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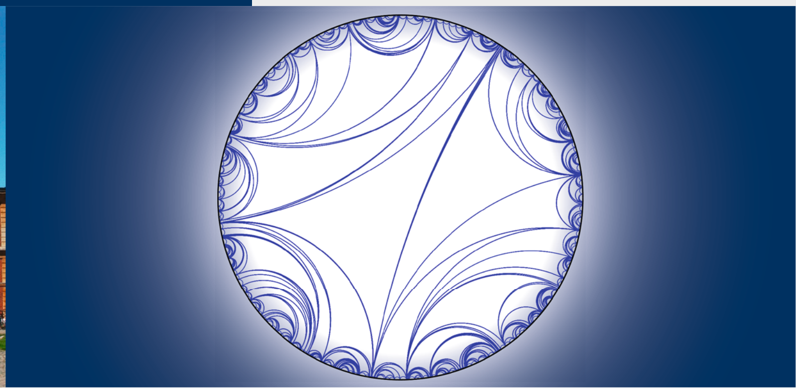

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Bibliografische Information der Deutschen Nationalbibliothek
Die Deutsche Nationalbibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliografie; detaillierte bibliografische Daten sind im Internet über http://dnb.dnb.de abrufbar.

Universitätsverlag Potsdam 2015
http://verlag.ub.uni-potsdam.de/

Am Neuen Palais 10, 14469 Potsdam
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E-Mail: verlag@uni-potsdam.de

Die Schriftenreihe **Preprints des Instituts für Mathematik der Universität Potsdam** wird herausgegeben vom Institut für Mathematik der Universität Potsdam.

ISSN (online) 2193-6943

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Titelabbildungen:
1. Karla Fritze | Institutsgebäude auf dem Campus Neues Palais
2. Nicolas Curien, Wendelin Werner | Random hyperbolic triangulation
Published at: http://arxiv.org/abs/1105.5089

Online veröffentlicht auf dem Publikationsserver der Universität Potsdam
URN urn:nbn:de:kobv:517-opus4-81537
http://nbn-resolving.de/urn:nbn:de:kobv:517-opus4-81537

ON THE CONVERGENCE OF CONTINUOUS NEWTON METHOD

AVIV GIBALI, DAVID SHOIKHET, AND NIKOLAI TARKHANOV

Abstract. In this paper we study the convergence of continuous Newton method for solving nonlinear equations with holomorphic mappings in complex Banach spaces. Our contribution is based on a recent progress in the geometric theory of spirallike functions. We prove convergence theorems and illustrate them by numerical simulations.

Contents

1. Introduction

Consider the classical problem of finding an approximate solution to a nonlinear equation

$$f(z) = 0$$

in a domain D in the complex plane $\mathbb{C}$, where $f : D \to \mathbb{C}$ is a holomorphic function in D. To this end one uses diverse modifications of the recurrence formula

$$z_{n+1} = z_n - \lambda_n \frac{f(z_n)}{f'(z_n)} \tag{1.1}$$

for $n = 0, 1, \ldots$, where z_0 is an initial approximation in D and $\lambda_n > 0$. For a suitably chosen sequence $\{\lambda_n\}$, formula (1.1) is often called the

2010 *Mathematics Subject Classification.* Primary 49M15; Secondary 58C15.
Key words and phrases. Newton method, spirallike function.

damped Newton method while for $\lambda_n \equiv 1$ it is called the classical Newton method, see [KA64].

We focus on the classical Newton method. The convergence of (1.1) is widely explored and depends on the specific choice of the initial point $z_0 \in D$.

The recurrence formula (1.1) displays immediately the initial boundary value problem

$$\left\{ \begin{array}{rcl} \dot{z} & = & -\dfrac{f(z)}{f'(z)}, \quad \text{if} \quad t > 0, \\ z(0) & = & z_0, \end{array} \right. \tag{1.2}$$

for a curve $z = z(t)$ in D starting at z_0 and leading to a solution $a \in D$ of $f(z) = 0$. In fact, if $\lambda \neq 0$, then for $f(a)$ to vanish it is necessary and sufficient that

$$-\lambda \frac{f(a)}{f'(a)} = 0,$$

which is equivalent to

$$a - \lambda \frac{f(a)}{f'(a)} = a.$$

The standard successive approximations for solving this equation look like

$$z_{n+1} = z_n - \lambda \frac{f(z_n)}{f'(z_n)}$$

for $n = 0, 1, \ldots$. On writing $z_n = z(n)$ and passing to a continuous parameter $t \in [0, \infty)$ we get

$$\frac{z(t + \Delta t) - z(t)}{\Delta t} = -\frac{f(z(t))}{f'(z(t))}$$

with $\Delta t = \lambda$. Taking the limit as $\Delta t \to 0$ yields (1.2), as desired.

The continuous version of the Newton method defined by (1.2) was earlier studied in [Gav58, KA64, Air99].

It is worth pointing out that the vector field on the right-hand side of (1.2) just amounts to (minus) the logarithmic derivative of f. The first integral of (1.2) is

$$f(z(t)) = f(z_0)e^{-t}$$

for $t \geq 0$, as is easy to check. Hence it follows that $f(z(t)) \to 0$, when $t \to \infty$. Notice that $f(z(t))$ describes the discrepancy of the approximate solution $z(t)$, provided that $z(t)$ converges to $a \in D$ as $t \to \infty$.

Summarising we conclude that the study of the continuous version of the Newton method consists of two main steps. The first of the two is to describe those initial data $z_0 \in D$ for which the initial boundary value problem (1.2) has a solution $z(t)$ defined for all $t \geq 0$. It is a general observation that the solution is unique whenever it exists. And the second step consists in studying the asymptotic behaviour of the global solution $z(t)$, as $t \to \infty$. This is precisely what the present paper is aimed at.

2. Spirallike mappings

Throughout this paper by D is meant a domain in a complex Banach space X endowed with a norm $\|\cdot\|$. We denote by $\mathrm{Hol}(D, X)$ the space of all holomorphic (i.e., Fréchet differentiable) mappings on D with values in X. For $f \in \mathrm{Hol}(D, X)$ we denote by $f'(x)$ the Fréchet derivative of f at a point $x \in X$. By definition, this is a bounded linear operator in X.

As usual, X^* stands for the dual space of X. By the Hahn-Banach theorem, for each $x \in X$ there is a functional $l_x \in X^*$ with the property that $l_x(x) = \|l_x\| \, \|x\|$. On normalising l_x one obtains a functional whose norm is $\|x\|$. Write x^* for any functional $l \in X^*$ satisfying $\Re l(x) = \|x\|^2 = \|l\|^2$, and $*x$ for the set of all functionals l with this property (cf. the Hodge star operator). Such a functional x^* is in general not unique. However, if X is a Hilbert space, then the element x^* is unique and it can be identified with x, which is due to the Riesz representation theorem.

A mapping $f \in \mathrm{Hol}(D, X)$ is said to be locally biholomorphic if for each $x \in D$ there are neighborhoods $U \subset D$ of x and V of $f(x)$, such that $f|_U$ is a bijective mapping of U onto V and its inverse is holomorphic. It is well known that $f \in \mathrm{Hol}(D, X)$ is locally biholomorphic on D if and only if, for each $x \in D$, the Fréchet derivative $f'(x)$ is a bijective mapping of X. By the inverse mapping theorem of Banach, the bijectivity of $f'(x)$ implies readily the boundedness of its inverse. For a finite dimensional space X, a mapping $f \in \mathrm{Hol}(D, X)$ is locally biholomorphic if and only if it is locally one-to-one. However, this fact no longer holds for general infinite dimensional spaces X (see for instance [HS81]).

A bounded linear operator A in X is called strongly accretive if there is a constant $k > 0$ with the property that $\Re \langle Ax, x^* \rangle \geq k \|x\|^2$ for all $x \in X$ and $x^* \in *x$. The following assertion characterises those bounded linear operators in X which have spectrum in the open right half-plane $\Re \lambda > 0$.

Lemma 2.1. *Suppose $A : X \to X$ is a bounded linear operator. The following are equivalent:*

1) The spectrum of the operator A lies in the open right half-plane $\Re\lambda > 0$.

2) The linear semigroup $\exp(-tA)$ converges to 0 in the operator norm, as $t \to \infty$.

3) There is an equivalent norm on X, such that A is strongly accretive with respect to the corresponding sesquilinear form.

Proof. The equivalence of 1) and 2) is actually a consequence of the spectral mapping theorem. However, we will need some addition details.

Denote by $\chi(A)$ the lower exponential index of A, that is

$$0 < \chi(A) := \inf_{\lambda \in \operatorname{sp} A} \Re\lambda = \lim_{t\to\infty} \frac{\log \|\exp(-tA)\|}{-t}, \tag{2.1}$$

where $\operatorname{sp} A$ stands for the spectrum of A (see [DK70]). Then for any $\lambda \in (0, \chi(A))$ there is $C > 0$ such that

$$\|\exp(-tA)\| \leq C \exp(-\lambda\|A\|)$$

for all $t \geq 0$. On setting

$$\|x\|_1 := \sup_{t\geq 0} \|\exp(-t(A-\lambda I))\, x\|$$

we conclude that $\|x\| \leq \|x\|_1 \leq C\,\|x\|$ for all $x \in X$, which is due to (2.1), and

$$\|\exp(-tA)x\|_1 \leq \exp(-\lambda\|A\|)\,\|x\|_1 \tag{2.2}$$

for $t \geq 0$. Hence it follows that

$$\Re\,\langle Ax, x^*\rangle_1 \geq \lambda\,\|x\|_1^2$$

for all $x \in X$. Using the Hille-Yosida exponential formula (see [Yos80]) one proves that the last estimate implies (2.2) (and so 2)), which completes the proof. □

Definition 2.2. Let A be a bounded linear operator in X with spectrum in the open right half-plane and D a convex domain in X containing the origin. A mapping $f \in \operatorname{Hol}(D, X)$ is called A-spirallike with respect to the origin if $\exp(-tA)f(x) \in f(D)$ for all $x \in D$ and $t \geq 0$.

For $A = \lambda I$ with $\Re\lambda > 0$ we say for short that f is λ-spirallike. If $A = I$ in the above definition, then f is called starlike with respect to the origin.

3. General results for Banach spaces

Suppose $f \in \mathrm{Hol}(D,X)$ is a locally biholomorphic mapping of D, such that the origin belongs to the closure of $f(D)$. When looking for an approximate solution of the nonlinear equation $f(x)=0$ in D, one can exploit similarly to (1.2) a continuous analogue of the classical Newton method

$$\left\{ \begin{array}{rcll} \dot{x} + (f'(x))^{-1} f(x) & = & 0, & \text{if} \quad t>0, \\ x(0) & = & x_0, & \end{array} \right. \tag{3.1}$$

where $x_0 \in D$ is an initial approximation. We slightly generalise it by considering

$$\left\{ \begin{array}{rcll} \dot{x} + (f'(x))^{-1} A\, f(x) & = & 0, & \text{if} \quad t>0, \\ x(0) & = & x_0, & \end{array} \right. \tag{3.2}$$

where A is a bounded linear operator in X.

Definition 3.1. The method (3.2) is called well defined on D if for any data $x_0 \in D$ the initial value problem has a unique solution $x = x(t)$, such that $x(t) \in D$ for all $t > 0$ and the discrepancy $f(x(t))$ tends to zero as $t \to \infty$.

The following theorem gives a criterion for the continuous version of the Newton method to be well defined.

Theorem 3.2. *Suppose that f is a biholomorphic mapping on a domain $D \subset X$ and A satisfies one of the equivalent conditions of Lemma 2.1. Then method (3.2) is well defined if and only if f is A-spirallike in X.*

Proof. Let the method defined by (3.2) is well defined. Given any $x_0 \in D$, the initial value problem (3.2) has a unique solution $x = x(t)$ with values in D and $f(x(t)) \to 0$ as $t \to \infty$. Set $y(t) = f(x(t))$. From the differential equation we get

$$\begin{array}{rcl} \dot{y} & = & f'(x)\,\dot{x} \\ & = & -f'(x)\,(f'(x)]^{-1}\,A\,f(x) \\ & = & -Ay \end{array}$$

for all $t > 0$. On the other hand, under our assumptions on A, the initial value problem

$$\left\{ \begin{array}{rcll} \dot{y} + Ay & = & 0, & \text{if} \quad t>0, \\ y(0) & = & y_0 & \end{array} \right.$$

has a unique solution $y(t) = \exp(-tA)y_0$ for each $y_0 \in f(D)$. Hence it follows that $\exp(-t\Gamma)y_0 = f(x(t)) \in f(D)$ for all $t > 0$. So, f is A-spirallike.

Conversely, if f is A-spirallike, then, for each $x_0 \in D$, the trajectory $x(t) = f^{-1}(\exp(-tA)f(x_0))$ with $t \geq 0$ is well defined and does not go beyond the domain D. A direct calculation shows that $x = x(t)$ satisfies the initial value problem (3.2). Moreover, $f(x(t)) = \exp(-tA)f(x_0)$ tends to zero uniformly with respect to x_0 on each ball inside D, as desired. □

One can show that, if f is a locally biholomorphic mapping vanishing at a point $a \in D$ and A a linear operator in X satisfying one of the equivalent conditions of Lemma 2.1, then f is actually biholomorphic provided the method given by (3.2) is well defined. In particular, f is A-spirallike.

4. A Nevanlinna type condition

Denote by $D = \mathbb{D}$ the unit disk around the origin in $\mathbb{C}$. In the one-dimensional case $X = \mathbb{C}$ a criterion for a mapping $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ to be starlike with respect to the origin is given by the familiar Nevanlinna condition

$$\Re\Big(z\frac{f'(z)}{f(z)}\Big) > 0$$

for all $z \in \mathbb{D}$. However, verifying such a condition might be hard because of its computational complexity. The following sufficient condition simplifies the use of Theorem 3.2.

Theorem 4.1. *Let f be a holomorphic function in $\mathbb{D}$ vanishing at the origin and satisfying*

$$\begin{array}{rcl} f'(z) & \neq & 0, \\ \Re\dfrac{f(z)f''(z)}{(f'(z))^2} & < & 1 \end{array} \tag{4.1}$$

for all $z \in \mathbb{D}$. Then f is starlike on $\mathbb{D}$.

Proof. It suffices to show that (4.1) implies the Nevanlinna condition, i.e.,

$$\Re g(z) > 0$$

for all $z \in \mathbb{D}$, where

$$g(z) := \frac{1}{z}\frac{f(z)}{f'(z)}.$$

To do this we consider the function $zg(z)$ and note that condition (4.1) is equivalent to

$$\Re\,(zg)'\,(z) > 0$$

for all $z \in \mathbb{D}$, for

$$(zg)'\,(z) = \frac{(f'(z))^2 - f(z)f''(z)}{(f'(z))^2}$$

and our claim is obvious.

Thus, we have to show that $\Re\,(g(z) + zg'(z)) > 0$ implies $\Re g(z) > 0$. Setting $z = re^{\imath\varphi}$ with $r \in [0,1)$ and $\varphi \in [0, 2\pi)$, we get

$$zg'(z) = r\frac{\partial}{\partial r}g$$

and so

$$\Re\,(g(z) + zg'(z)) = \Re g(re^{\imath\varphi}) + \Re\Big(r\frac{\partial}{\partial r}g\Big) > 0. \tag{4.2}$$

We first show that from (4.2) it follows that $\Re g(z) \geq 0$ for all $z \in \mathbb{D}$. Suppose, contrary to our claim, that there is $z_0 = r_0 e^{\imath\varphi_0}$ in $\mathbb{D}$, such that $\Re g(z_0) < 0$. From (4.2) we get $\Re g(0) > 0$. Hence, there is $r_1 \in (0, r_0)$ such that

$$\begin{array}{rcl} \Re g(r_1 e^{\imath\varphi_0}) & = & 0, \\ \Re g(r_0 e^{\imath\varphi_0}) & < & 0, \end{array}$$

and so one can find $r_2 \in (r_1, r_0)$ with the property that $\Re g(r_2 e^{\imath\varphi_0}) < 0$ and

$$\Re\Big(\frac{\partial}{\partial r}g\Big)(r_2 e^{\imath\varphi_0}) < 0$$

which contradicts (4.2). We thus conclude that $\Re g(z) \geq 0$ everywhere in $\mathbb{D}$.

If we assume that $\Re g(z_0) = 0$ for some $z_0 \in \mathbb{D}$, then it follows by the maximum principle for holomorphic functions that $g(z) = \imath c$ for all $z \in \mathbb{D}$, where c is a real constant. Hence, $\Re\,(g(z) + zg'(z)) = 0$, which is impossible. □

Example 4.2. Let f be a holomorphic function in $\mathbb{D}$ determined by the equation

$$f(z) = -f'(z)\,(z + 2\log(1-z))\,.$$

In this case we have

$$\Re\Big(z\frac{f'(z)}{f(z)}\Big) = -\Re\frac{z}{z + 2\log(1-z)}$$

and it is not clear how to see if the Nevanlinna condition holds. On the other hand, since

$$\Re\frac{f(z)f''(z)}{(f'(z))^2} = \Re\Big(1 - \Big(\frac{f(z)}{f'(z)}\Big)'\Big),$$

one easily verifies that

$$\Re \frac{f(z)f''(z)}{(f'(z))^2} = \Re \Big(1 - \frac{1+z}{1-z} \Big) < 1,$$

and so the continuous Newton method is well defined.

5. A canonical reduction

To clarify the remark after Theorem 3.2 we first consider a more general version of the continuous Newton method. Namely, let g be a holomorphic mapping of D to X (not necessarily locally biholomorphic) and let $h \in \mathrm{Hol}(D,X)$ have invertible total derivative $h'(x)$ at each point $x \in D$.

We study the behaviour of the solution $x = x(t)$ (if there is any) to the initial value problem

$$\left\{ \begin{array}{rcll} \dot{x} + (h'(x))^{-1} A\, g(x) & = & 0, & \text{if } \ t > 0, \\ x(0) & = & x_0 & \end{array} \right. \tag{5.1}$$

for large t, where $x_0 \in D$ is an initial approximation. If g is locally biholomorphic, one can choose $h = g =: f$, thus recovering the continuous Newton method of (3.2). In a sense the converse assertion holds also true.

Theorem 5.1. *Let g and h be holomorphic mappings on D and $h'(x)$ be invertible at each point $x \in D$. Suppose (5.1 is well defined on D, with $g(a) = 0$ and $A = h'(a)\,(g'(a))^{-1}$ for some $a \in D$. Then there is a biholomorphic mapping f on D, such that the method (3.1) is well defined on D and the solutions of (5.1) and (3.1) are the same and converge to a as $t \to \infty$.*

Proof. Given any $x_0 \in D$, let $x = x(t, x_0)$ be the solution of (5.1). We define the mapping f by

$$f(x_0) = \lim_{t \to \infty} e^t \,(x(t, x_0) - a)\,. \tag{5.2}$$

First we show that this limit exists for each $x_0 \in D$. For simplicity we set $a = 0$. Consider the mapping $Q \in \mathrm{Hol}(D,X)$ given by the formula $Q(x) := (h'(x))^{-1} A g(x)$. Since $Q(0) = 0$ and $Q'(0) = I$, the Taylor expansion of Q looks like

$$Q(x) = x + \sum_{k=k_0}^{\infty} P_k(x)$$

for x in a ball $B_r \subset D$ of radius $r > 0$ with centre at the origin, where $k_0 \geq 2$ and P_k are homogenous polynomials of degree k on X. By the

Schwarz lemma,

$$\|Q(x) - x\| \leq \frac{M}{r^{k_0}} \|x\|^{k_0},$$

where $M = \sup_{x \in D} \|Q(x) - x\|$ (see for instance [RS05]).

A simple calculation shows that $\Re \langle Q(x), x^* \rangle > 0$ for all $x \neq 0$ satisfying

$$\|x\| < \min \left\{ \left(\frac{M}{r^{k_0}} \right)^{\frac{1}{k_0 - 1}}, r \right\} = r_1.$$

This means that the ball B_{r_1} is invariant for the solution $x(t, \cdot)$ of (5.1), i.e., $\|x(t, x_0)\| < r_1$ for all $t \geq 0$ and $x_0 \in B_{r_1}$. Without loss of generality we can assume that $r_1 = 1$. Then it follows from Corollary 9.1 of [RS05] that

$$\|x(t, x_0)\| \leq e^{-t} \frac{\|x_0\|}{(1 - \|x_0\|)^2},$$

and so

$$\begin{aligned} \|e^t \left(Q(x(t, x_0)) - x(t, x_0) \right) \| &\leq e^t \frac{M}{r^{k_0}} \|x(t, x_0)\|^{k_0} \\ &\leq e^{(1-k_0)t} \frac{M}{r^{k_0}} \frac{\|x_0\|^{k_0}}{(1 - \|x_0\|)^{2k_0}} \\ &\to 0 \end{aligned}$$

since $k_0 \geq 2$. Setting now $y(t, x_0) = e^t x(t, x_0)$, we get

$$\dot{y}(t, x_0) = e^t \left(x(t, x_0) - Q(x(t, x_0)) \right) \to 0$$

as $t \to \infty$ for each $x_0 \in B_1$. Thus the limit (5.2)

$$\lim_{t \to \infty} e^t x(t, x_0) = \lim_{t \to \infty} y(t, x_0) =: f(x_0)$$

exists for all $x_0 \in B_1$.

The global convergence for all $x_0 \in D$ follows now from the fact that one can find a sufficiently large $T > 0$ with the property that $x(T, x_0) \in B_1$. Therefore, using the semigroup property one concludes that

$$\begin{aligned} \lim_{t \to \infty} e^{T+t} x(T + t, x_0) &= e^T \lim_{t \to \infty} e^t x(t, x(T, x_0)) \\ &= e^T f(x(T, x_0)). \end{aligned}$$

We have actually proved that

$$e^{-s} f(x_0) = f(x(s, x_0)) \in D$$

for any $s \geq 0$, which means that f is a starlike mapping. Moreover, on differentiating the latter equality in $s \geq 0$ we see that $x(s, x_0)$ satisfies (3.1), as desired. $\square$

6. A local continuous Newton method

In this section we study the following problem. Let $f \in \mathrm{Hol}(D, X)$ be a locally biholomorphic mapping satisfying $f(0) = 0$. A general question is whether there is a ball B_r in D such that the continuous Newton method is well defined on B_r. For example, in the one-dimensional case a well-known result due to Grunksy says that each univalent function f on the unit disk $\mathbb{D}$ is starlike on D_r with $0 < r \le \tanh(\pi/4)$, see [Gol66]. However, this is not longer true in higher dimensions, and so additional conditions are required. In [Suf73, Suf76] is was shown that a holomorphic mapping on the open unit ball $\mathbb{B} := \{x \in X : \|x\| < 1\}$ with $f(0) = 0$ is starlike if and only if $\Re \langle (f'(x))^{-1} f(x), x^* \rangle \ge 0$ for all $x^* \in *x$.

We consider a weaker condition on f, namely

$$\Re \langle (f'(x))^{-1} f(x), x^* \rangle \ge -m \|x\|^2 \tag{6.1}$$

for all $x^* \in *x$, where m is a nonnegative constant. We show that the answer to the above question is affirmative.

Other local problems are described as follows. Let λ be a complex number satisfying $\Re\lambda > 0$ and $\arg \lambda \in (0, \pi/2)$. Suppose $f : \mathbb{B} \to X$ is a locally biholomorphic mapping on $\mathbb{B}$, such that $f(0) = 0$ and the generalised continuous Newton method with $A = \lambda I$ is well defined. We ask whether the continuous Newton method is well defined on a possibly smaller ball. Conversely, if the continuous Newton method is well defined, is there a number $r \in (0, 1)$ depending on λ, such that the generalised continuous Newton method is well defined on the ball B_r?

To answer these question, we replace (6.1) by a more general condition. More precisely,

$$\Re \langle e^{\imath\varphi} (f'(x))^{-1} f(x), x^* \rangle \ge -m \|x\|^2 \tag{6.2}$$

for all $x^* \in *x$.

Theorem 6.1. *Let f be a locally biholomorphic mapping on $\mathbb{B}$ satisfying $f(0) = 0$. Suppose that condition (6.2) is fulfilled with some $m \ge 0$ and $-\pi/2 < \varphi < \pi/2$. Then, for each $0 < r < r_0$, the continuous Newton method given by (3.1) is well defined on B_r and it converges to the origin, where $r_0 = r_0(\varphi) \le 1$ is the unique root of the quadratic equation*

$$(1 - r^2) - 2r(1 - r\cos\varphi)(m + \cos\varphi) = 0 \tag{6.3}$$

in $(0, 1]$.

Proof. Denote $g(x) := (f'(x))^{-1} f(x)$. By assumption,

$$\Re \langle e^{\imath\varphi} g(x), x^* \rangle \geq -m \|x\|^2$$

for all $x^* \in *x$. Write $x = zv$ where $z \in \mathbb{C}$ and $\|v\| = \|v^*\| = 1$. Consider the function $h(z) = \langle g(zv), v^* \rangle$. We get

$$\Re \langle e^{\imath\varphi} g(zv), (zv)^* \rangle = \Re\, e^{\imath\varphi} h(z)\overline{z} \geq -m\, |z|^2.$$

From $h(0) = 0$ it follows that there is a holomorphic function Q on the disk $\mathbb{D}$, such that $h(z) = zQ(z)$. Then $h'(0) = Q(0) = 1$ and, by the above,

$$\Re(e^{\imath\varphi}\, |z|^2 Q(z)) \geq -m\, |z|^2$$

or $\Re(e^{\imath\varphi}\, Q(z)) \geq -m$ whenever $|z| < 1$. On applying an inequality of [KM07] we calculate

$$\begin{aligned} \Re\,(Q(z) - Q(0)) &= \Re \left(e^{-\imath\varphi}((e^{\imath\varphi}Q)(z) - (e^{\imath\varphi}Q)(0)) \right) \\ &\geq \frac{2r(1-r\cos\varphi)}{1-r^2} \left(\inf_{|\zeta|<1} \Re\,(e^{\imath\varphi}Q)(\zeta) - \Re\,(e^{\imath\varphi}Q)(0) \right) \\ &\geq \frac{2r(1-r\cos\varphi)}{1-r^2}\,(-m - \cos\varphi) \end{aligned}$$

for all $z \in B_r$ and $r \in (0, 1)$.

Since $\Re Q(0) = 1$ we get

$$\Re Q(z) \geq 1 + \frac{2r(1-r\cos\varphi)}{1-r^2}\,(-m - \cos\varphi)\,,$$

which can be equivalently rewritten as

$$F(r, \varphi) := (1 - r^2) - 2r(1 - r\cos\varphi)(m + \cos\varphi) \geq 0.$$

By assumption, $-\pi/2 < \varphi < \pi/2$, and so $F(0, \varphi) = 1 > 0$ and $F(1, \varphi) = -2(1 - \cos\varphi)(m + \cos\varphi) \leq 0$. Therefore, the equation $F(r, \varphi) = 0$ has a unique solution $r_0 = r_0(\varphi)$ in the interval $(0, 1]$. It follows that $F(r, \varphi) \geq 0$ for all $r \in (0, r_0]$. So, f is starlike on the ball B_r for each $0 < r \leq r_0$, as desired. □

For $\varphi = 0$ the formulation of Theorem 6.1 is especially simple.

Corollary 6.2. *Let f be a locally biholomorphic mapping on $\mathbb{B}$ vanishing at the origin. Assume that condition (6.1) holds for some $m \geq 0$. Then, for each $0 < r < 1/(1 + 2m)$, the continuous Newton method is well defined on B_r and it converges to the origin.*

Example 6.3. Let $f(z) = \dfrac{z}{1 - z - k}$, where $k \in [0, 1)$. In this case we have

$$(f'(z))^{-1} f(z) = \frac{1}{1 - k}\, z(1 - z - k).$$

Obviously,

$$\Re \langle (f'(z))^{-1} f(z), \overline{z} \rangle \geq -\frac{k}{1-k}\, |z|^2$$

which means that $m = k/(1-k)$ in (6.1). Thus, Theorem 6.1 applies, showing that the continuous Newton method with given f is well defined on B_r provided $r < r_0 = (1-k)/(1+k)$. Moreover, this method converges to the origin and the estimate

$$\frac{\|x(t)\|}{1 - \|x(t)\|^2} \leq e^{-t}\, \frac{\|x_0\|}{1 - \|x_0\|^2}$$

holds for all initial data $x_0 \in B_{r_0}$.

A computer simulation shows that for x_0 away from the ball B_{r_0} the trajectory fails to converge to the origin, see Fig. 1.

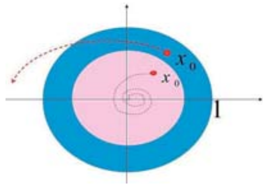

FIG. 1. The trajectory for two different x_0.

Choosing $m = 0$ in Corollary 6.2 and solving equation (6.3) we obtain in the same way

Corollary 6.4. *Let $f : \mathbb{B} \to X$ be a locally biholomorphic mapping on $\mathbb{B}$ satisfying $f(0) = 0$. Suppose that the generalised continuous Newton method corresponding to $A = \lambda I$, where $|\arg \lambda| < \pi/2$, is well defined. Then the continuous Newton method is well defined on the ball B_r whenever*

$$r \leq (\sqrt{2} \cos(\arg \lambda - \pi/4)^{-1} < 1.$$

Converse considerations lead us to the following result.

Theorem 6.5. *Assume that $f : \mathbb{B} \to X$ is a locally biholomorphic mapping on the unit ball, such that $f(0) = 0$ and the continuous Newton method is well defined. Then, for each $\varphi \in (-\pi/2, \pi/2)$ and r satisfying $0 < r \leq (1 - |\sin \varphi|)/\cos \varphi < 1$, the generalised continuous Newton method with $A = \lambda I$, where $\arg \lambda = \varphi$, is well defined on the smaller ball B_r.*

7. An example

In this section we consider an example mentioned in [Sis98]. As usual, $\mathbb{D}$ stands for the open unit disk in the complex plane. Consider the function

$$f(z) = \frac{z}{1-z},$$

then one verifies easily that

$$\begin{aligned} g(z) &= \frac{f(z)}{f'(z)} \\ &= z(1-z). \end{aligned}$$

Since $\Re g(z)\overline{z} \geq 0$ for all $z \in \mathbb{D}$, the continuous Newton method is well defined.

In Fig. 2 we present several trajectories of the analytic solution along with approximation by the continuous Newton method. In addition in Fig. 3 we present the difference in norm between two successive iterations.

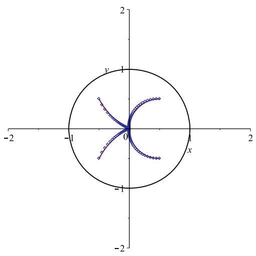

Fig. 2. Trajectories of the approximate solution (blue) starting from different x_0. In red are the exact solutions.

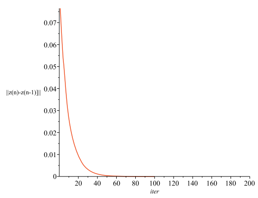

Fig. 3. Difference in norm between two successive iterations of the continuous Newton method.

If now we choose the same g but with $A = e^{-\imath(\pi/4)}$, we can see in Fig. 4 that the generalised continuous Newton method is not well defined. For instance, on taking $z_0 = (1+\imath)/\sqrt{2}$ we make certain that the solution is no longer invariant with respect to the whole unit disk. On the other hand, in Fig. 5 we observe that the solution is invariant for a small disc of radius $r_0 = \sqrt{2} - 1$.

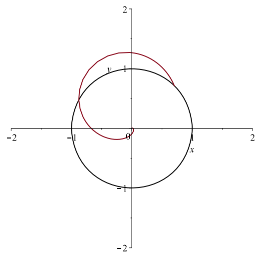

FIG. 4. The trajectory of the solution by the generalised continuous Newton method with $Ag(z) = e^{-\imath(\pi/4)} z(1-z)$.

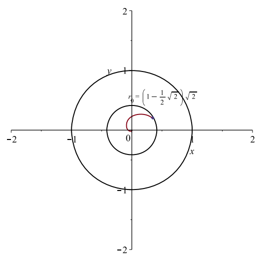

FIG. 5. The solution is invariant only for a small disk of radius $r_0 = \sqrt{2} - 1$.

ACKNOWLEDGEMENTS The second author gratefully acknowledges the support of the German Research Society (DFG), grant TA 289/12-1, and wishes to thank the University of Potsdam for the invitation and hospitality.

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