Since $D^2 = 0$ the differential is nilpotent.

A differential operator D^0 on an open subset $U \subset \mathbb{R}^n$ is called overdetermined, if there is a differential operator D^1 with $D^1 D^0 = 0$, i.e. if the sequence

$$0 \longrightarrow C^\infty(U, \mathbb{K}^{i_0}) \stackrel{D^0}{\longrightarrow} C^\infty(U, \mathbb{K}^{i_1}) \stackrel{D^1}{\longrightarrow} C^\infty(U, \mathbb{K}^{i_2}) \longrightarrow 0$$

is a complex. In this case $D^1 f = 0$ is a necessary condition for the inhomogeneous equation $D^0 u = f$ to be solvable.

Example 1.2.5 Let for example $U \subset \mathbb{R}^2$ and consider the sequence

$$0 \longrightarrow C^\infty(U,\mathbb{R}) \xrightarrow{D^0} C^\infty(U,\mathbb{R}^2) \xrightarrow{D^1} C^\infty(U,\mathbb{R}) \longrightarrow 0$$

with $D^0 u = \operatorname{grad} u = (\partial_x u, \partial_y u)^T$ and $D^1(f_1,f_2)^T = \partial_y f_2 - \partial_x f_1$. Then we obtain

$$D^1 D^0 u = (\partial_y \partial_x - \partial_x \partial_y) u = 0$$

due to the Schwarz lemma. Hence, the sequence is a complex.

Example 1.2.6 Similarly, in the case $U \subset \mathbb{R}^3$ the standard differential operators of vector analysis lead to the complex

$$0 \longrightarrow C^\infty(U,\mathbb{R}) \xrightarrow{\text{grad}} C^\infty(U,\mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(U,\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(U,\mathbb{R}) \longrightarrow 0.$$

Obviously, if $\{V^\cdot, D\}$ is a complex, then $\operatorname{im} D^{i-1} \subset \ker D^i$ is satisfied, i.e. $\operatorname{im} D^{i-1}$ is a subspace of $\ker D^i$. Hence, the quotient space

$$H^i(V^\cdot) := \ker D^i / \operatorname{im} D^{i-1},$$

the cohomology of the complex at step i, is well defined. Moreover, a complex $\{V^\cdot, D\}$ is called exact if its cohomology vanishes at each step.

Example 1.2.7 Let $A\colon V^0 \to V^1$ be a continuous linear map between topological vector spaces. Then A defines the so-called short complex

$$\{V^\cdot, A\}\colon\ 0 \longrightarrow V^0 \xrightarrow{A} V^1 \longrightarrow 0$$

whose cohomology is

$$\begin{array}{rclcl} H^0(V^\cdot) & = & \ker A/\{0\} & = & \ker A, \\ H^1(V^\cdot) & = & W/\operatorname{im} A & =: & \operatorname{coker} A. \end{array}$$

Consequently, if A is Fredholm, we obtain $\operatorname{ind} A = \dim H^0(V^\cdot) - \dim H^1(V^\cdot)$.

Suppose that $\{V^\cdot, D\}$ is a complex with finite dimensional cohomology. Then the Euler characteristic of the complex is defined by

$$\chi(V^\cdot) := \sum_i (-1)^i \dim H^i(V^\cdot).$$

Obviously, this is a generalisation of the index of an operator.

Example 1.2.8 Let $\{V^\cdot, D\}$ be a complex whose spaces $V^0, V^1, \ldots, V^N$ are of finite dimension. Then a familiar formula of linear algebra (sometimes called the Euler formula) implies

$$\chi(V^\cdot) = \sum_{i=0}^N (-1)^i \dim V^i.$$

Let X be a smooth manifold of dimension n and $i \in \{0, 1, \ldots, n\}$. We denote by $\Omega^i(X) := \mathcal{E}(X, \Lambda^i T^*X)$ the space of all differential forms of degree i on X with smooth coefficients. It should be noted that in order to consider differential forms with complex coefficients one complexifies the cotangent bundle $T^*_{\mathbb{C}}X = T^*X \otimes \mathbb{C}$ and sets

$$\Omega^i(X) = \mathcal{E}(X, \Lambda^i T^*_{\mathbb{C}}X).$$

Locally any $\omega \in \Omega^i(X)$ looks like

$$\omega(x) = \sum_{\substack{J=(j_1,\ldots,j_i)\\ 1\le j_1<\ldots<j_i\le n}} \omega_J(x)\, dx^J$$

for $x = (x^1, \ldots, x^n)$ in a coordinate patch U of X, where $dx^J = dx^{j_1} \wedge \ldots \wedge dx^{j_i}$ and $\omega_J \in C^\infty(U)$.

Remark 1.2.9 A manifold is said to be orientable, if there is an $\omega \in \Omega^n(X, \mathbb{R})$ such that $\omega(p) \neq 0$ for all $p \in X$.

The differential of a function $f \in C^\infty(X)$ is an element of $\Omega^1(X)$ given in local coordinates by $df := \partial_1 f dx_1 + \ldots + \partial_n f dx_n$. It extends to a sequence of linear mappings $d\colon \Omega^i(X) \to \Omega^{i+1}(X)$, for $i = 1, \ldots, n$, satisfying the product rule

$$d(\upsilon \wedge \omega) = d\upsilon \wedge \omega + (-1)^i \upsilon \wedge d\omega$$

whenever the degree of υ is i.

The map d is called the exterior or Cartan derivative. In a local chart $x = h(p)$ with $h : U \to \mathbb{R}^n$ this map is given by

$$d\omega(x) = \sum_{\substack{J=(j_1,\ldots,j_i)\\ 1\le j_1<\ldots<j_i\le n}} d\omega_J(x) \wedge dx^J$$

for a differential form ω of degree i. It is easy to verify that d is nilpotent, i.e. $d \circ d = 0$. A form ω is said to be closed, if $d\omega = 0$ holds. Suppose that there is a $\upsilon \in \Omega^{i-1}(X)$, such that $d\upsilon = \omega$. Then ω is said to be exact. From what has been said it follows that the sequence

$$\Omega^\cdot(X)\colon\ 0 \longrightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n(X) \longrightarrow 0$$

is actually a complex. It is called the de Rham complex whose cohomology at step i is

$$H^i_{\mathrm{dR}}(X) := H^i(\Omega^\cdot(X)).$$

The de Rham complex is a classical example of a complex and we will come back to this example frequently. For this reason let us study some properties of this complex.

Example 1.2.10 Suppose that X has m connected components. If $f \in C^\infty(X)$ satisfies $df = 0$ in X, then this function is constant on each connected component of X. Hence,

$$H^0_{\mathrm{dR}}(X) \cong \mathbb{K}^m,$$

where $\mathbb{K}$ is the corresponding field.

Obviously the dimension of de Rham cohomology $H^i_{\mathrm{dR}}(X)$ is determined by certain topological properties of the underlying manifold X. Let us specify this fact a bit more precise. To this end suppose that Y is another manifold and $f \in C^\infty(X, Y)$ is a smooth map. Then there is a linear map $T_x f : T_x X \to T_{f(x)} Y$ which induces a natural linear map

$$T_x^* f : \mathcal{L}(T_{f(x)} Y, \mathbb{R}) \to \mathcal{L}(T_x X, \mathbb{R}).$$

The pull back $f^* : \Omega^q(Y) \to \Omega^q(X)$ is defined pointwise by

$$(f^*\omega)(x) := (T_x^* f)\omega(f(x))$$

for $\omega \in \Omega^q(Y)$ and $x \in X$.

The pullback operator satisfies

$$f^*(d\omega) = d(f^*\omega). \tag{1.2.2}$$

Furthermore, $\mathrm{Id}_X^* = \mathrm{Id}_{\Omega^\cdot(X)}$ and $f^*(\upsilon \wedge \omega) = f^*\upsilon \wedge f^*\omega$ hold. If Z is another manifold and $g \in C^\infty(Y, Z)$, we obtain $(g \circ f)^* = f^* \circ g^*$.

From (1.2.2) we deduce that each smooth map $f : X \to Y$ induces a homomorphism of de Rham cohomology

$$Hf^* : H^i_{\mathrm{dR}}(Y) \to H^i_{\mathrm{dR}}(X)$$

given by $[\omega] \mapsto [f^*\omega]$. The properties of the pullback operator on forms extend to the pullback operator on cohomology. In particular, Hf^* is an isomorphism, if f is a diffeomorphism. Hence, the diffeomorphic manifolds posssess isomorphic cohomology.

Two maps $f, g \in C^\infty(X, Y)$ are said to be homotopic, if there is a family of maps $h \in C^\infty([0,1] \times X, Y)$ with the property that $h(0, \cdot) = f$ and $h(1, \cdot) = g$. In this case we write $f \simeq g$. The manifold X is called contractible, if the identity map Id_X is homotopic to a constant map $c : X \to X$. Note that $Hf^* = Hg^*$ holds if f and g are homotopic.

Example 1.2.11 Let X be contractible (e.g. $X = \mathbb{R}^n$). Then Example 1.2.10 and the Poincaré lemma imply

$$H^i_{\mathrm{dR}}(X) \cong \begin{cases} \mathbb{R}, & \text{if } i = 0, \\ \{0\}, & \text{if } i \neq 0, \end{cases}$$

and thus $\chi(\Omega^\cdot(X)) = 1$. Defining $\iota : \mathbb{R} \to \mathcal{E}(X)$ by $\iota(c)(x) := c$, we obtain the exact sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{\iota} \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(X) \longrightarrow 0.$$

Two manifolds X and Y are said to be homotopically equivalent, if there are maps $f \in C^\infty(X,Y)$ and $g \in C^\infty(Y,X)$, such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$ is satisfied.

Theorem 1.2.12 *Homotopically equivalent manifolds possess isomorphic de Rham cohomology.*

Example 1.2.13 We consider the punctured plane $\mathbb{R}^2 \setminus \{0\}$. Example 1.2.10 implies $H^0_{\mathrm{dR}}(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{K}$. In order to find the remaining cohomology groups we first mention that $\mathbb{S}^1$ and $\mathbb{R}^2 \setminus \{0\}$ are homotopically equivalent, which can be easily verified by using

$$\begin{array}{rrcllcl} f: & \mathbb{S}^1 & \to & \mathbb{R}^2 \setminus \{0\}, & x & \mapsto & x, \\ g: & \mathbb{R}^2 \setminus \{0\} & \to & \mathbb{S}^1, & x & \mapsto & x/|x|, \\ h: & [0,1] \times (\mathbb{R}^2 \setminus \{0\}) & \to & \mathbb{R}^2 \setminus \{0\}, & (t,x) & \mapsto & x/|x|^t. \end{array}$$

Since $\mathbb{S}^1$ is of dimension 1, we obtain $H^2_{\mathrm{dR}}(\mathbb{R}^2 \setminus \{0\}) \cong H^2_{\mathrm{dR}}(\mathbb{S}^1) \cong \{0\}$. Now, we consider the differential form $\omega \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$ given by

$$\omega(x,y) = \frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx.$$

Since ω is closed, the cohomology class $[\omega]$ is well defined. Using the parametrisation $\varphi \mapsto (\cos\varphi, \sin\varphi)$ of the unit circle with $\varphi \in [0, 2\pi]$ we get

$$\int_{\mathbb{S}^1} \omega = 2\pi,$$

hence, ω is not exact, for the integral does not vanish. The pullback $f^*\omega$ coincides with the restriction of ω to $\mathbb{S}^1$, which is $d\varphi$ in the above coordinate φ. Let $\upsilon \in \Omega^1(\mathbb{S}^1)$ be another 1-form. In the local coordinate the form υ looks like $\upsilon(\varphi) = c(\varphi)d\varphi$, where c is a smooth periodic function on $\mathbb{R}$. Setting

$$m = \frac{1}{2\pi}\int_0^{2\pi} c(\varphi)d\varphi$$

we define a smooth function u on $[0, 2\pi]$ by

$$u(\varphi) = \int_0^{\varphi} c(t)dt - m\varphi$$

for $\varphi \in [0, 2\pi]$. Obviously, $u(0) = u(2\pi) = 0$ and $du = \upsilon - m\omega$, i.e. the difference $\upsilon - m\omega$ is exact. Hence, $[\upsilon] = m[\omega]$ whence $H^1_{\mathrm{dR}}(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{R}$. On summarising we find

$$\chi(\Omega^\cdot(\mathbb{R}^2 \setminus \{0\})) = 1 - 1 + 0 = 0$$

for the Euler characteristic of the punctured plane.

Let us clarify the origin of the Euler characteristic. For this purpose we consider a polyhedron P with c corners, e edges and f faces. Then the integer

$$\chi(P) = c - e + f$$

is called the Euler characteristic of the polyhedron. Note that if P is convex then the well-known polyhedron formula of Euler $\chi(P) = 2$ holds. Further, let $\mathcal{S}$ be a compact closed surface of genus g. The Euler characteristic of $\mathcal{S}$ is given by

$$\chi(\mathcal{S}) = \begin{cases} c - e + f - 2g, & \text{if } S \text{ is orientable,} \\ c - e + f - g, & \text{if } S \text{ is not orientable,} \end{cases}$$

where c, e and f stand for the numbers of corners, edges and surfaces of an arbitrary triangulation of $\mathcal{S}$, respectively, cf. [CR96]. It turns out that this definition does not depend on the particular choice of triangulation. Moreover,

$$\chi(\mathcal{S}) = \chi(\Omega^{\cdot}(\mathcal{S})),$$

which is due to the Hodge theorem. For instance, we get $\chi(\Omega^{\cdot}(\mathbb{S}^2)) = 2$, where $\mathbb{S}^2$ is the 2-dimensional sphere in $\mathbb{R}^3$.

1.2.3 Characteristic objects

Let V be a $\mathbb{K}$-vector space of dimension n and $A \in \mathcal{L}(V)$. Then, all essential properties of A are encoded in the characteristic polynomial

$$p_A(\lambda) := \det(\lambda \operatorname{Id}_V - A) = \sum_{k=0}^{n} (-1)^{n-k} a_k \lambda^k$$

with $a_0 = 1$. The numbers a_k are certain characteristic numbers of the map A. In particular, $a_n = \det A$ and $a_1 = \operatorname{tr} A$ hold. Further examples of such numbers are the trace of an operator $A \in \mathfrak{S}_1(V)$ on a Hilbert space V and the Euler characteristic of a complex. Moreover, the winding number of Example 1.1.19 is a characteristic number. This is a very particular case of the more abstract mapping degree, which can be defined for different types of functions. We refer the reader to [Nir74] for explanations of the mapping degree of Leray and Schauder and local degree of selfmappings on a manifold.

Let F be a smooth complex vector bundle over a manifold X. By a connection on F is meant a first order differential operator

$$\partial_F : \mathcal{E}(X, F) \to \mathcal{E}(X, F \otimes T^* X)$$

satisfying the Leibniz rule

$$\partial_F(fu) = df\, u + f\, \partial_F u$$

for all $u \in \mathcal{E}(X, F)$ and $f \in C^\infty(X)$. The Leibniz rule allows one to extend the connection to the differential forms of arbitrary degree q with coefficients in F on X by requiring that

$$\partial_F(f \wedge u) = (df) \wedge u + (-1)^p f \wedge \partial_F u$$

holds for all $u \in \Omega^q(X,F) := \mathcal{E}(X, F \otimes \Lambda^q T^* X)$ and $f \in \Omega^p(X)$. We thus arrive at the sequence

$$\Omega^{\cdot}(X,F):\ 0 \longrightarrow \Omega^0(X,F) \xrightarrow{\partial_F} \Omega^1(X,F) \xrightarrow{\partial_F} \dots \xrightarrow{\partial_F} \Omega^n(X,F) \longrightarrow 0.$$

The operator $\Omega = \partial_F^2$ is a differential operator of order 0 and is called the curvature of the connection ∂_F. More precisely, this is a matrix with entries in differential forms of degree 2 on X.

If $f(z)$ is an analytic function in a neighbourhood of $z = 0$ then it expands in powers of z. On substituting $z = \Omega$ we define $f(\Omega)$, provided that the power series converges. This is the case, indeed, for Ω^k vanishes if $2k$ exceeds the dimension of X, and so the power series breaks. The characteristic classes of the bundle F are defined by using a curvature of F. They are actually independent modulo cohomology on the particular choice of the curvature ∂_F.

Example 1.2.14

$$\begin{aligned} \mathrm{ch}\,(F) &= \mathrm{tr}\ \exp \omega, \\ \mathcal{T}\,(F) &= \det \Big(\frac{\omega}{1 - \exp(-\omega)} \Big) \end{aligned}$$

with

$$\omega = -\frac{\Omega}{2\pi\imath}$$

are the Chern character and the Todd class of F, respectively, see for instance [Pal65], [Wel80].

The set of all equivalence classes of isomorphic vector bundles over X form an Abelian semigroup with operation $[F_1] + [F_2] := [F_1 \oplus F_2]$. Since from the equality $[E] + [F_1] = [E] + [F_2]$ it follows that $[F_1] = [F_2]$, which is sometimes referred to as cancelation rule, this semigroup can be extended to a group which is denoted by $K(X)$. This group forms actually a ring under the second operation $[E] \cdot [F] := [E \otimes F]$. By setting

$$\mathrm{ch}\,([F_1] \pm [F_2]) = \mathrm{ch}\,(F_1) \pm \mathrm{ch}\,(F_2)$$

the Chern character extends to $K(X)$. In particular, $\mathrm{ch}\,(E \otimes F) = \mathrm{ch}\,(E) \wedge \mathrm{ch}\,(F)$ is satisfied.

Next, let X be an arbitrary (not necessarily compact) manifold. We consider the compactification $X^+ = X \cup \{\infty\}$ and set $K^{\mathrm{comp}}(X) = \ker \iota^*$, where $\iota \colon \{\infty\} \to X^+$ is the inclusion of the point ∞. Suppose F_1 and F_2 are vector bundles of the same rank over X and $h : F_1 \restriction_{X \setminus K} \to F_2 \restriction_{X \setminus K}$ a bundle isomorphism away from a compact set $K \subset X$. Choose a relative compact neighbourhood X_1 of K in X and define $X_2 := X \setminus K$ and $Y := X_1 \cap X_2$. Let F be a vector bundle over X_1, such that $F_1 \restriction_{X_1} \oplus F$ is trivial. Then $F_2 \restriction_Y \oplus F \restriction_Y$ is trivial. Pick a trivialisation $t_2 : (F_2 \oplus F) \restriction_Y \to Y \times \mathbb{C}^N$ and set $t_1 := t_2 \circ (h \oplus \mathrm{Id})$. In this manner we obtain the vector bundles

$$\begin{aligned} G_1 &:= X_1 \times F \cup_{t_1} (X^+ \setminus K) \times \mathbb{C}^N, \\ G_2 &:= X_1 \times F \cup_{t_2} (X^+ \setminus K) \times \mathbb{C}^N \end{aligned}$$

over X^+. This allows us to introduce the difference bundle

$$\mathrm{d}(h) := [G_1] - [G_2] \in K^{\mathrm{comp}}(X)$$

with the Chern character $\mathrm{ch}\,(\mathrm{d}(h)) = \mathrm{ch}\,(G_1) - \mathrm{ch}\,(G_2)$.

1.3 Pseudodifferential operators

In this section we sketch a calculus of pseudodifferential operators on a compact manifold without boundary. We will follow the presentations of the books [Hoe85] and [Shu87].

1.3.1 Classical pseudodifferential operators

Let $m \in \mathbb{R}$. As usual, a function $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is called positively homogeneous of degree m, if

$$f(\lambda\xi) = \lambda^m f(\xi)$$

is satisfied for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and all $\lambda > 0$. This holds if and only if Euler's equation $\langle f'(\xi), \xi\rangle = m\, f(\xi)$ is satisfied for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

Let U be an open set in $\mathbb{R}^n$. We denote by $\mathcal{S}^m(U \times \mathbb{R}^n)$ the space of all smooth functions $a \in C^\infty(U \times \mathbb{R}^n)$ with the property that for each multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ and any compact set $K \subset U$ there exists a constant $c_{\alpha,\beta,K} > 0$, such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le c_{\alpha,\beta,K}\, \langle\xi\rangle^{m-|\beta|}$$

for all $(x,\xi) \in K \times \mathbb{R}^n$. The elements of $\mathcal{S}^m(U \times \mathbb{R}^n)$ are called symbols and those of

$$\mathcal{S}^{-\infty}(U \times \mathbb{R}^n) = \bigcap_m \mathcal{S}^m(U \times \mathbb{R}^n)$$

smoothing symbols.

To any symbol $a \in \mathcal{S}^m(U \times \mathbb{R}^n)$ we assign the canonical pseudodifferential operator $A = a(x, D)$ by

$$Au\,(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\imath\langle x,\xi\rangle} a(x,\xi)\hat{u}(\xi) d\xi$$

for $u \in \mathcal{D}(U)$, where $\hat{u}$ is the Fourier transform of u. Note that A maps $\mathcal{D}(U)$ continuously into $C^\infty(U)$. The function $\sigma(A) := a$ is called the symbol of A.

We now want to consider classical pseudodifferential operators. They form an important subclass of canonical pseudodifferential operators which is closed under basic operations. Classical pseudodifferential operators were introduced in 1965 by J. J. Kohn and L. Nirenberg who reinforced the theory of S. G. Michlin, A. P. Calderon, etc. The main property of this class is the existence of a principal symbol. More precisely, a symbol $a \in \mathcal{S}^m(U \times \mathbb{R}^n)$ is said to be classical (or multihomogeneous) if there is a sequence $\{a_{m-j}\}_{j=0,1,\ldots}$ of functions $a_{m-j} \in C^\infty(U \times (\mathbb{R}^n \setminus \{0\}))$

positively homogeneous of degree $m - j$ in ξ, such that

$$a - \chi \sum_{j=0}^{N} a_{m-j} \in \mathcal{S}^{m-N-1}(U \times \mathbb{R}^n)$$

for all $N = 0, 1, \ldots$, where $\chi \in C^\infty(\mathbb{R}^n)$ is a cut-off function with respect to $\xi = 0$. Obviously, all the components a_{m-j} are uniquely determined by a. A canonical pseudodifferential operator A on U is called classical if its symbol $\sigma(A)$ is classical. The set of all classical pseudodifferential operators of degree m on U is denoted by $\Psi^m_{\mathrm{cl}}(U)$. The component

$$\sigma^m(A) := a_m$$

is called the principal symbol of A. The set of all (classical) pseudodifferential operators will be denoted by $\Psi^m_{(\mathrm{cl})}(U)$.

Example 1.3.1 Any linear partial differential operator A of order m on U has the form

$$A(x, D) := \sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha,$$

where $A_\alpha \in C^\infty(U)$. This is a classical pseudodifferential operator with symbol $\sigma(A)(x, \xi) = A(x, \xi)$. The principal symbol of A is

$$\sigma^m(A)(x, \xi) = \sum_{|\alpha| = m} A_\alpha(x) \xi^\alpha.$$

Now, let us consider pseudodifferential operator that act on vector valued functions. An operator $A \in \mathcal{L}(\mathcal{D}(U, \mathbb{C}^k), C^\infty(U, \mathbb{C}^l))$ is called (classical) pseudodifferential operator of order m, if there are pseudodifferential operators $A_{i,j} \in \Psi^m_{(\mathrm{cl})}(U)$, such that

$$Au = (A_{i,j})_{\substack{i=1,\ldots,l \\ j=1,\ldots,k}} u$$

for all $u \in \mathcal{D}(U, \mathbb{C}^k)$. The principal symbol of A is defined to be the $(l \times k)$-matrix $\sigma^{(m)}(A) = \big(\sigma^{(m)}(A_{i,j})\big)$ and the set of (classical) operators will be denoted by $\Psi^m_{(\mathrm{cl})}(U; \mathbb{C}^k, \mathbb{C}^l)$. Note that the elements of $\Psi^{-\infty}(U; \mathbb{C}^k, \mathbb{C}^l)$ are called smoothing operators.

Suppose U and V are open sets in $\mathbb{R}^n$ and $f : U \to V$ a diffeomorphism. For an operator $A \in \Psi^m_{\mathrm{cl}}(U; \mathbb{C}^k, \mathbb{C}^l)$ the pushforward of A under f is given by

$$f_* A\,(u) := f_*(A\, f^* u) = (A(u \circ f)) \circ f^{-1}$$

for $u \in \mathcal{D}(V, \mathbb{C}^k)$.

Theorem 1.3.2 *Let $f : U \to V$ be a diffeomorphism and $A \in \Psi^m_{\mathrm{cl}}(U; \mathbb{C}^k, \mathbb{C}^l)$. Then $f_* A \in \Psi^m_{\mathrm{cl}}(V; \mathbb{C}^k, \mathbb{C}^l)$ holds and*

$$\sigma^m(f_* A)(f(x), \eta) = \sigma^m(A)(x, (f'(x))^T \eta)$$

for any $x \in U$ and $\eta \in \mathbb{R}^n \setminus \{0\}$, where $f'(x)$ is the Jacobi matrix of f at x.

Since $U \times \mathbb{R}^n \cong T^*U$, the principal symbol of a classical pseudodifferential operator can be thought of as a matrix-valued function away from the zero section of T^*U. The transformation rule of Theorem 1.3.2 shows that the principal symbol can be actually defined for pseudodifferential operators on a manifold as a function on the cotangent bundle with values in the bundle homomorphisms, see for instance [Shu87].

1.3.2 Pseudodifferential operators on manifolds

Let E and F be smooth vector bundles over a smooth manifold X and $m \in \mathbb{R}$. A map $A : \mathcal{D}(X, E) \to \mathcal{E}(X, F)$ is called a (classical) pseudodifferential operator of order m, if for each coordinate patch $U \subset X$ over which E and F are trivial, each choice of local trivialisations and each set $V \Subset U$, the operator A_V given by the commutative diagram

$$\begin{CD}
\mathcal{D}(X,E)\restriction_V @>{A}>> \mathcal{E}(X,F)\restriction_V \\
@A{\subset}AA @VV{\subset}V \\
\mathcal{D}(V,E) @>>> \mathcal{E}(V,F) \\
@A{\cong}AA @VV{\cong}V \\
\mathcal{D}(V,\mathbb{C}^k) @>{A_V}>> \mathcal{E}(V,\mathbb{C}^l)
\end{CD}$$

is a canonical pseudodifferential operator $A_V \in \Psi^m_{(\mathrm{cl})}(V; \mathbb{C}^k, \mathbb{C}^l)$. The space of all (classical) pseudodifferential operator between sections of vector bundles E and F on X is denoted by $\Psi^m_{(\mathrm{cl})}(X; E, F)$. The elements of $\Psi^{-\infty}(X; E, F)$ are called smoothing operators.

Let $\pi\colon T^*X \to X$ be the canonical projection and π^*E, π^*F the induced bundles over T^*X. In local coordinates we obtain

$$A_V u = \begin{pmatrix} A_{1,1} & \dots & A_{1,k} \\ & \dots & \\ A_{l,1} & \dots & A_{l,k} \end{pmatrix} u.$$

This allows us to define $\sigma^m(A)(x, \xi)$ in local coordinates by $\sigma^m(A_V)(x, \xi)$. From what is said after Theorem 1.3.2 we deduce that $\sigma^m(A)$ is a well-defined homogeneous function of degree m on $T^*X \setminus \{0\}$ with values in $\mathrm{Hom}(E, F)$,

$$\sigma^m(A)\colon \pi^*E \to \pi^*F.$$

Note that A possesses a kernel $K_A \in \mathcal{D}'(X \times X, F \boxtimes E')$ and the operators with smooth kernels are precisely the smoothing operators on X.

Example 1.3.3 Pseudodifferential operators on the unit circle $\mathbb{S}^1$ can be easily described using the Fourier series. Consider $A \in \Psi^m_{(\mathrm{cl})}(\mathbb{S}^1)$ and pick a smooth function u on $\mathbb{S}^1$. As mentioned in Example 1.2.13 we can think of u as a smooth 2π-periodic function on $\mathbb{R}$. Setting

$$c_n(u) := \int_0^{2\pi} e^{-\imath n x} u(x)\, dx$$

we find

$$Au(x) := \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{\imath nx} a(x,n) c_n(u)$$

for $x \in \mathbb{R}$, where $a \in \mathcal{S}^m(\mathbb{R} \times \mathbb{R})$ is 2π-periodic in x. The smoothing operators on $\mathcal{D}(\mathbb{S}^1)$ look like

$$Au(x) = \int_0^{2\pi} K(x,y) u(y)\, dy$$

with a smooth function $K \in C^\infty(\mathbb{R} \times \mathbb{R})$ which is 2π-periodic in both variables. For examples of pseudodifferential operators on other manifolds like higher-dimensional tori see for instance [Agr97].

Let $A \in \Psi^m_{\rm cl}(X; E, F)$. On endowing the bundles E and F by Riemannian metrics we introduce the formal adjoint operator $A^* \colon \mathcal{D}(X, F) \to \mathcal{E}(X, E)$ by requiring

$$(Au, g)_{L^2(X,F)} = (u, A^* g)_{L^2(X,E)}$$

for all $u \in \mathcal{D}(X, E)$ and $g \in \mathcal{D}(X, F)$. A priori it is by no means clear if any A^* exists.

Theorem 1.3.4 *Each pseudodifferential operator $A \in \Psi^m_{\rm cl}(X; E, F)$ possesses a formal adjoint $A^* \in \Psi^m_{\rm cl}(X; F, E)$ and $\sigma^m(A^*) = (\sigma^m(A))^*$ holds.*

On combining duality arguments and Theorem 1.3.4 one readily sees that any pseudodifferential operator $A \in \Psi^m_{\rm cl}(X; E, F)$ extends by continuity to a continuous mapping $A : \mathcal{E}'(X, E) \to \mathcal{D}'(X, F)$ of spaces of distribution sections, $\mathcal{E}'$ meaning distribution sections of compact support. Each section $u \in \mathcal{E}'(X, E)$ proves to be of finite order. Hence it follows that the space $\mathcal{E}'(X, E)$ is exhausted by the scale of Sobolev spaces $H^s(X, E)$ with $s \in \mathbb{R}$, but not $\mathcal{D}'(X, F)$ unless X is compact. Pick a formally selfadjoint operator $\Lambda_E \in \Psi^2_{\rm cl}(X; E)$ which is nonnegative and invertible on smooth sections of E. For example, one can choose $\Lambda_E = \partial_E^* \partial_E + \mathrm{Id}_E$ where ∂_E is a connection on the bundle E. For $s \in \mathbb{R}$, the Sobolev space $H^s(X, E)$ is defined to consist of all $u \in \mathcal{E}'(X, E)$, such that $\Lambda_E^{s/2} u \in L^2(X, E)$. This is a Hilbert space with scalar product

$$(u, v)_{H^s(X,E)} := (\Lambda_E^{s/2} u, \Lambda_E^{s/2} v)_{L^2(X,E)}$$

for $u, v \in H^s(X, E)$.

Theorem 1.3.5 *Let X be a compact closed manifold. For each $s \in \mathbb{R}$, any operator $A \in \Psi^m_{\rm cl}(X; E, F)$ maps $H^s(X, E)$ continuously into $H^{s-m}(X, F)$.*

Suppose X is a compact closed smooth manifold of dimension n (e.g. X is a sphere in $\mathbb{R}^{n+1}$). In this case pseudodifferential operators on X can be composed with each other thus giving rise to the simplest operator algebra

$$\bigcup_{m \in \mathbb{R}} \Psi^m_{\rm cl}(X).$$

Theorem 1.3.6 *Suppose X is a compact closed manifold and $A \in \Psi^l_{\mathrm{cl}}(X;E,F)$ and $B \in \Psi^m_{\mathrm{cl}}(X;F,G)$. Then $BA \in \Psi^{l+m}_{\mathrm{cl}}(X;E,G)$ and $\sigma^{l+m}(BA) = \sigma^m(B)\sigma^l(A)$ holds.*

On combining Theorem 1.3.5 with the Rellich-Kondrashov theorem on compact embeddings of Sobolev spaces we obtain the first defining property of the principal symbol mapping. Together with the multiplicativity property of Theorem 1.3.6 this allows one to identify the principal symbol mapping with the functor ϕ_Σ of Section 1.1.4.

Theorem 1.3.7 *Any operator $A \in \Psi^m_{\mathrm{cl}}(X;E,F)$ on a compact closed manifold X is a compact operator from $H^s(X,E)$ to $H^{s-m}(X,F)$ if $\sigma^m(A) = 0$ and it belongs to $\mathfrak{S}_p(H^s(X,E), H^s(X,F))$ if $p \geq 1$ and $m < -n/p$.*

By the above, each smoothing operator $A \in \Psi^{-\infty}_{\mathrm{cl}}(X,E)$ extends to a trace class operator in $H^s(X,E)$. The corresponding trace is given by

$$\mathrm{tr}(A) = \int_X \mathrm{tr}\, K_A(x,x)\, dx,$$

i.e. it does not depend on the particular choice of s.

1.3.3 Ellipticity

A pseudodifferential operator $A \in \Psi^m_{\mathrm{cl}}(X;E,F)$ is said to be elliptic (in the classical sense), if

$$\sigma^m(A)(x,\xi) \colon E_x \to F_x$$

is invertible for all $x \in X$ and $\xi \in T^*_x X \setminus \{0\}$. In particular, a canonical pseudodifferential operator $A \in \Psi^m_{\mathrm{cl}}(U;\mathbb{C}^k,\mathbb{C}^k)$ is elliptic if and only if $\det \sigma^m(A)(x,\xi) \neq 0$ is satisfied for all $(x,\xi) \in U \times (\mathbb{R}^n \setminus \{0\})$.

Example 1.3.8 In order to clarify the origin of this definition, we consider a linear differential operator A of order 2 with smooth real-valued coefficients on an open set $U \subset \mathbb{R}^2$, i.e.

$$A = \sum_{|\alpha| \leq 2} A_\alpha(x)\partial^\alpha.$$

The principal symbol is given by

$$\sigma^2(A)(x,\xi) = -A_{(2,0)}(x)\xi_1^2 - A_{(1,1)}(x)\xi_1\xi_2 - A_{(0,2)}(x)\xi_2^2 = -\left(M_A(x)\xi,\xi\right)$$

for each $(x,\xi) \in U \times (\mathbb{R}^2 \setminus \{0\})$, where

$$M_A(x) = \begin{pmatrix} A_{(2,0)}(x) & \frac{1}{2}A_{(1,1)}(x) \\ \frac{1}{2}A_{(1,1)}(x) & A_{(0,2)}(x) \end{pmatrix}.$$

Linear algebra shows that the symmetric matrix $M_A(x)$ satisfies $(M_A(x)\xi, \xi) \neq 0$ for all $\xi \neq 0$ if and only if $\det M_A(x) > 0$ holds. But this is fulfilled if and only if the set

$$\{\xi \in \mathbb{R}^2 |\ (M_A(x)\xi, \xi) = 1\}$$

is an ellipse.

Let $\Delta := \partial_1^2 + \ldots + \partial_n^2$ be the Laplace operator in $\mathbb{R}^n$. Its principal symbol is given by

$$\sigma^2(\Delta)(x, \xi) = -|\xi|^2 \tag{1.3.1}$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Hence, the Laplace operator is elliptic. The Cauchy-Riemann operator $(1/2)(\partial_1 + \imath\partial_2)$ is elliptic in $\mathbb{R}^2$, while the heat operator $\partial_1 - c^2\partial_2^2$ and the d'Alembert operator $\partial_1^2 - c^2\partial_2^2$ are not.

Example 1.3.9 Recall that $d^i : \Omega^i(X) \to \Omega^{i+1}(X)$ stands for the exterior derivative on a Riemannian manifold X. Since d^i are differential operators of first order, the formal adjoints d^{i*} are differential operators of first order as well. The operators

$$\Delta^i := d^{i-1}(d^{i-1})^* + (d^i)^* d^i \tag{1.3.2}$$

of $\Psi^2_{\mathrm{cl}}(X; \Lambda^i T^* X)$ are called the Hodge-Laplace operators. For $i = 0$, the operator Δ^0 is usually referred to as the Laplace-Beltrami operator on X. In the case $X = \mathbb{R}^n$ the Hodge-Laplace operators take especially simple form

$$\Delta^i = \begin{pmatrix} -\Delta & \ldots & 0 \\ & \ldots & \\ 0 & \ldots & -\Delta \end{pmatrix}$$

(a $(C_n^i \times C_n^i)$-matrix) and (1.3.1) yields

$$\sigma^2(\Delta^i)(x, \xi) = |\xi|^2\ E_{C_n^i}.$$

We thus deduce that the Hodge-Laplace operators are elliptic. This is still the case for arbitrary Riemannian manifolds, see [Wel80].

Assume that $A : \mathcal{D}(U, \mathbb{C}^k) \to C^\infty(U, \mathbb{C}^k)$ is a pseudodifferential operator on an open set $U \subset \mathbb{R}^n$ of the form

$$Au = (A_{i,j})_{\substack{i=1,\ldots,k \\ j=1,\ldots,k}}\ u$$

with entries $A_{i,j} \in \Psi^{l_i+m_j}_{\mathrm{cl}}(U)$. Then A is said to be elliptic in the sense of Douglis-Nirenberg if the matrix of principal symbols

$$\Big(\sigma^{l_i+m_j}(A_{i,j})(x, \xi)\Big)_{\substack{i=1,\ldots,k \\ j=1,\ldots,k}}$$

is invertible for each $(x, \xi) \in U \times (\mathbb{R}^n \setminus \{0\})$. This is a definition of ellipticity which is more general than the classical one. It may be extended to pseudodifferential operators acting on sections of vector bundles.

Example 1.3.10 The operator $A \in \mathcal{L}(\mathcal{D}(\mathbb{R}^2, \mathbb{C}^3), \mathcal{E}(\mathbb{R}^2, \mathbb{C}^3))$ given by

$$Au = \begin{pmatrix} 0 & \partial_1 & \partial_2 \\ \partial_1 & -1 & 0 \\ \partial_2 & 0 & -1 \end{pmatrix} u$$

is elliptic in the sense of Douglis-Nirenberg but not in the classical one.

From now on we assume that X is compact and closed. A pseudodifferential operator $P \in \Psi^{-m}(X; F, E)$ is called a parametrix for $A \in \Psi^m(X; E, F)$, if

$$\begin{array}{rcl} \mathrm{Id} - PA & \in & \Psi^{-1}(X; E), \\ \mathrm{Id} - AP & \in & \Psi^{-1}(X; F) \end{array}$$

holds.

Theorem 1.3.11 *For each elliptic operator $A \in \Psi^m_{\mathrm{cl}}(X; E, F)$ there is an operator $P \in \Psi^{-m}_{\mathrm{cl}}(X; F, E)$, such that*

$$\begin{array}{rcl} \mathrm{Id} - PA & \in & \Psi^{-\infty}(X; E), \\ \mathrm{Id} - AP & \in & \Psi^{-\infty}(X; F). \end{array} \tag{1.3.3}$$

A parametrix $P \in \Psi^{-m}_{\mathrm{cl}}(X; F, E)$ satisfying the equalities (1.3.3) is sometimes called a regulariser of A, cf. [Fed91]. Theorem 1.3.11 gains in interest if we realise that any formal parametrix is actually a parametrix in the sense of Hilbert spaces. In order to show this we first mention the so-called property of spectral invariance of the classical algebra of pseudodifferential operators on a compact closed smooth manifold.

Theorem 1.3.12 *Let $A \in \Psi^m_{\mathrm{cl}}(X; E, F)$ be elliptic and invertible on smooth sections. Then $A^{-1} \in \Psi^{-m}_{\mathrm{cl}}(X; F, E)$.*

Suppose $A \in \Psi^m_{\mathrm{cl}}(X; E, F)$ is an elliptic operator invertible on smooth sections. Then the extension $A : H^s(X, E) \to H^{s-m}(X, F)$ is invertible, too. This follows from the fact that the inverse A^{-1} on smooth sections is actually a pseudodifferential operator in $\Psi^{-m}_{\mathrm{cl}}(X; F, E)$, which is due to Theorem 1.3.12.

Theorem 1.3.13 *Suppose that $A \in \Psi^m_{\mathrm{cl}}(X; E, F)$ is elliptic. Then, for each $s \in \mathbb{R}$, the operator $A\colon H^s(X, E) \to H^{s-m}(X, F)$ is Fredholm and its null-space belongs to $C^\infty(X, E)$.*

Let $A \in \Psi^m_{\mathrm{cl}}(X; E, F)$ be elliptic. By the above, the extension of A to Sobolev spaces is a Fredholm operator and its index does not depend on s, cf. the arguments in Section 3.2.1. Thus, we can define the Fredholm index of an elliptic pseudodifferential operator to be the index of the corresponding Fredholm operators in Sobolev spaces. Moreover, the principal symbol induces an isomorphism of vector bundles $\sigma^m(A)\colon \pi^*E \to \pi^*F$ over $T^*X \setminus \{0\}$. In this way A determines an element $\mathrm{d}(\sigma^m(A))$

of the functor with compact support $K^{\mathrm{comp}}(T^*X)$. One can define the topological index $\mathrm{ind}_{\mathrm{top}}(A)$ of A by

$$\mathrm{ind}_{\mathrm{top}}(A) = \int_{T^*X} \mathrm{ch}\ (\mathrm{d}(\sigma^m(A)))\ \mathcal{T}\,(T_{\mathbb{C}}X),$$

where the orientation of T^*X is given by the differential form $d\xi_1 \wedge dx^1 \wedge \ldots \wedge d\xi_n \wedge dx^n$.

Theorem 1.3.14 *For an elliptic operator A on a compact closed manifold X the Fredholm and the topological indices coincide, i.e.* $\mathrm{ind}\,(A) = \mathrm{ind}_{\mathrm{top}}(A)$.

See [Pal65] for details on this important index formula of Atiyah-Singer, which is one of the most famous results of mathematics in the 20 th century.

Chapter 2

Sequences of compact curvature

This main chapter deals with quasicomplexes in the context of functional analysis, i.e. sequences of Hilbert spaces with compact curvature. We will introduce the Fredholm theory and characteristic numbers. The proofs are mostly algebraic and use the ideal structures. Note that we will introduce the theory of Fredholm comlexes not separately but obtain it as a special case of a more general theory. Moreover, the corresponding behavior of quasicomplexes in Banach spaces will be described in some remarks respectively.

2.1 Fredholm quasicomplexes

2.1.1 The concept of quasicomplexes

Let

$$\{V^\cdot, A\}:\ 0 \longrightarrow V^0 \stackrel{A^0}{\longrightarrow} V^1 \stackrel{A^1}{\longrightarrow} \ldots \stackrel{A^{N-1}}{\longrightarrow} V^N \longrightarrow 0 \tag{2.1.1}$$

be a sequence of Hilbert spaces with operators $A^i \in \mathcal{L}(V^i, V^{i+1})$.

Definition 2.1.1 The sequence (2.1.1) is called quasicomplex, if its curvature is compact at each step, i.e.

$$A^{i+1}A^i \in \mathcal{K}(V^i, V^{i+2})$$

for each $i = 0, 1, \ldots, N-2$.

For quasicomplexes the cohomology is no longer defined, since the image of A^{i-1} fails to lie in the null-space of the operator A^i in general. However, in order to define Fredholm quasicomplexes we may use the construction with the Calkin quotient spaces of Section 1.1.4. We choose an arbitrary Hilbert space Σ and consider the sequence

$$\phi_\Sigma(V^\cdot):\ 0 \longrightarrow \phi_\Sigma(V^0) \stackrel{\phi_\Sigma(A^0)}{\longrightarrow} \phi_\Sigma(V^1) \stackrel{\phi_\Sigma(A^1)}{\longrightarrow} \ldots \stackrel{\phi_\Sigma(A^{N-1})}{\longrightarrow} \phi_\Sigma(V^N) \longrightarrow 0,$$

which actually proves to be a complex. Indeed, since $A^{i+1}A^i$ is compact, we get

$$\phi_\Sigma(A^{i+1})\phi_\Sigma(A^i) = \phi_\Sigma(A^{i+1}A^i) = 0.$$

Definition 2.1.2 A quasicomplex $V^\cdot$ is said to be Fredholm, if $\phi_\Sigma(V^\cdot)$ is exact for each Hilbert space Σ.

Note that each (finite) sequence of linear maps between finite-dimensional Hilbert spaces is a Fredholm quasicomplex.

Definition 2.1.3 By a parametrix of a quasicomplex $\{V^\cdot, A\}$ is meant any sequence of linear maps $P^i \in \mathcal{L}(V^i, V^{i-1})$ satisfying

$$P^{i+1}A^i + A^{i-1}P^i = \mathrm{Id}_{V^i} - K^i \tag{2.1.2}$$

with $K^i \in \mathcal{K}(V^i)$ for all $i = 0, 1, \ldots, N$.

It is easily seen that Definition 2.1.3 coincides with the definition of a parametrix to a single operator A in Section 1.1.4 if $\{V^\cdot, A\}$ is a short complex.

Theorem 2.1.4 *A quasicomplex is Fredholm if and only if it possesses a parametrix.*

Proof. We proceed in a standard way, cf. for instance [AV95].

Necessity. Let $\{V^\cdot, A\}$ be Fredholm. If $i = N$, then from the exactness of $\phi_{V^N}(V^\cdot)$ at step N it follows that there are operators $P^N \in \mathcal{L}(V^N, V^{N-1})$ and $K^N \in \mathcal{K}(V^N)$, such that

$$A^{N-1}P^N = \mathrm{Id}_{V^N} - K^N$$

holds. We now proceed by induction. Suppose we have already found mappings $P^i, P^{i+1}, \ldots$ and $K^i, K^{i+1}, \ldots$, such that the equality (2.1.2) is satisfied. Note that

$$\begin{aligned} A^{i-1}(\mathrm{Id}_{V^{i-1}} - P^iA^{i-1}) &= A^{i-1} - (\mathrm{Id}_{V^i} - K^i - P^{i+1}A^i)A^{i-1} \\ &= K^iA^{i-1} + P^{i+1}A^iA^{i-1} \\ &\in \mathcal{K}(V^{i-1}, V^i) \end{aligned}$$

by (2.1.2). From the exactness of $\phi_{V^{i-1}}(V^\cdot)$ at step $i-1$ it follows that there are operators $P^{i-1} \in \mathcal{L}(V^{i-1}, V^{i-2})$ and $K^{i-1} \in \mathcal{K}(V^{i-1})$, such that

$$P^iA^{i-1} + A^{i-2}P^{i-1} = \mathrm{Id}_{V^{i-1}} - K^{i-1}$$

is satisfied.

Sufficiency. Assume $\{V^\cdot, A\}$ admits a parametrix. Then the identity mapping Id_{V^i} vanishes on the cohomology $H^i(\phi_\Sigma(V^\cdot))$, hence $\phi_\Sigma(V^\cdot)$ is exact for each Hilbert space Σ.

□

As is shown in [TW12], it also makes sense to consider sequences whose curvatures belong to some operator ideal $\mathcal{I} \subset \mathcal{K}$, where $\mathcal{K}$ is the class of all compact operators. Let us first prove that the operators of finite rank form the smallest nontrivial ideal, i.e. $\mathcal{F} \subset \mathcal{I}$ if $\mathcal{I} \neq \mathcal{O}$, cf. [Pie78].

Lemma 2.1.5 *Let $\mathcal{I} \neq \mathcal{O}$ be an operator ideal in the context of Hilbert spaces. Then $\mathcal{F}(V, W) \subset \mathcal{I}(V, W)$ is satisfied for arbitrary Hilbert spaces V and W.*

Proof. Since $\mathcal{I} \neq \mathcal{O}$, there is an operator $A \in \mathcal{I}(V,W)$ satisfying $Av \neq 0$ for an element $v \in V$. Assume $F \in \mathcal{F}(V,W)$ is an arbitrary operator of finite rank. Since F induces an isomorphism of $V/\ker F$ onto $\operatorname{im} F$, it follows that we can find bases $\{v_1,\dots,v_n\}$ of the orthogonal complement $(\ker F)^\perp$ of $\ker F$ and $\{w_1,\dots,w_n\}$ of $\operatorname{im} F$. Define the operators $P_j \in \mathcal{L}(V)$ and $Q_j \in \mathcal{L}(W)$ by

$$\begin{aligned} P_j v_k &= \delta_{j,k} v, \\ Q_j\, Av &= w_j \end{aligned}$$

and $P_j \equiv 0$ on $\ker F$, $Q_j \equiv 0$ on $(Av)^\perp$, respectively. It is easy to see that the operator

$$T := \sum_{j=1}^{n} Q_j A P_j$$

belongs to $\mathcal{I}(V,W)$ and has the same range as F. Hence it follows that

$$F = T \left(T \restriction_{(\ker F)^\perp} \right)^{-1} F$$

belongs to $\mathcal{I}(V,W)$, as desired.

□

Definition 2.1.6 The sequence (2.1.1) is called an $\mathcal{I}$-quasicomplex if the equality $A^{i+1}A^i \in \mathcal{I}(V^i, V^{i+2})$ is fulfilled for all $i = 0, 1, \dots, N-2$.

An $\mathcal{O}$-quasicomplex is obviously a complex and a $\mathcal{K}$-quasicomplex is just called a quasicomplex.

Lemma 2.1.7 *On perturbing the operators of a Fredholm $\mathcal{I}$-quasicomplex $\{V^\cdot, A\}$ by operators $K^i \in \mathcal{I}(V^i, V^{i+1})$ we obtain a Fredholm $\mathcal{I}$-quasicomplex.*

Proof. Let $\{P^i\}$ be a parametrix of $\{V^\cdot, A\}$, i.e.

$$P^{i+1}A^i + A^{i-1}P^i = \mathrm{Id}_{V^i} - C^i$$

with $C^i \in \mathcal{K}(V^i)$. Setting $B^i := A^i + K^i$ we obtain $B^{i+1}B^i \in \mathcal{I}(V^i, V^{i+2})$ and

$$\begin{aligned} P^{i+1}B^i + B^{i-1}P^i &= P^{i+1}A^i + A^{i-1}P^i + P^{i+1}K^i + K^{i-1}P^i \\ &= Id_{V^i} - \underbrace{(C^i - P^{i+1}K^i - K^{i-1}P^i)}_{\in \mathcal{K}(V^i)}, \end{aligned}$$

hence $\{P^i\}$ is a parametrix of $\{V^\cdot, B\}$. This implies that $\{V^\cdot, B\}$ is Fredholm.

□

All the above statements remain still valid in the case of Banach spaces. The only difference is that one has to replace the arbitrary Hilbert space Σ in the definition of Fredholm property by a Banach space.

2.1.2 Hodge theory

Let $\{V^\cdot, A\}$ be an $\mathcal{I}$-quasicomplex. Then

$$\{V^\cdot, A^*\}:\ 0 \longleftarrow V^0 \overset{A^{0*}}{\longleftarrow} V^1 \overset{A^{1*}}{\longleftarrow} \dots \overset{A^{N-1*}}{\longleftarrow} V^N \longleftarrow 0$$

is called the adjoint quasicomplex of $\{V^\cdot, A\}$. Since

$$A^{i*}A^{i+1*} = (A^{i+1}A^i)^*$$

belongs to $\mathcal{K}(V^{i+2}, V^i)$, the sequence $\{V^\cdot, A^*\}$ is a quasicomplex indeed.

The operators

$$\Delta^i := A^{i*}A^i + A^{i-1}A^{i-1*}$$

are said to be the Laplacians of the quasicomplex. These are obviously selfadjoint operators in V^i and it is easy to see that

$$A^i\Delta^i - \Delta^{i+1}A^i \in \mathcal{I}(V^i, V^{i+1}) \tag{2.1.3}$$

is satisfied. The following lemma describes the null-space of the Laplacians. Note that the elements of $\ker \Delta^i$ are generally called harmonic sections, cf. [Wel80].

Lemma 2.1.8 *The null-space of the Laplacian Δ^i is the intersection of the null-spaces of A^i and A^{i-1*}, i.e.* $\ker \Delta^i = \ker A^i \cap \ker A^{i-1*}$.

Proof. Assume $h \in V^i$ is such that $A^ih = A^{i-1*}h = 0$. Then by definition $\Delta^ih = 0$, and so $h \in \ker \Delta^i$. On the other hand, if $h \in \ker \Delta^i$ is fulfilled, then

$$\begin{aligned} 0 &= (\Delta^ih, h) \\ &= (A^{i-1}A^{i-1*}h + A^{i*}A^ih, h) \\ &= (A^{i-1*}h, A^{i-1*}h) + (A^ih, A^ih) \\ &= \|A^{i-1*}h\|^2 + \|A^ih\|^2 \end{aligned}$$

and thus $A^ih = A^{i-1*}h = 0$, as desired.

□

Suppose that the Laplacian Δ^i at step i is Fredholm. In this case, we denote by $H^i \in \mathcal{F}(V^i)$ the orthogonal projection of V^i onto the null-space of Δ^i and introduce the Green operator by

$$G^i := (\Delta^i \restriction_{(\ker \Delta^i)^\perp})^{-1}(\mathrm{Id}_{V^i} - H^i). \tag{2.1.4}$$

Then $\mathrm{Id}_{V^i} = H^i + \Delta^iG^i$ holds.

By Theorem 1.1.20, an operator $A \in \mathcal{L}(V^0, V^1)$ is Fredholm if and only if the short complex

$$\{V^\cdot, A\}:\ 0 \longrightarrow V^0 \overset{A}{\longrightarrow} V^1 \longrightarrow 0$$

is Fredholm, which is satisfied if and only if the cohomology is finite dimensional. The next lemma shows that this property holds true for complexes, see [RS82] and elsewhere. In particular, the Euler characteristic is well defined for any Fredholm complex.

Lemma 2.1.9 *Suppose $\{V^\cdot, D\}$ is a complex. Then, the following statements are equivalent:*

i) $\{V^\cdot, D\}$ *is Fredholm.*

ii) *The cohomology $H^i(V^\cdot)$ is of finite dimension at each step.*

iii) *Each Laplacian Δ^i of $\{V^\cdot, D\}$ is Fredholm.*

Proof.

i) $\Rightarrow$ *ii)* Suppose $\{V^\cdot, D\}$ is Fredholm. By Theorem 2.1.4, the complex possesses a parametrix P. For $h \in \ker D^i$ we find

$$D^{i-1}P^i h = (1 - K^i)h, \tag{2.1.5}$$

i.e. $\operatorname{im} D^{i-1} \supset (\mathrm{Id}_{V^i} - K^i) \ker D^i$. The equality (2.1.5) implies that the restriction of K^i to $\ker D^i$ is a selfmapping of $\ker D^i$. Hence, $(\mathrm{Id}_{V^i} - K^i) \ker D^i$ is a closed subspace od finite codimension in $\ker D^i$. Consequently, $\operatorname{im} D^{i-1}$ is a closed subspace od finite codimension in $\ker D^i$, i.e. $H^i(V^\cdot)$ is finite dimensional.

ii) $\Rightarrow$ *iii)* Suppose the cohomology is of finite dimension at each step. Since $\operatorname{im} D^i$ has finite codimension in $\ker D^{i+1}$ and $\ker D^{i+1}$ is a closed subspace of V^{i+1}, it follows that the image of D^i is closed in V^{i+1}. By duality, the image of D^{i*} is closed in V^i and we obtain the strong orthogonal decomposition

$$V^i = \ker D^i \oplus \operatorname{im} D^{i*}.$$

Obviously, the orthogonal complement of $\operatorname{im} D^{i-1}$ in $\ker D^i$ consists of all $h \in V^i$ satisfying $D^i h = 0$ and $D^{i-1*}h = 0$. This implies

$$\ker D^i = \ker \Delta^i \oplus \operatorname{im} D^{i-1}$$

and thus, $\ker \Delta^i \cong H^i(V^\cdot)$ is finite dimensional. Since Δ^i is selfadjoint, we obtain

$$V^i = \ker \Delta^i \oplus \operatorname{im} \Delta^{i*} = \ker \Delta^i \oplus \operatorname{im} \Delta^i.$$

Hence, $\operatorname{im} \Delta^i$ has finite codimension in V^i, and thus Δ^i is Fredholm.

iii) $\Rightarrow$ *i)* Let G^i be the Green operator for Δ^i. Since $\mathcal{I} = \mathcal{O}$ holds in our case, the relation (2.1.3) implies $\Delta^{i+1}D^i = D^i\Delta^i$. Multiplying this equation by G^{i+1} from the left and by G^i from the right we obtain

$$D^i G^i = G^{i+1} D^i,$$

since $H^{i+1}D^i = D^i H^i = 0$. Hence it follows that the operators

$$P^i := D^{i-1*}G^i$$

constitute a parametrix P for $\{V^\cdot, D\}$.

□

An inspection of the proof above shows that the cohomology of a Fredholm complex $\{V^\cdot, D\}$ satisfies $\ker \Delta^i \cong H^i(V^\cdot)$. Thus we obtain the formula

$$\chi(V^\cdot) = \sum_{i=0}^{N} (-1)^i \dim\ker \Delta^i,$$

which for a single Fredholm operator A transforms into

$$\operatorname{ind}(A) = \dim\ker A - \dim\ker A^*.$$

Remark 2.1.10 The proof of the implication $i) \Rightarrow ii)$ still works in the case of Banach spaces. However, $ii) \Rightarrow i)$ is no longer true. On may consult [EP96] about an example of a Koszul complex which has finite-dimensional cohomology, however, no parametrix can be found.

Example 2.1.11 The complex

$$0 \longrightarrow \ell^2(\mathbb{K}) \xrightarrow{D^0} \ell^2(\mathbb{K}) \xrightarrow{D^1} \ell^2(\mathbb{K}) \longrightarrow 0$$

with

$$\begin{aligned} D^0(x_1, x_2, x_3, x_4, \ldots) &:= (0, x_1, 0, x_2, \ldots), \\ D^1(x_1, x_2, x_3, x_4, \ldots) &:= (x_1, x_3, x_5, x_7, \ldots) \end{aligned}$$

is an exact sequence of Hilbert spaces and consequently Fredholm. The adjoint operators are given by

$$\begin{aligned} D^{0*}(y_1, y_2, y_3, y_4, \ldots) &:= (y_2, y_4, y_6, y_8, \ldots), \\ D^{1*}(y_1, y_2, y_3, y_4, \ldots) &:= (y_1, 0, y_2, 0, \ldots). \end{aligned}$$

The Laplacians and Green operators are the identity operators on $\ell^2(\mathbb{K})$. Hence, a parametrix is given by $P = \{D^{0*}, D^{1*}\}$.

If $\{V^\cdot, D\}$ is a Fredholm complex, then the associated Laplace operators are Fredholm and from $D^i G^i = G^{i+1} D^i$ we get immediately the Hodge decomposition

$$v = H^i v + D^{i-1} D^{i-1*} G^i v + D^{i*} G^{i+1} D^i v \tag{2.1.6}$$

for each $v \in V^i$. Note that the summands are pairwise orthogonal.

Corollary 2.1.12 *Let $\{V^\cdot, D\}$ be a Fredholm complex. Given a $w \in V^i$, the equation $D^{i-1} v = w$ has a solution $v \in V^{i-1}$ if and only if $D^i w = 0$ and $H^i w = 0$.*

Proof. Assume $v \in V^{i-1}$ is a solution of the equation. Then

$$\begin{aligned} D^i w &= D^i D^{i-1} v = 0, \\ H^i w &= H^i D^{i-1} v = 0. \end{aligned}$$

Conversely, suppose that $D^i w = 0$ and $H^i w = 0$ is satisfied. Then, the Hodge decomposition shows that $w = D^{i-1} D^{i-1*} G^i w$ and the desired assertion follows by setting $v := D^{i-1*} G^i w$.

□

2.1.3 Reduction to complexes

Theorem 2.1.13 *For every Fredholm $\mathcal{I}$-quasicomplex $\{V^\cdot, A\}$ there exist operators $D^i \in \mathcal{L}(V^i, V^{i+1})$ satisfying $D^i - A^i \in \mathcal{I}(V^i, V^{i+1})$ and $D^{i+1}D^i = 0$ for all i.*

Proof. We follow the sheme suggested in [Tar07] where the case $\mathcal{I} = \mathcal{K}$ was considered. By Theorem 2.1.4, the quasicomplex possesses a parametrix P. We start at step N and mention first that P^N is a left parametrix of the last operator D^{N-1}. Set $D^{N-1} = A^{N-1}$ and consider the Laplacian

$$\Delta^N = D^{N-1}D^{N-1*}.$$

Using the same arguments as in the proof of Lemma 2.1.9, we see that Δ^N is a Fredholm operator. By the abstract Hodge theory, there is a Green operator $G^N \in \mathcal{L}(V^N)$ satisfying

$$\mathrm{Id}_{V^N} = H^N + \Delta^N G^N,$$

where $H^N : V^N \to \ker \Delta^N$ is the orthogonal projection. In particular, the operator $\Phi^N = D^{N-1*}G^N$ is a special right parametrix for D^{N-1} in $\mathcal{L}(V^N, V^{N-1})$.

We now show that $\Pi^{N-1} = \mathrm{Id}_{V^N} - \Phi^N D^{N-1}$ is an orthogonal projection onto the kernel of D^{N-1}. Indeed, Π^{N-1} is the identity operator on $\ker D^{N-1}$ and

$$\begin{aligned} D^{N-1}\Pi^{N-1} &= D^{N-1} - \Delta^N G^N D^{N-1} \\ &= D^{N-1} - (\mathrm{Id}_{V^N} - H^N)D^{N-1} \\ &= H^N D^{N-1} \\ &= (D^{N-1*}H^N)^* \\ &= 0. \end{aligned}$$

From this the desired conclusion follows.

In order to construct D^{N-2} we consider the last fragment of the sequence, namely

$$V^{N-2} \stackrel{A^{N-2}}{\longrightarrow} V^{N-1} \stackrel{D^{N-1}}{\longrightarrow} V^N.$$

Set

$$D^{N-2} = \Pi^{N-1}A^{N-2},$$

then D^{N-2} satisfies

$$\begin{aligned} D^{N-1}D^{N-2} &= D^{N-1}\Pi^{N-1}A^{N-2} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} D^{N-2} &= (\mathrm{Id}_{V^{N-1}} - \Phi^N A^{N-1})A^{N-2} \\ &= A^{N-2} \end{aligned}$$

modulo operators in $\mathcal{I}(V^{N-2}, V^{N-1})$.

To construct D^{N-3} we can argue in the same way as in the construction of D^{N-2}. Namely, we consider the sequence

$$V^{N-3} \stackrel{A^{N-3}}{\longrightarrow} V^{N-2} \stackrel{D^{N-2}}{\longrightarrow} V^{N-1} \stackrel{D^{N-1}}{\longrightarrow} V^{N}$$

and the Laplacian

$$\Delta^{N-1} = D^{N-1*}D^{N-1} + D^{N-2}D^{N-2*}.$$

Since $D^{N-1} - A^{N-1}$ and $D^{N-2} - A^{N-2}$ belong to $\mathcal{K}$, we find

$$D^{N-2}P^{N-1} - P^{N}D^{N-1} = \mathrm{Id}_{V^{N-1}} - R^{N-1}$$

mit $R^{N-1} \in \mathcal{K}(V^{N-1})$. Using the same arguments as in the proof of lemma 2.1.9, we see that Δ^{N-1} is a Fredholm operator. This implies that the orthogonal projection $H^{N-1} : V^{N-1} \to \ker \Delta^{N-1}$ has finite rank. With a Green operator $G^{N-1} \in \mathcal{L}(V^{N-1})$ we find

$$\mathrm{Id}_{V^{N-1}} = H^{N-1} + \Delta^{N-1}G^{N-1}.$$

Set $\Phi^{N-2} = D^{N-2*}G^{N-1}$, then the pair $\{\Phi^{N-2}, \Phi^{N-1}\}$ is a special parametrix of the sequence at step $N-1$. We now show that

$$\Pi^{N-2} = Id_{V^{N-1}} - \Phi^{N-1}D^{N-2}$$

is an orthogonal projection onto the kernel of D^{N-2}. Indeed, Π^{N-2} is the identity operator on $\ker D^{N-2}$ and

$$\begin{aligned} D^{N-2}\Pi^{N-2} &= D^{N-2} - D^{N-2}\Phi^{N-1}D^{N-2} \\ &= D^{N-2} - (\mathrm{Id}_{V^{N-1}} - H^{N-1} - \Phi^{N}D^{N-1})D^{N-2} \\ &= 0 \end{aligned}$$

for $H^{N-1}D^{N-2} = (D^{N-2*}H^{N-1})^* = 0$.

Set $D^{N-3} = \Pi^{N-2}A^{N-3}$, then

$$\begin{aligned} D^{N-2}D^{N-3} &= D^{N-2}\Pi^{N-2}A^{N-3} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} D^{N-3} &= (\mathrm{Id}_{V^{N-2}} - \Phi^{N-1}D^{N-2})A^{N-3} \\ &= (\mathrm{Id}_{V^{N-2}} - \Phi^{N-1}A^{N-2})A^{N-3} \\ &= A^{N-3} \end{aligned}$$

modulo operators in $\mathcal{I}$. We now proceed by induction, thus completing the proof. □

The theorem holds true in the context of Banach spaces, cf. [EP96] for a proof. The advantage of using the Hilbert space method lies in the fact that we obtain explicit formulas to construct the reduced complex.

Theorem 2.1.14 *A quasicomplex $\{V^\cdot, A\}$ is Fredholm if and only if the Laplacians are Fredholm operators at each step.*

Proof.
Necessity. Assume $\{V^\cdot, A\}$ is Fredholm. We reduce the quasicomplex to a Fredholm complex $\{V^\cdot, D\}$. The associated Laplacians Δ^i_D are Fredholm due to Lemma 2.1.9. Since Δ^i_A and Δ^i_D differ merely by a compact operator, we conclude that Δ^i_A is Fredholm.

Sufficiency. Assume the Laplacians Δ^i are Fredholm. For $\mathcal{I} = \mathcal{O}$ the assertion follows from Lemma 2.1.9. Let $\mathcal{I} \neq \mathcal{O}$ and consider the Green operators G^i. Multiplying the operator in (2.1.3) by G^{i+1} from the left and by G^i from the right we obtain

$$A^i G^i - G^{i+1} A^i \in \mathcal{I}(V^i, V^{i+1}),$$

for $H^i \in \mathcal{I}(V^i)$, if $\mathcal{I} \neq \mathcal{O}$. Hence it follows that the operators

$$P^i := A^{i-1*} G^i$$

yield a parametrix P for $\{V^\cdot, A\}$.

□

Remark 2.1.15 Since the Laplacians of a Fredholm $\mathcal{I}$-quasicomplex are Fredholm, we can construct a special parametrix $P^i = A^{i-1*} G^i$ of the quasicomlex as mentioned. If $\mathcal{I} = \mathcal{O}$, this is an $\mathcal{F}$-parametrix of the complex. If $\mathcal{I} \neq \mathcal{O}$, then $\mathcal{F} \subset \mathcal{I}$ implies that P is a special $\mathcal{I}$-parametrix of the quasicomplex.

2.1.4 Euler characteristic

By the above, every Fredholm complex has finite-dimensional cohomology at each step and thus its Euler characteristic is well defined. In order to define the Euler characteristic of a Fredholm quasicomplex we use Theorem 2.1.13.

Definition 2.1.16 The Euler characteristic of a Fredholm quasicomplex $\{V^\cdot, A\}$ is defined to be

$$\chi(V^\cdot, A) := \chi(V^\cdot, D),$$

where $\{V^\cdot, D\}$ is a complex, such that $A^i - D^i \in \mathcal{K}(V^i, V^{i+1})$.

We have to show that the definition does not depend on the particular choice of the reduced complex. Since the complex $\{V^\cdot, D\}$ is Fredholm, the Laplacians Δ^i_D are Fredholm, too. We split $V = \oplus V^i$ into the sum

$$V = V^{\mathrm{even}} \oplus V^{\mathrm{odd}}$$

where $V^{\mathrm{even}} = \oplus V^{2i}$, $V^{\mathrm{odd}} = \oplus V^{2i+1}$ and consider

$$(D + D^*)_e : V^{\mathrm{even}} \to V^{\mathrm{odd}}$$

given by the block operator

$$\begin{pmatrix} D^0 & D^{1*} & 0 & 0 & \dots \\ 0 & D^2 & D^{3*} & 0 & \dots \\ 0 & 0 & D^4 & D^{5*} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

The Laplacians of this operator $(D+D^*)_e$ satisfy

$$\begin{aligned} (D+D^*)_e^*(D+D^*)_e &= \begin{pmatrix} \varDelta_D^0 & 0 & 0 & 0 & \dots \\ 0 & \varDelta_D^2 & 0 & 0 & \dots \\ 0 & 0 & \varDelta_D^4 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \\ (D+D^*)_e(D+D^*)_e^* &= \begin{pmatrix} \varDelta_D^1 & 0 & 0 & 0 & \dots \\ 0 & \varDelta_D^3 & 0 & 0 & \dots \\ 0 & 0 & \varDelta_D^5 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \end{aligned}$$

Hence, the Laplacians of $(D+D^*)_e$ are Fredholm. This implies that the block operator is Fredholm as well and satisfies

$$\begin{aligned} \operatorname{ind}(D+D^*)_e &= \dim\ker(D+D^*)_e^*(D+D^*)_e - \dim\ker(D+D^*)_e(D+D^*)_e^* \\ &= \sum_{i=0}^{N}(-1)^i \dim\ker \varDelta_D^i \\ &= \sum_{i=0}^{N}(-1)^i \dim H^i(V^\cdot, D) \\ &= \chi(V^\cdot, D). \end{aligned}$$

Since $(A+A^*)_e$ and $(D+D^*)_e$ differ by a mere compact operator, we conclude that

$$\chi(V^\cdot, A) = \chi(V^\cdot, D) = \operatorname{ind}(D+D^*)_e = \operatorname{ind}(A+A^*)_e.$$

This shows the independence of $\{V^\cdot, D\}$, as desired.

Corollary 2.1.17 *Assume $\{V^\cdot, A\}$ is a Fredholm quasicomplex. Then the associated block operator $(A+A^*)_e$ is Fredholm and satisfies* $\operatorname{ind}(A+A^*)_e = \chi(V^\cdot, A)$.

Theorem 2.1.18 *Let $\{V^\cdot, A\}$ be a Fredholm quasicomplex of length N. Then the adjoint quasicomplex $\{V^\cdot, A^*\}$ is also Fredholm and its Euler characteristic is evaluated by*

$$\chi(V^\cdot, A^*) = (-1)^N \chi(V^\cdot, A).$$

Proof. We reduce $\{V^\cdot, A\}$ to a Fredholm complex $\{V^\cdot, D\}$. Since

$$A^{i*} - D^{i*} = (A^i - D^i)^*$$

belongs to $\mathcal{K}(V^{i+1}, V^i)$ for each i, the adjoint complex $\{V^\cdot, D^*\}$ is a reduced complex for $\{V^\cdot, A^*\}$. Hence, $\{V^\cdot, A^*\}$ is Fredholm, for the complex $\{V^\cdot, D\}$ and its adjoint $\{V^\cdot, D^*\}$ have the same Laplacians. Since

$$H^i(V^\cdot, D) \cong \ker \Delta_D^i \cong H_i(V^\cdot, D^*)$$

holds, the sub i being due to the chain structure of $\{V^\cdot, D^*\}$, the assertion for the Euler characteristics follows immediately.

□

2.2 Quasimorphisms

2.2.1 Reduction to morphisms

Definition 2.2.1 An $\mathcal{I}$-quasimorphism of two sequences $\{V^\cdot, A\}$ and $\{W^\cdot, B\}$ is a sequence of linear maps $L^i \in \mathcal{L}(V^i, W^i)$, such that

$$L^{i+1}A^i - B^iL^i \in \mathcal{I}(V^i, W^{i+1})$$

is fulfilled for all $i = 0, 1, \dots, N-1$.

In other words, an $\mathcal{I}$-quasimorphism is a collection of selfmappings that makes the diagram

$$\begin{array}{ccccccccc} 0 & \to & V^0 & \overset{A^0}{\to} & V^1 & \overset{A^1}{\to} \dots & \overset{A^{N-1}}{\to} & V^N & \to & 0 \\ & & \downarrow L^0 & & \downarrow L^1 & & & \downarrow L^N & & \\ 0 & \to & W^0 & \overset{B^0}{\to} & W^1 & \overset{B^1}{\to} \dots & \overset{B^{N-1}}{\to} & W^N & \to & 0 \end{array}$$

commutative modulo operators of $\mathcal{I}$. As above, the $\mathcal{K}$-quasimorphisms are called quasimorphisms and $\mathcal{O}$-quasimorphisms are called morphisms. Note that the morphisms are also said to be cochain mappings.

Suppose L is a morphism from $\{V^\cdot, D_A\}$ to $\{W^\cdot, D_B\}$, i.e. $L^{i+1}D_A^i = D_B^iL^i$. It is well known that L induces a sequence of homomorphisms $HL^i \colon H^i(V^\cdot) \to H^i(W^\cdot)$ on the cohomology by

$$HL^i[v] := [L^iv]$$

for any $[v] \in H^i(V^\cdot)$.

Let us show that the homomorphisms HL^i are well defined. To this end we choose $v \in \ker D_A^i$. Then $D_B^i(L^iv) = L^{i+1}(D_A^iv) = L^{i+1}0 = 0$ is satisfied and thus $[L^iv] \in H^i(W^\cdot)$ is well defined. In order to show that the map does not depend on the particular choice of representatives, we pick $v_1, v_2 \in \ker D_A^i$ satisfying $[v_1] = [v_2]$, i.e. $v_1 - v_2 \in \operatorname{im} D_A^{i-1}$. Since L^i is linear, we obtain $v_1 - v_2 = D_A^{i-1}w$ with $w \in V^{i-1}$ and thus

$$L^i(v_1 - v_2) = L^i(D_A^{i-1}w) = D_B^{i-1}(L^{i-1}w) \in \operatorname{im} D_B^{i-1}.$$

Hence, $[L^iv_1] = [L^iv_2 + L^i(v_1 - v_2)] = [L^iv_2]$ is satisfied. Moreover, the linearity of HL^i is obvious.

Theorem 2.2.2 *Let $\{V^\cdot, A\}$ and $\{W^\cdot, B\}$ be Fredholm $\mathcal{I}$-quasicomplexes, L be an $\mathcal{I}$-quasimorphism of these quasicomplexes and $\{V^\cdot, D_A\}$, $\{W^\cdot, D_B\}$ any complexes with the property that $D_A^i - A^i \in \mathcal{I}(V^i, V^{i+1})$ and $D_B^i - B^i \in \mathcal{I}(W^i, W^{i+1})$. Then, there is a morphism $\widetilde{L}$ of $\{V^\cdot, D_A\}$ and $\{W^\cdot, D_B\}$ satisfying $\widetilde{L}^i - L^i \in \mathcal{I}(V^i, W^i)$.*

Proof. The case $\mathcal{I} = 0$ is trivial. For $\mathcal{I} \neq 0$, let P be an $\mathcal{I}$-parametrix of $\{V^\cdot, A\}$, i.e. $P^{i+1}A^i + A^{i-1}P^i = \mathrm{Id}_{V^i} - R^i$ with $R^i \in \mathcal{I}(V^i)$. Then it is easy to see that

$$\widetilde{L}^i := D_B^{i-1} L^{i-1} P^i + L^i P^{i+1} D_A^i$$

is a morphism of $\{V^\cdot, D_A\}$ and $\{W^\cdot, D_B\}$.

Setting

$$\begin{aligned} T^i &:= L^{i+1} D_A^i - D_B^i L^i \\ &= L^{i+1} A^i - B^i L^i + L^{i+1}(D_A^i - A^i) - (D_B^i - B^i) L^i \\ &\in \mathcal{I}(V^i, W^{i+1}) \end{aligned}$$

we obtain

$$\begin{aligned} L^i - \widetilde{L}^i &= L^i - (D_B^{i-1} L^{i-1} P^i + L^i P^{i+1} D_A^i) \\ &= L^i - L^i (D_A^{i-1} P^i + P^{i+1} D_A^i) + T^{i-1} P^i \\ &= L^i R^i + T^{i-1} P^i \\ &\in \mathcal{I}(V^i, W^i), \end{aligned}$$

as desired.

□

Assume L_1 and L_2 are quasimorphisms of sequences $\{V^\cdot, A\}$ and $\{W^\cdot, B\}$. Then we obtain a new quasimorphism by setting

$$L_1 + L_2 := (L_1^0 + L_2^0, \ldots, L_1^N + L_2^N).$$

Moreover, L_1 and L_2 are said to be homotopic if there exists a sequence $\{h^1, \ldots, h^N\}$ of bounded linear operators $h^i : V^i \to W^{i-1}$ with the property that

$$L_1^i - L_2^i = B^{i-1} h^i + h^{i+1} A^i$$

for all $i = 0, 1, \ldots, N$.

Obviously, an $\mathcal{I}$-parametrix of a quasicomplex $\{V^\cdot, A\}$ is a homotopy between the identity mappings $\{\mathrm{Id}_{V^0}, \ldots, \mathrm{Id}_{V^N}\}$ and a sequence $\{R^0, \ldots, R^N\}$, such that $R^i \in \mathcal{I}(V^i)$.

Lemma 2.2.3 *Assume L_1 and L_2 are homotopic morphisms of complexes $\{V^\cdot, D_A\}$ and $\{W^\cdot, D_B\}$. Then the induced maps HL_1^i and HL_2^i coincide.*

Proof. We find

$$[L_1^i v] = [L_2^i v] + [D_B^{i-1} h^i v] + [h^{i+1} D_A^i v] = [L_2^i v]$$

for all $v \in \ker D_A^i$, and the desired assertion follows.

□

2.2.2 The cone of a quasimorphism

If $\{V^\cdot, A\}$ and $\{W^\cdot, B\}$ are $\mathcal{I}$-quasicomplexes we may use an $\mathcal{I}$-quasimorphism L of them to construct a new $\mathcal{I}$-quasicomplex

$$\mathcal{C}(L):\ 0 \ \to \ \begin{array}{c} V^0 \\ \oplus \\ 0 \end{array} \ \stackrel{C^0}{\to} \ \begin{array}{c} V^1 \\ \oplus \\ W^0 \end{array} \ \stackrel{C^1}{\to} \ \ldots \ \stackrel{C^{N-1}}{\to} \ \begin{array}{c} V^N \\ \oplus \\ W^{N-1} \end{array} \ \stackrel{C^N}{\to} \ \begin{array}{c} 0 \\ \oplus \\ W^N \end{array} \ \to \ 0\ ,$$

where we set

$$C^i = \begin{pmatrix} -A^i & 0 \\ L^i & B^{i-1} \end{pmatrix}.$$

Indeed, since

$$C^{i+1}C^i = \begin{pmatrix} A^{i+1}A^i & 0 \\ B^iL^i - L^{i+1}A^i & B^iB^{i-1} \end{pmatrix}$$

is satisfied, $\mathcal{C}(L)$ is an $\mathcal{I}$-quasicomplex called the cone of the quasimorphism L, cf. [Spa66].

Theorem 2.2.4 *The cone $\mathcal{C}(L)$ associated to a quasimorphism L of Fredholm quasicomplexes is Fredholm.*

Proof. Suppose P_A and P_B are parametrices of $\{V^\cdot, A\}$ and $\{W^\cdot, B\}$, respectively, i.e.

$$\begin{array}{rcl} P_A^{i+1}A^i + A^{i-1}P_A^i &=& \mathrm{Id}_{V^i} - R_A^i, \\ P_B^{i+1}B^i + B^{i-1}P_B^i &=& \mathrm{Id}_{W^i} - R_B^i \end{array}$$

with $R_A^i \in \mathcal{K}(V^i)$ and $R_B^i \in \mathcal{K}(W^i)$. Setting

$$P^i = \begin{pmatrix} -P_A^i & 0 \\ P_B^{i-1}L^{i-1}P_A^i & P_B^{i-1} \end{pmatrix}$$

we obtain

$$\begin{array}{rl} & C^{i-1}P^i + P^{i+1}C^i \\ = & \begin{pmatrix} \mathrm{Id}_{V^i} - R_A^i & 0 \\ (R_B^{i-1} - P_B^iB^{i-1})L^{i-1}P_A^i + P_B^iL^i(A^{i-1}P_A^i + R_A^i) & \mathrm{Id}_{W^{i-1}} - R_B^{i-1} \end{pmatrix} \\ = & \mathrm{Id}_{V^i\oplus W^{i-1}} - R^i, \end{array}$$

where

$$\begin{array}{rcl} R^i &:=& \begin{pmatrix} R_A^i & 0 \\ -(R_B^{i-1} - P_B^iB^{i-1})L^{i-1}P_A^i - P_B^iL^i(A^{i-1}P_A^i + R_A^i) & R_B^{i-1} \end{pmatrix} \\ &=& \begin{pmatrix} R_A^i & 0 \\ -R_B^{i-1}L^{i-1}P_A^i + P_B^i(B^{i-1}L^{i-1} - L^iA^{i-1})P_A^i + P_B^iL^iR_A^i & R_B^{i-1} \end{pmatrix} \\ &\in& \mathcal{K}(V^i \oplus W^{i-1}). \end{array}$$

Hence, P is a parametrix of $\mathcal{C}(L)$, as desired.

□

An inspection of the proof shows that the constructed parametrix P is a special regulariser as in Remark 2.1.15, if we start with special regularisers P_A, P_B of the $\mathcal{I}$-quasicomlexes and if L is an $\mathcal{I}$-quasimorphism.

2.2.3 Quasiendomorphisms

We now turn to quasiendomorphisms of a sequence $\{V^{\cdot}, A\}$, i.e. sequences of linear operators $E^i \in \mathcal{L}(V^i)$ which make the diagram

$$\begin{array}{ccccccccc} 0 & \to & V^0 & \stackrel{A^0}{\to} & V^1 & \stackrel{A^1}{\to} & \dots & \stackrel{A^{N-1}}{\to} & V^N & \to & 0 \\ & & \downarrow E^0 & & \downarrow E^1 & & & & \downarrow E^N & & \\ 0 & \to & V^0 & \stackrel{A^0}{\to} & V^1 & \stackrel{A^1}{\to} & \dots & \stackrel{A^{N-1}}{\to} & V^N & \to & 0 \end{array}$$

commutative modulo operators of $\mathcal{I}$, i.e. $E^{i+1}A^i - A^iE^i \in \mathcal{I}(V^i, V^{i+1})$ holds for all $i = 0, 1, \dots, N-1$.

Example 2.2.5 Assume $L = \{L^0, \dots, L^N\}$ is an $\mathcal{I}$-quasimorphism of $\{V^{\cdot}, A\}$ and $\{W^{\cdot}, B\}$ and $M = \{M^0, \dots, M^N\}$ is an $\mathcal{I}$-quasimorphism of $\{W^{\cdot}, B\}$ and $\{V^{\cdot}, A\}$. Then,

$$ML := \{M^0L^0, \dots, M^NL^N\}$$

yields an $\mathcal{I}$-quasiendomorphism of $\{V^{\cdot}, A\}$.

The reduction Theorem 2.2.2 has the following simple form in the case of quasi-endomorphisms.

Theorem 2.2.6 *Suppose E is an $\mathcal{I}$-quasiendomorphism of a Fredholm $\mathcal{I}$-quasi-complex $\{V^{\cdot}, A\}$ and $\{V^{\cdot}, D\}$ is any complex satisfying $D^i - A^i \in \mathcal{I}(V^i, V^{i+1})$. Then there is an endomorphism $\widetilde{E}$ of $\{V^{\cdot}, D\}$ satisfying $\widetilde{E}^i - E^i \in \mathcal{I}(V^i)$.*

In order to reduce $\{E^i\}$, we can choose an arbitrary $\mathcal{I}$-parametrix P of $\{V^{\cdot}, A\}$ and set

$$\widetilde{E}^i := D^{i-1}E^{i-1}P^i + E^iP^{i+1}D^i.$$

Before studying an interesting example, we state a simple lemma of functional analysis.

Lemma 2.2.7 *Assume $A \in \mathcal{L}(V)$ is a selfadjoint operator on a Hilbert space V. Then $\ker A^n = \ker A$ is satisfied for all $n \in \mathbb{N}$.*

Proof. If $n = 1$, nothing is to show. Hence, we consider $n \geq 2$ and set $A^0 := \mathrm{Id}_V$. Obviously $\ker A \subset \ker A^n$ is satisfied. Indeed, if $Av = 0$ holds for some $v \in V$, then we get $A^nv = A^{n-1}(Av) = 0$. On the other hand, consider $v \in \ker A^n$, i.e. $A^nv = 0$. If $w \in V$ then

$$0 = (A^nv, w)_V = (A^{n-1}v, Aw)_V,$$

for A is selfadjoint. In particular,

$$\|A^{n-1}v\|^2 = (A^{n-1}v, AA^{n-2}v)_V = 0$$

and hence $v \in \ker A^{n-1}$. In the case $n = 2$ we are ready. Otherwise, we proceed by induction and find $v \in \ker A^{n-2}$, etc., $v \in \ker A$, as desired.

□

Assume $E = \{E^0, \dots, E^N\}$ is an $\mathcal{I}$-quasiendomorphism of an $\mathcal{I}$-quasicomplex $\{V^{\cdot}, A\}$. On setting $B^i := A^iE^i$ we obtain a new $\mathcal{I}$-quasicomplex $\{V^{\cdot}, B\}$, as is easy to check.

Example 2.2.8 The sequence of Laplacians $\{\Delta^0, \ldots, \Delta^N\}$ of an $\mathcal{I}$-quasicomplex $\{V^\cdot, A\}$ constitutes an $\mathcal{I}$-quasiendomorphism. We set $B^i := A^i \Delta^i$ and obtain an $\mathcal{I}$-quasicomplex $\{V^\cdot, B\}$. Suppose $\{V^\cdot, A\}$ is Fredholm. Then $\{V^\cdot, B\}$ is Fredholm, too. To see this, we write Δ^i_A for the Laplacians of $\{V^\cdot, A\}$ and reduce $\{V^\cdot, A\}$ to a complex $\{V^\cdot, D\}$ with Laplacians Δ^i_D. It is easy to verify that $\{V^\cdot, D\Delta_D\}$ is a reduced complex for $\{V^\cdot, B\}$ and

$$\Delta^i_{D\Delta_D} = (\Delta^i_D)^3$$

holds. Since Δ^i_D is a selfadjoint Fredholm operator, we get $\ker \Delta^i_{D\Delta_D} = \ker \Delta^i_D$ whence

$$\chi(V^\cdot, B) = \chi(V^\cdot, D\Delta_D) = \chi(V^\cdot, A).$$

One may ask if for any $\mathcal{I}$-quasiendomorphism E of an $\mathcal{I}$-quasicomplex $\{V^\cdot, A\}$ there is a reduced complex $\{V^\cdot, D_A\}$, such that E is an endomorphism of the complex, cf. [TW12]. This question was answered in [Esc13] by the following counterexample.

Example 2.2.9 The Toeplitz operators T_z and $T_{\bar{z}}$ on the unit circle are Fredholm and satisfy

$$0 \neq [T_{\bar{z}}, T_z] \in \mathfrak{S}_1(H^2(\mathbb{S})).$$

Suppose there exists an operator $C \in \mathfrak{S}_1(H^2(\mathbb{S}))$ such that

$$(T_{\bar{z}} + C)\, T_z = T_z\, (T_{\bar{z}} + C)\,.$$

Since the commutant of T_z consists of all Toeplitz operators T_g with $g \in H^\infty(\mathbb{S})$, we deduce that

$$T_{\bar{z}} + C = T_g$$

with some $g \in H^\infty(\mathbb{S})$. On the other hand, there are no nontrivial compact Toeplitz operators, i.e. $C = 0$. We thus arrive at the contradiction $\bar{z} = g \in H^\infty(\mathbb{S})$.

2.3 The Lefschetz number for quasicomplexes

2.3.1 Eulers identity

Suppose $E = \{E^0, \ldots, E^N\}$ is an endomorphism of a Fredholm complex $\{V^\cdot, D\}$. Then the mapping

$$HE^i : H^i(V^\cdot) \to H^i(V^\cdot),$$

given by $[v] \mapsto [E^i v]$, is an endomorphism of the finite-dimensional space $H^i(V^\cdot)$, and so the trace $\operatorname{tr} HE^i$ is well defined, for each i. The alternating sum

$$\mathscr{L}(E, D) := \sum_{i=0}^{N} (-1)^i \operatorname{tr} HE^i$$

is called the Lefschetz number of the endomorphism.

Example 2.3.1 Set $E^i = \mathrm{Id}_{V^i}$ for $i = 0, 1, \ldots, N$. Then $\mathrm{tr}\, HE^i = \dim H^i(V^\cdot)$ is satisfied and we obtain

$$\mathscr{L}(E) = \sum_{i=0}^{N} (-1)^i \dim H^i(V^\cdot) = \chi(V^\cdot).$$

If E and F are homotopic endomorphisms of a Fredholm complex $\{V^\cdot, D\}$ then Lemma 2.2.3 implies $\mathscr{L}(E, D) = \mathscr{L}(F, D)$.

It turns out that the Lefschetz number can be extended to $\mathfrak{S}_1$-quasiendomorphims of $\mathfrak{S}_1$-quasicomplexes. To show this we need an auxiliary result which is usually referred to as Euler's identity, see [AB67] or Theorem 19.1.15 in [Hoe85]. In order to show this for nonseparable Hilbert spaces, we prove an additional result.

Lemma 2.3.2 *Let U be a closed subspace of a Hilbert space V. Assume $A \in \mathfrak{S}_1(V)$ satisfies $U^\perp \subset \ker A$. Then*

$$\mathrm{tr}\, A = \mathrm{tr}\, (P_U A \restriction_U),$$

where P_U is the orthogonal projection onto U.

Proof. Without loss of generality we assume $\mathbb{K} = \mathbb{C}$, otherwise we complexify the space V. Set $\tilde{A} := P_U A \restriction_U$. Then $(\lambda \mathrm{Id}_V - A)^n v = 0$ is satisfied for some $v \in V \setminus \{0\}$ and some $\lambda \in \mathbb{C} \setminus \{0\}$ if and only if $v \in U$ and $(\lambda \mathrm{Id}_U - \tilde{A})^n v = 0$ holds. Hence, $\lambda \neq 0$ is an eigenvalue of A with algebraic multiplicity m if and only if λ is an eigenvalue of $\tilde{A}$ with algebraic multiplicity m. This implies readily $\mathrm{tr}\, A = \mathrm{tr}\, \tilde{A}$, as desired.

□

Lemma 2.3.3 *Let E be an endomorphism of a Fredholm complex $\{V^\cdot, D\}$ satisfying $E^i \in \mathfrak{S}_1(V^i)$ for all $i = 0, 1, \ldots, N$. Then*

$$\mathscr{L}(E) = \sum_{i=0}^{N} (-1)^i \, \mathrm{tr}\, E^i.$$

Proof. Using the Hodge decomposition (2.1.6), we find

$$\begin{aligned} E^i &= H^i E^i + D^{i-1} D^{i-1*} G^i E^i + D^{i*} G^{i+1} D^i E^i \\ &= H^i E^i + D^{i-1} D^{i-1*} G^i E^i + D^{i*} G^{i+1} E^{i+1} D^i, \end{aligned}$$

where all summands are trace class operators. The second and the third summands cancel each other in the alternating sum of traces. We thus obtain

$$\sum_{i=0}^{N} (-1)^i \, \mathrm{tr}\, E^i = \sum_{i=0}^{N} (-1)^i \, \mathrm{tr}\, H^i E^i = \sum_{i=0}^{N} (-1)^i \, \mathrm{tr}\, H^i H^i E^i = \sum_{i=0}^{N} (-1)^i \, \mathrm{tr}\, H^i E^i H^i.$$

We have to show that $\mathrm{tr}\, H^i E^i H^i = \mathrm{tr}\, HE^i$ holds, where HE^i is the linear map in $H^i(V^\cdot)$, which is induced by E^i. We set $T^i := H^i E^i H^i$. Then $(\ker \Delta^i)^\perp \subset \ker T^i$

holds and Lemma 2.3.2 implies that $\operatorname{tr} T^i = \operatorname{tr}\left(H^i E^i \restriction_{\ker \Delta^i}\right)$ is satisfied. Since $\ker \Delta^i \cong H^i(V^\cdot)$ holds, the traces of the operators HE^i and $H^i E^i \restriction_{\ker \Delta^i}$ coincide, as desired.

□

Note that Lemma 2.3.3 is valid not only for the trace class operators E^i but also for those operators E^i, for which the wave front calculus allows one to define the trace by restricting the Schwartz kernel to the diagonal, see Theorem 19.4.1 of [Hoe85].

2.3.2 Definition of Lefschetz number

The following definition is of crucial importance in this work. It stems from [TW12] by direct calculation.

Definition 2.3.4 Let $\{V^\cdot, A\}$ be a Fredholm $\mathfrak{S}_1$-quasicomplex and E a $\mathfrak{S}_1$-quasi-endomorphisms of this quasicomplex. Then the Lefschetz number is defined as

$$\mathscr{L}(E, A) = \mathscr{L}(\widetilde{E}, D) + \sum_{i=0}^{N} (-1)^i \operatorname{tr}(E^i - \widetilde{E}^i),$$

where $\{V^\cdot, D\}$ is a complex, such that $D^i - A^i \in \mathfrak{S}_1(V^i, V^{i+1})$, and $\widetilde{E}$ is an endomorphism of $\{V^\cdot, D\}$, such that $\widetilde{E}^i - E^i \in \mathfrak{S}_1(V^i)$.

Obviously, $\mathscr{L}(E, A)$ coincides with the classical Lefschetz number, if $\{V^\cdot, A\}$ is a Fredholm complex and E is an endomorphism of $\{V^\cdot, A\}$.

We have to show that the definition is independent of the particular choice of D and $\tilde{E}$. For this purpose we choose an arbitrary $\mathfrak{S}_1$-parametrix P. Then $\tilde{E}$ and

$$\widetilde{E}^i - D^{i-1}\widetilde{E}^{i-1}P^i - \widetilde{E}^i P^{i+1} D^i \in \mathfrak{S}_1(V^i)$$

are homotopic endomorphisms of $\{V^\cdot, D\}$. By Lemma 2.3.3,

$$\mathscr{L}(\widetilde{E}, D) = \sum_{i=0}^{N} (-1)^i \operatorname{tr}(\widetilde{E}^i - D^{i-1}\widetilde{E}^{i-1}P^i - \widetilde{E}^i P^{i+1} D^i)$$

and therefore

$$\begin{aligned}
\mathscr{L}(E, A) &= \sum_{i=0}^{N} (-1)^i \operatorname{tr}(E^i - D^{i-1}\widetilde{E}^{i-1}P^i - \widetilde{E}^i P^{i+1} D^i) \\
&= \sum_{i=0}^{N} (-1)^i \operatorname{tr}(E^i - D^{i-1}E^{i-1}P^i - E^i P^{i+1} D^i) \\
&= \sum_{i=0}^{N} (-1)^i \operatorname{tr}(E^i - A^{i-1}E^{i-1}P^i - E^i P^{i+1} A^i),
\end{aligned}$$

the second and third equalities being due to Theorem 1.1.14. Indeed, the differences of the right-hand sides and the left-hand sides of these equalities just amount to

$$\sum_{i=0}^{N-1}(-1)^i \operatorname{tr}\left((E^i-\widetilde{E}^i)P^{i+1}D^i - D^i(E^i-\widetilde{E}^i)P^{i+1}\right),$$
$$\sum_{i=0}^{N-1}(-1)^i \operatorname{tr}\left(E^iP^{i+1}(A^i-D^i) - (A^i-D^i)E^iP^{i+1}\right),$$

respectively, where each summand vanishes by Theorem 1.1.14. This shows the independence of $\tilde{E}$ and D.

Corollary 2.3.5 *Let $\{V^\cdot, A\}$ be a Fredholm $\mathfrak{S}_1$-quasicomplex and E a $\mathfrak{S}_1$-quasiendomorphism of this quasicomplex. Then*

$$\mathscr{L}(E,A) = \sum_{i=0}^{N}(-1)^i \operatorname{tr}\left(E^i - A^{i-1}E^{i-1}P^i - E^iP^{i+1}A^i\right)$$

for each $\mathfrak{S}_1$-parametrix P of $\{V^\cdot, A\}$.

The corollary above can also be used as a definition of the Lefschetz number. This was precisely our approach in [TW12].

Choosing $\widetilde{E}^i := D^{i-1}E^{i-1}P^i + E^iP^{i+1}D^i$ as in the proof of Theorem 2.2.6, we get $\mathscr{L}(\widetilde{E}, D) = 0$, for $\widetilde{E}$ and 0 are homotopic endomorphisms of $\{V^\cdot, D\}$. Hence it follows that

$$\mathscr{L}(E,A) = \sum_{i=0}^{N}(-1)^i \operatorname{tr}\left(E^i - \widetilde{E}^i\right)$$

in this special case.

Remark 2.3.6 The equivalence of Definition 2.3.4 and Corollary 2.3.5 was shown recently by J. Eschmeier in [Esc13]. Moreover, he proved Theorem 2.2.6 in the case of $\mathfrak{S}_p$-quasicomplexes in Banach spaces.

Let $\{E^0, E^1\}$ be a $\mathfrak{S}_1$-quasiendomorphism of a Fredholm operator $A \in \mathcal{L}(V^0, V^1)$. Then

$$\mathscr{L}(E,A) = \operatorname{tr}\left(E^0 - E^0PA\right) - \operatorname{tr}\left(E^1 - AE^0P\right)$$

holds. Here, P is a $\mathfrak{S}_1$-parametrix (regulariser) of the operator A. As far as we know, this formula was first suggested (in the case of endomorphisms) by B.V. Fedosov in [Fed91], who mentioned that the trace class perturbations of the operator A could not influence the Lefschetz number and so the formula made it possible to consider $\mathfrak{S}_1$-quasiendomorphisms. Inspired by this, we decided to use the formula of Corollary 2.3.5 to define the Lefschetz number in the context of quasicomplexes. Note that the independence of the particular choice of a parametrix is much more transparent for a single operator, for any two parametrices differ merely by an operator of the class $\mathfrak{S}_1(V^1, V^0)$.

In [Esc13] the formula above was applied to Toeplitz operators in order to generalise the Gokhberg-Krein index formula.

Example 2.3.7 Let $f, g \in C^\infty(\mathbb{S})$. Suppose $g \neq 0$ on $\mathbb{S}$. Then the Toeplitz operator T_g is Fredholm. Moreover,

$$\begin{aligned} \mathrm{Id}_{H^2(\mathbb{S})} - T_{g^{-1}} T_g &\in \mathfrak{S}_1(H^2(\mathbb{S})), \\ \mathrm{Id}_{H^2(\mathbb{S})} - T_g T_{g^{-1}} &\in \mathfrak{S}_1(H^2(\mathbb{S})) \end{aligned}$$

is satisfied. Hence, $E := \{T_f, T_f\}$ yields a $\mathfrak{S}_1$-quasiendomorphism of the Fredholm operator T_g. The Lefschetz number fulfills

$$\begin{aligned} \mathscr{L}(E, T_g) &= \mathrm{tr}\,(T_f - T_f T_{g^{-1}} T_g) - \mathrm{tr}\,(T_f - T_g T_f T_{g^{-1}}) \\ &= \mathrm{tr}\,[T_g, T_f T_{g^{-1}}] \\ &= \mathrm{tr}\,[T_g, T_{fg^{-1}}] \\ &= \frac{1}{2\pi\imath} \int_{\mathbb{S}} g\, d_z(fg^{-1}) \\ &= \frac{1}{2\pi\imath} \int_{\mathbb{S}} d_z f + gf\, d_z(g^{-1}) \\ &= -\frac{1}{2\pi\imath} \int_{\mathbb{S}} f\, \frac{d_z g}{g}. \end{aligned}$$

In particular, for $f = 1$ we obtain $\mathscr{L}(1, T_g) = -\deg(g, 0)$.

Example 2.3.8 We consider Hilbert-Schmidt operators S and E^0, E^1 in $\mathfrak{S}_2(V)$ and set $A := \mathrm{Id}_V - S$ and $E = \{E^0, E^1\}$. Assume $E^0 - E^1 \in \mathfrak{S}_1(V)$ is satisfied. We find

$$AE^0 - E^1 A = (E^0 - E^1 + E^0 S - SE^1) \in \mathfrak{S}_1(V),$$

and thus E is a $\mathfrak{S}_1$-quasiendomorphism of the short complex $\{V^\cdot, A\}$. Moreover, A is a Fredholm operator and a $\mathfrak{S}_1$-parametrix of A is given by $P := \mathrm{Id}_V + S$. We obtain the formula

$$\begin{aligned} \mathscr{L}(E, A) &= \mathrm{tr}\,(E^0 - E^0 PA) - \mathrm{tr}\,(E^1 - AE^0 P) \\ &= \mathrm{tr}\,(E^0 - E^1). \end{aligned}$$

2.3.3 Properties

It turns out that all essential properties of the classical Lefschetz number, which is defined for endomorphisms of complexes, survive if we consider quasiendomorphisms and quasicomplexes. This displays once again the canonical nature of our definition.

Theorem 2.3.9 *Let E, F be $\mathfrak{S}_1$-quasiendomorphisms of a Fredholm $\mathfrak{S}_1$-quasicomplex $\{V^\cdot, A\}$. The following assertions hold:*

i) $\mathscr{L}(E + F, A) = \mathscr{L}(E, A) + \mathscr{L}(F, A)$ *(additivity).*

ii) $\mathscr{L}(E, A) = \mathscr{L}(F, A)$, *if E and F are homotopic (homotopy invariance).*

iii) $\mathscr{L}(\mathrm{Id}_{V^\cdot}, A) = \chi(V^\cdot, A)$ *(normalisation).*

Proof. Since the trace is additive, the first assertion follows directly by Corollary 2.3.5.

We use the same corollary to show *ii*). Namely, choose a complex $\{V^\cdot, D\}$, such that $T^i := A^i - D^i \in \mathfrak{S}_1(V^i, V^{i+1})$. Set

$$\begin{aligned} G^i &:= E^i - F^i - T^{i-1}h^i - h^{i+1}T^i \\ &= D^{i-1}h^i + h^{i+1}D^i. \end{aligned}$$

Then G is an endomorphism of the complex $\{V^\cdot, D\}$ homotopic to 0, and we find

$$\begin{aligned} \mathscr{L}(E,A) - \mathscr{L}(F,A) &= \mathscr{L}(E,D) - \mathscr{L}(F,D) \\ &= \sum_{i=0}^{N}(-1)^i \operatorname{tr}(E^i - F^i - D^{i-1}(E^{i-1} - F^{i-1})P^i - (E^i - F^i)P^{i+1}D^i) \\ &= \sum_{i=0}^{N}(-1)^i \operatorname{tr}(G^i - D^{i-1}G^{i-1}P^i - G^iP^{i+1}D^i) \\ &= \mathscr{L}(G,D) \\ &= 0, \end{aligned}$$

the third equation being a consequence of Theorem 1.1.14.

Definition 2.3.4 implies in particular that $\mathscr{L}(E,A) = \mathscr{L}(E,D)$, and so we obtain immediately

$$\mathscr{L}(\mathrm{Id}_{V^\cdot}, A) = \mathscr{L}(\mathrm{Id}_{V^\cdot}, D) = \chi(V^\cdot, D) =: \chi(V^\cdot, A).$$

□

Example 2.3.10 Let $\{V^\cdot, A\}$ be a Fredholm $\mathfrak{S}_1$-quasicomplex. Then the sequence of Laplacians $\Delta = \{\Delta^0, \ldots, \Delta^N\}$ is a $\mathfrak{S}_1$-quasiendomorphism of $\{V^\cdot, A\}$ homotopic to $0 = \{0_{V^0}, \ldots, 0_{V^N}\}$ and we obtain $\mathscr{L}(\Delta, A) = 0$.

Assume $\{V^\cdot, A\}$ is a Fredholm $\mathfrak{S}_1$-quasicomplex. Then Theorem 2.3.9 and Corollary 2.3.5 imply the trace formula

$$\chi(V^\cdot, A) = \sum_{i=0}^{N}(-1)^i \operatorname{tr}(\mathrm{Id}_{V^i} - A^{i-1}P^i - P^{i+1}A^i)$$

for the Euler characteristic, cf. [Tar07]. In particular,

$$\operatorname{ind} A = \operatorname{tr}(\mathrm{Id}_{V^0} - PA) - \operatorname{tr}(\mathrm{Id}_{V^1} - AP)$$

for a Fredholm operator $A \in \mathcal{L}(V^0, V^1)$ and $\operatorname{ind} A = \operatorname{tr}[A, P]$, if $V^0 = V^1$, where P is a $\mathfrak{S}_1$-parametrix of the quasicomplex $\{V^\cdot, A\}$ and of the operator A, respectively.

Example 2.3.11 Let $\{V^\cdot, A\}$ and $\{W^\cdot, B\}$ be Fredholm quasicomplexes and let L be a quasimorphism of these quasicomplexes. We are going to compute the Euler characteristic of the associated cone $\mathcal{C}(L)$. To this end, we choose complexes $\{V^\cdot, D_A\}$ and $\{W^\cdot, D_B\}$ with the property that

$$\begin{array}{rcl} D_A^i - A^i & \in & \mathcal{K}(V^i, V^{i+1}), \\ D_B^i - B^i & \in & \mathcal{K}(W^i, W^{i+1}), \end{array}$$

respectively. Then there is a morphism $\widetilde{L}$ of $\{V^\cdot, D_A\}$ and $\{W^\cdot, D_B\}$ satisfying $\widetilde{L}^i - L^i \in \mathcal{K}(V^i, W^i)$. We obtain a new cone

$$\mathcal{C}(\widetilde{L}): \ 0 \ \to \ \begin{array}{c} V^0 \\ \oplus \\ 0 \end{array} \ \stackrel{\widetilde{C}^0}{\to} \ \begin{array}{c} V^1 \\ \oplus \\ W^0 \end{array} \ \stackrel{\widetilde{C}^1}{\to} \ \ldots \ \stackrel{\widetilde{C}^{N-1}}{\to} \ \begin{array}{c} V^N \\ \oplus \\ W^{N-1} \end{array} \ \stackrel{\widetilde{C}^N}{\to} \ \begin{array}{c} 0 \\ \oplus \\ W^N \end{array} \ \to \ 0$$

where

$$\widetilde{C}^i = \begin{pmatrix} -D_A^i & 0 \\ \widetilde{L}^i & D_B^{i-1} \end{pmatrix}.$$

Suppose P_A, P_B are $\mathfrak{S}_1$-parametrices of $\{V^\cdot, D_A\}$ and $\{W^\cdot, D_B\}$, respectively, i.e.

$$\begin{array}{rcl} P_A^{i+1} D_A^i + D_A^{i-1} P_A^i & = & \mathrm{Id}_{V^i} - R_A^i, \\ P_B^{i+1} D_B^i + D_B^{i-1} P_B^i & = & \mathrm{Id}_{W^i} - R_B^i \end{array}$$

with $R_A^i \in \mathfrak{S}_1(V^i)$ and $R_B^i \in \mathfrak{S}_1(W^i)$. For

$$P^i = \begin{pmatrix} -P_A^i & 0 \\ P_B^{i-1} \widetilde{L}^{i-1} P_A^i & P_B^{i-1} \end{pmatrix},$$

we get

$$\mathrm{Id}_{V^i \oplus W^{i-1}} - \widetilde{C}^{i-1} P^i - P^{i+1} \widetilde{C}^i = R^i$$

where

$$\begin{array}{rcl} R^i & = & \begin{pmatrix} R_A^i & 0 \\ -R_B^{i-1} \widetilde{L}^{i-1} P_A^i + P_B^i \widetilde{L}^i R_A^i & R_B^{i-1} \end{pmatrix} \\ & \in & \mathfrak{S}_1(V^i \oplus W^{i-1}). \end{array}$$

Hence it follows that

$$\begin{array}{rcl} \chi(\mathcal{C}(L)) & = & \chi(\mathcal{C}(\widetilde{L})) \\ & = & \displaystyle\sum_{i=0}^{N+1} (-1)^i \operatorname{tr} R^i \\ & = & \displaystyle\sum_{i=0}^{N+1} (-1)^i \left(\operatorname{tr} R_A^i + \operatorname{tr} R_B^{i-1}\right) \\ & = & \displaystyle\sum_{i=0}^{N} (-1)^i \operatorname{tr} R_A^i - \sum_{i=0}^{N} (-1)^i \operatorname{tr} R_B^i \\ & = & \chi(V^\cdot, D_A) - \chi(W^\cdot, D_B) \\ & = & \chi(V^\cdot, A) - \chi(W^\cdot, B). \end{array}$$

Theorem 2.3.12 *Let $L^i : V^i \to W^i$ and $M^i : W^i \to V^i$, with $i = 0, 1, \dots, N$, be $\mathfrak{S}_1$-quasimorphisms of Fredholm $\mathfrak{S}_1$-quasicomplexes $\{V^\cdot, A\}$ and $\{W^\cdot, B\}$, respectively. Then*

$$\mathscr{L}(FE, A) = \mathscr{L}(EF, B)$$

holds.

Proof. Choose reduced complexes $\{V^\cdot, D_A\}$, $\{W^\cdot, D_B\}$ and morphisms $\widetilde{L}$ and $\widetilde{M}$ of them, such that

$$\begin{array}{rcccl} T_L^i & := & \widetilde{L}^i - L^i & \in & \mathfrak{S}_1(V^i, W^i), \\ T_M^i & := & \widetilde{M}^i - M^i & \in & \mathfrak{S}_1(W^i, V^i). \end{array}$$

Then $\widetilde{M}\widetilde{L}$ is an endomorphism of $\{V^\cdot, D_A\}$ and $\widetilde{L}\widetilde{M}$ is an endomorphism of $\{W^\cdot, D_B\}$ satisfying

$$\begin{array}{rcl} \widetilde{M}^i\widetilde{L}^i - M^iL^i & \in & \mathfrak{S}_1(V^i), \\ \widetilde{L}^i\widetilde{M}^i - L^iM^i & \in & \mathfrak{S}_1(W^i). \end{array}$$

The equality

$$\operatorname{tr} H(\widetilde{M}\widetilde{L})^i = \operatorname{tr} H(\widetilde{M})^i H(\widetilde{L})^i = \operatorname{tr} H(\widetilde{L})^i H(\widetilde{M})^i = \operatorname{tr} H(\widetilde{L}\widetilde{M})^i$$

implies

$$\mathscr{L}(\tilde{M}\tilde{L}, D_A) = \mathscr{L}(\tilde{L}\tilde{M}, D_B).$$

Moreover, using

$$\begin{array}{rcl} \operatorname{tr}(M^iL^i - \widetilde{M}^i\widetilde{L}^i) & = & \operatorname{tr}(M^iL^i - (M^i + T_M^i)(L^i + T_L^i)) \\ & = & -\operatorname{tr}(T_M^iL^i) - \operatorname{tr}(M^iT_L^i) - \operatorname{tr}(T_M^iT_L^i) \\ & = & -\operatorname{tr}(L^iT_M^i) - \operatorname{tr}(T_L^iM^i) - \operatorname{tr}(T_L^iT_M^i) \\ & = & \operatorname{tr}(L^iM^i - (L^i + T_L^i)(M^i + T_M^i)) \\ & = & \operatorname{tr}(L^iM^i - \widetilde{L}^i\widetilde{M}^i) \end{array}$$

we obtain

$$\begin{array}{rcl} \mathscr{L}(ML, A) & = & \mathscr{L}(\widetilde{M}\widetilde{L}, D_A) + \displaystyle\sum_{i=0}^{N} (-1)^i \operatorname{tr}(M^iL^i - \widetilde{M}^i\widetilde{L}^i) \\ & = & \mathscr{L}(\widetilde{L}\widetilde{M}, D_B) + \displaystyle\sum_{i=0}^{N} (-1)^i \operatorname{tr}(L^iM^i - \widetilde{L}^i\widetilde{M}^i) \\ & = & \mathscr{L}(LM, B), \end{array}$$

as desired.

□

Chapter 3

Applications

In this chapter we apply the theory developed above to sequences of pseudodifferential operators acting in spaces of smooth sections of vector bundles. Note that these spaces are (Fréchet-Schwartz spaces and hence) no longer Banach, hence compact operators fail to be "small" in this context, for all bounded operators are compact. Following [KTT07] we call a pseudodifferential operator "small" if its principal symbol vanishes. According to this, we elaborate the theory of elliptic quasicomplexes and prove that the concept of ellipticity can be extended naturally to quasicomplexes of pseudodifferential operators. Unless otherwise stated we assume in the sequel that X is a smooth compact closed manifold and F^i smooth vector bundles over X.

3.1 Sequences of pseudodifferential operators

3.1.1 Elliptic quasicomplexes

We start with some pretty natural definitions.

Definition 3.1.1 By a quasicomplex of pseudodifferential operators on X is meant any sequence of the form

$$\mathcal{E}(X,F^\cdot):\ 0 \to \mathcal{E}(X,F^0) \stackrel{A^0}{\to} \mathcal{E}(X,F^1) \stackrel{A^1}{\to} \ldots \stackrel{A^{N-1}}{\to} \mathcal{E}(X,F^N) \to 0 \qquad (3.1.1)$$

with $A^i \in \Psi^{m_i}_{\mathrm{cl}}(X;F^i,F^{i+1})$ satisfying $A^{i+1}A^i \in \Psi^{m_{i+1}+m_i-1}(X;F^i,F^{i+2})$.

To any quasicomplex one assigns the sequence of principal symbols

$$\pi^*F^\cdot:\ 0 \to \pi^*F^0 \stackrel{\sigma^{m_0}(A^0)}{\to} \pi^*F^1 \stackrel{\sigma^{m_1}(A^1)}{\to} \ldots \stackrel{\sigma^{m_{N-1}}(A^{N-1})}{\to} \pi^*F^N \to 0$$

which is actually a complex of bundle homomorphisms, for

$$\begin{aligned} \sigma^{m_{i+1}}(A^{i+1})\,\sigma^{m_i}(A^i) &= \sigma^{m_{i+1}+m_i}(A^{i+1}A^i) \\ &= 0. \end{aligned}$$

Locally, we obtain the complex

$$F_x^{\cdot}:\ 0 \longrightarrow F_x^0 \stackrel{\sigma^{m_0}(A^0)(x,\xi)}{\longrightarrow} F_x^1 \stackrel{\sigma^{m_1}(A^1)(x,\xi)}{\longrightarrow} \dots \stackrel{\sigma^{m_{N-1}}(A^{N-1})(x,\xi)}{\longrightarrow} F_x^N \longrightarrow 0.$$

Definition 3.1.2 A quasicomplex $\mathcal{E}(X, F^{\cdot})$ is called elliptic if its symbol complex $F_x^{\cdot}$ is exact (away from the zero section of T^*X).

It should be noted that the principal symbol mapping plays actually the role of a functor.

As usual, by a parametrix of a quasicomplex $\mathcal{E}(X, F^{\cdot})$ is meant any sequence of pseudodifferential operators $P^i \in \Psi^{-m_{i-1}}(X; F^i, F^{i-1})$ satisfying

$$P^{i+1}A^i + A^{i-1}P^i = Id_{\mathcal{E}(X,F^i)} - S^i \tag{3.1.2}$$

with $S^i \in \Psi_{\mathrm{cl}}^{-1}(X; F^i)$ for all $i = 0, 1, \dots, N$.

Let us first study the case where the orders of the operators A^i in sequence (3.1.1) are the same, that is $m_i = m$ for all $i = 0, 1, \dots, N-1$.

Assume $\mathcal{E}(X, F^{\cdot})$ is a quasicomplex of pseudodifferential operators of the same order $A^i \in \Psi_{\mathrm{cl}}^m(X; F^i, F^{i+1})$. The operators

$$L^i = A^{i-1}A^{i-1*} + A^{i*}A^i$$

in $\Psi_{\mathrm{cl}}^{2m}(X; F^i)$ are called (formal) Laplacians of the quasicomplex.

Lemma 3.1.3 *A quasicomplex $\mathcal{E}(X, F^{\cdot})$ with operators of the same order is elliptic if and only if all the formal Laplacians L^i are elliptic.*

Proof. We consider the complex of principal symbols

$$F_x^{\cdot}:\ 0 \longrightarrow F_x^0 \stackrel{\sigma^{m}(A^0)(x,\xi)}{\longrightarrow} F_x^1 \stackrel{\sigma^{m}(A^1)(x,\xi)}{\longrightarrow} \dots \stackrel{\sigma^{m}(A^{N-1})(x,\xi)}{\longrightarrow} F_x^N \longrightarrow 0$$

for $(x, \xi) \in T^*X \setminus \{0\}$. This is a complex of Hilbert spaces. By the Hodge theory, $F_x^{\cdot}$ is exact if and only if the Laplacians

$$\sigma^m(A^{i-1})(x,\xi)(\sigma^m(A^{i-1})(x,\xi))^* + (\sigma^m(A^i)(x,\xi))^*\sigma^m(A^i)(x,\xi)$$

are isomorphisms. Since

$$\begin{aligned} \sigma^{2m}(L^i)(x,\xi) &= \sigma^{2m}(A^{i-1}A^{i-1*} + A^{i*}A^i)(x,\xi) \\ &= \sigma^m(A^{i-1})(x,\xi)(\sigma^m(A^{i-1})(x,\xi))^* + (\sigma^m(A^i)(x,\xi))^*\sigma^m(A^i)(x,\xi), \end{aligned}$$

this holds if and only if the (formal) Laplacians L^i are elliptic.

□

Example 3.1.4 We consider the de Rham complex

$$\Omega^{\cdot}(X):\ 0 \longrightarrow \Omega^0(X) \stackrel{d}{\longrightarrow} \Omega^1(X) \stackrel{d}{\longrightarrow} \dots \stackrel{d}{\longrightarrow} \Omega^n(X) \longrightarrow 0,$$

where n is the dimension of X. Since the operators d^i are of the same order, the (formal) Laplacians $L^i \in \Psi^2_{\mathrm{cl}}(X, \Lambda^i T^*_{\mathbb{C}} X)$ are elliptic of order 2. In fact, these are the Hodge-Laplace operators which are elliptic due to Example 1.3.9. Applying Lemma 3.1.3 we deduce that the de Rham complex is elliptic, i.e.

$$0 \longrightarrow \mathbb{C} \overset{\sigma^1(d^0)(x,\xi)}{\longrightarrow} \mathbb{C}^n \overset{\sigma^1(d^1)(x,\xi)}{\longrightarrow} \ldots \overset{\sigma^1(d^{N-1})(x,\xi)}{\longrightarrow} \mathbb{C} \longrightarrow 0$$

is exact away from the zero section of T^*X.

As mentioned, the differential d should be replaced by a connection ∂ if we consider differential forms with values in sections of a vector bundle over F. This leads to a natural example of an elliptic quasicomplex.

Example 3.1.5 Let F be a smooth vector bundle of rank k on X. Pick a connection ∂ on F. Consider the sequence

$$\Omega^{\cdot}(X, F): \ 0 \to \mathcal{E}(X, F) \overset{\partial^0}{\to} \Omega^1(X, F) \overset{\partial^1}{\to} \ldots \overset{\partial^{n-1}}{\to} \Omega^n(X, F) \to 0.$$

Since $\partial^{i+1}\partial^i$ is a differential operator of order 0, the sequence is a quasicomplex. The principal symbols of the (formal) Laplacians L^i_∂ are given by

$$\sigma^2(L^i_\partial)(x, \xi) = \mathrm{Id}_{F_x} \otimes \sigma^2(L^i_d)(x, \xi),$$

where L^i_d are the Hodge-Laplace operators. We thus conclude that $\sigma^2(L^i_\partial)(x, \xi)$ is invertible for all $(x, \xi) \in T^*X \setminus \{0\}$. Hence, $\Omega^{\cdot}(X, F)$ is elliptic.

Note that the quasicomplex of connections is a complex if and only if the associated bundle is trivial, cf. [Wel80].

3.1.2 Order reduction

We now turn to quasicomplexes with pseudodifferential operators of different orders m_i, in which case the formal Laplacians fail to be relevant to the study. To get rid of this irrelevance we use a familiar construction with order reduction isomorphisms. More precisely, for a fixed $s \in \mathbb{R}$, we choose invertible operators

$$R_i \in \Psi^{s-(m_0+\ldots+m_{i-1})}_{\mathrm{cl}}(X; F^i)$$

$(m_{-1} = 0)$ and set $\widetilde{A}^i = R_{i+1} A^i R_i^{-1}$. Then $\widetilde{A}^i \in \Psi^0_{\mathrm{cl}}(X; F^i, F^{i+1})$ holds and we obtain a quasicomplex

$$\{\mathcal{E}(X, F^{\cdot}), \widetilde{A}\}: \ 0 \to \mathcal{E}(X, F^0) \overset{\widetilde{A}^0}{\to} \mathcal{E}(X, F^1) \overset{\widetilde{A}^1}{\to} \ldots \overset{\widetilde{A}^{N-1}}{\to} \mathcal{E}(X, F^N) \to 0$$

of operators of order 0.

Lemma 3.1.6 *A quasicomplex $\{\mathcal{E}(X, F^{\cdot}), A\}$ is elliptic if and only if the reduced quasicomplex $\{\mathcal{E}(X, F^{\cdot}), \widetilde{A}\}$ is elliptic.*

Proof. Set

$$s_i = s - (m_0 + \ldots + m_{i-1}) \tag{3.1.3}$$

for $i = 0, 1, \ldots, N$, so that $s_0 := s$ and $s_{i+1} = s_i - m_i$. Since $R_i \in \Psi^{s_i}_{\mathrm{cl}}(X; F^i)$ is invertible, the inverse operator is available in $\Psi^{-s_i}_{\mathrm{cl}}(X; F^i)$, which is a consequence of spectral invariance of Theorem 1.3.12. Moreover, we get $\sigma^{-s_i}(R_i^{-1}) = (\sigma^{s_i}(R_i))^{-1}$ whence

$$\sigma^0(\widetilde{A}^i) = \sigma^{s_{i+1}}(R_{i+1})\, \sigma^{m_i}(A^i)\, (\sigma^{s_i}(R_i))^{-1}.$$

On using this equality we establish the assertion by a straightforward computation, as desired.

□

Lemma 3.1.7 *A family $\{P^i\}$ is a parametrix of $\mathcal{E}(X, F^\cdot)$, if and only if $\{Q^i\}$ with $Q^i = R_{i-1}^{-1} P^i R_i$ is a parametrix of the reduced quasicomplex $\{\mathcal{E}(X, F^\cdot), \widetilde{A}\}$.*

Proof. By

$$\begin{aligned}
\widetilde{A}^{i-1} Q^i + Q^{i+1} \widetilde{A}^i &= R_i^{-1} A^{i-1} R_{i-1} R_{i-1}^{-1} P^i R_i + R_i^{-1} P^{i+1} R_{i+1} R_{i+1}^{-1} A^i R_i \\
&= R_i^{-1} (A^{i-1} P^i + P^{i+1} A^i) R_i
\end{aligned}$$

the desired assertion follows.

□

Theorem 3.1.8 *Each elliptic quasicomplex $\mathcal{E}(X, F^\cdot)$ possesses a parametrix.*

Proof. We reduce the elliptic quasicomplex $\mathcal{E}(X, F^\cdot)$ to an elliptic quasicomplex $(\mathcal{E}(X, F^\cdot), \widetilde{A})$ with operators of order 0. By Lemma 3.1.3, the (formal) Laplacians Δ^i are elliptic. Applying Lemma 1.3.11 we deduce that there exists a parametrix $G^i \in \Psi^0_{\mathrm{cl}}(X; F^i)$ for Δ^i. We now proceed as in the proof of Theorem 2.1.14 and obtain that the operators $P^i = G^{i-1} \widetilde{A}^{i-1*}$ constitute a parametrix of $\{\mathcal{E}(X, F^\cdot), \widetilde{A}\}$. Hence, $\mathcal{E}(X, F^\cdot)$ possesses a parametrix due to Lemma 3.1.7.

□

An inspection of the proofs above shows that an elliptic quasicomplex with smoothing curvature admits a special parametrix satisfying (3.1.2) with smoothing operators $S^i \in \Psi^{-\infty}_{\mathrm{cl}}(X, F^i)$. Similar to the operator case, we will call such a parametrix a regulariser of the quasicomplex.

3.1.3 Extension to Sobolev spaces

We may extend the quasicomplex $\mathcal{E}(X, F^\cdot)$ to a quasicomplex with Sobolev spaces, i.e.

$$H^{s.}(X, F^\cdot)\colon\ 0 \to H^{s_0}(X, F^0) \stackrel{A^0}{\to} H^{s_1}(X, F^1) \stackrel{A^1}{\to} \ldots \stackrel{A^{N-1}}{\to} H^{s_N}(X, F^N) \to 0$$

where s_i are given by (3.1.3). From Theorem 1.3.7 it follows that $H^{s.}(X, F^\cdot)$ is a quasicomplex in the context of Hilbert spaces. Particularly, it is a $\mathfrak{S}_p$-quasicomplex if $p > n$.

Theorem 3.1.9 *Assume that* $\mathcal{E}(X, F^{\cdot})$ *is an elliptic quasicomplex. Then the extended quasicomplex* $H^{s_{\cdot}}(X, F^{\cdot})$ *is Fredholm.*

Proof. Since $\mathcal{E}(X, F^{\cdot})$ is elliptic, this quasicomplex possesses a (formal) parametrix. The extension of this parametrix to Sobolev spaces is a genuine parametrix of $H^{s_{\cdot}}(X, F^{\cdot})$ in the sense of Hilbert spaces, for the smoothing operators are compact. □

Note that there is no canonical choice for the scalar products in $H^{s_i}(X, F^i)$ while the norms are eqivalent. Let $\Lambda_{F^i} \in \Psi^2_{\mathrm{cl}}(X; F^i)$ be those invertible operators which define the norm in $H^s(X, F^i)$. We denote by $(A^i)^{\mathrm{adj}}$ the adjoint operator for $A^i \colon H^{s_i}(X, F^i) \to H^{s_{i+1}}(X, F^{i+1})$ in the sense of Hilbert spaces.

Theorem 3.1.10 *If* $A \in \Psi^m_{\mathrm{cl}}(X; E, F)$ *then* $A^{\mathrm{adj}} = \Lambda_E^{-s} A^* \Lambda_F^{s-m}$. *In particular,* $A^{\mathrm{adj}} \in \Psi^{-m}_{\mathrm{cl}}(X; F, E)$.

Proof. Since $\Lambda_E^{-s} \in \Psi^{-2s}_{\mathrm{cl}}(X; E)$ and $\Lambda_F^{s-m} \in \Psi^{2(s-m)}_{\mathrm{cl}}(X; F)$, it suffices to establish the equality $A^{\mathrm{adj}} = \Lambda_E^{-s} A^* \Lambda_F^{s-m}$ only. By the definition of Hilbert adjoint we obtain

$$\begin{aligned}
(Au, g)_{H^{s-m}(X,F)} &= (\Lambda_F^{(s-m)/2} Au, \Lambda_F^{(s-m)/2} g)_{L^2(X,F)} \\
&= (Au, \Lambda_F^{s-m} g)_{L^2(X,F)} \\
&= (u, A^* \Lambda_F^{s-m} g)_{L^2(X,E)} \\
&= (\Lambda_E^{s/2} u, \Lambda_E^{s/2} \Lambda_E^{-s} A^* \Lambda_F^{s-m} g)_{L^2(X,E)} \\
&= (u, \Lambda_E^{-s} A^* \Lambda_F^{s-m} g)_{H^s(X,E)}
\end{aligned}$$

for all $u \in \mathcal{E}(X, E)$ and $g \in \mathcal{E}(X, F)$. So the assertion follows by a familiar density argument. □

Theorem 3.1.10 shows readily that the Laplacians $\Delta^i := A^{i-1} A^{i-1\,\mathrm{adj}} + A^{i\,\mathrm{adj}} A^i$ of this quasicomplex belong to $\Psi^0_{\mathrm{cl}}(X; F^i)$.

Theorem 3.1.11 *A quasicomplex* $\mathcal{E}(X, F^{\cdot})$ *is elliptic if and only if the Laplacians* Δ^i *are elliptic.*

Proof. Set

$$R_i := \Lambda_{F^i}^{s_i/2}$$

and $\widetilde{A}^i := R_{i+1} A^i R_i^{-1}$. By definition, the diagram

$$\begin{array}{ccccccccccc}
0 & \to & H^{s_0}(X, F^0) & \stackrel{A^0}{\to} & H^{s_1}(X, F^1) & \stackrel{A^1}{\to} & \ldots & \stackrel{A^{N-1}}{\to} & H^{s_N}(X, F^N) & \to & 0 \\
 & & \downarrow R_0 & & \downarrow R_1 & & & & \downarrow R_N & & \\
0 & \to & L^2(X, F^0) & \stackrel{\widetilde{A}^0}{\to} & L^2(X, F^1) & \stackrel{\widetilde{A}^1}{\to} & \ldots & \stackrel{\widetilde{A}^{N-1}}{\to} & L^2(X, F^N) & \to & 0
\end{array}$$

is commutative. With

$$\begin{aligned}
A^{i\,\mathrm{adj}} &= \Lambda_{F^i}^{-s_i} A^{i*} \Lambda_{F^{i+1}}^{s_{i+1}} \\
&= R_i^{-2} A^{i*} R_{i+1}^2
\end{aligned}$$

and $A^{i*} = R_i \widetilde{A}^{i*} R_{i+1}^{-1}$ we find

$$\begin{aligned}\Delta^i &= (R_i^{-1} \widetilde{A}^{i-1} R_{i-1}) (R_{i-1}^{-1} \widetilde{A}^{i-1*} R_i) + (R_i^{-1} \widetilde{A}^{i*} R_{i+1}) (R_{i+1}^{-1} \widetilde{A}^i R_i) \\ &= R_i^{-1} \widetilde{\Delta}^i R_i,\end{aligned}$$

where $\widetilde{\Delta}^i = \widetilde{A}^{i-1}\widetilde{A}^{i-1*} + \widetilde{A}^{i*}\widetilde{A}^i$ are the Laplacians of the reduced quasicomplex. Since

$$\sigma^0(\Delta^i) = (\sigma^{s_i}(R_i))^{-1} \sigma^0(\widetilde{\Delta}^i) \sigma^{s_i}(R_i),$$

the pseudodifferential operator Δ^i is elliptic if and only is so is $\widetilde{\Delta}^i$. On applying Theorem 3.1.6 we get the desired assertion.

□

3.1.4 Reduction to complexes

Theorem 3.1.12 *For each elliptic quasicomplex $\{\mathcal{E}(X, F^\cdot), A\}$ there is an elliptic complex $\{\mathcal{E}(X, F^\cdot), D\}$, such that $D^i \in \Psi^{m_i}_{\mathrm{cl}}(X; F^i, F^{i+1})$ and $D^i = A^i$ modulo $\Psi^{m_i-1}(X; F^i, F^{i+1})$.*

Proof. Since the differential A of the quasicomplex is given by pseudodifferential operators on X, we can turn to the quasicomplex $\{H^{s\cdot}(X, F^\cdot), A\}$ of Hilbert spaces, where $s \in \mathbb{R}$ is any fixed number. Arguing as in the proof of Theorem 8.1 in [KTT07] we modify the quasicomplex $\{H^{s\cdot}(X, F^\cdot), A\}$ into a complex $\{H^{s\cdot}(X, F^\cdot), D\}$ whose differential D differs from A by a operator of order -1.

We start from the end of the quasicomplex

$$\ldots \stackrel{A^{N-2}}{\to} H^{s_{N-1}}(X, F^{N-1}) \stackrel{D^{N-1}}{\to} H^{s_N}(X, F^N) \to 0,$$

setting $D^{N-1} = A^{N-1}$. Since $\sigma^{m_{N-1}}(A^{N-1})$ is surjective, it follows that the Laplacian $\Delta^N = D^{N-1}(D^{N-1})^{\mathrm{adj}}$ is a selfadjoint elliptic pseudodifferential operator of order 0 in $H^{s_N}(X, F^N)$. By the Hodge theory for single operators, there is an operator $G^N \in \Psi^0(X; F^N)$ satisfying

$$\mathrm{Id}_{F^N} = H^N + \Delta^N G^N = H^N + D^{N-1} P^N,$$

where H^N stands for the orthogonal projection onto the finite-dimensional space $\ker \Delta^N = \ker (D^{N-1})^{\mathrm{adj}}$ and $P^N = (D^{N-1})^{\mathrm{adj}} G^N$. We set

$$\Pi^{N-1} = \mathrm{Id}_{F^{N-1}} - P^N D^{N-1}$$

thus obtaining a pseudodifferential operator in $\Psi^0(X; F^{N-1})$. We claim that Π^{N-1} is a projection onto $\ker D^{N-1}$. Indeed, $\Pi^{N-1} = \mathrm{Id}_{F^{N-1}}$ is valid on $\ker D^{N-1}$ and

$$\begin{aligned}\Pi^{N-1}\Pi^{N-1} &= (\mathrm{Id}_{F^{N-1}} - P^N D^{N-1})(\mathrm{Id}_{F^{N-1}} - P^N D^{N-1}) \\ &= \mathrm{Id}_{F^{N-1}} - 2 P^N D^{N-1} + P^N (D^{N-1} P^N) D^{N-1} \\ &= \mathrm{Id}_{F^{N-1}} - 2 P^N D^{N-1} + P^N (\mathrm{Id}_{F^N} - H^N) D^{N-1} \\ &= \Pi^{N-1},\end{aligned}$$

for $H^N D^{N-1} = ((D^{N-1})^{\mathrm{adj}} H^N)^{\mathrm{adj}} = 0$.

Next we set $D^{N-2} = \Pi^{N-1} A^{N-2}$. Then $D^{N-2} \in \Psi^{m_{N-2}}(X; F^{N-2}, F^{N-1})$ and $D^{N-1} D^{N-2} = 0$, for Π^{N-1} is a projection onto $\ker D^{N-1}$. Furthermore, we get

$$\begin{aligned} D^{N-2} &= A^{N-2} - P^N A^{N-1} A^{N-2} \\ &= A^{N-2} \end{aligned}$$

modulo $\Psi^{m_{N-2}-1}(X; F^{N-2}, F^{N-1})$, for the composition $A^{N-1} A^{N-2}$ is an operator of order $m_{N-1} + m_{N-2} - 1$.

Consider now a slightly modified quasicomplex

$$\ldots \stackrel{A^{N-3}}{\to} H^{s_{N-2}}(X, F^{N-2}) \stackrel{D^{N-2}}{\to} H^{s_{N-1}}(X, F^{N-1}) \stackrel{D^{N-1}}{\to} H^{s_N}(X, F^N) \to 0.$$

Since the symbol complex of the initial quasicomplex is exact and the operators D^i and A^i have the same principal symbol for $i = N-2, N-1$, the Laplacian

$$\Delta^{N-1} = D^{N-2}(D^{N-2})^{\mathrm{adj}} + (D^{N-1})^{\mathrm{adj}} D^{N-1}$$

is a selfadjoint elliptic operator of order 0 on $H^{s_{N-1}}(X, F^{N-1})$. Using the Hodge theory for complexes, we deduce that there is an operator $G^{N-1} \in \Psi^0(X; F^{N-1})$, such that

$$\begin{aligned} \mathrm{Id}_{F^{N-1}} &= H^{N-1} + D^{N-2}(D^{N-2})^{\mathrm{adj}} G^{N-1} + (D^{N-1})^{\mathrm{adj}} G^N D^{N-1} \\ &= H^{N-1} + D^{N-2} P^{N-1} + P^N D^{N-1} \end{aligned}$$

where H^{N-1} is the orthogonal projection onto the null-space of Δ^{N-1} which is $\ker(D^{N-2})^{\mathrm{adj}} \cap \ker D^{N-1}$, and $P^{N-1} = (D^{N-2})^{\mathrm{adj}} G^{N-1}$. Then, we claim that

$$\Pi^{N-2} = \mathrm{Id}_{F^{N-1}} - P^{N-1} D^{N-2}$$

is the orthogonal projection onto $\ker D^{N-2}$. Indeed, Π^{N-2} is the identity operator on $\ker D^{N-2}$. Moreover,

$$\begin{aligned} (\Pi^{N-2})^2 &= \Pi^{N-2} - P^{N-1} D^{N-2} + P^{N-1}(D^{N-2} P^{N-1}) D^{N-2} \\ &= \Pi^{N-2} - P^{N-1} D^{N-2} + P^{N-1}(\mathrm{Id}_{F^{N-1}} - H^{N-1} - P^N D^{N-1}) D^{N-2} \\ &= \Pi^{N-2} - P^{N-1} H^{N-1} D^{N-2} \\ &= \Pi^{N-2}, \end{aligned}$$

since $H^{N-1} D^{N-2} = ((D^{N-2})^{\mathrm{adj}} H^{N-1})^{\mathrm{adj}}$ vanishes. Introducing $D^{N-3} = \Pi^{N-2} A^{N-3}$ we thus obtain $D^{N-2} D^{N-3} = 0$ and

$$\begin{aligned} D^{N-3} &= A^{N-3} - P^{N-1} D^{N-2} A^{N-3} \\ &= A^{N-3} \end{aligned}$$

modulo $\Psi^{m_{N-3}-1}(X; F^{N-3}, F^{N-2})$, for

$$D^{N-2} A^{N-3} = A^{N-2} A^{N-3} + (D^{N-2} - A^{N-2}) A^{N-3}$$

is a operator of order $m_{N-2}+m_{N-3}-1$.

Continuing in this fashion, in a finite number of steps we obtain a complex of operators $D^i \in \Psi^{m_i}(X;F^i,F^{i+1})$, such that $D^i - A^i \in \Psi^{m_i-1}(X;F^i,F^{i+1})$ for all $i=0,1,\ldots,N-1$.

□

An inspection of the proofs above shows that for each elliptic quasicomplex $\{\mathcal{E}(X,F^\cdot),A\}$ with smoothing curvature there is an elliptic complex $\{\mathcal{E}(X,F^\cdot),D\}$, such that $D^i \in \Psi^{m_i}_{\mathrm{cl}}(X;F^i,F^{i+1})$ and $D^i = A^i$ modulo $\Psi^{-\infty}(X;F^i,F^{i+1})$.

It is worth pointing out that the desired complex $\{\mathcal{E}(X,F^\cdot),D\}$ is constructed within the framework of pseudodifferential calculus on X. I.e. D^i are pseudodifferential operators even in the case if the initial sequences of symbols stem from differential operators.

3.2 Characteristic numbers

3.2.1 The index formula

Definition 3.2.1 *Let $\mathcal{E}(X,F^\cdot)$ be an elliptic quasicomplex. We define the Euler characteristic of this quasicomplex by*

$$\chi(\mathcal{E}(X,F^\cdot),A) := \chi(H^{s.}(X,F^\cdot),D),$$

where $(H^{s.}(X,F^\cdot),D)$ is any complex with the properties listed in Theorem 3.1.12.

As defined above, the Euler characteristic is independent neither of $s\in\mathbb{R}$ nor of the choice of the differential D with $(D)^2=0$. To prove this it suffices to show that the cohomology of $\{H^{s.}(X,F^\cdot),D\}$ is independent of the choice of s and D. Since the Laplacians $\Delta^i = D^{i-1}(D^{i-1})^{\mathrm{adj}} + (D^i)^{\mathrm{adj}}D^i$ are elliptic pseudodifferential operators, Theorem 1.3.13 implies that the null-space of Δ^i belongs to $\mathcal{E}(X,F^i)$, i.e. is independent of s. Finally, the abstract Hodge theory yields

$$H^i(H^s(X,F^\cdot),D) \overset{\mathrm{top}}{\cong} \ker \Delta^i.$$

The proof is completed by observing that the Euler characteristic of an elliptic complex $\{\mathcal{E}(X,F^\cdot),D\}$ is uniquely determined by the principal symbols of D^i.

Using the Atiyah-Singer index formula [Pal65] we are able to evaluate the index of an elliptic quasicomplex.

Theorem 3.2.2 *Let $\mathcal{E}(X,F^\cdot)$ be an elliptic quasicomplex. Then*

$$\chi(\mathcal{E}(X,F^\cdot)) = \int_{T^*X} \mathrm{ch}\,(\mathrm{d}(A\oplus A^*)_e)\,\mathcal{T}\,(T_{\mathbb{C}}X)$$

holds.

Proof. Since the analytical index does only depend on the principal symbols, the assertion follows immediately from the Atiyah-Singer index theorem for elliptic operators. Namely, if $\{\mathcal{E}(X,F^\cdot),A\}$ is an elliptic complex over X, we split the space $V=\oplus\mathcal{E}(X,F^i)$ into the sum

$$V = V^{\text{even}} \oplus V^{\text{odd}}$$

with $V^{\text{even}}=\oplus\mathcal{E}(X,F^{2i})$ and $V^{\text{odd}}=\oplus\mathcal{E}(X,F^{2i+1})$ and consider the block operator

$$(A\oplus A^*)_e:\ \mathcal{E}(X,\oplus F^{2i})\to\mathcal{E}(X,\oplus F^{2i+1})$$

given by

$$\begin{pmatrix} A^0 & A^{1*} & 0 & 0 & \dots \\ 0 & A^2 & A^{3*} & 0 & \dots \\ 0 & 0 & A^4 & A^{5*} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Then $(A\oplus A^*)_e$ is elliptic and for the Euler characteristic of the complex we find

$$\chi(\mathcal{E}(X,F^\cdot)) = \int_{T^*X} \mathrm{ch}\,(\mathrm{d}(A\oplus A^*)_e)\,\mathcal{T}\,(T_{\mathbb{C}}X).$$

□

It should be noted that for elliptic quasicomplexes of pseudodifferential operators of different order the ellipticity of $(A\oplus A^*)_e$ is understood in the sense of Douglis-Nirenberg.

Example 3.2.3 We consider the connection quasicomplex of Example 3.1.5

$$\Omega^\cdot(X,F):\ 0\to\mathcal{E}(X,F)\stackrel{\partial^0}{\to}\Omega^1(X,F)\stackrel{\partial^1}{\to}\dots\stackrel{\partial^{n-1}}{\to}\Omega^n(X,F)\to 0.$$

Since the quasicomplex is elliptic, there is an elliptic complex of pseudodifferential operators

$$(\Omega^\cdot(X,F),D):\ 0\to\mathcal{E}(X,F)\stackrel{D^0}{\to}\Omega^1(X,F)\stackrel{D^1}{\to}\dots\stackrel{D^{n-1}}{\to}\Omega^n(X,F)\to 0$$

satisfying $\partial^i - D^i \in \Psi^0_{cl}(X;F^i,F^{i+1})$ for all $i=0,1,\dots,n$. The Euler characteristic of $\Omega^\cdot(X,F)$ is defined by

$$\chi(\Omega^\cdot(X,F)) := \chi(\Omega^\cdot(X,F),D).$$

Moreover, $\chi(\Omega^\cdot(X,F))=\mathrm{ind}\,(\partial+\partial^*)_e$, where

$$(\partial+\partial^*)_e:\ \mathcal{E}(X,\oplus(F\otimes\Lambda^{2i}T^*X))\to\mathcal{E}(X,\oplus(F\otimes\Lambda^{2i+1}T^*X))$$

is the block operator

$$\begin{pmatrix} \partial^0 & \partial^{1*} & 0 & 0 & \dots \\ 0 & \partial^2 & \partial^{3*} & 0 & \dots \\ 0 & 0 & \partial^4 & \partial^{5*} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Let A be defined by the commutative diagram

$$\begin{array}{ccc}
\mathcal{E}(X,\oplus(F\otimes\Lambda^{2i}T^*X)) & \stackrel{(\partial+\partial^*)_e}{\longrightarrow} & \mathcal{E}(X,\oplus(F\otimes\Lambda^{2i+1}T^*X)) \\
\uparrow\cong & & \uparrow\cong \\
\mathcal{E}(X,F\otimes(\oplus\Lambda^{2i}T^*X)) & \stackrel{A}{\longrightarrow} & \mathcal{E}(X,F\otimes(\oplus\Lambda^{2i+1}T^*X))
\end{array}$$

where the isomorphisms are canonical. We obtain

$$\mathrm{ind}(\partial+\partial^*)_e=\mathrm{ind}(A),$$

and the Atiyah-Singer index formula yields

$$\chi(\Omega^\cdot(X,F))=\int_{T^*X}\mathrm{ch}\,(\mathrm{d}(A))\,\mathcal{T}(TX)=\int_{T^*X}\mathrm{ch}\,(F)\,\mathrm{ch}\,(\mathrm{d}(d\oplus d^*)_e)\,\mathcal{T}(TX).$$

We thus find

$$\begin{array}{rcl}
\chi(\Omega^\cdot(X,F)) & = & k\displaystyle\int_{T^*X}\mathrm{ch}\,(\mathrm{d}(d\oplus d^*)_e)\,\mathcal{T}(TX) \\
 & = & k\,\chi(\Omega^\cdot(X))
\end{array}$$

which is $k\,\chi(X)$ by the Gauß-Bonnet theorem, see for instance [Pal65, Ch. V, § 2] and elsewhere. We have used the fact that the only nontrivial cohomology class of $\mathrm{ch}\,(\mathrm{d}(d\oplus d^*)_e)\,\mathcal{T}(TX)$ is that at degree $2n$, and the summand of degree 0 in $\mathrm{ch}\,F$ is $k=\mathrm{rank}\,F$.

3.2.2 Geometric quasiendomorphisms

Suppose that $\{\mathcal{E}(X,F^\cdot),A\}$ is a quasicomplex with smoothing curvature. By a quasiendomorphism of $\{\mathcal{E}(X,F^\cdot),A\}$ is meant a family $E=\{E^i\}$ of bounded linear selfmappings E^i of $\mathcal{E}(X,F^i)$, such that $E^{i+1}A^i=A^iE^i$ is satisfied modulo smoothing operators $\Psi^{-\infty}(X;F^i,F^{i+1})$ for all $i=0,1,\ldots,N-1$. As mentioned, there is a perturbation D of the differential A by smoothing operators, such that $\{\mathcal{E}(X,F^\cdot),D\}$ is a complex. Moreover, a slight change in the proof of Lemma 2.2.6 shows that there is an endomorphism $\widetilde{E}=\{\widetilde{E^i}\}$ of $\{\mathcal{E}(X,F^\cdot),D\}$ with the property that $\widetilde{E^i}-E^i\in\Psi^{-\infty}(X;F^i)$ for all $i=0,1,\ldots,N$. For a quasiendomorphism E of an elliptic quasicomplex $\{\mathcal{E}(X,F^\cdot),A\}$ we introduce the Lefschetz number $\mathscr{L}(E,A)$ by Definition 2.3.4, where the traces are evaluated in Sobolev spaces, as clarified above.

Clearly, this definition is independent of the particular choice of s. For the cohomology of the elliptic complex $\{H^{s\cdot}(X,F^\cdot),D\}$ does not depend on s and it just amounts to that of $\{\mathcal{E}(X,F^\cdot),D\}$. Hence, if every map $E^i\in\mathcal{L}(\mathcal{E}(X,F^i))$ extends to a bounded linear selfmap of $H^{s_i}(X,F^i)$ for s large enough, then the same is true for $\widetilde{E^i}$ and so the Lefschetz number $\mathscr{L}(\widetilde{E},D)$ can be also evaluated for the complex $\{H^{s\cdot}(X,F^\cdot),D\}$ of Hilbert spaces.

Let f be a smooth selfmap of the manifold X and f^*F^i the induced bundles. The maps $f^*:\mathcal{E}(X,F^i)\to\mathcal{E}(X,f^*F^i)$ given by $(f^*u)(x)=u(f(x))$ are linear. Moreover, we consider smooth bundle homomorphisms $h^i : f^*F^i \to F^i$. By abuse of notation, we use the same letter h^i to designate the induced maps $h^i : \mathcal{E}(X, f^*F^i) \to \mathcal{E}(X, F^i)$ of sections. Then the compositions $E^i := h^i \circ f^*$ are obviously selfmaps of $\mathcal{E}(X, F^i)$. More precisely, we define

$$E^i u(x) = h^i(x) u(f(x))$$

for any $u \in \mathcal{E}(X, F^i)$.

Definition 3.2.4 *The family $E = \{h^i \circ f^*\}$ is called a geometric quasiendomorphism of $\{\mathcal{E}(X, F^\cdot), A\}$ if the equality $A^i E^i = E^{i+1} A^i$ holds modulo smoothing operators for all $i = 0, 1, \ldots, N-1$.*

Obviously, the geometric quasiendomorphisms E to be considered fail to be of trace class, hence the Euler identity of Lemma 2.3.3 no longer applies. However, using the facts that $\widetilde{E}^i$ and

$$\widetilde{E}^i - D^{i-1}\widetilde{E}^{i-1}P^i - \widetilde{E}^i P^{i+1} D^i \in \mathcal{L}(\mathcal{E}(X, F^i))$$

are homotopic and $\mathrm{Id} - D^{i-1}P^i - P^{i+1}D^i \in \Psi^{-\infty}(X; F^i)$ holds, we can exploit Theorem 19.4.1 of [Hoe85] and obtain

$$\begin{aligned} \mathscr{L}(\widetilde{E}, D) &= \mathscr{L}(\widetilde{E} - D\widetilde{E}P - \widetilde{E}PD, D) \\ &= \sum_{i=0}^{N} (-1)^i \operatorname{tr}(\widetilde{E}^i - D^{i-1}\widetilde{E}^{i-1}P^i - \widetilde{E}^i P^{i+1} D^i) \\ &= \sum_{i=0}^{N} (-1)^i \operatorname{tr}(\widetilde{E}^i - A^{i-1}E^{i-1}P^i - E^i P^{i+1} A^i), \end{aligned}$$

where the traces are evaluated by restricting the (not necessarily smooth) kernels. Hence, we can compute the Lefschetz number by the explicit formula of Corollary 2.3.5

$$\mathscr{L}(E, A) := \sum_{i=0}^{N} (-1)^i \operatorname{tr}(E^i - A^{i-1}E^{i-1}P^i - E^i P^{i+1} A^i)$$

where P is a parametrix of the quasicomplex $\{\mathcal{E}(X, F^\cdot), A\}$, cf. Lemma 7.2 of [ST00].

3.2.3 The fixed point formula

Denote by $\mathrm{Fix}\,(f) := \{p \in X : f(p) = p\}$ the set of fixed points of a smooth selfmapping f of X. An isolated fixed point is said to be simple, if $\det(1 - f'(p)) \neq 0$ is fulfilled. The following theorem presents a natural generalisation of the Lefschetz fixed point formula for elliptic complexes on a compact closed manifold due to [AB67].

Theorem 3.2.5 *Assume $E = \{h^i \circ f^*\}_{i=0,1,\ldots,N}$ is a geometric quasiendomorphism of an elliptic quasicomplex $\{\mathcal{E}(X, F^\cdot), A\}$ with smoothing curvature and f has only simple fixed points. Then*

$$\mathscr{L}(E, A) = \sum_{p \in \mathrm{Fix}(f)} \iota(p),$$

where

$$\iota(p) = \frac{\sum (-1)^i \operatorname{tr} h^i(p)}{|\det(\mathrm{Id} - f'(p))|}.$$

Proof. The proof follows the scheme suggested by Fedosov in [Fed91]. We pick a partition of unity $\{\phi_\nu\}$ on X with the property that each ϕ_ν either vanishes or is equal to 1 in a neighbourhood of any fixed point of f.

Let further ψ_0 be a function of compact support on T^*X, such that $\psi_0(\xi) \equiv 1$ near $\xi = 0$, and let $\psi_\infty = 1 - \psi_0$. In local coordinates on X, we introduce operators $\Psi_{0,\nu}$ and $\Psi_{\infty,\nu}$ by

$$\begin{aligned} \Psi_{0,\nu} u &= \mathscr{F}^{-1}_{\xi \mapsto x} \psi_0(\hbar\xi) \mathscr{F}_{x \mapsto \xi} (\phi_\nu u), \\ \Psi_{\infty,\nu} u &= \mathscr{F}^{-1}_{\xi \mapsto x} \psi_\infty(\hbar\xi) \mathscr{F}_{x \mapsto \xi} (\phi_\nu u), \end{aligned}$$

$\mathscr{F}$ being the Fourier transform and $\hbar$ an arbitrary positive constant. These operators decompose the identity operator. Moreover, the operators $\Psi_{0,\nu}$ are smoothing and hence of trace class on each Sobolev space. We can assert, by the Lidskii theorem, that

$$\operatorname{tr} A^i E^i P^{i+1} \Psi_{0,\nu} = \operatorname{tr} E^i P^{i+1} \Psi_{0,\nu} A^i$$

whence

$$\begin{aligned} &\sum_{i=0}^{N} (-1)^i \operatorname{tr} (E^i - A^{i-1} E^{i-1} P^i - E^i P^{i+1} A^i) \\ &= \sum_\nu \sum_{i=0}^{N} (-1)^i \operatorname{tr} E^i \Psi_{0,\nu} \\ &+ \sum_\nu \sum_{i=0}^{N} (-1)^i \operatorname{tr} (E^i - A^{i-1} E^{i-1} P^i - E^i P^{i+1} A^i) \Psi_{\infty,\nu} \\ &- \sum_\nu \sum_{i=0}^{N-1} (-1)^i \operatorname{tr} E^i P^{i+1} \left[A^i, \Psi_{0,\nu}\right], \end{aligned} \tag{3.2.1}$$

$\left[A^i, \Psi_{0,\nu}\right]$ being the commutator of A^i and $\Psi_{0,\nu}$.

In a local chart close to a fixed point of f, the operator $E^i \Psi_{0,\nu}$ is given by the iterated integral

$$E^i \Psi_{0,\nu} u\,(x) = \frac{1}{(2\pi\hbar)^n} \iint e^{\frac{\imath}{\hbar}\langle \xi, f(x) - y\rangle} h^i(x) \psi_0(\xi)\, \phi_\nu(y) u(y)\, dy d\xi,$$

and consequently

$$\operatorname{tr} E^i \Psi_{0,\nu} = \frac{1}{(2\pi\hbar)^n} \iint e^{\frac{\imath}{\hbar}\langle \xi, f(x) - x\rangle} \operatorname{tr} h^i(x)\, \psi_0(\xi) \phi_\nu(x)\, d\xi dx.$$

For $\hbar \to 0$, the limit of the integral on the right hand side of this equality can be evaluated by the method of stationary phase. For this purpose we consider the phase function

$$\varphi(x, \xi) = \langle \xi, f(x) - x\rangle.$$

The stationary points are just the points where $\xi = 0$ and $f(x) - x = 0$. Moreover, the gradient of φ has the form

$$\varphi'(x,\xi) = \big(\langle \xi, f'(x) - \mathrm{Id}\rangle, f(x) - x\big)\,,$$

where Id stands actually for the unity $(n \times n)$-matrix. An easy computation shows that the Hesse matrix of φ at a stationary point $(p, 0)$ just amounts to

$$\varphi''(p,0) = \begin{pmatrix} 0 & f'(p) - \mathrm{Id} \\ f'(p) - \mathrm{Id} & 0 \end{pmatrix}.$$

We thus obtain

$$\sqrt{|\det \varphi''(p,0)|} = |\det(f'(p) - \mathrm{Id})|.$$

Since $\lambda \in \mathbb{R}$ is an eigenvalue of $\varphi''(p,0)$ if and only if $-\lambda$ is an eigenvalue, the signature of this matrix is zero. Now, we apply Lemma 1.1.7 to the integral above. In the principal part independent of $\hbar$ the contribution of a fixed point p is equal to

$$\frac{\operatorname{tr} h^i(p)}{|\det(\mathrm{Id} - f'(p))|}.$$

On the other hand, the remaining terms on the right side of (3.2.1) are oscillatory integrals whose phase function has no critical points. Indeed,

$$\begin{aligned} \big[A^i, \varPsi_{0,\nu}\big] &= \big[A^i, \varPsi_{0,\nu} - \mathrm{Id}\big] \\ &= -\big[A^i, \varPsi_{\infty,\nu}\big] \end{aligned}$$

close to each fixed point and the function ψ_∞ vanishes in a neighbourhood of $\xi = 0$. Hence it follows that the remaining summands in (3.2.1) are rapidly decreasing as $\hbar \to 0$. Since the left hand side of (3.2.1) is actually independent of $\hbar$, we arrive at the desired formula.

□

Example 3.2.6 Assume X is oriented and $\mathcal{E}(X, F^\cdot) := \Omega^\cdot(X)$ is the de Rham complex of X. Then $\iota(p)$ proves to be the topological degree of the (local) mapping $x \mapsto x - f(x)$ at p, i.e.

$$\iota(p) = \frac{\det(\mathrm{Id} - f'(p))}{|\det(\mathrm{Id} - f'(p))|} = \pm 1.$$

3.3 Outlook

We have studied elliptic quasicomplexes on compact closed manifolds. The paper [KTT07] treats elliptic quasicomplexes on compact manifolds with boundary. The underlying theory of operators in the Boutet de Monvel calculus is much more technical, cf. [dM71]. The theory of quasicomplexes is of great interest also on other singular spaces like the stratified manifolds. As but one familiar example we mention the Dolbeaux complex on a complex manifold. Depending on context there might exist diverse index and Lefschetz fixed point formulas, see for instance [Gil79] and [BS91].

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