Hyperequational Theory for partial algebras

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This work is dedicated to my parents

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S. Busaman

Contents

Introduction

In Mathematics and its applications there exist operations that when inputting some values no outputs exist. Those operations are called *partial operations* and operations where the output exists for every input are called *total operations*. Let $O^n(A)$ be the set of all n-ary total operations on the set A and let $P^n(A)$ be the set of all n-ary partial operations on A. Let $O(A) := \bigcup\limits_{n=1}^{\infty} O^n(A)$ and let $P(A) := \bigcup\limits_{n=1}^{\infty} P^n(A)$. We have $O(A) \subseteq P(A)$. A *partial algebra* $\mathcal{A} := (A; (f_i^A)_{i\in I})$ is a pair consisting of a set A and a sequence of partial operations $(f_i^A)_{i\in I}$ which assigns to every element of the index set I an n_i-ary partial operation f_i^A defined on A. To every $i \in I$ we assign a natural number n_i which we call *arity* of f_i^A. Let $(n_i)_{i\in I}$ be the sequence of arities where f_i^A is n_i-ary. The sequence $\tau = (n_i)_{i\in I}$ is called *type* of the partial algebra $\mathcal{A}$. Let $Alg(\tau)$ be the set of all total algebras of type τ and let $PAlg(\tau)$ be the set of all partial algebras of type τ. We have $Alg(\tau) \subseteq PAlg(\tau)$.

The concepts of a strong identity and a strong regular identity were introduced by B. Staruch and B. Staruch in [48]. An equation $s \approx t$ of type τ is called a *strong identity* in the partial algebra $\mathcal{A}$ (in symbols $\mathcal{A} \underset{s}{\models} s \approx t$) if the right hand side is defined whenever the left hand side is defined and conversely and both are equal. An equation $s \approx t$ of type τ is called a *strong regular identity* in the partial algebra $\mathcal{A}$ (in symbols $\mathcal{A} \underset{sr}{\models} s \approx t$) if the equation $s \approx t$ is a strong identity in $\mathcal{A}$ and the variables occurring in the term s are equal to the variables occurring in the term t. Let $K \subseteq PAlg(\tau)$ be a class of partial algebras of type τ and let $\Sigma \subseteq W_\tau(X)^2$ be a set of equations. Consider the connection between $PAlg(\tau)$ and $W_\tau(X)^2$ given by the following two operators:

$$Id^{sr} : \mathcal{P}(PAlg(\tau)) \to \mathcal{P}(W_\tau(X)^2) \qquad \text{and}$$

$$Mod^{sr} : \mathcal{P}(W_\tau(X)^2) \to \mathcal{P}(PAlg(\tau)) \qquad \text{with}$$

$$\begin{aligned} Id^{sr}K &:= \{s \approx t \in W_\tau(X)^2 \mid \forall \mathcal{A} \in K\ (\mathcal{A} \underset{sr}{\models} s \approx t)\} \qquad \text{and} \\ Mod^{sr}\Sigma &:= \{\mathcal{A} \in PAlg(\tau) \mid \forall\ s \approx t \in \Sigma\ (\mathcal{A} \underset{sr}{\models} s \approx t)\}. \end{aligned}$$

Let $V \subseteq PAlg(\tau)$ be a class of partial algebras. The class V is called a *strong regular variety* of partial algebras if $V = Mod^{sr}Id^{sr}V$.

B. Staruch and B. Staruch proved in [48] that a class K is a strong regular variety of partial algebras of type τ iff K is closed under closed homomorphic images, initial segments, closed subalgebras, direct products and the pin operator which describes the one-point extension of partial to total algebras.

The concept of a strong regular equational theory was introduced by B. Staruch and B. Staruch in [48]. A set of regular equations $\Sigma \subseteq W_\tau(X)^2$ is called a *strong regular equational theory* if there is a class of partial algebras $K \subseteq PAlg(\tau)$ such that $\Sigma = Id^{sr}K$.

A strong identity $s \approx t$ in the partial algebra $\mathcal{A}$ of type τ is called a *strong hyperidentity* of $\mathcal{A}$ if, for every substitution of terms of appropriate arity for the operation symbols in $s \approx t$, the resulting strong identity holds in $\mathcal{A}$. This leads to the definition of a map $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ such that $\sigma(f_i)$ is an n_i-ary term of type τ. Any such mapping σ is called a *hypersubstitution* of type τ. This concept was first introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert in [30]. Any hypersubstitution σ uniquely determines a mapping, denoted by $\widehat{\sigma}$, on the set of all terms of type τ. Using such induced maps the binary operation $\circ_h$ can be defined by $(\sigma \circ_h \sigma')(f_i) := \widehat{\sigma}[\sigma'(f_i)]$ for all $i \in I$. Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ. Indeed, $(Hyp(\tau); \circ_h, \sigma_{id})$ forms a monoid where σ_{id} maps f_i to $f_i(x_1, \ldots, x_{n_i})$. Regular hypersubstitutions were defined in [34] as hypersubstitutions with the property that for every fundamental operation f_i of arity n_i, all the variables $x_1, \ldots, x_{n_i}$ occur in the term $\sigma(f_i)$ for all $i \in I$. Let $Hyp_R(\tau)$ be the set of all regular hypersubstitutions of type τ. Then $\mathcal{H}yp_R(\tau) := (Hyp_R(\tau); \circ_h, \sigma_{id})$ forms a monoid.

As D. Welke proved in [49] a necessary condition for $\widehat{\sigma}[s] \approx \widehat{\sigma}[t]$ to be a strong regular identity in a partial algebra $\mathcal{A}$ whenever $s \approx t$ is a strong regular identity in $\mathcal{A}$ is that σ is regular. So, to define strong regular hyperidentities we will consider only regular hypersubstitutions.

Let $\mathcal{M}$ be a submonoid of $\mathcal{H}yp_R(\tau)$ and let $\mathcal{A}$ be a partial algebra of type τ. Then a strong regular identity $s \approx t$ of $\mathcal{A}$ is called a *strong regular M-hyperidentity* of $\mathcal{A}$ (in symbols $\mathcal{A} \underset{srMh}{\models} s \approx t$) if for every regular hypersubstitution $\sigma_R \in M$ the equation $\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t]$ is also a strong regular identity of $\mathcal{A}$. In the case, if $M = Hyp_R(\tau)$, strong regular M-hyperidentities are called *strong regular hyperidentities*.

Let $K \subseteq PAlg(\tau)$ be a class of partial algebras of type τ and let $\Sigma \subseteq W_\tau(X)^2$ be a set of equations. Consider the connection between $PAlg(\tau)$ and $W_\tau(X)^2$ given by the following two operators:

$$H_M Id^{sr} : \mathcal{P}(PAlg(\tau)) \to \mathcal{P}(W_\tau(X)^2) \qquad \text{and}$$

$$H_M Mod^{sr} : \mathcal{P}(W_\tau(X)^2) \to \mathcal{P}(PAlg(\tau)) \qquad \text{with}$$

$$\begin{array}{lcll} H_M Id^{sr} K & := & \{s \approx t \in W_\tau(X)^2 \mid \forall \mathcal{A} \in K \ (\mathcal{A} \underset{srMh}{\models} s \approx t)\} & \text{and} \\ H_M Mod^{sr} \Sigma & := & \{\mathcal{A} \in PAlg(\tau) \mid \forall\ s \approx t \in \Sigma \ (\mathcal{A} \underset{srMh}{\models} s \approx t)\}. & \end{array}$$

The concept of a strong regular M-hyperequational theory was introduced by D. Welke in [49]. A set of regular equations $\Sigma \subseteq W_\tau(X)^2$ is called a *strong regular M-hyperequational theory* if there is a class of partial algebras $K \subseteq PAlg(\tau)$ such that $\Sigma = H_M Id^{sr} K$.

For $M = Hyp_R(\tau)$ we speak of strong regular hyperequational theories, $HId^{sr}K$.

One of the most interesting concepts in this area is the concept of a solid strong regular variety. Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a partial algebra of type τ and $\sigma_R \in Hyp_R(\tau)$. We let

$$\sigma_R(\mathcal{A}) := (A; (\sigma_R(f_i)^{\mathcal{A}})_{i\in I}),$$

which is called *derived algebra* of type τ.

Let $\mathcal{M}$ be a submonoid of $\mathcal{H}yp_R(\tau)$. We introduce two operators χ_M^E and χ_M^A. Let $\Sigma \subseteq W_\tau(X) \times W_\tau(X)$ be a set of regular equations, $s \approx t \in \Sigma$, we let

$$\chi_M^E[s \approx t] := \{\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t] \mid \sigma_R \in M\} \qquad \text{and}$$

$$\chi_M^E[\Sigma] := \bigcup_{s \approx t \in \Sigma} \chi_M^E[s \approx t].$$

For any partial algebra $\mathcal{A}$ of type τ and $K \subseteq PAlg(\tau)$, we let

$$\chi_M^A[\mathcal{A}] := \{\sigma_R(\mathcal{A}) \mid \sigma_R \in M\} \qquad \text{and}$$

$$\chi_M^A[K] := \bigcup_{\mathcal{A} \in K} \chi_M^A[\mathcal{A}].$$

A strong regular variety V of type τ is called *M-solid* if $V = \chi_M^A[V]$ and if $M = Hyp_R(\tau)$, then V is called *solid*.

One of the aims of this thesis is to study M-solid strong regular varieties of partial algebras for different submonoids and subsemigroups M of $Hyp_R(\tau)$.
Our work goes in two directions. At first we want to transfer definitions, concepts and results of the theory of hyperidentities and solid varieties from the total to the partial case.
1) The concept of an n-full term of type τ was considered in [18]. Using n-full terms we define strong regular n-full identities in partial algebras of type τ. We use the concept of strong regular n-full satisfaction to define the relation R_{rnf} which is a subrelation of the relation R_s defined by strong satisfaction. As a subrelation of R_s the relation R_{rnf} is Galois-closed (see e.g. [28]). All n-ary n-full terms of type τ form with respect to superposition of terms an algebraic structure $n - clone^{nF}(\tau)$ which satisfies the axioms of a Menger algebra of rank n and the set of all strong regular n-full identities of a strong regular variety forms a congruence relation on $n - clone^{nF}(\tau)$. The concept of an n-full hypersubstitution of type τ was considered in [18]. We give the definition of a regular n-full hypersubstitution of type τ and define the concept of a strong regular n-full hyperidentity for partial algebras. We use the concept of a regular n-full hypersubstitution of type τ to define the operators χ^A_{RNF} and χ^E_{RNF} and prove that $(\chi^A_{RNF}, \chi^E_{RNF})$ forms a conjugate pair of additive operators. These operators are in general not closure operators. Therefore the fixed points under χ^A_{RNF} are characterized only by three instead of four equivalent conditions in the case of closure operators ([27]).
2) We consider strongly full varieties as a special case of strong regular n-full varieties. Using strongly full terms we define the concept of a strongly full identity in a partial algebra of type $\tau_n = (n_i)_{i \in I}$ with $n_i = n$ for all $i \in I$. All strongly full terms of type τ_n form with respect to superposition of terms an algebraic structure $clone^{SF}(\tau_n)$ which satisfies the axioms of a Menger algebra of rank n and the set of all strongly full n-ary identities $Id^{SF}_n V$ of a strongly full variety V forms a congruence relation on $clone^{SF}(\tau_n)$. We give the definition of a strongly full hyperidentity. This concept is a special case of a strong regular n-full hyperidentity. Then we consider the quotient algebra $clone^{SF} V := clone^{SF}(\tau_n)/Id^{SF}_n V$ and study the relationship between strongly full hyperidentities in V and identities in $clone^{SF} V$. A strongly

full variety V of partial algebras of type τ_n is called $n-SF-solid$ if every identity $s \approx t \in Id_n^{SF}V$ is satisfied as a strongly full hyperidentity in V. In [19] the concept of an $\mathcal{O}$-solid variety and of i-closedness for total algebras were defined. Now we define an $\mathcal{O}^{SF}$-solid strongly full variety and of I^{SF}-closedness for partial algebras.
3) The concepts of unsolid and fluid varieties were considered in [46], [20], [21], and [22]. We will be interested in unsolid and fluid strong varieties of partial algebras. In [40] an equivalence relation $\sim_V$ on $Hyp(\tau)$ with respect to a variety V was defined by $\sigma_1 \sim_V \sigma_2$ iff $\sigma_1(f_i) \approx \sigma_2(f_i) \in IdV$ for all operation symbols $f_i, i \in I$, and in [22] an equivalence relation $\sim_{V-iso}$ on $Hyp(\tau)$ with respect to a variety V was defined by $\sigma_1 \sim_{V-iso} \sigma_2$ iff $\forall \mathcal{A} \in V(\sigma_1(\mathcal{A}) \cong \sigma_2(\mathcal{A}))$. We will be also interested in equivalence relations $\sim_V$ and $\sim_{V-iso}$ on $Hyp_R^C(\tau)$ (the set of all regular C-hypersubstitutions of type τ).
4) The concepts of M-solid quasivarieties and M-hyperquasi-equational theories were considered in [14]. We will be interested in M-solid strong quasivarieties of partial algebras and strong M-hyperquasi-equational theories for partial algebras.
The second direction of our work is to follow ideas which are typical for the partial case.
1) The collection of all clones of partial operations defined on a fixed set A, $|A| > 1$, forms a complete atomic and dually atomic lattice. The maximal elements of this lattice were determined in [43] and [44]. The minimal clones are determined in [16], [37], [38] and [45] modulo to the knowledge of minimal total clones. But, the determination of all minimal total clones is yet open. Here we determine all minimal partial clones with a special property which is called a strong solidifyability.
2) A hypersubstitution of type τ is a total mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$. Then we extend the concept of a hypersubstitution of type τ to a partial hypersubstitution of type τ and we define the concept of a regular partial hypersubstitution of type τ. On the basis of regular hypersubstitutions we develop the whole theory of conjugate pairs of additive closure operators.

This work consists of nine chapters.

Chapter 1 presents some basic concepts on partial algebras and some basic concepts from Universal Algebra which are needed.

In Chapter 2, we give an example that the set $W_\tau(X_n)^{\mathcal{A}}$ (the set of all n-ary term operations on the partial algebra $\mathcal{A}$) is different from the set of all partial operations generated by $\{f_i^A | i \in I\}$ using superposition and we introduce another kind of terms, so-called C-terms. In Section 2.2, the concept of a strong identity (by usual terms) which was introduced in [48] is used to define model classes and the corresponding Galois connection. In [5], it was proved that a class K is a strong variety iff $K = \mathbf{H}_c\mathbf{S}_c\mathbf{P}_{filt}K \cup \{\underline{\emptyset}\}$ where $\underline{\emptyset}$ is the empty algebra. The concept of a strong identity (by C-terms) which was introduced in [2] is used to define model classes and the corresponding Galois connection. In [2], it was proved that a class K is a strong variety iff $K = \mathbf{H}_c\mathbf{S}_c\mathbf{P}_{filt}K$. In Section 2.3, the concept of a strong regular identity (by usual terms) which was introduced in [48] is used to define model classes and the corresponding Galois connection. We show that in the case of C-terms strong identities can be replaced with strong regular identities.

In Chapter 3, the concept of a regular hypersubstitution which was introduced in [49] is used to define strong regular M-hyperidentities and M-solid strong regular varieties where M is a submonoid of the monoid of all regular hypersubstitutions. In Section 3.2, the concept of a regular C-hypersubstitution which was introduced in [49] is used to define strong M-hyperidentities and M-solid strong varieties where M is a submonoid of regular C-hypersubstitutions.

In Chapter 4, we prove that the relation R_{rnf} is a Galois-closed subrelation of R_s and we show that the set of all strong regular n-full identities of a strong regular variety is a congruence relation on the Menger algebra $n - clone^{nF}(\tau)$ of rank n. Further, we define the operators χ^A_{RNF} and χ^E_{RNF} which are only monotone and additive and we show that the set of all fixed points of these operators are characterized only by three instead of four equivalent conditions for the case of closure operators.

In Chapter 5, we prove that the algebra $(P^n(A); S^{n,A})$ is a Menger algebra of rank n where $S^{n,A}$ is the superposition operation of partial operations and we show that $Id_n^{sF}V$ is a congruence relation on the Menger algebra $clone^{sF}(\tau_n)$ of rank n. Using this result, we consider the quotient algebra $clone^{SF}V := clone^{SF}(\tau_n)/Id_n^{SF}V$ and we prove that $s \approx t$ is a strongly full hyperidentity in V iff $s \approx t$ is an identity in

$clone^{SF}V$ where V is a strongly full variety of partial algebras. We define the concept of an $n - SF - solid$ strongly full variety and we prove that V is $n - SF - solid$ iff $clone^{SF}V$ is free with respect to itself, freely generated by the independent set $\{[f_i(x_1, \ldots, x_n)]_{Id_n^{SF}V} \mid i \in I\}$. At the end of this chapter, we define the concepts of $I^{SF} - closedness$ and $\mathcal{O}^{SF} - solid$ strongly full variety and we prove that V is $I^{SF} - closed$ iff it is $\mathcal{O}^{SF} - solid$ where $V = Mod^{SF}Id_n^{SF}V$.

In Chapter 6, we show that $\sim_V$ and $\sim_{V-iso}$ are right congruences on $Hyp_R^C(\tau)$. We use the concept of a V-proper hypersubstitution and of an inner hypersubstitution to define the concepts of unsolid and fluid strong varieties and we prove that if V is a fluid strong variety and $[\sigma_{id}]_{\sim_V} = [\sigma_{id}]_{\sim_{V-iso}}$, then V is unsolid. Furthermore, we generalize unsolid and fluid strong varieties to n-fluid and n-unsolid strong varieties and we show that if V is n-fluid and $\sim_V |_{P(V)} = \sim_{V-iso} |_{P(V)}$ then V is k-unsolid for $k \geq n$ where $P(V)$ is the set of all V-proper hypersubstitutions of type τ. Finally, we give an example of an n-unsolid strong variety of partial algebras.

In Chapter 7, we prove that an M-solid strong quasivariety satisfies four equivalent conditions and we prove that a strong M-hyperquasi-equational theory satisfies four equivalent conditions.

In Chapter 8, we prove that strong varieties of different types are equivalent if and only if their clones of all term operations of different types are isomorphic. We study minimal partial clones (see [3]) and we define the concept of a strongly solidifyable partial clone. After this, we characterize minimal partial clones which are strongly solidifyable.

Finally in Chapter 9 we prove that the set of all regular partial hypersubstitutions forms a submonoid of the set of all partial hypersubstitutions. Next, we consider only regular partial hypersubstitutions of type $\tau = (n)$, $n \in \mathbb{N}^+$ and we prove that the extension of a partial hypersubstitution is injective if and only if the partial hypersubstitution is a regular partial hypersubstitution of type $\tau = (n)$ when $n \geq 2$. At the end of this chapter, we define the concept of a $PHyp_R(\tau)$-solid strong regular variety of partial algebras and we prove that a $PHyp_R(\tau)$-solid strong regular variety satisfies four equivalent conditions.

Chapter 1

Basic Concepts

In this chapter, certain basic notions and results are presented. In Section 1.1, we recall the definition of partial algebras, homomorphisms, subalgebras and different kinds of products. For more details we refer to [2], [4], [5]. In Section 1.2 and Section 1.3, we recall the definition of Galois connections, conjugate pairs of additive closure operators and give a brief discussion about their properties (see [1], [24], [27], [28]).

1.1 Partial Algebras and Superposition of Partial Operations

Let A be a non-empty set and $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of natural numbers. We define $A^0 = \{\emptyset\}$, and $A^n = \{(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in A\}$ if $n \in \mathbb{N}^+$ ($\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$). Let $P^n(A) := \{f^A : A^n \multimap\!\!\!\to A\}$ be the set of all *n-ary partial operations* defined on the set A. If $n = 0$, then we suppose that $A \neq \emptyset$. Let $P(A) := \bigcup\limits_{n=1}^{\infty} P^n(A)$ be the set of all partial operations on A.

If $f^A \in P^n(A)$ is a partial operation, then

$$dom f^A := \{(a_1, \ldots, a_n) \mid \exists a \in A \ (f^A(a_1, \ldots, a_n) = a)\} \subseteq A^n,$$

$$Im f^A := \{a \in A \mid \exists (a_1, \ldots, a_n) \in dom f^A \ (a = f^A(a_1, \ldots, a_n))\} \subseteq A$$

and

$$graph f^A := \{(a_1, \ldots, a_n, a) \mid (a_1, \ldots, a_n) \in dom f^A \ (f^A(a_1, \ldots, a_n) = a)\} \subseteq A^{n+1}.$$

Let $O(A) \subset P(A)$ be the set of all total operations defined on A, i.e. $O(A) := \bigcup\limits_{n=1}^{\infty} O^n(A)$ with $O^n(A) := \{f^A \in P^n(A) \mid domf^A = A^n\}$.

If $f : A \multimap B$ and $g : B \multimap C$, then the *composition* $g \circ f$ of f and g is the partial function:

$$g \circ f : A \multimap C$$

$$dom\ g \circ f := \{a \in A \mid a \in domf \text{ and } f(a) \in domg\}.$$

Special n-ary (total) operations are the *projections* to the i-th argument, where $1 \leq i \leq n$:

$$e_i^{n,A} : A^n \to A$$

$$e_i^{n,A}(a_1, \ldots, a_n) := a_i.$$

Let $D \subseteq A^n$ be an n-ary relation on A. Then for every positive integer n and each $1 \leq i \leq n$ we denote by $e_{i,D}^{n,A}$ the *n-ary i-th partial projection* defined by

$$e_{i,D}^{n,A}(a_1, \ldots, a_n) = a_i$$

for all $(a_1, \ldots, a_n) \in D$.

Let $J_A := \{e_{i,D}^{n,A} \mid 1 \leq i \leq n \text{ and } D = A^n\}$ be the set of all total projections defined on A and let J_A^n be the set of all total n-ary projections defined on A.

For $n, m \in \mathbb{N}^+$ we define the *superposition* operation

$$S_n^{m,A} : P^m(A) \times (P^n(A))^m \to P^n(A)$$

$$S_n^{m,A}(f^A, g_1^A, \ldots, g_m^A)(a_1, \ldots, a_n) := f^A(g_1^A(a_1, \ldots, a_n), \ldots, g_m^A(a_1, \ldots, a_n)).$$

Here $(a_1, \ldots, a_n) \in domS_n^{m,A}(f^A, g_1^A, \ldots, g_m^A)$ iff $(a_1, \ldots, a_n) \in \bigcap\limits_{j=1}^{m} domg_j^A$ and for $b_j = g_j^A(a_1, \ldots, a_n)$, we have $(b_1, \ldots, b_m) \in domf^A$, i.e. $domS_n^{m,A}(f^A, g_1^A, \ldots, g_m^A) := \{(a_1, \ldots, a_n) \in A^n \mid (a_1, \ldots, a_n) \in \bigcap\limits_{j=1}^{m} domg_j^A \text{ and } (b_1, \ldots, b_m) \in domf^A\}$.

A *partial clone* C on A is a superposition closed subset of $P(A)$ containing J_A. A *proper partial clone* is a partial clone C containing at least an n-ary operation f^A with $domf^A \neq A^n$. If $C \subseteq O(A)$ then C is called a *total clone*.

Partial clones can be regarded as subalgebras of the heterogeneous algebra

$$((P^n(A))_{n\in\mathbb{N}^+}; (S_n^{m,A})_{m,n\in\mathbb{N}^+}, (J_A^n)_{n\in\mathbb{N}^+})$$

where $\mathbb{N}^+$ is the set of all positive integers.
This remark shows that the set of all partial clones on A, ordered by inclusion, forms an algebraic lattice $\mathcal{L}_{P(A)}$ in which arbitrary infimum is the set-theoretical intersection. For $F \subseteq P(A)$ by $\langle F\rangle$ we denote by the least partial clone containing F.

Any mapping $\varphi = (\varphi^{(n)})_{n\in\mathbb{N}^+} : C \to C'$ from a clone $C \subseteq P(A)$ into $C' \subseteq P(B)$ is a *clone homomorphism* if
(i) arity (f)= arity $\varphi(f)$ for $f \in C$,
(ii) $\varphi(e_i^{n,A}) = e_i^{n,B}$ $(1 \leq i \leq n \in \mathbb{N}^+)$,
(iii) $\varphi(S_m^{n,A}(f^A, g_1^A, \ldots, g_n^A)) = S_m^{n,B}(\varphi(f^A), \varphi(g_1^A), \ldots, \varphi(g_n^A))$ for $f^A \in C^{(n)}$ and $g_1^A, \ldots, g_n^A \in C^{(m)}$.
(Here $\varphi(f^A)$ means $\varphi^{(n)}(f^A)$ where f^A is n-ary).

Let $(f_i)_{i\in I}$ be a sequence of operation symbols, where I is an index set. To each f_i we assign an integer $n_i > 0$ as its arity . A *type* τ is the sequence of arities of f_i for all $i \in I$. We always write $\tau := (n_i)_{i\in I}$.

Let $\tau = (n_i)_{i\in I}$ be a type with the sequence of operation symbols $(f_i)_{i\in I}$. A *partial algebra of type* τ is an ordered pair $\mathcal{A} := (A; (f_i^A)_{i\in I})$, where A is a non-empty set and $(f_i^A)_{i\in I}$ is a sequence of partial operations on A indexed by a non-empty index set I such that to each n_i-ary operation symbol f_i there is a corresponding n_i-ary operation f_i^A on A. (If $n_i > 0$ for all $i \in I$, we can also consider the empty algebra, i.e. $A = \emptyset$).

The set A is called the *universe* of $\mathcal{A}$ and the sequence $(f_i^A)_{i\in I}$ is called the *sequence of fundamental operations* of $\mathcal{A}$. We sometimes write $\mathcal{A} := (A; (f_i^A)_{i\in I})$. We denote by $PAlg(\tau)$ the class of all partial algebras of type τ.

Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ and $\mathcal{B} = (B; (f_i^B)_{i\in I})$ be partial algebras and let $B \subseteq A$. A partial algebra $\mathcal{B}$ is called a *weak subalgebra* of the partial algebra $\mathcal{A}$, if

$$\text{graph} f_i^B \subseteq \text{graph} f_i^A.$$

A partial algebra $\mathcal{B}$ is called a *relative subalgebra* of the partial algebra $\mathcal{A}$, if

$$\mathrm{graph} f_i^B = \mathrm{graph} f_i^A \cap B^{n_i+1}.$$

A partial algebra $\mathcal{B}$ is called a *closed subalgebra* of the partial algebra $\mathcal{A}$, if

$$\mathrm{graph} f_i^B = \mathrm{graph} f_i^A \cap (B^{n_i} \times A).$$

A relative subalgebra $\mathcal{B}$ of a partial algebra $\mathcal{A}$ is an *initial segment* in $\mathcal{A}$ iff for all $i \in I$ and for all $(a_1, \ldots, a_{n_i}) \in A^{n_i}$ if $f_i^A(a_1, \ldots, a_{n_i}) \in B$ then $a_j \in B$ for all j with $1 \leq j \leq n_i$.

Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ and $\mathcal{B} = (B; (f_i^B)_{i\in I})$ be partial algebras. A function $h : A \to B$ is called a *homomorphism* of $\mathcal{A}$ into $\mathcal{B}$ iff for all f_i, $i \in I$ and for all $(a_1, \ldots, a_{n_i}) \in A^{n_i}$ and $a \in A$ the following holds:

$$\text{if } f_i^A(a_1, \ldots, a_{n_i}) = a, \text{ then } f_i^B(h(a_1), \ldots, h(a_{n_i})) = h(a).$$

A homomorphism $h : A \to B$ is called a *full homomorphism* of $\mathcal{A}$ into $\mathcal{B}$ iff for all f_i, $i \in I$ and for all $(a_1, \ldots, a_{n_i}) \in A^{n_i}$ and $a \in A$ the following holds:
if $(h(a_1), \ldots, h(a_{n_i})) \in dom f_i^B$ and $f_i^B(h(a_1), \ldots, h(a_{n_i})) = h(a)$, then there exists $(a'_1, \ldots, a'_{n_i}) \in A^{n_i}$, such that $(h(a'_1), \ldots, h(a'_{n_i})) = (h(a_1), \ldots, h(a_{n_i}))$ and $(a'_1, \ldots, a'_{n_i}) \in dom f_i^A$.
A homomorphism $h : A \to B$ is called a *closed homomorphism* of $\mathcal{A}$ into $\mathcal{B}$ iff for all f_i, $i \in I$ and for all $(a_1, \ldots, a_{n_i}) \in A^{n_i}$ the following holds:

$$\text{if } (h(a_1), \ldots, h(a_{n_i})) \in dom f_i^B, \text{ then } (a_1, \ldots, a_{n_i}) \in dom f_i^A.$$

Let $(\mathcal{A}_j)_{j\in J}$ be a family of partial algebras of type τ, then the *direct product* $\prod\limits_{j\in J} \mathcal{A}_j$ is a partial algebra with $\prod\limits_{j\in J} A_j$ as its universe and the operations $f_i^{\prod_{j\in J} A_j}$ defined for every $i \in I$ as follows

$$f_i^{\prod_{j\in J} A_j}((a_{1j})_{j\in J}, \ldots, (a_{n_ij})_{j\in J}) := (f_i^{A_j}(a_{1j}, \ldots, a_{n_ij}))_{j\in J}.$$

This means, the left hand side is defined iff for all $j \in J$, we have $(a_{1j}, \ldots, a_{n_ij}) \in dom f_i^{A_j}$.

Let $(\mathcal{A}_j)_{j\in J}$ be a family of partial algebras of type τ where $J = \{1, \ldots, n\}$. $\mathcal{A} = \prod\limits_{j\in J}\Big|_F \mathcal{A}_j$ is called *filter product* of $(\mathcal{A}_j)_{j\in J}$ if

$$([\underline{a}_1]_{\theta_F}, \ldots, [\underline{a}_n]_{\theta_F}) \in dom f_i^A \text{ iff } (a_{1j}, \ldots, a_{nj}) \in dom f_i^{A_j} \text{ where } \{j \in J\} \in F$$

and $\prod\limits_{j\in J}\Big|_F A_j := \{[\underline{a}]_{\theta_F} \mid \underline{a} \in \prod\limits_{j\in J} A_j\}$.

1.2 Closure Operators and Galois Connections

Lattices form important examples of universal algebras (see [25]).

An ordered pair $(L, \leq)$ is called a *partially ordered set* if L is a non-empty set and $\leq$ is a partial order on L, i.e. a relation $\leq$ satisfying the reflexive law, the anti-symmetric law and the transitive law. A partially ordered set $(L, \leq)$ is called a *lattice* if for every $a, b \in L$ both $sup\{a, b\}$ (*supremum* of a and b) and $inf\{a, b\}$ (*infimum* of a and b) exist in L. Let M be a non-empty subset of L. Then $\mathcal{M} := (M, \leq)$ is called *sublattice* of $\mathcal{L} := (L, \leq)$ if $a, b \in M$ implies $sup\{a, b\} \in M$ and $inf\{a, b\} \in M$, a partially ordered set $(L, \leq)$ is called a *complete lattice* if for every nonempty subset A of L both $supA$ and $infA$ exist in L.

Note that the lattice $(L, \leq)$ can be considered as an algebra of type $\tau = (2, 2)$. Indeed, we define two binary operations, denoted by $\vee$ and $\wedge$, the so-called *join* and *meet*, respectively, by: $a \vee b := sup\{a, b\}$ and $a \wedge b := inf\{a, b\}$ for all $a, b \in L$. This algebra satisfies a list of axioms containing the associative laws, the commutative laws, the idempotent laws for both operations and the absorption laws, i.e. for all $a, b \in L$, we get $a \vee (a \wedge b) = a = a \wedge (a \vee b)$. Conversely every algebra of type $\tau = (2, 2)$ satisfying these axioms is a lattice in the first sense.

Let A be a non-empty set and $\mathcal{P}(A)$ be the power set of A. A mapping $\gamma : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ is called a *closure operator* on A if for any $X, Y \in \mathcal{P}(A)$, the following conditions hold:

(i) $X \subseteq \gamma(X)$ (extensivity);

(ii) $X \subseteq Y \Rightarrow \gamma(X) \subseteq \gamma(Y)$ (monotonicity);

(iii) $\gamma(\gamma(X)) = \gamma(X)$ (idempotency).

A subset X of A is called a *closed set* with respect to the closure operator γ if $\gamma(X) = X$. Let H_γ denote the set of all closed sets with respect to the closure operator γ, the so-called *closure system* with respect to γ. In fact, H_γ forms a complete lattice.

Proposition 1.2.1 ([6]) *Let $\gamma : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ be a closure operator on A. Then H_γ is a complete lattice with respect to set inclusion. For any set $\{H_i \in H_\gamma \mid i \in I\}$,* *the meet and join operators are defined by*

$$\begin{aligned}\bigwedge\{H_i \in H_\gamma \mid i \in I\} &:= \bigcap_{i\in I} H_i,\\ \bigvee\{H_i \in H_\gamma \mid i \in I\} &:= \bigcap\{H \in H_\gamma \mid H \supseteq \bigcup_{i\in I} H_i\} = \gamma(\bigcup_{i\in I} H_i).\end{aligned}$$

The concept of a closure operator is closely connected to the next concept of a Galois connection.

A *Galois connection* between sets A and B is a pair (μ, ι) of mappings $\mu : \mathcal{P}(A) \to \mathcal{P}(B)$ and $\iota : \mathcal{P}(B) \to \mathcal{P}(A)$ such that for any $X, X' \in \mathcal{P}(A)$ and $Y, Y' \in \mathcal{P}(B)$ the following conditions are fulfilled:

(i) $X \subseteq X' \Rightarrow \mu(X) \supseteq \mu(X')$ and $Y \subseteq Y' \Rightarrow \iota(Y) \supseteq \iota(Y')$;

(ii) $X \subseteq \iota\mu(X)$ and $Y \subseteq \mu\iota(Y)$.

Proposition 1.2.2 ([27]) *Let (μ, ι) with $\mu : \mathcal{P}(A) \to \mathcal{P}(B)$ and $\iota : \mathcal{P}(B) \to \mathcal{P}(A)$ be a Galois connection between sets A and B. Then*

(i) *$\mu\iota\mu = \mu$ and $\iota\mu\iota = \iota$;*

(ii) *$\iota\mu$ and $\mu\iota$ are closure operators on A and B, respectively;*

(iii) *the closed sets under $\iota\mu$ are exactly the sets of the form $\iota(Y)$ for $Y \subseteq B$ and the closed sets under $\mu\iota$ are exactly the sets of the form $\mu(X)$ for $X \subseteq A$;*

(iv) *$\mu(\bigcup_{i\in I} X_i) = \bigcap_{i\in I} \mu(X_i)$, where $X_i \subseteq A$ for all $i \in I$;*

(v) *$\iota(\bigcup_{i\in I} Y_i) = \bigcap_{i\in I} \iota(Y_i)$, where $Y_i \subseteq B$ for all $i \in I$.*

Note that any relation $R \subseteq A \times B$ between sets A and B induces a Galois connection (μ_R, ι_R) between A and B as follows:
We can define the mappings $\mu_R : \mathcal{P}(A) \to \mathcal{P}(B)$ and $\iota_R : \mathcal{P}(B) \to \mathcal{P}(A)$ by

$$\begin{aligned}\mu_R(X) &:= \{y \in B \mid \forall x \in X((x, y) \in R)\},\\ \iota_R(Y) &:= \{x \in A \mid \forall y \in Y((x, y) \in R)\}.\end{aligned}$$

Conversely, for any Galois connection (μ, ι) between sets A and B, we define a relation $R_{\mu,\iota}$ by

$$R_{\mu,\iota} = \bigcup\{X \times \mu(X) \mid X \subseteq A\}.$$

In fact, there is a one-to-one correspondence between Galois connections and relations between sets A and B.

Now we want to describe a way starting from a relation $R \subseteq A \times B$ and the induced Galois connection (μ, ι) to obtain a certain subrelation of R which induces a new Galois connection.

Let R and R' be relations between sets A and B. Let (μ, ι) and (μ', ι') be the Galois connections between A and B induced by R and R', respectively. The relation R' is called a *Galois-closed subrelation* of R if,

(i) $R' \subseteq R$ and

(ii) $\forall T \subseteq A, \forall S \subseteq B\ (\mu'(T) = S\ \wedge\ \iota'(S) = T) \Rightarrow (\mu(T) = S\ \wedge\ \iota(S) = T)$.

The following are equivalent characterizations of Galois-closed subrelations.

Proposition 1.2.3 ([28]) *Let $R' \subseteq R$ be relations between sets A and B. Then the following are equivalent:*

(i) *R' is a Galois-closed subrelation of R;*

(ii) *For any $T \subseteq A$, if $\iota'\mu'(T) = T$ then $\mu(T) = \mu'(T)$, and for any $S \subseteq B$, if $\mu'\iota'(S) = S$ then $\iota(S) = \iota'(S)$;*

(iii) *For all $T \subseteq A$ and for all $S \subseteq B$ the equations $\iota'\mu'(T) = \iota\mu'(T)$ and $\mu'\iota'(S) = \mu\iota'(S)$ are satisfied.*

From this definition, we can prove the following characterization of complete sublattices of a complete lattice.

Theorem 1.2.4 ([1]) *Let $R \subseteq A \times B$ be a relation between sets A and B, with the induced Galois connection (μ, ι). Let $\mathcal{H}_{\iota\mu}$ be the corresponding lattice of closed subsets of A .*

(i) *If $R' \subseteq A \times B$ is a Galois-closed subrelation of R, then the class $\mathcal{U}_{R'} := \mathcal{H}_{\iota'\mu'}$ is a complete sublattice of $\mathcal{H}_{\iota\mu}$, where (μ', ι') is the Galois connection induced by the relation R'.*

(ii) *If* $\mathcal{U}$ *is a complete sublattice of* $\mathcal{H}_{\iota\mu}$*, then the relation*

$$\mathcal{R}_{\mathcal{U}} := \cup\{T \times \mu(T) \mid T \in \mathcal{U}\}$$

is a Galois-closed subrelation of R*.*

(iii) *For any Galois-closed subrelation* R' *of* R *and any complete sublattice* $\mathcal{U}$ *of* $\mathcal{H}_{\iota\mu}$ *we have* $\mathcal{U}_{R_{\mathcal{U}}} = \mathcal{U}$ and $R_{\mathcal{U}_{R'}} = R'$.

Let A be a non-empty set and $\mathcal{P}(A)$ be the power set of A. A mapping $\kappa : \mathcal{P}(A) \to \mathcal{P}(A)$ is called a *kernel operator* on A if for any $M, N \in \mathcal{P}(A)$, the following conditions hold:

(i) $\kappa(M) \subseteq M$ (intensivity);

(ii) $M \subseteq N \Rightarrow \kappa(M) \subseteq \kappa(N)$ (monotonicity);

(iii) $\kappa(\kappa(M)) = \kappa(M)$ (idempotency).

A *kernel system* on A is defined as a subset $\mathcal{K} \subseteq \mathcal{P}(A)$ with the property that for all $\mathcal{B} \subseteq \mathcal{K}$, the set $\bigcap \mathcal{B}$ is in $\mathcal{K}$.

1.3 Conjugate Pairs of Additive Closure Operators

In this part we will define a particular pair $\gamma := (\gamma_1, \gamma_2)$ of closure operators with respect to a given relation $R \subseteq A \times B$ and after this we define a subrelation $R_\gamma \subseteq R$ of R via γ and study the interconnections between Galois connections induced by R_γ and by R.

A closure operator $\gamma : \mathcal{P}(A) \to \mathcal{P}(A)$ on a set A is said to be *additive* if for all subsets T of A

$$\gamma(T) = \bigcup_{a \in T} \gamma(a)$$

(here we write $\gamma(a)$ instead of $\gamma(\{a\})$).

Let $\gamma_1 : \mathcal{P}(A) \to \mathcal{P}(A)$, $\gamma_2 : \mathcal{P}(B) \to \mathcal{P}(B)$ be closure operators on a set A and on a set B, respectively. Let $R \subseteq A \times B$ be a given relation between A and B. Then

(γ_1, γ_2) is called a *conjugate pair* with respect to R if for any $t \in A$ and for any $s \in B$

$$\gamma_1(t) \times \{s\} \subseteq R \Leftrightarrow \{t\} \times \gamma_2(s) \subseteq R.$$

If (γ_1, γ_2) is a conjugate pair of additive closure operators with respect to a relation $R \subseteq A \times B$ then for any $T \subseteq A$ and for any $S \subseteq B$ we have

$$\gamma_1(T) \times S \subseteq R \Leftrightarrow T \times \gamma_2(S) \subseteq R.$$

Let $\gamma := (\gamma_1, \gamma_2)$ be a conjugate pair of additive closure operators, with respect to a relation $R \subseteq A \times B$. Let R_γ be the following relation between A and B:

$$R_\gamma := \{(t, s) \in A \times B \mid \gamma_1(t) \times \{s\} \subseteq R\}.$$

Theorem 1.3.1 ([24]) *Let $\gamma := (\gamma_1, \gamma_2)$ be a conjugate pair of additive closure operators with respect to a given relation $R \subseteq A \times B$. Let (μ, ι), $(\mu_\gamma, \iota_\gamma)$ be the Galois connections between A and B induced by R and by R_γ, respectively.*
Then for any $T \subseteq A$ and for any $S \subseteq B$ we have

$$\begin{array}{llcllllcl}
(1) & \mu_\gamma(T) & = & \mu\gamma_1(T), & (1') & \iota_\gamma(S) & = & \iota\gamma_2(S), \\
(2) & \mu_\gamma(T) & \subseteq & \mu(T), & (2') & \iota_\gamma(S) & \subseteq & \iota(S), \\
(3) & \gamma_2\mu_\gamma(T) & = & \mu_\gamma(T), & (3') & \gamma_1\iota_\gamma(S) & = & \iota_\gamma(S), \\
(4) & \gamma_1\iota\mu_\gamma(T) & = & \iota\mu_\gamma(T), & (4') & \gamma_2\mu\iota_\gamma(S) & = & \mu\iota_\gamma(S), \\
(5) & \mu_\gamma\iota_\gamma(S) & = & \mu\iota\gamma_2(S), & (5') & \iota_\gamma\mu_\gamma(T) & = & \iota\mu\gamma_1(T), \\
(6) & \mu_\gamma\iota_\gamma(S) & = & \mu\iota_\gamma(S), & (6') & \iota_\gamma\mu_\gamma(T) & = & \iota\mu_\gamma(T).
\end{array}$$

Theorem 1.3.2 ([24]) *Let $\gamma := (\gamma_1, \gamma_2)$ be a conjugate pair of additive closure operators with respect to a given relation $R \subseteq A \times B$, and let (μ, ι), $(\mu_\gamma, \iota_\gamma)$ be the Galois connections between A and B induced by R and by R_γ, respectively. Then*

I. For any $T \subseteq A$ with $\iota\mu(T) = T$ and for any $S \subseteq B$ with $\mu\iota(S) = S$ the following conditions (1)-(4) and (1')-(4'), respectively, are equivalent:

$$\begin{array}{llcllllcl}
(1) & T & = & \iota_\gamma\mu_\gamma(T), & (1') & S & = & \mu_\gamma\iota_\gamma(S), \\
(2) & \gamma_1(T) & = & T, & (2') & \gamma_2(S) & = & S, \\
(3) & \mu(T) & = & \mu_\gamma(T), & (3') & \iota(S) & = & \iota_\gamma(S), \\
(4) & \gamma_2\mu(T) & = & \mu(T), & (4') & \gamma_1\iota(S) & = & \iota(S).
\end{array}$$

II. For any $T \subseteq A$ and for any $S \subseteq B$ the following conditions are true:

$$\begin{array}{lllllll}
(1) & \gamma_1(T) & \subseteq & \iota\mu(T) & \Leftrightarrow & \iota\mu(T) & = & \iota_\gamma\mu_\gamma(T), \\
(2) & \gamma_1(T) & \subseteq & \iota\mu(T) & \Leftrightarrow & \gamma_1\iota\mu(T) & \subseteq & \iota\mu(T), \\
(3) & \gamma_2(S) & \subseteq & \mu\iota(S) & \Leftrightarrow & \mu\iota(S) & = & \mu_\gamma\iota_\gamma(S), \\
(4) & \gamma_2(S) & \subseteq & \mu\iota(S) & \Leftrightarrow & \gamma_2\mu\iota(S) & \subseteq & \mu\iota(S).
\end{array}$$

Theorem 1.3.3 ([24]) *Let $\gamma := (\gamma_1, \gamma_2)$ be a conjugate pair of additive closure operators with respect to a given relation $R \subseteq A \times B$. Let (μ, ι), $(\mu_\gamma, \iota_\gamma)$ be the Galois connections between A and B induced by R and by R_γ, respectively. Then $\mathcal{H}_{\mu_\gamma\iota_\gamma}$, the class of all closed sets under the closure operator $\mu_\gamma\iota_\gamma$, is a complete sublattice of $\mathcal{H}_{\mu\iota}$ and $\mathcal{H}_{\iota_\gamma\mu_\gamma}$ is a complete sublattice of $\mathcal{H}_{\iota\mu}$.*

Chapter 2

Strong Regular Varieties

In this chapter we study strong regular varieties of partial algebras. In Section 2.1 we define terms, the superposition of terms and term operations of partial algebras (see [49], [2], [47]). Since the set of all term operations of a partial algebra induced by usual terms is different from the set of all partial operations produced by the set of all fundamental operations of the partial algebra, we introduce another kind of terms, so-called C-terms which were first introduced by W. Craig ([15]) (see also [2], [49]). Then we define different kinds of strong identities in partial algebras, study the corresponding Galois connections and model classes.

2.1 Terms, Superposition of Terms and Term Operations

First we recall the usual definition of terms. Let $n \in \mathbb{N}^+$ and $X_n = \{x_1, \ldots, x_n\}$ be an $n-$element set. The set X_n is called an *alphabet* and its elements are called *variables*. To every *operation symbol* f_i, we assign a natural number $n_i \geq 1$, the arity of f_i. Let $\tau = (n_i)_{i \in I}$ be a type such that the set of operation symbols $\{f_i \mid i \in I\}$ is disjoint with X_n. An $n-$*ary term* of type τ is inductively defined as follows:

(i) every variable $x_j \in X_n$ is an $n-$ary term of type τ;

(ii) if $t_1, \ldots, t_{n_i}$ are $n-$ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an $n-$ary term of type τ.

The set $W_\tau(X_n)$ of all n-ary terms of type τ is the smallest set containing $x_1, \ldots, x_n$ that is closed under finite application of (ii). The set of all terms of type τ over the alphabet $X := \{x_1, x_2, \ldots\}$ is defined as disjoint union $W_\tau(X) := \bigcup\limits_{n=1}^{\infty} W_\tau(X_n)$.

By using step (ii) in the definition of terms of type τ, the term algebra

$$\mathcal{F}_\tau(X) := (W_\tau(X), (\bar{f}_i)_{i\in I})$$

of type τ, the so-called *absolutely free algebra*, can be defined by

$$\bar{f}_i(t_1, \ldots, t_{n_i}) := f_i(t_1, \ldots, t_{n_i})$$

for each operation symbol f_i and $t_1, \ldots, t_{n_i} \in W_\tau(X)$.

As for partial operations we can also define a superposition of terms. Clones of terms are subsets of $W_\tau(X)$ which are closed under the operation of superposition of terms and contain all variables. For each pair of natural numbers m and n greater than zero, the superposition operation S^n_m maps one n-ary term and n m-ary terms to an m-ary term, so that

$$S^n_m : W_\tau(X_n) \times (W_\tau(X_m))^n \;\to W_\tau(X_m).$$

The operation S^n_m is defined inductively, by setting
$S^n_m(x_j, t_1, \ldots, t_n) := t_j$ for any variable $x_j \in X_n$, and
$S^n_m(f_r(s_1, \ldots, s_{n_r}), t_1, \ldots, t_n) := f_r(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_r}, t_1, \ldots, t_n))$.

Using these operations, we form the heterogeneous or multi-based algebra

$$clone\,\tau \;:=\; (\,(W_\tau(X_n))_{n>0};\; (S^n_m)_{n,m>0}, (x_i)_{i\leq n, n\geq 1}\,).$$

It is well-known and easy to check that this algebra satisfies the clone axioms

(C1) $\overline{S^p_m}(\tilde{Z}, \overline{S^n_m}(\tilde{Y}_1, \tilde{X}_1, \ldots, \tilde{X}_n), \ldots, \overline{S^n_m}(\tilde{Y}_p, \tilde{X}_1, \ldots, \tilde{X}_n))$
$\qquad \approx \overline{S^n_m}(\overline{S^p_n}(\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_p), \tilde{X}_1, \ldots, \tilde{X}_n)$,

(C2) $\overline{S^m_m}(\lambda_j, \tilde{X}_1, \ldots, \tilde{X}_m) \;\approx\; \tilde{X}_j$, for $1 \leq j \leq m$,

(C3) $\overline{S^m_m}(\tilde{X}_j, \lambda_1, \ldots, \lambda_m) \;\approx\; \tilde{X}_j$, for $1 \leq j \leq m$,

where $\overline{S^p_m}$ and $\overline{S^n_m}$ are operation symbols corresponding to the operations S^p_m, S^n_m of $clone\,\tau$, $\lambda_1, \ldots, \lambda_m$ are nullary operation symbols and $\tilde{Z}$, $\tilde{Y}_1, \ldots, \tilde{Y}_p$, $\tilde{X}_1, \ldots, \tilde{X}_m$ are variables. The algebra $clone\,\tau$ is also called a *Menger system*.

Since the set $W_\tau(X_n)$ of all n-ary terms of type τ is closed under the superposition operation $S^n := S^n_n$, there is a homogeneous analogue of this structure. The algebra $(W_\tau(X_n); S^n, x_1, \ldots, x_n)$ is an algebra of type $\tau = (n{+}1, 0, \ldots, 0)$, which still satisfies the clone axioms above for the case that $p = m = n$. Such an algebra is called a *unitary Menger algebra of rank n*. An algebra $(W_\tau(X_n), S^n)$ of type $\tau = (n+1)$ is called a *Menger algebra of rank n* if it satisfies the axiom (C1).

Let $t \in W_\tau(X_n)$ for $n \in \mathbb{N}^+$. To each partial algebra $\mathcal{A} = (A; (f_i^A)_{i\in I})$ of type τ we obtain a partial operation $t^{\mathcal{A}}$, called the *n-ary term operation* induced by t as follows:

(i) If $t = x_j \in X_n$ then $t^{\mathcal{A}} = x_j^{\mathcal{A}} := e_j^{n,A}$, where $e_j^{n,A}$ is the n-ary total projection on the j-th component.

(ii) Now assume that $t = f_i(t_1, \ldots, t_{n_i})$ where f_i is an n_i-ary operation symbol, and assume also that $t_1^{\mathcal{A}}, \ldots, t_{n_i}^{\mathcal{A}}$ are the term operations induced by the terms $t_1, \ldots, t_{n_i}$, and that the $t_j^{\mathcal{A}}(a_1, \ldots, a_n)$ are defined, with values $t_j^{\mathcal{A}}(a_1, \ldots, a_n) = b_j$, for $1 \leq j \leq n_i$. If $f_i^A(b_1, \ldots, b_{n_i})$ is defined, then $t^{\mathcal{A}}(a_1, \ldots, a_n)$ is defined and $t^{\mathcal{A}}(a_1, \ldots, a_n) = S_n^{n_i,A}(f_i^A, t_1^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, t_{n_i}^{\mathcal{A}}(a_1, \ldots, a_n))$.

Let $W_\tau(X_n)^{\mathcal{A}}$ be the set of all n-ary term operations of type τ.

Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a partial algebra of a given type τ. To every partial algebra $\mathcal{A}$ we assign the partial clone generated by $\{f_i^A \mid i \in I\}$, denoted by $T(\mathcal{A})$. The set $T(\mathcal{A})$ is called *clone of all term operations* of the algebra $\mathcal{A}$.

Example 2.1.1 *Let $\mathcal{A} = (\{0,1\}; f^A)$ be a partial algebra of type (1). Let f^A be the partial operation defined by*

$$f^A(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ \text{not defined} & \text{if} \quad x = 1. \end{cases}$$

Let $t^{\mathcal{A}}$ be the term operation induced by a term $t \in W_{(1)}(X)$. Then $t^{\mathcal{A}} \in J_A \cup \{f^A, c^1_\infty\}$ when the symbol c^1_∞ is used to express that $f^A(x)$ is an unary constant nowhere defined. But the operation

$$g^A = S_2^{2,A}(e_1^{2,A}, e_1^{2,A}, S_2^{1,A}(f^A, e_2^{2,A}))$$

is different from f^A, c^1_∞ and elements of J_A. We have $g^A \in T(\mathcal{A})$ but $g^A \notin W_{(1)}(X)^{\mathcal{A}}$.

Since the set $W_\tau(X_n)^{\mathcal{A}}$ is different from the set of all partial operations generated by $\{f_i^{\mathcal{A}} \mid i \in I\}$ we need a new definition of terms over partial algebras of type τ which overcomes this problem.
Let X be an alphabet and let $\{f_i \mid i \in I\}$ be a set of operation symbols of type τ, where each f_i has the arity n_i and $X \cap \{f_i \mid i \in I\} = \emptyset$. We need additional symbols $\varepsilon_j^k \notin X$, for every $k \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ and $1 \leq j \leq k$. Let $X_n = \{x_1, \ldots, x_n\}$ be an n-element alphabet. The set of *n-ary C-terms* of type τ over X_n is defined inductively as follows:

(i) Every $x_j \in X_n$ is an n-ary C-term of type τ.

(ii) If $w_1, \ldots, w_k$ are n-ary C-terms of type τ, then $\varepsilon_j^k(w_1, \ldots, w_k)$ is an n-ary C-term of type τ for all $1 \leq j \leq k$ and all $k \in \mathbb{N}^+$.

(iii) If $w_1, \ldots, w_{n_i}$ are n-ary C-terms of type τ and if f_i is an n_i-ary operation symbol, then $f_i(w_1, \ldots, w_{n_i})$ is an n-ary C-term of type τ.

Let $W_\tau^C(X_n)$ be the set of all n-ary C-terms of type τ defined in this way. Then $W_\tau^C(X) := \bigcup\limits_{n=1}^{\infty} W_\tau^C(X_n)$ denotes the set of all C-terms of this type. Note that here the use of the superscript C shall distinguish these sets from the analogous ones in the total case; the letter C was used since Craig in [15] suggested the addition of the extra constant terms ε_j^k.

Every n-ary C-term $w \in W_\tau^C(X_n)$ induces an n-ary C-term operation $w^{\mathcal{A}}$ of any partial algebra $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ of type τ. For $a_1, \ldots, a_n \in A$, the value $w^{\mathcal{A}}(a_1, \ldots, a_n)$ is defined in the following inductive way:

(i) If $w = x_j$ then $w^{\mathcal{A}} = x_j^{\mathcal{A}} = e_j^{n,A}$, where $e_j^{n,A}$ is as usual the n-ary total projection on the j-th component.

(ii) If $w = \varepsilon_j^k(w_1, \ldots, w_k)$ and we assume that $w_1^{\mathcal{A}}, \ldots, w_k^{\mathcal{A}}$ are the C-term operations induced by the terms $w_1, \ldots, w_k$ and that the $w_i^{\mathcal{A}}(a_1, \ldots, a_n)$ are defined for $1 \leq i \leq k$, then $w^{\mathcal{A}}(a_1, \ldots, a_n)$ is defined and $w^{\mathcal{A}}(a_1, \ldots, a_n) = w_j^{\mathcal{A}}(a_1, \ldots, a_n)$.

(iii) Now assume that $w = f_i(w_1, \ldots, w_{n_i})$ where f_i is an n_i-ary operation symbol, and assume that the $w_j^{\mathcal{A}}(a_1, \ldots, a_n)$ are defined, with values $w_j^{\mathcal{A}}(a_1, \ldots, a_n) = b_j$ for $1 \le j \le n_i$. If $f_i^A(b_1, \ldots, b_{n_i})$ is defined, then $w^{\mathcal{A}}(a_1, \ldots, a_n)$ is defined and $w^{\mathcal{A}}(a_1, \ldots, a_n) = S_n^{n_i, A}(f_i^A, w_1^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, w_{n_i}^{\mathcal{A}}(a_1, \ldots, a_n))$.

Let $W_\tau^C(X_n)^{\mathcal{A}}$ be the set of all n-ary term operations induced by the terms from $W_\tau^C(X_n)$ on the partial algebra $\mathcal{A}$ and let $W_\tau^C(X)^{\mathcal{A}} := \bigcup\limits_{n=1}^{\infty} W_\tau^C(X_n)^{\mathcal{A}}$.

Note that for C-terms we have $T(\mathcal{A}) = W_\tau^C(X)^{\mathcal{A}}$ (see [2]).

Now we show that arbitrary term operations induced by C-terms satisfy the same compatibility condition as fundamental operations of $\mathcal{A}$.

Lemma 2.1.2 *Let $\varphi : T(\mathcal{A}) \to T(\mathcal{B})$ be a clone homomorphism defined by $\varphi(f_i^A) = f_i^{\mathcal{B}}$ for all $i \in I$. Then $\varphi(t^{\mathcal{A}}) = t^{\mathcal{B}}$ for all $t \in W_\tau^C(X)$.*

The Lemma can be proved by induction on the complexity of the term $t \in W_\tau^C(X)$ (see [12]).

On the sets $W_\tau^C(X_n)$ we may introduce the following superposition operations. Let $w_1, \ldots, w_m$ be n-ary C-terms and let t be an m-ary C-term. Then we define an n-ary C-term $\overline{S}_n^m(t, w_1, \ldots, w_m)$ inductively by the following steps:

(i) For $t = x_j$, $1 \le j \le m$ (m-ary variable), we define
$\overline{S}_n^m(x_j, w_1, \ldots, w_m) = w_j$.

(ii) For $t = \varepsilon_j^k(s_1, \ldots, s_k)$ we set
$\overline{S}_n^m(t, w_1, \ldots, w_m) = \varepsilon_j^k(\overline{S}_n^m(s_1, w_1, \ldots, w_m), \ldots, \overline{S}_n^m(s_k, w_1, \ldots, w_m))$,
where $s_1, \ldots, s_k$ are m-ary, for all $k \in \mathbb{N}^+$ and $1 \le j \le k$.

(iii) For $t = f_i(s_1, \ldots, s_{n_i})$ we set
$\overline{S}_n^m(t, w_1, \ldots, w_m) = f_i(\overline{S}_n^m(s_1, w_1, \ldots, w_m), \ldots, \overline{S}_n^m(s_{n_i}, w_1, \ldots, w_m))$,
where $s_1, \ldots, s_{n_i}$ are m-ary.

This defines an operation

$$\overline{S}_n^m : W_\tau^C(X_m) \times (W_\tau^C(X_n))^m \longrightarrow W_\tau^C(X_n),$$

which describes the superposition of C-terms.

The *C-term clone* of type τ is the heterogeneous algebra

$$clone\tau^C := ((W_\tau^C(X_n))_{n>0}; (\overline{S}_n^m)_{n,m>0}, (x_j)_{j\leq m, m\geq 1}).$$

Let $T^n(\mathcal{A})$ be the set of all n-ary term operations of a partial algebra $\mathcal{A} = (A; (f_i^A)_{i\in I})$. Then $T(\mathcal{A}) = ((T^n(\mathcal{A}))_{n\in\mathbb{N}^+}; (S_n^{m,A})_{n,m\in\mathbb{N}^+}, (e_j^{n,A})_{n\in\mathbb{N}^+, 1\leq j\leq n})$ is also a partial clone, it is the partial clone generated by the fundamental operations of the algebra $\mathcal{A}$.

We define a family $\varphi = (\varphi^{(n)})_{n\in\mathbb{N}^+}$ of mappings, $\varphi^{(n)} : W_\tau^C(X_n) \to T^n(\mathcal{A})$, by setting $\varphi^{(n)}(t) = t^{\mathcal{A}}$, the n-ary term operation induced by t. It is easy to see that φ has the following properties ([49]):
(i) $\varphi^{(n)}(x_j) = e_j^{n,A}$, $1 \leq j \leq n$, $n \in \mathbb{N}^+$,
(ii) $\varphi^{(n)}(\overline{S}_n^m(s, t_1, \ldots, t_m)) \mid_D = S_n^{m,A}(\varphi^{(m)}(s), \varphi^{(n)}(t_1), \ldots, \varphi^{(n)}(t_m)) \mid_D$, for $n \in \mathbb{N}^+$, where D is the intersection of the domains of all $t_j^{\mathcal{A}}$, $1 \leq j \leq m$, where s is m-ary, and $t_1, \ldots, t_m$ are n-ary.

2.2 Strong Varieties

Let τ be a type. An ordered pair $(t_1, t_2) \in W_\tau(X)^2$ is called an *equation* of type τ; we usually write $t_1 \approx t_2$.

An equation $t_1 \approx t_2 \in W_\tau(X)^2$ is called a *strong identity* in a partial algebra $\mathcal{A}$ (in symbols $\mathcal{A} \underset{s}{\models} t_1 \approx t_2$) iff $t_1^{\mathcal{A}}$ is defined whenever $t_2^{\mathcal{A}}$ is defined and conversely and $t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$ on the common domain, i.e. the induced partial term operations $t_1^{\mathcal{A}}$ and $t_2^{\mathcal{A}}$ are equal.

Let $K \subseteq PAlg(\tau)$ be a class of partial algebras of type τ and $\Sigma \subseteq W_\tau(X)^2$. Consider the connection between $PAlg(\tau)$ and $W_\tau(X)^2$ given by the following two operators:

$$Id^s : \mathcal{P}(PAlg(\tau)) \to \mathcal{P}(W_\tau(X)^2) \qquad \text{and}$$

$$Mod^s : \mathcal{P}(W_\tau(X)^2) \to \mathcal{P}(PAlg(\tau)) \qquad \text{with}$$

$$\begin{array}{lcl} Id^s K & := & \{s \approx t \in W_\tau(X)^2 \mid \forall \mathcal{A} \in K \ (\mathcal{A} \underset{s}{\models} s \approx t)\} \qquad \text{and} \\ Mod^s \Sigma & := & \{\mathcal{A} \in PAlg(\tau) \mid \forall\ s \approx t \in \Sigma \ (\mathcal{A} \underset{s}{\models} s \approx t)\}. \end{array}$$

Clearly, the pair (Mod^s, Id^s) is a Galois connection between $PAlg(\tau)$ and $W_\tau(X)^2$.

As usual for a Galois connection, we have two closure operators $Mod^s Id^s$ and $Id^s Mod^s$ and their sets of fixed points, i.e. the sets

$$\{\Sigma \subseteq W_\tau(X)^2 \mid Id^s Mod^s \Sigma = \Sigma\} \qquad \text{and} \qquad \{K \subseteq PAlg(\tau) \mid Mod^s Id^s K = K\},$$

form two complete lattices $\mathcal{E}^s(\tau)$, $\mathcal{L}^s(\tau)$.

Let $V \subseteq PAlg(\tau)$ be a class of partial algebras. The class V is called a *strong variety* of partial algebras iff there is a set $\Sigma \subseteq W_\tau(X)^2$ of strong identities in V such that $V = Mod^s \Sigma$.

In [4] P. Burmeister introduced the concept of an ECE-equation. By [5], page 67, ECE-equations and strong equations are equivalent if the empty algebra is excluded. Therefore we have the following Birkhoff-type characterization of strong varieties.

Theorem 2.2.1 ([5], p. 199) *Let K be a class of partial algebras of type τ. Then a class K is a strong variety iff $K = \boldsymbol{H}_c \boldsymbol{S}_c \boldsymbol{P}_{filt} K \cup \{\underline{\emptyset}\}$ where $\underline{\emptyset}$ is the empty algebra. (i.e. K is closed under closed homomorphic images, closed subalgebras, and filtered products of partial algebras from $K \cup \{\underline{\emptyset}\}$).*

Now we consider equations consisting of C-terms. As for usual terms we define: An equation $t_1 \approx t_2 \in W_\tau^C(X)^2$ is called a *strong identity* in a partial algebra $\mathcal{A}$ (in symbols $\mathcal{A} \underset{s}{\models} t_1 \approx t_2$) iff $t_1^{\mathcal{A}}$ is defined whenever $t_2^{\mathcal{A}}$ is defined and conversely and $t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$ on the common domain, i.e. the induced partial term operations $t_1^{\mathcal{A}}$ and $t_2^{\mathcal{A}}$ are equal.

Let $K \subseteq PAlg(\tau)$ be a class of partial algebras of type τ and $\Sigma \subseteq W_\tau^C(X)^2$. Consider the connection between $PAlg(\tau)$ and $W_\tau^C(X)^2$ given by the following two operators:

$$Id^s : \mathcal{P}(PAlg(\tau)) \to \mathcal{P}(W_\tau^C(X)^2) \qquad \text{and}$$

$$Mod^s : \mathcal{P}(W_\tau^C(X)^2) \to \mathcal{P}(PAlg(\tau)) \qquad \text{with}$$

$$\begin{array}{lcl} Id^s K & := & \{s \approx t \in W_\tau^C(X)^2 \mid \forall \mathcal{A} \in K \ (\mathcal{A} \underset{s}{\models} s \approx t)\} \qquad \text{and} \\ Mod^s \Sigma & := & \{\mathcal{A} \in PAlg(\tau) \mid \forall\ s \approx t \in \Sigma \ (\mathcal{A} \underset{s}{\models} s \approx t)\}. \end{array}$$

Clearly, the pair (Mod^s, Id^s) is a Galois connection between $PAlg(\tau)$ and $W_\tau^C(X)^2$. We have two closure operators $Mod^s Id^s$ and $Id^s Mod^s$ and their sets of fixed points.

Let $V \subseteq PAlg(\tau)$ be a class of partial algebras. The class V is called a *strong variety* of partial algebras iff there is a set $\Sigma \subseteq W_\tau^C(X)^2$ of strong identities in V such that $V = Mod^s\Sigma$.

Theorem 2.2.2 ([2]) *Let K be a class of partial algebras of type τ. Then a class K is a strong variety iff $K = \boldsymbol{H}_c\boldsymbol{S}_c\boldsymbol{P}_{filt}K$ (i.e. K is closed under closed homomorphic images, closed subalgebras, and filtered products of partial algebras from K).*

2.3 Strong Regular Varieties

For a term $t \in W_\tau(X)$ we denote the set of all variables in t by Var(t).

An equation $p \approx q \in W_\tau(X)^2$ of terms is called *regular* if in p and q the same variables occur i.e. if Var(p)=Var(q).

Let $W_\tau^r(X)^2 \subseteq W_\tau(X)^2$ be the set of all regular equations of type τ.

An equation $s \approx t \in W_\tau(X)^2$ is called a *strong regular identity* in a partial algebra $\mathcal{A}$ (in symbols $\mathcal{A} \underset{sr}{\models} s \approx t$) iff $\mathcal{A} \underset{s}{\models} s \approx t$ and Var(s)=Var(t).

Let $K \subseteq PAlg(\tau)$ be a class of partial algebras of type τ and $\Sigma \subseteq W_\tau(X)^2$. Consider the connection between $PAlg(\tau)$ and $W_\tau(X)^2$ given by the following two operators:

$$Id^{sr} : \mathcal{P}(PAlg(\tau)) \to \mathcal{P}(W_\tau(X)^2) \qquad \text{and}$$

$$Mod^{sr} : \mathcal{P}(W_\tau(X)^2) \to \mathcal{P}(PAlg(\tau)) \qquad \text{with}$$

$$\begin{aligned} Id^{sr}K \quad &:= \quad \{s \approx t \in W_\tau(X)^2 \mid \forall \mathcal{A} \in K\ (\mathcal{A} \underset{sr}{\models} s \approx t)\} \qquad \text{and} \\ Mod^{sr}\Sigma \quad &:= \quad \{\mathcal{A} \in PAlg(\tau) \mid \forall\ s \approx t \in \Sigma\ (\mathcal{A} \underset{sr}{\models} s \approx t)\}. \end{aligned}$$

Clearly, the pair (Mod^{sr}, Id^{sr}) is a Galois connection between $PAlg(\tau)$ and $W_\tau(X)^2$. Again we have two closure operators $Mod^{sr}Id^{sr}$ and $Id^{sr}Mod^{sr}$ and their sets of fixed points, i.e. the sets

$$\{\Sigma \subseteq W_\tau(X)^2 \mid Id^{sr}Mod^{sr}\Sigma = \Sigma\} \qquad \text{and} \qquad \{K \subseteq PAlg(\tau) \mid Mod^{sr}Id^{sr}K = K\},$$

form two complete lattices $\mathcal{E}^{sr}(\tau)$, $\mathcal{L}^{sr}(\tau)$.

Let $V \subseteq PAlg(\tau)$ be a class of partial algebras. The class V is called a *strong regular variety* of partial algebras iff there is a set $\Sigma \subseteq W_\tau^r(X)^2$ of strong regular identities in V such that $V = Mod^{sr}\Sigma$.

To obtain a Birkhoff-type characterization for strong regular varieties we introduce the following *pin* operator $\perp$.

For a partial algebra $\mathcal{A} = (A : (f_i^A)_{i\in I})$, let $\mathcal{A}^\perp = (A \cup \{\perp\}; (f_i^{A^\perp})_{i\in I})$ when $\perp \notin A$ and

$$f_i^{A^\perp}(a_1, \ldots, a_{n_i}) = \begin{cases} f_i^A(a_1, \ldots, a_{n_i}) & \text{if} \quad (a_1, \ldots, a_{n_i}) \in dom f_i^A \\ \perp & \text{otherwise.} \end{cases}$$

The operation $f_i^{A^\perp}$ is called *one-point extension* of f_i^A.

Let $K \subseteq PAlg(\tau)$ and $K^\perp = \{\mathcal{A}^\perp | \mathcal{A} \in K\}$. Moreover

$$K^{\perp_0} = K,\ K^{\perp_{n+1}} = (K^{\perp_n})^\perp \text{ for all } n \in \mathbb{N}.$$

Now we can define the pin operator on K.

$$\perp K := \bigcup_{n=0}^{\infty} K^{\perp_n}.$$

Theorem 2.3.1 ([48]) *Let K be a class of partial algebras of type τ. Then a class K is a strong regular variety iff $K = \boldsymbol{H}_c\boldsymbol{In}\boldsymbol{S}_c\boldsymbol{P}\perp(K)$ (i.e. K is closed under closed homomorphic images, initial segments, closed subalgebras, direct products and the pin operator applied on partial algebras from K).*

Proposition 2.3.2 *Let $s, t \in W_\tau^C(X)$ and $\mathcal{A} \in PAlg(\tau)$. If $s \approx t \in Id^s\mathcal{A}$ then there exist $s', t' \in W_\tau^C(X)$ and $s' \approx t' \in Id^{sr}\mathcal{A}$ (i.e. $Var(s') = Var(t')$ and $s' \approx t' \in Id^s\mathcal{A}$).*

Proof. Let $s \approx t \in Id^s\mathcal{A}$. Since

$$\varepsilon_1^2(s,t) \approx s \approx t \approx \varepsilon_2^2(s,t) \in Id^s\mathcal{A}.$$

Let $s' = \varepsilon_1^2(s,t)$ and $t' = \varepsilon_2^2(s,t)$. We have $Var(s') = Var(t')$ and $s' \approx t' \in Id^s\mathcal{A}$. Then $s' \approx t' \in Id^{sr}\mathcal{A}$. ∎

Because of Proposition 2.3.2 in the case of C-terms instead of strong identities we can always consider strong regular identities.

Chapter 3

Hyperidentities

This chapter shall motivate the study of hyperidentities. We first define the concepts of hypersubstitutions, regular hypersubstitutions, strong regular M-hyperidentities and M-solid strong regular varieties on the basis of terms from $W_\tau(X)$. Secondly, we give the definition of M-solid strong varieties considering terms from $W_\tau^C(X)$.

3.1 Hyperidentities and M-solid Strong Regular Varieties

We consider mappings from the set of all operation symbols of type τ into the set of all terms of type τ. Such mappings are called *hypersubstitutions* of type τ if they preserve the arities. This means that to each n_i-ary operation symbol of type τ, we assign an n_i-ary term from $W_\tau(X)$. Hypersubstitutions σ can be extended to mappings $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ which are defined on the set $W_\tau(X)$ of all terms of type τ by the following inductive definition:

(i) $\widehat{\sigma}[x] := x$ for every variable $x \in X$;

(ii) $\widehat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S_n^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$ for all terms $t_1, \ldots, t_{n_i} \in W_\tau(X_n)$.

As Welke proved in [49], a necessary condition for $\widehat{\sigma}[s] \approx \widehat{\sigma}[t]$ to be a strong regular identity in a partial algebra $\mathcal{A}$ whenever $s \approx t$ is a strong regular identity in $\mathcal{A}$ is that $\widehat{\sigma}$ maps terms of the form $f_i(x_1, \ldots, x_{n_i})$ to terms t with $Var(t) = \{x_1, \ldots, x_{n_i}\}$. So to define strong regular hyperidentities we will consider only such hypersubstitutions.

Let σ be a hypersubstitution. We say that the hypersubstitution σ is a *regular hypersubstitution* if $Var(\sigma(f_i)) = \{x_1, \ldots, x_{n_i}\}$ for all $i \in I$.
Let $Hyp_R(\tau)$ denote the set of all regular hypersubstitutions of type τ and let σ_R denote some member of $Hyp_R(\tau)$.

On $Hyp_R(\tau)$ we define a binary operation by

$$\sigma_{R_1} \circ_h \sigma_{R_2} := \widehat{\sigma}_{R_1} \circ \sigma_{R_2}.$$

From [49] follows that for any two regular hyperstitutions of type τ we have $(\sigma_{R_1} \circ_h \sigma_{R_2})^{\widehat{}} = \widehat{\sigma}_{R_1} \circ \widehat{\sigma}_{R_2}$ (this equation is valid for arbitrary hypersubstitutions).

Theorem 3.1.1 ([49]) *The algebra $\mathcal{H}yp_R(\tau) := (Hyp_R(\tau); \circ_h, \sigma_{id})$ is a monoid with $\sigma_{id}(f_i) = f_i(x_1, \ldots, x_{n_i})$ for all $i \in I$.*

Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a partial algebra of type $\tau = (n_i)_{i\in I}$, and let $\sigma_R \in Hyp_R(\tau)$ be a regular hypersubstitution. We want to consider the *derived algebra* $\sigma_R(\mathcal{A}) = (A; (\sigma_R(f_i)^{\mathcal{A}})_{i\in I})$, where $\sigma_R(f_i)^{\mathcal{A}}$ is the term operation induced by the term $\sigma_R(f_i)$ on the algebra $\mathcal{A}$. For regular hypersubstitutions we have the following important feature.

Lemma 3.1.2 ([49]) *Let σ_R be a regular hypersubstitution of type τ and let $\mathcal{A}$ be a partial algebra of type τ. For a term $t \in W_\tau(X)$ we denote by $t^{\sigma_R(\mathcal{A})}$ the term operation induced by t in the algebra $\sigma_R(\mathcal{A})$, and by $\widehat{\sigma}_R[t]^{\mathcal{A}}$ the term operation induced by $\widehat{\sigma}_R[t]$ in the algebra $\mathcal{A}$. Then for every term $t \in W_\tau(X)$ we have*

$$\widehat{\sigma}_R[t]^{\mathcal{A}} = t^{\sigma_R(\mathcal{A})}.$$

Let $\mathcal{M}$ be a submonoid of $\mathcal{H}yp_R(\tau)$. We introduce two operators χ_M^E and χ_M^A. Let $\Sigma \subseteq W_\tau(X) \times W_\tau(X)$ be regular equations, $s \approx t \in \Sigma$, we let

$$\chi_M^E[s \approx t] := \{\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t] \mid \sigma_R \in M\} \qquad \text{and}$$

$$\chi_M^E[\Sigma] := \bigcup_{s\approx t\in\Sigma} \chi_M^E[s \approx t].$$

For any partial algebra $\mathcal{A}$ of type τ and $K \subseteq PAlg(\tau)$, we let

$$\chi_M^A[\mathcal{A}] := \{\sigma_R(\mathcal{A}) \mid \sigma_R \in M\} \qquad \text{and}$$

$$\chi_M^A[K] := \bigcup_{\mathcal{A}\in K} \chi_M^A[\mathcal{A}].$$

Now we can define the concept of a strong regular M-hyperidentity of a partial algebra of type τ.

Let $\mathcal{M}$ be a submonoid of $\mathcal{H}yp_R(\tau)$ and let $\mathcal{A}$ be a partial algebra of type τ. Then a strong regular identity $s \approx t$ of $\mathcal{A}$ is called a *strong regular M-hyperidentity* of $\mathcal{A}$ if for every regular hypersubstitution $\sigma_R \in M$ the equation $\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t]$ is also a strong regular identity of $\mathcal{A}$. We write

$$\mathcal{A} \underset{srMh}{\models} s \approx t :\Leftrightarrow \forall \sigma_R \in M(\mathcal{A} \underset{sr}{\models} \widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t]).$$

A strong regular identity is called a *strong regular M-hyperidentity* of a class K of partial algebras of type τ if it holds as strong regular M-hyperidentity in every partial algebra in K. In the case, if $M = Hyp_R(\tau)$, strong regular M-hyperidentities are called *strong regular hyperidentities*.

The relation

$$\underset{srMh}{\models} := \{(\mathcal{A}, s \approx t) \in PAlg(\tau) \times W_\tau(X)^2 | \ \forall \sigma_R \in M(\mathcal{A} \underset{sr}{\models} \widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t])\}$$

induces the Galois connection $(H_M Id^{sr}, H_M Mod^{sr})$ defined on subclasses K of $PAlg(\tau)$ and regular equations Σ of identities in $W_\tau(X)^2$ as follows:

$$\begin{aligned} H_M Id^{sr} K &:= \{s \approx t \in W_\tau(X)^2 \mid \forall \mathcal{A} \in K(\mathcal{A} \underset{srMh}{\models} s \approx t)\}; \\ H_M Mod^{sr} \Sigma &:= \{\mathcal{A} \in PAlg(\tau) \mid \forall s \approx t \in \Sigma(\mathcal{A} \underset{srMh}{\models} s \approx t)\}. \end{aligned}$$

A set Σ of identities in $W_\tau(X)^2$ is called a *strong regular M-hyperequational theory* if there is a class K of partial algebras of type τ such that $\Sigma = H_M Id^{sr} K$.

A class K of partial algebras of type τ is called a *strong regular M-hyperequational class* if there is a set of identities Σ such that $K = H_M Mod^{sr}\Sigma$.

Corollary 3.1.3 ([49]) *For every submonoid $\mathcal{M} \subseteq \mathcal{H}yp_R(\tau)$ the operators χ_M^A and χ_M^E form a conjugate pair of additive closure operators with respect to the relation $\underset{sr}{\models}$.*

Let $V \subseteq PAlg(\tau)$ be a strong regular variety of partial algebras, so that $V = Mod^{sr}\Sigma$ for some regular equation $\Sigma \subseteq W_\tau(X) \times W_\tau(X)$. Then V is said to be *M-solid* if $\chi^A_M[V] = V$.

Theorem 3.1.4 ([49]) *Let $V \subseteq PAlg(\tau)$ be a strong regular variety of partial algebras and let $\Sigma \subseteq W_\tau(X)^2$ be a strong regular equational theory. Then the following propositions* (i)-(iv) *and* (i')-(iv') *are equivalent:*
(i) V is a strong regular M-hyperequational class, i.e. $V = H_M Mod^{sr} H_M Id^{sr} V$.
(ii) V is M-solid, i.e. $\chi^A_M[V] = V$.
(iii) $Id^{sr}V = H_M Id^{sr}V$, i.e. every strong regular identity of V is a strong regular M-hyperidentity of V.
(iv) $\chi^E_M[Id^{sr}V] = Id^{sr}V$.
And the following conditions are also equivalent
(i') $\Sigma = H_M Id^{sr} H_M Mod^{sr}\Sigma$.
(ii') $\chi^E_M[\Sigma] = \Sigma$.
(iii') $Mod^{sr}\Sigma = H_M Mod^{sr}\Sigma$.
(iv') $\chi^A_M[Mod^{sr}\Sigma] = Mod^{sr}\Sigma$.

Theorem 3.1.5 ([49]) *For any $K \subseteq PAlg(\tau)$ and for any set of regular equations $\Sigma \subseteq W_\tau(X)^2$ the following conditions hold:*

(i) $\chi^A_M[K] \subseteq Mod^{sr}Id^{sr}K \Leftrightarrow Mod^{sr}Id^{sr}K = H_M Mod^{sr} H_M Id^{sr}K$
and
(ii) $\chi^E_M[\Sigma] \subseteq Id^{sr}Mod^{sr}\Sigma \Leftrightarrow Id^{sr}Mod^{sr}\Sigma = H_M Id^{sr} H_M Mod^{sr}\Sigma$.

3.2 Hyperidentities and M-solid Strong Varieties

In this section we recall some basis facts on regular hypersubstitutions, strong hyperidentities and solid strong varieties of partial algebras using C-terms. For more details see [26], [27] and [49].

Let $\{f_i \mid i \in I\}$ be a set of operation symbols of type τ and $W^C_\tau(X)$ be the set of all C-terms of this type. A mapping $\sigma : \{f_i \mid i \in I\} \longrightarrow W^C_\tau(X)$ which maps each n_i-ary fundamental operation f_i to a C-term of arity n_i is called a *C-hypersubstitution* of type τ.

Any C-hypersubstitution σ of type τ can be extended to a map $\widehat{\sigma} : W_\tau^C(X) \longrightarrow W_\tau^C(X)$ defined for all C-terms, in the following way:

(i) $\widehat{\sigma}[x_j] = x_j$ for every $x_j \in X_n$,

(ii) $\widehat{\sigma}[\varepsilon_j^k(s_1, \ldots, s_k)] = \overline{S}_n^k(\varepsilon_j^k(x_1, \ldots, x_k), \widehat{\sigma}[s_1], \ldots, \widehat{\sigma}[s_k])$, where $s_1, \ldots, s_k \in W_\tau^C(X_n)$,

(iii) $\widehat{\sigma}[f_i(t_1, \ldots, t_{n_i})] = \overline{S}_n^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$, where $t_1, \ldots, t_{n_i} \in W_\tau^C(X_n)$.

The C-hypersubstitution σ is called *regular* if $Var(\sigma(f_i)) = \{x_1, \ldots, x_{n_i}\}$, for all $i \in I$.

Let $Hyp_R^C(\tau)$ be the set of all regular C-hypersubstitutions of type τ and let σ_R denote some member of $Hyp_R^C(\tau)$.

Lemma 3.2.1 ([49]) Let $\sigma_{R_1}, \sigma_{R_2} \in Hyp_R^C(\tau)$. Then $(\widehat{\sigma}_{R_2} \circ \sigma_{R_1})\widehat{\ } = \widehat{\sigma}_{R_2} \circ \widehat{\sigma}_{R_1}$, where $\circ$ is the usual composition of functions.

Now we define a product of C-hypersubstitutions in the usual way, by $\sigma_{R_1} \circ_h \sigma_{R_2} := \widehat{\sigma}_{R_1} \circ \sigma_{R_2}$ and obtain:

Theorem 3.2.2 ([49]) The algebra $\mathcal{H}yp_R^C(\tau) := (Hyp_R^C(\tau); \circ_h, \sigma_{id})$ with $\sigma_{id}(f_i) = f_i(x_1, \ldots, x_{n_i})$ is a monoid.

Let $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ be a partial algebra of type $\tau = (n_i)_{i \in I}$, and let $\sigma_R \in Hyp_R^C(\tau)$. We want to consider the *derived algebra* $\sigma_R(\mathcal{A}) = (A; (\sigma_R(f_i)^{\mathcal{A}})_{i \in I})$, where $\sigma_R(f_i)^{\mathcal{A}}$ is the term operation induced by the term $\sigma_R(f_i)$ on the algebra $\mathcal{A}$.

Lemma 3.2.3 ([49]) Let σ_R be a regular C-hypersubstitution of type τ and let $\sigma_R(\mathcal{A}) = (A; (\sigma_R(f_i)^{\mathcal{A}})_{i \in I})$. For a term $t \in W_\tau^C(X)$ we denote by $t^{\sigma_R(\mathcal{A})}$ the term operation induced by t on the algebra $\sigma_R(\mathcal{A})$, and by $\widehat{\sigma}_R[t]^{\mathcal{A}}$ the term operation induced by $\widehat{\sigma}_R[t]$ on the algebra $\mathcal{A}$. Then for every term $t \in W_\tau^C(X)$ we have

$$\widehat{\sigma}_R[t]^{\mathcal{A}} = t^{\sigma_R(\mathcal{A})}.$$

Lemma 3.2.4 *Let $\sigma_{R_1}, \sigma_{R_2} \in Hyp_R^C(\tau)$ and $\mathcal{A} \in PAlg(\tau)$. Then $\sigma_{R_1}(\sigma_{R_2}(\mathcal{A})) = (\sigma_{R_2} \circ_h \sigma_{R_1})(\mathcal{A})$.*

Proof. We have

$$\begin{array}{rcl}\sigma_{R_1}(\sigma_{R_2}(\mathcal{A})) & = & (A; (\sigma_{R_1}(f_i)^{\sigma_{R_2}(\mathcal{A})})_{i\in I}) \\ & = & (A; (\widehat{\sigma}_{R_2}[\sigma_{R_1}(f_i)]^{\mathcal{A}})_{i\in I}) \\ & = & (A; ((\sigma_{R_2} \circ_h \sigma_{R_1})(f_i)^{\mathcal{A}})_{i\in I}) \\ & = & (\sigma_{R_2} \circ_h \sigma_{R_1})(\mathcal{A}).\end{array}$$

■

(Remark that for the fundamental operations of the derived algebra $\sigma(\mathcal{A})$ we have $f_i^{\sigma(\mathcal{A})} = \sigma(f_i)^{\mathcal{A}}$. For $\sigma_1(\sigma_2(\mathcal{A}))$ this gives $f_i^{\sigma_1(\sigma_2(\mathcal{A}))} = \sigma_1(f_i)^{\sigma_2(\mathcal{A})} = \widehat{\sigma}_2(\sigma_1(f_i))^{\mathcal{A}}$ by Lemma 3.2.3.)

Let $\mathcal{M}$ be a submonoid of $\mathcal{H}yp_R^C(\tau)$. We introduce two operators χ_M^E and χ_M^A. For any equation $s \approx t \in W_\tau^C(X) \times W_\tau^C(X)$ and any set $\Sigma \subseteq W_\tau^C(X) \times W_\tau^C(X)$, we let

$$\chi_M^E[s \approx t] := \{\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t] \mid \sigma_R \in M\} \qquad \text{and}$$

$$\chi_M^E[\Sigma] := \bigcup_{s\approx t\in\Sigma} \chi_M^E[s \approx t].$$

For any partial algebra $\mathcal{A}$ of type τ and $K \subseteq PAlg(\tau)$, we let

$$\chi_M^A[\mathcal{A}] := \{\sigma_R(\mathcal{A}) \mid \sigma_R \in M\} \qquad \text{and}$$

$$\chi_M^A[K] := \bigcup_{\mathcal{A}\in K} \chi_M^A[\mathcal{A}].$$

Proposition 3.2.5 *Let $\mathcal{A} \in PAlg(\tau)$ and $s \approx t \in W_\tau^C(X)^2$. Then*

$$\chi_M^A[\mathcal{A}] \models_s s \approx t \text{ iff } \mathcal{A} \models_s \chi_M^E[s \approx t].$$

Proof. We have

$$\begin{array}{rcl}\chi_M^A[\mathcal{A}] \models_s s \approx t & \Leftrightarrow & \forall \sigma_R \in M(\sigma_R(\mathcal{A}) \models_s s \approx t) \\ & \Leftrightarrow & \forall \sigma_R \in M(s^{\sigma_R(\mathcal{A})} = t^{\sigma_R(\mathcal{A})}) \\ & \Leftrightarrow & \forall \sigma_R \in M(\widehat{\sigma}_R[s]^{\mathcal{A}} = \widehat{\sigma}_R[t]^{\mathcal{A}}) \\ & \Leftrightarrow & \forall \sigma_R \in M(\mathcal{A} \models_s \widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t]) \\ & \Leftrightarrow & \mathcal{A} \models_s \chi_M^E[s \approx t].\end{array}$$

■

Now we can define the concept of a strong M-hyperidentity of a partial algebra of type τ.

Let $\mathcal{M}$ be a submonoid of $\mathcal{H}yp_R^C(\tau)$ and let $\mathcal{A}$ be a partial algebra of type τ. Then a strong identity $s \approx t$ of $\mathcal{A}$ is called a *strong M-hyperidentity* of $\mathcal{A}$ if for every regular C-hypersubstitution $\sigma_R \in M$ the equation $\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t]$ is also a strong identity of $\mathcal{A}$. We write

$$\mathcal{A} \underset{sMh}{\models} s \approx t :\Leftrightarrow \forall \sigma_R \in M(\mathcal{A} \underset{s}{\models} \widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t]).$$

A strong identity is called a *strong M-hyperidentity* of a class K of partial algebras of type τ if it holds as strong M-hyperidentity in every partial algebra in K. In the case, if $M = Hyp_R^C(\tau)$, strong M-hyperidentities are called *strong hyperidentities*.

The relation

$$\underset{sMh}{\models} := \{(\mathcal{A}, s \approx t) \in PAlg(\tau) \times W_\tau^C(X)^2 | \ \forall \sigma_R \in M(\mathcal{A} \underset{s}{\models} \widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t])\}$$

induces the Galois connection $(H_M Id^s, H_M Mod^s)$ defined on subclasses K of $PAlg(\tau)$ and for sets $\Sigma \subseteq W_\tau^C(X)^2$ as follows:

$$\begin{aligned} H_M Id^s K &:= \{s \approx t \in W_\tau^C(X)^2 \mid \forall \mathcal{A} \in K(\mathcal{A} \underset{sMh}{\models} s \approx t)\}; \\ H_M Mod^s \Sigma &:= \{\mathcal{A} \in PAlg(\tau) \mid \forall s \approx t \in \Sigma(\mathcal{A} \underset{sMh}{\models} s \approx t)\}. \end{aligned}$$

A set $\Sigma \subseteq W_\tau^C(X)^2$ is called a *strong M-hyperequational theory* if there is a class K of partial algebras of type τ such that $\Sigma = H_M Id^s K$.

A class K of partial algebras of type τ is called a *strong M-hyperequational class* if there is a set Σ such that $K = H_M Mod^s \Sigma$.

Corollary 3.2.6 ([49]) *For every submonoid $\mathcal{M} \subseteq \mathcal{H}yp_R^C(\tau)$ the operators χ_M^A and χ_M^E form a conjugate pair of additive closure operators with respect to the relation $\underset{sMh}{\models}$.*

Let $V \subseteq PAlg(\tau)$ be a strong variety of partial algebras, so that $V = Mod^s\Sigma$ for some set $\Sigma \subseteq W_\tau^C(X) \times W_\tau^C(X)$. Then V is said to be *M-solid* if $\chi_M^A[V] = V$. If $M = Hyp_R^C(\tau)$, then V is called *solid*.

Theorem 3.2.7 ([49]) *Let $V \subseteq PAlg(\tau)$ be a strong variety of partial algebras and let $\Sigma \subseteq W_\tau^C(X)^2$ be a strong equational theory. Then the following propositions* (i)-(iv) *and* (i')-(iv') *are equivalent:*

(i) V is a strong M-hyperequational class, i.e. $V = H_M Mod^s H_M Id^s V$.

(ii) V is M-solid, i.e. $\chi_M^A[V] = V$.

(iii) $Id^s V = H_M Id^s V$, i.e. every strong identity of V is a strong M-hyperidentity of V.

(iv) $\chi_M^E[Id^s V] = Id^s V$.

And the following conditions are also equivalent

(i') $\Sigma = H_M Id^s H_M Mod^s \Sigma$.

(ii') $\chi_M^E[\Sigma] = \Sigma$.

(iii') $Mod^s \Sigma = H_M Mod^s \Sigma$.

(iv') $\chi_M^A[Mod^s \Sigma] = Mod^s \Sigma$.

Theorem 3.2.8 ([49]) *For any $K \subseteq PAlg(\tau)$ and for any $\Sigma \subseteq W_\tau^C(X)^2$ the following conditions hold:*

(i) $\chi_M^A[K] \subseteq Mod^s Id^s K \Rightarrow Mod^s Id^s K = H_M Mod^s H_M Id^s K$

and

(ii) $\chi_M^E[\Sigma] \subseteq Id^s Mod^s \Sigma \Rightarrow Id^s Mod^s \Sigma = H_M Id^s H_M Mod^s \Sigma.$

Chapter 4

Strong Regular n-full Varieties

This chapter refers to [18]. The chapter is divided into three sections. In Section 4.1 we define strong regular n-full identities in partial algebras of type τ and study the connections between the relations R_s, R_{sr} and R_{rnf}. In Section 4.2 and Section 4.3 we will characterize strong regular varieties of partial algebras where every strong regular n-full identity is a strong regular n-full hyperidentity.

4.1 Regular n-full Identities in Partial Algebras

N-full terms were studied in [18] and are defined in the following way:

Let $n \in \mathbb{N}^+$ and let $\tau = (n_i)_{i \in I}$ be a type with corresponding operation symbols $(f_i)_{i \in I}$ for some index set I. We define an *n-full term* as follows:

(i) $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)})$ is an n-full term of type τ for every function $\alpha \in H_{n_i,n}$ where $H_{n_i,n}$ is the set of all functions from the set $\{1, \ldots, n_i\}$ into the set $\{1, \ldots, n\}$.

(ii) If $t_1, \ldots, t_n$ are n-full terms of type τ, then $f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n_i)})$ is an n-full term of type τ for every $\alpha \in H_{n_i,n}$.

Let $W_\tau^{nF}(X_n)$ be the set of all n-full terms of type τ.

For every partial algebra $\mathcal{A}$ of type τ and every n-full term t the n-full term operation $t^{\mathcal{A}}$ on $\mathcal{A}$ is defined as follows:

(i) If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)})$ for $\alpha \in H_{n_i,n}$, then $t^{\mathcal{A}}(a_1, \ldots, a_n) = (f_i^A)_\alpha(a_1, \ldots, a_n) := f_i^A(a_{\alpha(1)}, \ldots, a_{\alpha(n_i)})$ for $(a_{\alpha(1)}, \ldots, a_{\alpha(n_i)}) \in dom f_i^A$.

(ii) If $t = f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n_i)})$ and assume that $t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}$ are the term operations induced by the terms $t_1, \ldots, t_n$ and that $t_j^{\mathcal{A}}(a_1, \ldots, a_n)$ are defined, with values $t_j^{\mathcal{A}}(a_1, \ldots, a_n) = b_j$ for $1 \leq j \leq n$. If $f_i^A(b_{\alpha(1)}, \ldots, b_{\alpha(n_i)})$ where $b_{\alpha(1)}, \ldots, b_{\alpha(n_i)} \in \{b_1, \ldots, b_n\}$ is defined, then $t^{\mathcal{A}}(a_1, \ldots, a_n)$ is defined and

$$\begin{aligned} t^{\mathcal{A}}(a_1, \ldots, a_n) &= [f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n_i)})]^{\mathcal{A}}(a_1, \ldots, a_n) \\ &= f_i^A(t_{\alpha(1)}^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, t_{\alpha(n_i)}^{\mathcal{A}}(a_1, \ldots, a_n)). \end{aligned}$$

The superposition of n-full terms is defined as follows:

For $n \in \mathbb{N}^+$, we define an operation: $S^n : W_\tau^{nF}(X_n)^{n+1} \to W_\tau^{nF}(X_n)$ as follows:

$$\begin{aligned} &\text{(i) } S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)}), t_1, \ldots, t_n) &:=\ & f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n_i)}), \\ &\text{(ii) } S^n(f_i(s_{\alpha(1)}, \ldots, s_{\alpha(n_i)}), t_1, \ldots, t_n) &:=\ & f_i(S^n(s_{\alpha(1)}, t_1, \ldots, t_n), \ldots, \\ &&& S^n(s_{\alpha(n_i)}, t_1, \ldots, t_n)), \end{aligned}$$

where $s_1, \ldots, s_n \in W_\tau^{nF}(X_n)$ and $\alpha \in H_{n_i,n}$.

$$\begin{aligned} &\text{Let} \quad n - clone^{nF}(\tau) &:=\ & (W_\tau^{nF}(X_n); S^n) \\ &\text{and} \quad clone^{nF}(\tau) &:=\ & ((W_\tau^{nF}(X_n))_{n>0}; (S^n)_{n>0}). \end{aligned}$$

Proposition 4.1.1 ([18]) *The algebra $n - clone^{nF}(\tau)$ is a Menger algebra of rank n.*

Clearly, the operation S^n can also be defined on the set $W_\tau(X_n)$ of all $n-ary$ terms. This gives an algebra $(W_\tau(X_n); S^n)$ which also satisfies (C1). In [18] was proved that $n - clone^{nF}(\tau)$ is a subalgebra of $(W_\tau(X_n); S^n)$.

Now we consider the following set of equations: $W_\tau^{nF}(X_n)^2 \cap W_\tau^r(X_n)^2 := W_\tau^{RNF}(X_n)^2$.

An equation $s \approx t \in W_\tau(X)^2$ is called a *regular n-full identity* in a partial algebra $\mathcal{A}$ (in symbols $\mathcal{A} \underset{rnf}{\models} s \approx t$) iff $\mathcal{A} \underset{s}{\models} s \approx t$ and $s \approx t \in W_\tau^{RNF}(X_n)^2$.

Let $K \subseteq PAlg(\tau)$ be a class of partial algebras of type τ and $\Sigma \subseteq W_\tau(X)^2$. Consider the connection between $PAlg(\tau)$ and $W_\tau(X)^2$ given by the following two operators:

$$Id^{rnf} : \mathcal{P}(PAlg(\tau)) \to \mathcal{P}(W_\tau(X)^2) \qquad \text{and}$$

$$Mod^{rnf} : \mathcal{P}(W_\tau(X)^2) \to \mathcal{P}(PAlg(\tau)) \qquad \text{with}$$

$$\begin{aligned} Id^{rnf}K &:= \{s \approx t \in W_\tau(X)^2 \mid \forall \mathcal{A} \in K\ (\mathcal{A} \underset{rnf}{\models} s \approx t)\} \qquad \text{and} \\ Mod^{rnf}\Sigma &:= \{\mathcal{A} \in PAlg(\tau) \mid \forall\ s \approx t \in \Sigma\ (\mathcal{A} \underset{rnf}{\models} s \approx t)\}. \end{aligned}$$

Clearly, the pair (Mod^{rnf}, Id^{rnf}) is a Galois connection between $PAlg(\tau)$ and $W_\tau(X)^2$. As usual for a Galois connection, we have two closure operators $Mod^{rnf}Id^{rnf}$ and $Id^{rnf}Mod^{rnf}$ and their sets of fixed points, i.e. the sets

$$\{\Sigma \subseteq W_\tau(X)^2 \mid Id^{rnf}Mod^{rnf}\Sigma = \Sigma\} \text{ and } \{K \subseteq PAlg(\tau) \mid Mod^{rnf}Id^{rnf}K = K\},$$

form two complete lattices $\mathcal{E}^{rnf}(\tau)$, $\mathcal{L}^{rnf}(\tau)$.

Let $V \subseteq PAlg(\tau)$ be a class of partial algebras. The class V is called a *strong ragular n-full variety* of partial algebras iff there is a set $\Sigma \subseteq W_\tau(X)^2$ of regular n-full identities in V such that $V = Mod^{rnf}\Sigma$.

Let $\Sigma \subseteq W_\tau(X_n)^2$ and consider a mapping $RNF^E : \mathcal{P}(W_\tau(X_n)^2) \to \mathcal{P}(W_\tau(X_n)^2)$ defined by $RNF^E : \Sigma \longmapsto RNF^E(\Sigma) := \Sigma \cap W_\tau^{RNF}(X_n)^2$.

Proposition 4.1.2 *RNF^E has the properties of a kernel operator on $W_\tau(X_n)^2$.*

Proof. (i) We prove that the operator RNF^E is intensive.
Since $\Sigma \cap W_\tau^{RNF}(X_n)^2 \subseteq \Sigma$ then $RNF^E(\Sigma) \subseteq \Sigma$.
(ii) We prove that the operator RNF^E is monotone.
Let $\Sigma_1, \Sigma_2 \subseteq W_\tau(X_n)^2$ and $\Sigma_1 \subseteq \Sigma_2$. Then $\Sigma_1 \cap W_\tau^{RNF}(X_n)^2 \subseteq \Sigma_2 \cap W_\tau^{RNF}(X_n)^2$ and $RNF^E(\Sigma_1) \subseteq RNF^E(\Sigma_2)$.
(iii) We prove that the operator RNF^E is idempotent.
We have $RNF^E(RNF^E(\Sigma)) = RNF^E(\Sigma \cap W_\tau^{RNF}(X_n)^2) = (\Sigma \cap W_\tau^{RNF}(X_n)^2) \cap W_\tau^{RNF}(X_n)^2 = \Sigma \cap W_\tau^{RNF}(X_n)^2 = RNF^E(\Sigma)$. ∎

Let $V \subseteq PAlg(\tau)$ and consider a mapping $RNF^A : \mathcal{P}(PAlg(\tau)) \to \mathcal{P}(PAlg(\tau))$ defined by $RNF^A : V \longmapsto RNF^A(V) := Mod^{sr}(Id^{sr}V \cap W_\tau^{RNF}(X_n)^2)$.

Proposition 4.1.3 *RNF^A has the properties of a closure operator on $PAlg(\tau)$.*

Proof. (i) We prove at first that the operator RNF^A is extensive.
Since $Id^{sr}V \cap W_\tau^{RNF}(X_n)^2 \subseteq Id^{sr}V$, then $V \subseteq Mod^{sr}Id^{sr}V \subseteq Mod^{sr}(Id^{sr}V \cap W_\tau^{RNF}(X_n)^2) = RNF^A(V)$.
(ii) We prove that the operator RNF^A is monotone. Let $V_1 \subseteq V_2$ then $Id^{sr}V_2 \subseteq Id^{sr}V_1$ and $Id^{sr}V_2 \cap W_\tau^{RNF}(X_n)^2 \subseteq Id^{sr}V_1 \cap W_\tau^{RNF}(X_n)^2$. So $RNF^A(V_1) =$

$Mod^{sr}(Id^{sr}V_1 \cap W_\tau^{RNF}(X_n)^2) \subseteq Mod^{sr}(Id^{sr}V_2 \cap W_\tau^{RNF}(X_n)^2) = RNF^A(V_2)$.

(iii) We prove that the operator RNF^A is idempotent. From (i) and (ii), we have $RNF^A(V) \subseteq RNF^A(RNF^A(V))$. Since $Id^{sr}Mod^{sr}$ is a closure operator, we have

$Id^{sr}V \cap W_\tau^{RNF}(X_n)^2 \subseteq Id^{sr}Mod^{sr}(Id^{sr}V \cap W_\tau^{RNF}(X_n)^2)$

$\Rightarrow Id^{sr}V \cap W_\tau^{RNF}(X_n)^2 \subseteq Id^{sr}Mod^{sr}(Id^{sr}V \cap W_\tau^{RNF}(X_n)^2) \cap W_\tau^{RNF}(X_n)^2$

$\Rightarrow Mod^{sr}(Id^{sr}Mod^{sr}(Id^{sr}V \cap W_\tau^{RNF}(X_n)^2) \cap W_\tau^{RNF}(X_n)^2) \subseteq Mod^{sr}(Id^{sr}V \cap W_\tau^{RNF}(X_n)^2)$

$\Rightarrow RNF^A(RNF^A(V)) \subseteq RNF^A(V)$. ■

Now we want to study the connections between the relations

$R_s := \{(\mathcal{A}, s \approx t) \in PAlg(\tau) \times W_\tau(X)^2 \mid \mathcal{A} \underset{s}{\models} s \approx t\}$,

$R_{sr} := \{(\mathcal{A}, s \approx t) \in PAlg(\tau) \times W_\tau(X)^2 \mid \mathcal{A} \underset{sr}{\models} s \approx t\}$ and

$R_{rnf} := \{(\mathcal{A}, s \approx t) \in PAlg(\tau) \times W_\tau(X)^2 \mid \mathcal{A} \underset{rnf}{\models} s \approx t\}$.

We have:

Proposition 4.1.4 *The relation R_{sr} is a Galois-closed subrelation of R_s.*

Proof. Clearly, $R_{sr} \subseteq R_s$. Let $K \subseteq PAlg(\tau)$ and $\Sigma \subseteq W_\tau(X)^2$ such that $Id^{sr}K = \Sigma$ and $Mod^{sr}\Sigma = K$. We will show that $Id^sK = \Sigma$ and $Mod^s\Sigma = K$. From $\Sigma = Id^{sr}K$ we have that all identities in Σ are regular and thus $Mod^s\Sigma = Mod^{sr}\Sigma$. Since $K = Mod^{sr}\Sigma$, then $K = Mod^s\Sigma$. From $\Sigma = Id^{sr}K$ and $Id^{sr}K \subseteq Id^sK$ there follows $\Sigma \subseteq Id^sK$. $K = Mod^{sr}\Sigma$ means that $\mathcal{A} \underset{sr}{\models} s \approx t$ for all $\mathcal{A} \in K$ and for all $s \approx t \in \Sigma$. Then

$s \approx t \in Id^sK$

$\Rightarrow \quad s \approx t \in Id^sMod^{sr}\Sigma$ by $K = Mod^{sr}\Sigma$

$\Rightarrow \quad Mod^{sr}\Sigma \underset{s}{\models} s \approx t$

$\Rightarrow \quad \mathcal{A} \underset{sr}{\models} s \approx t$ for all $\mathcal{A} \in K$

$\Rightarrow \quad s \approx t \in Id^{sr}K$

Since $Id^{sr}K = \Sigma$, then $s \approx t \in \Sigma$ and therefore $Id^sK \subseteq \Sigma$. ■

Proposition 4.1.5 *The relation R_{rnf} is a Galois-closed subrelation of R_s.*

Proof. Clearly, $R_{rnf} \subseteq R_s$. Let $K \subseteq PAlg(\tau)$ and $\Sigma \subseteq W_\tau(X)^2$ such that $Id^{rnf}K = \Sigma$ and $Mod^{rnf}\Sigma = K$. We will show that $Id^sK = \Sigma$ and $Mod^s\Sigma = K$. From $\Sigma = Id^{rnf}K$ we have that all identities in Σ are members of $W_\tau^{RNF}(X_n)^2$ and $Mod^s\Sigma = \{\mathcal{A} \in PAlg(\tau) \mid \forall s \approx t \in \Sigma\ (\mathcal{A} \underset{s}{\models} s \approx t)\}$. So $Mod^s\Sigma = Mod^{rnf}\Sigma = \{\mathcal{A} \in PAlg(\tau) \mid \forall s \approx t \in \Sigma\ (\mathcal{A} \underset{s}{\models} s \approx t, s \approx t \in W_\tau^{RNF}(X_n)^2)\}$. Since $K = Mod^{rnf}\Sigma$, then $K = Mod^s\Sigma$. From $\Sigma = Id^{rnf}K$ and $Id^{rnf}K \subseteq Id^sK$ there follows $\Sigma \subseteq Id^sK$. The equation $K = Mod^{rnf}\Sigma$ means that $\mathcal{A} \underset{rnf}{\models} s \approx t$ (i.e. $\mathcal{A} \underset{s}{\models} s \approx t$ and $s \approx t \in W_\tau^{RNF}(X_n)^2$) for all $\mathcal{A} \in K$ and for all $s \approx t \in \Sigma$. Then

$$
\begin{aligned}
& s \approx t \in Id^sK \\
& \quad \Rightarrow \quad s \approx t \in Id^sMod^{rnf}\Sigma \text{ by } K = Mod^{rnf}\Sigma \\
& \quad \Rightarrow \quad Mod^{rnf}\Sigma \underset{s}{\models} s \approx t \\
& \quad \Rightarrow \quad \mathcal{A} \underset{s}{\models} s \approx t \text{ for all } \mathcal{A} \in K = Mod^{rnf}\Sigma \text{ and } s \approx t \in W_\tau^{RNF}(X_n)^2 \\
& \quad \Rightarrow \quad \mathcal{A} \underset{rnf}{\models} s \approx t \text{ for all } \mathcal{A} \in K \\
& \quad \Rightarrow \quad s \approx t \in Id^{rnf}K.
\end{aligned}
$$

Since $Id^{rnf}K = \Sigma$, then $s \approx t \in \Sigma$ and therefore $Id^sK \subseteq \Sigma$. ∎

If R' is a Galois-closed subrelation of R, then the complete lattice obtained from R' is a complete sublattice of the complete lattice obtained from R and any complete sublattices of the original lattice arise in this way (see e.g. [28]).

4.2 Clones of n-full Terms over a Strong Variety

Now we prove that $Id^{rnf}V$ is a congruence relation on the Menger algebra $n-clone^{nF}(\tau)$ of rank n.

Theorem 4.2.1 *Let V be a strong regular n-full variety of partial algebras of type τ and let $Id^{rnf}V$ be the set of all regular n-full identities satisfied in V. Then $Id^{rnf}V$ is a congruence relation on $n-clone^{nF}(\tau)$.*

Proof. Clearly, $Id^{rnf}V$ is an equivalence relation on $n-clone^{nF}(\tau)$. At first we prove by induction on the complexity of the n-full term t that from $t_1 \approx s_1, \ldots, t_n \approx s_n \in Id^{rnf}V$ follows $S^n(t, t_1, \ldots, t_n) \approx S^n(t, s_1, \ldots, s_n) \in Id^{rnf}V$.

a) If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)})$ for some $\alpha \in H_{n_i,n}$ then

$$\begin{array}{rcl} S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)}), t_1, \ldots, t_n) & = & f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n_i)}) \quad \text{and} \\ S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)}), s_1, \ldots, s_n) & = & f_i(s_{\alpha(1)}, \ldots, s_{\alpha(n_i)}). \end{array}$$

Since $t_{\alpha(j)} \approx s_{\alpha(j)} \in Id^{rnf}V;\ j = 1, \ldots, n_i$, then

$f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n_i)}) \approx f_i(s_{\alpha(1)}, \ldots, s_{\alpha(n_i)}) \in Id^{rnf}V$ and

$$\begin{aligned} S^n(t, t_1, \ldots, t_n) &= S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)}), t_1, \ldots, t_n) \\ &\approx S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)}), s_1, \ldots, s_n) = S^n(t, s_1, \ldots, s_n) \in Id^{rnf}V. \end{aligned}$$

b) If $t = f_i(l_{\alpha(1)}, \ldots, l_{\alpha(n_i)})$ where $l_1, \ldots, l_n \in W_\tau^{nF}(X_n)$, for some $\alpha \in H_{n_i,n}$ and if we assume that $S^n(l_{\alpha(j)}, t_1, \ldots, t_n) \approx S^n(l_{\alpha(j)}, s_1, \ldots, s_n) \in Id^{rnf}V$ for $j = 1, \ldots, n_i$, then

$$\begin{array}{rl} & S^n(f_i(l_{\alpha(1)}, \ldots, l_{\alpha(n_i)}), t_1, \ldots, t_n) \\ = & f_i(S^n(l_{\alpha(1)}, t_1, \ldots, t_n), \ldots, S^n(l_{\alpha(n_i)}, t_1, \ldots, t_n)) \\ \approx & f_i(S^n(l_{\alpha(1)}, s_1, \ldots, s_n), \ldots, S^n(l_{\alpha(n_i)}, s_1, \ldots, s_n)) \\ = & S^n(f_i(l_{\alpha(1)}, \ldots, l_{\alpha(n_i)}), s_1, \ldots, s_n) \in Id^{rnf}V. \end{array}$$

The next step consists in showing that for n-full terms $s_1, \ldots, s_n$ we have

$$t \approx s \in Id^{rnf}V \ \Rightarrow S^n(t, s_1, \ldots, s_n) \approx S^n(s, s_1, \ldots, s_n) \in Id^{rnf}V.$$

Since $t \approx s \in Id^{rnf}V$ and $s_1, \ldots, s_n \in W_\tau^{nF}(X_n)$ we have $(S^n(t, s_1, \ldots, s_n), S^n(s, s_1, \ldots, s_n)) \in W_\tau^{RNF}(X_n)^2$. Since $t \approx s \in Id^{rnf}V$ and $Id^{rnf}V \subseteq Id^{sr}V$ we have $t \approx s \in Id^{sr}V$ and $S^n(t, s_1, \ldots, s_n) \approx S^n(s, s_1, \ldots, s_n) \in Id^{sr}V$ by [49]. Therefore we get $S^n(t, s_1, \ldots, s_n) \approx S^n(s, s_1, \ldots, s_n) \in Id^{rnf}V$.

Assume now that $t \approx s, t_1 \approx s_1, \ldots, t_n \approx s_n \in Id^{rnf}V$. Then $S^n(t, t_1, \ldots, t_n) \approx S^n(t, s_1, \ldots, s_n) \approx S^n(s, s_1, \ldots, s_n) \approx S^n(s, t_1, \ldots, t_n) \in Id^{rnf}V$. Then $Id^{rnf}V$ is a congruence relation on $n - clone^{nF}(\tau)$. ■

The quotient algebra $n - clone^{rnF}V := n - clone^{nF}(\tau)/Id^{rnf}V$ is also a Menger algebra of rank n.

In the next section we need an additional definition. For any n-full term $t \in W_\tau^{nF}(X_n)$ we denote by t_α the term which is formed from t by applying a mapping $\alpha : \{1, \ldots, n\} \to \{1, \ldots, n\}$ to the variables in t. This can be defined inductively by the following two steps (see [18]):

(i) If $t = f_i(x_{\beta(1)}, \ldots, x_{\beta(n_i)})$ and for some mapping $\beta \in H_{n_i,n}$, then $t_\alpha = f_i(x_{\alpha(\beta(1))}, \ldots, x_{\alpha(\beta(n_i))})$;

(ii) If $t = f_i(t_{\beta(1)}, \ldots, t_{\beta(n_i)})$ where $t_1, \ldots, t_n \in W_\tau^{nF}(X_n)$ and $\beta \in H_{n_i,n}$, then $t_\alpha = f_i((t_{\beta(1)})_\alpha, \ldots, (t_{\beta(n_i)})_\alpha)$.

Clearly, the term t_α is an n-full term.

Lemma 4.2.2 ([18]) *Let* $t, t_1, \ldots, t_n \in W_\tau^{nF}(X_n)$ *and let* $\alpha : \{1, \ldots, n\} \to \{1, \ldots, n\}$. *Then*

$$S^n(t, t_{\alpha(1)}, \ldots, t_{\alpha(n)}) = S^n(t_\alpha, t_1, \ldots, t_n).$$

4.3 N-full Hypersubstitutions and Hyperidentities

Let $n \geq 1$ be a natural number. An *NF-hypersubstitution* of type τ is a mapping from the set $\{f_i \mid i \in I\}$ of $n_i - ary$ operation symbols of type τ to the set $W_\tau^{nF}(X_n)$ of all n-full terms of type τ with the additional condition that for $n > n_i$ the image $\sigma(f_i)$ has to be n_i-ary (and therefore also n-ary).

Any NF-hypersubstitution σ induces a mapping $\widehat{\sigma}$ on the set $W_\tau^{nF}(X_n)$ of all $n_i - ary$ terms of the type, as follows

Let σ be an NF-hypersubstitution of type τ. Then σ induces a mapping $\widehat{\sigma} : W_\tau^{nF}(X_n) \longrightarrow W_\tau^{nF}(X_n)$, by setting (see[18]):

(i) $\widehat{\sigma}[t] := (\sigma(f_i))_{\alpha'}$ if $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)})$ where $\alpha' \in H_{n,n}$ is defined by $\alpha'(j) := \alpha(j)$ if $1 \leq j \leq min(n, n_i)$ and $\alpha'(j) = n$, otherwise, for some $\alpha \in H_{n_i,n}$.

(ii) $\widehat{\sigma}[t] := S^n(\sigma(f_i), \widehat{\sigma}[t_{\alpha'(1)}], \ldots, \widehat{\sigma}[t_{\alpha'(n)}])$ if $t = f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n_i)})$ where $t_1, \ldots, t_n \in W_\tau^{nF}(X_n)$ and for some $\alpha \in H_{n_i,n}$.

Let σ be an NF-hypersubstitution. We say that the NF-hypersubstitution σ is an *RNF-hypersubstitution* if $Var(\widehat{\sigma}[f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)})]) = \{x_{\alpha(1)}, \ldots, x_{\alpha(n_i)}\}$ for all $i \in I$ and for some $\alpha \in H_{n_i,n}$.

Let $Hyp^{RNF}(\tau)$ denote the set of all RNF-hypersubstitutions of type τ and let σ_{rnf} denote some member of $Hyp^{RNF}(\tau)$.

Proposition 4.3.1 *Let σ_{rnf} be a RNF-hypersubstitution of type τ. Then $Var(\widehat{\sigma}_{rnf}[t]) = Var(t)$ for all $t \in W_\tau^{nF}(X_n)$.*

Proof. We will give a proof by induction on the complexity of the term t.
(i) If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)})$ for some $\alpha \in H_{n_i,n}$ then

$$Var(t) = \{x_{\alpha(1)}, \ldots, x_{\alpha(n_i)}\} = Var(\widehat{\sigma}_{rnf}[t]).$$

(ii) If $t = f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n_i)})$ where $t_1, \ldots, t_n \in W_\tau^{nF}(X_n)$ for some $\alpha \in H_{n_i,n}$ and if we assume that $Var(t_j) = Var(\widehat{\sigma}_{rnf}[t_j]); j = 1, \ldots, n$, then

$$Var(t) = \bigcup_{k=1}^{n_i} Var(t_{\alpha(k)}) = \bigcup_{k=1}^{n_i} Var(\widehat{\sigma}_{rnf}[t_{\alpha(k)}]) = Var(\widehat{\sigma}_{rnf}[t]).$$

■

Lemma 4.3.2 *The extension $\widehat{\sigma}_{rnf}$ of an RNF-hypersubstitution σ_{rnf} of type τ is an endomorphism of the algebra $n - clone^{nF}(\tau)$.*

Proof. Let $t, t_1, \ldots, t_n \in W_\tau^{nF}(X_n)$. We will show that

$$\widehat{\sigma}_{rnf}[S^n(t, t_1, \ldots, t_n)] = S^n(\widehat{\sigma}_{rnf}[t], \widehat{\sigma}_{rnf}[t_1], \ldots, \widehat{\sigma}_{rnf}[t_n]).$$

(i) If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)})$ for some $\alpha \in H_{n_i,n}$, then

$$\begin{aligned}
\widehat{\sigma}_{rnf}[S^n(t, t_1, \ldots, t_n)] &= \widehat{\sigma}_{rnf}[S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)}), t_1, \ldots, t_n] \\
&= \widehat{\sigma}_{rnf}[f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n_i)})] \\
&= S^n(\sigma_{rnf}(f_i), \widehat{\sigma}_{rnf}[t_{\alpha'(1)}], \ldots, \widehat{\sigma}_{rnf}[t_{\alpha'(n)}]) \\
&= S^n((\sigma_{rnf}(f_i))_{\alpha'}, \widehat{\sigma}_{rnf}[t_1], \ldots, \widehat{\sigma}_{rnf}[t_n]) \\
&= S^n(\widehat{\sigma}_{rnf}[f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)})], \widehat{\sigma}_{rnf}[t_1], \ldots, \widehat{\sigma}_{rnf}[t_n]) \\
&= S^n(\widehat{\sigma}_{rnf}[t], \widehat{\sigma}_{rnf}[t_1], \ldots, \widehat{\sigma}_{rnf}[t_n]).
\end{aligned}$$

(ii) If $t = f_i(u_{\alpha(1)}, \ldots, u_{\alpha(n_i)})$ where $u_1, \ldots, u_n \in W_\tau^{nF}(X_n)$ for some $\alpha \in H_{n_i,n}$ and if we assume that $\widehat{\sigma}_{rnf}[S^n(u_{\alpha(j)}, t_1, \ldots, t_n)] = S^n(\widehat{\sigma}_{rnf}[u_{\alpha(j)}], \widehat{\sigma}_{rnf}[t_1], \ldots, \widehat{\sigma}_{rnf}[t_n])$ for all $j = 1, \ldots, n_i$, then

$\widehat{\sigma}_{rnf}[S^n(t, t_1, \ldots, t_n)]$

$= \widehat{\sigma}_{rnf}[S^n(f_i(u_{\alpha(1)}, \ldots, u_{\alpha(n_i)}), t_1, \ldots, t_n)]$

$= \widehat{\sigma}_{rnf}[f_i(S^n(u_{\alpha(1)}, t_1, \ldots, t_n), \ldots, S^n(u_{\alpha(n_i)}, t_1, \ldots, t_n))]$

$= S^n(\sigma_{rnf}(f_i), \widehat{\sigma}_{rnf}[S^n(u_{\alpha'(1)}, t_1, \ldots, t_n)], \ldots, \widehat{\sigma}_{rnf}[S^n(u_{\alpha'(n)}, t_1, \ldots, t_n)])$

$= S^n(\sigma_{rnf}(f_i), S^n(\widehat{\sigma}_{rnf}[u_{\alpha'(1)}], \widehat{\sigma}_{rnf}[t_1], \ldots, \widehat{\sigma}_{rnf}[t_n]), \ldots, S^n(\widehat{\sigma}_{rnf}[u_{\alpha'(n)}], \widehat{\sigma}_{rnf}[t_1], \ldots,$

$$\widehat{\sigma}_{rnf}[t_n]))$$
$$= S^n(S^n(\sigma_{rnf}(f_i), \widehat{\sigma}_{rnf}[u_{\alpha'(1)}], \ldots, \widehat{\sigma}_{rnf}[u_{\alpha'(n)}]), \widehat{\sigma}_{rnf}[t_1], \ldots, \widehat{\sigma}_{rnf}[t_n])$$
$$= S^n(\widehat{\sigma}_{rnf}[f_i(u_{\alpha(1)}, \ldots, u_{\alpha(n_i)})], \widehat{\sigma}_{rnf}[t_1], \ldots, \widehat{\sigma}_{rnf}[t_n])$$
$$= S^n(\widehat{\sigma}_{rnf}[t], \widehat{\sigma}_{rnf}[t_1], \ldots, \widehat{\sigma}_{rnf}[t_n]). \qquad \blacksquare$$

On $Hyp^{RNF}(\tau)$ we define a binary operation by

$$\sigma_{rnf_1} \circ_h \sigma_{rnf_2} := \widehat{\sigma}_{rnf_1} \circ \sigma_{rnf_2}.$$

From ([26]) follows that for any two hyperstitutions of type τ we have $(\sigma_1 \circ_h \sigma_2)^\frown = \widehat{\sigma}_1 \circ \widehat{\sigma}_2$.

Proposition 4.3.3 *For $n_i \leq n$ and let $\sigma_{rnf_{id}}(f_i) = f_i(x_1, \ldots, x_{n_i})$. Then $\widehat{\sigma}_{rnf_{id}}[t] = t$ for all $t \in W_\tau^{nF}(X_n)$.*

Proof. We will give a proof by induction on the complexity of the term t.
(i) If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)})$ for some $\alpha \in H_{n_i,n}$, then

$$\widehat{\sigma}_{rnf_{id}}[t] = (\sigma_{rnf_{id}}(f_i))_{\alpha'} = (f_i(x_1, \ldots, x_{n_i}))_{\alpha'} = f_i(x_{\alpha'(1)}, \ldots, x_{\alpha'(n_i)}) = t.$$

(ii) If $t = f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n_i)})$ where $t_1, \ldots, t_n \in W_\tau^{nF}(X_n)$ for some $\alpha \in H_{n_i,n}$ and if we assume that $\widehat{\sigma}_{rnf_{id}}[t_j] = t_j; j = 1, \ldots, n$, then

$$\begin{aligned} \widehat{\sigma}_{rnf_{id}}[t] &= S^n(\sigma_{rnf_{id}}(f_i), \widehat{\sigma}_{rnf_{id}}[t_{\alpha(1)}], \ldots, \widehat{\sigma}_{rnf_{id}}[t_{\alpha(n_i)}]) \\ &= S^n(f_i(x_1, \ldots, x_{n_i}), t_{\alpha(1)}, \ldots, t_{\alpha(n_i)}) \\ &= f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n_i)}) \\ &= t. \end{aligned}$$

$\blacksquare$

Theorem 4.3.4 *The algebra $\mathcal{H}yp^{RNF}(\tau) := (Hyp^{RNF}(\tau); \circ_h)$ is a semigroup.*

Proof. We have to prove that the product of two RNF-hypersubstitutions of type τ belongs to the set of all RNF-hypersubstitutions of type τ. Let $\sigma_{rnf_1}, \sigma_{rnf_2} \in Hyp^{RNF}(\tau)$. Then

$$\begin{aligned} &Var((\sigma_{rnf_1} \circ_h \sigma_{rnf_2})^\frown[f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)})]) \\ &\quad= Var(\widehat{\sigma}_{rnf_1}[\widehat{\sigma}_{rnf_2}[f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)})]]) \\ &\quad= Var(\widehat{\sigma}_{rnf_2}[f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)})]) \\ &\quad= Var(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)})) \\ &\quad= \{x_{\alpha(1)}, \ldots, x_{\alpha(n_i)}\} \text{ by Proposition 4.3.1.} \end{aligned}$$

$\blacksquare$

Remark 4.3.5 The semigroup $\mathcal{H}yp^{RNF}(\tau)$ in general has no identity element.

Consider the following case:

Let $\mathcal{A} \in PAlg(\tau)$ and $Hyp^{RNF}(\tau)$ be the subsemigroup of $Hyp(\tau)$. Let $t_1, t_2 \in W_\tau^{nF}(X_n)$. Then $t_1 \approx t_2 \in Id^{sr}\mathcal{A}$ is called a *strong regular n-full hyperidentity* (SRNF-hyperidentity) in $\mathcal{A}$ (in symbols $\mathcal{A} \underset{sRNFh}{\models} t_1 \approx t_2$) if for all $\sigma_{rnf} \in Hyp^{RNF}(\tau)$ we have $\widehat{\sigma}_{rnf}[t_1] \approx \widehat{\sigma}_{rnf}[t_2] \in Id^{sr}\mathcal{A}$.

Let $K \subseteq PAlg(\tau)$ be a class of partial algebras of type τ and $\Sigma \subseteq W_\tau^{nF}(X_n)^2$. Consider the connection between $PAlg(\tau)$ and $W_\tau^{nF}(X_n)^2$ given by the following two operators.

$$H_{RNF}Id^{sr} : \mathcal{P}(PAlg(\tau)) \to \mathcal{P}(W_\tau^{nF}(X_n)^2) \qquad \text{and}$$

$$H_{RNF}Mod^{sr} : \mathcal{P}(W_\tau^{nF}(X_n)^2) \to \mathcal{P}(PAlg(\tau)) \qquad \text{with}$$

$$\begin{array}{lcl} H_{RNF}Id^{sr}K & := & \{s \approx t \in W_\tau^{nF}(X_n)^2 \mid \forall \mathcal{A} \in K \ (\mathcal{A} \underset{sRNFh}{\models} s \approx t)\} \quad \text{and} \\ H_{RNF}Mod^{sr}\Sigma & := & \{\mathcal{A} \in PAlg(\tau) \mid \forall\ s \approx t \in \Sigma\ (\mathcal{A} \underset{sRNFh}{\models} s \approx t)\}. \end{array}$$

Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a partial algebra of type τ and $\mathcal{H}yp^{RNF}(\tau)$ be the subsemigroup of $\mathcal{H}yp(\tau)$, then we define the *derived algebra* $\sigma_{rnf}(\mathcal{A}) := (A; (\sigma_{rnf}(f_i)^{\mathcal{A}})_{i\in I})$ for $\sigma_{rnf} \in Hyp^{RNF}(\tau)$.

Lemma 4.3.6 *Let $t \in W_\tau^{nF}(X_n)$, $\mathcal{A} \in PAlg(\tau)$ and $\sigma_{rnf} \in Hyp^{RNF}(\tau)$. Then*

$$(\widehat{\sigma}_{rnf}[t])^{\mathcal{A}}|D = t^{\sigma_{rnf}(\mathcal{A})}|D.$$

where D is the common domain of both sides.

Proof. We will give a proof by induction on the complexity of the term t.
(i) If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)})$ for some $\alpha \in H_{n_i,n}$, then

$$\begin{aligned} (\widehat{\sigma}_{rnf}[t])^{\mathcal{A}} &= ((\sigma_{rnf}(f_i))_{\alpha'})^{\mathcal{A}} \\ &= ((\sigma_{rnf}(f_i))^{\mathcal{A}})_{\alpha'} \\ &= (f_i^{\sigma_{rnf}(\mathcal{A})})_{\alpha'} = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n_i)})^{\sigma_{rnf}(\mathcal{A})}. \end{aligned}$$

(ii) If $t = f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n_i)})$ where $t_1, \ldots, t_n \in W_\tau^{nF}(X_n)$ for some $\alpha \in H_{n_i,n}$ and if we assume that $\widehat{\sigma}_{rnf}[t_j]^{\mathcal{A}} \mid_D = t_j^{\sigma_{rnf}(\mathcal{A})} \mid_D$ for $j = 1, \ldots, n$ and $D = \bigcap\limits_{j=1}^{n} dom\widehat{\sigma}_{rnf}[t_j]^{\mathcal{A}}$, then

$$\begin{array}{rcl}
(\widehat{\sigma}_{rnf}[t]^{\mathcal{A}}) \mid_D & = & [S^n(\sigma_{rnf}(f_i), \widehat{\sigma}_{rnf}[t_{\alpha'(1)}], \ldots, \widehat{\sigma}_{rnf}[t_{\alpha'(n)}])]^{\mathcal{A}} \mid_D \\
& = & S^{n,A}(\sigma_{rnf}(f_i)^{\mathcal{A}}, \widehat{\sigma}_{rnf}[t_{\alpha'(1)}]^{\mathcal{A}}, \ldots, \widehat{\sigma}_{rnf}[t_{\alpha'(n)}]^{\mathcal{A}}) \mid_D \\
& = & S^{n,A}(\sigma_{rnf}(f_i)^{\mathcal{A}}, \widehat{\sigma}_{rnf}[t_{\alpha'(1)}]^{\mathcal{A}} \mid_D, \ldots, \widehat{\sigma}_{rnf}[t_{\alpha'(n)}]^{\mathcal{A}} \mid_D) \\
& = & S^{n,\sigma_{rnf}(\mathcal{A})}((f_i)^{\sigma_{rnf}(\mathcal{A})}, t_{\alpha'(1)}^{\sigma_{rnf}(\mathcal{A})} \mid_D, \ldots, t_{\alpha'(n)}^{\sigma_{rnf}(\mathcal{A})} \mid_D) \\
& = & S^{n,\sigma_{rnf}(\mathcal{A})}((f_i)^{\sigma_{rnf}(\mathcal{A})}, t_{\alpha'(1)}^{\sigma_{rnf}(\mathcal{A})}, \ldots, t_{\alpha'(n)}^{\sigma_{rnf}(\mathcal{A})}) \mid_D \\
& = & [f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n_i)})]^{\sigma_{rnf}(\mathcal{A})} \mid_D \\
& = & t^{\sigma_{rnf}(\mathcal{A})} \mid_D.
\end{array}$$

∎

Let $\mathcal{A}$ be a partial algebra of type τ and $\mathcal{H}yp^{RNF}(\tau)$ be the subsemigroup of $\mathcal{H}yp(\tau)$. Then

$$\chi^A_{RNF} : \mathcal{P}(PAlg(\tau)) \to \mathcal{P}(PAlg(\tau)) \text{ and}$$

$$\chi^E_{RNF} : \mathcal{P}(W^{nF}_\tau(X_n)^2) \to \mathcal{P}(W^{nF}_\tau(X_n)^2)$$

by

$$\begin{array}{rcll}
\chi^A_{RNF}[\mathcal{A}] & := & \{\sigma_{rnf}(\mathcal{A}) \mid \sigma_{rnf} \in Hyp^{RNF}(\tau)\} & \text{and} \\
\chi^E_{RNF}[s \approx t] & := & \{\widehat{\sigma}_{rnf}[s] \approx \widehat{\sigma}_{rnf}[t] \mid \sigma_{rnf} \in Hyp^{RNF}(\tau)\}. &
\end{array}$$

For $K \subseteq PAlg(\tau)$ be a class of partial algebras of type τ and $\Sigma \subseteq W^{nF}_\tau(X_n)^2$ we define $\chi^A_{RNF}[K] := \bigcup\limits_{\mathcal{A}\in K} \chi^A_{RNF}[\mathcal{A}]$ and $\chi^E_{RNF}[\Sigma] := \bigcup\limits_{s\approx t\in\Sigma} \chi^E_{RNF}[s \approx t]$.

Proposition 4.3.7 *For any $K, K' \subseteq PAlg(\tau)$ and $\Sigma, \Sigma' \subseteq W^{nF}_\tau(X_n)^2$ the following conditions hold:*

(i) *the operators χ^A_{RNF} and χ^E_{RNF} are additive operators on $PAlg(\tau)$ and $W^{nF}_\tau(X_n)^2$ respectively,*

(ii) $\Sigma \subseteq \Sigma' \Rightarrow \chi^E_{RNF}[\Sigma] \subseteq \chi^E_{RNF}[\Sigma']$,

(iii) $\chi^E_{RNF}[\chi^E_{RNF}[\Sigma]] \subseteq \chi^E_{RNF}[\Sigma]$,

(iv) $K \subseteq K' \Rightarrow \chi^A_{RNF}[K] \subseteq \chi^A_{RNF}[K']$,

(v) $\chi^A_{RNF}[\chi^A_{RNF}[K]] \subseteq \chi^A_{RNF}[K]$

and $(\chi^A_{RNF}, \chi^E_{RNF})$ *forms a conjugate pair with respect to the relation*

$$R := \{(\mathcal{A}, s \approx t) \in PAlg(\tau) \times W^{nF}_\tau(X_n)^2 \mid \mathcal{A} \underset{sr}{\models} s \approx t\}$$

i.e. for all $\mathcal{A} \in PAlg(\tau)$ *and for all* $s \approx t \in W^{nF}_\tau(X_n)^2$, *we have* $\chi^A_{RNF}[\mathcal{A}] \underset{sr}{\models} s \approx t$ *iff* $\mathcal{A} \underset{sr}{\models} \chi^E_{RNF}[s \approx t]$.

Proof. (i) It is clear from the definition that both, χ^A_{RNF} and χ^E_{RNF}, are additive operators.
(ii) Suppose $\Sigma \subseteq \Sigma' \subseteq W^{nF}_\tau(X_n)^2$, then

$$\chi^E_{RNF}[\Sigma] := \bigcup_{s\approx t\in\Sigma} \chi^E_{RNF}[s \approx t] \subseteq \bigcup_{s\approx t\in\Sigma'} \chi^E_{RNF}[s \approx t] =: \chi^E_{RNF}[\Sigma'].$$

(iii) Suppose $\sigma_{rnf_1}, \sigma_{rnf_2} \in Hyp^{RNF}(\tau)$ are two arbitrary RNF-hypersubstitutions and assume that $\widehat{\sigma}_{rnf_1}[\widehat{\sigma}_{rnf_2}[s]] \approx \widehat{\sigma}_{rnf_1}[\widehat{\sigma}_{rnf_2}[t]]$ is an identity from $\chi^E_{RNF}[\chi^E_{RNF}[\Sigma]]$. Let $\sigma_{rnf} \in Hyp^{RNF}(\tau)$ be a RNF-hypersubstitution with $\sigma_{rnf} := \sigma_{rnf_1} \circ_h \sigma_{rnf_2}$. Since $Hyp^{RNF}(\tau)$ is a semigroup it follows that $\sigma_{rnf} \in Hyp^{RNF}(\tau)$. Then we have $\widehat{\sigma}_{rnf}[s] = (\sigma_{rnf_1} \circ_h \sigma_{rnf_2})\widehat{\ }[s] = \widehat{\sigma}_{rnf_1}[\widehat{\sigma}_{rnf_2}[s]] \approx \widehat{\sigma}_{rnf_1}[\widehat{\sigma}_{rnf_2}[t]] = (\sigma_{rnf_1} \circ_h \sigma_{rnf_2})\widehat{\ }[t] = \widehat{\sigma}_{rnf}[t]$, i.e. $\widehat{\sigma}_{rnf}[s] \approx \widehat{\sigma}_{rnf}[t] \in \chi^E_{RNF}[\Sigma]$.
(iv) and (v) can be proved in a similar way.
Finally, we need to show that $\chi^A_{RNF}[\mathcal{A}] \underset{sr}{\models} s \approx t$ iff $\mathcal{A} \underset{sr}{\models} \chi^E_{RNF}[s \approx t]$. Indeed, we have

$$\begin{array}{ll} \chi^A_{RNF}[\mathcal{A}] \underset{sr}{\models} s \approx t & \\ \quad\Leftrightarrow & \forall \sigma_{rnf} \in Hyp^{RNF}(\tau)(\sigma_{rnf}(\mathcal{A}) \underset{sr}{\models} s \approx t) \\ \quad\Leftrightarrow & \forall \sigma_{rnf} \in Hyp^{RNF}(\tau)\ (s^{\sigma_{rnf}(\mathcal{A})}|D = t^{\sigma_{rnf}(\mathcal{A})}|D) \\ \quad\Leftrightarrow & \forall \sigma_{rnf} \in Hyp^{RNF}(\tau)\ (\widehat{\sigma}_{rnf}[s]^{\mathcal{A}}|D = \widehat{\sigma}_{rnf}[t]^{\mathcal{A}}|D) \\ & \text{by Lemma 4.3.6 (where } D \text{ is the common domain)} \\ \quad\Leftrightarrow & \forall \sigma_{rnf} \in Hyp^{RNF}(\tau)\ (\mathcal{A} \underset{sr}{\models} \widehat{\sigma}_{rnf}[s] \approx \widehat{\sigma}_{rnf}[t]) \\ \quad\Leftrightarrow & \mathcal{A} \underset{sr}{\models} \chi^E_{RNF}[s \approx t]. \end{array}$$

■

Theorem 4.3.8 *For all* $K \subseteq PAlg(\tau)$ *and for all* $\Sigma \subseteq W^{nF}_\tau(X_n)^2$, *the following properties hold:*

(i) $H_{RNF}Id^{sr}K = Id^{sr}\chi^{A}_{RNF}[K]$,

(ii) $\chi^{E}_{RNF}[H_{RNF}Id^{sr}K] \subseteq H_{RNF}Id^{sr}K$,

(iii) $\chi^{E}_{RNF}[H_{RNF}Id^{sr}K] \subseteq Id^{sr}K$,

(iv) $\chi^{A}_{RNF}[Mod^{sr}H_{RNF}Id^{sr}K] \subseteq H_{RNF}Mod^{sr}H_{RNF}Id^{sr}K$,

(v) $Mod^{sr}Id^{sr}\chi^{A}_{RNF}[K] \subseteq H_{RNF}Mod^{sr}H_{RNF}Id^{sr}K$, *and dually,*

(i') $H_{RNF}Mod^{sr}\Sigma = Mod^{sr}\chi^{E}_{RNF}[\Sigma]$,

(ii') $\chi^{A}_{RNF}[H_{RNF}Mod^{sr}\Sigma] \subseteq H_{RNF}Mod^{sr}\Sigma$,

(iii') $\chi^{A}_{RNF}[H_{RNF}Mod^{sr}\Sigma] \subseteq Mod^{sr}\Sigma$,

(iv') $\chi^{E}_{RNF}[Id^{sr}H_{RNF}Mod^{sr}\Sigma] \subseteq H_{RNF}Id^{sr}H_{RNF}Mod^{sr}\Sigma$,

(v') $Id^{sr}Mod^{sr}\chi^{E}_{RNF}[\Sigma] \subseteq H_{RNF}Id^{sr}H_{RNF}Mod^{sr}\Sigma$.

Proof. We will prove only (i)-(v), the proofs of the other propositions being dual.

(i) $H_{RNF}Id^{sr}K$
$= \{s \approx t \in W^{nF}_{\tau}(X_n)^2 \mid \forall \mathcal{A} \in K\ (\mathcal{A} \underset{sRNFh}{\models} s \approx t)\}$
$= \{s \approx t \in W^{nF}_{\tau}(X_n)^2 \mid \forall \mathcal{A} \in K, \forall \sigma_{rnf} \in Hyp^{RNF}(\tau)\ (\mathcal{A} \underset{sr}{\models} \widehat{\sigma}_{rnf}[s] \approx \widehat{\sigma}_{rnf}[t])\}$
$= \{s \approx t \in W^{nF}_{\tau}(X_n)^2 \mid \forall \mathcal{A} \in K, \forall \sigma_{rnf} \in Hyp^{RNF}(\tau)\ (\sigma_{rnf}(\mathcal{A}) \underset{sr}{\models} s \approx t)\}$
$= Id^{sr}\chi^{A}_{RNF}[K]$.

(ii) Let $s \approx t \in \chi^{E}_{RNF}[H_{RNF}Id^{sr}K]$ then $s \approx t \in \chi^{E}_{RNF}[u \approx v]$ for some $u \approx v \in H_{RNF}Id^{sr}K$. By (i) we have $u \approx v \in Id^{sr}\chi^{A}_{RNF}[K]$ (i.e. $\chi^{A}_{RNF}[K] \underset{sr}{\models} u \approx v$) but $\chi^{A}_{RNF}[\chi^{A}_{RNF}[K]] \subseteq \chi^{A}_{RNF}[K]$ and then $\chi^{A}_{RNF}[\chi^{A}_{RNF}[K]] \underset{sr}{\models} u \approx v$ and $\chi^{A}_{RNF}[K] \underset{sr}{\models} \chi^{E}_{RNF}[u \approx v]$. Since $s \approx t \in \chi^{E}_{RNF}[u \approx v]$ we have $\chi^{A}_{RNF}[K] \underset{sr}{\models} s \approx t$ and $x \approx y \in Id^{sr}\chi^{A}_{RNF}[K]$. By (i) we have $s \approx t \in H_{RNF}Id^{sr}K$ and

$\chi^E_{RNF}[H_{RNF}Id^{sr}K] \subseteq H_{RNF}Id^{sr}K$.

(iii) $\chi^E_{RNF}[H_{RNF}Id^{sr}K] \subseteq Id^{sr}K$ by (ii) since $H_{RNF}Id^{sr}K \subseteq Id^{sr}K$.

(iv) From (ii) we obtain

$$\begin{array}{lll} & \chi^E_{RNF}[H_{RNF}Id^{sr}K] & \subseteq & H_{RNF}Id^{sr}K \\ \Rightarrow & Mod^{sr}H_{RNF}Id^{sr}K & \subseteq & Mod^{sr}\chi^E_{RNF}[H_{RNF}Id^{sr}K] \\ \Rightarrow & \chi^A_{RNF}[Mod^{sr}H_{RNF}Id^{sr}K] & \subseteq & \chi^A_{RNF}[Mod^{sr}\chi^E_{RNF}[H_{RNF}Id^{sr}K]]. \end{array}$$

Further we get

$$\begin{array}{llll} & \chi^A_{RNF}[Mod^{sr}H_{RNF}Id^{sr}K] & \subseteq & \chi^A_{RNF}[Mod^{sr}\chi^E_{RNF}[H_{RNF}Id^{sr}K]] \\ = & \chi^A_{RNF}[H_{RNF}Mod^{sr}H_{RNF}Id^{sr}K] & \subseteq & H_{RNF}Mod^{sr}H_{RNF}Id^{sr}K \text{ by (i') and (ii').} \end{array}$$

(v) From (i) we obtain

$$\begin{array}{llll} & Id^{sr}\chi^A_{RNF}[K] & = & H_{RNF}Id^{sr}K \\ \Rightarrow & Mod^{sr}Id^{sr}\chi^A_{RNF}[K] & = & Mod^{sr}H_{RNF}Id^{sr}K. \end{array}$$

From (ii) we get

$$\begin{array}{llll} & \chi^E_{RNF}[H_{RNF}Id^{sr}K] & \subseteq & H_{RNF}Id^{sr}K \\ \Rightarrow & Mod^{sr}H_{RNF}Id^{sr}K & \subseteq & Mod^{sr}\chi^E_{RNF}[H_{RNF}Id^{sr}K]. \end{array}$$

Then $Mod^{sr}Id^{sr}\chi^A_{RNF}[K]$

$$\begin{array}{ll} = & Mod^{sr}H_{RNF}Id^{sr}K \\ \subseteq & Mod^{sr}\chi^E_{RNF}[H_{RNF}Id^{sr}K] \\ = & H_{RNF}Mod^{sr}H_{RNF}Id^{sr}K \text{ by (i').} \end{array}$$

■

Theorem 4.3.9 *The operators $\chi^A_{RNF}, \chi^E_{RNF}$ satisfy the conditions in Proposition 4.3.7. For any $K \subseteq PAlg(\tau)$ with $Mod^{sr}Id^{sr}K = K$ and for any $\Sigma \subseteq W^{nF}_\tau(X_n)^2$ with $Id^{sr}Mod^{sr}\Sigma = \Sigma$ the following conditions* (i)-(iii) *and* (i')-(iii'), *respectively, are equivalent:*

(i) $Mod^{sr}H_{RNF}Id^{sr}K = K$,

(ii) $\chi^A_{RNF}[K] \subseteq K$,

(iii) $Id^{sr}K = H_{RNF}Id^{sr}K$,

(i') $Id^{sr}H_{RNF}Mod^{sr}\Sigma = \Sigma$,

(ii') $\chi^E_{RNF}[\Sigma] \subseteq \Sigma$,

(iii') $Mod^{sr}\Sigma = H_{RNF}Mod^{sr}\Sigma$.

Proof. We will only show the equivalence of (i),(ii) and (iii), the other equivalences can be shown analogously.

(i)⇒(ii) $\chi^A_{RNF}[K]$
$$\begin{aligned}&\subseteq Mod^{sr}Id^{sr}\chi^A_{RNF}[K]\\ &= Mod^{sr}H_{RNF}Id^{sr}K \text{ by Theorem 4.3.8(i)}\\ &= K \text{ by (i).}\end{aligned}$$

(ii)⇒(iii) From (ii) we have $Id^{sr}K \subseteq Id^{sr}\chi^A_{RNF}[K]$. Then $Id^{sr}K \subseteq Id^{sr}\chi^A_{RNF}[K] = H_{RNF}Id^{sr}K$ by Theorem 4.3.8(i). The converse inclusion is clear.

(iii)⇒(i) From (iii) we have

$$Mod^{sr}H_{RNF}Id^{sr}K \subseteq Mod^{sr}Id^{sr}K = K.$$

■

Theorem 4.3.10 *The operators $\chi^A_{RNF}, \chi^E_{RNF}$ satisfy the conditions in Proposition 4.3.7. Then for all $K \subseteq PAlg(\tau)$ and $\Sigma \subseteq W^{nF}_\tau(X_n)^2$, we have:*

(i)	$\chi^A_{RNF}[K] \subseteq Mod^{sr}Id^{sr}K$	$\Leftrightarrow$	$Mod^{sr}H_{RNF}Id^{sr}K \subseteq Mod^{sr}Id^{sr}K,$
(ii)	$H_{RNF}Mod^{sr}H_{RNF}Id^{sr}K = K$	$\Rightarrow$	$Mod^{sr}Id^{sr}K = K,$
(iii)	$H_{RNF}Mod^{sr}H_{RNF}Id^{sr}K \subseteq Mod^{sr}Id^{sr}K$	$\Rightarrow$	$\chi^A_{RNF}[K] \subseteq Mod^{sr}Id^{sr}K,$
(iv)	$\chi^A_{RNF}[Mod^{sr}Id^{sr}K] \subseteq Mod^{sr}Id^{sr}K$	$\Rightarrow$	$\chi^A_{RNF}[K] \subseteq Mod^{sr}Id^{sr}K,$
(i')	$\chi^E_{RNF}[\Sigma] \subseteq Id^{sr}Mod^{sr}\Sigma$	$\Leftrightarrow$	$Id^{sr}H_{RNF}Mod^{sr}\Sigma \subseteq Id^{sr}Mod^{sr}\Sigma,$
(ii')	$H_{RNF}Id^{sr}H_{RNF}Mod^{sr}\Sigma = \Sigma$	$\Rightarrow$	$Id^{sr}Mod^{sr}\Sigma = \Sigma,$
(iii')	$H_{RNF}Id^{sr}H_{RNF}Mod^{sr}\Sigma \subseteq Id^{sr}Mod^{sr}\Sigma$	$\Rightarrow$	$\chi^E_{RNF}[\Sigma] \subseteq Id^{sr}Mod^{sr}\Sigma,$
(iv')	$\chi^E_{RNF}[Id^{sr}Mod^{sr}\Sigma] \subseteq Id^{sr}Mod^{sr}\Sigma$	$\Rightarrow$	$\chi^E_{RNF}[\Sigma] \subseteq Id^{sr}Mod^{sr}\Sigma.$

Proof. (i) Assume $\chi^A_{RNF}[K] \subseteq Mod^{sr}Id^{sr}K$, then $Mod^{sr}Id^{sr}\chi^A_{RNF}[K] \subseteq Mod^{sr}Id^{sr}Mod^{sr}Id^{sr}K = Mod^{sr}Id^{sr}K$. From Theorem 4.3.8 (i), $H_{RNF}Id^{sr}K = Id^{sr}\chi^A_{RNF}[K]$ we get $Mod^{sr}H_{RNF}Id^{sr}K = Mod^{sr}Id^{sr}\chi^A_{RNF}[K]$. Therefore $Mod^{sr}H_{RNF}Id^{sr}K \subseteq Mod^{sr}Id^{sr}K$. Conversely, assume $Mod^{sr}H_{RNF}Id^{sr}K \subseteq Mod^{sr}Id^{sr}K$ and since $\chi^A_{RNF}[K] \subseteq Mod^{sr}Id^{sr}\chi^A_{RNF}[K]$ we get $\chi^A_{RNF}[K] \subseteq Mod^{sr}Id^{sr}\chi^A_{RNF}[K] = Mod^{sr}H_{RNF}Id^{sr}K \subseteq Mod^{sr}Id^{sr}K$ by Theorem 4.3.8 (i).

(ii) Assume $H_{RNF}Mod^{sr}H_{RNF}Id^{sr}K = K$, then $K = H_{RNF}Mod^{sr}H_{RNF}Id^{sr}K = Mod^{sr}\chi^E_{RNF}[H_{RNF}Id^{sr}K] \supseteq Mod^{sr}Id^{sr}K$ by Theorem 4.3.8 (i') and (iii). But $K \subseteq Mod^{sr}Id^{sr}K$ and then $Mod^{sr}Id^{sr}K = K$.

(iii) Assume $H_{RNF}Mod^{sr}H_{RNF}Id^{sr}K \subseteq Mod^{sr}Id^{sr}K$ and since $\chi^A_{RNF}[K] \subseteq Mod^{sr}Id^{sr}\chi^A_{RNF}[K]$ we get $\chi^A_{RNF}[K] \subseteq Mod^{sr}Id^{sr}\chi^A_{RNF}[K] \subseteq H_{RNF}Mod^{sr}H_{RNF}Id^{sr}K \subseteq Mod^{sr}Id^{sr}K$ by Theorem 4.3.8 (v).

(iv) Assume $\chi^A_{RNF}[Mod^{sr}Id^{sr}K] \subseteq Mod^{sr}Id^{sr}K$ and since $K \subseteq Mod^{sr}Id^{sr}K$, we get $\chi^A_{RNF}[K] \subseteq \chi^A_{RNF}[Mod^{sr}Id^{sr}K] \subseteq Mod^{sr}Id^{sr}K$. The proofs of (i'),(ii'),(iii') and (iv') are similar to the proofs of (i),(ii),(iii) and (iv), respectively. ■

Chapter 5

Strongly Full Varieties

In this chapter we consider a special case of strong regular n-full varieties. In Section 5.1 and Section 5.2, we define the concepts of strongly full terms and strongly full varieties. In Section 5.3 we give the definition of $clone^{SF}V$, of $n-SF-solid$ varieties and we show that V is $n-SF-solid$ if and only if $clone^{SF}V$ is free with respect to itself. In Section 5.4 we examine the connection between a strongly full variety V of partial algebras and the class $\{\mathcal{T}^{SF}(\mathcal{A})|\mathcal{A} \in V\}$ of n-ary strongly full term operations of its algebras.

5.1 Strongly full Terms

In the sequel we will consider a so-called n-ary type $\tau_n = (n, \ldots, n, \ldots)$ where all operation symbols are n-ary for $n \geq 1,\ n \in \mathbb{N}^+$.

Let $(f_i)_{i\in I}$ be an indexed set of n-ary operation symbols and let $X_n = \{x_1, \ldots, x_n\}$ be a set of variables. Then *n-ary strongly full* terms of type τ_n are defined inductively by the following steps:

(i) $f_i(x_1, \ldots, x_n)$ is a strongly full term of type τ_n,

(ii) If $t_1, \ldots, t_n$ are strongly full terms of type τ_n, then for every operation symbol f_i the term $f_i(t_1, \ldots, t_n)$ is strongly full.

Let $W^{SF}_{\tau_n}(X_n)$ be the set of all strongly full n-ary terms of type τ_n.

If we define $\overline{f_i} : W^{SF}_{\tau_n}(X_n)^n \to W^{SF}_{\tau_n}(X_n)$ by $\overline{f_i}(t_1, \ldots, t_n) := f_i(t_1, \ldots, t_n)$, then we get an algebra $\mathcal{F}^{SF}_{\tau_n}(X_n) = (W^{SF}_{\tau_n}(X_n); (\overline{f_i})_{i\in I})$ of type τ_n.

Another way to define operations on $W_{\tau_n}^{SF}(X_n)$ is to consider the so-called superposition operation S^n on $W_{\tau_n}^{SF}(X_n)$ defined by

$$
\begin{array}{lll}
\text{(i) } S^n(f_i(x_1,\ldots,x_n),t_1,\ldots,t_n) & := & f_i(t_1,\ldots,t_n),\\
\text{(ii) } S^n(f_i(s_1,\ldots,s_n),t_1,\ldots,t_n) & := & f_i(S^n(s_1,t_1,\ldots,t_n),\ldots,S^n(s_n,t_1,\ldots,t_n)).
\end{array}
$$

The operation $S^n : W_{\tau_n}^{SF}(X_n)^{n+1} \to W_{\tau_n}^{SF}(X_n)$ has the arity $n+1$. This gives an algebra $clone^{SF}\tau_n := (W_{\tau_n}^{SF}(X_n); S^n)$ of type $\tau = (n+1)$. (We should denote $clone^{SF}\tau_n$ better by $n - clone^{SF}\tau_n$, but for abbreviation we write $clone^{SF}\tau_n$).

Then we can prove:

Proposition 5.1.1 *The algebra $clone^{SF}\tau_n$ is a Menger algebra of rank n.*

It can be proved by induction on the complexity of the term that $clone^{SF}\tau_n$ satisfies the axiom $(C1)$ (see [8]).

Another way to obtain a Menger algebra is to consider the superposition of partial operations. The operation $S^{n,A}$ is a total operation defined on sets of partial operations. Then we prove:

Theorem 5.1.2 *The algebra $(P^n(A); S^{n,A})$ is a Menger algebra of rank n.*

Proof. Let $f^A, g_1^A, \ldots, g_n^A, h_1^A, \ldots, h_n^A \in P^n(A)$. At first we will prove that $(a_1,\ldots,a_n) \in domS^{n,A}(S^{n,A}(f^A, g_1^A,\ldots,g_n^A), h_1^A,\ldots,h_n^A)$ iff

$(a_1,\ldots,a_n) \in domS^{n,A}(f^A, S^{n,A}(g_1^A, h_1^A,\ldots,h_n^A),\ldots,S^{n,A}(g_n^A,h_1^A,\ldots,h_n^A))$.

Indeed, we have

$$
\begin{array}{ll}
\multicolumn{2}{l}{domS^{n,A}(S^{n,A}(f^A, g_1^A,\ldots,g_n^A), h_1^A,\ldots,h_n^A)}\\
= & \{(a_1,\ldots,a_n) \mid (a_1,\ldots,a_n) \in \bigcap\limits_{j=1}^{n} domh_j^A \text{ and if } h_j^A(a_1,\ldots,a_n) = b_j,\\
& \text{then } (b_1,\ldots,b_n) \in domS^{n,A}(f^A, g_1^A,\ldots,g_n^A) \text{ and } (b_1,\ldots,b_n) \in \bigcap\limits_{k=1}^{n} domg_k^A\\
& \text{and if } g_k^A(b_1,\ldots,b_n) = c_k, \text{ then } (c_1,\ldots,c_n) \in domf^A\}\\
= & \{(a_1,\ldots,a_n) \mid (a_1,\ldots,a_n) \in \bigcap\limits_{j=1}^{n} domh_j^A \text{ and if } h_j^A(a_1,\ldots,a_n) = b_j,\\
& \text{then } (b_1,\ldots,b_n) \in \bigcap\limits_{k=1}^{n} domg_k^A \text{ and if } g_k^A(b_1,\ldots,b_n) = c_k,\\
& \text{then } (c_1,\ldots,c_n) \in domf^A\}
\end{array}
$$

$$\begin{aligned}
&= \{(a_1,\ldots,a_n) \mid (a_1,\ldots,a_n) \in \bigcap_{k=1}^{n} domS^{n,A}(g_k^A, h_1^A,\ldots,h_n^A) \text{ and} \\
&\quad \text{if } g_k^A(h_1^A(a_1,\ldots,a_n),\ldots,h_n^A(a_1,\ldots,a_n)) = c_k, \text{ then } (c_1,\ldots,c_n) \in domf^A\} \\
&= domS^{n,A}(f^A, S^{n,A}(g_1^A,h_1^A,\ldots,h_n^A),\ldots,S^{n,A}(g_n^A,h_1^A,\ldots,h_n^A)).
\end{aligned}$$

Now we prove that if both sides are defined, then they are equal.

$S^{n,A}(S^{n,A}(f^A, g_1^A,\ldots,g_n^A), h_1^A,\ldots,h_n^A)(a_1,\ldots,a_n)$
$=S^{n,A}(f^A, S^{n,A}(g_1^A,h_1^A,\ldots,h_n^A),\ldots,S^{n,A}(g_n^A,h_1^A,\ldots,h_n^A))(a_1,\ldots,a_n)$. ■

Every $n-ary$ strongly full term $t \in W_{\tau_n}^{SF}(X_n)$ induces an n-ary term operation $t^{\mathcal{A}}$ on any partial algebra $\mathcal{A} = (A;(f_i^A)_{i\in I})$ of type τ_n, in the following inductive way:

(i) If $t = f_i(x_1,\ldots,x_n)$ then $t^{\mathcal{A}} := [f_i(x_1,\ldots,x_n)]^{\mathcal{A}} = f_i^A$.

(ii) If $t = f_i(t_1,\ldots,t_n)$ and assume that $t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}$ are the term operations induced by the terms $t_1,\ldots,t_n$ and that $t_j^{\mathcal{A}}(a_1,\ldots,a_n)$ are defined, with values $t_j^{\mathcal{A}}(a_1,\ldots,a_n) = b_j$, for $1 \le j \le n$. If $f_i^A(b_1,\ldots,b_n)$ is defined, then $t^{\mathcal{A}}(a_1,\ldots,a_n)$ is defined and
$$\begin{aligned}
t^{\mathcal{A}}(a_1,\ldots,a_n) &= [f_i(t_1,\ldots,t_n)]^{\mathcal{A}}(a_1,\ldots,a_n) \\
&:= S^{n,A}(f_i^A, t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}})(a_1,\ldots,a_n) \\
&= f_i^A(t_1^{\mathcal{A}}(a_1,\ldots,a_n),\ldots,t_n^{\mathcal{A}}(a_1,\ldots,a_n)).
\end{aligned}$$

Let $W_{\tau_n}^{SF}(X_n)^{\mathcal{A}}$ be the set of all n-ary term operations of the partial algebra $\mathcal{A}$ induced by strongly full n-ary terms.

Theorem 5.1.3 *The algebra $(W_{\tau_n}^{SF}(X_n)^{\mathcal{A}}; S^{n,A})$ is a Menger algebra of rank n.*

The theorem can be proved by induction on the complexity of terms. We have to prove that $(W_{\tau_n}^{SF}(X_n)^{\mathcal{A}}; S^{n,A})$ satisfies the axiom $(C1)$ (see [8]).

5.2 Strongly full Varieties of Partial Algebras

Let $\mathcal{A} = (A;(f_i^A)_{i\in I})$ be a partial algebra of type τ_n and let $s \approx t$ be an equation of strongly full $n-ary$ terms $s,t \in W_{\tau_n}^{SF}(X_n)$.

The equation $s \approx t$ is called a *strongly full identity* satisfied in the partial algebra $\mathcal{A}$ if $s^{\mathcal{A}} = t^{\mathcal{A}}$ for the term operations $s^{\mathcal{A}}$ and $t^{\mathcal{A}}$ induced by the terms s and t,

respectively. In this case we write $\mathcal{A} \underset{sf}{\models} s \approx t$. (This is, $s \approx t$ is a strongly full identity iff the right hand side is defined whenever the left hand side is defined and both are equal).
By $Id^{SF}\mathcal{A}$ we denote the set of all strongly full identities satisfied in $\mathcal{A}$. For a class $K \subseteq PAlg(\tau_n)$ of partial algebras of type τ_n we write $Id^{SF}K$. If $\Sigma \subseteq W^{SF}_{\tau_n}(X_n)^2$ is a set of strongly full equations, then we can ask for the class of all partial algebras satisfying every $s \approx t \in \Sigma$ as a strongly full identity and call this class $Mod^{SF}\Sigma$. Let $K \subseteq PAlg(\tau_n)$ be a class of partial algebras of type τ_n and $\Sigma \subseteq W^{SF}_{\tau_n}(X_n)^2$. Consider the connection between $PAlg(\tau_n)$ and $W^{SF}_{\tau_n}(X_n)^2$ given by the following two operators.

$$Id^{SF} : \mathcal{P}(PAlg(\tau_n)) \to \mathcal{P}(W^{SF}_{\tau_n}(X_n)^2) \text{ and}$$

$$Mod^{SF} : \mathcal{P}(W^{SF}_{\tau_n}(X_n)^2) \to \mathcal{P}(PAlg(\tau_n)) \text{ with}$$

$Id^{SF}K := \{(s,t) \in W^{SF}_{\tau_n}(X_n)^2 \mid \forall \mathcal{A} \in K \ (\mathcal{A} \underset{sf}{\models} s \approx t)\}$ and
$Mod^{SF}\Sigma := \{\mathcal{A} \in PAlg(\tau_n) \mid \forall\ (s,t) \in \Sigma\ (\mathcal{A} \underset{sf}{\models} s \approx t)\}$.
Clearly, the pair (Mod^{SF}, Id^{SF}) is a Galois connection between $PAlg(\tau_n)$ and $W^{SF}_{\tau_n}(X_n)^2$.
As usual for a Galois connection, we have two closure operators $Mod^{SF}Id^{SF}$ and $Id^{SF}Mod^{SF}$ and their sets of fixed points, i.e. the sets
$\{\Sigma \subseteq W^{SF}_{\tau_n}(X_n)^2 \mid Id^{SF}Mod^{SF}\Sigma = \Sigma\}$ and $\{K \subseteq PAlg(\tau_n) \mid Mod^{SF}Id^{SF}K = K\}$, form two complete lattices $\mathcal{E}^{SF}(\tau_n)$, $\mathcal{L}^{SF}(\tau_n)$.

Let $V \subseteq PAlg(\tau_n)$ be a class of partial algebras. The class V is called a *strongly full variety* of partial algebras if $V = Mod^{SF}Id^{SF}V$. Let $\Sigma \subseteq W^{SF}_{\tau_n}(X_n)^2$ be a set of strongly full equations of type τ_n. Then Σ is called a *strongly full equational theory* if $\Sigma = Id^{SF}Mod^{SF}\Sigma$. (For more information on strong varieties of partial algebras see e.g. [48]).

Then from the property of a Galois connection, we have

Proposition 5.2.1 *V is a strongly full variety of partial algebras iff there exists a set $\Sigma \subseteq W^{SF}_{\tau_n}(X_n)^2$ such that $V = Mod^{SF}\Sigma$.*

We notice that strongly full terms, strongly full identities and strongly full varieties can be considered for arbitrary types τ.
Let $Id_n^{SF}K$ be the intersection of $Id^{SF}K$ and $W_{\tau_n}^{SF}(X_n)^2$.

Lemma 5.2.2 *Let $K \subseteq PAlg(\tau_n)$. Then $Id_n^{SF}K$ is a congruence relation on $\mathcal{F}_{\tau_n}^{SF}(X_n)$.*

Proof. Clearly, $Id_n^{SF}K$ is an equivalence relation on $\mathcal{F}_{\tau_n}^{SF}(X_n)$. The next step is to show that $\overline{f_i}$ is compatible with $Id_n^{SF}K$. Let $s_1 \approx t_1, \ldots, s_n \approx t_n \in Id_n^{SF}K$. Then $s_1 \approx t_1, \ldots, s_n \approx t_n \in Id_n^{SF}\mathcal{A}$ for all $\mathcal{A} \in K$ because of $Id_n^{SF}K = \bigcap\limits_{\mathcal{A}\in K} Id_n^{SF}\mathcal{A}$ and $doms_i^{\mathcal{A}} = domt_i^{\mathcal{A}},\ s_i^{\mathcal{A}}|_{doms_i^{\mathcal{A}}} = t_i^{\mathcal{A}}|_{domt_i^{\mathcal{A}}}$ for all $i = 1, \ldots, n$.
Let $D = \bigcap\limits_{i=1}^{n} dom\ s_i^{\mathcal{A}} = \bigcap\limits_{i=1}^{n} dom\ t_i^{\mathcal{A}}$ and

$$D' = \{(a_1, \ldots, a_n) \in D \text{ and } (s_1^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, s_n^{\mathcal{A}}(a_1, \ldots, a_n)) \in domf_i^A \text{ and } (t_1^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, t_n^{\mathcal{A}}(a_1, \ldots, a_n)) \in domf_i^A\}.$$

We have

$$f_i^A(s_1^{\mathcal{A}}|_{doms_1^{\mathcal{A}}}, \ldots, s_n^{\mathcal{A}}|_{doms_n^{\mathcal{A}}})|_{D'} = f_i^A(t_1^{\mathcal{A}}|_{domt_1^{\mathcal{A}}}, \ldots, t_n^{\mathcal{A}}|_{domt_n^{\mathcal{A}}})|_{D'}$$

$$\begin{array}{lll} \Rightarrow\ f_i^A(s_1^{\mathcal{A}}|_D, \ldots, s_n^{\mathcal{A}}|_D)|_{D'} & = & f_i^A(t_1^{\mathcal{A}}|_D, \ldots, t_n^{\mathcal{A}}|_D)|_{D'} \\ \Rightarrow\ f_i^A(s_1^{\mathcal{A}}, \ldots, s_n^{\mathcal{A}})|_{D'} & = & f_i^A(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})|_{D'} \\ \Rightarrow\ [f_i(s_1, \ldots, s_n)]^{\mathcal{A}}|_{D'} & = & [f_i(t_1, \ldots, t_n)]^{\mathcal{A}}|_{D'} \\ \Rightarrow\ [\overline{f_i}(s_1, \ldots, s_n)]^{\mathcal{A}}|_{D'} & = & [\overline{f_i}(t_1, \ldots, t_n)]^{\mathcal{A}}|_{D'} \end{array}$$

then $\overline{f_i}(s_1, \ldots, s_n) \approx \overline{f_i}(t_1, \ldots, t_n) \in Id_n^{SF}\mathcal{A}$ for all $\mathcal{A} \in K$.
So $\overline{f_i}(s_1, \ldots, s_n) \approx \overline{f_i}(t_1, \ldots, t_n) \in Id_n^{SF}K$. ∎

Now we prove that $Id_n^{SF}V$ is also a congruence relation on the algebra $clone^{SF}\tau_n$.

Lemma 5.2.3 *Let $s, t_1, \ldots, t_n \in W_{\tau_n}^{SF}(X_n)$ and $\mathcal{A} \in PAlg(\tau_n)$. Then $[S^n(s, t_1, \ldots, t_n)]^{\mathcal{A}}|_D = S^{n,A}(s^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})|_D$ where $D = \bigcap\limits_{i=1}^{n} dom\ t_i^{\mathcal{A}}$.*

The Lemma can be proved by induction on the complexity of terms (see [8]).

Theorem 5.2.4 *Let V be a strongly full variety of partial algebras of type τ_n and let $Id_n^{SF}V$ be the set of all strongly full n-ary identities satisfied in V. Then $Id_n^{SF}V$ is a congruence relation on $clone^{SF}\tau_n$.*

Proof. At first we prove by induction on the complexity of the term t that from $t_1 \approx s_1, \ldots, t_n \approx s_n \in Id_n^{SF}V$ follows $S^n(t, t_1, \ldots, t_n) \approx S^n(t, s_1, \ldots, s_n) \in Id_n^{SF}V$.

(a) If $t = f_i(x_1, \ldots, x_n)$ with $S^n(f_i(x_1, \ldots, x_n), t_1, \ldots, t_n) = f_i(t_1, \ldots, t_n)$ and $S^n(f_i(x_1, \ldots, x_n), s_1, \ldots, s_n) = f_i(s_1, \ldots, s_n)$, then $f_i(t_1, \ldots, t_n) \approx f_i(s_1, \ldots, s_n) \in Id_n^{SF}V$ and $S^n(t, t_1, \ldots, t_n) \approx S^n(t, s_1, \ldots, s_n) \in Id_n^{SF}V$.

(b) If $t = f_i(l_1, \ldots, l_n)$ and if we assume that $S^n(l_j, t_1, \ldots, t_n) \approx S^n(l_j, s_1, \ldots, s_n) \in Id_n^{SF}V$ for $j = 1, \ldots, n$, then

$$\begin{aligned} S^n(f_i(l_1, \ldots, l_n), t_1, \ldots, t_n) &= f_i(S^n(l_1, t_1, \ldots, t_n), \ldots, S^n(l_n, t_1, \ldots, t_n)) \\ &\approx f_i(S^n(l_1, s_1, \ldots, s_n), \ldots, S^n(l_n, s_1, \ldots, s_n)) \\ &= S^n(f_i(l_1, \ldots, l_n), s_1, \ldots, s_n) \in Id_n^{SF}V. \end{aligned}$$

The next step consists in showing that for strongly full terms $s_1, \ldots, s_n$ we have

$$t \approx s \in Id_n^{SF}V \ \Rightarrow S^n(t, s_1, \ldots, s_n) \approx S^n(s, s_1, \ldots, s_n) \in Id_n^{SF}V.$$

From $t \approx s \in Id_n^{SF}V = \bigcap\limits_{\mathcal{A} \in V} Id_n^{SF}\mathcal{A}$ we get $domt^A = doms^A$ and $t^A|_{domt^A} = s^A|_{doms^A}$. We will show that $S^n(t, s_1, \ldots, s_n) \approx S^n(s, s_1, \ldots, s_n) \in Id_n^{SF}V$. Let $D = \bigcap\limits_{i=1}^{n} dom\ s_i^{\mathcal{A}}$. Consider the following cases :

case 1. Let $t = f_i(x_1, \ldots, x_n)$ and $s = f_j(x_1, \ldots, x_n)$. Then $t^{\mathcal{A}} = f_i^A$ and $s^{\mathcal{A}} = f_j^A$.

Since $t \approx s \in Id_n^{SF}\mathcal{A}$, we have $f_i^A|_{D'} = f_j^A|_{D'}$ with

$D' = \{(a_1, \ldots, a_n) \in D$ and $(s_1^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, s_n^{\mathcal{A}}(a_1, \ldots, a_n)) \in dom\ f_i^A$ and $(s_1^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, s_n^{\mathcal{A}}(a_1, \ldots, a_n)) \in dom\ f_j^A\}$. Then

$$\begin{aligned} [S^n(t, s_1, \ldots, s_n)]^{\mathcal{A}}|_D &= S^{n,A}(f_i^A, s_1^{\mathcal{A}}, \ldots, s_n^{\mathcal{A}})|_D \\ &= S^{n,A}(f_j^A, s_1^{\mathcal{A}}, \ldots, s_n^{\mathcal{A}})|_D \\ &= [S^n(s, s_1, \ldots, s_n)]^{\mathcal{A}}|_D. \end{aligned}$$

case 2. Let $t = f_i(x_1, \ldots, x_n)$ and $s = f_j(l_1, \ldots, l_n)$ then $t^{\mathcal{A}} = f_i^A$ and $s^{\mathcal{A}} = S^{n,A}(f_j^A, l_1^{\mathcal{A}}, \ldots, l_n^{\mathcal{A}})$. Since $t \approx s \in Id_n^{SF}\mathcal{A}$, we have $f_i^A|_{D'} = S^{n,A}(f_j^A, l_1^{\mathcal{A}}, \ldots, l_n^{\mathcal{A}})|_{D'}$ with

$D' = \{(a_1, \ldots, a_n) \in D$ and $(s_1^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, s_n^{\mathcal{A}}(a_1, \ldots, a_n)) \in dom\ f_i^A$ and

$(s_1^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, s_n^{\mathcal{A}}(a_1, \ldots, a_n)) \in domS^{n,A}(f_j^A, l_1^{\mathcal{A}}, \ldots, l_n^{\mathcal{A}})\}$.

$$\begin{aligned}\text{Then } [S^n(t,s_1,\ldots,s_n)]^{\mathcal{A}}|_D &= S^{n,A}(f_i^A, s_1^{\mathcal{A}},\ldots,s_n^{\mathcal{A}})|_D\\ &= S^{n,A}(S^{n,A}(f_j^A, l_1^{\mathcal{A}},\ldots,l_n^{\mathcal{A}}), s_1^{\mathcal{A}},\ldots,s_n^{\mathcal{A}})|_D\\ &= [S^n(s,s_1,\ldots,s_n)]^{\mathcal{A}}|_D.\end{aligned}$$

case 3. For $t = f_i(t_1,\ldots,t_n)$ and $s = f_j(x_1,\ldots,x_n)$ we can give a similar proof as in case 2.

case 4. Let $t = f_i(t_1,\ldots,t_n)$ and $s = f_j(l_1,\ldots,l_n)$ then $t^{\mathcal{A}} = S^{n,A}(f_i^A, t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}})$ and $s^{\mathcal{A}} = S^{n,A}(f_j^A, l_1^{\mathcal{A}},\ldots,l_n^{\mathcal{A}})$. Since $t \approx s$ is $n-ary$ strongly full identity $\mathcal{A}$, we have $S^{n,A}(f_i^A, t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}})|_{D'} = S^{n,A}(f_j^A, l_1^{\mathcal{A}},\ldots,l_n^{\mathcal{A}}))|_{D'}$ with

$D' = \{(a_1,\ldots,a_n) \in D$ and $(s_1^{\mathcal{A}}(a_1,\ldots,a_n),\ldots,s_n^{\mathcal{A}}(a_1,\ldots,a_n)) \in dom\ S^{n,A}(f_i^A, t_1^{\mathcal{A}}$

$,\ldots,t_n^{\mathcal{A}})$ and $(s_1^{\mathcal{A}}(a_1,\ldots,a_n),\ldots,s_n^{\mathcal{A}}(a_1,\ldots,a_n)) \in dom\ S^{n,A}(f_j^A, l_1^{\mathcal{A}},\ldots,l_n^{\mathcal{A}})\}$.

$$\begin{aligned}\text{Then } [S^n(t,s_1,\ldots,s_n)]^{\mathcal{A}}|_D &= S^{n,A}([f_i(t_1,\ldots,t_n)]^{\mathcal{A}}, s_1^{\mathcal{A}},\ldots,s_n^{\mathcal{A}})|_D\\ &= S^{n,A}(S^{n,A}(f_i^A, t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}), s_1^{\mathcal{A}},\ldots,s_n^{\mathcal{A}})|_D\\ &= S^{n,A}(S^{n,A}(f_j^A, l_1^{\mathcal{A}},\ldots,l_n^{\mathcal{A}}), s_1^{\mathcal{A}},\ldots,s_n^{\mathcal{A}})|_D\\ &= S^{n,A}([f_j(l_1,\ldots,l_n)]^{\mathcal{A}}, s_1^{\mathcal{A}},\ldots,s_n^{\mathcal{A}})|_D\\ &= [S^n(s,s_1,\ldots,s_n)]^{\mathcal{A}}|_D.\end{aligned}$$

Therefore in all cases we get $S^n(t,s_1,\ldots,s_n) \approx S^n(s,s_1,\ldots,s_n) \in Id_n^{SF}\mathcal{A}$ for all $\mathcal{A} \in V$. Assume now that $t \approx s, t_1 \approx s_1,\ldots,t_n \approx s_n \in Id_n^{SF}V$. Then

$$S^n(t,t_1,\ldots,t_n) \approx S^n(t,s_1,\ldots,s_n) \approx S^n(s,s_1,\ldots,s_n) \approx S^n(s,t_1,\ldots,t_n) \in Id_n^{SF}V.$$

Clearly, $Id_n^{SF}V$ is an equivalence relation on $clone^{SF}\tau_n$. Then $Id_n^{SF}V$ is a congruence relation on $clone^{SF}\tau_n$. ■

The quotient algebra $clone^{SF}V := clone^{SF}\tau_n/Id_n^{SF}V$ belongs also to the variety V_{M_n} of Menger algebras of rank n. (Again we write $clone^{SF}V$ instead of $n - clone^{SF}V$)

5.3 Hypersubstitutions and Clone Substitutions

Now we consider a mapping from the set of operation symbols $\{f_i \mid i \in I\}$ to the set of all strongly full terms of type τ_n.

A *strongly full hypersubstitution* of type τ_n is a mapping from the set $\{f_i \mid i \in I\}$ of n-ary operation symbols of type τ_n to the set $W_{\tau_n}^{SF}(X_n)$ of all n-ary terms of type τ_n.

Any strongly full hypersubstitution σ induces a mapping $\widehat{\sigma}$ on the set $W^{SF}_{\tau_n}(X_n)$ of all n-ary terms of the type, as follows.

Let σ be a strongly full hypersubstitution of type τ_n. Then σ induces a mapping $\widehat{\sigma} : W^{SF}_{\tau_n}(X_n) \longrightarrow W^{SF}_{\tau_n}(X_n)$, by setting

(i) $\widehat{\sigma}[f_i(x_1, \ldots, x_n)] := \sigma(f_i)$

(ii) $\widehat{\sigma}[f_i(t_1, \ldots, t_n)] := S^n(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])$.

Let $Hyp^{SF}\tau_n$ be the set of all strongly full hypersubstitutions of type τ_n.

Remark 5.3.1 $Hyp^{SF}\tau_n \subseteq Hyp_R(\tau_n)$.

Theorem 5.3.2 *The extension $\widehat{\sigma}$ of a strongly full hypersubstitution σ of type τ_n is an endomorphism of $clone^{SF}\tau_n$.*

The Theorem can be proved by induction on the complexity of the term (see [8]).

On $Hyp^{SF}\tau_n$ we define a binary operation by $\sigma_1 \circ_h \sigma_2 := \widehat{\sigma_1} \circ \sigma_2$ and let σ_{id} be the strongly full hypersubstitution defined by $\sigma_{id}(f_i) := f_i(x_1, \ldots, x_n)$. Clearly, $\widehat{\sigma}_{id}[t] = t$ for all $t \in W^{SF}_{\tau_n}(X_n)$. It is easy to see that the set $Hyp^{SF}\tau_n$ together with the binary operation $\circ_h$ and with σ_{id} forms a monoid $(Hyp^{SF}\tau_n; \circ_h, \sigma_{id})$. For more background on hypersubstitutions see e.g. [26].

Now we consider mappings from the generating system $F_{\tau_n} := \{f_i(x_1, \ldots, x_n) \mid i \in I\}$ to $W^{SF}_{\tau_n}(X_n)$.

A *substitution* of $(W^{SF}_{\tau_n}(X_n); S^n)$ is a mapping $su : \{f_i(x_1, \ldots, x_n) \mid i \in I\} \rightarrow W^{SF}_{\tau_n}(X_n)$ and the extension of a substitution su is a mapping $\overline{su} : W^{SF}_{\tau_n}(X_n) \rightarrow W^{SF}_{\tau_n}(X_n)$ defined by $\overline{su}(f_i(x_1, \ldots, x_n)) = su(f_i(x_1, \ldots, x_n))$ and

$$\overline{su}(f_i(t_1, \ldots, t_n)) \ = S^n(\overline{su}(f_i(x_1, \ldots, x_n)), \overline{su}(t_1), \ldots, \overline{su}(t_n)).$$

Now we want to prove that every substitution $su : \ F_{\tau_n} \rightarrow W^{SF}_{\tau_n}(X_n)$ can be uniquely extended to an endomorphism.

Let V_{M_n} be the variety of all Menger algebras of rank n. Let $\{X_i \mid i \in I\}$ be a new set of variables. This set is indexed with the index set I for the set of operation symbols of type τ_n. Let $\mathcal{F}_{V_{M_n}}(\{X_i \mid i \in I\})$ be the free algebra with respect to the variety V_{M_n}, freely generated by $\{X_i \mid i \in I\}$. Then we have:

Theorem 5.3.3 *The algebra $clone^{SF}\tau_n$ is isomorphic to the free algebra $\mathcal{F}_{V_{M_n}}(\{X_i \mid i \in I\})$, freely generated by the set F_{τ_n}.*

Proof. We define a map $\varphi : W^{SF}_{\tau_n}(X_n) \longrightarrow \mathcal{F}_{V_{M_n}}(\{X_i \mid i \in I\})$ inductively as follows:

(1) $\varphi(f_i(x_1, \ldots, x_n)) := X_i,\ i \in I$,

(2) $\varphi(f_i(t_1, \ldots, t_n)) := \widetilde{S^n}(X_i, \varphi(t_1), \ldots, \varphi(t_n))$.

We prove the homomorphism property $\varphi(S^n(t, s_1, \ldots, s_n)) = \widetilde{S^n}(\varphi(t), \varphi(s_1), \ldots, \varphi(s_n))$ by induction on the complexity of the term t.

(i) If $t = f_i(x_1, \ldots, x_n)$ then $\varphi(S^n(t, s_1, \ldots, s_n)) = \varphi(f_i(s_1, \ldots, s_n))$
$= \widetilde{S^n}(X_i, \varphi(s_1), \ldots, \varphi(s_n)) = \widetilde{S^n}(\varphi(t), \varphi(s_1), \ldots, \varphi(s_n))$.

(ii) If $t = f_i(t_1, \ldots, t_n)$ and if we assume that
$\varphi(S^n(t_j, s_1, \ldots, s_n)) = \widetilde{S^n}(\varphi(t_j), \varphi(s_1), \ldots, \varphi(s_n))$ for all $j = 1, \ldots, n$,
then $\varphi(S^n(t, s_1, \ldots, s_n))$
$= \varphi(f_i(S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)))$
$= \widetilde{S^n}(X_i, \varphi(S^n(t_1, s_1, \ldots, s_n)), \ldots, \varphi(S^n(t_n, s_1, \ldots, s_n)))$
$= \widetilde{S^n}(X_i, \widetilde{S^n}(\varphi(t_1), \varphi(s_1), \ldots, \varphi(s_n)), \ldots, \widetilde{S^n}(\varphi(t_n), \varphi(s_1), \ldots, \varphi(s_n)))$
$= \widetilde{S^n}(\widetilde{S^n}(X_i, \varphi(t_1), \ldots, \varphi(t_n)), \varphi(s_1), \ldots, \varphi(s_n))$
$= \widetilde{S^n}(\varphi(f_i(t_1, \ldots, t_n)), \varphi(s_1), \ldots, \varphi(s_n))$
$= \widetilde{S^n}(\varphi(t), \varphi(s_1), \ldots, \varphi(s_n))$.

Thus φ is a homomorphism. It maps the generating set $F_{\tau_n} = \{f_i(x_i, \ldots, x_n) \mid i \in I\}$ of the algebra $clone^{SF}\tau_n$ onto the set $\{X_i \mid i \in I\}$, since $\varphi(f_i(x_i, \ldots, x_n)) = X_i$ for every $i \in I$. Furthermore, since $\{X_i \mid i \in I\}$ is a free independent set ([36]), we have

$$X_i = X_j \Rightarrow i = j \Rightarrow f_i(x_1, \ldots, x_n) = f_j(x_1, \ldots, x_n).$$

Thus φ is a bijection between the generating sets of $clone^{SF}\tau_n$ and $\mathcal{F}_{V_{M_n}}(\{X_i \mid i \in I\})$. Altogether, φ is an isomorphism. ■

As a consequence we get:

Corollary 5.3.4 *The extension of a substitution is an endomorphism of the algebra $clone^{SF}\tau_n$.*

For two substitutions $su_1, su_2 \in Subst$ we define the product $su_1 \odot su_2$ by $su_1 \odot su_2 := \overline{su_1} \circ su_2$, where $\overline{su_1}$ is the extension of su_1.

Now we want to prove that the monoid of all strongly full hypersubstitutions of type τ_n is isomorphic to the endomorphism monoid $End(clone^{SF}\tau_n)$. To do so we need the following equations for the identity hypersubstitution, for substitutions su, su_1, su_2 and its extensions:

(i) $\overline{su} = (su \circ \sigma_{id})^{\frown}$ and

(ii) $(su_1 \odot su_2) \circ \sigma_{id} = (su_1 \circ \sigma_{id})^{\frown} \circ (su_2 \circ \sigma_{id})$.

Clearly, $su \circ \sigma_{id}$ is a hypersubstitution. If $t = f_i(x_1, \ldots, x_n)$ then $\overline{su}(f_i(x_1, \ldots, x_n)) = su(f_i(x_1, \ldots, x_n)) = su(\sigma_{id}(f_i)) = (su \circ \sigma_{id})(f_i) = (su \circ \sigma_{id})^{\frown}(f_i(x_1, \ldots, x_n))$.

If $t = f_i(t_1, \ldots, t_n)$ and if we assume that $\overline{su}(t_j) = (su \circ \sigma_{id})^{\frown}(t_j);\ j = 1, \ldots, n$, then

$$\begin{array}{rcl} (su \circ \sigma_{id})^{\frown}(f_i(t_1, \ldots, t_n)) & = & S^n((su \circ \sigma_{id})(f_i), (su \circ \sigma_{id})^{\frown}(t_1), \ldots, (su \circ \sigma_{id})^{\frown}(t_n)) \\ & = & S^n((su \circ \sigma_{id})(f_i), \overline{su}(t_1), \ldots, \overline{su}(t_n)) \\ & = & S^n(\overline{su}(f_i(x_1, \ldots, x_n)), \overline{su}(t_1), \ldots, \overline{su}(t_n)) \\ & = & \overline{su}(S^n(f_i(x_1, \ldots, x_n), t_1, \ldots, t_n)) \\ & = & \overline{su}(f_i(t_1, \ldots, t_n)). \end{array}$$

For the second equation, consider $((su_1 \odot su_2) \circ \sigma_{id})(f_i) = ((\overline{su_1} \circ su_2) \circ \sigma_{id})(f_i)$ $= (\overline{su_1} \circ su_2)((\sigma_{id})(f_i)) = (\overline{su_1} \circ su_2)(f_i(x_1, \ldots, x_n)) = \overline{su_1}(su_2(f_i(x_1, \ldots, x_n)))$ $= (su_1 \circ \sigma_{id})^{\frown}(su_2(f_i(x_1, \ldots, x_n))) = (su_1 \circ \sigma_{id})^{\frown}(su_2 \circ \sigma_{id}(f_i)) = ((su_1 \circ \sigma_{id})^{\frown} \circ (su_2 \circ \sigma_{id}))(f_i)$.

Let $id_{\mathcal{F}_{\tau_n}}$ be the identity mapping on $\mathcal{F}_{\tau_n}$. Then we have

Proposition 5.3.5 *The monoids* $(Subst; \odot, id_{F_{\tau_n}})$ *and* $(Hyp^{SF}\tau_n; \circ_h, \sigma_{id})$ *are isomorphic.*

Proof. We consider the mapping $\eta : (Subst; \odot, id_{F_{\tau_n}}) \rightarrow (Hyp^{SF}\tau_n; \circ_h, \sigma_{id})$ defined by $\eta(su) = su \circ \sigma_{id}$. Clearly, η is well-defined and injective since from $su \circ \sigma_{id} = su' \circ \sigma_{id}$ by multiplication with σ_{id}^{-1} from the right hand side there follows $su = su'$. The mapping η is surjective since for $\sigma \in Hyp^{SF}\tau_n$ and $\sigma \circ \sigma_{id}^{-1} \in Subst$, we have $(\sigma \circ \sigma_{id}^{-1}) \circ \sigma_{id} = \sigma$. Therefore, η is a bijection. Let $su_1, su_2 \in Subst$, then $(su_1 \odot su_2) \circ \sigma_{id} = (su_1 \circ \sigma_{id})^{\frown} \circ (su_2 \circ \sigma_{id}) = \eta(su_1)^{\frown} \circ \eta(su_2) = \eta(su_1) \circ_h \eta(su_2)$. This shows that η is a homomorphism. ∎

Clearly, the monoids $(Hyp^{SF}\tau_n; \circ_h, \sigma_{id})$ and $(End(clone^{SF}\tau_n); \circ, id_{W^{SF}_{\tau_n}(X_n)})$ are isomorphic.

Let V be a strongly full variety of partial algebras. Then $t_1 \approx t_2 \in Id^{SF}V$ is called a *strongly full hyperidentity* in V if for any $\sigma \in Hyp^{SF}\tau_n$ we have $\widehat{\sigma}[t_1] \approx \widehat{\sigma}[t_2] \in Id^{SF}V$. Let $HId^{SF}V$ be the set of all strongly full hyperidentities in V.

Theorem 5.3.6 *Let V be a strongly full variety of partial algebras and let $t_1 \approx t_2 \in Id^{SF}_n V$. Then $t_1 \approx t_2$ is an identity in $clone^{SF}V$ iff $t_1 \approx t_2$ is a strongly full hyperidentity in V.*

Proof. Let $t_1 \approx t_2 \in Id^{SF}_n V$ be an identity in $clone^{SF}V$. This means, for every valuation mapping $v : \{f_i(x_1, \ldots, x_n) \mid i \in I\} \to clone^{SF}V$, we have $\overline{v}(t_1) = \overline{v}(t_2)$. Let σ be any strongly full hypersubstitution. We will show that $\widehat{\sigma}[t_1] \approx \widehat{\sigma}[t_2] \in Id^{SF}_n V$. We denote by $nat : W^{SF}_{\tau_n}(X_n) \to W^{SF}_{\tau_n}(X_n)/Id^{SF}_n V$ the natural mapping. Clearly, $\eta := \sigma \circ \sigma^{-1}_{id}$ is a clone substitution, $\eta \in Subst$, and $v := nat \circ \eta$ is a valuation mapping which is uniquely determined and has the extension $\overline{v} = nat \circ \overline{\eta}$. Then

$$
\begin{array}{lrcl}
\multicolumn{4}{l}{\overline{v}(t_1) = \overline{v}(t_2)} \\
\Rightarrow & (nat \circ \overline{\eta})(t_1) & = & (nat \circ \overline{\eta})(t_2) \\
\Rightarrow & (nat \circ \overline{\sigma \circ \sigma^{-1}_{id}})(t_1) & = & (nat \circ \overline{\sigma \circ \sigma^{-1}_{id}})(t_2) \\
\Rightarrow & (nat \circ \widehat{\sigma})(t_1) & = & (nat \circ \widehat{\sigma})(t_2) \\
\Rightarrow & [\widehat{\sigma}(t_1)]_{Id^{SF}_n V} & = & [\widehat{\sigma}(t_2)]_{Id^{SF}_n V} \\
\Rightarrow & \widehat{\sigma}(t_1) & \approx & \widehat{\sigma}(t_2) \in Id^{SF}_n V.
\end{array}
$$

Conversely, let $t_1 \approx t_2$ be a strongly full hyperidentity in V. This means that for every $\sigma \in Hyp^{SF}\tau_n$, we have $\widehat{\sigma}[t_1] \approx \widehat{\sigma}[t_2] \in Id^{SF}_n V$. To show that $t_1 \approx t_2$ is an identity in $clone^{SF}V$, we will show that $\overline{v}(t_1) = \overline{v}(t_2)$ for every valuation mapping $v : \{f_i(x_1, \ldots, x_n) \mid i \in I\} \to clone^{SF}V$. Consider a mapping $\eta : \{f_i(x_1, \ldots, x_n) \mid i \in I\} \to clone^{SF}\tau_n$ such that $v = nat \circ \eta$. That means, using a choice function $\phi : clone^{SF}V \to clone^{SF}\tau_n$ for every $f_i(x_1, \ldots, x_n)$ we select from the class $[v(f_i(x_1, \ldots, x_n))]_{Id^{SF}_n V}$ a uniquely determined element from $clone^{SF}\tau_n$ as image of $f_i(x_1, \ldots, x_n)$ under η. Then η is well-defined since from $f_i(x_1, \ldots, x_n) = f_j(x_1, \ldots, x_n)$ there follows $[f_i(x_1, \ldots, x_n)]_{Id^{SF}_n V} = [f_j(x_1, \ldots, x_n)]_{Id^{SF}_n V}$ and then the choice function ϕ selects exactly one element $\eta(f_i(x_1, \ldots, x_n))$ from this class.

Therefore $\eta(f_i(x_1,\ldots,x_n)) = \eta(f_j(x_1,\ldots,x_n))$. The extension $\overline{v}$ of v is uniquely determined and we have $\overline{v} = nat \circ \overline{\eta}$. Then $\sigma := \eta \circ \sigma_{id} \in Hyp^{SF}\tau_n$ and

$$\begin{array}{lrcl}
\multicolumn{4}{l}{(\eta \circ \sigma_{id})\hat{}[t_1] \approx (\eta \circ \sigma_{id})\hat{}[t_2] \in Id_n^{SF}V} \\
\Rightarrow & \overline{\eta}(t_1) & \approx & \overline{\eta}(t_2) \in Id_n^{SF}V, \text{ by (i) before Proposition 5.3.5} \\
\Rightarrow & [\overline{\eta}(t_1)]_{Id_n^{SF}V} & = & [\overline{\eta}(t_2)]_{Id_n^{SF}V} \\
\Rightarrow & (nat \circ \overline{\eta})(t_1) & = & (nat \circ \overline{\eta})(t_2) \\
\Rightarrow & \overline{v}(t_1) & = & \overline{v}(t_2).
\end{array}$$

■

Let V be a strongly full variety of partial algebras and let $Id_n^{SF}V$ be the set of all $n-ary$ strongly full identities satisfied in V. If every identity $s \approx t \in Id_n^{SF}V$ is a strongly full hyperidentity in V, then V is called $n - SF - solid$.

Proposition 5.3.7 *Let V be a strongly full variety of partial algebras of type τ_n. Then V is $n - SF - solid$ iff $Id_n^{SF}V$ is a fully invariant congruence relation on $clone^{SF}\tau_n$.*

Proof. Let V be $n - SF - solid$, let $t_1 \approx t_2 \in Id_n^{SF}V$ and let $\overline{\varphi} : clone^{SF}\tau_n \to clone^{SF}\tau_n$ be an endomorphism of $clone^{SF}\tau_n$, which extends $\varphi : \{f_i(x_1,\ldots,x_n) \mid i \in I\} \to clone^{SF}\tau_n$. Then we have

$$\overline{\varphi}(t_1) = (\varphi \circ \sigma_{id})\hat{}[t_1] \approx (\varphi \circ \sigma_{id})\hat{}[t_2] = \overline{\varphi}(t_2) \in Id_n^{SF}V$$

since $\varphi \circ \sigma_{id}$ is a strongly full hypersubstitution with $\overline{\varphi} = (\varphi \circ \sigma_{id})\hat{}$ (see (i) before Proposition 5.3.5). Therefore $Id_n^{SF}V$ is fully invariant.
If conversely $Id_n^{SF}V$ is fully invariant, $t_1 \approx t_2 \in Id_n^{SF}V$ and let $\sigma \in Hyp^{SF}\tau_n$, then $\hat{\sigma}[t_1] \approx \hat{\sigma}[t_2] \in Id_n^{SF}V$ since by Theorem 5.3.2 the extension of a strongly full hypersubstitution is an endomorphism of $clone^{SF}\tau_n$. This shows that every identity $t_1 \approx t_2 \in Id_n^{SF}V$ is satisfied as a strongly full hyperidentity and then V is $n - SF - solid$. ■

Theorem 5.3.8 *Let V be a strongly full variety of partial algebras. Then V is $n - SF - solid$ iff $clone^{SF}V$ is free with respect to itself, freely generated by the independent set $\{[f_i(x_1,\ldots,x_n)]_{Id_n^{SF}V} \mid i \in I\}$, meaning that every mapping $u : \{[f_i(x_1,\ldots,x_n)]_{Id_n^{SF}V} \mid i \in I\} \to clone^{SF}V$ can be extended to an endomorphism $\overline{u} : clone^{SF}V \to clone^{SF}V$.*

Proof. Let $clone^{SF}V$ be free with respect to itself. Using the equivalence from Theorem 5.3.6, we will show that V is $n-SF-solid$ if every identity $t_1 \approx t_2 \in Id_n^{SF}V$ is also an identity in $clone^{SF}V$. Let $t_1 \approx t_2 \in Id_n^{SF}V$. To show that $t_1 \approx t_2$ is an identity in $clone^{SF}V$, we will show that $\overline{v}(t_1) = \overline{v}(t_2)$ for any mapping $v : \{f_i(x_1, \ldots, x_n) \mid i \in I\} \to clone^{SF}V$. Given v, we define a mapping $\alpha_v : \{[f_i(x_1, \ldots, x_n)]_{Id_n^{SF}V} \mid i \in I\} \to clone^{SF}V$ by $\alpha_v([f_i(x_1, \ldots, x_n)]_{Id_n^{SF}V}) = v(f_i(x_1, \ldots, x_n))$ i.e. by $\alpha_v \circ nat = v$. Since

$$\begin{array}{rrcl} & [f_i(x_1, \ldots, x_n)]_{Id_n^{SF}V} & = & [f_j(x_1, \ldots, x_n)]_{Id_n^{SF}V} \\ \Rightarrow & i & = & j \\ \Rightarrow & f_i(x_1, \ldots, x_n) & = & f_j(x_1, \ldots, x_n) \\ \Rightarrow & v(f_i(x_1, \ldots, x_n)) & = & v(f_j(x_1, \ldots, x_n)) \\ \Rightarrow & \alpha_v([f_i(x_1, \ldots, x_n)]_{Id_n^{SF}V}) & = & \alpha_v([f_j(x_1, \ldots, x_n)]_{Id_n^{SF}V}), \end{array}$$

the mapping α_v is well-defined and because of the freeness of $clone^{SF}V$ it can be uniquely extened to $\overline{\alpha}_v : clone^{SF}V \to clone^{SF}V$ with $\overline{\alpha_v} \circ nat = \overline{v}$ (Here we use that $\{[f_i(x_1, \ldots, x_n)]_{Id_n^{SF}V} \mid i \in I\}$ is a free independent generating set of $clone^{SF}V$). Since the set $\{f_i(x_1, \ldots, x_n) \mid i \in I\}$ generates the free algebra $clone^{SF}\tau_n$, the mapping v can be uniquely extended to a homomorphism $\overline{v} : clone^{SF}\tau_n \to clone^{SF}V$. Then we have

$$\begin{array}{rcl} t_1 \approx t_2 \in Id_n^{SF}V & \Rightarrow & [t_1]_{Id_n^{SF}V} = [t_2]_{Id_n^{SF}V} \\ & \Rightarrow & \overline{\alpha}_v([t_1]_{Id_n^{SF}V}) = \overline{\alpha}_v([t_2]_{Id_n^{SF}V}) \\ & \Rightarrow & (\overline{\alpha}_v \circ nat)(t_1) = (\overline{\alpha}_v \circ nat)(t_2) \\ & \Rightarrow & \overline{v}(t_1) = \overline{v}(t_2), \end{array}$$

showing that $t_1 \approx t_2 \in Id_n^{SF}clone^{SF}V$.

For the converse, we show that when V is $n - SF - solid$, any mapping $\alpha : \{[f_i(x_1, \ldots, x_n)]_{Id_n^{SF}V} \mid i \in I\} \to clone^{SF}V$ can be extended to an endomorphism of $clone^{SF}V$. We consider the mapping $\alpha \circ nat : \{f_i(x_1, \ldots, x_n) \mid i \in I\} \to clone^{SF}V$ which is a valuation map. So we have $\overline{(\alpha \circ nat)}(f_i(x_1, \ldots, x_n)) = (\alpha \circ nat)(f_i(x_1, \ldots, x_n))$. We define the map $\overline{\alpha} : clone^{SF}V \to clone^{SF}V$ by $\overline{(\alpha \circ nat)}(t) = \overline{\alpha}([t]_{Id_n^{SF}V})$.

We will prove that

(i) $\overline{\alpha}$ is well-defined. Let $t_1, t_2 \in W_{\tau_n}^{SF}(X_n)$, it follows from $[t_1]_{Id_n^{SF}V} = [t_2]_{Id_n^{SF}V}$ that $t_1 \approx t_2 \in Id_n^{SF}V$. Since V is $n - SF - solid$ and with $\alpha \circ nat : \{(f_i(x_1, \ldots, x_n) \mid i \in I\} \to clone^{SF}V$, we have $\overline{(\alpha \circ nat)}(t_1) = \overline{(\alpha \circ nat)}(t_2) \Rightarrow \overline{\alpha}([t_1]_{Id_n^{SF}V}) = \overline{\alpha}([t_2]_{Id_n^{SF}V})$. This shows that $\overline{\alpha}$ is well-defined.

(ii) $\overline{\alpha}$ is an endomorphism.

Consider $\overline{\alpha}(\overline{S^n}([t_0]_{Id_n^{SF}V},\ldots,[t_n]_{Id_n^{SF}V}))$
$= \overline{\alpha}[S^n(t_0,\ldots,t_n)]_{Id_n^{SF}V}$
$= \overline{(\alpha \circ nat)}(S^n(t_0,\ldots,t_n))$
$= \overline{S^n}(\overline{(\alpha \circ nat)}(t_0),\ldots,\overline{(\alpha \circ nat)}(t_n))$
$= \overline{S^n}(\overline{\alpha}[t_0]_{Id_n^{SF}V},\ldots,\overline{\alpha}[t_n]_{Id_n^{SF}V})$.

(iii) $\overline{\alpha}$ extends α. Indeed, we have

$$\begin{aligned}
\overline{\alpha}([f_i(x_1,\ldots,x_n)]_{Id_n^{SF}V}) &= \overline{(\alpha \circ nat)}(f_i(x_1,\ldots,x_n)) \\
&= (\alpha \circ nat)(f_i(x_1,\ldots,x_n)) \\
&= \alpha(nat(f_i(x_1,\ldots,x_n))) \\
&= \alpha([f_i(x_1,\ldots,x_n)]_{Id_n^{SF}V}).
\end{aligned}$$

■

Let $\mathcal{T}^{SF}(\mathcal{A})$:=$(W_{\tau_n}^{SF}(X_n)^{\mathcal{A}}; S^{n,A})$ be the Menger algebra of all n-ary strongly full term operations of the partial algebra $\mathcal{A}$. Let $\mathcal{A}$ be a partial algebra and $\mathcal{C}$ be a submonoid of $(Subst; \odot, id_{F_{\tau_n}})$. Then $t_1 \approx t_2$ is called a $\mathcal{C}$-identity in $\mathcal{T}^{SF}(\mathcal{A})$ iff $\overline{\eta}(t_1) = \overline{\eta}(t_2)$ for every $\eta \in C$.

Proposition 5.3.9 *Let $\mathcal{A}$ be a partial algebra and let $\psi^{-1}(M)$ be the submonoid of $(Subst; \odot, id_{F_{\tau_n}})$ corresponding to the submonoid $\mathcal{M}$ of $Hyp^{SF}\tau_n$. Then $t_1 \approx t_2$ is a strong M-hyperidentity in $\mathcal{A}$ iff $t_1 \approx t_2$ is an $\psi^{-1}(M)$-identity in $\mathcal{T}^{SF}(\mathcal{A})$.*

Proof. Let $t_1 \approx t_2$ be a atrong M-hyperidentity in $\mathcal{A}$. This means that for every $\sigma \in M$ we have $\hat{\sigma}[t_1] \approx \hat{\sigma}[t_2] \in Id^{SF}\mathcal{A}$. Let $\eta \in \psi^{-1}(M)$. By Proposition 5.3.5, we have $\psi(\eta) = \eta \circ \sigma_{id} \in M$ and $\overline{\eta} = (\eta \circ \sigma_{id})^\frown$. Then $\overline{\eta}(t_1) = (\eta \circ \sigma_{id})^\frown[t_1] \approx (\eta \circ \sigma_{id})^\frown[t_2] = \overline{\eta}(t_2) \in Id^{SF}\mathcal{A}$ and $\overline{\eta}(t_1)^{\mathcal{A}} = \overline{\eta}(t_2)^{\mathcal{A}}$. So $\overline{\eta}(t_1) \approx \overline{\eta}(t_2) \in Id\ \mathcal{T}^{SF}(\mathcal{A})$. Conversely, let $t_1 \approx t_2$ be an $\psi^{-1}(M)$-identity in $\mathcal{T}^{SF}(\mathcal{A})$. This means that for every $\eta \in \psi^{-1}(M)$ we have $\overline{\eta}(t_1) \approx \overline{\eta}(t_2) \in Id\ \mathcal{T}^{SF}(\mathcal{A})$. Let $\sigma \in M$. By Proposition 5.3.5, we have $\sigma \circ \sigma_{id}^{-1} \in \psi^{-1}(M)$ because of $\psi(\sigma \circ \sigma_{id}^{-1}) = (\sigma \circ \sigma_{id}^{-1}) \circ \sigma_{id} = \sigma \in M$ and $\overline{\sigma \circ \sigma_{id}^{-1}} = \hat{\sigma}$. Then $\hat{\sigma}[t_1] = (\overline{\sigma \circ \sigma_{id}^{-1}})(t_1) \approx (\overline{\sigma \circ \sigma_{id}^{-1}})(t_2) = \hat{\sigma}[t_2] \in Id\ \mathcal{T}^{SF}(\mathcal{A})$ and

$$\hat{\sigma}[t_1] \approx \hat{\sigma}[t_2] \in Id\ \mathcal{T}^{SF}(\mathcal{A}) \Rightarrow \hat{\sigma}[t_1]^{\mathcal{A}} = \hat{\sigma}[t_2]^{\mathcal{A}} \Rightarrow \hat{\sigma}[t_1] \approx \hat{\sigma}[t_2] \in Id^{SF}\mathcal{A}.$$

■

5.4 $I^{SF}-closed$ and $V^{SF}-closed$ Varieties

Let V be a strongly full variety of partial algebras of type τ_n. Then V is called $I^{SF}-closed$ if whenever $\mathcal{A} \in V$ and $\mathcal{T}^{SF}(\mathcal{A}) \cong \mathcal{T}^{SF}(\mathcal{B})$, then also $\mathcal{B} \in V$.

We consider the following set of strongly full hypersubstitutions of type τ_n:

$$\mathcal{O}^{SF} := \{\sigma \mid \sigma \in Hyp^{SF}\tau_n \text{ and } \widehat{\sigma} \text{ is surjective }\}.$$

It is easy to see that $\mathcal{O}^{SF}$ is a submonoid of $Hyp^{SF}\tau_n$.

Let $\mathcal{A}$ be a partial algebra of type τ_n. A congruence $\theta \in Con\mathcal{A}$ is said to be *weakly invariant* if for every $\rho \in Con\mathcal{A}$ the following condition is satisfied : if there exists a full homomorphism from $\mathcal{A}/\theta$ onto $\mathcal{A}/\rho$, then $\theta \subseteq \rho$. Let $\mathcal{A}$ be a partial algebra, and let θ and ρ be any congruences on $\mathcal{A}$. From the second isomorphism theorem (see [4]), it always follows from $\theta \subseteq \rho$ that there exists a surjective full homomorphism $\mathcal{A}/\theta \to \mathcal{A}/\rho$ such that $\mathcal{A}/\rho$ is isomorphic to $(\mathcal{A}/\theta)/(\rho/\theta)$ (see [4]).

A set $\mathcal{C}$ of congruences of a partial algebra $\mathcal{A}$ of type τ_n is said to be *isomorphically closed* if whenever $\theta \in \mathcal{C}$ and $\mathcal{A}/\theta \cong \mathcal{A}/\rho$ it follows that $\rho \in \mathcal{C}$.

Theorem 5.4.1 *Let $\mathcal{A}$ be a partial algebra of type τ_n. Then we have:*

(i) A congruence θ on $\mathcal{A}$ is weakly invariant iff the principal filter $[\theta)$ generated by θ in $Con\mathcal{A}$ is isomorphically closed.

(ii) Every weakly invariant congruence on $\mathcal{A}$ is invariant under all surjective full endomorphisms of $\mathcal{A}$.

Proof. (i) First let θ be weakly invariant. Let ρ and β be congruences on $\mathcal{A}$ such that $\rho \in [\theta)$ and $\mathcal{A}/\rho \cong \mathcal{A}/\beta$. Since $\theta \subseteq \rho$, it follows from the second isomorphism theorem that there is a surjective full homomorphism from $\mathcal{A}/\theta$ onto $\mathcal{A}/\rho$. But then by composition there is also a surjective full homomorphism from $\mathcal{A}/\theta$ onto $\mathcal{A}/\beta$, and since θ is weakly invariant, we have $\theta \subseteq \beta$. Thus $\beta \in [\theta)$, showing that $[\theta)$ is isomorphically closed.

Conversely, let $[\theta)$ be isomorphically closed. To show that θ is weakly invariant, we consider $\rho \in Con\mathcal{A}$ for which there is a surjective full homomorphism $\psi : \mathcal{A}/\theta \to \mathcal{A}/\rho$. Using the natural surjective full homomorphism $nat\theta : \mathcal{A} \to \mathcal{A}/\theta$,

we get a surjective full homomorphism $\psi \circ nat\theta : \mathcal{A} \to \mathcal{A}/\rho$ and by the first homomorphism theorem $\mathcal{A}/\rho \cong \mathcal{A}/ker(\psi \circ nat\theta)$. Clearly $\theta = kernat\theta \subseteq ker(\psi \circ nat\theta)$ and since $[\theta)$ is isomorphically closed, we have $\rho \in [\theta)$ and $\theta \subseteq \rho$.
(ii) Let θ be a weakly invariant congruence on $\mathcal{A}$ and let $\phi : \mathcal{A} \to \mathcal{A}$ by any surjective full endomorphism. Then $nat\theta \circ \phi : \mathcal{A} \to \mathcal{A}/\theta$ is a surjective full homomorphism and by the first homomorphism theorem $\mathcal{A}/\theta \cong \mathcal{A}/ker(nat\theta \circ \phi)$. But $[\theta)$ is isomorphically closed and by (i) we have $ker(nat\theta \circ \phi) \in [\theta)$. So $\theta \subseteq ker(nat\theta \circ \phi)$ and from this we get

$$\begin{aligned}
(u,v) \in \theta &\Rightarrow (u,v) \in ker(nat\theta \circ \phi)\\
&\Rightarrow (nat\theta \circ \phi)(u) = (nat\theta \circ \phi)(v) \text{ and } u,v \in dom(nat\theta \circ \phi)\\
&\Rightarrow nat\theta(\phi(u)) = nat\theta(\phi(v)) \text{ and } \phi(u), \phi(v) \in dom(nat\theta)\\
&\Rightarrow (\phi(u), \phi(v)) \in ker(nat\theta) = \theta.
\end{aligned}$$

Then θ is invariant under all surjective full endomorphisms of $\mathcal{A}$. ■

Proposition 5.4.2 *Let $\mathcal{A}$ be a partial algebra of type τ_n. Then the set $Id_n^{SF}\mathcal{A}$ of its $n-ary$ identities is a congruence on $clone^{SF}\tau_n$, and the quotient algebra $clone^{SF}\tau_n/Id_n^{SF}\mathcal{A}$ is isomorphic to the term clone $\mathcal{T}^{SF}(\mathcal{A})$.*

Proof. By Theorem 5.2.4 the relation $Id_n^{SF}\mathcal{A}$ is a congruence on $clone^{SF}\tau_n$. We define $\varphi : clone^{SF}\tau_n/Id_n^{SF}\mathcal{A} \to T^{SF}(\mathcal{A})$ by $\varphi([t]_{Id_n^{SF}\mathcal{A}}) := t^{\mathcal{A}}$. We get

$$\begin{aligned}
[s]_{Id_n^{SF}\mathcal{A}} = [t]_{Id_n^{SF}\mathcal{A}} &\Rightarrow s \approx t \in Id_n^{SF}\mathcal{A}\\
&\Rightarrow s^{\mathcal{A}}|_{doms^{\mathcal{A}}} = t^{\mathcal{A}}|_{domt^{\mathcal{A}}} \text{ and } doms^{\mathcal{A}} = domt^{\mathcal{A}}\\
&\Rightarrow s^{\mathcal{A}} = t^{\mathcal{A}}\\
&\Rightarrow \varphi([s]_{Id_n^{SF}\mathcal{A}}) = \varphi([t]_{Id_n^{SF}\mathcal{A}})
\end{aligned}$$

and the mapping φ is well-defined.

Then we have

$$\begin{aligned}
\varphi([s]_{Id_n^{SF}\mathcal{A}}) = \varphi([t]_{Id_n^{SF}\mathcal{A}}) &\Rightarrow s^{\mathcal{A}} = t^{\mathcal{A}}\\
&\Rightarrow s^{\mathcal{A}}|_{doms^{\mathcal{A}}} = t^{\mathcal{A}}|_{domt^{\mathcal{A}}} \text{ and } doms^{\mathcal{A}} = domt^{\mathcal{A}}\\
&\Rightarrow s \approx t \in Id_n^{SF}\mathcal{A}\\
&\Rightarrow [s]_{Id_n^{SF}\mathcal{A}} = [t]_{Id_n^{SF}\mathcal{A}}
\end{aligned}$$

and the mapping φ is injective.

Clearly, φ is surjective.

We prove the homomorphism property $\varphi(\overline{S^n}([s]_{Id_n^{SF}\mathcal{A}}, [t_1]_{Id_n^{SF}\mathcal{A}}, \ldots, [t_n]_{Id_n^{SF}\mathcal{A}}))|_D = S^{n,A}(\varphi([s]_{Id_n^{SF}\mathcal{A}}), \varphi([t_1]_{Id_n^{SF}\mathcal{A}}), \ldots, \varphi([t_n]_{Id_n^{SF}\mathcal{A}}))|_D$; $D = \bigcap_{j=1}^{n} domt_j^{\mathcal{A}}$.

Consider $\varphi(\overline{S^n}([s]_{Id_n^{SF}\mathcal{A}}, [t_1]_{Id_n^{SF}\mathcal{A}}, \ldots, [t_n]_{Id_n^{SF}\mathcal{A}}))|_D$

$= \varphi([S^n(s,t_1,\ldots,t_n)]_{Id_n^{SF}\mathcal{A}})|_D$
$= (S^n(s,t_1,\ldots,t_n))^{\mathcal{A}}|_D$
$= S^{n,A}(s^A,t_1^A,\ldots,t_n^A)|_D$
$= S^{n,A}(\varphi([s]_{Id_n^{SF}\mathcal{A}}),\varphi([t_1]_{Id_n^{SF}\mathcal{A}}),\ldots,\varphi([t_n]_{Id_n^{SF}\mathcal{A}}))|_D.$
Altogether, φ is an isomorphism. ■

For any congruence θ on $clone^{SF}\tau_n$, we can define the usual quotient algebra $(W_{\tau_n}^{SF}(X_n)/\theta;(f_i^\star)_{i\in I})$, whose operations $f_i^\star$ are defined by $f_i^\star([t_1]_\theta,\ldots,[t_n]_\theta) = [f_i(t_1,\ldots,t_n)]_\theta$. In the unary case $n=1$, the congruence $Id_n^{SF}\mathcal{A}$ is called the *Myhill-congruence* on $\mathcal{A}$, and the corresponding quotient algebra is called the *Myhill-algebra* ([39]). We now generalize this to $n-ary$ algebras.

For any congruence θ on $clone^{SF}\tau_n$, the quotient algebra $M(\theta) := (W_{\tau_n}^{SF}(X_n)/\theta;(f_i^\star)_{i\in I})$ is called the Myhill-algebra of θ. For any partial algebra $\mathcal{A}$ of type τ_n, the Myhill-algebra of $Id_n^{SF}\mathcal{A}$ is denoted by $M(\mathcal{A})$. For any $n-ary$ strongly full variety V, we set $Id_n^{SF}V = \cap\{Id_n^{SF}\mathcal{A} \mid \mathcal{A}\in V\}$; this is also a congruence on $clone^{SF}\tau_n$, whose quotient algebra $M(V)$ is called the Myhill-algebra of $Id_n^{SF}V$.

Proposition 5.4.3 *For every congruence θ on $clone^{SF}\tau_n$ we have*

$$\mathcal{T}^{SF}(M(\theta)) \cong clone^{SF}\tau_n/\theta.$$

In particular, $\mathcal{T}^{SF}(M(\mathcal{A})) \cong \mathcal{T}^{SF}(\mathcal{A})$.

Proof. $\mathcal{T}^{SF}(M(\theta))$ is the clone generated by $\{f_i^\star \mid i\in I\}$. We define a mapping $\varphi^\star : F_{\tau_n} \to \{f_i^\star \mid i\in I\}$, by $\varphi^\star(f_i(x_1,\ldots,x_n)) = f_i^\star$ for all $i\in I$. Since $clone^{SF}\tau_n$ is freely generated by the set F_{τ_n}, the mapping $\varphi^\star$ can be extended to a homomorphism $\overline{\varphi}^\star$, which is surjective since $\overline{\varphi}^\star(\langle\{f_i(x_1,\ldots,x_n) \mid i\in I\}\rangle) = \langle\overline{\varphi}^\star(\{f_i(x_1,\ldots,x_n) \mid i\in I\})\rangle = \langle\{f_i^\star \mid i\in I\}\rangle = T^{SF}(M(\theta))$ and by the first homomorphism theorem, we have $\mathcal{T}^{SF}(M(\theta)) \cong clone^{SF}\tau_n/ker\overline{\varphi}^\star$. We consider a mapping $\overline{\varphi} : W_{\tau_n}(X_n) \to T^{(n)}(M(\theta))$ whose restriction to $W_{\tau_n}^{SF}(X_n)$ is equal to $\overline{\varphi}^\star$, i.e. $\overline{\varphi}|_{W_{\tau_n}^{SF}(X_n)} = \overline{\varphi}^\star$. Since $clone^{SF}\tau_n$ is a subalgebra of $n-clone\tau_n = (W_{\tau_n}(X_n);S^n)$, we have $T^{SF}(M(\theta)) \subseteq T^{(n)}(M(\theta))$ and then $\overline{\varphi}^\star(f_i(x_1,\ldots,x_n)) = \overline{\varphi}|_{clone^{SF}\tau_n}(f_i(x_1,\ldots,x_n)) = \overline{\varphi}(f_i(x_1,\ldots,x_n)) = [f_i(x_1,\ldots,x_n)]_\theta$ (*) by [19]. We will show that $\theta = ker\overline{\varphi}^{\star}$.

case 1. If $(f_i(x_1,\ldots,x_n), f_j(x_1,\ldots,x_n)) \in ker\overline{\varphi}^{\star}$, then

$\overline{\varphi}^{\star}(f_i(x_1,\ldots,x_n)) = \overline{\varphi}^{\star}(f_j(x_1,\ldots,x_n))$

$\Leftrightarrow [f_i(x_1,\ldots,x_n)]_\theta = [f_j(x_1,\ldots,x_n)]_\theta$ by (*)

$\Leftrightarrow (f_i(x_1,\ldots,x_n), f_j(x_1,\ldots,x_n)) \in \theta$.

case 2. If $(f_i(x_1,\ldots,x_n), f_j(t_1,\ldots,t_n)) \in ker\overline{\varphi}^{\star}$ and we assume that $\overline{\varphi}^{\star}(t_k) = [t_k]_\theta;\ k = 1,\ldots,n$, then

$$\begin{aligned}
[f_i(x_1,\ldots,x_n)]_\theta &= \overline{\varphi}^{\star}(f_i(x_1,\ldots,x_n))\\
&= \overline{\varphi}^{\star}(f_j(t_1,\ldots,t_n))\\
&= \overline{\varphi}^{\star}(S^n(f_j(x_1,\ldots,x_n), t_1,\ldots,t_n))\\
&= \overline{S^n}^{\star}(\overline{\varphi}^{\star}(f_j(x_1,\ldots,x_n)), \overline{\varphi}^{\star}(t_1),\ldots,\overline{\varphi}^{\star}(t_n))\\
&= \overline{S^n}^{\star}([f_j(x_1,\ldots,x_n)]_\theta, [t_1]_\theta,\ldots,[t]_\theta)\\
&= [S^n(f_j(x_1,\ldots,x_n), t_1,\ldots,t_n)]_\theta = [f_j(t_1,\ldots,t_n)]_\theta.
\end{aligned}$$

In the same way, we show

$(f_i(x_1,\ldots,x_n), f_j(t_1,\ldots,t_n)) \in \theta \Rightarrow (f_i(x_1,\ldots,x_n), f_j(t_1,\ldots,t_n)) \in ker\overline{\varphi}^{\star}$.

case 3. In a similar way, we show

$(f_i(s_1,\ldots,s_n), f_j(x_1,\ldots,x_n)) \in ker\overline{\varphi}^{\star} \Leftrightarrow (f_i(s_1,\ldots,s_n), f_j(x_1,\ldots,x_n)) \in \theta$

case 4. If $(f_i(s_1,\ldots,s_n), f_j(t_1,\ldots,t_n)) \in ker\overline{\varphi}^{\star}$ and we assume that $\overline{\varphi}^{\star}(s_k) = [s_k]_\theta, \overline{\varphi}^{\star}(t_k) = [t_k]_\theta;\ k = 1,\ldots,n$, then

$$\begin{aligned}
&\overline{\varphi}^{\star}(f_i(s_1,\ldots,s_n)) = \overline{\varphi}^{\star}(f_j(t_1,\ldots,t_n))\\
&\Leftrightarrow \overline{\varphi}^{\star}(S^n(f_i(x_1,\ldots,x_n), s_1,\ldots,s_n))\\
&\qquad = \overline{\varphi}^{\star}(S^n(f_j(x_1,\ldots,x_n), t_1,\ldots,t_n))\\
&\Leftrightarrow \overline{S^n}^{\star}(\overline{\varphi}^{\star}(f_i(x_1,\ldots,x_n)), \overline{\varphi}^{\star}(s_1),\ldots,\overline{\varphi}^{\star}(s_n))\\
&\qquad = \overline{S^n}^{\star}(\overline{\varphi}^{\star}(f_j(x_1,\ldots,x_n)), \overline{\varphi}^{\star}(t_1),\ldots,\overline{\varphi}^{\star}(t_n))\\
&\Leftrightarrow \overline{S^n}^{\star}([f_i(x_1,\ldots,x_n)]_\theta, [s_1]_\theta,\ldots,[s_n]_\theta)\\
&\qquad = \overline{S^n}^{\star}([f_j(x_1,\ldots,x_n)]_\theta, [t_1]_\theta,\ldots,[t_n]_\theta)\\
&\Leftrightarrow [S^n(f_i(x_1,\ldots,x_n), s_1,\ldots,s_n)]_\theta\\
&\qquad = [S^n(f_j(x_1,\ldots,x_n), t_1,\ldots,t_n)]_\theta\\
&\Leftrightarrow [f_i(s_1,\ldots,s_n)]_\theta\\
&\qquad = [f_j(t_1,\ldots,t_n)]_\theta.\\
&\Leftrightarrow (f_i(s_1,\ldots,s_n), f_j(t_1,\ldots,t_n)) \in \theta.
\end{aligned}$$

Then the isomorphic $\mathcal{T}^{SF}(M(\mathcal{A}))\cong\mathcal{T}^{SF}(\mathcal{A})$ follows from our result. ∎

Theorem 5.4.4 *Let V be a strongly full variety of partial algebras of type τ_n. Then V is $I^{SF}-closed$ iff V satisfies the following two properties:*
(i) $\mathcal{A}\in V$ iff $M(\mathcal{A})\in V$,
(ii) $Id_n^{SF}V$ is weakly invariant.

Proof. Suppose first that V is $I^{SF}-closed$. Property (i) follows from the $I^{SF}-closedness$ and the result from Proposition 5.4.3 that for any $\mathcal{A}\in V$, we have $\mathcal{T}^{SF}(M(\mathcal{A}))\cong\mathcal{T}^{SF}(\mathcal{A})$. By Theorem 5.4.1, we can prove that (ii) holds by showing that $[Id_n^{SF}V)$ is isomorphically closed. For this, let $\alpha\in[Id_n^{SF}V)$, then $Id_n^{SF}V\subseteq\alpha$. Let θ be a congruence on $clone^{SF}\tau_n$ such that $clone^{SF}\tau_n/\alpha\cong clone^{SF}\tau_n/\theta$. Since $Id_n^{SF}V=\bigcap\limits_{\mathcal{A}\in V}Id_n^{SF}\mathcal{A}$, we have $\Delta_{\mathcal{F}_{\tau_n}^{SF}(X_n)/Id_n^{SF}V}=Id_n^{SF}V/Id_n^{SF}V=\bigcap\limits_{\mathcal{A}\in V}Id_n^{SF}\mathcal{A}/Id_n^{SF}V=\bigcap\limits_{\mathcal{A}\in V}(Id_n^{SF}\mathcal{A}/Id_n^{SF}V)$ and $M(V)=\mathcal{F}_{\tau_n}^{SF}(X_n)/Id_n^{SF}V$ is isomorphic to a subdirect product of $M(\mathcal{A})\in V$, and thus $M(V)\in V$. From $Id_n^{SF}V\subseteq\alpha$ follows that we have a surjective homomorphism
$M(V)=\mathcal{F}_{\tau_n}^{SF}(X_n)/Id_n^{SF}V\to(\mathcal{F}_{\tau_n}^{SF}(X_n)/Id_n^{SF}V)/(\alpha/Id_n^{SF}V)\cong\mathcal{F}_{\tau_n}^{SF}(X_n)/\alpha=M(\alpha)$. But V is a strongly full variety, so $M(\alpha)\in V$. Furthermore by Proposition 5.4.3, we have

$$\mathcal{T}^{SF}(M(\alpha))\cong clone^{SF}\tau_n/\alpha\cong clone^{SF}\tau_n/\theta\cong\mathcal{T}^{SF}(M(\theta)),$$

and since V is $I^{SF}-closed$, this gives $M(\theta)\in V$. This means that $Id_n^{SF}V\subseteq Id_n^{SF}M(\theta)$, and we can finish the proof by showing that $Id_n^{SF}M(\theta)=\theta$, so that $\theta\in[Id_n^{SF}V)$. The equality $Id_n^{SF}M(\theta)=\theta$ holds because of

$$s\approx t\in Id_n^{SF}M(\theta)\Leftrightarrow[s]_\theta=[t]_\theta\Leftrightarrow(s,t)\in\theta.$$

Conversely, we assume that the strongly full variety of partial algebras V of type τ_n satisfies (i) and (ii). From (i) we get $M(\mathcal{A})\in V$ for all $\mathcal{A}\in V$, since $Id_n^{SF}V=\bigcap\limits_{\mathcal{A}\in V}Id_n^{SF}\mathcal{A}$. Then we have $\Delta_{\mathcal{F}_{\tau_n}^{SF}(X_n)/Id_n^{SF}V}=Id_n^{SF}V/Id_n^{SF}V=\bigcap\limits_{\mathcal{A}\in V}Id_n^{SF}\mathcal{A}/Id_n^{SF}V=\bigcap\limits_{\mathcal{A}\in V}(Id_n^{SF}\mathcal{A}/Id_n^{SF}V)$ and $M(V)=\mathcal{F}_{\tau_n}^{SF}(X_n)/Id_n^{SF}V$ is isomorphic to a subdirect product of $M(\mathcal{A})\in V$, and thus $M(V)\in V$. To establish that V is $I^{SF}-closed$, let $\mathcal{B}$ and $\mathcal{C}$ be any two partial algebras, and suppose that $\mathcal{T}^{SF}(\mathcal{B})\cong\mathcal{T}^{SF}(\mathcal{C})$ and

$\mathcal{B} \in V$. It follows from Proposition 5.4.2 that

$$clone^{SF}\tau_n/Id_n^{SF}\mathcal{B} \cong \mathcal{T}^{SF}(\mathcal{B}) \cong \mathcal{T}^{SF}(\mathcal{C}) \cong clone^{SF}\tau_n/Id_n^{SF}\mathcal{C},$$

and since $\mathcal{B} \in V$ we have $Id_n^{SF}V \subseteq Id_n^{SF}\mathcal{B}$. By (ii) $Id_n^{SF}V$ is weakly invariant, so we get $Id_n^{SF}V \subseteq Id_n^{SF}\mathcal{C}$. But $M(V) = \mathcal{F}_{\tau_n}^{SF}(X_n)/Id_n^{SF}V \to (\mathcal{F}_{\tau_n}^{SF}(X_n)/Id_n^{SF}V)/(Id_n^{SF}\mathcal{C}/Id_n^{SF}V) \cong \mathcal{F}_{\tau_n}^{SF}(X_n)/Id_n^{SF}\mathcal{C} = M(\mathcal{C})$ is a surjective homomorphism. Since V is a strongly full variety, then we have $M(\mathcal{C}) \in V$. By (i) we get $\mathcal{C} \in V$, establishing that V is $I^{SF} - closed$. ■

Theorem 5.4.5 *Let V be a strongly full variety of partial algebras of type τ_n which is the model class of its $n-ary$ strongly full identities, that is, let $V = Mod^{SF}Id_n^{SF}V$. Then V is $I^{SF} - closed$ iff it is $\mathcal{O}^{SF} - solid$.*

Proof. First assume that V is $\mathcal{O}^{SF} - solid$, so that every $s \approx t \in Id_n^{SF}V$ is an $\mathcal{O}^{SF} - hyperidentity$ in V (i.e. $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id_n^{SF}V$ for all $\sigma \in \mathcal{O}^{SF}$). Let $\mathcal{A} \in V$ and let $\mathcal{T}^{SF}(\mathcal{A})$ be isomorphic to $\mathcal{T}^{SF}(\mathcal{B})$. Then

$\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id_n^{SF}\mathcal{A}$ for all $\sigma \in \mathcal{O}^{SF}$

$\Rightarrow$ $s \approx t$ is an $\mathcal{O}^{SF} - hyperidentity$ in $\mathcal{A}$

$\Rightarrow$ $s \approx t$ is an $\psi^{-1}(\mathcal{O}^{SF}) - identity$ in $\mathcal{T}^{SF}(\mathcal{A})$ (by Proposition 5.3.9)

$\Rightarrow$ $s \approx t$ is an $\psi^{-1}(\mathcal{O}^{SF}) - identity$ in $\mathcal{T}^{SF}(\mathcal{B})$

$\Rightarrow$ $s \approx t$ is an $\mathcal{O}^{SF} - hyperidentity$ in $\mathcal{B}$ (by Proposition 5.3.9).

Then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id_n^{SF}\mathcal{B}$ for all $\sigma \in \mathcal{O}^{SF}$. So $Id_n^{SF}V \subseteq Id_n^{SF}B$ and $\mathcal{B} \in V$.

Conversely, assume that V is $I^{SF} - closed$. Then by Theorem 5.4.4 and Theorem 5.4.1, we know that $Id_n^{SF}V$ is both, weakly invariant and invariant under all surjective endomorphisms of $clone^{SF}\tau_n$. Then for any identity $s \approx t \in Id_n^{SF}V$ and any surjective endomorphism $\overline{\eta} : clone^{SF}\tau_n \to clone^{SF}\tau_n$, we have $\overline{\eta}(s) \approx \overline{\eta}(t) \in Id_n^{SF}V$. Given $\sigma \in \mathcal{O}^{SF}$, then $\sigma \in Hyp^{SF}\tau_n$ and $\hat{\sigma}$ is surjective. But $\overline{\eta} = \overline{\sigma \circ \sigma_{id}^{-1}} = \hat{\sigma}$. Then $\hat{\sigma}(s) \approx \hat{\sigma}(t) \in Id_n^{SF}V$ for all $\sigma \in \mathcal{O}^{SF}$. This shows that V is $\mathcal{O}^{SF} - solid$. ■

A strongly full variety of partial algebras V of type τ_n is said to be $S^{SF} - closed$ if for every partial algebra $\mathcal{B}$ of type τ_n, whenever $\mathcal{A} \in V$ and $\mathcal{T}^{SF}(\mathcal{B})$ is isomorphic to a subalgebra of $\mathcal{T}^{SF}(\mathcal{A})$, it follows that $\mathcal{B} \in V$. The class V is said to be $V^{SF} - closed$ if for every partial algebra $\mathcal{B}$ of type τ_n, whenever $\mathcal{A} \in V$ and $Id\, \mathcal{T}^{SF}(\mathcal{B}) \supseteq Id\, \mathcal{T}^{SF}(\mathcal{A})$ it follows that $\mathcal{B} \in V$.

Proposition 5.4.6 *Let V be a strongly full variety of partial algebras of type τ_n. If V is $V^{SF} - closed$, then it is both, $I^{SF} - closed$ and $S^{SF} - closed$.*

Proof. Let V be $V^{SF} - closed$, and let $\mathcal{A} \in V$ and $\mathcal{T}^{SF}(\mathcal{B}) \cong \mathcal{T}^{SF}(\mathcal{A})$. Then $Id\ \mathcal{T}^{SF}(\mathcal{B}) = Id\ \mathcal{T}^{SF}(\mathcal{A})$, and so $\mathcal{B} \in V$ since V is $V^{SF} - closed$. Similarly, if $\mathcal{T}^{SF}(\mathcal{B})$ is isomorphic to a subalgebra of $\mathcal{T}^{SF}(\mathcal{A})$, then $Id\ \mathcal{T}^{SF}(\mathcal{B}) \supseteq Id\ \mathcal{T}^{SF}(\mathcal{A})$ and so $\mathcal{B} \in V$ since V is $V^{SF} - closed$. This shows that V is both, $I^{SF} - closed$ and $S^{SF} - closed$. ∎

Theorem 5.4.7 *Let V be a strongly full variety of partial algebras of type τ_n and assume that $V = Mod^{SF} Id_n^{SF} V$. Then V is $V^{SF} - closed$ iff it is $\mathcal{O}^{SF} - solid$.*

Proof. Let V be $\mathcal{O}^{SF} - solid$, $\mathcal{A} \in V$ and $Id\ \mathcal{T}^{SF}(\mathcal{B}) \supseteq Id\ \mathcal{T}^{SF}(\mathcal{A})$. From the fact that V is $\mathcal{O}^{SF} - solid$ for all $\sigma \in \mathcal{O}^{SF}$ we obtain $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id_n^{SF} V$ where $s \approx t \in Id_n^{SF} V$. Since $Id_n^{SF} V \subseteq Id_n^{SF} \mathcal{A}$ we have :

$\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id_n^{SF} \mathcal{A}$ for all $\sigma \in \mathcal{O}^{SF}$

$\Rightarrow$ $s \approx t$ is an $\mathcal{O}^{SF} - hyperidentity$ in $\mathcal{A}$

$\Rightarrow$ $s \approx t$ is an $\psi^{-1}(\mathcal{O}^{SF}) - identity$ in $\mathcal{T}^{SF}(\mathcal{A})$ by Proposition 5.3.9

$\Rightarrow$ $s \approx t$ is an $\psi^{-1}(\mathcal{O}^{SF}) - identity$ in $\mathcal{T}^{SF}(\mathcal{B})$ by $Id\ \mathcal{T}^{SF}(\mathcal{B}) \supseteq Id\ \mathcal{T}^{SF}(\mathcal{A})$

$\Rightarrow$ $s \approx t$ is an $\mathcal{O}^{SF} - hyperidentity$ in $\mathcal{B}$ by Proposition 5.3.9.

Then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id_n^{SF} \mathcal{B}$ for all $\sigma \in \mathcal{O}^{SF}$. So $Id_n^{SF} V \subseteq Id_n^{SF} B$ and $\mathcal{B} \in V$.

Assume conversely that V is $V^{SF} - closed$ and let $\mathcal{A} = (A; (f_i^A)_{i \in I})$ be a partial algebra in V. For any hypersubstitution σ, we consider the derived algebra $\sigma(\mathcal{A}) = (A; (\sigma(f_i)^A)_{i \in I})$. Since the operations $\sigma(f_i)^{\mathcal{A}}$ are term operations of $\mathcal{A}$, we have $\mathcal{T}^{SF}(\sigma(\mathcal{A})) \subseteq \mathcal{T}^{SF}(\mathcal{A})$ and therefore $Id\, \mathcal{T}^{SF}(\sigma(\mathcal{A})) \supseteq Id\, \mathcal{T}^{SF}(\mathcal{A})$. Since V is $V^{SF} - closed$, we have $\sigma(\mathcal{A}) \in V$. This shows that any derived algebra, formed from an algebra in V, belongs to V, which is known to be equivalent to the solidity of V. ∎

Chapter 6

Unsolid and Fluid Strong Varieties

In this chapter, we generalize some results of the papers [20], [21], [22] and [46] to the partial case. In Section 6.1, we define the concepts of V-proper hypersubstitutions and inner hypersubstitutions. In Section 6.2, we use the concepts of V-proper hypersubstitutions and inner hypersubstitutions to define the concepts of unsolid and fluid strong varieties. We generalize unsolid and fluid strong varieties to n-fluid and n-unsolid strong varieties. Finally, we give an example of n-unsolid strong variety of partial algebras.

6.1 V-proper Hypersubstitutions

Now we consider regular C-hypersubstitutions which preserve all strong identities of a strong variety of partial algebras.

Let V be a strong variety of partial algebras of type τ. A regular hypersubstitution $\sigma_R \in Hyp_R^C(\tau)$ is called a *V-proper hypersubstitution* if for every $s \approx t \in Id^sV$ we get $\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t] \in Id^sV$.

We use $P(V)$ for the set of all V-proper hypersubstitutions of type τ.

Proposition 6.1.1 *The algebra $(P(V); \circ_h, \sigma_{id})$ is a submonoid of the algebra $(Hyp_R^C(\tau); \circ_h, \sigma_{id})$.*

Proof. Clearly, $\sigma_{id} \in P(V)$. If $\sigma_{R_1}, \sigma_{R_2} \in P(V)$, then for every $s \approx t \in Id^sV$ we have $\widehat{\sigma}_{R_2}[s] \approx \widehat{\sigma}_{R_2}[t] \in Id^sV$ and $\widehat{\sigma}_{R_1}[\widehat{\sigma}_{R_2}[s]] \approx \widehat{\sigma}_{R_1}[\widehat{\sigma}_{R_2}[t]] \in Id^sV$. This means that $(\widehat{\sigma}_{R_1} \circ \widehat{\sigma}_{R_2})[s] \approx (\widehat{\sigma}_{R_1} \circ \widehat{\sigma}_{R_2})[t] \in Id^sV$ and we get that $(\sigma_{R_1} \circ_h \sigma_{R_2})\widehat{\ }[s] \approx (\sigma_{R_1} \circ_h \sigma_{R_2})$

$\hat{}[t] \in Id^sV$. Therefore $\sigma_{R_1} \circ_h \sigma_{R_2} \in P(V)$, and we have that $P(V)$ is a submonoid of $\mathcal{H}yp_R^C(\tau)$. ■

Let V be a strong variety of partial algebras of type τ. Two regular C-hypersubstitutions $\sigma_{R_1}, \sigma_{R_2} \in Hyp_R^C(\tau)$ are called *V-equivalent* iff $\sigma_{R_1}(f_i) \approx \sigma_{R_2}(f_i) \in Id^sV$ for all $i \in I$. In this case we write $\sigma_{R_1} \sim_V \sigma_{R_2}$.

Theorem 6.1.2 *Let V be a strong variety of partial algebras of type τ, and let $\sigma_{R_1}, \sigma_{R_2} \in Hyp_R^C(\tau)$. Then the following are equivalent:*
(i) $\sigma_{R_1} \sim_V \sigma_{R_2}$.
(ii) *For all $t \in W_\tau^C(X)$ the equation $\widehat{\sigma}_{R_1}[t] \approx \widehat{\sigma}_{R_2}[t]$ is an identity from Id^sV.*
(iii) *For all $\mathcal{A} \in V$ we have $\sigma_{R_1}(\mathcal{A}) = \sigma_{R_2}(\mathcal{A})$.*

Proof. (i)$\Rightarrow$ (ii). The implication can be proved by induction on the complexity of the term t (see [11]).

(ii)$\Rightarrow$ (iii). We consider the term $t = f_i(x_1, \ldots, x_{n_i})$ for $i \in I$. Then $\widehat{\sigma}_{R_1}[f_i(x_1, \ldots, x_{n_i})] \approx \widehat{\sigma}_{R_2}[f_i(x_1, \ldots, x_{n_i})] \in Id^s\mathcal{A}$ for all $\mathcal{A} \in V$ by (ii) and we get $\widehat{\sigma}_{R_1}[f_i(x_1, \ldots, x_{n_i})]^{\mathcal{A}} = \widehat{\sigma}_{R_2}[f_i(x_1, \ldots, x_{n_i})]^{\mathcal{A}}$ for all $i \in I$ and all $\mathcal{A} \in V$. Thus $\sigma_{R_1}(\mathcal{A}) = \sigma_{R_2}(\mathcal{A})$.

(iii)$\Rightarrow$(i). Here we have $\widehat{\sigma}_{R_1}[f_i(x_1, \ldots, x_{n_i})]^{\mathcal{A}} = \widehat{\sigma}_{R_2}[f_i(x_1, \ldots, x_{n_i})]^{\mathcal{A}}$ for all $i \in I$ and all $\mathcal{A} \in V$. Therefore $\sigma_{R_1}(f_i) \approx \sigma_{R_2}(f_i) \in Id^s\mathcal{A}$ for all $\mathcal{A} \in V$. So, $\sigma_{R_1} \sim_V \sigma_{R_2}$. ■

Proposition 6.1.3 *Let V be a strong variety of partial algebras of type τ. Then the relation $\sim_V$ is a right congruence on $Hyp_R^C(\tau)$.*

Proof. Let $\sigma_{R_1} \sim_V \sigma_{R_2}$ and $\sigma_R \in Hyp_R^C(\tau)$. By Theorem 6.1.2 (ii) we have

$$(\sigma_{R_1} \circ_h \sigma_R)(f_i) = \widehat{\sigma}_{R_1}[\sigma_R(f_i)] \approx \widehat{\sigma}_{R_2}[\sigma_R(f_i)] = (\sigma_{R_2} \circ_h \sigma_R)(f_i) \in Id^sV.$$

So, $\sigma_{R_1} \circ_h \sigma_R \sim_V \sigma_{R_2} \circ_h \sigma_R$. This shows that $\sim_V$ is a right congruence. ■

In general, $\sim_V$ is not a left congruence. But if V is solid, then it is a congruence.

Proposition 6.1.4 *Let V be a strong variety of partial algebras of type τ. If $\sigma_{R_1} \sim_V \sigma_{R_2}$ and $\widehat{\sigma}_{R_1}[s] \approx \widehat{\sigma}_{R_1}[t] \in Id^sV$, then $\widehat{\sigma}_{R_2}[s] \approx \widehat{\sigma}_{R_2}[t] \in Id^sV$ when $\sigma_{R_1}, \sigma_{R_2} \in Hyp_R^C(\tau)$ and $s, t \in W_\tau^C(X)$.*

Proof. Assume that $\sigma_{R_1} \sim_V \sigma_{R_2}$ and $\widehat{\sigma}_{R_1}[s] \approx \widehat{\sigma}_{R_1}[t] \in Id^sV$. By Theorem 6.1.2, we have $\widehat{\sigma}_{R_1}[s] \approx \widehat{\sigma}_{R_2}[s] \in Id^sV$ and $\widehat{\sigma}_{R_1}[t] \approx \widehat{\sigma}_{R_2}[t] \in Id^sV$. Thus $\widehat{\sigma}_{R_2}[s] \approx \widehat{\sigma}_{R_2}[t] \in Id^sV$. ■

As a corollary we get

Corollary 6.1.5 *The set $P(V)$ is a union of equivalence classes with respect to $\sim_V$. (In this case one say that $P(V)$ is saturated with respect to $\sim_V$).*

Now we consider the equivalence class of the identity hypersubstitution.

A regular C-hypersubstitution $\sigma_R \in Hyp_R^C(\tau)$ is called an *inner hypersubstitution* of a strong variety V of partial algebras of type τ if for every $i \in I$,

$$\widehat{\sigma}_R[f_i(x_1, \ldots, x_{n_i})] \approx f_i(x_1, \ldots, x_{n_i}) \in Id^sV.$$

Let $P_0(V)$ be the set of all inner hypersubstitutions of V.
By definition, $P_0(V)$ is the equivalence class $[\sigma_{id}]_{\sim_V}$.

Proposition 6.1.6 *If $\sigma_R \in P_0(V)$, then $\widehat{\sigma}_R[t] \approx t \in Id^sV$ for $t \in W_\tau^C(X)$.*

The Proposition can be proved by induction on the complexity of terms (see [11]).

Proposition 6.1.7 *The algebra $(P_0(V); \circ_h, \sigma_{id})$ is a submonoid of $(P(V); \circ_h, \sigma_{id})$.*

Proof. Clearly, $\sigma_{id} \in P_0(V)$. Assume that $\sigma_{R_1}, \sigma_{R_2} \in P_0(V)$. Then

$$\begin{array}{rcll} (\sigma_{R_1} \circ_h \sigma_{R_2})\widehat{\ }[f_i(x_1, \ldots, x_{n_i})] & = & \widehat{\sigma}_{R_1}[\widehat{\sigma}_{R_2}[f_i(x_1, \ldots, x_{n_i})]] & \\ & \approx & \widehat{\sigma}_{R_1}[f_i(x_1, \ldots, x_{n_i})] & \text{by Proposition 6.1.6} \\ & \approx & f_i(x_1, \ldots, x_{n_i}) & \text{by Proposition 6.1.6} \\ & \in & Id^sV. & \end{array}$$

Therefore $\sigma_{R_1} \circ_h \sigma_{R_2} \in P_0(V)$. Thus $P_0(V)$ is a monoid. By Proposition 6.1.6, we have $P_0(V) \subseteq P(V)$. So, the algebra $(P_0(V); \circ_h, \sigma_{id})$ is a submonoid of $(P(V); \circ_h, \sigma_{id})$. ■

Now we show that the compatibility condition from the definition of a closed homomorphism for partial algebras transfers from fundamental operations to arbitrary term operations.

Lemma 6.1.8 *Let $\mathcal{A} \in PAlg(\tau)$ and $t^{\mathcal{A}}$ be the n-ary term operation on A induced by the n-ary term $t \in W_\tau^C(X)$. If $\mathcal{B} \in PAlg(\tau)$ and if $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a surjective closed homomorphism, then for all $a_1, \ldots, a_n \in A$,*

$$\varphi(t^{\mathcal{A}}(a_1, \ldots, a_n)) = t^{\mathcal{B}}(\varphi(a_1), \ldots, \varphi(a_n)).$$

The Lemma can be proved by induction on the complexity of terms (see [11]).

Lemma 6.1.9 *Let $\mathcal{A}, \mathcal{B} \in PAlg(\tau)$ and $\sigma_R \in Hyp_R^C(\tau)$. If $h : \mathcal{A} \to \mathcal{B}$ is a surjective closed homomorphism, then $h : \sigma_R(\mathcal{A}) \to \sigma_R(\mathcal{B})$ is a closed homomorphism.*

Proof. From Lemma 6.1.8, for the term $\sigma_R(f_i)$ we have $h(f_i^{\sigma_R(\mathcal{A})}(a_1, \ldots, a_n)) = h(\sigma_R(f_i)^{\mathcal{A}}(a_1, \ldots, a_n)) = \sigma_R(f_i)^{\mathcal{B}}(h(a_1), \ldots, h(a_n)) = f_i^{\sigma_R(\mathcal{B})}(h(a_1), \ldots, h(a_n))$. This shows that $h : \sigma_R(\mathcal{A}) \to \sigma_R(\mathcal{B})$ is a closed homomorphism. ■

Lemma 6.1.10 *Let $\mathcal{A}, \mathcal{B} \in PAlg(\tau)$ and $\sigma_R \in Hyp_R^C(\tau)$. If $f : \mathcal{A} \to \mathcal{B}$ is an isomorphism, then f is also an isomorphism from $\sigma_R(\mathcal{A})$ to $\sigma_R(\mathcal{B})$.*

Proof. Since $f : \mathcal{A} \to \mathcal{B}$ is bijective, the mapping $f : \sigma_R(\mathcal{A}) \to \sigma_R(\mathcal{B})$ is also bijective because partial algebras and their derived algebras have the same universes and by Lemma 6.1.9, we have $\sigma_R(\mathcal{A}) \cong \sigma_R(\mathcal{B})$. ■

Let V be a strong variety of partial algebras of type τ and $\sigma_{R_1}, \sigma_{R_2} \in Hyp_R^C(\tau)$. Then we define

$$\sigma_{R_1} \sim_{V-iso} \sigma_{R_2} \quad \text{iff} \quad \sigma_{R_1}(\mathcal{A}) \cong \sigma_{R_2}(\mathcal{A}) \quad \text{for all } \mathcal{A} \in V.$$

Clearly, $\sim_V \subseteq \sim_{V-iso}$. If $V = PAlg(\tau)$, then we use $\sim_{iso}$ instead of $\sim_{PAlg(\tau)-iso}$.

Proposition 6.1.11 *Let V be a strong variety of partial algebras of type τ. Then*
(i) *the relation $\sim_{V-iso}$ is a right congruence on $Hyp_R^C(\tau)$;*
(ii) *if V is a solid variety then $\sim_{V-iso}$ is a congruence on $Hyp_R^C(\tau)$.*

Proof. (i) Let $\sigma_{R_1} \sim_{V-iso} \sigma_{R_2}$ and $\sigma_R \in Hyp_R^C(\tau)$. Then $\sigma_{R_1}(\mathcal{A}) \cong \sigma_{R_2}(\mathcal{A})$ and $\sigma_R(\sigma_{R_1}(\mathcal{A})) \cong \sigma_R(\sigma_{R_2}(\mathcal{A}))$ for all $\mathcal{A} \in V$ by Lemma 6.1.10. We have

$$(\sigma_{R_1} \circ_h \sigma_R)(\mathcal{A}) = \sigma_R(\sigma_{R_1}(\mathcal{A})) \cong \sigma_R(\sigma_{R_2}(\mathcal{A})) = (\sigma_{R_2} \circ_h \sigma_R)(\mathcal{A}).$$

So, $\sigma_{R_1} \circ_h \sigma_R \sim_{V-iso} \sigma_{R_2} \circ_h \sigma_R$. This shows that $\sim_{V-iso}$ is a right congruence.

(ii) Assume that V is solid. Then $\sigma_R(\mathcal{A}) \in V$ for all $\sigma_R \in Hyp_R^C(\tau)$ for $\mathcal{A} \in V$. From $\sigma_{R_1} \sim_{V-iso} \sigma_{R_2}$ implies that $\sigma_{R_1}(\sigma_R(\mathcal{A})) \cong \sigma_{R_2}(\sigma_R(\mathcal{A}))$ for all $\mathcal{A} \in V$. We have

$$(\sigma_R \circ_h \sigma_{R_1})(\mathcal{A}) = \sigma_{R_1}(\sigma_R(\mathcal{A})) \cong \sigma_{R_2}(\sigma_R(\mathcal{A})) = (\sigma_R \circ_h \sigma_{R_2})(\mathcal{A}).$$

So, $\sigma_R \circ_h \sigma_{R_1} \sim_{V-iso} \sigma_R \circ_h \sigma_{R_2}$. This shows that $\sim_{V-iso}$ is a left congruence and (i) shows that it is a congruence. ■

Proposition 6.1.12 *If $V = PAlg(\tau)$, then $\sim_{iso}$ is a congruence on $Hyp_R^C(\tau)$.*

Proof. Since $V = PAlg(\tau)$ is a solid variety, the claim follows from Proposition 6.1.11. ■

Proposition 6.1.13 *The equivalence class $P_0^{V-iso}(V) = [\sigma_{id}]_{\sim_{V-iso}}$ is a submonoid of $(Hyp_R^C(\tau); \circ_h, \sigma_{id})$.*

Proof. Clearly, $\sigma_{id} \in P_0^{V-iso}(V)$. Next, we will show that $P_0^{V-iso}(V)$ is closed under the operation $\circ_h$. Let $\sigma_{R_1}, \sigma_{R_2} \in P_0^{V-iso}(V)$. Then $\sigma_{R_1} \sim_{V-iso} \sigma_{id}$ and $\sigma_{R_2} \sim_{V-iso} \sigma_{id}$ implies that $\sigma_{R_1}(\mathcal{A}) \cong \mathcal{A}$ and $\sigma_{R_2}(\mathcal{A}) \cong \mathcal{A}$ for all $\mathcal{A} \in V$.

$$\begin{array}{llll} \text{We have } (\sigma_{R_1} \circ_h \sigma_{R_2})(\mathcal{A}) & = & \sigma_{R_2}(\sigma_{R_1}(\mathcal{A})) & \text{by Lemma 3.2.3} \\ & \cong & \sigma_{R_2}(\mathcal{A}) & \text{by } \sigma_{R_1} \in P_0^{V-iso}(V) \\ & \cong & \mathcal{A} & \text{by } \sigma_{R_2} \in P_0^{V-iso}(V). \end{array}$$

Then $(\sigma_{R_1} \circ_h \sigma_{R_2}) \sim_{V-iso} \sigma_{id}$. Therefore $\sigma_{R_1} \circ_h \sigma_{R_2} \in P_0^{V-iso}(V)$. So, $P_0^{V-iso}(V)$ is a submonoid of $\mathcal{H}yp_R^C(\tau)$. ■

Proposition 6.1.14 *Let V be a strong variety of partial algebras of type τ, $s \approx t \in Id^sV$ for $s, t \in W_\tau^C(X_n)$ and let $\sigma_{R_1}, \sigma_{R_2} \in Hyp_R^C(\tau)$. If $\sigma_{R_1} \sim_{V-iso} \sigma_{R_2}$ and $\widehat{\sigma}_{R_1}[s] \approx \widehat{\sigma}_{R_1}[t] \in Id^sV$, then $\widehat{\sigma}_{R_2}[s] \approx \widehat{\sigma}_{R_2}[t] \in Id^sV$.*

Proof. Assume that $\sigma_{R_1} \sim_{V-iso} \sigma_{R_2}$ and $\widehat{\sigma}_{R_1}[s] \approx \widehat{\sigma}_{R_1}[t] \in Id^sV$. Then $\sigma_{R_1}(\mathcal{A}) \cong \sigma_{R_2}(\mathcal{A})$ for all $\mathcal{A} \in V$. We get that there is an isomorphism φ from $\sigma_{R_1}(\mathcal{A})$ to $\sigma_{R_2}(\mathcal{A})$. Let $b_1, \ldots, b_n \in A$. Then there are elements $a_1, \ldots, a_n \in A$ such that $\varphi(a_1) = b_1, \ldots, \varphi(a_n) = b_n$.

We have

$$\begin{array}{rcl}
dom(\widehat{\sigma}_{R_2}[s]^{\mathcal{A}}) & = & \{(b_1, \ldots, b_n) \mid \widehat{\sigma}_{R_2}[s]^{\mathcal{A}}(b_1, \ldots, b_n) \text{ exists } \} \\
& = & \{(b_1, \ldots, b_n) \mid \widehat{\sigma}_{R_2}[s]^{\mathcal{A}}(\varphi(a_1), \ldots, \varphi(a_n)) \text{ exists } \} \\
& = & \{(b_1, \ldots, b_n) \mid \varphi(\widehat{\sigma}_{R_1}[s]^{\mathcal{A}}(a_1, \ldots, a_n)) \text{ exists } \} \\
& & \text{since } \varphi \text{ is an isomorphism from } \sigma_{R_1}(\mathcal{A}) \text{ to } \sigma_{R_2}(\mathcal{A}) \\
& = & \{(b_1, \ldots, b_n) \mid \varphi(\widehat{\sigma}_{R_1}[t]^{\mathcal{A}}(a_1, \ldots, a_n)) \text{ exists } \} \\
& & \text{since } \widehat{\sigma}_{R_1}[s] \approx \widehat{\sigma}_{R_1}[t] \in Id^s\mathcal{A} \text{ for all } \mathcal{A} \in V \\
& = & \{(b_1, \ldots, b_n) \mid \widehat{\sigma}_{R_2}[t]^{\mathcal{A}}(\varphi(a_1), \ldots, \varphi(a_n)) \text{ exists } \} \\
& = & \{(b_1, \ldots, b_n) \mid \widehat{\sigma}_{R_2}[t]^{\mathcal{A}}(b_1, \ldots, b_n) \text{ exists } \} \\
& = & dom(\widehat{\sigma}_{R_2}[t]^{\mathcal{A}})
\end{array}$$

and

$$\begin{array}{rcl}
\widehat{\sigma}_{R_2}[s]^{\mathcal{A}}(b_1, \ldots, b_n) & = & \widehat{\sigma}_{R_2}[s]^{\mathcal{A}}(\varphi(a_1), \ldots, \varphi(a_n)) \\
& = & \varphi(\widehat{\sigma}_{R_1}[s]^{\mathcal{A}}(a_1, \ldots, a_n)) \\
& = & \varphi(\widehat{\sigma}_{R_1}[t]^{\mathcal{A}}(a_1, \ldots, a_n)) \\
& = & \widehat{\sigma}_{R_2}[t]^{\mathcal{A}}(\varphi(a_1), \ldots, \varphi(a_n)) \\
& = & \widehat{\sigma}_{R_2}[t]^{\mathcal{A}}(b_1, \ldots, b_n).
\end{array}$$

Then $\widehat{\sigma}_{R_2}[s] \approx \widehat{\sigma}_{R_2}[t] \in Id^s\mathcal{A}$ for all $\mathcal{A} \in V$. So, $\widehat{\sigma}_{R_2}[s] \approx \widehat{\sigma}_{R_2}[t] \in Id^sV$. ■

As a corollary we get

Corollary 6.1.15 *The set $P(V)$ is a union of equivalence classes with respect to $\sim_{V-iso}$. (i.e. $P(V)$ is saturated with respect to $\sim_{V-iso}$).*

Remark 6.1.16 $P_0(V) \subseteq P_0^{V-iso}(V) \subseteq P(V)$.

6.2 Unsolid and Fluid Strong Varieties

For a solid strong variety every strong identity is closed under all regular hypersubstitutions. At the other extreme is the case where the strong identities are closed only under the identity hypersubstitution.

A strong variety V of partial algebras of type τ is said to be *unsolid* if $P(V) = P_0(V)$ and V is said to be *completely unsolid* if $P(V) = P_0(V) = \{\sigma_{id}\}$.

A strong variety V of partial algebras of type τ is said to be *iso-unsolid* if $P(V) = P_0^{V-iso}(V)$ and V is said to be *completely iso-unsolid* if $P(V) = P_0^{V-iso}(V) = \{\sigma_{id}\}$.

Proposition 6.2.1 *Let V be a strong variety of partial algebras of type τ. Then*
(i) *If V is unsolid, then V is iso-unsolid.*
(ii) *V is completely unsolid iff V is completely iso-unsolid.*

Proof. (i) The claim follows from the definitions and Remark 6.1.16.
(ii) If V is completely unsolid then V is completely iso-unsolid by Remark 6.1.16. Conversely, assume that V is completely iso-unsolid. Then $P(V) = P_0^{V-iso}(V) = \{\sigma_{id}\}$. Since $P_0(V) \subseteq P(V)$ and $P(V) \neq \emptyset$, we get $P_0(V) = \{\sigma_{id}\}$. So, V is completely unsolid. ■

A strong variety V of partial algebras of type τ is said to be *fluid* if, for every partial algebra $\mathcal{A} \in V$ and every regular C-hypersubstitution $\sigma_R \in Hyp_R^C(\tau)$, there holds

$$\sigma_R(\mathcal{A}) \in V \Rightarrow \sigma_R(\mathcal{A}) \cong \mathcal{A}.$$

We denote by $\sigma_R(V)$ the class of all algebras $\sigma_R(\mathcal{A})$ with $\mathcal{A} \in V$. As an easy consequence of the definition we have the following result:

Proposition 6.2.2 *If a strong variety V of partial algebras of type τ is fluid then for every regular C-hypersubstitution $\sigma_R \in Hyp_R^C(\tau)$, there holds*

$$\sigma_R(V) \subseteq V \Rightarrow \forall \mathcal{A} \in V(\sigma_R(\mathcal{A}) \cong \mathcal{A}).$$

Proposition 6.2.3 *Let V be a strong variety of partial algebras of type τ. Then for all $\sigma_R \in Hyp_R^C(\tau)$, we have $\sigma_R(V) \subseteq V$ iff $\sigma_R \in P(V)$.*

Proof. Assume that $\sigma_R(V) \subseteq V$. Let $s \approx t \in Id^sV$. Then $Id^sV \subseteq Id^s\sigma_R(V)$ and we have $s \approx t \in Id^s\sigma_R(V)$. So, $\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t] \in Id^sV$ by Proposition 3.2.5. Therefore $\sigma_R \in P(V)$. Conversely, we assume that $\sigma_R \in P(V)$. Let $\mathcal{A} \in \sigma_R(V)$ and $s \approx t \in Id^sV$. Then $\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t] \in Id^sV$ by $\sigma_R \in P(V)$ and $s \approx t \in Id^s\sigma_R(V)$ by Proposition 3.2.5. Since $\mathcal{A} \in \sigma_R(V)$ we have $s \approx t \in Id^s\mathcal{A}$ and $\mathcal{A} \in V$. So, $\sigma_R(V) \subseteq V$. ■

This shows that if a strong variety V of partial algebras of type τ is fluid, then for every regular hypersubstitution $\sigma_R \in Hyp_R^C(\tau)$, there holds

$$\sigma_R \in P(V) \Rightarrow \forall \mathcal{A} \in V(\sigma_R(\mathcal{A}) \cong \mathcal{A}).$$

Proposition 6.2.4 *Let V be a fluid strong variety of partial algebras of type τ. Then $P(V) = [\sigma_{id}]_{\sim_{V-iso}}$.*

Proof. Let $\sigma_R \in P(V)$. Then $\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t] \in Id^sV$ for all $s \approx t \in Id^sV$ implies that $\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t] \in Id^s\mathcal{A}$ for all $\mathcal{A} \in V$. By Proposition 3.2.5, we have $s \approx t \in Id^s\sigma_R(\mathcal{A})$. So, $\sigma_R(\mathcal{A}) \in V$ for all $\mathcal{A} \in V$ and for all $\sigma_R \in Hyp_R^C(\tau)$. Since V is fluid, we have $\sigma_R(\mathcal{A}) \cong \mathcal{A}$ and this implies that $\sigma_R \sim_{V-iso} \sigma_{id}$. Therefore $\sigma_R \in [\sigma_{id}]_{\sim_{V-iso}}$. Thus $P(V) \subseteq [\sigma_{id}]_{\sim_{V-iso}}$ but $[\sigma_{id}]_{\sim_{V-iso}} \subseteq P(V)$. So, $P(V) = [\sigma_{id}]_{\sim_{V-iso}}$. ■

Proposition 6.2.5 *Let V be solid variety of partial algebras of type τ. Then V is fluid iff $P(V) = [\sigma_{id}]_{\sim_{V-iso}}$.*

Proof. By Proposition 6.2.4, we have that if V is fluid then $P(V) = [\sigma_{id}]_{\sim_{V-iso}}$. Conversely, we assume that $P(V) = [\sigma_{id}]_{\sim_{V-iso}}$. Let $\sigma_R \in Hyp_R^C(\tau)$. Since V is solid, we get $\sigma_R(\mathcal{A}) \in V$ for all $\mathcal{A} \in V$. Next, we will show that $\sigma_R \in P(V)$. Suppose that $\sigma_R \notin P(V)$. Then there is an identity $s \approx t \in Id^sV$ such that $\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t] \notin Id^sV$ and this implies that there exists $\mathcal{A} \in V$ such that $\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t] \notin Id^s\mathcal{A}$. By Proposition 3.2.5, we get $s \approx t \notin Id^s\sigma_R(\mathcal{A})$ and $\sigma_R(\mathcal{A}) \notin V$ which is a contradiction. So, $\sigma_R \in P(V) = [\sigma_{id}]_{\sim_{V-iso}}$ and $\sigma_R \sim_{V-iso} \sigma_{id}$. Therefore $\sigma_R(\mathcal{A}) \cong \mathcal{A}$ for all $\mathcal{A} \in V$. Then V is fluid. ■

Let V be a fluid strong variety of partial algebras of type τ and assume W is a subvariety of V. Clearly, W is also fluid since, for all $\mathcal{A} \in W \subseteq V$ and $\sigma_R \in Hyp_R^C(\tau)$, we have

$$\sigma_R(\mathcal{A}) \in W \Rightarrow \sigma_R(\mathcal{A}) \cong \mathcal{A}.$$

Therefore, we have the following :

Proposition 6.2.6 *Every subvariety of a fluid strong variety of partial algebras of type τ is fluid.*

Proposition 6.2.7 *If V is a fluid strong variety of partial algebras of type τ and $[\sigma_{id}]_{\sim_V} = [\sigma_{id}]_{\sim_{V-iso}}$, then V is unsolid.*

Proof. Assume that V is fluid and $[\sigma_{id}]_{\sim_V} = [\sigma_{id}]_{\sim_{V-iso}}$. Let $\sigma_R \in P(V)$. Since V is fluid, we get $\sigma_R(\mathcal{A}) \cong \mathcal{A}$ for all $\mathcal{A} \in V$ (i.e. $\sigma_R \sim_{V-iso} \sigma_{id}$). Therefore $\sigma_R \in [\sigma_{id}]_{\sim_{V-iso}} = [\sigma_{id}]_{\sim_V}$ (i.e. $\sigma_R \sim_V \sigma_{id}$) and we have $\sigma_R \in P_0(V)$. So $P(V) \subseteq P_0(V)$, but since $P_0(V) \subseteq P(V)$ then $P(V) = P_0(V)$. Therefore V is unsolid. ∎

Proposition 6.2.8 *Let V be a strong variety of partial algebras of type τ. Then $\sim_V |_{P(V)}$ is a congruence relation on the algebra $(P(V); \circ_h, \sigma_{id})$.*

Proof. Let $\sigma_{R_1}, \sigma_{R_2} \in P(V)$ such that $\sigma_{R_1} \sim_V |_{P(V)}\ \sigma_{R_2}$ and let $\sigma_R \in P(V)$. Then $\sigma_R(\mathcal{A}) \in V$ for all $\mathcal{A} \in V$.

We show that $\sim_V |_{P(V)}$ is a right-congruence.

$\sigma_{R_1} \sim_V |_{P(V)}\ \sigma_{R_2}$ implies that $\sigma_{R_1}(\mathcal{A}) = \sigma_{R_2}(\mathcal{A})$ for all $\mathcal{A} \in V$ and we get that $\sigma_R(\sigma_{R_1}(\mathcal{A})) = \sigma_R(\sigma_{R_2}(\mathcal{A}))$ since σ_R is a function. So, $\sigma_{R_1} \circ_h \sigma_R \sim_V \sigma_{R_2} \circ_h \sigma_R$ but $\sigma_{R_1} \circ_h \sigma_R, \sigma_{R_2} \circ_h \sigma_R \in P(V)$ because $P(V)$ is a monoid. Therefore $\sigma_{R_1} \circ_h \sigma_R \sim_V |_{P(V)}$ $\sigma_{R_2} \circ_h \sigma_R$.

We show that $\sim_V |_{P(V)}$ is a left-congruence.

$\sigma_R(\mathcal{A}) \in V$ and $\sigma_{R_1} \sim_V |_{P(V)}\ \sigma_{R_2}$ imply that $\sigma_{R_1}(\sigma_R(\mathcal{A})) = \sigma_{R_2}(\sigma_R(\mathcal{A}))$. So, $\sigma_R \circ_h \sigma_{R_1} \sim_V \sigma_R \circ_h \sigma_{R_2}$ but $\sigma_R \circ_h \sigma_{R_1}, \sigma_R \circ_h \sigma_{R_2} \in P(V)$ because $P(V)$ is a monoid. Therefore $\sigma_R \circ_h \sigma_{R_1} \sim_V |_{P(V)}\ \sigma_R \circ_h \sigma_{R_2}$.

So, $\sim_V |_{P(V)}$ is a congruence relation. ∎

Proposition 6.2.9 *Let V be a strong variety of partial algebras of type τ. Then $\sim_{V-iso} |_{P(V)}$ is a congruence relation on the algebra $(P(V); \circ_h, \sigma_{id})$.*

Proof. Let $\sigma_{R_1}, \sigma_{R_2} \in P(V)$ such that $\sigma_{R_1} \sim_{V-iso} |_{P(V)}\ \sigma_{R_2}$ and let $\sigma_R \in P(V)$. Then $\sigma_R(\mathcal{A}) \in V$ for all $\mathcal{A} \in V$.

We show that $\sim_{V-iso} |_{P(V)}$ a right-congruence.

$\sigma_{R_1} \sim_{V-iso} |_{P(V)}\ \sigma_{R_2}$ implies that $\sigma_{R_1}(\mathcal{A}) \cong \sigma_{R_2}(\mathcal{A})$ for all $\mathcal{A} \in V$ and by Lemma 6.1.10, we get that $\sigma_R(\sigma_{R_1}(\mathcal{A})) \cong \sigma_R(\sigma_{R_2}(\mathcal{A}))$. So, $\sigma_{R_1} \circ_h \sigma_R \sim_{V-iso} \sigma_{R_2} \circ_h \sigma_R$ but $\sigma_{R_1} \circ_h \sigma_R, \sigma_{R_2} \circ_h \sigma_R \in P(V)$ because $P(V)$ is a monoid. Therefore $\sigma_{R_1} \circ_h \sigma_R$

$\sim_{V-iso}|_{P(V)}\ \sigma_{R_2}\circ_h\sigma_R$.
We show that $\sim_{V-iso}|_{P(V)}$ is a left-congruence.
Since $\sigma_R(\mathcal{A})\cong\sigma_R(\mathcal{A})$ and $\sigma_R(\mathcal{A})\in V$ then $\sigma_{R_1}(\sigma_R(\mathcal{A}))\cong\sigma_{R_2}(\sigma_R(\mathcal{A}))$. So, $\sigma_R\circ_h\sigma_{R_1}\sim_{V-iso}\sigma_R\circ_h\sigma_{R_2}$ but $\sigma_R\circ_h\sigma_{R_1},\sigma_R\circ_h\sigma_{R_2}\in P(V)$ because $P(V)$ is a monoid. Therefore $\sigma_R\circ_h\sigma_{R_1}\sim_{V-iso}|_{P(V)}\ \sigma_R\circ_h\sigma_{R_2}$.
So, $\sim_{V-iso}|_{P(V)}$ is a congruence relation. ∎

6.3 n-fluid and n-unsolid Strong Varieties

The concepts of fluid and unsolid strong varieties of partial algebras can be generalized in the following way:

Let $1\leq n\in\mathbb{N}^+$. A strong variety V of partial algebras of type τ is called *n-fluid*, if there are $\sigma_{R_1},\ldots,\sigma_{R_n}\in P(V)$ with $\sigma_{R_i}\not\sim_{V-iso}\sigma_{R_j}$ for all $1\leq i\neq j\leq n$ such that for all $\mathcal{A}\in V$ and for all $\sigma_R\in Hyp_R^C(\tau)$ the following implication holds:
$(*)$ If $\sigma_R(\mathcal{A})\in V$, then there is a $k\in\{1,\ldots,n\}$ with $\sigma_R(\mathcal{A})\cong\sigma_{R_k}(\mathcal{A})$.

Proposition 6.3.1 *Let V be an n-fluid strong variety of partial algebras of type τ. Then $|P(V)/_{\sim_{V-iso}|_{P(V)}}|\geq n$.*

Proof. Since V is n-fluid, there are $\sigma_{R_1},\ldots,\sigma_{R_n}\in P(V)$ with $\sigma_{R_i}\not\sim_{V-iso}\sigma_{R_j}$ for all $1\leq i\neq j\leq n$ such that condition $(*)$ is satisfied. Since $[\sigma_{R_i}]_{\sim_{V-iso}|_{P(V)}}\subseteq P(V)$ for all $i\in\{1,\ldots,n\}$ we have $[\sigma_{R_1}]_{\sim_{V-iso}|_{P(V)}}\cup\ldots\cup[\sigma_{R_n}]_{\sim_{V-iso}|_{P(V)}}\subseteq P(V)$ and $|P(V)/_{\sim_{V-iso}|_{P(V)}}|\geq n$. ∎

A strong variety V of partial algebras of type τ is called *n-unsolid* iff $|P(V)/_{\sim_V|_{P(V)}}|=n$.

By this definition, we have that if V is n-unsolid, then $P(V)=[\sigma_{R_1}]_{\sim_V|_{P(V)}}\cup\ldots\cup[\sigma_{R_n}]_{\sim_V|_{P(V)}}$. where $\sigma_{R_i}\not\sim_V\sigma_{R_j}$ for all $1\leq i\neq j\leq n$. But $[\sigma_{R_i}]_{\sim_V|_{P(V)}}\subseteq[\sigma_{R_i}]_{\sim_{V-iso}|_{P(V)}}\subseteq P(V)$ for all $i\in\{1,\ldots,n\}$. So $P(V)=[\sigma_{R_1}]_{\sim_{V-iso}|_{P(V)}}\cup\ldots\cup[\sigma_{R_n}]_{\sim_{V-iso}|_{P(V)}}$. We have that if V is n-unsolid then $P(V)=[\sigma_{R_1}]_{\sim_V|_{P(V)}}\cup\ldots\cup[\sigma_{R_n}]_{\sim_V|_{P(V)}}=[\sigma_{R_1}]_{\sim_{V-iso}|_{P(V)}}\cup\ldots\cup[\sigma_{R_n}]_{\sim_{V-iso}|_{P(V)}}$.

The following concept generalizes that of an n-fluid variety.

Proposition 6.3.2 *Let $1 \leq n \in \mathbb{N}$ and V be a strong variety of partial algebras of type τ with $\sim_V|_{P(V)} = \sim_{V-iso}|_{P(V)}$. If V is n-fluid then V is k-unsolid for $k \geq n$.*

Proof. Assume that V is n-fluid. Then we have $|P(V)/_{\sim_{V-iso}|_{P(V)}}| \geq n$. Since $\sim_V|_{P(V)} = \sim_{V-iso}|_{P(V)}$ we get $|P(V)/_{\sim_{V-iso}|_{P(V)}}| = |P(V)/_{\sim_V|_{P(V)}}| = k$, i.e. V is k-unsolid.

■

6.4 Examples

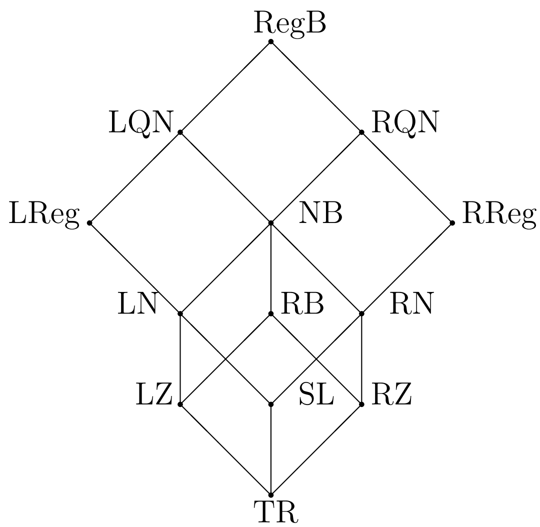
Let B be the strong regular variety

$$B = Mod^{sr}\{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1^2 \approx x_1\},$$

i.e., the class of all partial algebras of type (2) which satisfy the associative and the idempotent law as strong identities. Both equations are regular (i.e. the both sides of the equation have the same variables occurring). We denote by $\sigma_t \in Hyp_R^C(2)$ the regular C-hypersubstitution which maps the binary operation symbol f to the term $t \in W_{(2)}^C(\{x_1, x_2\})$. Instead of $f(x_1, x_2)$ we write simply x_1x_2. The set $Hyp_R^C(2)/_{\sim_B}$ consists precisely of the following classes of hypersubstitutions: $[\sigma_{\varepsilon_1^2(x_1,x_2)}]_{\sim_B}$, $[\sigma_{\varepsilon_2^2(x_1,x_2)}]_{\sim_B}$, $[\sigma_{x_1x_2}]_{\sim_B}$, $[\sigma_{x_2x_1}]_{\sim_B}$, $[\sigma_{x_1x_2x_1}]_{\sim_B}$, $[\sigma_{x_2x_1x_2}]_{\sim_B}$. We will be particularly interested in the following strong regular subvarieties of the strong regular variety B:

$TR = Mod^{sr}\{\varepsilon_1^2(x_1, x_2) \approx \varepsilon_2^2(x_1, x_2)\}$,
$LZ = Mod^{sr}\{x_1x_2 \approx \varepsilon_1^2(x_1, x_2)\}$,
$RZ = Mod^{sr}\{x_1x_2 \approx \varepsilon_2^2(x_1, x_2)\}$,
$SL = Mod^{sr}\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ {x_1}^2 \approx x_1,\ x_1x_2 \approx x_2x_1\}$,
$RB = Mod^{sr}\{x_1(x_2x_3) \approx (x_1x_2)x_3 \approx \varepsilon_1^2(x_1, x_2)x_3,\ {x_1}^2 \approx x_1\}$,
$NB = Mod^{sr}\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ {x_1}^2 \approx x_1,\ x_1x_2x_3x_4 \approx x_1x_3x_2x_4\}$,
$RegB = Mod^{sr}\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ {x_1}^2 \approx x_1,\ x_1x_2x_1x_3x_1 \approx x_1x_2x_3x_1\}$,
$LN = Mod^{sr}\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ {x_1}^2 \approx x_1,\ x_1x_2x_3 \approx x_1x_3x_2\}$,
$RN = Mod^{sr}\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ {x_1}^2 \approx x_1,\ x_1x_2x_3 \approx x_2x_1x_3\}$,
$LReg = Mod^{sr}\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ {x_1}^2 \approx x_1,\ x_1x_2 \approx x_1x_2x_1\}$,

$RReg = Mod^{sr}\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ {x_1}^2 \approx x_1,\ x_1x_2 \approx x_2x_1x_2\}$,
$LQN = Mod^{sr}\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ {x_1}^2 \approx x_1,\ x_1x_2x_3 \approx x_1x_2x_1x_3\}$,
$RQN = Mod^{sr}\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ {x_1}^2 \approx x_1,\ x_1x_2x_3 \approx x_1x_3x_2x_3\}$.
All these varieties are strong regular varieties of partial algebras.
These varieties are given in the following diagram:



This is not the lattice of all strong subvarieties of B since we consider strong regular ones.

A strong regular variety V of partial algebras of type (2) is called *dual solid* if from $s \approx t \in Id^{sr}V$ there follows $\hat{\sigma}_{x_2x_1}[s] \approx \hat{\sigma}_{x_2x_1}[t] \in Id^{sr}V$.

Then we have the following results:

Theorem 6.4.1 *1. TR, LZ, RZ, SL are unsolid.*

2. $LN, RN, LReg, RReg$ are 2-unsolid.

3. B, RB, LQN, RQN are 4-unsolid.

4. NB and $RegB$ are 6-unsolid.

5. All dual solid varieties different from $TR, SL, NB,$ and $RegB$ are 4-unsolid.

6. *Any strong regular variety $V \subseteq B$ other than $LZ, RZ, LN, RN, LReg, RReg, LQN, RQN$ which is not dual-solid is* 3*-unsolid.*

Proof. 1. It is easy to see that TR, LZ, RZ are unsolid. Further, $Hyp_R^C(2) = [\sigma_{\varepsilon_1^2(x_1,x_2)}]_{\sim SL} \cup [\sigma_{\varepsilon_2^2(x_1,x_2)}]_{\sim SL} \cup [\sigma_{x_1x_2}]_{\sim SL}$, where $\sigma_{x_1x_2} \in P(SL)$. The application of $\sigma_{\varepsilon_1^2(x_1,x_2)}$ to $x_1x_2 \approx x_2x_1 \in Id^{sr}SL$ provides $x_1 \approx x_2 \notin Id^{sr}SL$ and the application of $\sigma_{\varepsilon_2^2(x_1,x_2)}$ to $x_1x_2 \approx x_2x_1$ provides $x_2 \approx x_1 \notin Id^{sr}SL$. This shows that $\sigma_{\varepsilon_1^2(x_1,x_2)}, \sigma_{\varepsilon_2^2(x_1,x_2)} \notin P(SL)$. Consequently, $|P(SL)| = |[\sigma_{x_1x_2}]_{\sim SL}| = 1$, i.e. SL is unsolid.
2. It is easy to see that $Hyp_R^C(2) = [\sigma_{\varepsilon_1^2(x_1,x_2)}]_{\sim LN} \cup [\sigma_{\varepsilon_2^2(x_1,x_2)}]_{\sim LN} \cup [\sigma_{x_1x_2}]_{\sim LN} \cup [\sigma_{x_2x_1}]_{\sim LN}$, where $\sigma_{\varepsilon_1^2(x_1,x_2)}, \sigma_{x_1x_2} \in P(LN)$. If we apply $\hat{\sigma}_{\varepsilon_2^2(x_1,x_2)}$ to $x_1x_2x_3 \approx x_1x_3x_2$ we obtain $x_3 \approx x_2$ which is not satisfied in LN and applying $\hat{\sigma}_{x_2x_1}$ to $x_1x_2x_3 \approx x_1x_3x_2$ gives $x_3x_2x_1 \approx x_2x_3x_1$ which is also not satisfied. Therefore $P(LN)/_{\sim_{LN|P(LN)}} = \{[\sigma_{\varepsilon_1^2(x_1,x_2)}]_{\sim LN}, [\sigma_{x_1x_2}]_{\sim LN}\}$, i.e. LN is 2-unsolid. Similarly we can show that RN is 2-unsolid. For $LReg$ and $RReg$ we show in a similar way that these strong varieties are 2-unsolid.
3. It is easy to check that $Hyp_R^C(2) = [\sigma_{\varepsilon_1^2(x_1,x_2)}]_{\sim B} \cup [\sigma_{\varepsilon_2^2(x_1,x_2)}]_{\sim B} \cup [\sigma_{x_1x_2}]_{\sim B} \cup [\sigma_{x_2x_1}]_{\sim B} \cup [\sigma_{x_1x_2x_1}]_{\sim B} \cup [\sigma_{x_2x_1x_2}]_{\sim B}$, where $\sigma_{\varepsilon_1^2(x_1,x_2)}, \sigma_{\varepsilon_2^2(x_1,x_2)}, \sigma_{x_1x_2}, \sigma_{x_2x_1} \in P(B)$. The application of $\sigma_{x_1x_2x_1}$ to the associative law provides $x_1x_2x_1x_3x_1x_2x_1 \approx x_1x_2x_3x_2x_1 \notin Id^{sr}B$ and the application of $\sigma_{x_2x_1x_2}$ to the associative law provides $x_3x_2x_1x_2x_3 \approx x_3x_2x_3x_1x_3x_2x_3 \notin Id^{sr}B$. This shows that $\sigma_{x_1x_2x_1}, \sigma_{x_2x_1x_2} \notin P(B)$. Consequently, $|P(B)/_{\sim_{B|P(B)}}| = |\{[\sigma_{\varepsilon_1^2(x_1,x_2)}]_{\sim B}, [\sigma_{\varepsilon_2^2(x_1,x_2)}]_{\sim B}, [\sigma_{x_1x_2}]_{\sim B}, [\sigma_{x_2x_1}]_{\sim B}\}| = 4$, i.e. B is 4-unsolid. Further we have $Hyp_R^C(2) = [\sigma_{\varepsilon_1^2(x_1,x_2)}]_{\sim RB} \cup [\sigma_{\varepsilon_2^2(x_1,x_2)}]_{\sim RB} \cup [\sigma_{x_1x_2}]_{\sim RB} \cup [\sigma_{x_2x_1}]_{\sim RB}$, where $\sigma_{\varepsilon_1^2(x_1,x_2)}, \sigma_{\varepsilon_2^2(x_1,x_2)}, \sigma_{x_1x_2}, \sigma_{x_2x_1} \in P(RB)$ and $|P(RB)/_{\sim_{B|P(RB)}}| = |\{[\sigma_{\varepsilon_1^2(x_1,x_2)}]_{\sim RB}, [\sigma_{\varepsilon_2^2(x_1,x_2)}]_{\sim RB}, [\sigma_{x_1x_2}]_{\sim RB}, [\sigma_{x_2x_1}]_{\sim RB}\}| = 4$, i.e. RB is 4-unsolid. In a similar one proves that LQN as well as RQN are 4-unsolid.
4. It is easy to check that $Hyp_R^C(2) = [\sigma_{\varepsilon_1^2(x_1,x_2)}]_{\sim NB} \cup [\sigma_{\varepsilon_2^2(x_1,x_2)}]_{\sim NB} \cup [\sigma_{x_1x_2}]_{\sim NB} \cup [\sigma_{x_2x_1}]_{\sim NB} \cup [\sigma_{x_1x_2x_1}]_{\sim NB} \cup [\sigma_{x_2x_1x_2}]_{\sim NB}$. All these hypersubstitutions are NB-proper, i.e. NB is solid. This gives $|P(NB)/_{\sim NB}| = 6$, i.e., NB is 6-unsolid. In a similar way one proves that $RegB$ is 6-unsolid.
5. Let now V be a dual solid variety different from TR, SL, RB, NB and $RegB$. Then we have $Hyp_R^C(2) = [\sigma_{\varepsilon_1^2(x_1,x_2)}]_{\sim V} \cup [\sigma_{\varepsilon_2^2(x_1,x_2)}]_{\sim V} \cup [\sigma_{x_1x_2}]_{\sim V} \cup [\sigma_{x_2x_1}]_{\sim V} \cup$

$[\sigma_{x_1x_2x_1}]_{\sim_V} \cup [\sigma_{x_2x_1x_2}]_{\sim_V}$. Since V is dual solid, the hypersubstitutions $\sigma_{x_1x_2}$ and $\sigma_{x_2x_1}$ are V-proper. As a consequence of $V \neq TR, SL$ and since V is dual solid we have $\sigma_{\varepsilon_1^2(x_1,x_2)}, \sigma_{\varepsilon_2^2(x_1,x_2)} \in P(V)$. The application of $\sigma_{x_1x_2x_1}$ to the associative law provides $x_1x_2x_3x_2x_1 \approx x_1x_2x_1x_3x_1x_2x_1$. From this equation we derive $x_1x_2x_3x_1 \approx x_1x_2x_1x_3x_1$ in the following way

$$\begin{aligned}
x_1x_2x_3x_1 &\approx x_1x_2x_3x_3x_2x_3x_1 \\
&\approx x_1x_2x_3x_1x_3x_1x_2x_3x_1 \\
&\approx x_1x_2x_3x_1x_3x_1x_2x_1x_3x_1 \\
&\approx x_1x_2x_1x_3x_1x_3x_1x_2x_1x_3x_1 \\
&\approx x_1x_2x_1x_3x_1x_2x_1x_3x_1 \\
&\approx x_1x_2x_1x_3x_1x_1x_2x_1x_3x_1 \\
&\approx x_1x_2x_1x_3x_1.
\end{aligned}$$

This shows $V \subseteq RegB$. But TR, SL, RB, NB and $RegB$ are the only dual solid subvarieties of $RegB$. Since V is different from these varieties we have $\sigma_{x_1x_2x_1} \notin P(V)$. The same argument shows $\sigma_{x_2x_1x_2} \notin P(V)$. Since $RB \subseteq V$ the set $Id^{sr}V$ of all strong regular identities satisfied in V consists only of outermost identities and this shows $|P(V)/_{\sim_{V|P(V)}}| = |\{[\sigma_{\varepsilon_1^2(x_1,x_2)}]_{\sim_V}, [\sigma_{\varepsilon_2^2(x_1,x_2)}]_{\sim_V}, [\sigma_{x_1x_2}]_{\sim_V}, [\sigma_{x_2x_1}]_{\sim_V}\}| = 4$, i.e. V is 4-unsolid.

6. Finally if V is not a dual solid variety different from $LZ, RZ, LN, RN, LReg, RReg, LQN, RQN$, then $Hyp_R^C(2) = [\sigma_{\varepsilon_1^2(x_1,x_2)}]_{\sim_V} \cup [\sigma_{\varepsilon_2^2(x_1,x_2)}]_{\sim_V} \cup [\sigma_{x_1x_2}]_{\sim_V} \cup [\sigma_{x_2x_1}]_{\sim_V} \cup [\sigma_{x_1x_2x_1}]_{\sim_V} \cup [\sigma_{x_2x_1x_2}]_{\sim_V}$. We can prove that $\sigma_{x_2x_1}, \sigma_{x_1x_2x_1}, \sigma_{x_2x_1x_2} \notin P(V)$. Then $|P(V)/_{\sim_{V|P(V)}}| = |\{[\sigma_{\varepsilon_1^2(x_1,x_2)}]_{\sim_V}, [\sigma_{\varepsilon_2^2(x_1,x_2)}]_{\sim_V}, [\sigma_{x_1x_2}]_{\sim_V}\}| = 3$, i.e. V is 3-unsolid. ■

Chapter 7

M-solid Strong Quasivarieties

In this chapter we study strong quasivarieties of partial algebras. We first define the concepts of strong quasi-identities and strong quasivarieties. Secondly, we develop the theory of M-solid strong quasivarieties on the basis of two Galois-connections and a pair of additive closure operators. Finally, we use a different definition of a strong M-hyperquasi-identitiy to define weakly M-solid strong quasivarieties.

7.1 Introduction

A *quasi-equation* of type τ is a first order formula of the form

$$e : \forall x_1, \ldots, x_s(s_1 \approx t_1 \wedge s_2 \approx t_2 \wedge \ldots \wedge s_n \approx t_n \Rightarrow u \approx v)$$

where $s_1, \ldots, s_n, t_1, \ldots, t_n, u, v \in W_\tau(X)$ and where $\wedge, \Rightarrow$ are the binary propositional connectives conjunction and implication.
For abbreviation with $e' : s_1 \approx t_1 \wedge s_2 \approx t_2 \wedge \ldots \wedge s_n \approx t_n$ and $e'' : u \approx v$ we write

$$e : \forall x_1, \ldots, x_s(e' \Rightarrow e'').$$

Then the quasi-equation e is satisfied in the partial algebra $\mathcal{A}$ as a strong quasi-identity if from $s_1^{\mathcal{A}} = t_1^{\mathcal{A}} \wedge \ldots \wedge s_n^{\mathcal{A}} = t_n^{\mathcal{A}}$ it follows $u^{\mathcal{A}} = v^{\mathcal{A}}$ ($s^{\mathcal{A}} = t^{\mathcal{A}}$ means that the induced partial term operation $s^{\mathcal{A}}$ is defined whenever the induced partial term operation $t^{\mathcal{A}}$ is defined and both are equal). In this case we write $\mathcal{A} \underset{sq}{\models} e$.

Using the relation $\underset{sq}{\models}$ for every class K of partial algebras of type τ and for every set $Q\Sigma$ of quasi-equations (i.e. implications of the form $e' \Rightarrow e''$) we form the sets

$$QId^sK \quad := \quad \{e \in Q\Sigma \mid \forall \mathcal{A} \in K \ (\mathcal{A} \underset{sq}{\models} e)\} \quad \text{and}$$
$$QMod^sQ\Sigma \quad := \quad \{\mathcal{A} \in PAlg(\tau) \mid \forall\, e \in Q\Sigma \ (\mathcal{A} \underset{sq}{\models} e)\}.$$

Let $QV \subseteq PAlg(\tau)$ be a class of partial algebras. The class QV is called a *strong quasivariety* of partial algebras if $QV = QMod^sQId^sQV$.

In [5] Burmeister considered a different kind of quasi-identities based on *QE*-equations and its model theory. In the next section we study quasi-identities considering C-terms.

7.2 Strong Quasi-identities

In this section, we define strong quasi-identities using terms from $W_\tau^C(X)$.

A *quasi-equation* of type τ is a first order formula of the form

$$ce : \forall x_1, \ldots, x_s(s_1 \approx t_1 \wedge s_2 \approx t_2 \wedge \ldots \wedge s_n \approx t_n \Rightarrow u \approx v)$$

where $s_1, \ldots, s_n, t_1, \ldots, t_n, u, v \in W_\tau^C(X)$ and where $\wedge, \Rightarrow$ are the binary propositional connectives conjunction and implication.
For abbreviation with $ce' : s_1 \approx t_1 \wedge s_2 \approx t_2 \wedge \ldots \wedge s_n \approx t_n$ and $ce'' : u \approx v$ we write

$$ce : \forall x_1, \ldots, x_s(ce' \Rightarrow ce'').$$

Then the quasi-equation ce is satisfied in the partial algebra $\mathcal{A}$ as a *strong quasi-identity* if from $s_1^{\mathcal{A}} = t_1^{\mathcal{A}} \wedge \ldots \wedge s_n^{\mathcal{A}} = t_n^{\mathcal{A}}$ it follows $u^{\mathcal{A}} = v^{\mathcal{A}}$. In this case we write $\mathcal{A} \underset{sq}{\models} ce$.

Let $CQ\Sigma$ be a set of quasi-equations (i.e. implications of the form $ce' \Rightarrow ce''$). Let $Q\tau$ denote the set of all quasi-equations of type τ and let $K \subseteq PAlg(\tau)$ be a class of partial algebras of type τ. Consider the connection between $PAlg(\tau)$ and $Q\tau$ given by the following two operators:

$$QId^s : \mathcal{P}(PAlg(\tau)) \to \mathcal{P}(Q\tau) \text{ and}$$

$$QMod^s : \mathcal{P}(Q\tau) \to \mathcal{P}(PAlg(\tau)) \text{ with}$$

$$QId^sK \quad := \quad \{ce \in CQ\Sigma \mid \forall \mathcal{A} \in K \ (\mathcal{A} \underset{sq}{\models} ce)\} \quad \text{and}$$
$$QMod^sCQ\Sigma \quad := \quad \{\mathcal{A} \in PAlg(\tau) \mid \forall\, ce \in CQ\Sigma \ (\mathcal{A} \underset{sq}{\models} ce)\}.$$

Clearly, the pair $(QMod^s, QId^s)$ is a Galois connection between $PAlg(\tau)$ and $Q\tau$, i.e it satisfies the following properties:

$$K_1 \subseteq K_2 \Rightarrow QId^sK_2 \subseteq QId^sK_1, CQ\Sigma_1 \subseteq CQ\Sigma_2 \Rightarrow QMod^sCQ\Sigma_2 \subseteq QMod^sCQ\Sigma_1$$

and

$$K \subseteq QMod^sQId^sK, CQ\Sigma \subseteq QId^sQMod^sCQ\Sigma.$$

The products $QMod^sQId^s$ and QId^sQMod^s are closure operators and their fixed points form complete lattices.

Let $QV \subseteq PAlg(\tau)$ be a class of partial algebras. The class QV is called a *strong quasivariety* of partial algebras if $QV = QMod^sQId^sQV$.

7.3 Strong Hyperquasi-identities

In [14] hyperquasi-identities for total algebras were introduced. We want to generalize this approach to partial algebras but instead of terms from $W_\tau(X)$ as in [14] we will use terms from $W_\tau^C(X)$.

Let $\mathcal{A}$ be a partial algebra of type τ and let M be a submonoid of the monoid $Hyp_R^C(\tau)$. Then the quasi-equation

$$ce := (s_1 \approx t_1 \wedge ... \wedge s_n \approx t_n \Rightarrow u \approx v)$$

of type τ in $\mathcal{A}$ is a *strong M-hyperquasi-identity* in $\mathcal{A}$ if for every regular C-hypersubstitution $\sigma_R \in M$, the formulas

$$\widehat{\sigma}_R[ce] := (\widehat{\sigma}_R[s_1] \approx \widehat{\sigma}_R[t_1] \wedge ... \wedge \widehat{\sigma}_R[s_n] \approx \widehat{\sigma}_R[t_n] \Rightarrow \widehat{\sigma}_R[u] \approx \widehat{\sigma}_R[v])$$

are strong quasi-identities in $\mathcal{A}$. For $M = Hyp_R^C(\tau)$, we speak simply of a *strong hyperquasi-identity* in $\mathcal{A}$.

A strong quasivariety V of type τ is called *M-solid* if $\chi_M^A[V] = V$. If ce is a strong M-hyperquasi-identity in $\mathcal{A}$ or in V, we will write $\mathcal{A} \underset{sMhq}{\models} ce$ or $V \underset{sMhq}{\models} ce$, respectively.

Example 7.3.1 *Consider the strong regular quasivariety V of type $\tau = (2)$ defined by the following strong quasi-identities:*

(S1) $x(yz) \approx (xy)z$,

(S2) $x^2 \approx x$,

(S3) $xyx \approx \varepsilon_1^2(x,y)$,

(S4) $xy \approx yx \Rightarrow \varepsilon_1^2(x,y) \approx \varepsilon_2^2(x,y)$.

Because of (S1), (S2), (S3) we have to consider exactly the following binary terms over V:

$$t_1(x,y) = \varepsilon_1^2(x,y), t_2(x,y) = \varepsilon_2^2(x,y), t_3(x,y) = xy, t_4(x,y) = yx$$

and the regular hypersubstitutions $\sigma_{t_i}, i = 1, \ldots, 4$ which map the binary operation symbol f to the terms $t_i, i = 1, \ldots, 4$. It is easy to see that the application of each of these regular hypersubstitutions to (S1), (S2), (S3), (S4) gives a strong identity or a strong quasi-identity which is satisfied in V. This is enough to show that V is a solid strong quasivariety.

As usual, the relation $\underset{sMhq}{\models}$ induces a Galois-connection. For any set $CQ\Sigma$ of quasi-equations of type τ and for any class K of partial algebras of type τ we define:

$$\begin{array}{lcl} H_M QMod^s CQ\Sigma & := & \{\mathcal{A} \in PAlg(\tau) \mid \forall ce \in CQ\Sigma(\mathcal{A} \underset{sMhq}{\models} ce)\}, \\ H_M QId^s K & := & \{ce \in CQ\Sigma \mid \forall \mathcal{A} \in K(\mathcal{A} \underset{sMhq}{\models} ce)\}. \end{array}$$

The products $H_M QMod^s H_M QId^s$ and $H_M QId^s H_M QMod^s$ are closure operators. The fixed points with respect to these closure operators form two complete lattices. For a quasi-equation ce, we define $\chi_M^{QE}[ce] := \{\widehat{\sigma}_R[ce] \mid \sigma_R \in M\}$, and for a set $CQ\Sigma$ of quasi-equations we set $\chi_M^{QE}[CQ\Sigma] := \bigcup_{ce \in CQ\Sigma} \chi_M^{QE}[ce]$. Then the following lemma is very easy to prove.

Lemma 7.3.2 *Let M be a submonoid of $Hyp_R^C(\tau)$. Then the pair (χ_M^A, χ_M^{QE}) is a pair of additive closure operators having the property $\chi_M^A[\mathcal{A}] \underset{sq}{\models} ce \Leftrightarrow \mathcal{A} \underset{sq}{\models} \chi_M^{QE}[ce]$ for any quasi-equation ce (a conjugate pair).*

Proof. By definition, χ_M^A and χ_M^{QE} are additive closure operators. We will use that for every term $t \in W_\tau^C(X)$, for every regular C-hypersubstitution σ_R and for every partial algebra $\mathcal{A}$, we have $t^{\sigma_R(\mathcal{A})} = \widehat{\sigma}_R[t]^{\mathcal{A}}$ ([49]). Further we have

$$
\begin{aligned}
&\chi_M^A[\mathcal{A}] \underset{sq}{\models} \; ce \\
&\quad \Leftrightarrow \chi_M^A[\mathcal{A}] \underset{sq}{\models} \; (s_1 \approx t_1 \wedge ... \wedge s_n \approx t_n \Rightarrow u \approx v) \\
&\quad \Leftrightarrow \forall \sigma_R \in M \; (\sigma_R(\mathcal{A}) \underset{sq}{\models} \; (s_1 \approx t_1 \wedge ... \wedge s_n \approx t_n \Rightarrow u \approx v)) \\
&\quad \Leftrightarrow \forall \sigma_R \in M \; (s_1^{\sigma_R(\mathcal{A})} = t_1^{\sigma_R(\mathcal{A})} \wedge \ldots \wedge s_n^{\sigma_R(\mathcal{A})} = t_n^{\sigma_R(\mathcal{A})} \Rightarrow u^{\sigma_R(\mathcal{A})} = v^{\sigma_R(\mathcal{A})}) \\
&\quad \Leftrightarrow \forall \sigma_R \in M \; (\widehat{\sigma}_R[s_1]^{\mathcal{A}} = \widehat{\sigma}_R[t_1]^{\mathcal{A}} \wedge \ldots \wedge \widehat{\sigma}_R[s_n]^{\mathcal{A}} = \widehat{\sigma}_R[t_n]^{\mathcal{A}} \Rightarrow \widehat{\sigma}_R[u]^{\mathcal{A}} = \widehat{\sigma}_R[v]^{\mathcal{A}}) \\
&\quad \Leftrightarrow \forall \sigma_R \in M \; (\mathcal{A} \underset{sq}{\models} \; (\widehat{\sigma}_R[s_1] \approx \widehat{\sigma}_R[t_1] \wedge \ldots \wedge \widehat{\sigma}_R[s_n] \approx \widehat{\sigma}_R[t_n] \Rightarrow \widehat{\sigma}_R[u] \approx \widehat{\sigma}_R[v])) \\
&\quad \Leftrightarrow \forall \sigma_R \in M (\mathcal{A} \underset{sq}{\models} \; \widehat{\sigma}_R[ce]) \\
&\quad \Leftrightarrow \mathcal{A} \underset{sq}{\models} \; \chi_M^{QE}[ce].
\end{aligned}
$$
■

If $CQ\Sigma$ is a set of quasi-equations of type τ, then classes of the form $H_M QMod^s CQ\Sigma$ are called *strong M-hyperquasi-equational* classes and the fixed points under $H_M QId^s H_M QMod^s$ are called *strong M-hyperquasi-equational theories*. Therefore we can characterize M-solid strong quasivarieties by the following conditions:

Theorem 7.3.3 *Let M be a submonoid of $Hyp_R^C(\tau)$. Then for every strong quasivariety $QV \subseteq PAlg(\tau)$ the following conditions are equivalent:*

(i) *QV is a strong M-hyperquasi-equational class.*

(ii) *QV is M-solid, i.e. $\chi_M^A[QV] = QV$.*

(iii) *$QId^s QV = H_M QId^s QV$, i.e. every strong quasi-identity in QV is a strong M-hyperidentity in QV.*

(iv) *$\chi_M^{QE}[QId^s QV] = QId^s QV$, $QId^s QV$ is closed under the operator χ_M^{QE}.*

Proof. (i)$\Rightarrow$ (ii): Since χ_M^A is a closure operator, the inclusion $QV \subseteq \chi_M^A[QV]$ is clear and we have only to show the opposite inclusion. Assume that $\mathcal{B} \in \chi_M^A[QV]$. Then there is a regular C-hypersubstitution $\sigma_R \in M$ and a partial algebra $\mathcal{A} \in QV$

such that $\mathcal{B} = \sigma_R(\mathcal{A})$. Since QV is a strong M-hyperquasi-equational class, there is a set $CQ\Sigma$ of quasi-equations such that $QV = H_M QMod^s CQ\Sigma$ and $\mathcal{A} \in QV$ means that for all regular C-hypersubstitutions $\sigma_R \in M$ and for all $ce \in CQ\Sigma$, we have $\mathcal{A} \underset{sq}{\models} \widehat{\sigma}_R[ce]$. By the conjugate property from Lemma 7.3.2 we have that $\sigma_R(\mathcal{A}) \underset{sq}{\models} ce$ and therefore $\sigma_R(\mathcal{A}) \in QMod^s CQ\Sigma = QV$ since QV is a strong quasivariety.

(ii)$\Rightarrow$ (iii): From $\chi_M^A[QV] = QV$ implies that $QId^s\chi_M^A[QV] = QId^sQV$. Because of

$$\begin{aligned} QId^s\chi_M^A[QV] &= \{ce \mid \forall\sigma_R \in M, \forall\mathcal{A} \in QV(\sigma_R(\mathcal{A}) \underset{sq}{\models} ce)\} \\ &= \{ce \mid \forall\sigma_R \in M, \forall\mathcal{A} \in QV(\mathcal{A} \underset{sq}{\models} \widehat{\sigma}_R[ce])\} \\ &= H_M QId^sQV \end{aligned}$$

we have $H_M QId^sQV = QId^sQV$.

(iii)$\Rightarrow$ (iv): The inclusion $QId^sQV \subseteq \chi_M^{QE}[QId^sQV]$ follows from the property of χ_M^{QE}. We only have to show the opposite inclusion. Let $\sigma_R \in M$ and $ce \in QId^sQV$. Then $\widehat{\sigma}_R[ce] \in QId^sQV$ since $QId^sQV = H_M QId^sQV$.

(iv)$\Rightarrow$ (i): From $\chi_M^{QE}[QId^sQV] = QId^sQV$ by applying the operator $QMod^s$ on both sides we obtain the equation

$$QV = QMod^sQId^sQV = QMod^s(\chi_M^{QE}[QId^sQV]).$$

Considering the right hand side, we get

$$\begin{aligned} QMod^s(\chi_M^{QE}[QId^sQV]) &= \{\mathcal{A} \in PAlg(\tau) \mid \forall ce \in QId^sQV, \forall\sigma_R \in M \ (\mathcal{A} \underset{sq}{\models} \widehat{\sigma}_R[ce])\} \\ &= H_M QMod^sQId^sQV \end{aligned}$$

and therefore with $CQ\Sigma = QId^sQV$ we have shown that QV is a strong M-hyperquasi-equational class. ■

The following theorem is a consequence of the general theory of conjugate pairs of additive closure operators (see [34]).

Theorem 7.3.4 *Let M be a submonoid of $Hyp_R^C(\tau)$. Then for every strong quasi-equational theory $CQ\Sigma$, the following conditions are equivalent:*

(i) *$CQ\Sigma$ is a strong M-hyperquasi-equational theory, i.e. there is a class QV of partial algebras of type τ such that $CQ\Sigma = H_M QId^sQV$.*

(ii) $\chi_M^{QE}[CQ\Sigma] = CQ\Sigma$.

(iii) $QMod^s CQ\Sigma = H_M QMod^s CQ\Sigma$.

(iv) $\chi_M^A[QMod^s CQ\Sigma] = QMod^s CQ\Sigma$.

Proof. The proof goes in a similar way as in ([14]). ∎

7.4 Weakly M-solid Strong Quasivarieties

Now we define a different concept of M-hypersatisfaction of a quasi-equation. This leads us to weakly M-solid strong quasivarieties. We will use the operator χ_M^E introduced in Section 7.3.

Let $\mathcal{A}$ be a partial algebra of type τ, let $\mathcal{M}$ be a monoid of regular C-hypersubstitutions, and let $ce := (s_1 \approx t_1 \wedge ... \wedge s_n \approx t_n \Rightarrow u \approx v)$ be a quasi-equation of type τ. Then ce is called a *weakly strong M-hyperquasi-identity* in $\mathcal{A}$ if the implication:

$$\chi_M^E[\{s_1 \approx t_1 \wedge \ldots \wedge s_n \approx t_n\}] \Rightarrow \chi_M^E[u \approx v]$$

is satisfied in $\mathcal{A}$. In this case we write $\mathcal{A} \underset{wsMhq}{\models} ce$. If every partial algebra $\mathcal{A}$ of a class QV has this property, we write $QV \underset{wsMhq}{\models} ce$.

Proposition 7.4.1 *If ce is a strong M-hyperquasi-identity in the class QV of partial algebras of type τ, then ce is a weakly strong M-hyperquasi-identity in QV but not conversely.*

Proof. If ce is a strong M-hyperquasi-identity in QV then for every $\sigma_R \in M$ we have $\widehat{\sigma}_R[ce] \in QId^s QV$. Therefore we have

$$\forall \sigma_R \in M((\widehat{\sigma}_R[s_1] \approx \widehat{\sigma}_R[t_1] \wedge \ldots \wedge \widehat{\sigma}_R[s_n] \approx \widehat{\sigma}_R[t_n] \Rightarrow \widehat{\sigma}_R[u] \approx \widehat{\sigma}_R[v]) \in QId^s QV).(*)$$

Using the rules of the predicate calculus from $(*)$ we get,
$(\forall \sigma_R \in M(\widehat{\sigma}_R[s_1] \approx \widehat{\sigma}_R[t_1] \wedge \ldots \wedge \widehat{\sigma}_R[s_n] \approx \widehat{\sigma}_R[t_n])$

$\Rightarrow \forall \sigma_R \in M(\widehat{\sigma}_R[u] \approx \widehat{\sigma}_R[v])) \subseteq QId^sQV$

and this means

$$(\chi_M^E[s_1 \approx t_1 \wedge \ldots \wedge s_n \approx t_n] \Rightarrow \chi_M^E[u \approx v]) \subseteq QId^sQV(**)$$

and therefore ce is satisfied as a weakly strong M-hyperquasi-identity in QV.
The converse is not true since it could be possible to find a regular C-hypersubstitution $\sigma_{R_1} \in M$ with

$$\widehat{\sigma}_{R_1}[s_1] \approx \widehat{\sigma}_{R_1}[t_1] \wedge \ldots \wedge \widehat{\sigma}_{R_1}[s_n] \approx \widehat{\sigma}_{R_1}[t_n] \Rightarrow \widehat{\sigma}_{R_1}[u] \approx \widehat{\sigma}_{R_1}[v] \notin QId^sQV$$

even if $(**)$ is satisfied. ■

Using this new concept we define:

A strong quasivariety QV of partial algebras of type τ is *weakly M-solid* if every $ce \in QId^sQV$ is a weakly strong M-hyperquasi-identity in QV. Our next aim is to characterize weakly M-solid strong quasivarieties.
In the usual way the relation $\underset{wsMhq}{\models}$ induces a Galois connection if we define:

$$\begin{array}{lcl} WH_MQMod^sCQ\Sigma & := & \{\mathcal{A} \in PAlg(\tau) \mid \forall ce \in CQ\Sigma(\mathcal{A} \underset{wsMhq}{\models} ce)\}, \\ WH_MQId^sK & := & \{ce \in Q\tau \mid \forall \mathcal{A} \in K(\mathcal{A} \underset{wsMhq}{\models} ce)\}. \end{array}$$

For sets $CQ\Sigma \subseteq Q\tau$ of quasi-equations and classes $QV \subseteq PAlg(\tau)$ of partial algebras of type τ. Then the pair (WH_MQMod^s, WH_MQId^s) is a Galois-connection between the power sets $\mathcal{P}(PAlg(\tau))$ and $\mathcal{P}(Q\tau)$ and the fixed points of the closure operators $WH_MQMod^sWH_MQId^s$ and $WH_MId^sWH_MQMod^s$ form two complete lattices which are dually isomorphic.
We are going to show that strong quasivarieties which are fixed points with respect to $WH_MQMod^sWH_MQId^s$ are weakly M-solid.

Proposition 7.4.2 *If QV is a strong quasivariety of partial algebras of type τ and $WH_MQMod^sWH_MQId^sQV = QV$ then QV is weakly M-solid.*

Proof. From the definition we get
$QV = WH_MQMod^sWH_MQId^sQV$
$= \{\mathcal{A} \in PAlg(\tau) \mid \forall ce \in QId^sQV(\mathcal{A} \underset{wsMhq}{\models} ce)\}$
and this means that every strong quasi-identity in QV is weakly M-solid. ■

If we compare M-solid and weakly M-solid strong quasivarieties, we obtain:

Proposition 7.4.3 *Every M-solid strong quasivariety of type τ is also weakly M-solid.*

Proof. If QV is M-solid, then by definition we have $\chi_M^A[QV] = QV$. The application of Theorem 7.3.3 gives $QId^sQV = H_MQId^sQV \subseteq WH_MQId^sQV$ by Proposition 7.4.1. But this means by definition of weakly M-solid strong quasivarieties that QV is weakly M-solid. ■

The fixed points with respect to the closure operator $WH_MQMod^sWH_MQId^s$ form also a complete lattice and Proposition 7.4.3 shows that this complete lattice contains the complete lattice of all M-solid strong quasivarieties of partial algebras of type τ. This does not yet mean that the complete lattice of M-solid strong quasivarieties is a complete sublattice of the complete lattice of weakly M-solid strong quasivarieties. We want to show that the lattice of all weakly M-solid strong quasivarieties is a complete sublattice of the complete lattice of all strong quasivarieties. A way to characterize complete sublattices of a complete lattice is via Galois-closed subrelations.

We want to apply Theorem 1.2.4 and prove at first.

Lemma 7.4.4 $\underset{wsMhq}{\models}$ *is a Galois closed subrelation of* $\underset{sq}{\models}$.

Proof. Let $\mathcal{A}$ be a partial algebra of type τ and let ce be a quasi-equation of type τ such that $(\mathcal{A}, ce) \in \underset{wsMhq}{\models}$. Then $\mathcal{A} \underset{wsMhq}{\models} ce$ and by definition of weakly strong M-hyperquasi-identity we have $\mathcal{A} \underset{sq}{\models} ce$. Therefore $\underset{wsMhq}{\models} \subseteq \underset{sq}{\models}$.
Assume that $K = WH_MQMod^sCQ\Sigma$ and $CQ\Sigma = WH_MQId^sK$ where $K \subseteq PAlg(\tau)$. If $\mathcal{A} \in K$, then $\mathcal{A} \underset{wsMhq}{\models} CQ\Sigma$, i.e. for all $ce \in CQ\Sigma$ we have $\mathcal{A} \underset{wsMhq}{\models} ce$. But then also $\mathcal{A} \underset{sq}{\models} ce$ by definition of weakly strong M-hyperquasi-identity, therefore $\mathcal{A} \in QMod^sCQ\Sigma$ and $K \subseteq QMod^sCQ\Sigma$.
Conversely, if $\mathcal{A} \in QMod^sCQ\Sigma$, then for every $ce \in CQ\Sigma$ we have $\mathcal{A} \underset{sq}{\models} ce$ and because of $CQ\Sigma = WH_MQId^sK$ also $\mathcal{A} \underset{wsMhq}{\models} ce$ for every $ce \in CQ\Sigma$ and this means

$\mathcal{A} \in WH_M QId^s K = K$. Altogether we have $K = QMod^s CQ\Sigma$.
From $ce \in CQ\Sigma = WH_M QId^s K$ it follows $\mathcal{A} \underset{wsMhq}{\models} ce$ for all $\mathcal{A} \in K$. But then by definition of a weakly strong M-hyperquasi-identity, $\mathcal{A} \underset{sq}{\models} ce$ and this means $ce \in QId^s K$ and thus $CQ\Sigma \subseteq QId^s K$. If $ce \in QId^s K$, then for all $\mathcal{A} \in K = WH_M QMod^s CQ\Sigma$ we have $\mathcal{A} \underset{sq}{\models} ce$, therefore $\mathcal{A} \underset{wsMhq}{\models} ce$ and $ce \in WH_M QId^s K = CQ\Sigma$. This shows that $QId^s K \subseteq CQ\Sigma$ and altogether $CQ\Sigma = QId^s K$. ■

As a consequence we have

Corollary 7.4.5 *For every monoid $\mathcal{M}$ of regular hypersubstitutions the lattice of all weakly M-solid strong quasivarieties is a complete sublattice of the complete lattice of all strong quasivarieties of type τ.*

Proof. This follows with Lemma 7.4.4 from Theorem 1.2.4. ■

The next step is to define the following operator χ_M^{wQE} on sets of quasi-equations. Let $ce : ce' \Rightarrow ce''$ be a quasi-equation. Then

$$\chi_M^{wQE}[ce] := \chi_M^{QE}[ce'] \Rightarrow \chi_M^{QE}[ce''].$$

For sets $CQ\Sigma$ of quasi-equations we define: $\chi_M^{wEQ}[CQ\Sigma] = \bigcup\limits_{ce \in CQ\Sigma} \chi_M^{wQE}[ce]$.

This operator has the following properties:

Proposition 7.4.6 *The operator $\chi_M^{wQE} : \mathcal{P}(Q\tau) \to \mathcal{P}(Q\tau)$ is monotone and idempotent, but in general not extensive.*

Proof. By definition the operator χ_M^{wQE} is additive and therefore monotone. We show the idempotency. Let $CQ\Sigma \subseteq Q\tau$ and $ce \in CQ\Sigma$. Then $\chi_M^{wQE}[ce] = \chi_M^{QE}[ce'] \Rightarrow \chi_M^{QE}[ce'']$ if ce is the implication $ce' \Rightarrow ce''$. Then

$$\begin{array}{rcl} \chi_M^{wQE}[\chi_M^{wQE}[ce]] & = & \chi_M^{QE}[\chi_M^{QE}[ce']] \Rightarrow \chi_M^{QE}[\chi_M^{QE}[ce'']] \\ & = & \chi_M^{QE}[ce'] \Rightarrow \chi_M^{QE}[ce''] \\ & = & \chi_M^{wQE}[ce] \end{array}$$

for every $ce \in CQ\Sigma$ since the operator χ_M^{QE} is idempotent. Since χ_M^{wQE} is additive, we obtain the idempotency. ■

Finally we want to give an example showing that a strong quasivariety can satisfy an implication as a weakly strong M-hyperquasi-identity, but not as a strong M-hyperquasi-identity.
We consider the strong regular quasivariety V of type $\tau = (2)$ defined by

(i) $x(yz) \approx (xy)z$,

(ii) $x^2 \approx x$,

(iii) $xyuv \approx xuyv$,

(iv) $xy \approx yx \Rightarrow \varepsilon_1^2(x, y) \approx \varepsilon_2^2(x, y)$.

There are exactly the following binary terms over $QV : \varepsilon_1^2(x, y), \varepsilon_2^2(x, y), xy, yx, xyx,$ yxy. We prove that (iv) is a weakly strong hyperquasi-identity in QV. That means, for every partial algebra $\mathcal{A} \in QV$ we have to prove

$$(\mathcal{A} \underset{shq}{\models} xy \approx yx) \Rightarrow (\mathcal{A} \underset{shq}{\models} \varepsilon_1^2(x, y) \approx \varepsilon_2^2(x, y)).$$

This becomes clear because of $\mathcal{A} \underset{shq}{\models} xy \approx yx \Leftrightarrow \forall \sigma_R(\mathcal{A} \underset{sq}{\models} \widehat{\sigma}_R[xy] \approx \widehat{\sigma}_R[yx] \Leftrightarrow$ $\mathcal{A} \underset{sq}{\models} \varepsilon_1^2(x, y) \approx \varepsilon_2^2(x, y) \wedge \mathcal{A} \underset{sq}{\models} \varepsilon_2^2(x, y) \approx \varepsilon_1^2(x, y) \wedge \mathcal{A} \underset{sq}{\models} xy \approx yx \wedge \mathcal{A} \underset{sq}{\models} yx \approx xy \wedge$ $\mathcal{A} \underset{sq}{\models} xyx \approx yxy \wedge \mathcal{A} \underset{sq}{\models} yxy \approx xyx)$. The implication $xy \approx yx \Rightarrow \varepsilon_1^2(x, y) \approx \varepsilon_2^2(x, y)$ is satisfied as a weakly strong hyperquasi-identity also in the case if $\mathcal{A} \underset{shq}{\models} xy \approx yx$ is wrong, for instance, if $\mathcal{A} \underset{shq}{\models} \varepsilon_1^2(x, y) \approx \varepsilon_2^2(x, y)$ is not satisfied and if $\mathcal{A} \underset{shq}{\models} \varepsilon_1^2(x, y) \approx$ $\varepsilon_2^2(x, y)$ is satisfied. In this case $\mathcal{A}$ has more than one element and is commutative. But then $xy \approx yx \Rightarrow \varepsilon_1^2(x, y) \approx \varepsilon_2^2(x, y)$ is not a strong quasi-identity in $\mathcal{A}$ and $xy \approx yx \Rightarrow \varepsilon_1^2(x, y) \approx \varepsilon_2^2(x, y)$ is not satisfied as a strong hyperquasi-identity.

Chapter 8

Solidifyable Minimal Partial Clones

In this chapter, we generalize some results of the paper [23] to minimal partial clones. The chapter is divided into three sections. In Section 8.1 we define the concept of equivalence of strong varieties of different types and we show that strong varieties of different types are equivalent if and only if their clones of all term operations of different types are isomorphic. In Section 8.2 we study minimal partial clones in ([3]). In Section 8.3 we define the concept of a strongly solidifyable partial clone and we want to find properties of minimal partial clones which are strongly solidifyable.

8.1 Equivalent Strong Varieties of Partial Algebras

The concept of a hypersubstitution can be generalized to a mapping which assigns operation symbols of one type to terms of a different type (see [49]).

Let $\tau = (f_i)_{i\in I}$, $\tau' = (g_j)_{j\in J}$ be arbitrary types. A mapping

$$ {}_{\tau}^{\tau'}\sigma : \{f_i \mid i \in I\} \to W_{\tau'}^{C}(X), $$

(with arity f_i=arity $\sigma(f_i)$), which assigns to every n_i-ary operation symbol f_i of type τ an n_i-ary term $\sigma(f_i) \in W_{\tau'}^{C}(X)$, is called a (τ, τ')-*hypersubstitution*.

The (τ, τ')-hypersubstitution ${}_{\tau}^{\tau'}\sigma$ is called *regular* if $Var({}_{\tau}^{\tau'}\sigma(f_i)) = \{x_1, \ldots, x_{n_i}\}$ for all operation symbols f_i of type τ.

Let $Hyp_R^C(\tau,\tau')$ denote the set of all regular (τ,τ')-hypersubstitutions and let ${}_{\tau}^{\tau'}\sigma_R$ be some member of $Hyp_R^C(\tau,\tau')$.
Any regular (τ,τ')-hypersubstitution ${}_{\tau}^{\tau'}\sigma_R$ can be extended to a map

$$ {}_{\tau}^{\tau'}\widehat{\sigma}_R : W_{\tau}^C(X) \to W_{\tau'}^C(X) $$

defined for all terms, in the following way:

(i) ${}_{\tau}^{\tau'}\widehat{\sigma}_R[x_j] = x_j$ for $x_j \in X$;

(ii) ${}_{\tau}^{\tau'}\widehat{\sigma}_R[\varepsilon_j^k(t_1,\ldots,t_k)] = \varepsilon_j^k({}_{\tau}^{\tau'}\widehat{\sigma}_R[t_1],\ldots,{}_{\tau}^{\tau'}\widehat{\sigma}_R[t_k])$;

(iii) ${}_{\tau}^{\tau'}\widehat{\sigma}_R[f_i(t_1,\ldots,t_{n_i})] = \overline{S'}_n^{n_i}({}_{\tau}^{\tau'}\sigma_R(f_i),{}_{\tau}^{\tau'}\widehat{\sigma}_R[t_1],\ldots,{}_{\tau}^{\tau'}\widehat{\sigma}_R[t_{n_i}])$.

Lemma 8.1.1 ([49]) *Let* ${}_{\tau}^{\tau'}\sigma_R \in Hyp_R^C(\tau,\tau')$. *Then*

$$ {}_{\tau}^{\tau'}\widehat{\sigma}_R[\overline{S}_n^m(t,t_1,\ldots,t_m)] = \overline{S'}_n^m({}_{\tau}^{\tau'}\widehat{\sigma}_R[t],{}_{\tau}^{\tau'}\widehat{\sigma}_R[t_1],\ldots,{}_{\tau}^{\tau'}\widehat{\sigma}_R[t_m]). $$

Since the extension ${}_{\tau}^{\tau'}\widehat{\sigma}_R$ of the regular (τ,τ')-hypersubstitution ${}_{\tau}^{\tau'}\sigma_R$ preserves arities, every extension ${}_{\tau}^{\tau'}\widehat{\sigma}_R$ defines a family of mappings

$$ {}_{\tau}^{\tau'}\widehat{\sigma}_R = (\eta^{(n)} : W_{\tau}^C(X_n) \to W_{\tau'}^C(X_n))_{n\in\mathbb{N}^+}. $$

Theorem 8.1.2 ([49]) *The extension* ${}_{\tau}^{\tau'}\widehat{\sigma}_R$ *of a regular* (τ,τ')*-hypersubstitution* ${}_{\tau}^{\tau'}\sigma_R$ *defines a homomorphism* $(\eta^{(n)})_{n\in\mathbb{N}^+} : Clone\tau^c \to Clone\tau'^c$ *where*
$Clone\tau^c := ((W_{\tau}^C(X_n))_{n\in\mathbb{N}^+}; (\overline{S}_n^m)_{m,n\in\mathbb{N}^+}, (e_j^k)_{k\in\mathbb{N}^+,1\leq j\leq k})$ *and*
$Clone\tau'^c := ((W_{\tau'}^C(X_n))_{n\in\mathbb{N}^+}; (\overline{S'}_n^m)_{m,n\in\mathbb{N}^+}, (e_j'^k)_{k\in\mathbb{N}^+,1\leq j\leq k})$.

Using our new concept of a hypersubstitution we can define a relation between strong varieties of partial algebras of different types (see [49]).

Let $V \subseteq PAlg(\tau)$ and $V' \subseteq PAlg(\tau')$ be strong varieties of type τ and τ', respectively. Then V and V' are called *equivalent*, $V \sim V'$, if there exist a regular (τ,τ')-hypersubstitution ${}_{\tau}^{\tau'}\sigma_R$ and a regular (τ',τ)-hypersubstitution ${}_{\tau'}^{\tau}\sigma_R$ such that for all $t,t_1,t_2 \in W_{\tau}^C(X)$ and $t',t_1',t_2' \in W_{\tau'}^C(X)$:
(a) $V \underset{s}{\models} t_1 \approx t_2 \Rightarrow V' \underset{s}{\models} {}_{\tau}^{\tau'}\widehat{\sigma}_R[t_1] \approx {}_{\tau}^{\tau'}\widehat{\sigma}_R[t_2]$;
(a') $V' \underset{s}{\models} t_1' \approx t_2' \Rightarrow V \underset{s}{\models} {}_{\tau'}^{\tau}\widehat{\sigma}_R[t_1'] \approx {}_{\tau'}^{\tau}\widehat{\sigma}_R[t_2']$;

(b) $V \underset{s}{\models} {}^{\tau}_{\tau'}\widehat{\sigma}_R[{}^{\tau'}_{\tau}\widehat{\sigma}_R[t]] \approx t$;

(b′) $V' \underset{s}{\models} {}^{\tau'}_{\tau}\widehat{\sigma}_R[{}^{\tau}_{\tau'}\widehat{\sigma}_R[t']] \approx t'$.

Lemma 8.1.3 *Let ${}^{\tau'}_{\tau}\sigma_{R_1}$ and ${}^{\tau'}_{\tau}\sigma_{R_2}$ be regular (τ,τ')-hypersubstitutions and $\mathcal{A} \in PAlg(\tau')$. If ${}^{\tau'}_{\tau}\sigma_{R_1}(f_i)^{\mathcal{A}} =^{\tau'}_{\tau} \sigma_{R_2}(f_i)^{\mathcal{A}}$ for all $i \in I$, then ${}^{\tau'}_{\tau}\widehat{\sigma}_{R_1}[t]^{\mathcal{A}} =^{\tau'}_{\tau} \widehat{\sigma}_{R_2}[t]^{\mathcal{A}}$ for $t \in W^C_{\tau}(X)$.*

The Lemma can be proved by induction on the complexity of terms (see [12]).

Lemma 8.1.4 *For every mapping $h : \{f_i \mid i \in I\} \to T(\mathcal{A})$, $\mathcal{A} \in PAlg(\tau')$, which maps the n_i-ary operation symbol f_i of type τ to an n_i-ary term operation from $T(\mathcal{A})$, there exists a regular (τ,τ')-hypersubstitution ${}^{\tau'}_{\tau}\sigma_R$ such that $h(f_i) =^{\tau'}_{\tau} \sigma_R(f_i)^{\mathcal{A}}$ for all $i \in I$.*

Proof. Let a mapping $h : \{f_i \mid i \in I\} \to T(\mathcal{A})$ i.e. $h(f_i) = t_i^{\mathcal{A}}$ when $t_i \in W^C_{\tau'}(X_{n_i})$ be given. Then we can consider a regular (τ,τ')-hypersubstitution ${}^{\tau'}_{\tau}\sigma_R : \{f_i \mid i \in I\} \to W^C_{\tau'}(X)$ defined by ${}^{\tau'}_{\tau}\sigma_R(f_i) = t_i$, for $i \in I$ and we get that $h(f_i) = t_i^{\mathcal{A}} =^{\tau'}_{\tau} \sigma_R(f_i)^{\mathcal{A}}$ for $i \in I$. ∎

Lemma 8.1.5 *If $\mathcal{A} \in PAlg(\tau)$, $\mathcal{B} \in PAlg(\tau')$, then for every clone homomorphism $\gamma : T(\mathcal{A}) \to T(\mathcal{B})$ there exists a regular (τ,τ')-hypersubstitution ${}^{\tau'}_{\tau}\sigma_R$ such that $\gamma(t^{\mathcal{A}}) =^{\tau'}_{\tau} \widehat{\sigma}_R[t]^{\mathcal{B}}$ for every $t \in W^C_{\tau}(X)$.*

Proof. Let $\mathcal{A} \in PAlg(\tau)$, $\mathcal{B} \in PAlg(\tau')$ and $\gamma : T(\mathcal{A}) \to T(\mathcal{B})$ be a clone homomorphism. Since γ preserves the arity, we can consider a mapping $h : \{f_i \mid i \in I\} \to T(\mathcal{B})$ with $h(f_i) = \gamma(f_i^A)$, for $i \in I$ which preserves the arity and by Lemma 8.1.4, we have a regular (τ,τ')-hypersubstitution ${}^{\tau'}_{\tau}\sigma_R$ such that $h(f_i) =^{\tau'}_{\tau} \sigma_R(f_i)^{\mathcal{B}}$, for $i \in I$. Then we get that $\gamma(f_i^A) =^{\tau'}_{\tau} \sigma_R(f_i)^{\mathcal{B}}$, for $i \in I$. We want to show that $\gamma(t^{\mathcal{A}}) =^{\tau'}_{\tau} \widehat{\sigma}_R[t]^{\mathcal{B}}$ for $t \in W^C_{\tau}(X)$ which can be proved by induction on the complexity of the term t (see [12]). ∎

Proposition 8.1.6 *Let $\mathcal{A} \in PAlg(\tau)$, $\mathcal{B} \in PAlg(\tau')$ be partial algebras and let $V := V(\mathcal{A})$ and $V' := V(\mathcal{B})$ be the strong varieties generated by $\mathcal{A}$ and by $\mathcal{B}$, respectively. Then we have $V \sim V'$ iff $T(\mathcal{A}) \cong T(\mathcal{B})$, i.e. if the clones $T(\mathcal{A})$ and $T(\mathcal{B})$ are isomorphic.*

Proof. Let $\tau = (f_i)_{i\in I}$, $\tau' = (g_j)_{j\in J}$. Let $V \sim V'$. Then there are regular hypersubstitutions ${}^{\tau'}_{\tau}\sigma_R$, ${}^{\tau}_{\tau'}\sigma_R$ satisfying properties $(a) - (b')$ of the definition of $V \sim V'$. Then $\gamma : T(\mathcal{A}) \to T(\mathcal{B})$ with $t^{\mathcal{A}} \mapsto {}^{\tau'}_{\tau} \widehat{\sigma}_R[t]^{\mathcal{B}}$ is well-defined (because of $s^{\mathcal{A}} = t^{\mathcal{A}} \Rightarrow {}^{\tau'}_{\tau} \widehat{\sigma}_R[s]^{\mathcal{B}} = {}^{\tau'}_{\tau} \widehat{\sigma}_R[t]^{\mathcal{B}}$) and by Lemma 8.1.1 we get that γ is a clone homomorphism. Moreover, γ is injective by properties (a') and (b) since

$${}^{\tau'}_{\tau}\widehat{\sigma}_R[s]^{\mathcal{B}} = {}^{\tau'}_{\tau} \widehat{\sigma}_R[t]^{\mathcal{B}} \Rightarrow {}^{\tau}_{\tau'} \widehat{\sigma}_R[{}^{\tau'}_{\tau}\widehat{\sigma}_R[s]]^{\mathcal{A}} = {}^{\tau}_{\tau'} \widehat{\sigma}_R[{}^{\tau'}_{\tau}\widehat{\sigma}_R[t]]^{\mathcal{A}} \Rightarrow s^{\mathcal{A}} = t^{\mathcal{A}},$$

and γ is surjective by property (b') since

$$t'^{\mathcal{B}} = {}^{\tau'}_{\tau} \widehat{\sigma}_R[{}^{\tau}_{\tau'}\widehat{\sigma}_R[t']]^{\mathcal{B}} = \gamma({}^{\tau}_{\tau'}\widehat{\sigma}_R[t']^{\mathcal{A}}).$$

Conversely, let $T(\mathcal{A}) \cong T(\mathcal{B})$ and let $\gamma : T(\mathcal{A}) \to T(\mathcal{B})$ be a clone isomorphism. Then there exist $t_i \in W^C_{\tau'}(X_{n_i})$, $s_j \in W^C_{\tau}(X_{n_j})$ such that $\gamma(f_i^{\mathcal{A}}) = t_i^{\mathcal{B}}$, $\gamma^{-1}(g_j^{\mathcal{B}}) = s_j^{\mathcal{A}}$. We define the regular hypersubstitutions ${}^{\tau'}_{\tau}\sigma_R : f_i \mapsto t_i$, ${}^{\tau}_{\tau'}\sigma_R : g_j \mapsto s_j$. By Lemma 8.1.5 we have $\gamma(t^{\mathcal{A}}) = {}^{\tau'}_{\tau} \widehat{\sigma}_R[t]^{\mathcal{B}}$, $\gamma^{-1}(t'^{\mathcal{B}}) = {}^{\tau}_{\tau'} \widehat{\sigma}_R[t']^{\mathcal{A}}$ for $t \in W^C_{\tau}(X)$ and $t' \in W^C_{\tau'}(X)$. We are going to show that ${}^{\tau'}_{\tau}\sigma_R$, ${}^{\tau}_{\tau'}\widehat{\sigma}_R$ fulfil properties $(a) - (b')$, what implies $V \sim V'$.
(a) $V \underset{s}{\models} s \approx t \Rightarrow s^{\mathcal{A}} = t^{\mathcal{A}} \Rightarrow {}^{\tau'}_{\tau} \widehat{\sigma}_R[s]^{\mathcal{B}} = \gamma(s^{\mathcal{A}}) = \gamma(t^{\mathcal{A}}) = {}^{\tau'}_{\tau} \widehat{\sigma}_R[t]^{\mathcal{B}} \Rightarrow V \underset{s}{\models}$ ${}^{\tau'}_{\tau}\widehat{\sigma}_R[s] \approx {}^{\tau'}_{\tau} \widehat{\sigma}_R[t]$.
Analogously we obtain for (a') (using γ^{-1} instead of γ):

$$(b) {}^{\tau}_{\tau'}\widehat{\sigma}_R[{}^{\tau'}_{\tau}\widehat{\sigma}_R[t]]^{\mathcal{A}} = \gamma^{-1}({}^{\tau'}_{\tau}\widehat{\sigma}_R[t]^{\mathcal{B}}) = \gamma^{-1}(\gamma(t^{\mathcal{A}})) = t^{\mathcal{A}},$$

i.e. $V \underset{s}{\models} {}^{\tau}_{\tau'}\widehat{\sigma}_R[{}^{\tau'}_{\tau}\widehat{\sigma}_R[t]] \approx t$.
In a similar way we conclude for (b'). ∎

8.2 Minimal Partial Clones

The next concept which we have to introduce is the concept of a totally symmetric and totally reflexive relation: (see [3])

A relation $R \subseteq A^n$ on the set A is called *totally symmetric* if for all permutations s on $\{1, \ldots, n\}$

$$(a_1, \ldots, a_n) \in R \Leftrightarrow (a_{s(1)}, \ldots, a_{s(n)}) \in R$$

and *totally reflexive* if $R \supseteq \iota_n$ where ι_n is defined by

$$\iota_n := \{(a_1, \ldots, a_n) \in A^n \mid a_i = a_j \text{ and } 1 \leq i < j \leq n\}.$$

R is called trivial if $R = A^n$.

A binary totally reflexive and totally symmetric relation is reflexive and symmetric in the usual sense.

Let A be a finite set. The lattice $\mathcal{L}_{P(A)}$ of all partial clones is atomic ([3]). There are only finitely many minimal partial clones (atoms). In [3] all of them are determined up to the knowledge of the minimal clones in the lattice $\mathcal{L}_{O(A)}$ of all total clones. Unfortunately, in general the total minimal clones are unknown. Lots of work has been done to determine all minimal clones of total operations defined on a finite set ([16], [45]). We will use the following theorem:

Theorem 8.2.1 ([3]) *The lattice $\mathcal{L}_{P(A)}$ of all partial clones on a finite set A is atomic and contains a finite number of atoms. $C \in \mathcal{L}_{P(A)}$ is a minimal partial clone iff C is a minimal total clone or C is generated by a proper partial projection with a nontrivial totally reflexive and totally symmetric domain.*

Example 8.2.2 *For a set F of operations defined on the same set let $\langle F\rangle$ be the clone generated by F. For the two-element set $A = \{0,1\}$ the total minimal clones are the following ones ([42]): $\langle\wedge\rangle$, $\langle\vee\rangle$, $\langle x+y+z\rangle$, $\langle m\rangle$, $\langle c_0^1\rangle$, $\langle c_1^1\rangle$, $\langle N\rangle$, where $\wedge, \vee, N$ denote the conjunction, disjunction and negation. The symbol $+$ denotes the addition modulo 2 and c_0^1, c_1^1 are the unary constant functions with the value 0 and 1, respectively. We denote by m a ternary function defined by $m(x,y,z) = (x\wedge y)\vee(y\wedge z)\vee(x\wedge z)$. Remark that we write $\langle\wedge\rangle$ instead of $\langle\{\wedge\}\rangle$. Since for $n > 2$ every totally symmetric and totally reflexive relation on $\{0,1\}$ is trivial, we have exactly the following proper partial minimal clones on $\{0,1\}$: $\langle e^2_{1,\{(00),(11)\}}\rangle$, $\langle e^1_{1,\{0\}}\rangle$, $\langle e^1_{1,\{1\}}\rangle$, $\langle e^1_{1,\emptyset}\rangle$. Altogether we have 11 minimal partial clones of functions defined on the set $\{0,1\}$.*

In [16] all total minimal clones on a three-element set are determined. There are 84 total minimal clones on $\{0,1,2\}$. Further we have exactly the proper partial minimal clones generated by unary partial projections with the domains $\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\},\emptyset$, and the proper *partial minimal clones generated by binary projections with the domains* $\{(0,0),(1,1),(2,2)\}$, $\{(0,0),(1,1),(2,2),(0,1),(1,0)\}$, $\{(0,0),(1,1),(2,2),(0,2),(2,0)\}$, $\{(0,0),(1,1),(2,2),(1,2),(2,1)\}$, $\{(0,0),(1,1),(2,2),(0,1),(1,0),(0,2),(2,0)\}$, $\{(0,0),(1,1),(2,2),(0,1),(1,0),(1,2),(2,1)\}$, $\{(0,1),(1,0),(0,2),(2,0)\}$. *Since for* $n > 3$ *every totally symmetric and totally reflexive relation on* $\{0,1,2\}$ *is trivial. We have to consider totally symmetric and totally reflexive at most ternary relations. Since the relations have to be totally symmetric by identification of variables one obtains binary proper partial projections except in the case that the domain is* $\{(0,0,0),(1,1,1),(2,2,2)\}$. *In this case by identification of variables one obtains the proper partial binary projection with domain* $\{(0,0),(1,1),(2,2)\}$. *Altogether we have* 98 *partial minimal clones on* $\{0,1,2\}$.

For $|A| > 4$ *not all total minimal clones are known. By* [45] *each total minimal clone can be generated by an operation* f *of one of the following types:*

(1) f *is unary and* $f^2 = f$ *or* $f^p = id$ *for some prime number* p,

(2) f *is binary and idempotent,*

(3) f *is a ternary majority function* ($f(x,x,y) = f(x,y,x) = f(y,x,x) = x$),

(4) f *is the ternary operation* $x + y + z$ *in a Boolean group,*

(5) f *is a semiprojection (i.e.* $ar\, f = n \geq 3$ *and there exists an element* $i \in \{1,\ldots,n\}$ *such that* $f(a_1,\ldots,a_n) = a_i$ *whenever* $a_1,\ldots,a_n$ *are not pairwise different).*

8.3 Strongly Solidifyable Partial Clones

A partial algebra $\mathcal{A}$ is called *strongly solid* if every strong identity is a strong hyperidentity of $\mathcal{A}$.

Example 8.3.1 *Consider the three-element partial algebra* $\mathcal{A} = (\{0,1,2\}; f^A)$ *of type* (1) *with* $dom f^A = \{1,2\}$ *and* $f^A(1) = 1$, $f^A(2) = 0$. *Every strong identity of* $\mathcal{A}$ *can be derived from the strong identity* $f^2(x) \approx f^3(x) \quad (f^n(x) = f(\ldots(f(x))\ldots))$. *The unary terms over* $\mathcal{A}$ *are* $\varepsilon_1^1(x)$, $f(x)$ *and* $f^2(x)$. *Each of them fulfils* $f^2(x) = f^3(x)$. *That means,* $f^2(x) = f^3(x)$ *is a strong hyperidentity and since all strong identities of* $\mathcal{A}$ *can be derived from* $f^2(x) = f^3(x)$ *every strong identity is a strong hyperidentity and* $\mathcal{A}$ *is strongly solid.*

Now we give some conditions under which $\mathcal{A}$ is not strongly solid.

Proposition 8.3.2 *Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a partial algebra with $|A| \geq 2$. Then $\mathcal{A}$ is not strongly solid if it satisfies one of the following conditions:*

(i) *There is a binary commutative operation under the fundamental operations,*

(ii) *there is a total constant operation under the fundamental operations,*

(iii) *there is a nowhere defined (discrete) operation under the fundamental operations,*

(iv) *$\mathcal{A}$ satisfies a strong identity $s \approx t$ with $Left(s) \neq Left(t)$ or $Right(s) \neq Right(t)$, where $Left(s)$ and $Right(s)$ denote the first and the last vaiable, respectively occurring in the term s.*

(v) *$\mathcal{A}$ satisfies a strong identity of the form*

$$f(x_{s_1(1)}, \ldots, x_{s_1(n)}) \approx f(x_{s_2(1)}, \ldots, x_{s_2(n)})$$

with mappings $s_1, s_2 : \{1, \ldots, n\} \to \{1, \ldots, n\}, n \geq 2$, such that $s_1(i) \neq s_2(i)$ for all $i = 1, \ldots, n$.

Proof. We show that $\mathcal{A}$ is not strongly solid indicating a strong identity which is not a strong hyperidentity.
(i) Let f^A be a binary commutative fundamental operation of $\mathcal{A}$. Commutativity of f^A means: $f(x, y) \approx f(y, x)$ is a strong identity. The strong identity $f(x, y) \approx f(y, x)$ is not a strong hyperidentity. This becomes clear if we substitute for the binary operation symbol f in $f(x, y)$, $f(y, x)$ the term $\varepsilon_1^2(x, y)$.
(ii),(iii) A total, constant or nowhere defined unary operation f^A satisfies the strong identity $f(x) \approx f(y)$. The strong identity $f(x) \approx f(y)$ is not a strong hyperidentity. This is evident if we substitute for f in $f(x) \approx f(y)$ the term $\varepsilon_1^1(x)$. If f^A is an n-ary total, constant or nowhere defined operation and $n > 1$, then $f(x_1, x_2 \ldots, x_n) \approx f(x_2, x_1, \ldots, x_n)$ is a strong identity but not a strong hyperidentity. We see this if we substitute for the n-ary operation symbol f in $f(x_1, x_2 \ldots, x_n) \approx f(x_2, x_1, \ldots, x_n)$ the term $\varepsilon_1^n(x_1, \ldots, x_n)$.

(iv) This becomes clear if we substitute for all n-ary operation symbols occurring in terms s, t the term $\varepsilon_1^n(x_1, \ldots, x_n)$ (or the term $\varepsilon_n^n(x_1, \ldots, x_n)$ in the second case in which $Right(s) \neq Right(t)$).
(v) In this case we get the proof substituting for all n-ary operation symbols ($n > 1$) in $f(x_{s_1(1)}, \ldots, x_{s_1(n)}) \approx f(x_{s_2(1)}, \ldots, x_{s_2(n)})$ the term $\varepsilon_j^n(x_1, \ldots, x_n)$ for $j = 1, \ldots, n$. ■

A partial clone $C \subseteq P(A)$ is called *strongly solidifyable* if there exists a strongly solid algebra $\mathcal{A}$ with $C = T(\mathcal{A})$.

From Proposition 8.3.2, we get some criterions for partial clones to be not strongly solidifyable.

Proposition 8.3.3 *Let $C \subseteq P(A)$ be a partial clone, $|A| \geq 2$. If C satisfies one of the following conditions* (1)-(4)*, then C is not strongly solidifyable.*

(1) *C contains a binary commutative operation,*

(2) *C contains a total constant operation,*

(3) *C contains a nowhere defined operation,*

(4) *there exists an $f^A \in C^{(n)}, n \geq 2$, and mappings $s_1, s_2 : \{1, \ldots, n\} \to \{1, \ldots, n\}, n \geq 2$, such that $s_1(i) \neq s_2(i)$ for all $i = 1, \ldots, n$ and $f(x_{s_1(1)}, \ldots, x_{s_1(n)}) \approx f(x_{s_2(1)}, \ldots, x_{s_2(n)})$ is a strong identity in $\mathcal{A}$.*

Proof. If $\mathcal{A}$ is a partial algebra such that $T(\mathcal{A}) = C$, and if C has one of the properties (1) - (4), then $T(\mathcal{A})$ has the same property. We can assume that $\mathcal{A}$ has one of the operations requested in conditions (1) - (4) under its fundamental operations. By Proposition 8.3.2 the partial algebra $\mathcal{A}$ cannot be strongly solid. ■

Since clones of partial operations are total algebras, we can characterize solidifyable clones in the same way as it was done in [23] for clones of total algebras.

Theorem 8.3.4 *C is strongly solidifyable iff C is a free algebra, freely generated by $\{f_i^A \mid i \in I\}$.*

Proof. Assume that C is strongly solidifyable. Then there exists a strongly solid partial algebra $\mathcal{A} = (A; (f^A)_{i\in I})$ such that $C = T(\mathcal{A})$. Let $F^{n,A} := \{f_j^A \mid j \in I$ and f_j^A is n-ary $\}$. Consider an arbitrary sequence $\varphi := (\varphi^{(n)})_{n\in\mathbb{N}^+}$ of mappings with $\varphi^{(n)} : F^{n,A} \to T^n(\mathcal{A})$. For every $n \in \mathbb{N}^+$ and every n-ary f_j^A, there are n-ary C-term operations $t_j^{\mathcal{A}} \in T^n(\mathcal{A})$ with $\varphi^{(n)}(f_j^A) = t_j^{\mathcal{A}}$. This allows us to define a regular C-hypersubstitution σ_R with $\sigma_R(f_j) = t_j$, $j \in I$. Then we have $\varphi^{(n)}(f_j^A) = \sigma_R(f_j)^{\mathcal{A}}$, $j \in I$. Let $\overline{\varphi^{(n)}}(t^{\mathcal{A}}) = \widehat{\sigma}_R[t]^{\mathcal{A}}$ for any $t \in W_\tau^C(X_n)$. Then $(\overline{\varphi^{(n)}})_{n\in\mathbb{N}^+}$ is the extension of $(\varphi^{(n)})_{n\in\mathbb{N}^+}$ since $\overline{\varphi^{(n)}}(f_i^A) = \widehat{\sigma}_R[f_i(x_1,\ldots,x_{n_i})]^{\mathcal{A}} = \sigma_R(f_i)^{\mathcal{A}}$ and $\overline{\varphi} = (\overline{\varphi^{(n)}})_{n\in\mathbb{N}^+}$ is an endomorphism because of

$$\begin{array}{ll}
\multicolumn{2}{l}{\overline{\varphi^{(n)}}(S_m^{n,A}(t^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}))} \\
= & \overline{\varphi^{(n)}}(\overline{S}_m^n(t, t_1, \ldots, t_n)^{\mathcal{A}}) \\
= & \widehat{\sigma}_R[\overline{S}_m^n(t, t_1, \ldots, t_n)]^{\mathcal{A}} \\
= & \overline{S}_m^n(\widehat{\sigma}_R[t], \widehat{\sigma}_R[t_1], \ldots, \widehat{\sigma}_R[t_n])^{\mathcal{A}} \qquad \text{by Lemma 8.1.1} \\
= & S_m^{n,A}(\widehat{\sigma}_R[t]^{\mathcal{A}}, \widehat{\sigma}_R[t_1]^{\mathcal{A}}, \ldots, \widehat{\sigma}_R[t_n]^{\mathcal{A}}) \\
= & S_m^{n,A}(\overline{\varphi^{(n)}}(t^{\mathcal{A}}), \overline{\varphi^{(n)}}(t_1^{\mathcal{A}}), \ldots, \overline{\varphi^{(n)}}(t_n^{\mathcal{A}})) \quad \text{for every } n \geq 1.
\end{array}$$

Therefore any mapping $(\varphi^{(n)})_{n\in\mathbb{N}^+}$ can be extended to an endomorphism of C and C is a free algebra, freely generated by $\{f_i^A \mid i \in I\}$.

Conversely, let C be a free algebra, freely generated by $\{f_i^A \mid i \in I\}$ (i.e. for every map $\varphi : \{f_i^A \mid i \in I\} \to C$ there is a homomorphism (clone homomorphism) $\overline{\varphi} : \langle\{f_i^A \mid i \in I\}\rangle \to C$). Then we have that $C = \langle\{f_i^A \mid i \in I\}\rangle = T(\mathcal{A})$, where $\mathcal{A} = (A; (f_i^A)_{i\in I})$ is a partial algebra. The next step is to show that $\mathcal{A}$ is strongly solid. Let $\sigma_R : \{f_i \mid i \in I\} \to W_\tau^C(X)$ be a regular C-hypersubstitution. Consider a mapping $\gamma : \{f_i^A \mid i \in I\} \to C = T(\mathcal{A})$ with $\gamma(f_i^A) = \sigma_R(f_i)^{\mathcal{A}}$. Then γ can be extended to a clone endomorphism $\overline{\gamma} : \langle\{f_i^A \mid i \in I\}\rangle \to C$ and by Lemma 8.1.5 for every term $t \in W_\tau^C(X)$ we have

$$\begin{array}{lll}
s \approx t \in Id^s\mathcal{A} & \Rightarrow & s^{\mathcal{A}} = t^{\mathcal{A}} \\
& \Rightarrow & \overline{\gamma}(s^{\mathcal{A}}) = \overline{\gamma}(t^{\mathcal{A}}) \\
& \Rightarrow & \widehat{\sigma}_R[s]^{\mathcal{A}} = \widehat{\sigma}_R[t]^{\mathcal{A}} \\
& \Rightarrow & \widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t] \in Id^s\mathcal{A}.
\end{array}$$

Therefore $\mathcal{A}$ is strongly solid. ∎

Proposition 8.3.5 *Let $C, C' \subseteq P(A)$ be clones of partial algebras. If $C \cong C'$ and C is strongly solidifyable then C' is also strongly solidifyable.*

Proof. Since C is strongly solidifyable, there is a partial algebra $\mathcal{A} = (A; (f_i^A)_{i\in I})$

such that $C = T(\mathcal{A}) = \langle\{f_i^A \mid i \in I\}\rangle$. Since $C \cong C'$, there is an isomorphism $\varphi : T(\mathcal{A}) \to C'$ which maps the generating system of $T(\mathcal{A})$ to a generating system of C'. Therefore $C' = \langle\{\varphi(f_i^A) \mid i \in I\}\rangle$ and we get that C' is a free algebra, freely generated by $\{\varphi(f_i^A) \mid i \in I\}$. By Theorem 8.3.4, we have that C' is strongly solidifyable. ■

From the definition of strongly solidifyable clones, from Proposition 8.1.6 and Proposition 8.3.5, we have that

Corollary 8.3.6 *If $\mathcal{A}$ is strongly solid and $V(\mathcal{A}) \sim V(\mathcal{B})$, then $\mathcal{B}$ is strongly solid.*

Now we want to determine all strongly solidifyable partial clones generated by a single unary operation f^A. A partial algebra $\mathcal{A} = (A; f^A)$, $(|A| \geq 2)$, where f^A is a unary operation on A is called *mono-unary*. Every strong identity of a mono-unary partial algebra has the form

$$f^k(x) \approx f^l(x) \quad (k, l \in \{0, 1, \ldots\})$$

or

$$f^k(x) \approx f^k(y) \quad (k \in \{1, 2, \ldots\}).$$

Obviously, identities of the second form cannot be strong hyperidentities because when substituting for the unary operation symbol the term $\varepsilon_1^1(x)$ we would get $\varepsilon_1^1(x) \approx \varepsilon_1^1(y)$ (i.e. $x \approx y$) in contradiction to $|A| > 1$.

For a partial operation $f^A : A \multimap A$ let $Imf^A := \{f^A(a) \mid a \in domf^A\}$ be the image of f^A and let $\lambda(f^A)$ denote the least non-negative m such that $Im(f^A)^m = Im(f^A)^{m+1}$.

Example 8.3.7 *1. Consider the three-element partial algebra $\mathcal{A} = (\{0, 1, 2\}; f^A)$ of type (1) with $domf^A = \{1, 2\}$ and $f^A(1) = 0$, $f^A(2) = 1$. Then we have*

	f^A	$(f^A)^2$	$(f^A)^3$
0	–	–	–
1	0	–	–
2	1	0	–

and $\lambda(f^A) = 3$.

2. *Consider the three-element partial algebra* $\mathcal{A} = (\{0,1,2\}; f^A)$ *of type* (1) *with* $domf^A = \{0,2\}$ *and* $f^A(0) = 0$, $f^A(2) = 0$. *Then we have*

	f^A	$(f^A)^2$
0	0	0
1	–	–
2	0	0

and $\lambda(f^A) = 1$. *Then* $|Im(f^A)^{\lambda(f^A)}| = |Im(f^A)^1| = 1$.

Corollary 8.3.8 *The partial clone generated by the mono-unary partial operation* f^A *contains a constant iff* $|Im(f^A)^{\lambda(f^A)}| = 1$.

Then we have:

Proposition 8.3.9 *A mono-unary partial algebra* $\mathcal{A} = (A; f^A)$, $(|A| \geq 2)$, *is strongly solid iff* $|Im(f^A)^{\lambda(f^A)}| > 1$ *(i.e.* $T(\mathcal{A})$ *contains no constant and no nowhere defined partial operation).*

Proof. Assume $|Im(f^A)^{\lambda(f^A)}| > 1$. Then the powers $(f^A)^m$ are not constant and not nowhere defined operations. Every strong identity of $\mathcal{A}$ is of the form $f^k(x) \approx f^l(x)$. The powers $(f^A)^m$ and the identity operation are the only unary operations of $T(\mathcal{A})$ and satisfy this identity since

$$((f^A)^m)^k(x) = ((f^A)^k)^m(x) = ((f^A)^l)^m(x) = ((f^A)^m)^l(x).$$

Thus every strong identity is a strong hyperidentity, i.e. $\mathcal{A}$ is strongly solid. If $|Im(f^A)^{\lambda(f^A)}| \leq 1$ then $(f^A)^{\lambda(f^A)}$ is a nowhere defined operation or $(f^A)^{\lambda(f^A)}$ is constant. In this case $f^k(x) \approx f^k(y)$ is a strong identity in $\mathcal{A}$ but not a strong hyperidentity in $\mathcal{A}$. This becomes clear when substituting for the unary operation symbols the term $\varepsilon_1^1(x)$. Then we get $\varepsilon_1^1(x) \approx \varepsilon_1^1(y)$ (i.e. $x \approx y$), a contradiction to $|A| > 1$. ■

If we want to determine all solidifyable minimal partial clones following Theorem 8.2.1 we have to investigate the proper partial minimal clones, i.e. the clones generated by a proper partial projection with a nontrivial totally reflexive and totally symmetric domain. We can restrict our investigation to one projection $e^n_{i,D}$ for every totally reflexive and totally symmetric domain D and every n since $e^n_{j,D} \in \langle e^n_{i,D} \rangle$ and $e^n_{i,D} \in \langle e^n_{j,D} \rangle$ for each $1 \leq i, j \leq n$ and thus $\langle e^n_{i,D} \rangle = \langle e^n_{j,D} \rangle$.

We consider the following cases:

(i) $2 < n \leq |A|$.

Choose $i = 1$. Then $\tilde{e}^n_{1,D}(x_1, x_2, x_3, x_4, \ldots, x_n) \approx \tilde{e}^n_{1,D}(x_1, x_3, x_2, x_4, \ldots, x_n)$ where $\tilde{e}^n_{1,D}$ is an operation symbol corresponding to the operation $e^n_{1,D}$, is a strong identity of the algebra $\mathcal{A} = (A; e^n_{1,D})$. Indeed, if $(x_1, x_2, x_3, x_4, \ldots, x_n) \in dom\ e^n_{1,D} (= D)$, then $(x_1, x_3, x_2, x_4, \ldots, x_n) \in D$ since D is totally symmetric and conversely. Further, in the case that both sides are defined, the values agree. The equation $f(x_1, x_2, x_3, x_4, \ldots, x_n) \approx f(x_1, x_3, x_2, x_4, \ldots, x_n)$ is not a strong hyperidentity of $\mathcal{A} = (A; e^n_{1,D})$ since when substituting for the operation symbol f the term $\varepsilon^n_2(x_1, \ldots, x_n)$ we would get $e^{n,A}_2(a_1, \ldots, a_n) \neq e^{n,A}_3(a_1, \ldots, a_n)$ because of $|A| > 2$. This means that $\mathcal{A}$ is not strongly solid. In a similar way for any other $1 < i \leq n$ and any totally symmetric and totally reflexive $D \subseteq A^n$ we get that $(A; e^n_{i,D})$ is not strongly solid. Therefore, the clones $\langle e^n_{i,D} \rangle$ with $n > 2$ and $1 \leq i \leq n$ are not strongly solidifyable.

(ii) $2 = n \leq |A|$.

Let $D \neq \iota_2$, i.e. D is different from the diagonal $\iota_2 = \{(a, a) \mid a \in A\}$. Now we consider the equation

$$\tilde{e}^2_{1,D}(x_1, \tilde{e}^2_{1,D}(x_1, x_2)) \approx \tilde{e}^2_{1,D}(x_1, \tilde{e}^2_{1,D}(x_2, x_1)).$$

Assume that the left hand side is defined, i.e. $(x_1, x_2) \in D$. Then $\tilde{e}^2_{1,D}(x_1, x_2) \approx x_1$ and $(x_1, x_1) \in D$ because of the reflexivity of D. Since D is symmetric we get $(x_2, x_1) \in D$ and therefore $\tilde{e}^2_{1,D}(x_2, x_1) \approx x_2$. From $(x_1, x_2) \in D$ we get that the right hand side is defined. In the same way we get that the left hand side is defined whenever the right hand side is defined and both sides agree. On the other hand, $f(x_1, f(x_1, x_2)) \approx f(x_1, f(x_2, x_1))$ is not a strong hyperidentity of $\mathcal{A} = (A; e^2_{1,D})$

since when we substitute the operation symbol f by the term $\varepsilon_2^2(x_1, x_2)$ we would get $e_2^{2,A}(a_1, a_2) = e_1^{2,A}(a_1, a_2)$ i.e. A would be a one-element set. If D is the diagonal ι_2 we have no contradiction. In this case $e_{1,D}^2$ is commutative and by Proposition 8.3.2(i) we conclude that $\mathcal{A}$ is not strongly solid. In a similar way we get also that $\langle e_{2,D}^2\rangle$ is not strongly solidifyable and therefore clones of the form $\langle e_{i,D}^2\rangle$ when $i \in \{1, 2\}, D = \iota_2$, are not strongly solidifyable.

(iii) $n = 1$.

At first we consider the case that $D \neq \emptyset$. Then all strong identities of the algebra $(A; e_D^1)$ can be derived from the strong identity $\tilde{e}_D^1(x_1) \approx [\tilde{e}_D^1]^2(x_1)$. Clearly, the equation $f(x_1) \approx f^2(x_1)$ is a strong hyperidentity of $\mathcal{A} = (A; e_D^1)$. If $D = \emptyset$, then e_D^1 is the discrete unary function satisfying the strong identity $\tilde{e}_D^1(x_1) \approx \tilde{e}_D^1(x_2)$ for all $x_1, x_2 \in A$. The equation $f(x_1) \approx f(x_2)$ is not a strong hyperidentity. This is evident if we substitute for f in $f(x_1) \approx f(x_2)$ the term $\varepsilon_1^1(x)$.

Together with Theorem 8.2.1 we get our result:

Theorem 8.3.10 *A minimal partial clone C of partial operations on A (A finite, $|A| \geq 2$) is strongly solidifyable iff C has one of the following forms*

(1) *C is generated by a unary operation f^A different from the unary empty function and satisfying $(f^A)^2 = f^A$ or $(f^A)^p = id$ where p is a prime number, id the identity operation on A and C contains no constant operation.*

(2) *C is generated by a binary operation g^A which fulfils the identities*

$$g(x_1, x_1) \approx x_1,\ g(g(x_1, x_2), x_3) \approx g(x_1, g(x_2, x_3)) \approx g(x_1, x_3).$$

Proof. We consider two cases:

case 1. C is generated by a proper partial projection with a nontrivial totally reflexive and totally symmetric domain. Then by the remarks before Theorem 8.3.10, C cannot be strongly solidifyable;

case 2. C is a total minimal clone. Then C is generated by an operation f of one of the types (1) - (5):

(1) f is unary and $f^2 = f$ or $f^p = id$ for some prime number p. Similar to Proposition

8.3.9, we get that $\mathcal{A}$ is a solid algebra and C is strongly solidifyable.

(2) The operation f is binary and idempotent. If the binary operation f satisfies $f(x_1, x_1) \approx x_1$ and $f(x_1, f(x_2, x_3)) \approx f(x_1, x_3)$, then $\langle f \rangle$ is the clone of a rectangular band and since rectangular bands are solid, $\langle f \rangle$ is strongly solidifyable. Conversely, assume that C is minimal, strongly solidifyable and of type (2). Then there exists a solid algebra $\mathcal{A}$ with $C = T(\mathcal{A})$. We may assume that the type of $\mathcal{A} = (A; f^A)$ is (n) since C is minimal and is generated by only one operation which is not a projection. By identification of variables, we get a binary operation $g(x_1, x_2) := f(x_1, x_2, \ldots, x_2)$ which belongs to C. Clearly, g cannot be a projection, otherwise $\mathcal{A}$ satisfies the identity $g(x_1, x_2) \approx x_1$ or the identity $g(x_1, x_2) \approx x_2$. This contradicts the solidity of $\mathcal{A}$. Therefore $\langle g \rangle = C$ and then $(A; g^A)$ is also solid. Let t be an arbitrary binary term over $(A; g^A)$ such that $leftmost(t) = rightmost(t) = x_1$. Assume that $t^{\mathcal{A}}$ is not a projection, then $t^{\mathcal{A}}$ generates C. This means, we can obtain g^A from $t^{\mathcal{A}}$ by superposition and then the term t can be produced by g and variables x_1, x_2 and this gives an equation of the form $g(x_1, x_2) \approx t(x_1, x_2, \ldots, x_2, x_1)$. Since $\mathcal{A}$ is a solid algebra, this cannot be an identity in $\mathcal{A}$ and thus $t^{\mathcal{A}}$ is a projection and the term t satisfies $t(x_1, x_2, \ldots, x_2, x_1) \approx x_1$. Therefore g satisfies the identities $g(x_1, x_1) \approx x_1$ and $g(x_1, g(x_2, x_1)) \approx x_1$.

(3) f is a ternary majority function ($f(x_1, x_1, x_2) \approx f(x_1, x_2, x_1) \approx f(x_2, x_1, x_1) \approx x_1$). Then the identity $f(x_2, x_1, x_1) \approx x_1$ is not a hyperidentity of $\mathcal{A} = (A; f^A)$ since when we substitute for the operation symbol the term $\varepsilon_1^3(x_1, x_2, x_3)$, we get a contradiction.

(4) f is the ternary operation $x_1 + x_2 + x_3$ in a Boolean group. Then we have that $x_1 + x_1 + x_2 \approx x_2 \approx x_2 + x_1 + x_1$ is an identity. The identity $x_1 + x_1 + x_2 \approx x_2$ is not a hyperidentity. This becomes clear if we substitute for the operation symbol the term $\varepsilon_1^3(x_1, x_2, x_3)$.

(5) f is a semiprojection (i.e. ar $f = n \geq 3$ and there exists an element $i \in \{1, \ldots, n\}$ such that $f(x_1, \ldots, x_n) = x_i$ whenever $x_1, \ldots, x_n$ are not pairwise different). Then we have that $f(x_1, x_2, \ldots, x_n) = x_i = f(x_2, x_1, \ldots, x_n)$ where $i \in \{1, \ldots, n\}$. So, the identity $f(x_1, x_2, \ldots, x_n) \approx f(x_2, x_1, \ldots, x_n)$ is not a hyperidentity since when we substitute for the operation symbol the term $\varepsilon_1^n(x_1, \ldots, x_n)$, we get $x_1 \approx x_2$. ∎

In ([23]) was introduced the concept of the degree of representability $degr(C)$ for a clone of total functions. We generalize this concept to clones of partial operations.

Let $C \subseteq P(A)$ be a clone of partial operations. Then the degree of representability $degr(C)$ is the smallest cardinality $|A'|$ such that there is a clone $C' \subseteq P(A')$ with $C \cong C'$.

Proposition 8.3.11 *Let C be a strongly solidifyable minimal partial clone.*
(i) *If $C = \langle f \rangle$, $f^2 = f$ and $dom\ f \subset A$ then $degr(C) = 2$.*
(ii) *If $C = \langle f \rangle$, $f^2 = f$ and $dom\ f = A$ then $degr(C) = 3$.*
(iii) *If $C = \langle f \rangle$, $f^p = id$ then $degr(C) = p$, where p is a prime number.*
(iv) *If $C = \langle f \rangle$ and f is binary then $degr(C) = 4$.*

Proof. (i) If $f^2 = f$ and $dom\ f \subset A$ then $C \cong T(\mathcal{A})$ where $\mathcal{A} = (\{0,1\}; f_0)$ with $f_0(0) = 0$ and $dom\ f_0 = \{0\}$ since in each case the Cayley table of the clone has the form

$$\begin{array}{c|cc} & id & f \\ \hline id & id & f \\ f & f & f \end{array}$$

and thus $C^{(1)} \cong T^{(1)}(\mathcal{A})$. Since C and $T(\mathcal{A})$ are generated by its unary functions we get

$$\langle C^{(1)} \rangle = C \cong T(\mathcal{A}) = \langle T^{(1)}(\mathcal{A}) \rangle.$$

(ii), (iii) and (iv) were proved in ([23]). ∎

Chapter 9

Partial Hyperidentities

In this chapter, we extend the concept of a hypersubstitution to partial hypersubstitutions. In Section 9.1, we define the concepts of partial hypersubstitutions, regular partial hypersubstitutions and we show that set of all regular partial hypersubstitutions forms a submonoid of the set of all partial hypersubstitutions. In Section 9.2, we consider only regular partial hypersubstitutions of type $\tau = (n)$, $n \in \mathbb{N}^+$, and we show that the extension of a partial hypersubstitution is injective if and only if the partial hypersubstitution is a regular partial hypersubstitutions of type $\tau = (n)$ when $n \geq 2$. In Section 9.3 and Section 9.4, we define the concept of a $PHyp_R(\tau)$-solid strong regular variety of partial algebras.

9.1 The Monoid of Partial Hypersubstitutions

Studying partial algebras there is also some interest in partial mappings which are compatible with the partial operations. Such partial homomorphisms were studied for instance in ([7]). It is quite natural to extend the concept of a hypersubstitution to partial ones.

Let $\{f_i \mid i \in I\}$ be a set of operation symbols, indexed by the set I and $W_\tau(X)$ be the set of all terms of type τ. A *partial hypersubstitution* σ on $\{f_i \mid i \in I\}$ of type τ is a partial function

$$\sigma : \{f_i \mid i \in I\} \multimap\!\!\rightarrow W_\tau(X)$$

with the property : $f_i \in dom\sigma \Rightarrow$ arity (f_i) =arity $(\sigma(f_i)) = n_i$.
If $dom\sigma = \{f_i \mid i \in I\}$, then σ is called a (total) hypersubstitution.

If $dom\sigma = \phi$, then σ is a called a discrete hypersubstitution.

Now we introduce a partial superposition operation $\acute{S}^{n_i}_m$, so that

$$\acute{S}^{n_i}_m : W_\tau(X_{n_i}) \times W_\tau(X_m)^{n_i} \multimap\!\!\longrightarrow W_\tau(X_m)$$

which is defined iff at all $n_i + 1$ inputs of $\acute{S}^{n_i}_m$ we have terms of the corresponding arities.

Every partial hypersubstitution σ of type τ induces a partial mapping $\widehat{\sigma}$: $W_\tau(X) \multimap\!\!\longrightarrow W_\tau(X)$ in the following canonical way:

(i) $\widehat{\sigma}[x_j] := x_j$ for all $x_j \in X$.

(ii) If $t_1, \ldots, t_{n_i} \in W_\tau(X_m)$ and $t_1, \ldots, t_{n_i} \in dom\widehat{\sigma}$ and if $f_i \in dom\sigma$, then $\widehat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := \acute{S}^{n_i}_m(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$.

Let $PHyp(\tau)$ be the set of all partial hypersubstitutions of type τ. On this set we introduce a binary operation, denoted by $\circ_p$, by $\sigma_1 \circ_p \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$ where $\circ$ is the usual composition of functions and $dom(\sigma_1 \circ_p \sigma_2) = \{f_i \mid i \in I,\ f_i \in dom\sigma_2$ and $\sigma_2(f_i) \in dom\widehat{\sigma}_1\}$.

Example 9.1.1 *Let f, g be binary operation symbols and let t_1, t_2 be the following terms: $t_1 = f(x_1, x_2)$ and $t_2 = g(x_2, x_1)$. Let $\sigma \in PHyp(2,2)$ be defined by $\sigma(f) = f(x_1, x_2)$ and let $\sigma(g)$ be not defined. Then we have $\widehat{\sigma}[\acute{S}^2_2(x_1, t_1, t_2)] = \widehat{\sigma}[f(x_1, x_2)] = f(x_1, x_2)$. But $\acute{S}^2_2(\widehat{\sigma}[x_1], \widehat{\sigma}[t_1], \widehat{\sigma}[t_2])$ is not defined and therefore $\widehat{\sigma}[\acute{S}^2_2(x_1, t_1, t_2)] \neq \acute{S}^2_2(\widehat{\sigma}[x_1], \widehat{\sigma}[t_1], \widehat{\sigma}[t_2])$.*

Lemma 9.1.2 *Let $\widehat{\sigma}$ be the extension of the partial hypersubstitution σ of type τ. If $\acute{S}^{n_i}_m(\widehat{\sigma}[t], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$ is defined, then*

$$\widehat{\sigma}[\acute{S}^{n_i}_m(t, t_1, \ldots, t_{n_i})] = \acute{S}^{n_i}_m(\widehat{\sigma}[t], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}]).$$

Proof. We will give a proof by induction on the complexity of the term t.
(i) If $t = x_i \in X$, then
$\acute{S}^{n_i}_m(\widehat{\sigma}[t], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$ exists by assumption

$$\begin{array}{lll}
\Rightarrow & \acute{S}^{n_i}_m(\widehat{\sigma}[x_i], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}]) & \text{exists} \\
\Rightarrow & \widehat{\sigma}[t_i] & \text{exists since } \widehat{\sigma}[x_i] = x_i \\
\Rightarrow & \widehat{\sigma}[\acute{S}^{n_i}_m(t, t_1, \ldots, t_{n_i})] & \text{exists since } t = x_i \\
\Rightarrow & \acute{S}^{n_i}_m(t, t_1, \ldots, t_{n_i}) \in dom\widehat{\sigma} &
\end{array}$$

and $\widehat{\sigma}[\acute{S}^{n_i}_m(t, t_1, \ldots, t_{n_i})] = \widehat{\sigma}[t_i] = \acute{S}^{n_i}_m(\widehat{\sigma}[x_i], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}]) = \acute{S}^{n_i}_m(\widehat{\sigma}[t], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$.

(ii) If $t = f_i(s_1, \ldots, s_{n_i})$ and if we assume that $\acute{S}^{n_i}_m(\widehat{\sigma}[s_j], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$ is defined and $\widehat{\sigma}[\acute{S}^{n_i}_m(s_j, t_1, \ldots, t_{n_i})] = \acute{S}^{n_i}_m(\widehat{\sigma}[s_j], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$ for $j = 1, \ldots, n_i$, then

$$\begin{array}{lll}
\multicolumn{2}{l}{\acute{S}^{n_i}_m(\widehat{\sigma}[t], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}]) \quad \text{exists by assumption}} & \\
\Rightarrow & \acute{S}^{n_i}_m(\widehat{\sigma}[f_i(s_1, \ldots, s_{n_i})], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}]) & \text{exists} \\
\Rightarrow & \acute{S}^{n_i}_m(\acute{S}^{n_i}_m(\sigma(f_i), \widehat{\sigma}[s_1], \ldots, \widehat{\sigma}[s_{n_i}]), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}]) & \text{exists} \\
\Rightarrow & \acute{S}^{n_i}_m(\sigma(f_i), \acute{S}^{n_i}_m(\widehat{\sigma}[s_1], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}]), \ldots, \acute{S}^{n_i}_m(\widehat{\sigma}[s_{n_i}], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])) & \text{exists} \\
\Rightarrow & \acute{S}^{n_i}_m(\sigma(f_i), \widehat{\sigma}[\acute{S}^{n_i}_m(s_1, t_1, \ldots, t_{n_i})], \ldots, \widehat{\sigma}[\acute{S}^{n_i}_m(s_{n_i}, t_1, \ldots, t_{n_i})]) & \text{exists} \\
\Rightarrow & \widehat{\sigma}[f_i(\acute{S}^{n_i}_m(s_1, t_1, \ldots, t_{n_i}), \ldots, \acute{S}^{n_i}_m(s_{n_i}, t_1, \ldots, t_{n_i}))] & \text{exists} \\
\Rightarrow & \widehat{\sigma}[\acute{S}^{n_i}_m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_{n_i})] & \text{exists} \\
\Rightarrow & \acute{S}^{n_i}_m(t, t_1, \ldots, t_{n_i}) \in dom\widehat{\sigma} &
\end{array}$$

and we can prove that $\widehat{\sigma}[\acute{S}^{n_i}_m(t, t_1, \ldots, t_{n_i})] = \acute{S}^{n_i}_m(\widehat{\sigma}[t], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$ in a similar way as in the total case. ∎

Lemma 9.1.3 *Let $\sigma_1, \sigma_2 \in PHyp(\tau)$. Then $(\sigma_1 \circ_p \sigma_2)\widehat{}\,[t] = (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[t]$ for $t \in W_\tau(X)$.*

Proof. We will give a proof by induction on the complexity of the term t.

(i) If $t = x_j \in X$ and since $(\sigma_1 \circ_p \sigma_2)\widehat{}\,, \widehat{\sigma}_1, \widehat{\sigma}_2$ are partial functions from $W_\tau(X)$ into $W_\tau(X)$ which are defined on variables, we have $x_j \in dom(\sigma_1 \circ_p \sigma_2)\widehat{}\,, x_j \in dom\widehat{\sigma}_1$ and $x_j \in dom\widehat{\sigma}_2$. Then $x_j \in dom(\sigma_1 \circ_p \sigma_2)\widehat{}\,,\ x_j \in dom(\widehat{\sigma}_1 \circ \widehat{\sigma}_2)$ and $(\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[x_j] = \widehat{\sigma}_1[\widehat{\sigma}_2[x_j]] = \widehat{\sigma}_1[x_j] = x_j = (\sigma_1 \circ_p \sigma_2)\widehat{}\,[x_j]$ for all $x_j \in X$.

(ii) If $t = f_i(t_1, \ldots, t_{n_i})$ and if we assume that $t_j \in dom(\sigma_1 \circ_p \sigma_2)\widehat{}\,$, $t_j \in dom(\widehat{\sigma}_1 \circ \widehat{\sigma}_2)$ and $(\sigma_1 \circ_p \sigma_2)\widehat{}\,[t_j] = (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[t_j]$ for $j = 1, \ldots, n_i$, then

$t \in dom(\sigma_1 \circ_p \sigma_2)\widehat{}$

$\Leftrightarrow f_i \in dom(\sigma_1 \circ_p \sigma_2)$ and $t_1, \ldots, t_{n_i} \in dom(\sigma_1 \circ_p \sigma_2)\widehat{}$

$\Leftrightarrow f_i \in dom\sigma_2$ and $\sigma_2(f_i) \in dom\widehat{\sigma}_1$ and $t_1, \ldots, t_{n_i} \in dom(\widehat{\sigma}_1 \circ \widehat{\sigma}_2)$

$\Leftrightarrow f_i \in dom\sigma_2$ and $\sigma_2(f_i) \in dom\widehat{\sigma}_1$ and $t_1, \ldots, t_{n_i} \in dom\widehat{\sigma}_2$ and $\widehat{\sigma}_2[t_1], \ldots, \widehat{\sigma}_2[t_{n_i}] \in dom\widehat{\sigma}_1$

$\Leftrightarrow f_i(t_1, \ldots, t_{n_i}) \in dom\widehat{\sigma}_2$ and $\acute{S}^{n_i}_m(\sigma_2(f_i), \widehat{\sigma}_2[t_1], \ldots, \widehat{\sigma}_2[t_{n_i}]) \in dom\widehat{\sigma}_1$

$\Leftrightarrow f_i(t_1, \ldots, t_{n_i}) \in dom\widehat{\sigma}_2$ and $\widehat{\sigma}_2[f_i(t_1, \ldots, t_{n_i})] \in dom\widehat{\sigma}_1$
$\Leftrightarrow f_i(t_1, \ldots, t_{n_i}) \in dom(\widehat{\sigma}_1 \circ \widehat{\sigma}_2)$
and $(\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[t] = \widehat{\sigma}_1[\widehat{\sigma}_2[t]]$

$$\begin{aligned}
&= \widehat{\sigma}_1[\acute{S}^{n_i}_m(\sigma_2(f_i), \widehat{\sigma}_2[t_1], \ldots, \widehat{\sigma}_2[t_{n_i}])] \\
&= \acute{S}^{n_i}_m(\widehat{\sigma}_1(\sigma_2(f_i)), \widehat{\sigma}_1[\widehat{\sigma}_2[t_1]], \ldots, \widehat{\sigma}_1[\widehat{\sigma}_2[t_{n_i}]]) \\
&= \acute{S}^{n_i}_m((\sigma_1 \circ_p \sigma_2)(f_i), (\sigma_1 \circ_p \sigma_2)\widehat{}\,[t_1], \ldots, (\sigma_1 \circ_p \sigma_2)\widehat{}\,[t_{n_i}]) \\
&= (\sigma_1 \circ_p \sigma_2)\widehat{}\,[f_i(t_1, \ldots, t_{n_i})] \\
&= (\sigma_1 \circ_p \sigma_2)\widehat{}\,[t].
\end{aligned}$$

■

Lemma 9.1.4 *Let $\sigma_1, \sigma_2, \sigma_3 \in PHyp(\tau)$. Then $((\sigma_1 \circ_p \sigma_2) \circ_p \sigma_3)(f_i) = (\sigma_1 \circ_p (\sigma_2 \circ_p \sigma_3))(f_i)$ for every $i \in I$.*

Proof. At first we show that $dom((\sigma_1 \circ_p \sigma_2) \circ_p \sigma_3) = dom(\sigma_1 \circ_p (\sigma_2 \circ_p \sigma_3))$.
We have $f_i \in dom((\sigma_1 \circ_p \sigma_2) \circ_p \sigma_3)$
$\Leftrightarrow f_i \in dom\sigma_3$ and $\sigma_3(f_i) \in dom(\sigma_1 \circ_p \sigma_2)\widehat{}$
$\Leftrightarrow f_i \in dom\sigma_3$ and $\sigma_3(f_i) \in dom(\widehat{\sigma}_1 \circ \widehat{\sigma}_2)$ by Lemma 9.1.3
$\Leftrightarrow f_i \in dom\sigma_3$ and $\sigma_3(f_i) \in dom\widehat{\sigma}_2$ and $\widehat{\sigma}_2[\sigma_3(f_i)] \in dom\widehat{\sigma}_1$
$\Leftrightarrow f_i \in dom(\sigma_2 \circ_p \sigma_3)$ and $(\sigma_2 \circ_p \sigma_3)(f_i) \in dom\widehat{\sigma}_1$
$\Leftrightarrow f_i \in dom(\sigma_1 \circ_p (\sigma_2 \circ_p \sigma_3))$.
The next step is to prove that $((\sigma_1 \circ_p \sigma_2) \circ_p \sigma_3)(f_i) = (\sigma_1 \circ_p (\sigma_2 \circ_p \sigma_3))(f_i)$. This can be done in a similar way as in the total case when we assume that $f_i \in dom((\sigma_1 \circ_p \sigma_2) \circ_p \sigma_3)$ and $f_i \in dom(\sigma_1 \circ_p (\sigma_2 \circ_p \sigma_3))$. ■

Let σ_{id} be the partial hypersubstitution defined by $\sigma_{id}(f_i) := f_i(x_1, \ldots, x_{n_i})$ for all $i \in I$.

Lemma 9.1.5 *Let $t \in W_\tau(X)$. Then $\widehat{\sigma}_{id}[t] = t$.*

This is clear since σ_{id} by definition is the total identity hypersubstitution.

Lemma 9.1.6 *Let $\sigma \in PHyp(\tau)$. Then $\sigma \circ_p \sigma_{id} = \sigma = \sigma_{id} \circ_p \sigma$.*

Proof. We will prove that $\sigma_{id} \circ_p \sigma = \sigma$.

We have
$$\begin{aligned} dom(\sigma_{id} \circ_p \sigma) &= \{f_i \mid i \in I \text{ and } (\sigma_{id} \circ_p \sigma)(f_i) \text{ exists}\} \\ &= \{f_i \mid i \in I \text{ and } \widehat{\sigma}_{id}[\sigma(f_i)] \text{ exists}\} \\ &= \{f_i \mid i \in I \text{ and } \sigma(f_i) \text{ exists}\} \\ &= dom\sigma \end{aligned}$$
and by Lemma 9.1.5, we have $(\sigma_{id} \circ_p \sigma)(f_i) = \sigma(f_i)$ where $f_i \in dom(\sigma_{id} \circ_p \sigma)$ and $f_i \in dom\sigma$. The second equation can be proved similarly. ■

Theorem 9.1.7 *The algebra $\mathcal{PH}yp(\tau) := (PHyp(\tau); \circ_p, \sigma_{id})$ is a monoid.*

The partial hypersubstitution $\sigma \in PHyp(\tau)$ is called *regular* if the following implication is satisfied : if $f_i \in dom\sigma$, then $Var(\sigma(f_i)) = \{x_1, \ldots, x_{n_i}\}$.

Let $PHyp_R(\tau)$ denote the set of all regular partial hypersubstitutions of type τ and let σ_R be some member of $PHyp_R(\tau)$.

Proposition 9.1.8 *Let σ_R be a regular partial hypersubstitution of type τ. If $t \in dom\widehat{\sigma}_R$, then $Var(\widehat{\sigma}_R[t]) = Var(t)$.*

Proof. We will give a proof by induction on the complexity of the term $t \in dom\widehat{\sigma}_R$.

(i) If $t = x_j \in X$, then $Var(\widehat{\sigma}_R[t]) = \{x_j\} = Var(t)$.

(ii) If $t = f_i(t_1, \ldots, t_{n_i})$ and if we assume that $Var(\widehat{\sigma}_R[t_j]) = Var(t_j)$ for $j = 1, \ldots, n_i$, then
$$\begin{aligned} Var(\widehat{\sigma}_R[t]) &= Var(\acute{S}^{n_i}_m(\sigma_R(f_i), \widehat{\sigma}_R[t_1], \ldots, \widehat{\sigma}_R[t_{n_i}])) \\ &= \bigcup_{j=1}^{n_i} Var(\widehat{\sigma}_R[t_j]) \\ &= \bigcup_{j=1}^{n_i} Var(t_j) \\ &= Var(t). \end{aligned}$$
■

Theorem 9.1.9 *The algebra $\mathcal{PH}yp_R(\tau) := (PHyp_R(\tau); \circ_p, \sigma_{id})$ is a submonoid of $(PHyp(\tau); \circ_p, \sigma_{id})$.*

Proof. We have to prove that the product of two regular partial hypersubstitutions of type τ belongs to the set of all regular partial hypersubstitutions of type τ.

Let $\sigma_{R_1}, \sigma_{R_2} \in PHyp_R(\tau)$.

We have $Var((\sigma_{R_1} \circ_p \sigma_{R_2})(f_i)) = Var(\widehat{\sigma}_{R_1}[\sigma_{R_2}(f_i)])$

$$= Var(\sigma_{R_2}(f_i))$$
$$= \{x_1, \ldots, x_{n_i}\}$$

and clearly, σ_{id} is a regular partial hypersubstitution. Then $(PHyp_R(\tau); \circ_p, \sigma_{id})$ is a submonoid of $(PHyp(\tau); \circ_p, \sigma_{id})$. ∎

9.2 Regular Partial Hypersubstitutions

Now we consider a type which has only one n-ary operation symbol for $n \geq 1$.

Lemma 9.2.1 *If $f \in dom\sigma$ then $t \in dom\widehat{\sigma}$ for all $t \in W_{(n)}(X)$.*

Proof. We will give a proof by induction on the complexity of the term t.

(i) The proposition is clear if $t = x_j$ since $x_j \in dom\widehat{\sigma}$.

(ii) If $t = f(t_1, \ldots, t_n)$ and if we assume that $t_j \in W_{(n)}(X_m)$ and $t_j \in dom\widehat{\sigma}$ for $j = 1, \ldots, n$, then $t \in dom\widehat{\sigma}$ because $\widehat{\sigma}[t] = \acute{S}^n_m(\sigma(f), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])$ exists. ∎

If t is not a variable, then the converse is also true.

Lemma 9.2.2 *If $t \in W_{(n)}(X) \setminus X$ then $f \in dom\sigma$ iff $t \in dom\widehat{\sigma}$.*

Proof. Assume that $t \in dom\widehat{\sigma}$ and $t \in W_{(n)}(X) \setminus X$ then $t = f(t_1, \ldots, t_n) \in dom\widehat{\sigma}$, i.e. $\widehat{\sigma}[t] = \acute{S}^n_m(\sigma(f), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])$ exists. Therefore, $f \in dom\sigma$. ∎

Lemma 9.2.3 *If $f \in dom\sigma$ then $f \in dom\sigma^l$ for all $l \in \mathbb{N}^+$.*

Proof. We will give a proof by induction on l.

For $l = 1$, everything is clear.

For $l = k$, we assume that $f \in dom\sigma^{k-1}$, then $\sigma^k(f) = \widehat{\sigma}[\sigma^{k-1}(f)]$ exists and $f \in dom\sigma^k$ by Lemma 9.2.1.

Therefore, we have $f \in dom\sigma^l$ for all $l \in \mathbb{N}^+$. ∎

Now we will prove that for a regular partial hypersubstitution σ_R the mapping $\widehat{\sigma}_R$ is injective. We need the concept of the *depth* of a term. The depth is defined inductively by the following steps:

(i) $depth(x_j) := 0$, if $x_j \in X$,

(ii) $depth(f_i(t_1, \ldots, t_{n_i})) := max\{depth(t_1), \ldots, depth(t_{n_i})\} + 1$.

Proposition 9.2.4 *If $\sigma_R \in PHyp_R(n)$, $n \geq 2$, and $\widehat{\sigma}_R[t] = \widehat{\sigma}_R[t']$ for $t, t' \in W_{(n)}(X)$, then $t = t'$.*

Proof. Since $n \geq 2$, the regular partial hypersubstitution σ_R maps the n-ary operation symbol f to a term which uses at least two variables and therefore $depth(\sigma_R(f)) \geq 1$. We will give a proof by induction on the complexity (depth) of the term t.
(i) If $t = x_j \in X$ and $f \in dom\sigma_R$, then $t \in dom\widehat{\sigma}_R$ and $\widehat{\sigma}_R[t] = x_j = \widehat{\sigma}_R[t']$. Since for $t' = f(t'_1, \ldots, t'_n)$ we have $0 = depth(\widehat{\sigma}_R[t]) = depth(\widehat{\sigma}_R[t']) = depth(\acute{S}^n_m(\sigma_R(f), \widehat{\sigma}_R[t'_1], \ldots, \widehat{\sigma}_R[t'_n])) \geq 1$, a contradiction. Therefore, t' is also a variable and $t' = x_j$ i.e. $t = t'$.
(ii) If $t = x_j \in X$ and $f \notin dom\sigma_R$, then $t \in dom\widehat{\sigma}_R$ and $\widehat{\sigma}_R[t] = x_j$ therefore $\widehat{\sigma}_R[t']$ exists because of $\widehat{\sigma}_R[t] = \widehat{\sigma}_R[t']$ and $\widehat{\sigma}_R[t'] = x_j$, thus $t' = x_i$ (since, if $t' = f(t'_1, \ldots, t'_n)$ then $\widehat{\sigma}_R[t']$ does not exist) i.e. $t = t'$.
(iii) If $t = f(t_1, \ldots, t_n)$ and $f \in dom\sigma_R$ and if we assume that from $\widehat{\sigma}_R[t_j] = \widehat{\sigma}_R[t'_j]$ follows $t_j = t'_j$ for $j = 1, \ldots, n$, then $\widehat{\sigma}_R[t] = \acute{S}^n_m(\sigma_R(f), \widehat{\sigma}_R[t_1], \ldots, \widehat{\sigma}_R[t_n]) = \widehat{\sigma}_R[t'] = \acute{S}^n_m(\sigma_R(f), \widehat{\sigma}_R[t'_1], \ldots, \widehat{\sigma}_R[t'_n])$. Since $\sigma_R(f)$ uses all variables $x_1, \ldots, x_n$ this is true only if $\widehat{\sigma}_R[t_j] = \widehat{\sigma}_R[t'_j]$ for $j = 1, \ldots, n$ and this means $t = f(t_1, \ldots, t_n) = f(t'_1, \ldots, t'_n) = t'$.
(iv) If $t = f(t_1, \ldots, t_n)$ and $f \notin dom\sigma_R$, then $t \notin dom\widehat{\sigma}_R$ and $\widehat{\sigma}[t]$ does not exist, therefore $\widehat{\sigma}_R[t] \neq \widehat{\sigma}_R[t']$, thus $\widehat{\sigma}_R[t] = \widehat{\sigma}_R[t']$ implies $t = t'$. ∎

Corollary 9.2.5 *Let σ be a partial hypersubstitution of type (n), $n \geq 2$. Then the extension $\widehat{\sigma}$ is injective iff $\sigma \in PHyp_R(n)$.*

Proof. Let $\widehat{\sigma}$ be injective. We will prove that $\sigma \in PHyp_R(n)$. We can consider the following cases:

(i) Let $f \notin dom\sigma$. Then the implication $f \in dom\sigma \Rightarrow Var(\sigma(f)) = \{x_1, \ldots, x_n\}$ is satisfied and $\sigma \in PHyp_R(n)$ by the definition of regular partial hypersubstitutions.

(ii) Let $f \in dom\sigma$ (then $t \in dom\widehat{\sigma}$ for all $t \in W_{(n)}(X)$). Assume that $\sigma \notin PHyp_R(n)$. Then $Var(\sigma(f)) = \{x_{k_1}, \ldots, x_{k_l}\} \subset \{x_1, \ldots, x_n\}$. If $t = f(t_1, \ldots, t_n)$, $t' = f(t'_1, \ldots, t'_n)$ and $t_{k_1} = t'_{k_1}, \ldots, t_{k_l} = t'_{k_l}$, but $t_j \neq t'_j$ for at least one $j \in \{1, \ldots, n\} \setminus \{k_1, \ldots, k_l\}$ then $\widehat{\sigma}[t_{k_1}] = \widehat{\sigma}[t'_{k_1}], \ldots, \widehat{\sigma}[t_{k_l}] = \widehat{\sigma}[t'_{k_l}]$, and $\widehat{\sigma}[t] = \acute{S}^n_m(\sigma(f), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) = \acute{S}^n_m(\sigma(f), \widehat{\sigma}[t'_1], \ldots, \widehat{\sigma}[t'_n]) = \widehat{\sigma}[t']$, but $t \neq t'$. This shows that $\sigma \in PHyp_R(n)$ must hold if $\widehat{\sigma}$ is injective. If conversely, $\sigma \in PHyp_R(n)$ and $\widehat{\sigma}[t] = \widehat{\sigma}[t']$, then by Proposition 9.2.4 we have $t = t'$ and hence $\widehat{\sigma}$ is injective. ∎

For a term $t \in W_\tau(X)$ we denote the first operation symbol (from the left) occurring in t by $firstops(t)$. Now we ask for injective partial hypersubstitutions if τ is an arbitrary type $\tau = (n_i)_{i \in I}$. We consider the following subset of $PHyp_R(\tau)$.

$PHypreg(\tau) := PHyp_R(\tau) \cap \{\sigma \in PHyp(\tau) \mid firstops(\sigma(f_i)) = f_i$ for $f_i \in dom\sigma$ and for $i \in I\}$.

Lemma 9.2.6 *If $firstops(t) = f_i$, if $\sigma \in PHypreg(\tau)$ and $t \in dom\widehat{\sigma}$, then $firstops(\widehat{\sigma}[t]) = firstops(t)$.*

Proof. Since $firstops(t) = f_i$, we can assume that $t = f_i(t_1, \ldots, t_{n_i})$ and $\widehat{\sigma}[t]$ exists since $t \in dom\widehat{\sigma}$. We have

$$\begin{aligned}
firstops(\widehat{\sigma}[t]) &= firstops(\acute{S}^{n_i}_m(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])) \\
&= firstops(f_i(\acute{S}^{n_i}_m(s_1, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}]), \ldots, \acute{S}^{n_i}_m(s_{n_i}, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}]))) \\
&\quad \text{where } \sigma(f_i) := f_i(s_1, \ldots, s_{n_i}) \\
&= f_i \\
&= firstops(t).
\end{aligned}$$ ∎

Proposition 9.2.7 *$(PHypreg(\tau); \circ_p, \sigma_{id})$ is a submonoid of $(PHyp(\tau); \circ_p, \sigma_{id})$.*

Proof. Clearly, $\sigma_{id} \in PHypreg(\tau)$. We have to prove that $\sigma_1 \circ_p \sigma_2 \in PHypreg(\tau)$ for $\sigma_1, \sigma_2 \in PHypreg(\tau)$. We get $firstops((\sigma_1 \circ_p \sigma_2)(f_i)) = firstops(\widehat{\sigma}_1[\sigma_2(f_i)]) =$

$firstops(\sigma_2(f_i)) = f_i$ by Lemma 9.2.6. Since $PHyp_R(\tau)$ is a submonoid of $PHyp(\tau)$ we have that $(PHypreg(\tau); \circ_p, \sigma_{id})$ is a submonoid of $(PHyp(\tau); \circ_p, \sigma_{id})$. ■

Proposition 9.2.8 *Let $\tau = (n_i)_{i \in I}$, $n_i \geq 1$, be an arbitrary type and assume that $\sigma \in PHypreg(\tau)$. If $\widehat{\sigma}[t] = \widehat{\sigma}[t']$, we have $t = t'$.*

Proof. We will give a proof by induction on the complexity of the term t.
(i) If $t = x_j \in X$ and $f_i \in dom\sigma$, then $\widehat{\sigma}[t] = x_i = \widehat{\sigma}[t']$ and $Var(\widehat{\sigma}[t']) = \{x_i\}$. Therefore $depth(\widehat{\sigma}[t']) = 0$ and $t' = x_i$.
(ii) If $t = x_j \in X$ and $f_i \notin dom\sigma$, then $t \in dom\widehat{\sigma}$ and $\widehat{\sigma}[t] = x_i$ therefore $\widehat{\sigma}[t']$ exists and $\widehat{\sigma}[t'] = x_i$, thus $t' = x_i$ (because if $t' = f_i(t'_1, \ldots, t'_{n_i})$ then $\widehat{\sigma}[t']$ does not exist) i.e. $t = t'$.
(iii) If $t = f_i(t_1, \ldots, t_{n_i})$ and $t \in dom\widehat{\sigma}$, then $\widehat{\sigma}[t] = \acute{S}^{n_i}_m(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}]) = \widehat{\sigma}[t']$, therefore $t' \notin X$. Let $t' = f_j(t'_1, \ldots, t'_{n_j})$ then we have $\widehat{\sigma}[t] = \acute{S}^{n_i}_m(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}]) = \acute{S}^{n_j}_m(\sigma(f_j), \widehat{\sigma}[t'_1], \ldots, \widehat{\sigma}[t'_{n_j}]) = \widehat{\sigma}[t']$. Since $firstops(\widehat{\sigma}[t]) = firstops(\widehat{\sigma}[t'])$ we get that $f_i = f_j$ and then $i = j$. We assume that from $\widehat{\sigma}[t_i] = \widehat{\sigma}[t'_i]$ follows $t_i = t'_i$, $i = 1, \ldots, n_i$. Since σ is a regular partial hypersubstitution, from $\widehat{\sigma}[t] = \widehat{\sigma}[t']$ we obtain $\widehat{\sigma}[t_i] = \widehat{\sigma}[t'_i]$, $i = 1, \ldots, n_i$ and can apply the hypothesis. Altogether, $t = t'$.
(iv) If $t = f_i(t_1, \ldots, t_{n_i})$ and $t \notin dom\widehat{\sigma}$, then $\widehat{\sigma}[t]$ does not exist therefore $\widehat{\sigma}[t] \neq \widehat{\sigma}[t']$, thus the implication $\widehat{\sigma}[t] = \widehat{\sigma}[t'] \Rightarrow t = t'$ is true. ■

Now we consider one more submonoid of $PHyp_R(\tau)$.

Let $PHyp_{BR}(\tau) := PHyp_R(\tau) \cap \{\sigma \in PHyp(\tau) \mid ops(\sigma(f_i)) = \{f_i\}$ for $f_i \in dom\sigma\}$.

Lemma 9.2.9 *If $\sigma \in PHyp_{BR}(\tau)$ and $t \in dom\widehat{\sigma}$, then $ops(\widehat{\sigma}[t]) = ops(t)$.*

Proof. We will give a proof by induction on the complexity of the term t.
(i) If $t = x_j \in X$, then $ops(\widehat{\sigma}[t]) = ops(t)$ since $\widehat{\sigma}[t] = t$.
(ii) If $t = f_i(t_1, \ldots, t_{n_i})$ and if we assume that $ops(\widehat{\sigma}[t_j]) = ops(t_j)$, $j = 1, \ldots, n_i$,

then

$$\begin{aligned}
ops(\widehat{\sigma}[t]) &= ops(\acute{S}^{n_i}_m(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])) \\
&= ops(\sigma(f_i)) \cup \bigcup_{j=1}^{n_i} ops(\widehat{\sigma}[t_j]) \\
&= \{f_i\} \cup \bigcup_{j=1}^{n_i} ops(t_j) \\
&= ops(f_i(t_1, \ldots, t_{n_i})) \\
&= ops(t).
\end{aligned}$$

■

Proposition 9.2.10 *$(PHyp_{BR}(\tau); \circ_p, \sigma_{id})$ is a submonoid of $(PHyp(\tau); \circ_p, \sigma_{id})$.*

Proof. Clearly, $\sigma_{id} \in PHyp_{BR}(\tau)$. We have to prove that $\sigma_1 \circ_p \sigma_2 \in PHyp_{BR}(\tau)$ for $\sigma_1, \sigma_2 \in PHyp_{BR}(\tau)$. One has $ops((\sigma_1 \circ_p \sigma_2)(f_i)) = ops(\widehat{\sigma}_1[\sigma_2(f_i)]) = ops(\sigma_2(f_i)) = \{f_i\}$ by Lemma 9.2.9. Therefore, $(PHyp_{BR}(\tau); \circ_p, \sigma_{id})$ is a submonoid of $(PHyp(\tau); \circ_p, \sigma_{id})$. ■

Example 9.2.11 *Let f, g be binary operation symbols. We define hypersubstitutions σ_1, σ_2 by $\sigma_1(f) = f(g(x_1, x_2), x_1)$, $\sigma_2(f) = f(f(x_1, x_2), x_1)$ and $\sigma_1(g) = g(f(x_1, x_2), x_1)$ and $\sigma_2(g) = g(g(x_1, x_2), x_1)$. We have $\sigma_1, \sigma_2 \in PHypreg(2, 2)$ but $\sigma_1 \notin PHyp_{BR}(2, 2)$. Therefore $PHyp_{BR}(\tau) \subset PHypreg(\tau)$.*

Then we have

Corollary 9.2.12 *$(PHyp_{BR}(\tau); \circ_p, \sigma_{id})$ is a proper submonoid of $(PHypreg(\tau); \circ_p, \sigma_{id})$.*

9.3 $PHyp_R(\tau)$-solid Varieties

Let $\mathcal{A} = (A; (f_i^A)_{i \in I})$ be a partial algebra of type τ. If for an arbitrary partial hypersubstitution σ_R we have $f_i \notin dom\sigma_R$, i.e., if the term $\sigma_R(f_i)$ is not defined, then the induced term operation $\sigma_R(f_i)^{\mathcal{A}}$ on the algebra $\mathcal{A}$ is a nowhere defined operation. In the same way, if f_i occurs in the term t, then $\widehat{\sigma}_R[t]$ is not defined and $\widehat{\sigma}_R[t]^{\mathcal{A}}$ is the nowhere defined operation. If $\sigma_R(f_i)$ is defined, then we define the term operation $\widehat{\sigma}_R[t]^{\mathcal{A}}$ in the usual way.

Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a partial algebra of type τ and $\sigma_R \in PHyp_R(\tau)$, then we define $\sigma_R(\mathcal{A}) := (A; (\sigma_R(f_i)^{\mathcal{A}})_{i\in I})$ where $\sigma_R(f_i)^{\mathcal{A}}$ is an n_i-ary partial operation on A. If $\sigma_R(f_i)$ is not defined then $\sigma_R(f_i)^{\mathcal{A}}$ is the nowhere defined n_i-ary operation on A.

Lemma 9.3.1 *Let t be a term from $W_\tau(X)$ and let $\mathcal{A} \in PAlg(\tau)$ and $\sigma_R \in PHyp_R(\tau)$. Then*

$$\widehat{\sigma}_R[t]^{\mathcal{A}} \mid_D = t^{\sigma_R(\mathcal{A})} \mid_D$$

where D is the common domain of both sides.

Proof. We will give a proof by induction on the complexity of the term t.
(i) If $t = x_j \in X$ because of $x_j \in dom\widehat{\sigma}_R$ for all $\sigma_R \in PHyp_R(\tau)$, we have $\widehat{\sigma}_R[t]^{\mathcal{A}} = \widehat{\sigma}_R[x_j]^{\mathcal{A}} = e_j^{n_i,A}$ is defined and $\widehat{\sigma}_R[t]^{\mathcal{A}} = \widehat{\sigma}_R[x_j]^{\mathcal{A}} = e_j^{n_i,A} = e_j^{\sigma_R(\mathcal{A})} = x_j^{\sigma_R(\mathcal{A})} = t^{\sigma_R(\mathcal{A})}$.
(ii) If $t = f_i(t_1, \ldots, t_{n_i})$ and if we assume that $\widehat{\sigma}_R[t]^{\mathcal{A}}$ is defined and $\widehat{\sigma}_R[t_j]^{\mathcal{A}} \mid_D = t_j^{\sigma_R(\mathcal{A})} \mid_D$ for $j = 1, \ldots, n_i$ where $D = \bigcap\limits_{j=1}^{n_i} dom\widehat{\sigma}_R[t_j]^{\mathcal{A}}$, then

$$\begin{array}{rcl}
\widehat{\sigma}_R[t]^{\mathcal{A}} \mid_D & = & \widehat{\sigma}_R[f_i(t_1, \ldots, t_{n_i})]^{\mathcal{A}} \mid_D \\
& = & [\acute{S}_m^{n_i}(\sigma_R(f_i), \widehat{\sigma}_R[t_1], \ldots, \widehat{\sigma}_R[t_{n_i}])]^{\mathcal{A}} \mid_D \\
& = & S_m^{n_i,A}(\sigma_R(f_i)^{\mathcal{A}}, \widehat{\sigma}_R[t_1]^{\mathcal{A}}, \ldots, \widehat{\sigma}_R[t_{n_i}]^{\mathcal{A}}) \mid_D \\
& = & S_m^{n_i,A}(\sigma_R(f_i)^{\mathcal{A}}, \widehat{\sigma}_R[t_1]^{\mathcal{A}} \mid_D, \ldots, \widehat{\sigma}_R[t_{n_i}]^{\mathcal{A}} \mid_D) \\
& = & S_m^{n_i,\sigma_R(\mathcal{A})}(f_i^{\sigma_R(\mathcal{A})}, t_1^{\sigma_R(\mathcal{A})} \mid_D, \ldots, t_{n_i}^{\sigma_R(\mathcal{A})} \mid_D) \\
& = & S_m^{n_i,\sigma_R(\mathcal{A})}(f_i^{\sigma_R(\mathcal{A})}, t_1^{\sigma_R(\mathcal{A})}, \ldots, t_{n_i}^{\sigma_R(\mathcal{A})}) \mid_D \\
& = & f_i(t_1, \ldots, t_{n_i})^{\sigma_R(\mathcal{A})} \mid_D \\
& = & t^{\sigma_R(\mathcal{A})} \mid_D.
\end{array}$$

This shows also that the domain of $\widehat{\sigma}_R[t]^{\mathcal{A}}$ is equal to the domain of $t^{\sigma_R(\mathcal{A})}$.
(iii) Assume $t = f_i(t_1, \ldots, t_{n_i})$ and that $\sigma_R(f_i)$ is not defined. By definition, $\widehat{\sigma}_R[t]^{\mathcal{A}}$ is nowhere defined and $t^{\sigma_R(\mathcal{A})} = S_m^{n_i,\sigma_R(\mathcal{A})}(f_i^{\sigma_R(\mathcal{A})}, t_1^{\sigma_R(\mathcal{A})}, \ldots, t_{n_i}^{\sigma_R(\mathcal{A})}) = S_m^{n_i,\sigma_R(\mathcal{A})}(\sigma_R(f_i)^{\mathcal{A}}, t_1^{\sigma_R(\mathcal{A})}, \ldots, t_{n_i}^{\sigma_R(\mathcal{A})})$ is nowhere defined because $\sigma_R(f_i)$ is not defined. Therefore $\widehat{\sigma}_R[t]^{\mathcal{A}} = t^{\sigma_R(\mathcal{A})}$. ■

Let $\mathcal{A} \in PAlg(\tau)$ and let $\mathcal{PH}yp_R(\tau)$ be the submonoid of $\mathcal{PH}yp(\tau)$. Let $t_1, t_2 \in W_\tau(X)$. Then $t_1 \approx t_2 \in Id^{sr}\mathcal{A}$ is called a $PHyp_R(\tau)$-*hyperidentity* in $\mathcal{A}$ (in symbols $\mathcal{A} \underset{srPh}{\models} t_1 \approx t_2$) if for all $\sigma_R \in PHyp_R(\tau)$ we have $\widehat{\sigma}_R[t_1] \approx \widehat{\sigma}_R[t_2] \in Id^{sr}\mathcal{A}$.

Let $K \subseteq PAlg(\tau)$ be a class of partial algebras of type τ and let $\Sigma \subseteq W_\tau(X)^2$. Consider the connection between $PAlg(\tau)$ and $W_\tau(X)^2$ given by the following two operators

$$Id^{sr}_{Ph} : \mathcal{P}(PAlg(\tau)) \to \mathcal{P}(W_\tau(X)^2) \qquad \text{and}$$

$$Mod^{sr}_{Ph} : \mathcal{P}(W_\tau(X)^2) \to \mathcal{P}(PAlg(\tau)) \qquad \text{with}$$

$$\begin{array}{lcl} Id^{sr}_{Ph}K & := & \{s \approx t \in W_\tau(X)^2 \mid \forall \mathcal{A} \in K \ (\mathcal{A} \underset{srPh}{\models} s \approx t)\} \quad \text{and} \\ Mod^{sr}_{Ph}\Sigma & := & \{\mathcal{A} \in PAlg(\tau) \mid \forall\ s \approx t \in \Sigma \ (\mathcal{A} \underset{srPh}{\models} s \approx t)\}. \end{array}$$

Clearly, the pair $(Mod^{sr}_{Ph}, Id^{sr}_{Ph})$ is a Galois connection between $PAlg(\tau)$ and $W_\tau(X)^2$. Again we have two closure operators $Mod^{sr}_{Ph}Id^{sr}_{Ph}$ and $Id^{sr}_{Ph}Mod^{sr}_{Ph}$ and their sets of fixed points.

Let $\mathcal{A}$ be a partial algebra of type τ and let $PHyp_R(\tau)$ be the monoid of all regular hypersubstitutions. Then we consider the operators
$\chi^A_{Ph} : \mathcal{P}(PAlg(\tau)) \to \mathcal{P}(PAlg(\tau))$ and $\chi^E_{Ph} : \mathcal{P}(W_\tau(X)^2) \to \mathcal{P}(W\tau(X)^2)$

defined by

$$\chi^A_{Ph}[\mathcal{A}] := \{\sigma_R(\mathcal{A}) \mid \sigma_R \in PHyp_R(\tau)\} \qquad \text{and}$$

$$\chi^E_{Ph}[s \approx t] := \{\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t] \mid \sigma_R \in PHyp_R(\tau)\}.$$

For $K \subseteq PAlg(\tau)$ a class of partial algebras of type τ and for $\Sigma \subseteq W_\tau(X)^2$ a set of equations we define $\chi^A_{Ph}[K] := \bigcup\limits_{\mathcal{A} \in K} \chi^A_{Ph}[\mathcal{A}]$ and $\chi^E_{Ph}[\Sigma] := \bigcup\limits_{s \approx t \in \Sigma} \chi^E_{Ph}[s \approx t]$.

Proposition 9.3.2 *For any $K, K' \subseteq PAlg(\tau)$ and $\Sigma, \Sigma' \subseteq W_\tau(X)^2$ the following conditions hold:*

(i) the operators χ^A_{Ph} and χ^E_{Ph} are additive operators on $PAlg(\tau)$ and on $W_\tau(X)^2$ respectively, i.e. we have

$$\begin{array}{lcl} \text{(ii) } \Sigma \subseteq \chi^E_{Ph}[\Sigma], & & \\ \text{(iii) } \Sigma \subseteq \Sigma' & \Rightarrow & \chi^E_{Ph}[\Sigma] \subseteq \chi^E_{Ph}[\Sigma'], \\ \text{(iv) } \chi^E_{Ph}[\chi^E_{Ph}[\Sigma]] & = & \chi^E_{Ph}[\Sigma], \\ \text{(v) } K \subseteq \chi^A_{Ph}[K], & & \\ \text{(vi) } K \subseteq K' & \Rightarrow & \chi^A_{Ph}[K] \subseteq \chi^A_{Ph}[K'], \\ \text{(vii) } \chi^A_{Ph}[\chi^A_{Ph}[K]] & = & \chi^A_{Ph}[K] \end{array}$$

and $(\chi^A_{Ph}, \chi^E_{Ph})$ forms a conjugate pair with respect to the relation

$R := \{(\mathcal{A}, s \approx t) \in PAlg(\tau) \times W_\tau(X)^2 \mid (\mathcal{A} \underset{sr}{\models} s \approx t)\}$ i.e. for all $\mathcal{A} \in PAlg(\tau)$ and for all $s \approx t \in W_\tau(X)^2$, we have $\chi^A_{Ph}[\mathcal{A}] \underset{sr}{\models} s \approx t$ iff $\mathcal{A} \underset{sr}{\models} \chi^E_{Ph}[s \approx t]$.

Proof. (i) It is clear from the definition that both, χ^A_{Ph} and χ^E_{Ph}, are additive operators.

(ii) Let $s \approx t \in \Sigma$. Since $s, t \in dom\widehat{\sigma}_{id}$ by Lemma 9.1.5 we have $\widehat{\sigma}_{id}[s] \approx \widehat{\sigma}_{id}[t]$ for all $s \approx t \in \Sigma$ and we get $\Sigma \subseteq \chi^E_{Ph}[\Sigma]$.

(iii) Suppose $\Sigma \subseteq \Sigma' \subseteq W_\tau(X)^2$, then

$$\begin{array}{rcl}
\chi^E_{Ph}[\Sigma] & = & \bigcup\limits_{s\approx t\in\Sigma} \chi^E_{Ph}[s \approx t] \\
& = & \bigcup\limits_{s\approx t\in\Sigma} \{\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t] \mid \sigma_R \in PHyp_R(\tau) \text{ (we have } s, t \in dom\widehat{\sigma}_R)\} \\
& \subseteq & \bigcup\limits_{s\approx t\in\Sigma'} \{\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t] \mid \sigma_R \in PHyp_R(\tau) \text{ (we have } s, t \in dom\widehat{\sigma}_R)\} \\
& = & \bigcup\limits_{s\approx t\in\Sigma'} \chi^E_{Ph}[s \approx t] \quad = \quad \chi^E_{Ph}[\Sigma'].
\end{array}$$

(iv) Suppose $\sigma_{R_1}, \sigma_{R_2} \in PHyp_R(\tau)$ are arbitrary two regular partial hypersubstitutions and $\widehat{\sigma}_{R_1}[\widehat{\sigma}_{R_2}[s]] \approx \widehat{\sigma}_{R_1}[\widehat{\sigma}_{R_2}[t]]$. Then $s, t \in dom(\widehat{\sigma}_{R_1} \circ \widehat{\sigma}_{R_2}))$ is an equation from $\chi^E_{Ph}[\chi^E_{Ph}[\Sigma]]$. Let $\sigma_R \in PHyp_R(\tau)$ be a regular partial hypersubstitution with $\sigma_R := \sigma_{R_1} \circ_p \sigma_{R_2}$. Since $PHyp_R(\tau)$ is a monoid, it follows that $\sigma_R \in PHyp_R(\tau)$ and $PHyp_R(\tau)$ is a submonoid of $PHyp(\tau)$ we have $\widehat{\sigma}_R = (\sigma_{R_1} \circ_p \sigma_{R_2})\widehat{\ } = \widehat{\sigma}_{R_1} \circ \widehat{\sigma}_{R_2}$. Since $s, t \in dom(\widehat{\sigma}_{R_1} \circ \widehat{\sigma}_{R_2})$ we have $s, t \in dom\widehat{\sigma}_R$. Then we have $\widehat{\sigma}_R[s] = (\sigma_{R_1} \circ_p \sigma_{R_2})\widehat{\ }[s] = \widehat{\sigma}_{R_1}[\widehat{\sigma}_{R_2}[s]] \approx \widehat{\sigma}_{R_1}[\widehat{\sigma}_{R_2}[t]] = (\sigma_{R_1} \circ_p \sigma_{R_2})\widehat{\ }[t] = \widehat{\sigma}_R[t]$, i.e. $\widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t] \in \chi^E_{Ph}[\Sigma]$. By (ii) and (iii), we have $\chi^E_{Ph}[\Sigma] \subseteq \chi^E_{Ph}[\chi^E_{Ph}[\Sigma]]$. Therefore, $\chi^E_{Ph}[\chi^E_{Ph}[\Sigma]] = \chi^E_{Ph}[\Sigma]$.

(v) Let $\mathcal{A} \in K$. Since $f_i \in dom\sigma_{id}$ for all $i \in I$ then $\sigma_{id}(f_i)^{\mathcal{A}} = f_i(x_1, \ldots, x_{n_i})^{\mathcal{A}} = f_i^A$ is defined because f_i^A is a partial operation and $\sigma_{id}(\mathcal{A}) = \mathcal{A}$. Therefore, we get $K \subseteq \chi^A_{Ph}[K]$.

(vi) Suppose $K \subseteq K' \subseteq PAlg(\tau)$, then

$$\begin{array}{rcl}
\chi^A_{Ph}[K] & = & \bigcup\limits_{\mathcal{A}\in K} \chi^A_{Ph}[\mathcal{A}] \\
& = & \bigcup\limits_{\mathcal{A}\in K} \{\sigma_R(\mathcal{A}) \mid \sigma_R \in PHyp_R(\tau)\} \\
& = & \bigcup\limits_{\mathcal{A}\in K} \{(A; (\sigma_R(f_i)^{\mathcal{A}})_{i\in I}) \mid \sigma_R(f_i)^{\mathcal{A}} \text{ is defined and } \sigma_R \in PHyp_R(\tau)\} \\
& \subseteq & \bigcup\limits_{\mathcal{A}\in K'} \{(A; (\sigma_R(f_i)^{\mathcal{A}})_{i\in I}) \mid \sigma_R(f_i)^{\mathcal{A}} \text{ is defined and } \sigma_R \in PHyp_R(\tau)\}
\end{array}$$

$$
\begin{aligned}
&= \bigcup_{\mathcal{A}\in K'} \{\sigma_R(\mathcal{A}) \mid \sigma_R \in PHyp_R(\tau)\} \\
&= \bigcup_{\mathcal{A}\in K'} \chi^A_{Ph}[\mathcal{A}] \quad = \quad \chi^A_{Ph}[K'].
\end{aligned}
$$

(vii) Suppose $\sigma_{R_1}, \sigma_{R_2} \in PHyp_R(\tau)$ are two arbitrary regular partial hypersubstitutions and $\sigma_{R_1}(\sigma_{R_2}(\mathcal{A})) \in \chi^A_{Ph}[\chi^A_{Ph}[K]]$ for all $\mathcal{A} \in K$, then $(\sigma_{R_2} \circ_p \sigma_{R_1})(f_i)^{\mathcal{A}}$ is defined. Let $\sigma_R \in PHyp_R(\tau)$ be a regular partial hypersubstitution with $\sigma_R := \sigma_{R_2} \circ_p \sigma_{R_1}$. Since $PHyp_R(\tau)$ is a monoid it follows that $\sigma_R \in PHyp_R(\tau)$. Then $\sigma_R(f_i)^{\mathcal{A}}$ is defined and $(\sigma_{R_2} \circ_p \sigma_{R_1})(f_i)^{\mathcal{A}} = \sigma_R(f_i)^{\mathcal{A}}$. Hence $\sigma_{R_1}(\sigma_{R_2}(\mathcal{A})) \in \chi^A_{Ph}[K]$ for all $\mathcal{A} \in K$ and $\chi^A_{Ph}[\chi^A_{Ph}[K]] \subseteq \chi^A_{Ph}[K]$. By (v) and (vi), we have $\chi^A_{Ph}[K] \subseteq \chi^A_{Ph}[\chi^A_{Ph}[K]]$. Therefore, we have $\chi^A_{Ph}[\chi^A_{Ph}[K]] = \chi^A_{Ph}[K]$.

Finally, we need to show that $\chi^A_{Ph}[\mathcal{A}] \underset{sr}{\models} s \approx t$ iff $\mathcal{A} \underset{sr}{\models} \chi^E_{Ph}[s \approx t]$.

Now we get

$$
\begin{aligned}
\chi^A_{Ph}[\mathcal{A}] \underset{sr}{\models} s \approx t &\Leftrightarrow \forall \sigma_R \in PHyp_R(\tau), (\sigma_R(\mathcal{A}) \underset{sr}{\models} s \approx t) \\
&\Leftrightarrow \forall \sigma_R \in PHyp_R(\tau), (s^{\sigma_R(\mathcal{A})} \mid_D = t^{\sigma_R(\mathcal{A})} \mid_D) \\
&\Leftrightarrow \forall \sigma_R \in PHyp_R(\tau), (\widehat{\sigma}_R[s]^{\mathcal{A}} \mid_D = \widehat{\sigma}_R[t]^{\mathcal{A}} \mid_D) \\
&\qquad \text{by Lemma 9.3.1 (where } D \text{ is the common domain)} \\
&\Leftrightarrow \forall \sigma_R \in PHyp_R(\tau)(\mathcal{A} \underset{sr}{\models} \widehat{\sigma}_R[s] \approx \widehat{\sigma}_R[t]) \\
&\Leftrightarrow \mathcal{A} \underset{sr}{\models} \chi^E_{Ph}[s \approx t].
\end{aligned}
$$

■

Now we have a Galois connection and a conjugate pair of additive closure operator and may apply the theory developed e.g. in ([34]). Without proofs we will give the following results. (The proofs can be found in [34].)

Theorem 9.3.3 *For all $V \subseteq PAlg(\tau)$ and $\Sigma \subseteq W_\tau(X)^2$, the following properties hold:*

(i) $Id^{sr}_{Ph}V = Id^{sr}\chi^A_{Ph}[V]$;

(ii) $Id^{sr}_{Ph}V \subseteq Id^{sr}V$;

(iii) $\chi^E_{Ph}[Id^{sr}_{Ph}V] = Id^{sr}_{Ph}V$;

(iv) $\chi^A_{Ph}[Mod^{sr}Id^{sr}_{Ph}V] = Mod^{sr}Id^{sr}_{Ph}V$;

(v) $Id^{sr}_{Ph}Mod^{sr}_{Ph}\Sigma = Id^{sr}Mod^{sr}\chi^E_{Ph}[\Sigma]$; and dually,

(i') $Mod^{sr}_{Ph}\Sigma = Mod^{sr}\chi^{E}_{Ph}[\Sigma]$;
(ii') $Mod^{sr}_{Ph}\Sigma \subseteq Mod^{sr}\Sigma$;
(iii') $\chi^{A}_{Ph}[Mod^{sr}_{Ph}\Sigma] = Mod^{sr}_{Ph}\Sigma$;
(iv') $\chi^{E}_{Ph}[Id^{sr}Mod^{sr}_{Ph}\Sigma] = Id^{sr}Mod^{sr}_{Ph}\Sigma$;
(v') $Mod^{sr}_{Ph}Id^{sr}_{Ph}V = Mod^{sr}Id^{sr}\chi^{A}_{Ph}[V]$.

Let V be a strong regular variety of partial algebras of type τ. Then V is said to be $PHyp_R(\tau)$*-solid* if $\chi^{A}_{Ph}[V] = V$.

$PHyp_R(\tau)$-solid varieties of partial algebras can be characterized as follows:

Theorem 9.3.4 *Let $V \subseteq PAlg(\tau)$ be a strong regular variety of partial algebras and let $\Sigma \subseteq W_\tau(X)^2$ be a strong regular equational theory (i.e. $V = Mod^{sr}Id^{sr}V$ and $\Sigma = Id^{sr}Mod^{sr}\Sigma$). Then the following propositions* (i)-(iv) and (i')-(iv') are equivalent:
(i) $V = Mod^{sr}_{Ph}Id^{sr}_{Ph}V$,
(ii) $\chi^{A}_{Ph}[V] = V$;
(iii) $Id^{sr}V = Id^{sr}_{Ph}V$;
(iv) $\chi^{E}_{Ph}[Id^{sr}V] = Id^{sr}V$.
and the following are also equivalent
(i') $\Sigma = Id^{sr}_{Ph}Mod^{sr}_{Ph}\Sigma$,
(ii') $\chi^{E}_{Ph}[\Sigma] = \Sigma$;
(iii') $Mod^{sr}\Sigma = Mod^{sr}_{Ph}\Sigma$;
(iv') $\chi^{A}_{Ph}[Mod^{sr}\Sigma] = Mod^{sr}\Sigma$.

9.4 Applications

As an example we want to determine all $PHyp_R(2)$-solid varieties of semigroups. Varieties of total semigroups can be characterized as $V = Mod\ \Sigma$ where Σ is a set of equations containing the associative law and V consists precisely of all semigroups satisfying all equations from Σ as identities. As usual, we denote by IdV the set of all identities satisfied in V. We need the following varieties of semigroups:
$C{:=}Mod\{(xy)z \approx x(yz),\ xy \approx yx\}$-the variety of commutative semigroups,
$SL{:=}Mod\{(xy)z \approx x(yz),\ x^2 \approx x,\ xy \approx yx\}$-the variety of semilattices,

$Z{:=}Mod\{xy \approx zt\}$-the variety of zero-semigroups (or of constant semigroups),
$NB{:=}Mod\{(xy)z \approx x(yz),\ x^2 \approx x,\ xyzt \approx xzyt\}$-the variety of normal bands,
$RB{:=}Mod\{(xy)z \approx x(yz),\ x^2 \approx x,\ xyz \approx xz\}$-the variety of rectangular bands,
$RegB{:=}Mod\{(xy)z \approx x(yz),\ x^2 \approx x,\ xyzx \approx xyxzx\}$-the variety of regular bands,
$V_{RS}^{nrec}{:=}Mod\{(xy)z \approx x(yz), x^2y^2z \approx x^2yx^2yz, xy^2z^2 \approx xyz^2yz^2, xyzyx \approx xyxzxyx,$
$x^2y^3 \approx y^2x^3, y^3x^2 \approx x^3y^2\}$,
$V_{PC}{:=}Mod\{x(yz) \approx (xy)z, xy \approx yx, x^2y \approx xy^2\}$-the greatest regular-solid variety of commutative semigroups ([34]),
$V_{RS}{:=}Mod\{(xy)z \approx x(yz), xyxzxyx \approx xyzyx, (x^2y)^2z \approx x^2y^2z, xy^2z^2 \approx x(yz^2)^2\}$
-the greatest regular-solid variety of semigroups ([34]).

Regular-solid varieties of semigroups were characterized in [34] by the following theorem:

Theorem 9.4.1 ([34]) *Let V be a variety of semigroups. Then V is regular-solid iff V is self-dual and one of the following statements is true:*

(1) $Z \vee RB \subseteq V \subseteq V_{RS}$;

(2) $V \subseteq V_{RS}^{nrec}$ *and* $V \nsubseteq Mod\{(xy)z \approx x(yz), xy^4 \approx y^2xy^2\}$;

(3) $V \subseteq V_{RS}^{nrec} \cap Mod\{(xy)z \approx x(yz), xy^4 \approx x^2y^3\}$ *and* $V \nsubseteq C$;

(4) $V \subseteq V_{RC}$;

(5) $V \in \{RB, NB, RegB\}$.

We have to check which of these varieties satisfy strong identities which are not satisfied after applying the nowhere defined hypersubstitution. Since we have only one operation symbol, this can only happen if there is an identity of the form $t \approx x$ for a variable x and a term t different from x. Such identities are called non-normal and a variety of semigroups is called *normal* if it satisfies only normal identities. For more background on normal varieties see e.g. [29].

Therefore we have:

Lemma 9.4.2 *A variety of semigroups is $PHyp_R(2)$-solid iff it is regular-solid and normal.*

Using this lemma we obtain:

Theorem 9.4.3 *A variety of semigroups is $PHyp_R(2)$-solid iff it is regular-solid and different from $RB, NB, RegB$, and SL.*

Proof. It is easy to see that the set IdZ of all identities satisfied in the variety Z of all zero-semigroups is precisely the set of all normal equations of type $\tau = (2)$. That means, if V is regular-solid and $Z \subseteq V$, then V is $PHyp_R(2)$-solid. This happens in the first case of Theorem 9.4.1. If in the cases (2) or (3) V is a non-trivial subvariety of V_{RS}^{nrec} which does not contain the variety Z of all zero-semigroups, then there is an identity $t \approx x$ in V. From this identity we can derive an identity $x^k \approx x$ for $k \geq 2$. From the identity $x^2y^3 \approx y^2x^3 \in IdV$ we can derive $x^7 \approx x^8$ and from this identity and from $x^k \approx x$ we get the idempotent identity. The identity $x^2y^3 \approx y^2x^3$ provides the commutative law and then $V = SL$. If in case (4) V is a non-trivial subvariety of V_{RC} which does not contain the variety Z of all zero-semigroups, then from the identity $t \approx x$ in V we derive again $x^k \approx x$ for $k \geq 2$. From $x^2y \approx xy^2$ we derive the $x^4 \approx x^5$ and from both we derive the idempotent law and then the commutative law is also satisfied. This shows $V = SL$ in case (4). ■

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