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International Research Training Group (IRTG) 1740

Synchronization of oscillatory networks in terms of global variables

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Dissertation

zur Erlangung des akademischen Grades
"doctor rerum naturalium"
(Dr. rer. nat.)
in der Wissenschaftsdisziplin "Theoretische Physik"

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät
der Universität Potsdam
von

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am
July 7, 2015
betreut durch

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Published online at the
Institutional Repository of the University of Potsdam:
URN urn:nbn:de:kobv:517-opus4-78182
http://nbn-resolving.de/urn:nbn:de:kobv:517-opus4-78182

Abstract

Synchronization of large ensembles of oscillators is an omnipresent phenomenon observed in different fields of science like physics, engineering, life sciences, etc. The most simple setup is that of globally coupled phase oscillators, where all the oscillators contribute to a global field which acts on all oscillators. This formulation of the problem was pioneered by Winfree and Kuramoto. Such a setup gives a possibility for the analysis of these systems in terms of global variables. In this work we describe nontrivial collective dynamics in oscillator populations coupled via mean fields in terms of global variables. We consider problems which cannot be directly reduced to standard Kuramoto and Winfree models.

In the first part of the thesis we adopt a method introduced by Watanabe and Strogatz. The main idea is that the system of identical oscillators of particular type can be described by a low-dimensional system of global equations. This approach enables us to perform a complete analytical analysis for a special but vast set of initial conditions. Furthermore, we show how the approach can be expanded for some nonidentical systems. We apply the Watanabe-Strogatz approach to arrays of Josephson junctions and systems of identical phase oscillators with leader-type coupling.

In the next parts of the thesis we consider the self-consistent mean-field theory method that can be applied to general nonidentical globally coupled systems of oscillators both with or without noise. For considered systems a regime, where the global field rotates uniformly, is the most important one. With the help of this approach such solutions of the self-consistency equation for an arbitrary distribution of frequencies and coupling parameters can be found analytically in the parametric form, both for noise-free and noisy cases. We apply this method to deterministic Kuramoto-type model with generic coupling and an ensemble of spatially distributed oscillators with leader-type coupling. Furthermore, with the proposed self-consistent approach we fully characterize rotating wave solutions of noisy Kuramoto-type model with generic coupling and an ensemble of noisy oscillators with bi-harmonic coupling.

Whenever possible, a complete analysis of global dynamics is performed and compared with direct numerical simulations of large populations.

Allgemeinverständliche Zusammenfassung

Die Synchronisation einer großen Menge von Oszillatoren ist ein omnipräsentes Phänomen, das in verschiedenen Forschungsgebieten wie Physik, Ingenieurwissenschaften, Medizin und Weiteren beobachtet wird. In der einfachsten Situation ist von einer Menge Phasenoszillatoren jeder mit dem Anderen gekoppelt und trägt zu einem gemeinsamen Feld (dem sogenannten mean field) bei, das auf alle Oszillatoren wirkt. Dieser Formulierung wurde von Winfree und Kuramoto der Weg bereitet und sie birgt die Möglichkeit einer Analyse des Systems mithilfe von globalen Variablen. In dieser Arbeit beschreiben wir mithilfe globaler Variablen die nicht-triviale kollektive Dynamik von Oszillatorpopulationen, welche mit einem mean field verbunden sind. Wir beschäftigen uns mit Problemen die nicht direkt auf die Standardmodelle von Kuramoto und Winfree reduziert werden können.

Im ersten Teil der Arbeit verwenden wir eine Methode die auf Watanabe und Strogatz zurückgeht. Die Hauptidee ist, dass ein System von identischen Oszillatoren eines bestimmten Typs durch ein niedrig-dimensionales System von globalen Gleichungen beschrieben werden kann. Dieser Ansatz versetzt uns in die Lage eine vollständige analytische Untersuchung für eine spezielle jedoch große Menge an Anfangsbedingungen durchzuführen. Wir zeigen des Weiteren wie der Ansatz auf nicht-identische Systeme erweitert werden kann. Wir wenden die Methode von Watanabe und Strogatz auf Reihen von Josephson-Kontakten und auf identische Phasenoszillatoren mit einer Anführer-Kopplung an.

Im nächsten Teil der Arbeit betrachten wir eine selbst-konsistente mean-field-Methode, die auf allgemeine nicht-identische global gekoppelte Phasenoszillatoren mit oder ohne Rauschen angewendet werden kann. Für die betrachteten Systeme gibt es ein Regime, in dem die globalen Felder gleichförmig rotieren. Dieses ist das wichtigste Regime. Es kann mithilfe unseres Ansatzes als Lösung einer Selbstkonsistenzgleichung für beliebige Verteilungen der Frequenzen oder Kopplungsstärken gefunden werden. Die Lösung liegt in einer analytischen, parametrischen Form sowohl für den Fall mit Rauschen, als auch für den Fall ohne Rauschen, vor. Die Methode wird auf ein deterministisches System der Kuramoto-Art mit generischer Kopplung und auf ein Ensemble von räumlich verteilten Oszillatoren mit Anführer-Kopplung angewendet. Zuletzt sind wir in der Lage, die Rotierende-Wellen-Lösungen der Kuramoto-artigen Modelle mit generischer Kopplung, sowie ein Ensemble von verrauschten Oszillatoren mit bi-harmonischer Kopplung, mithilfe des von uns vorgeschlagenen selbst-konsistenten Ansatzes vollständig zu charakterisieren.

Wann immer es möglich war, wurde eine vollständige Untersuchung der globalen Dynamik durchgeführt und mit numerischen Ergebnissen von großen Populationen verglichen.

Contents

1. Introduction

Synchronization in networks of a large number of coupled limit-cycle oscillators is a well-known phenomenon observed in many different systems of any nature, from biology to social sciences [1]. When such networks can be approximately considered fully connected, what means that each unit is connected to each other, it is possible to treat such a system as a set of oscillators coupled through a common global field (this case will be treated further in this work). Another frequently used assumption is that a coupling is weak, what implies that the amplitudes of the oscillators are relatively constant and the interaction takes place through the phases of the oscillators [2]. Therefore an approximate phase model can be formulated that represents the dynamics of the original system near a limit cycle. These approximate phase models are usually used to explain synchronization phenomena and collective behavior that appears in natural systems. A concrete form of a coupling function depends on the original model and should be obtained in the process of the phase reduction. There are two main directions on how to proceed with theoretical analysis. The first one is to take a "real" system and to perform a phase reduction process thus obtain a specific coupling function and to analyze results associated with this particular system. The second direction is to take an abstract system with "general" coupling function consistent of a limited amount of harmonics and to try to build some "general" theory of synchronization that afterwards can be applied to particular systems. We follow here the second approach.

Probably the most popular phase model was introduced by Kuromoto [2] and later extended by Sakaguchi and Kuramoto [3]. The model consists of globally coupled phase oscillators with one-harmonic (sine) coupling function of phase difference. For nonidentical elements the transition to synchrony appears at a critical value of the coupling strength depending on the width of the distribution of natural frequencies [4]. Nowadays more complex systems of coupled phase oscillators are considered. Distributions of coupling coefficients and phase shifts are added (particularly this case is treated below) or different coupling functions are considered. Even for the simplest Kuramoto model it is difficult to obtain analytical results. Depending on the particular type of the system (different dynamics of each oscillator and/or different coupling functions) different approaches should be applied. As it has been already mentioned, in this work, we will focus on the description of globally-coupled networks. Global coupling in some cases gives a possibility to derive the equations (or at least solutions) for the parameters of a global field. We call these parameters global variables. We will concentrate on presenting methods to derive the expressions for these global variables, applied to different systems.

We will start with a special case when the dynamics of global variables can be described by a low-dimensional system of equations. This approach is based on the seminal papers by Watanabe and Strogatz (WS) [5, 6] where a special variable transformation was suggested. With this variable transformation the dynamics of a system of identical oscillators of a particular type and of any size is fully described by a low-dimensional system of equations for global variables. In the second chapter we will present the application of the WS approach on two examples: arrays of Josephson junctions and systems of identical phase oscillators with leader-type coupling. Also we will show a setup of nonidentical oscillators where this approach can be applied. This is the case when the system splits into the groups of identical elements (the units inside each group are identical but differ from the elements of the other group). Similar setup is used to link [7] the WS theory and the Ott-Antonsen (OA) approach, where the ensemble with a distribution of natural frequencies can be analyzed. In this work we will show another system of nonidentical elements, which can be described by integro-differential equation. Yet, this approach cannot be applied to general nonidentical systems or systems with more complex types of phase equations.

In general, in order to find solutions in terms of global variables for the system of nonidentical oscillators, one should exploit another approach (similar to that of the original Kuramoto work) for derivation of the global equation and its solution. It is essentially the well-known self-consistent mean-field theory method that is used for the analysis of a large number of interacting units. The effect of all units acting on a particular unit can be represented by a single averaged field. This is exactly the case of globally coupled oscillators, when the coupling is explicitly represented through some global field. Then in the thermodynamic limit an equation for a distribution function of the phases should be solved. The solution usually can be only found self-consistently in an implicit form. Also, in most cases it is difficult to analyze the global dynamics as well as stability of the obtained solutions. This approach can be applied both for purely deterministic systems and for noise-driven oscillators. These cases will be treated separately. In the third chapter the application of the self-consistent approach to a noise-free case will be explained on two examples: Kuramoto-type model with generic coupling and an ensemble of spatially distributed oscillators with leader-type coupling. In the forth chapter we will show how the self-consistent method can be applied to noisy systems based on noisy Kuramoto-type model with generic coupling and an ensemble of noisy oscillators with bi-harmonic coupling.

The applicability of the methods described above strictly depends on the particular system. For some systems the methods can be extended so they become applicable for more complicated setups, for example the WS approach in some cases can be used to treat inhomogeneous ensembles but it is not applicable for general inhomogeneity. Furthermore, it is not always possible to obtain the dynamics for the global variables, so that analytical solutions should be found from self-consistent global field approach. However, sometimes it is possible to perform further analytical analysis. For example in some cases stability analysis can be performed. Also a way how one applies these methods (different reparametrization, etc) depends on a concrete system. That is why

we do not give a detailed general description of these methods, but rather illustrate them directly on particular examples. That gives us a possibility to explain not only the methods themselves but also their application and further additional analysis when possible.

2. The Watanabe-Strogatz approach.

2.1. The Watanabe-Strogatz theory

Before we consider an application of the WS theory, we are going to present a derivation of the global equations in terms of new notations that differs from the original WS notations [5, 6]. This new notations were firstly presented in [7].

As mentioned in the introduction, the WS theory is applicable to specific types of systems of coupled identical phase oscillators. Specific type means that the equation for the dynamics of each phase has specific form

$$\dot{\varphi_k} = f(t) + \mathrm{Im}\left(F(t)e^{-\mathrm{i}\varphi_k}\right), \tag{2.1}$$

with arbitrary real $f(t)$ and arbitrary complex $F(t)$. The coupling, that is represented through a global field, can enter either $f(t)$ or $F(t)$ or both.

With WS approach it is possible to formulate exact three-dimensional system of equations for the network of the globally coupled phase equations (2.1) of any size with the help of specific variable transformation. This transformation is essentially Möbius transformation [8] (see Appendix A for a different approach) in the form

$$e^{\mathrm{i}\varphi_k} = \frac{z + e^{\mathrm{i}(\psi_k+\Psi)}}{1 + z^* e^{\mathrm{i}(\psi_k+\Psi)}}, \tag{2.2}$$

where $z = z(t)$ is a complex and $\Psi = \Psi(t)$ is a real global variables and ψ_k are N constants. The fact that they are constants will be proven below.

In order to obtain equations for the global variables, the equation (2.1) should be rewritten for $e^{\mathrm{i}\varphi_k}$:

$$\frac{\partial}{\partial t}\left(e^{\mathrm{i}\varphi_k}\right) = \mathrm{i}f(t)e^{\mathrm{i}\varphi_k} + \frac{1}{2}F(t) - \frac{e^{\mathrm{i}2\varphi_k}}{2}F^*(t). \tag{2.3}$$

Then, we substitute (2.2) into (2.3) and obtain the following:

$$\begin{aligned}
&\dot{z} + \left[\dot{z}z^* - z\dot{z}^* + \mathrm{i}(\dot{\psi_k} + \dot{\Psi})(1 - |z|^2)\right] e^{\mathrm{i}(\psi_k+\Psi)} - \dot{z}^* e^{2\mathrm{i}(\psi_k+\Psi)} = \\
&= \mathrm{i}f\left[z + (1 + |z|^2)e^{\mathrm{i}(\psi_k+\Psi)} + z^* e^{2\mathrm{i}(\psi_k+\Psi)}\right] + \\
&\quad + \frac{F}{2}[1 + 2z^* e^{\mathrm{i}(\psi_k+\Psi)} + z^{*2} e^{2\mathrm{i}(\psi_k+\Psi)}] - \frac{F^*}{2}[z^2 + 2ze^{\mathrm{i}(\psi_k+\Psi)} + e^{2\mathrm{i}(\psi_k+\Psi)}]
\end{aligned} \tag{2.4}$$

By grouping together the terms $e^{n\mathrm{i}(\psi_k+\Psi)}$, $n=0,1,2$, it is possible to rewrite Eq. (2.4) as:

$$\begin{aligned}\dot{z}+\left[\dot{z}z^*-z\dot{z}^*+\mathrm{i}(\dot{\psi_k}+\dot{\Psi})(1-|z|^2)\right]e^{\mathrm{i}(\psi_k+\Psi)}-\dot{z}^*e^{2\mathrm{i}(\psi_k+\Psi)}=\\ =\mathrm{i}fz+\frac{F}{2}-\frac{F^*}{2}z^2+\\ +\left[\mathrm{i}f(1+|z|^2)+(z^*F-zF^*)\right]e^{\mathrm{i}(\psi_k+\Psi)}+\\ +\left[\mathrm{i}fz^*+\frac{F}{2}z^{*2}-\frac{F^*}{2}\right]e^{2\mathrm{i}(\psi_k+\Psi)}\end{aligned} \tag{2.5}$$

Then if the global variables $z(t)$ and $\Psi(t)$ satisfy equations (2.6), the Eq. (2.5) is valid for any ψ_k for every moment of time.

$$\begin{aligned}\dot{z}&=\mathrm{i}f(t)z+\frac{F(t)}{2}-\frac{F(t)^*}{2}z^2,\\ \dot{\Psi}&=f(t)+\mathrm{Im}(z^*F(t)).\end{aligned} \tag{2.6}$$

Next we are going to prove that ψ_k are constants. So we need to calculate the time derivative $\dot{\psi}_k$. First we express $e^{\mathrm{i}\psi_k}$ via the inverse WS transformation

$$e^{\mathrm{i}\psi_k}=e^{-\mathrm{i}\Psi}\frac{-z+e^{\mathrm{i}\varphi_k}}{1-z^*e^{\mathrm{i}\varphi_k}}. \tag{2.7}$$

Then we take time derivative of $e^{\mathrm{i}\psi_k}$

$$\begin{aligned}\frac{\partial}{\partial t}\left(e^{\mathrm{i}\psi_k}\right)=\mathrm{i}\dot{\psi_k}e^{\mathrm{i}\psi_k}=&-\mathrm{i}\dot{\Psi}e^{-\mathrm{i}\Psi}\frac{-z+e^{\mathrm{i}\varphi_k}}{1-z^*e^{\mathrm{i}\varphi_k}}+\\ &+e^{-\mathrm{i}\Psi}\frac{(\dot{e^{\mathrm{i}\varphi_k}}-\dot{z})(1-z^*e^{\mathrm{i}\varphi_k})-(e^{\mathrm{i}\varphi_k}-z)(-\dot{z^*}e^{\mathrm{i}\varphi_k}-z^*\dot{e^{\mathrm{i}\varphi_k}})}{(1-z^*e^{\mathrm{i}\varphi_k})^2}.\end{aligned} \tag{2.8}$$

Eq. (2.8) can be rewritten in the following way:

$$\mathrm{i}\dot{\psi_k}e^{\mathrm{i}\psi_k}=\frac{e^{-\mathrm{i}\Psi}}{(1-z^*e^{\mathrm{i}\varphi_k})^2}\left[\mathrm{i}\dot{\Psi}z-\dot{z}+(1-|z|^2)\dot{e^{\mathrm{i}\varphi_k}}+(\dot{z}z^*-z\dot{z^*}-\mathrm{i}\dot{\Psi}(1+|z|^2))e^{\mathrm{i}\varphi_k}+(\mathrm{i}\dot{\Psi}z^*+\dot{z^*})e^{2\mathrm{i}\varphi_k}\right]. \tag{2.9}$$

Then we insert Eq. (2.3) for $\dfrac{\partial}{\partial t}\left(e^{\mathrm{i}\varphi_k}\right)$ into Eq. (2.9).

$$\begin{aligned}\mathrm{i}\dot{\psi_k}e^{\mathrm{i}\psi_k}=\frac{e^{-\mathrm{i}\Psi}}{(1-z^*e^{\mathrm{i}\varphi_k})^2}\Bigg[&\mathrm{i}\dot{\Psi}z-\dot{z}+(1-|z|^2)\left(\mathrm{i}f(t)e^{\mathrm{i}\varphi_k}+\frac{1}{2}F(t)-\frac{e^{\mathrm{i}2\varphi_k}}{2}F^*(t)\right)+\\ &+(\dot{z}z^*-z\dot{z^*}-\mathrm{i}\dot{\Psi}(1+|z|^2))e^{\mathrm{i}\varphi_k}+(\mathrm{i}\dot{\Psi}z^*+\dot{z^*})e^{2\mathrm{i}\varphi_k}\Bigg].\end{aligned} \tag{2.10}$$

Then, by regrouping we obtain:

$$\begin{aligned}\mathrm{i}\dot{\psi_k}e^{\mathrm{i}\psi_k} =&\frac{e^{-\mathrm{i}\Psi}}{(1-z^*e^{\mathrm{i}\varphi_k})^2}\Bigg[\mathrm{i}\dot{\Psi}z-\dot{z}+(1-|z|^2)\frac{F}{2}+\\&+\left[\dot{z}z^*-z\dot{z^*}-\mathrm{i}\dot{\Psi}(1+|z|^2)+\mathrm{i}f(1-|z|^2)\right]e^{\mathrm{i}\varphi_k}+\\&+\left(\mathrm{i}\dot{\Psi}z^*+\dot{z^*}-(1-|z|^2)\frac{F^*}{2}\right)e^{2\mathrm{i}\varphi_k}\Bigg].\end{aligned}\tag{2.11}$$

It is easy to see that after the substitution of the global equations (2.6) all the coefficients in (2.11) at $e^{n\mathrm{i}\varphi_k}$, $n=0,1,2$ becomes zero, and thus $\dot{\psi_k}=0$.

Up to this point we have proved that if the dynamics of the global variables $z(t)$ and $\Psi(t)$ satisfies the equations (2.6) then these global variables describes the dynamics of the original system (2.1) and the original phases can be found via the WS transformation (2.2) at any moment of time, where ψ_k are indeed constants and they are determined by the initial conditions of the original system. Since we have added three more variables (one complex and one real variable) in order to determine unique set of constants ψ_k and initial conditions $z(0)$ and $\Psi(0)$ for global variables, we have to add three additional constraints. We choose this constraints to be $\sum_i e^{\mathrm{i}\psi_i}=\sum_i \cos 2\psi_i=0$ (see Appendix B for detailed discussion).

Another problem is how to represent a coupling through the new variables. As it was stressed out before, this method is applicable to global (all-to-all) coupling. So the oscillators create a global field and then this field acts on each oscillator as an "external" forcing. Note that a global field can enter either to the function $f(t)$ or $F(t)$ or both. In order to represent it through the new variables one should substitute the original phases φ_k (usually phases are included as $e^{\mathrm{i}\varphi_k}$) with the WS transformation (2.2). Here we will focus on the most popular case when a global field consists from the order parameter Z (2.12) multiplied by or added to some complex variable.

$$Z=\frac{1}{N}\sum_{j=1}^{N}e^{\mathrm{i}\varphi_j}.\tag{2.12}$$

So we need to express the order parameter Z in global variables. After substituting the WS transformation (2.2) we obtain (see [7] for details)

$$Z=\frac{1}{N}\sum_{j=1}^{N}e^{\mathrm{i}\varphi_j}=\frac{1}{N}\sum_{j=1}^{N}\frac{z+e^{\mathrm{i}(\psi_j+\Psi)}}{1+z^*e^{\mathrm{i}(\psi_j+\Psi)}}.\tag{2.13}$$

The expression (2.13) is rather complex and not applicable for analytical analysis. But there is a special case when this expression becomes extremely simple. First let us use an identity

$$\left(1+z^*e^{\mathrm{i}(\psi_j+\Psi)}\right)^{-1}=\sum_{l=0}^{\infty}(-z^*)^l e^{\mathrm{i}l(\psi_j+\Psi)}.\tag{2.14}$$

Using the identity (2.14) we rewrite the expression (2.13) for Z

$$Z = \frac{1}{N}\sum_{j=1}^{N}\left(z + e^{\mathrm{i}(\psi_j+\Psi)}\right)\sum_{l=0}^{\infty}(-z^*)^l e^{\mathrm{i}l(\psi_j+\Psi)}, \tag{2.15}$$

or

$$Z = z\left[1 + \left(1-|z|^{-2}\right)\sum_{l=1}^{\infty}(-z^*)^l \frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}l(\psi_j+\Psi)}\right], \tag{2.16}$$

Then, in the thermodynamic limit (the number of oscillators goes to infinity), there is one special configuration of constants ψ (the index has been dropped because constants now have continuous distribution) when this expression is simple. Such a configuration is a uniform distribution of constants ψ. In this case the order parameter Z is equal to the global variable z (due to the fact that the sums over j in (2.16) become integrals over the distribution and in the case of the uniform distribution these integrals vanish). Note that the requirement of the uniform distribution of constants ψ is a restriction on initial conditions, but it does not mean that the initial conditions should be also uniformly distributed (because $z(0)$ not necessary should be equal to zero (see Appendix B for details)).

Remarkably, in this case the variable Ψ does not enter the equation for z, thus the dynamics of the global variable z fully describes the dynamics of the original system (2.1). Further in this chapter we will stick with this particular case of the uniform distribution of constants ψ and by speaking about the system (2.6) of global variables we will mean only the first complex equation for z, since the equation for Ψ is irrelevant within the framework of our choice of constants ψ.

2.2. Josephson junctions

In this section we will present an application of the WS approach to the arrays of Josephson junctions (published in [9]). Let us consider the system of equations for the Josephson junction series array with a LCR load [10, 11, 12]. The equations for the junction phases ϕ_i and the load capacitor charge Q are

$$\begin{aligned} &\frac{\hbar}{2er}\frac{d\varphi_i}{dt} + I_c \sin\varphi_i = I - \frac{dQ}{dt}\,, \quad i = 1, ..., N\,, \\ &L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = \frac{\hbar}{2e}\sum_{i=1}^{N}\frac{d\varphi_i}{dt}\,. \end{aligned} \tag{2.17}$$

Here N is the number of junctions, I_c is the junction's critical current and r is the junction's resistance, L, C, R are the parameters of the LCR-load. After reparametrization

according to

$$\begin{aligned} \omega_c = 2erI_c/\hbar, \quad t^* = \omega_c t, \\ Q^* = \omega_c L^* Q/I_c, \quad I^* = I/I_c, \\ R^* = R/rN, \quad L^* = \omega_c L/rN, \quad C^* = N\omega_c rC\ , \end{aligned} \tag{2.18}$$

we obtain the dimensionless equations

$$\dot{\varphi}_i = I - \frac{1}{L^*}\dot{Q}^* - \sin\varphi_i, \tag{2.19a}$$

$$\ddot{Q}^* + \frac{R^*}{L^*}\dot{Q}^* + \frac{Q^*}{L^*C^*} = \frac{1}{N}\sum_{i=1}^{N}\dot{\varphi}_i\ . \tag{2.19b}$$

Moreover, it is convenient to substitute the expression for $\dot{\varphi}_i$ from (2.19a) into the equation for the load (2.19b) and introduce new parameters $\epsilon = 1/L^*$, $\gamma = (R^*+1)/L^*$ and $\omega_0 = 1/\sqrt{L^*C^*}$. So, the following system is obtained (dropping the asterixes for simplicity)

$$\begin{aligned} \dot{\varphi}_i = I - \epsilon\dot{Q} - \sin\varphi_i, \\ \ddot{Q} + \gamma\dot{Q} + \omega_0^2 Q = I - \frac{1}{N}\sum_{i=1}^{N}\sin\varphi_i. \end{aligned} \tag{2.20}$$

The sum over sines can be replaced by the imaginary value of the Kuramoto complex order parameter Z (2.12).

$$\begin{aligned} Z = \frac{1}{N}\sum_{i=1}^{N}(\cos\varphi_i + \mathrm{i}\sin\varphi_i)\ , \\ \mathrm{Im}(Z) = \frac{1}{N}\sum_{i=1}^{N}\sin\varphi_i\ , \end{aligned} \tag{2.21}$$

Also the equation for the junction's phases φ_i can be rewritten in the form (2.1)

$$\dot{\varphi}_i = I - \epsilon\dot{Q} + \mathrm{Im}(e^{-\mathrm{i}\varphi_i}) \tag{2.22}$$

for what the WS approach is applicable. Thus the dynamics of the system of Josephson junctions can be described by the system (2.6). So we need to insert $f = I - \epsilon\dot{Q}$ and $F = 1$ into the equations (2.6) for the WS global variables. From now on we will consider the thermodynamic limit $N \to \infty$ and the uniform distribution of constants ψ. In this case the closed system of equations that describes the dynamics of the array reads

$$\begin{aligned} \dot{Z} = \mathrm{i}(I - \epsilon\dot{Q})Z + \frac{1}{2} - \frac{Z^2}{2}, \\ \ddot{Q} + \gamma\dot{Q} + \omega_0^2 Q = I - \mathrm{Im}(Z). \end{aligned} \tag{2.23}$$

Further analysis is rather straightforward. At a steady state regime $\dot{Q} = 0$, thus the coupling vanishes, and the only stationary solution is the steady state that describes

an asynchronous regime with $Z_0 = \mathrm{i}(I - \sqrt{I^2-1})$, $Q_0 = \omega_0^{-2}\sqrt{I^2-1}$. Stability of this solution is determined by the fourth-order characteristic equation

$$\begin{aligned} \lambda^4 + \gamma\lambda^3 + (\omega_0^2 + I^2 - 1)\lambda^2 + \\ +[(\gamma - \epsilon)(I^2-1) + \epsilon I\sqrt{I^2-1}\,]\lambda + \omega_0^2(I^2-1) = 0. \end{aligned} \tag{2.24}$$

The stability border can be easily found by inserting $\lambda = \mathrm{i}\omega$ in the characteristic equation (2.24)

$$\omega_0^2 = (I^2-1) + \frac{\epsilon}{\gamma}\sqrt{I^2-1}(I - \sqrt{I^2-1}). \tag{2.25}$$

The dynamics of the phase Φ in fully synchronous regime in (2.23) with $|Z| = 1$ is determined by the system

$$\begin{aligned} \ddot{Q} + \gamma\dot{Q} + \omega_0^2 Q &= I - \sin\Phi\,, \\ \dot{\Phi} &= I - \epsilon\dot{Q} - \sin\Phi\,. \end{aligned} \tag{2.26}$$

The stability of the synchronous solution was analyzed by finding the largest multiplier of the numerical solution of Eq. (2.26). Combining this result with the stability border (2.25) of asynchronous regime, we obtain the intersecting domains of stability of the asynchronous and synchronous states, that form the region of bistability, see Fig. 2.1.

In Fig. 2.2 another illustration of the bistability is presented, showing the stable part of the asynchronous state Z_0 and stable synchronous regime $|Z| = 1$ as a function of the parameter I. In this figure we also show numerical solution when the distribution of constants ψ_i is not uniform. The initial conditions were prepared specially in order to have nonuniform distribution of constants ψ_i as described in ref. [7], appendix C. The simulation was done for an ensemble of 100 junctions. For this case there is no asynchronous steady state Z_0, but the asynchrony state demonstrates an oscillating variable $Z(t)$, minimum and maximum values of which are marked with squares. The synchronous regime, $|Z| = 1$, is the same, what means that the system loses its information on the constants ψ_i as synchrony establishes.

Additionally, we would like to point out that we found only the fully synchronous and the fully asynchronous states in the system (2.23). No partial synchrony like it was described in [13] was observed. In a partial synchronous state there exists a nonzero mean field, although the oscillators are not perfectly synchronous. They perform a collective behavior forming a bunch. Unlike the similar setup in [13] where the load was nonlinear, here the load was considered to be linear. We cannot prove that partial synchrony is impossible for linear loads, but our numerical simulation, together with the study of Ref. [14], did not show such regimes.

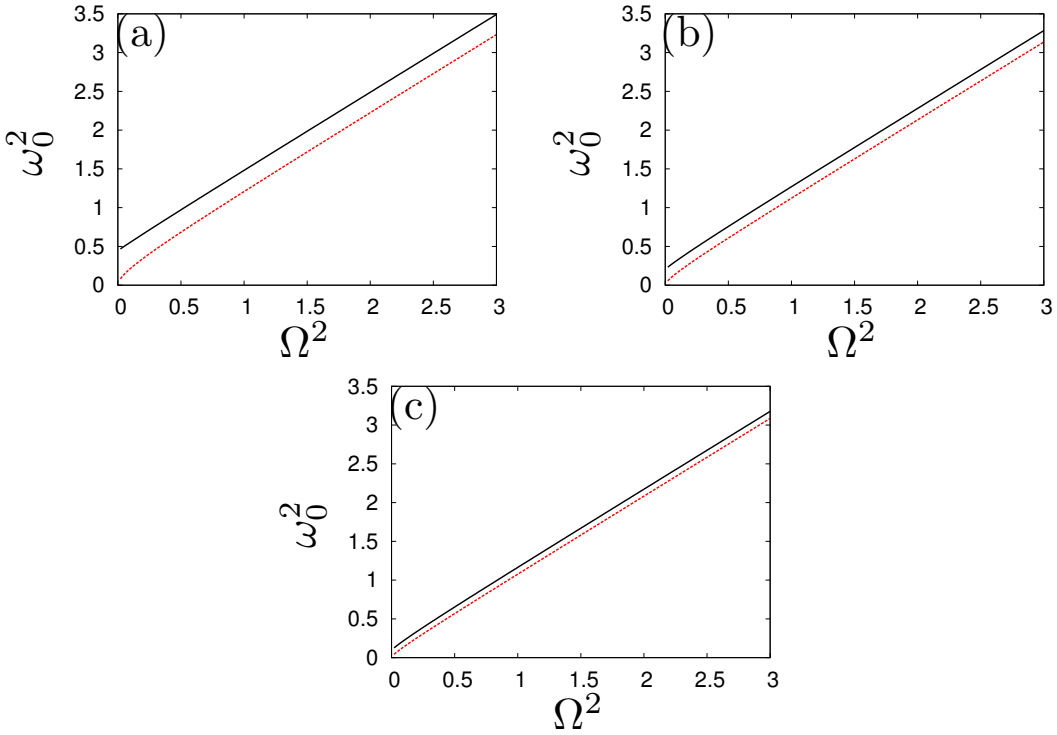


Figure 2.1.: (Color online) Domains of stability of synchronous (above lower dashed line) and asynchronous (below upper solid line) states on the plane of parameters (ω_0^2, Ω^2), where $\Omega = \sqrt{I^2 - 1}$ is the natural frequency of the junctions. Here $\epsilon = 0.5$, and $\gamma = 1.0$ (a), 1.7 (b), 2.7 (c). From [9].

2.3. A system of identical phase oscillators with leader-type coupling

In this section we apply WS approach to the system of phase oscillators with a leader-type coupling. Such a network structure is often called star network, the simplest small-world network. In our setup all the phase oscillators φ_k are identical having frequency ω and are coupled to the leader oscillator ϕ with coupling strength A and phase shift α. At the same time the leader ϕ has its own frequency ω_0 and is coupled to every other oscillators φ_j with coupling coefficient B and phase shift β:

$$\begin{aligned} \dot{\varphi_k} &= \omega + A\sin(\phi - \varphi_k - \alpha), \\ \dot{\phi} &= \omega_0 + \frac{1}{N}\sum_{j=1}^{N} B\sin(\varphi_j - \beta - \phi). \end{aligned} \tag{2.27}$$

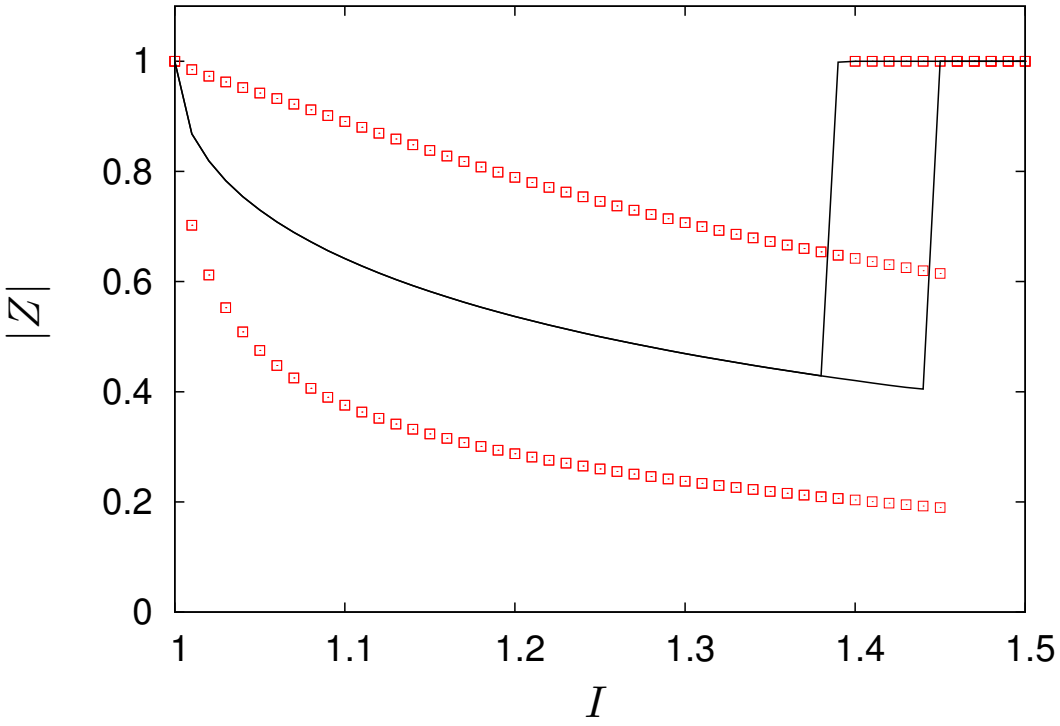


Figure 2.2.: Dependence of the order parameter $|Z|$ on the current I for 100 junctions. Line: uniform distribution of constants ψ_i, squares: nonuniform distributions. From [9].

The system (2.27) can be rewritten in terms of mean-field

$$\begin{aligned} \dot{\varphi_k} &= \omega + \mathrm{Im}(Ae^{\mathrm{i}(\phi-\varphi_k-\alpha)}), \\ \dot{\phi} &= \omega_0 + \mathrm{Im}(\tilde{G}(t)e^{-\mathrm{i}\phi}), \\ \tilde{G}(t) &= \frac{1}{N}\sum_{j=1}^{N} Be^{\mathrm{i}(\varphi_j-\beta)}. \end{aligned} \tag{2.28}$$

It is convenient to make a variable transformation to the phase differences $\Delta\varphi_k$ between the oscillators φ_k and the leader ϕ taking into account the phase shift α

$$\Delta\varphi_k = \varphi_k - \phi + \alpha. \tag{2.29}$$

Then, the equations for $\Delta\varphi_k$ and ϕ are

$$\begin{aligned} \Delta\dot{\varphi}_k &= -\dot{\phi} + \omega + \mathrm{Im}(Ae^{-\mathrm{i}\Delta\varphi_k}), \\ \dot{\phi} &= \omega_0 + \mathrm{Im}(G(t)), \\ G(t) &= \frac{1}{N}\sum_{j=1}^{N} Be^{\mathrm{i}(\Delta\varphi_j-\alpha-\beta)}. \end{aligned} \tag{2.30}$$

The expression for the leader dynamics can be directly introduced to the equations for $\Delta\varphi_k$ and thus we obtain the closed system

$$\begin{aligned} \Delta\dot{\varphi}_k &= \omega - \omega_0 - \mathrm{Im}\,(G(t)) + \mathrm{Im}(Ae^{-\mathrm{i}\Delta\varphi_k}), \\ G(t) &= Be^{-\mathrm{i}(\alpha+\beta)}\frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}\Delta\varphi_j}, \end{aligned} \tag{2.31}$$

that allows us to use the Watanabe-Strogatz ansatz.

Comparing with the system (2.1) we see that in our case $f(t) = \omega - \omega_0 - \mathrm{Im}\,(G(t))$ and $F(t) = A$. We consider below the problem in the thermodynamic limit $N \to \infty$ and uniform distribution of constants of motion ψ when the order parameter is equal to z. In this case from (2.6) it follows that Ψ does not enter the equation for z, so we obtain an equation for z that describes the system (2.31)

$$\dot{z} = \mathrm{i}\left(\Delta\omega - B\,\mathrm{Im}(ze^{-\mathrm{i}\delta})\right) z - A\frac{z^2-1}{2}, \tag{2.32}$$

where $\Delta\omega = \omega - \omega_0$ and $\delta = \alpha + \beta$.

For further analysis it is appropriate to represent the complex variable $z = \rho e^{\mathrm{i}\Delta\Phi}$ in polar form. Thus

$$\begin{aligned} \dot{\rho} &= A\frac{1-\rho^2}{2}\mathrm{Re}(e^{\mathrm{i}\Delta\Phi}), \\ \Delta\dot{\Phi} &= \Delta\omega + (B\sin\delta)\rho\,\mathrm{Re}(e^{\mathrm{i}\Delta\Phi}) - \frac{A + (A + 2B\cos\delta)\rho^2}{2\rho}\,\mathrm{Im}(e^{\mathrm{i}\Delta\Phi}). \end{aligned} \tag{2.33}$$

Note that Eqs. (2.33) are invariant under the following transformation of variables and parameters $\Delta\Phi \to -\Delta\Phi$, $\Delta\omega \to -\Delta\omega$ and $\delta \to -\delta$.

We start the analysis of (2.33) with finding its steady states. From the first equation in (2.33) it follows that there are two types of steady states when $\dot{\rho} = 0$: synchronous with $\rho = 1$ and asynchronous with $\cos\Delta\Phi = 0$. The synchronous steady state gives

$$\begin{aligned} &\rho = 1, \\ &\Delta\dot{\Phi} = \Delta\omega - \sqrt{A^2 + B^2 + 2AB\cos\delta}\;\mathrm{Im}\left(e^{\mathrm{i}\left(\Delta\Phi + \arcsin\frac{A+B\cos\delta}{\sqrt{A^2+B^2+2AB\cos\delta}} - \frac{\pi}{2}\right)}\right). \end{aligned} \tag{2.34}$$

From (2.34) it follows that the steady solution $\Delta\dot{\Phi} = 0$ exists only if $|\Delta\omega| \le \sqrt{A^2 + B^2 + 2AB\cos\delta}$.

By rescaling time it is convenient to reduce the number of parameters. Eq. (2.34) suggests that the most convenient rescaling is

$$t' = t\sqrt{A^2 + B^2 + 2AB\cos\delta}. \tag{2.35}$$

This rescaling is quite general except for two special cases when $\cos\delta = -1$ and $B = A$ (see Appendix C) or $A = B = 0$, the later is just the case of uniformly rotating uncoupled phase oscillators that does not present any interest. So in new parametrization Eq. (2.34) becomes

$$\begin{aligned} \rho &= 1, \\ \dot{\Delta\Phi} &= \Delta x - \operatorname{Im}\left(e^{\mathrm{i}\left(\Delta\Phi + \xi - \frac{\pi}{2}\right)}\right), \end{aligned} \tag{2.36}$$

where

$$\Delta x = \frac{\Delta\omega}{\sqrt{A^2 + B^2 + 2AB\cos\delta}} \quad \text{and} \quad \sin\xi = \frac{A + B\cos\delta}{\sqrt{A^2 + B^2 + 2AB\cos\delta}}. \tag{2.37}$$

Thus the steady solutions of Eq. (2.36) have the following phases

$$\Delta\Phi_{s1} = \frac{\pi}{2} + \arcsin\Delta x - \xi, \quad \Delta\Phi_{s2} = -\frac{\pi}{2} - \arcsin\Delta x - \xi. \tag{2.38}$$

In the new parametrization Eq. (2.33) becomes

$$\begin{aligned} \dot{\rho} &= g\frac{1-\rho^2}{2}\operatorname{Re}(e^{\mathrm{i}\Delta\Phi}), \\ \dot{\Delta\Phi} &= \Delta x + (\cos\xi)\rho\operatorname{Re}(e^{\mathrm{i}\Delta\Phi}) - \frac{g + (2\sin\xi - g)\rho^2}{2\rho}\operatorname{Im}(e^{\mathrm{i}\Delta\Phi}), \end{aligned} \tag{2.39}$$

where $g = \frac{A}{\sqrt{A^2+B^2+2AB\cos\delta}} \geq 0$. Note that analogous to Eqs. (2.33), Eqs. (2.39) are invariant to the following transformation of variables and parameters $\Delta\Phi \to -\Delta\Phi$, $\Delta x \to -\Delta x$ and $\cos\xi \to -\cos\xi$. Due to this symmetry we can consider only the case when $\cos\xi \geq 0$.

The asynchronous steady states could be found from

$$\begin{aligned} \Delta\Phi &= \pm\pi/2, \\ 0 &= \Delta x \mp \frac{g + (2\sin\xi - g)\rho^2}{2\rho}. \end{aligned} \tag{2.40}$$

Eq (2.40) gives two asynchronous steady solutions:

$$z_{a1,2} = \mathrm{i}\frac{\Delta x \pm \sqrt{\Delta x^2 - g(2\sin\xi - g)}}{2\sin\xi - g}. \tag{2.41}$$

It is convenient to rewrite Eq. (2.41) as

$$z_{a1,2} = \operatorname{sign}(\Delta x)\,\mathrm{i}\frac{|\Delta x| \pm \sqrt{\Delta x^2 - g(2\sin\xi - g)}}{2\sin\xi - g}, \tag{2.42}$$

note that here the cases split, depending on the value of $(2\sin\xi - g)$. If $|2\sin\xi - g| > g$, because $\rho = |z| \leq 1$, z_{a1} solution exists only if $|\Delta x| \leq |\sin\xi|$. And if $|2\sin\xi - g| \leq g$,

also because $\rho = |z| \leq 1$, z_{a2} solution exists only if $|\Delta x| \geq \sin\xi$. If $2\sin\xi - g = 0$, the asynchronous steady solutions are

$$z_{a1,2} = \pm\,\mathrm{sign}(\Delta x)\,\mathrm{i}\frac{g}{2|\Delta x|}, \tag{2.43}$$

but the condition on $|\Delta x| \geq \sin\xi = g/2$ is still the same.

Note that the expression $2\sin\xi - g$ is equal to $\frac{A+2B\cos\delta}{\sqrt{A^2+B^2+2AB\cos\delta}}$; if $\cos\delta \geq 0$ this expression is always positive and $g < 1$ and $\sin\xi > g$. If $\cos\delta < 0$ the sign of this expression depends on the sign of $A + 2B\cos\delta$ but $\sin\xi < g$.

2.3.1. Stability analysis

In order to study stability of the asynchronous steady solutions (2.42) we linearize the system around corresponding fixed point. The linearized system reads

$$\begin{aligned}
\dot{a}_{1,2} &= \mathrm{sign}(\Delta x)\left(-(\cos\xi)\,\frac{|\Delta x| \pm \sqrt{\Delta x^2 - g(2\sin\xi - g)}}{2\sin\xi - g}\,a_{1,2} \pm \sqrt{\Delta x^2 - g(2\sin\xi - g)}\,b_{1,2}\right),\\
\dot{b}_{1,2} &= \mathrm{sign}(\Delta x)\left(|\Delta x| - (\sin\xi)\,\frac{|\Delta x| \pm \sqrt{\Delta x^2 - g(2\sin\xi - g)}}{2\sin\xi - g}\right)a_{1,2},
\end{aligned} \tag{2.44}$$

where $a_{1,2} = \mathrm{Re}(z_{1,2})$ and $b_{1,2} = \mathrm{Im}(z_{1,2}) - \mathrm{Im}(z_{a1,2})$ respectively. Despite the fact that it is difficult to find explicit expressions for eigenvalues of the linear system (2.44), it is possible to find regions of parameters when they are positive or negative, thus to analyze stability of asynchronous solutions.

There is one truly remarkable case when $\cos\xi = 0$ or $\sin\delta = 0$. In this case, eigenvalues of the linear system (2.44) for z_{a2} are purely imaginary, what gives a possibility for the fixed point z_{a2} to be neutrally stable (see next section for details).

For two synchronous fixed points (2.38): $z_{s1} = e^{\mathrm{i}\Delta\Phi_{s1}}$ and $z_{s2} = e^{\mathrm{i}\Delta\Phi_{s2}}$, the corresponding linearized system reads

$$\begin{aligned}
\dot{a}_{1,2} =& \left[\mp\sqrt{1-\Delta x^2} + (\sin\xi - g)(-\Delta x\cos\xi \pm \sqrt{1-\Delta x^2}\sin\xi)\right]a_{1,2} +\\
&+ \left[(\sin\xi - g)(\pm\sqrt{1-\Delta x^2}\cos\xi + \Delta x\sin\xi)\right]b_{1,2},\\
\dot{b}_{1,2} =& \left[\cos\xi(-\Delta x\cos\xi \pm \sqrt{1-\Delta x^2}\sin\xi)\right]a_{1,2} +\\
&+ \left[-\sin\xi(-\Delta x\cos\xi \pm \sqrt{1-\Delta x^2}\sin\xi)\right]b_{1,2},
\end{aligned} \tag{2.45}$$

where $a_{1,2} = \mathrm{Re}(z_{1,2}) - \mathrm{Re}(z_{s1,2}) = \mathrm{Re}(z_{1,2}) - (-\Delta x \cos\xi \pm \sqrt{1-\Delta x^2}\sin\xi)$ and $b_{1,2} = \mathrm{Im}(z_{1,2}) - \mathrm{Im}(z_{s1,2}) = \mathrm{Im}(z_{1,2}) - (\pm\sqrt{1-\Delta x^2}\cos\xi + \Delta x \sin\xi)$ respectively.

Linear system (2.45) has two eigenvalues:

$$\begin{aligned} \lambda^1_{s1,2} &= g(\Delta x \cos\xi \mp \sqrt{1-\Delta x^2}\sin\xi), \\ \lambda^2_{s1,2} &= \mp\sqrt{1-\Delta x^2}. \end{aligned} \tag{2.46}$$

Their signs depend on the values of the parameters. We will outline all possible steady solutions together with their stability below.

2.3.2. Reversible case when phase shift $\cos\xi = \sin\delta = 0$

As shown by the stability analysis above, when $\cos\xi = 0$, z_{a2} can be neutrally stable for any Δx. The neutral stability can be proved by the fact that when $\cos\xi = \sin\delta = 0$, Eq. (2.32) is invariant to the variable transformation symmetrical with respect to the imaginary axis: $\mathrm{Im}(z) \to \mathrm{Im}(z)$, $\mathrm{Re}(z) \to -\mathrm{Re}(z)$ and $t \to -t$. Thus any trajectory that crosses $\mathrm{Im}(z)$ axes two times is a closed curve (Fig. 2.3). That makes the asynchronous steady solution neutrally stable. Thus it is difficult to determine it by direct numerical simulation of the system of oscillators coupled through a leader without phase shift (star networks). Which caused a debate around hysteretic transitions between asynchronous and synchronous regimes and the nature of asynchronous regime with non-zero order parameter (see [16] for detailed description of the problem).

2.3.3. The presentation of the solutions for all values of the parameters

Since we consider only the case when $\cos\xi > 0$ (for $\cos\xi = 0$ see previous section) and thus $\sin\xi \neq 1$, the solution with stability for $\Delta x > 0$ [$\Delta x < 0$] is

(i) (Fig. 2.4) $(2\sin\xi - g) > g$, note that $1 > \sin\xi > g \geq 0$ and

$$\left(|\Delta x| - (\sin\xi)\,\frac{|\Delta x| \pm \sqrt{\Delta x^2 - g(2\sin\xi - g)}}{2\sin\xi - g} \right) > 0. \tag{2.47}$$

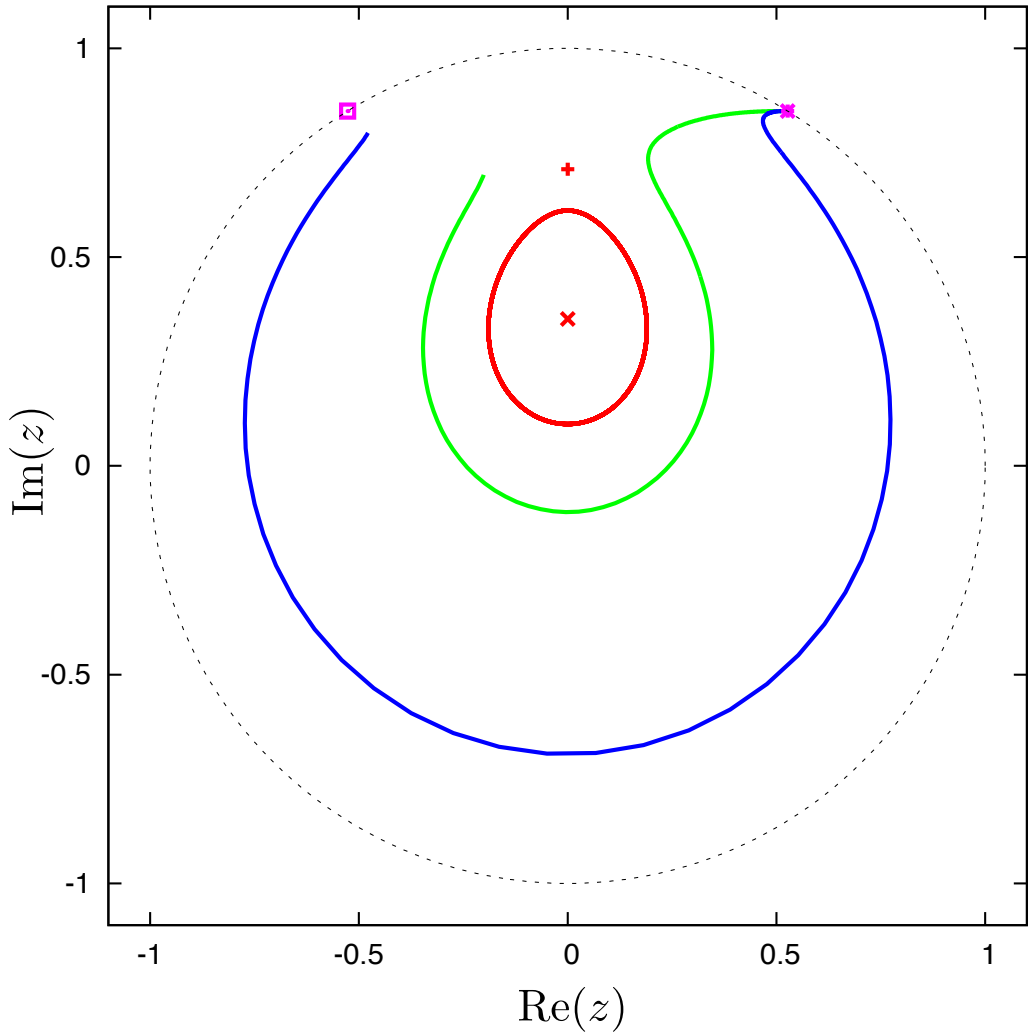


Figure 2.3.: The phase diagram for the case $\sin\xi = 1$. $\Delta x = 0.85$ and $g = 0.4$.

$$
\begin{aligned}
&z_{s1} - sink(stable)node\ [sink(stable)node] \quad \text{and} \\
&z_{s2} - source(unstable)node\ [source(unstable)node], \quad \text{if } |\Delta x| < \sqrt{g(2\sin\xi - g)}\,, \\
&\\
&z_{a1} - saddle\ [saddle], \quad z_{a2} - stable\ [unstable] \quad \text{and} \\
&z_{s1} - sink(stable)node\ [sink(stable)node] \quad \text{and} \\
&z_{s2} - source(unstable)node\ [source(unstable)node], \qquad \text{if } \sqrt{g(2\sin\xi - g)} \le |\Delta x| \le \sin\xi, \\
&\\
&z_{a2} - stable\ [unstable] \quad \text{and} \\
&z_{s1} - saddle\ [sink(stable)node] \quad \text{and} \\
&z_{s2} - source(unstable)node\ [saddle], \qquad \text{if } \sin\xi < |\Delta x| \le 1, \\
&\\
&z_{a2} - stable\ [unstable], \quad \text{and} \\
&|z| = 1,\ \arg(z) = \Delta\Phi(t),\ unstable\ [stable]\ limit\ cicle \quad \text{if } 1 < |\Delta x|.
\end{aligned}
\tag{2.48}
$$

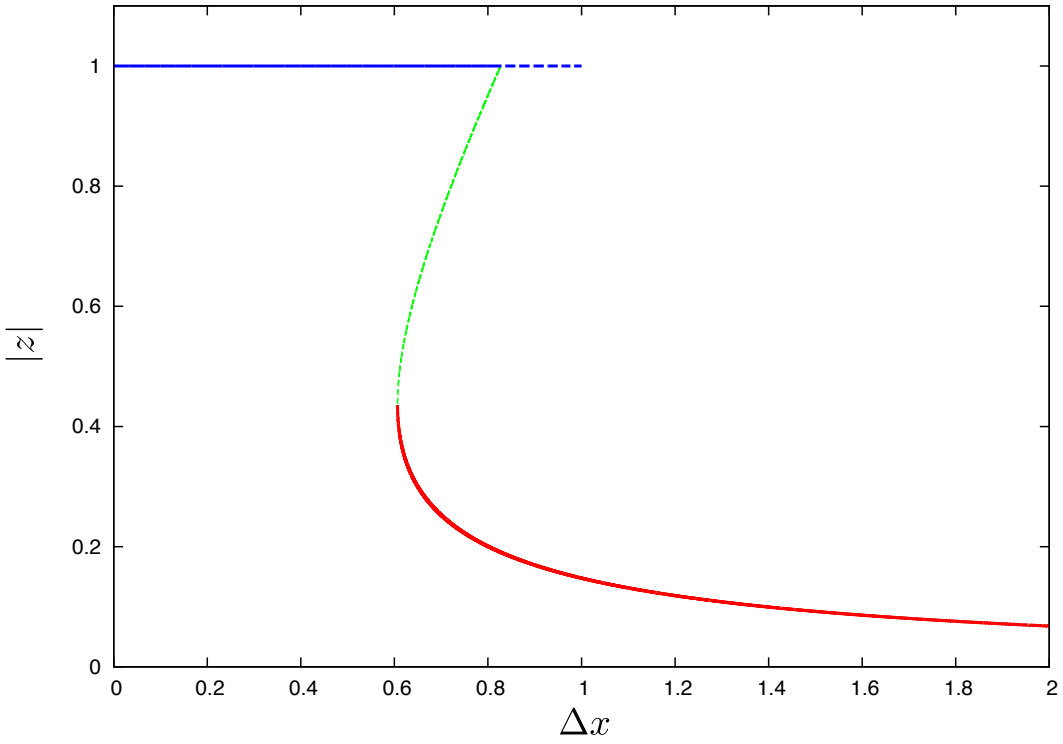


Figure 2.4.: The dependence of the order parameter $|z|$ on the relative frequency mismatch Δx, for the case $\sin\xi > g \geq 0$

(ii) (Fig. 2.5) $-g \leq (2\sin\xi - g) \leq g$, so $g \geq \sin\xi \geq 0$ and

$$\left(|\Delta x| - (\sin\xi)\, \frac{|\Delta x| \pm \sqrt{\Delta x^2 - g(2\sin\xi - g)}}{2\sin\xi - g} \right) > 0 \tag{2.49}$$

$$\begin{aligned} &z_{s1} - sink(stable)node\ [sink(stable)node] \quad \text{and} \\ &z_{s2} - source(unstable)node\ [source(unstable)node], \qquad \text{if } |\Delta x| < \sin\xi, \end{aligned}$$

$$\begin{aligned} &z_{a2} - stable\ [unstable] \quad \text{and} \\ &z_{s1} - saddle\ [sink(stable)node] \quad \text{and} \\ &z_{s2} - source(unstable)node\ [saddle], \qquad \text{if } \sin\xi \leq |\Delta x| \leq 1, \end{aligned} \tag{2.50}$$

$$\begin{aligned} &z_{a2} - stable\ [unstable], \quad \text{and} \\ &|z| = 1,\ \arg(z) = \Delta\Phi(t),\ unstable\ [stable]\ limit\ cicle \quad \text{if } 1 < |\Delta x|. \end{aligned}$$

(iii) (Fig. 2.6) $(2\sin\xi - g) < -g$, thus $\sin\xi < 0$ and

$$\begin{aligned} &\left(|\Delta x| - (\sin\xi)\, \frac{|\Delta x| + \sqrt{\Delta x^2 - g(2\sin\xi - g)}}{2\sin\xi - g} \right) < 0 \\ &\left(|\Delta x| - (\sin\xi)\, \frac{|\Delta x| - \sqrt{\Delta x^2 - g(2\sin\xi - g)}}{2\sin\xi - g} \right) > 0 \end{aligned} \tag{2.51}$$

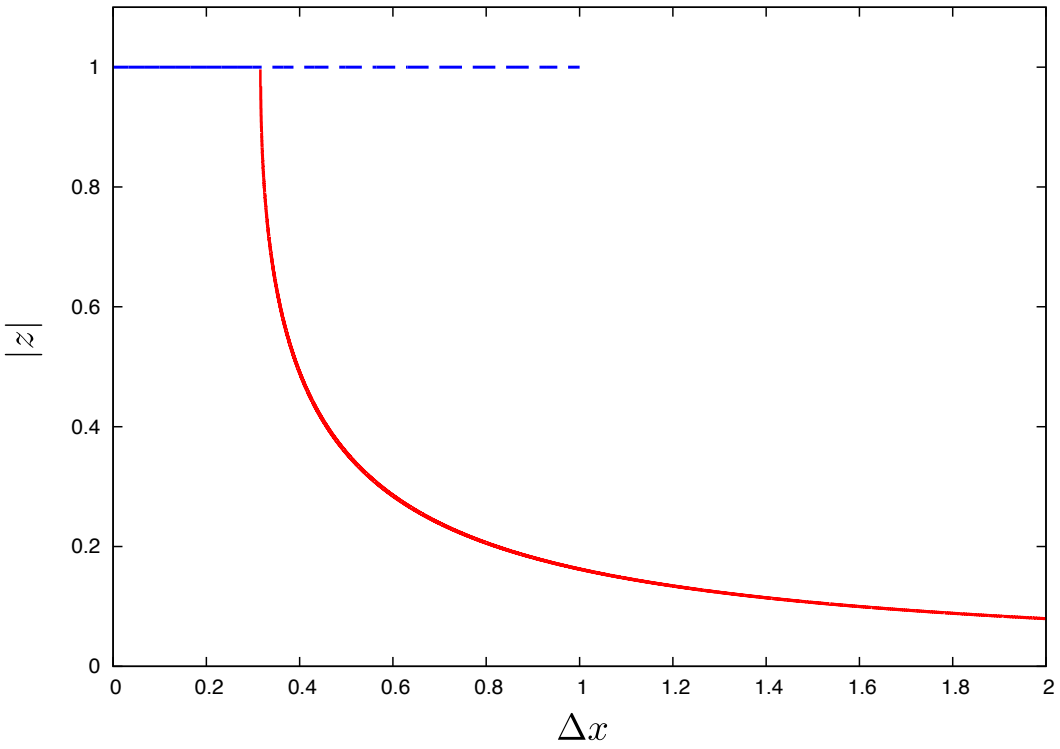


Figure 2.5.: The dependence of the order parameter $|z|$ on the relative frequency mismatch Δx, for the case $g \geq \sin\xi \geq 0$

$$
\begin{aligned}
&z_{a1} - \textit{unstable}\ [\textit{stable}], \quad z_{a2} - \textit{stable}\ [\textit{unstable}] \quad \text{and} \\
&z_{s1} - \textit{saddle}\ [\textit{saddle}] \quad \text{and} \\
&z_{s2} - \textit{saddle}\ [\textit{saddle}], \qquad \text{if } |\Delta x| \leq |\sin\xi|, \\
\\
&z_{a2} - \textit{stable}\ [\textit{unstable}] \quad \text{and} \\
&z_{s1} - \textit{saddle}\ [\textit{sink(stable)node}] \quad \text{and} \\
&z_{s2} - \textit{source(unstable)node}\ [\textit{saddle}], \qquad \text{if } |\sin\xi| < |\Delta x| \leq 1, \\
\\
&z_{a2} - \textit{stable}\ [\textit{unstable}], \quad \text{and} \\
&|z| = 1,\ \arg(z) = \Delta\Phi(t),\ \textit{unstable}\ [\textit{stable}]\ \textit{limit cicle} \quad \text{if } 1 < |\Delta x|.
\end{aligned}
\tag{2.52}
$$

Where

$$
\begin{aligned}
z_{s1} &= e^{\mathrm{i}\left(\frac{\pi}{2} + \arcsin\Delta x - \xi\right)}, \\
z_{s2} &= e^{\mathrm{i}\left(-\frac{\pi}{2} - \arcsin\Delta x - \xi\right)},
\end{aligned}
\tag{2.53}
$$

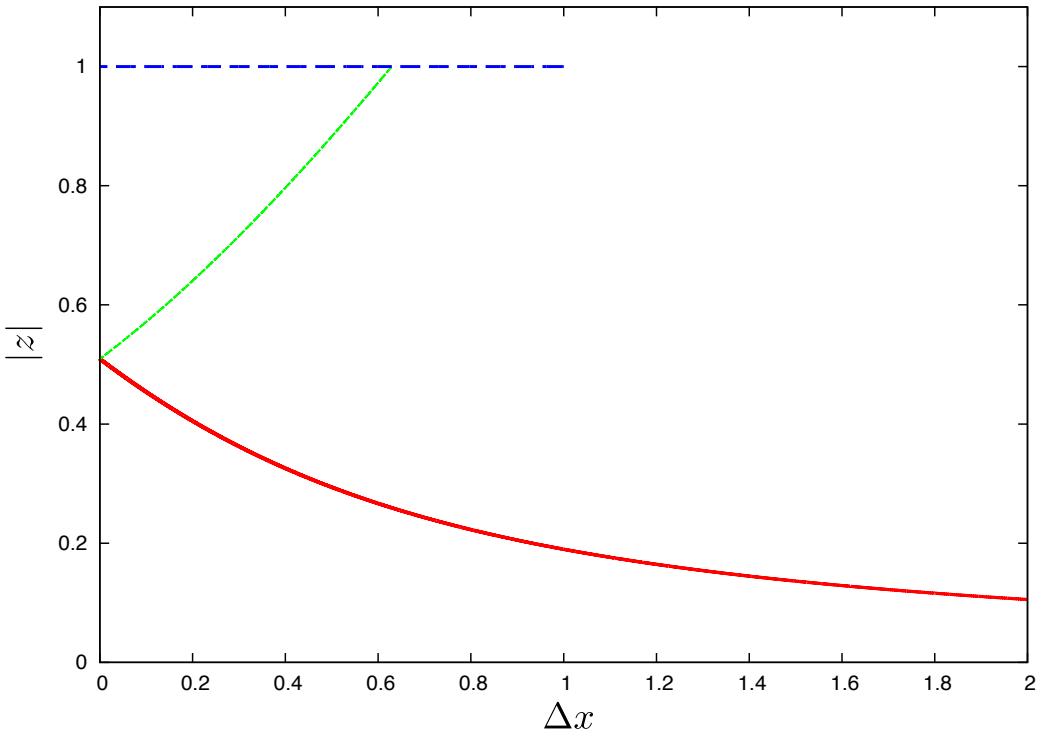


Figure 2.6.: The dependence of the order parameter $|z|$ on the relative frequency mismatch Δx, for the case $\sin\xi < 0$

and if $2\sin\xi - g \neq 0$

$$\begin{aligned} z_{a1} &= \operatorname{sign}(\Delta x)\,\mathrm{i}\frac{|\Delta x| + \sqrt{\Delta x^2 - g(2\sin\xi - g)}}{2\sin\xi - g}, \\ z_{a2} &= \operatorname{sign}(\Delta x)\,\mathrm{i}\frac{|\Delta x| - \sqrt{\Delta x^2 - g(2\sin\xi - g)}}{2\sin\xi - g}, \end{aligned} \tag{2.54}$$

or if $2\sin\xi - g = 0$

$$\begin{aligned} z_{a1} &= \operatorname{sign}(\Delta x)\,\mathrm{i}\frac{g}{2|\Delta x|}, \\ z_{a2} &= -\operatorname{sign}(\Delta x)\,\mathrm{i}\frac{g}{2|\Delta x|}. \end{aligned} \tag{2.55}$$

We obtain three main regions of parameters with three different transitions from asynchronous steady solution to synchronous one. We show the transition depending on the relative frequency mismatch Δx between individual oscillator's frequency and the frequency of the leader. Stability of the solutions depends on the sign of frequency mismatch Δx.

In the first case (Fig. 2.4) there is a hysteretic transition due to the existence of unstable fixed point (2.48). If $\Delta x > 0$ there are stable asynchronous steady solution that exists for large absolute values $|\Delta x|$ and stable synchronous steady solution that exists for

small $|\Delta x|$. These solutions coexist for a bounded region of $|\Delta x|$ forming hysteresis. If the frequency mismatch $\Delta x < 0$, the synchronous steady solution is still stable and also synchronous limit cycle solution becomes stable for large $|\Delta x|$ and asynchronous steady solution becomes unstable.

In the second case (Fig. 2.5) there is no hysteresis (2.50). In this case there is only one asynchronous steady solution existing for large values $|\Delta x|$ that is stable if $\Delta x > 0$ and unstable if $\Delta x < 0$. If $\Delta x > 0$ the synchronous solution is stable for small values of $|\Delta x|$, if $\Delta x < 0$ both synchronous (steady and limit cycle) solutions are steady for all $|\Delta x|$.

In the third case (Fig. 2.6) the transition is not hysteretic (2.52). For $\Delta x > 0$ there is only one stable steady solution that is asynchronous one. And for $\Delta x < 0$ for small $|\Delta x|$ there is stable asynchronous steady solution that transforms to synchronous steady solution for larger $|\Delta x|$, that then with further increase of $|\Delta x|$ becomes stable synchronous limit cycle.

2.3.4. Additional mean-field coupling

The analytical approach described above can be partially applied for the system of identical oscillators coupled not only through a leader but also via a Kuramoto-Sakaguchi mean-field. The system (2.31) with additional mean-field $H(t)$ becomes

$$\begin{aligned}
\dot{\Delta\varphi_k} &= \omega - \omega_0 - \mathrm{Im}\,(G(t)) + \mathrm{Im}(Ae^{-\mathrm{i}\Delta\varphi_k}) + \mathrm{Im}(H(t)e^{-\mathrm{i}\Delta\varphi_k}),\\
G(t) &= Be^{-\mathrm{i}(\alpha+\beta)}\frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}\Delta\varphi_j},\\
H(t) &= Ce^{-\mathrm{i}\gamma}\frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}\Delta\varphi_j}.
\end{aligned} \tag{2.56}$$

The WS approach can be also applied for the new system (2.56), so that according to (2.6)

$$\dot{z} = \mathrm{i}(\Delta\omega - B\,\mathrm{Im}(ze^{-\mathrm{i}\delta}))z - A\frac{z^2-1}{2} + \frac{C}{2}(e^{-\mathrm{i}\gamma} - e^{\mathrm{i}\gamma}|z|^2)z, \tag{2.57}$$

Then we perform the similar analysis together with the same rescaling of time (2.35) and reparameterization (2.37) as in the previous case. In the new notation Eq. (2.57) becomes

$$\begin{aligned}
\dot{\rho} &= \frac{1-\rho^2}{2}(g\mathrm{Re}(e^{\mathrm{i}\Delta\Phi}) + q\cos\gamma\,\rho),\\
\dot{\Delta\Phi} &= \Delta x - q\frac{1+\rho^2}{2}\sin\gamma + (\cos\xi)\rho\,\mathrm{Re}(e^{\mathrm{i}\Delta\Phi}) - \frac{g + (2\sin\xi - g)\rho^2}{2\rho}\,\mathrm{Im}(e^{\mathrm{i}\Delta\Phi}),
\end{aligned} \tag{2.58}$$

where $q = \frac{C}{\sqrt{B^2+A^2+2BA\cos\delta}} \geq 0$. Note that analogous to Eqs. (2.33), Eqs. (2.58) are invariant to the following transformation of variables and parameters $\Delta\Phi \to -\Delta\Phi$, $\Delta x \to -\Delta x$ and $\cos\xi \to -\cos\xi$, $\gamma \to -\gamma$. Thus, as before we will consider only the case when $\cos\xi \geq 0$.

The synchronous steady solutions with $\rho = |z| = 1$ of Eq. (2.58) are

$$\Delta\Phi_{s1} = \frac{\pi}{2} + \arcsin(\Delta x - q\sin\gamma) - \xi, \quad \Delta\Phi_{s2} = -\frac{\pi}{2} - \arcsin(\Delta x - q\sin\gamma) - \xi. \tag{2.59}$$

The incoherent steady solutions should be found from the following equations

$$\cos\Delta\Phi = -\frac{q\cos\gamma\,\rho}{g},$$
$$0 = \Delta x - q\frac{1+\rho^2}{2}\sin\gamma - (\cos\xi)\rho\,\frac{q\cos\gamma\,\rho}{g} \mp \frac{g + (2\sin\xi - g)\rho^2}{2\rho}\sqrt{1 - \left(\frac{q\cos\gamma\,\rho}{g}\right)^2}. \tag{2.60}$$

The system of equations (2.60) for ρ and $\Delta\Phi$ is rather complex for the analytical analysis, but it is clear that there are two main limiting cases. The first is the case with big C (or reparameterized q), this means that the dynamics of the system is mostly influenced by the mean-field. This case qualitatively coincides with the well studied case when $B = A = 0$ with two synchronous fixed points (one stable and one unstable with $|z| = 1$) and one asynchronous fixed point (stability of which depends on the coupling parameters and frequency mismatch). The second case is when the influence of the mean field is relatively small, or the coupling strength C (or q) is small. The qualitative picture for this case coincides with the limit $C = q = 0$ considered in the main part of this section. The quantitative results can be obtained numerically. Note that our approach is still useful here because for the numerical analysis the reduced system (2.58) is much simpler then the original one.

2.4. Nonidentical oscillators

Although the WS approach in its original form can be applied only to identical oscillators, in some organization of distributions of parameters this approach can help to reduce the number of equations and to obtain some analytical results. Here we will present such networks with nonidentical elements that can be treated with the help of the WS approach on the example of nonidentical Josephson junction array (published in [9]). So we will start with the formulation of the model.

2.4.1. The formulation of the model

In the model of the junctions (2.17) there are two individual parameters that can differ: the critical current I_c and the resistance r (cf. [11, 12]). The Watanabe-Strogatz

approach can be applied if the junctions are organized in groups, each of the size P, and the parameters of all junctions in a group are identical: the critical current is $I_c(1+\xi_k)$ and the resistance is $r(1+\eta_k)$, where index $k = 1, \ldots, M$ counts the groups. The total number of junctions is $N = MP$. Thus we write the equations for the junctions in a form

$$\begin{gathered} \dot{\varphi}_{ki} = (1+\eta_k)[I - \epsilon\dot{Q} - (1+\xi_k)\sin\varphi_{ki}] \\ \ddot{Q} + \gamma\dot{Q} + \omega_0^2 Q = I - \frac{1}{N}\sum_{k=1}^{M}(1+\eta_k)(1+\xi_k)\sum_{i=1}^{P}\sin\varphi_{ki}. \end{gathered} \tag{2.61}$$

Next, we apply the Watanabe-Strogatz ansatz to each group of the identical junctions, and obtain a system

$$\begin{aligned} \ddot{Q} + \gamma\dot{Q} + \omega_0^2 Q &= I - \langle(1+\eta_k)(1+\xi_k)\mathrm{Im}(Z_k)\rangle, \\ \dot{Z}_k &= (1+\eta_k)\left(\mathrm{i}(I - \epsilon\dot{Q})Z_k + (1+\xi_k)\frac{1-Z_k^2}{2}\right), \end{aligned} \tag{2.62}$$

where average $\langle\rangle$ is taken over all groups. Next we take a thermodynamic limit of an infinite number of groups $M \to \infty$, then instead of M WS variables Z_k we obtain a continuous function $Z(\eta, \xi)$. Then (2.62) becomes an integro-differential equation with the distribution function $W(\eta, \xi)$ of the parameters ξ, η (cf. [7]):

$$\begin{gathered} \ddot{Q} + \gamma\dot{Q} + \omega_0^2 Q = I - \\ - \iint d\eta\, d\xi\; W(\eta,\xi)\,(1+\eta)(1+\xi)\mathrm{Im}(Z(\eta,\xi))\;, \\ \dot{Z}(\eta,\xi) = (1+\eta)\left(\mathrm{i}(I - \epsilon\dot{Q})Z + (1+\xi)\frac{1-Z^2}{2}\right). \end{gathered} \tag{2.63}$$

2.4.2. Asynchronous state and its stability

As in the case of identical junctions the asynchronous state is the steady state of the system (2.63):

$$\begin{aligned} Z_0(\eta,\xi) &= \mathrm{i}\frac{I - \sqrt{I^2 - (1+\xi)^2}}{1+\xi}, \\ Q_0 &= \omega_0^{-2}\iint d\eta\, d\xi\; W(\eta,\xi)\,(1+\eta)\sqrt{I^2 - (1+\xi)^2}\;, \end{aligned} \tag{2.64}$$

where we assume $\langle\xi\rangle = \langle\eta\rangle = 0$. Remarkably, the parameter η (responsible for nonidentity of the junction resistances) does not enter the expression for Z_0, only the parameter ξ (nonidentity of the junction critical currents) enters the expression of asynchronous state. But the stability of the asynchronous state depends on distributions of η and ξ. We consider two cases of possible sources of diversity separately.

(i) Disorder in resistances only. That means that $W(\eta, \xi) = \delta(\xi)W_\mu(\eta)$ where we assume that W_μ is a uniform distribution in the interval $(-\mu, \mu)$. In order to analyze the

stability of the asynchronous state we linearize the integral equation (2.63) around the steady solution (2.64), and discretize the integral using 500 nodes. Then we calculate the eigenvalues of the resulting matrix. Fig. 2.7a shows the results for the maximal eigenvalue. It is clear to see that, as the external current I increases, the asynchronous state loses its stability nearly at the same critical value as for identical junctions (expression (2.25)), but with further increase of I the asynchronous state becomes stable again. The region of instability decreases as the value of μ increases.

(ii) Disorder in critical currents only. Similarly to the previous case $W(\eta, \xi) = \delta(\eta)W_\zeta(\xi)$, where ζ is the width of the uniform distribution. Next, the same procedure was performed and the stability eigenvalues were found. They are shown in Fig. 2.7b. The same qualitative picture was obtained: both sources of nonidentity result in a bounded (in terms of the external current I) domain of instability of the asynchronous state.

The main result of the calculations presented in Fig. 2.7 show, that the main effect of disorder in arrays is the stability of the asynchronous state for large values of current I, and the instability appears only in some closed area (which decreases with increase of diversity). The appearing synchrony regimes in nonidentical arrays are presented in the next subsection.

2.4.3. Numerical simulations

The results of numerical study of the nonidentical arrays of Josephson junctions are shown in Figs. 2.8. As above, we consider two cases when one of the parameters, η for individual resistance or ξ for individual critical current, has a distribution. For numerical simulations we use the discrete representation (2.62) with additional very small viscous term $\sim (Z_{k+1} + Z_{k-1} - 2Z_k)$ (it was added in order to avoid spurious non-smooth solutions) in the equation for Z_k that gives numerical stabilization of the integro-differential equation.

As a characterization of synchrony, the average over the array order parameter $z = M^{-1}\sum_k Z_k$ was used. It is plotted vs. parameter I in Fig. 2.8. As explained above, this parameter attains the fixed point (cf. Eq. (2.64)) in the asynchronous state , while in the synchronous state it performs oscillations around some mean value (the synchrony is not complete, $|z| < 1$, due to diversity of the array). The transition to synchrony and back is hysteretic both for small and large values of I, as shown on panels (b),(c),(e), and (f) of Fig. 2.8.

In order to show how the synchronous and asynchronous states appear on the macroscopic and the microscopic level, we illustrate the dynamics of the load fields $Q, \dot{Q}$ (solid and dashed curves, respectively) and of the Josephson phases φ_i in Fig. 2.9. Panels (a,b) show complete synchrony of identical junctions, when all the phases coincide. Panels (c,d) show synchronous state of junctions with a distribution of their resistances η_k, when the phases are not identical, but form a bunch, rotating with the

same frequency. In panels (e,f) we show an asynchronous state for bigger difference of the resistances, when the load fields $Q, \dot{Q}$ perform no oscillation what means effective absence of the coupling, and the phases of the junctions have different frequencies and thus diverge.

2.5. Summary

In this chapter we have presented the Watanabe-Strogatz (WS) theory and its applications. By virtue of this method we obtain a closed low-dimensional system of equations for global variables that fully describes the original system. The application to arrays of Josephson junctions proves the presence of the hysteric transitions from asynchrony to synchrony. The application to identical star networks gives the possibility for the full analytical analysis for described set of initial conditions. Such networks also show hysteretic transitions to synchrony in some region of the parameter space. For the application to nonidentical systems, a certain type of inhomogeneity is required, in which the system consists of the groups, such that the units inside each group are identical but differ from the units from another group.

The setup when a network can be divided into groups of identical elements is used in many cases for the analysis of inhomogeneous networks. For example, such setup has been used in Ott-Antonsen theory [15] (see [7] for details) where they considered the Cauchy distribution of natural frequencies. However, in the Ott-Antonsen case the integral over the distribution can be performed beforehand with the help of Cauchy's residue theorem. Therefore a low-dimensional system similar to (2.63) is not integro-differential, but simply is a complex differential equation, where instead of a parameter distribution the single value obtained from the Cauchy's residue theorem is substituted.

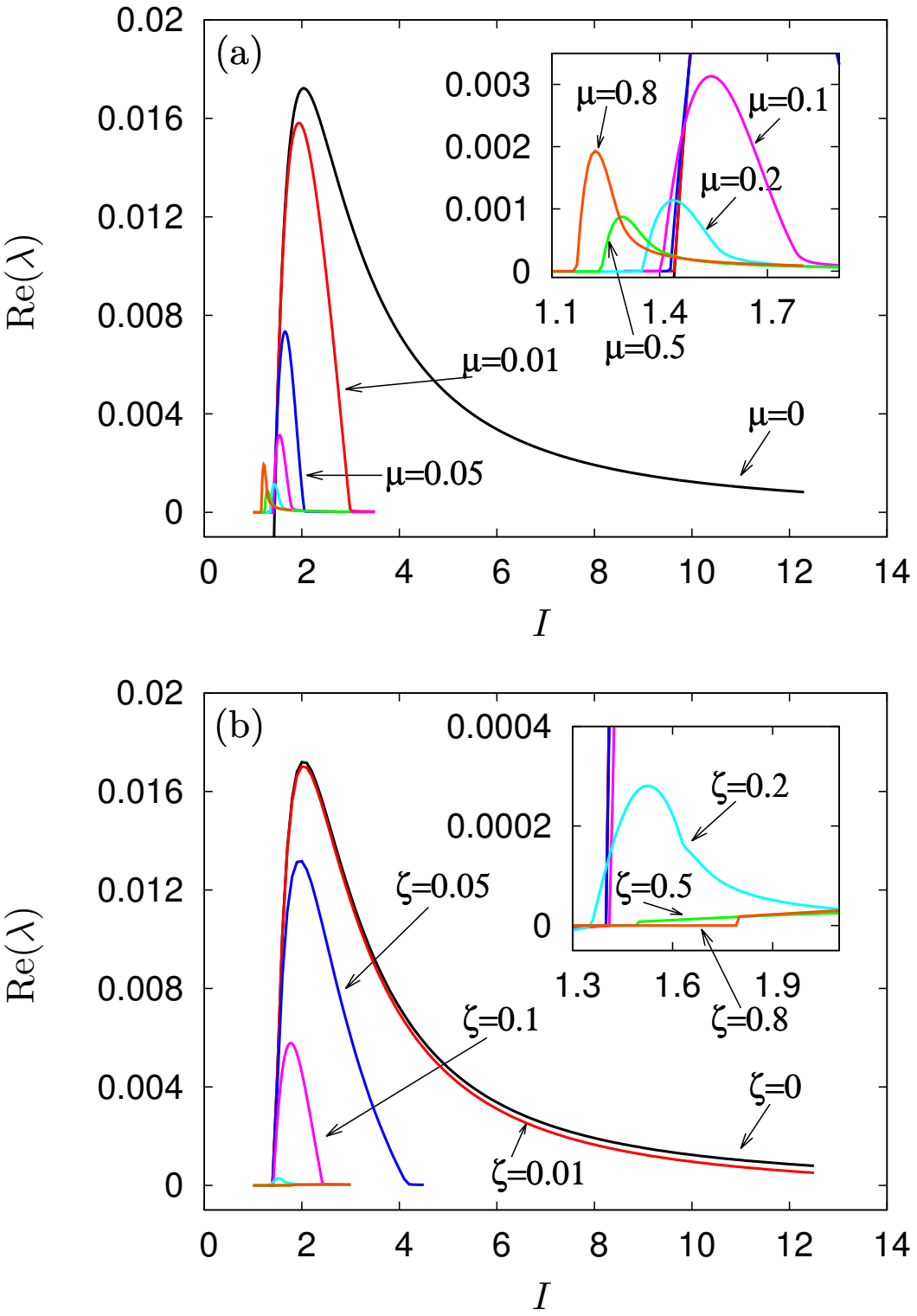


Figure 2.7.: (Color online) Real part of the maximum eigenvalue λ as a function of the dimensionless current I for the different values (shown on the panels) of μ (panel (a)) and ζ (panel (b)). From [9].

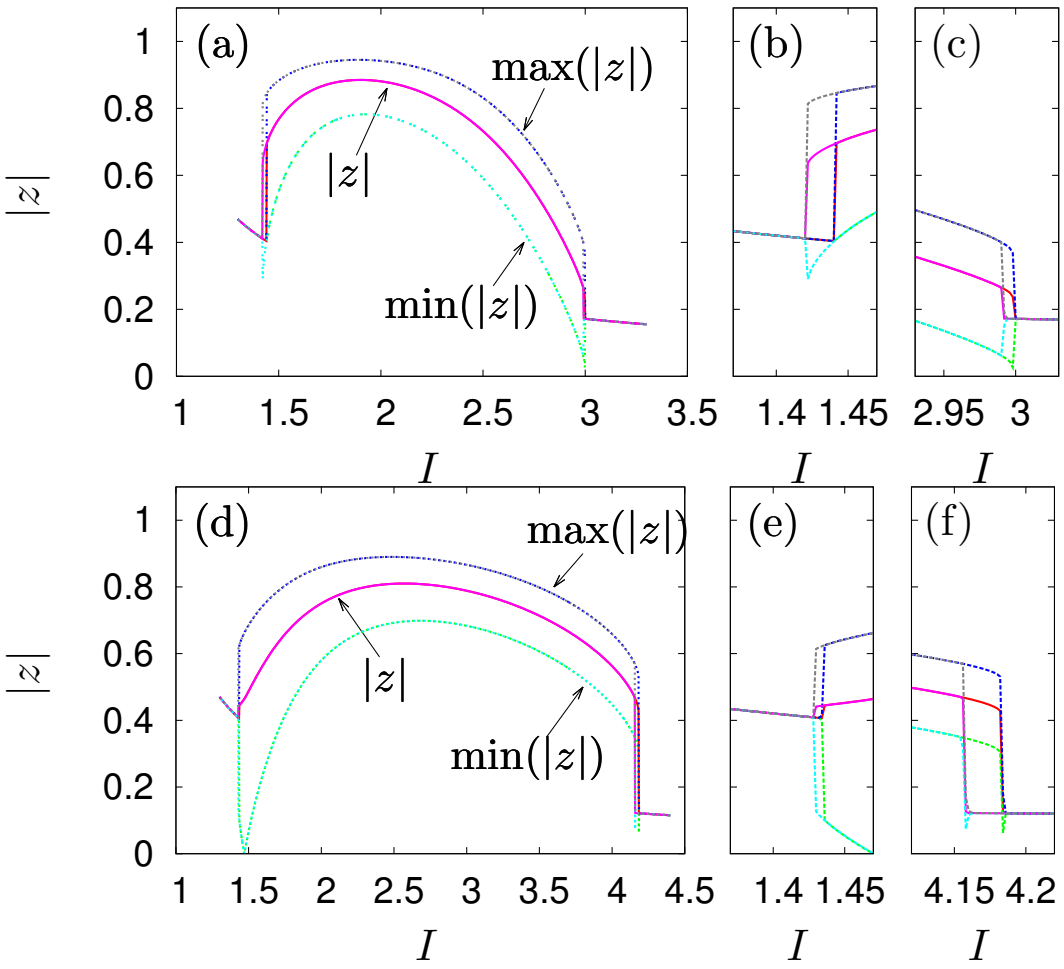


Figure 2.8.: (Color online) Panels (a),(d): Dependence of the averaged order parameter $|z|$ on current I, $\mu = 0.01$, $\zeta = 0$ and $\mu = 0$, $\zeta = 0.05$ respectively. Three lines show the maximal (upper dashed line), the average (solid line), and the minimal (lower dashed line) value of variations of $|z|$ in time, in the asynchronous states these lines coincide. Panels (c), (d), (e) and (f) show enlargements of the regions near the synchrony-asynchrony transitions, to demonstrate the hysteresis. From [9].

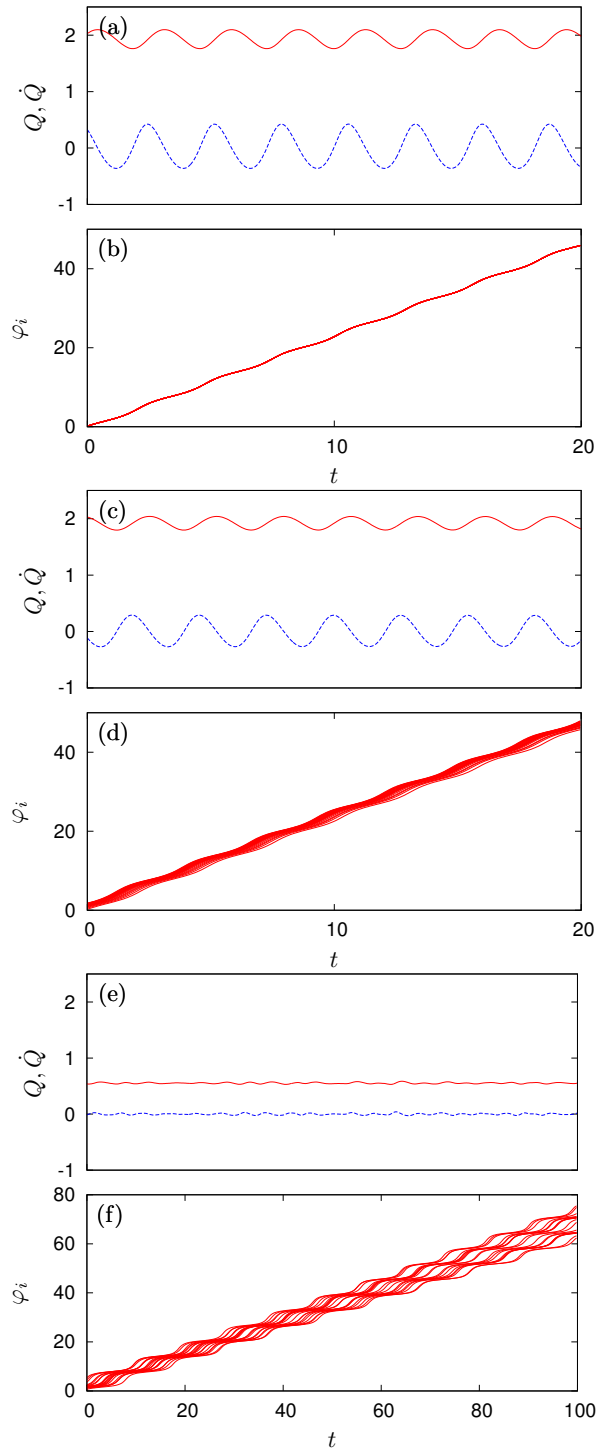


Figure 2.9.: (Color online) Results of simulations of an ensemble of 200 junctions for $\omega_0^2 = 1.2$, $\epsilon = 0.5$, $\zeta = 0$. In panels (b,d,f) we show only 20 phases out of 200, for better clarity. Panels (a,b): full synchrony for $\mu = 0$, $I = 2.5$. Panels (c,d): synchronous state in disordered array for $\mu = 0.01$, $I = 2.5$. Panels (e,f): asynchronous state for array with large disorder $\mu = 0.1$, $I = 1.2$. From [9].

3. Self-consistent approach without noise

3.1. Schematic description of the method

The type of oscillators diversity described above (the case when a system splits into groups of identical elements) is not general and therefore the WS approach is not suitable for all (or even large enough class of) inhomogeneous systems of coupled phase oscillators. Following the original Kuramoto approach, the solution for a general system can be obtained by finding a self-consistent solution of the global equation for the probability density functions in the thermodynamic limit.

Several remarks have to be added beforehand. First, a global coupling is necessary for this approach. Second, it will be not always possible to check the stability of the solutions or to find out the dynamics and general time-dependent solutions. Basically, we will just look for the solutions of special type, namely stationary and traveling wave solutions.

Let us consider general system of phase equations with global coupling and without noise

$$\dot{\varphi_k} = f(H(p,t), \varphi_k, x_k, p), \tag{3.1}$$

where x_k is a general vector of system's distributed parameters and p is a vector of common non-distributed (the same for all oscillators) parameters (they can enter both oscillator's dynamic equation and global field). A global field is represented by $H(p,t)$, where

$$H(p,t) = \frac{1}{N}\sum_{j=1}^{N} h(\varphi_j(t), x_j, p). \tag{3.2}$$

The function f has to be invariant to a rotation of all phases by an arbitrary angle α so that

$$f\left(\frac{1}{N}\sum_{j=1}^{N} h\big(\varphi_j, x_j, p\big), \varphi_k, x_k, p\right) = f\left(\frac{1}{N}\sum_{j=1}^{N} h\big(\varphi_j + \alpha, x_j, p\big), \varphi_k + \alpha, x_k, p\right). \tag{3.3}$$

Note that in general a system can have several global fields (see [17, 18] for an example of the case with two global fields). Here we will schematically explain the self-consistent method based on the example of one global field.

In the thermodynamic limit ($N \to \infty$) the index of φ and x can be dropped because they are considered to be continuous functions, where $\varphi(t)$ has time-dependent conditional probability density function $\rho(\varphi, t \,|\, x)$ and x is described through a joint distribution density $g(x)$. Time-dependent conditional probability density function $\rho(\varphi, t \,|\, x)$ should satisfy continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \varphi}\Big(f(H(p,t), \varphi, x, p)\rho\Big) = 0. \tag{3.4}$$

And the expression for the global field $H(p,t)$ reads

$$H(p,t) = \int g(x) \int_0^{2\pi} \rho(\varphi, t \,|\, x) h(\varphi, x, p) d\varphi \, dx. \tag{3.5}$$

In general it is rather difficult to find time-dependent solutions of (3.4). But of particular importance are synchronous solutions when the phases rotate uniformly. These solutions are traveling wave solutions rotating with constant common frequency Ω (note that a stationary solution is included in this type of solutions when $\Omega = 0$). First, we go to the rotating with Ω reference frame by introducing new variable $\Delta\varphi = \varphi - \Phi(t)$, where $\dot{\Phi} = \text{const} = \Omega$. Taking into account (3.3) the equation for $\Delta\varphi$ satisfies

$$\dot{\Delta\varphi} = f(H'(p,t), \Delta\varphi, x, p) - \Omega, \tag{3.6}$$

where an expression for the new global field $H'(p,t)$ reads

$$H'(p,t) = \int g(x) \int_0^{2\pi} \rho(\Delta\varphi, t \,|\, x) h(\Delta\varphi, x, p) d\Delta\varphi \, dx. \tag{3.7}$$

We are looking for solutions such that a distribution of the phases $\Delta\varphi$ is stationary, so $\dot{\rho}(\Delta\varphi, t \,|\, x) = 0$. Then the new global field (3.7) is also stationary $H'(p,t) = H'(p)$. The equation for the stationary density $\rho(\Delta\varphi, t \,|\, x) = \rho(\Delta\varphi \,|\, x)$ reads

$$\frac{\partial}{\partial \Delta\varphi}\left(\Big[f(H'(p), \Delta\varphi, x, p) - \Omega\Big]\rho\right) = 0. \tag{3.8}$$

The solution of (3.8) depends on particular values of the distributed parameters, here we will symbolically denote it as dependence on x. For those x, when there exists $\Delta\varphi_0(x)$ such that $f(H'(p), \Delta\varphi_0(x), x, p) - \Omega = 0$, the phases are locked and the solution of (3.8) is $\rho(\Delta\varphi \,|\, x) = \delta(\Delta\varphi - \Delta\varphi_0(x))$. For those x, when $f(H'(p), \Delta\varphi, x, p) - \Omega \neq 0$, the phases rotate keeping stationary distribution $\rho(\Delta\varphi \,|\, x) = C(x)|f(H'(p), \Delta\varphi, x, p) - \Omega|^{-1}$, where $C(x)$ is a normalization constant. Substituting these solutions into the equation for the global field $H'(t)$ (3.7) we obtain the self-consistent problem

$$\begin{aligned} H'(p) = Q(p)e^{\mathrm{i}\Delta\Theta(p)} = &\int_{\text{locked}} g(x) h(\Delta\varphi_0(x), x, p) dx + \\ &+ \int_{\text{rotating}} g(x) C(x) \int_0^{2\pi} |f(H'(p), \Delta\varphi, x, p) - \Omega|^{-1} h(\Delta\varphi, x, p) d\Delta\varphi \, dx. \end{aligned} \tag{3.9}$$

At this point it is convenient to treat Q, $\Delta\Theta$ and Ω not as unknown, but as auxiliary parameters and represent via them a set of non-distributed parameters $p = F(Q, \Delta\Theta, \Omega)$. By doing so we find the values of the non-distributed parameters p that gives the solutions with certain values of Q, $\Delta\Theta$ and Ω. Thus we are able to find traveling wave solutions for any given set of non-distributed parameters.

Above we have outlined the scheme of the self-consistent approach. A detailed method of applying this approach and consequent results strictly depend on a particular type of a system. We will analyze two examples: Kuramoto-type model with generic coupling (published in [19]) and ensembles of spatially distributed oscillators with a leader-type coupling (nonidentical star-type networks).

3.2. Kuramoto-type model with generic coupling

Let us consider a generic system of the Kuramoto-type phase oscillators $\theta_i(t)$ with individual frequencies ω_i, coupled through the mean field illustrated in Fig. 3.1. Every oscillator j contributes to the mean field $H(t)$ with individual coupling parameters: phase shift β_j and coupling constant B_j. On the other hand the mean field $H(t)$ acts on each oscillator i with different phase shift α_i and a coupling strength A_i.

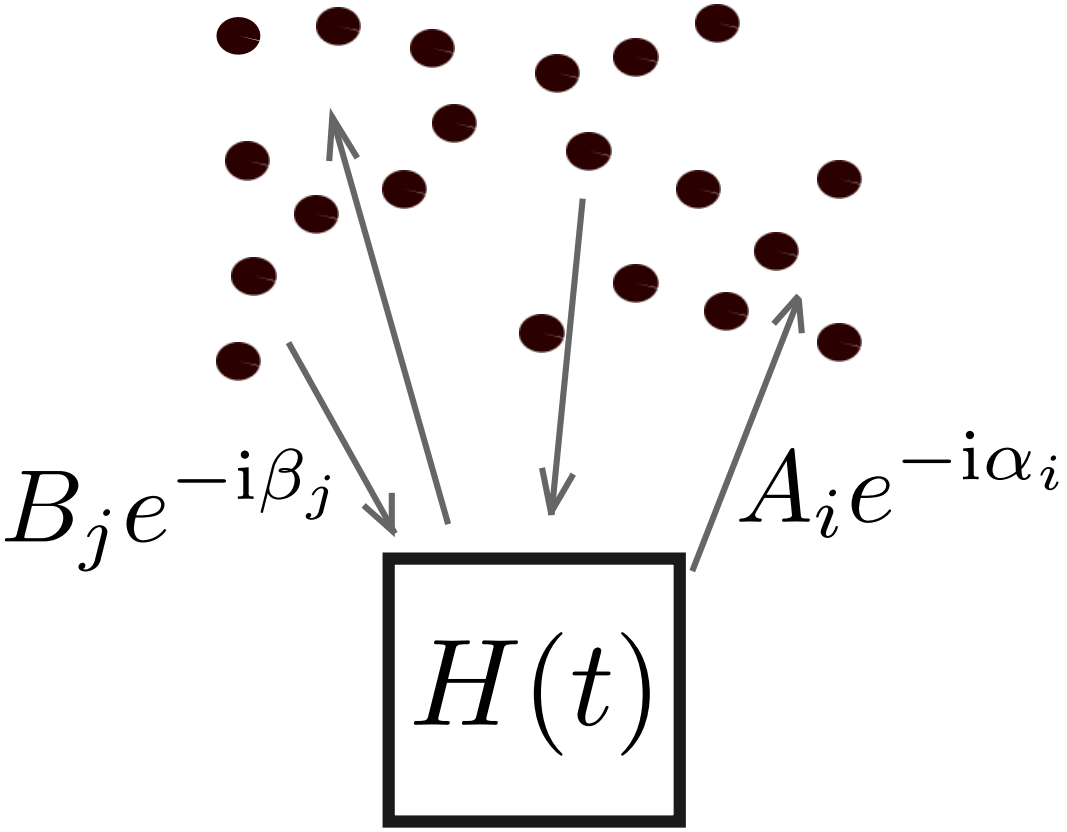


Figure 3.1.: Configuration of the network, coupled via the mean field $H(t)$. From [19].

The additional overall coupling strength ε is introduced for convenience (e.g, by normalizing one or both of the coupled coefficients A_i, B_j; also for definiteness we assume $A_i, B_j > 0$ because the sign of the coupling can be absorbed to the phase shifts β_j, α_i) and the overall phase shift δ as well (e.g., by normalizing the shifts β_j, α_i). In this formulation the equations of motions of the oscillators read

$$\dot{\theta}_i = \omega_i + A_i \frac{\varepsilon}{N} \sum_{j=1}^{N} B_j \sin(\theta_j - \beta_j - \theta_i + \alpha_i - \delta). \tag{3.10}$$

The system (3.10) can be rewritten in terms of the mean field $H(t)$:

$$\begin{aligned} \dot{\theta}_i &= \omega_i + A_i \operatorname{Im}\left(H(t) e^{-\mathrm{i}(\theta_i - \alpha_i)}\right), \\ H(t) &= \frac{\varepsilon e^{-\mathrm{i}\delta}}{N} \sum_{j=1}^{N} B_j e^{\mathrm{i}(\theta_j - \beta_j)}. \end{aligned} \tag{3.11}$$

A transformation of phases $\varphi_i = \theta_i - \alpha_i$ helps to reduce the number of parameters. Then the equations for φ_i are:

$$\begin{aligned} \dot{\varphi}_i &= \omega_i + A_i \operatorname{Im}\left(H(t) e^{-\mathrm{i}\varphi_i}\right), \\ H(t) &= \frac{\varepsilon e^{-\mathrm{i}\delta}}{N} \sum_{j=1}^{N} B_j e^{\mathrm{i}(\varphi_j - \psi_j)}, \end{aligned} \tag{3.12}$$

where $\psi_j = \beta_j - \alpha_j$.

This model combines together all the models of mean-field coupled Kuramoto-type phase oscillators. (i) The standard Kuramoto-Sakaguchi model [3] (all the parameters of the coupling $A_i, B_i, \beta_i, \alpha_i$ are constant). (ii) The case when there are only parameters A_i, α_i and ω_i and they have specific form has been considered previously in refs. [20, 21]. (iii) Also, the case with double delta distribution of A_i has been studied in ref. [22]. (iv) The case $\alpha_i = \beta_i = 0$ was considered in ref. [23]. In ref. [24] the system (3.10) was studied. Self-consistent approach is a natural way to obtain the solution for global variables. Below we formulate the self-consistent equation for this model and present its explicit solution.

We would like to mention that the complex mean field $H(t)$ is different from the "classical" Kuramoto order parameter $N^{-1} \sum_j e^{\mathrm{i}\varphi_j}$ and its absolute value can be larger than one, depending on the parameters of the system. This mean field acts as the forcing on the oscillators, and therefore it serves as a natural order parameter for this model.

3.2.1. Self-consistency condition and its solution

For the mean field $H(t)$ in the thermodynamic limit, a self-consistent equation can be formulated. In the thermodynamic limit the quantities ω, A, B and ψ have a joint

distribution density $g(x) = g(\omega, A, B, \psi)$, where x is a general vector of parameters. Below we will derive all the equation in a general form, but for the calculation we will consider two specific cases: (i) the quantities ω, A and B and ψ are independent, then g is a product of corresponding independent distribution densities; and (ii) the coupling parameters A, B, and ψ are determined by a geometrical position of an oscillator and thus depend on this position, parametrized by a scalar parameter x, while the frequency ω is distributed independently of x.

Introducing the conditional probability density function $\rho(\varphi, t \,|\, x)$, we can rewrite the system (4.11) as

$$\begin{aligned} \dot{\varphi} &= \omega + A \operatorname{Im}\left(H(t) e^{-\mathrm{i}\varphi}\right) = \omega + A\, Q \sin(\Theta - \varphi), \\ H(t) &= Q e^{\mathrm{i}\Theta} = \varepsilon e^{-\mathrm{i}\delta} \int g(x) B e^{-\mathrm{i}\psi} \int_0^{2\pi} \rho(\varphi, t \,|\, x) e^{\mathrm{i}\varphi} d\varphi \, dx. \end{aligned} \tag{3.13}$$

It is more convenient to write equations for $\Delta\varphi = \varphi - \Theta$, with the corresponding conditional probability density function $\rho(\Delta\varphi, t \,|\, x) = \rho(\varphi - \Theta, t \,|\, x)$:

$$\frac{d}{dt}\Delta\varphi = \omega - \dot{\Theta} - A\, Q \sin(\Delta\varphi), \tag{3.14}$$

$$Q = \varepsilon e^{-\mathrm{i}\delta} \int g(x) B e^{-\mathrm{i}\psi} \int_0^{2\pi} \rho(\Delta\varphi, t \,|\, x) e^{\mathrm{i}\Delta\varphi} d\Delta\varphi \, dx. \tag{3.15}$$

The continuity equation for the conditional probability density function $\rho(\Delta\varphi, t \,|\, x)$ follows from (3.14):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \Delta\varphi}\left(\left[\omega - \dot{\Theta} - A\, Q \sin(\Delta\varphi)\right] \rho\right) = 0. \tag{3.16}$$

A priori we cannot exclude complex regimes in Eq. (3.16), but the particular important regimes are the simplest synchronous states where the mean field $H(t)$ rotates uniformly (this corresponds to the classical Kuramoto solution). Therefore, we look for such solutions that the phase Θ of the mean field $H(t)$ rotates with a constant (yet unknown) frequency Ω. Correspondingly, the distribution of phase differences $\Delta\varphi$ is stationary in the rotating with Ω reference frame (such a solution is often called traveling wave):

$$\dot{\Theta} = \Omega, \quad \dot{\rho}(\Delta\varphi, t \,|\, x) = 0. \tag{3.17}$$

Thus, the equation for the stationary density $\rho(\Delta\varphi, t \,|\, x) = \rho(\Delta\varphi \,|\, x)$ reads:

$$\frac{\partial}{\partial \Delta\varphi}\left(\left[\omega - \Omega - A\, Q \sin(\Delta\varphi)\right] \rho\right) = 0. \tag{3.18}$$

The solution of Eq. (3.18) depends on the value of the parameter A. There are locked phases when $|A| > |\Omega - \omega|/Q$ so $\omega - \Omega - A\, Q \sin(\Delta\varphi) = 0$ and rotated phases when $|A| < |\Omega - \omega|/Q$ such that $\rho = C(A, \omega)|\omega - \Omega - A\, Q \sin(\Delta\varphi)|^{-1}$.

It is convenient to denote

$$F(\Omega,Q)=\int g(x)Be^{-\mathrm{i}\psi}\int_0^{2\pi}\rho(\Delta\varphi,t\,|\,x)e^{\mathrm{i}\Delta\varphi}d\Delta\varphi\,dx\;. \tag{3.19}$$

After the introduction of the solution of (3.18) to the function (3.19), the integral over parameter x splits into two integrals:

$$\begin{aligned}F(\Omega,Q)=&\int_{|A|>|\Omega-\omega|/Q}g(x)Be^{-\mathrm{i}\psi}\,e^{\mathrm{i}\Delta\varphi(A,\omega)}dx+\\&+\int_{|A|<|\Omega-\omega|/Q}g(x)Be^{-\mathrm{i}\psi}\,C(A,\omega)\int_0^{2\pi}\frac{e^{\mathrm{i}\Delta\varphi}\,d\Delta\varphi}{|\omega-\Omega-A\,Q\sin(\Delta\varphi)|}\,dx\,,\end{aligned} \tag{3.20}$$

where in the first integral

$$\sin(\Delta\varphi(A,\omega))=-\frac{\Omega-\omega}{A\,Q}\,,$$

and in the second one

$$C(A,\omega)=\left(\int_0^{2\pi}\frac{d\Delta\varphi}{|\omega-\Omega-A\,Q\sin(\Delta\varphi)|}\right)^{-1}\,.$$

Integrations over $\Delta\varphi$ in (3.20) can be performed explicitely:

$$\begin{aligned}C(A,\omega)=\left(\int_0^{2\pi}\frac{d\Delta\varphi}{|\omega-\Omega-A\,Q\sin(\Delta\varphi)|}\right)^{-1}&=\frac{\sqrt{(\Omega-\omega)^2-A^2Q^2}}{2\pi}\,,\\\int_0^{2\pi}\frac{e^{\mathrm{i}\Delta\varphi}\,d\Delta\varphi}{|\omega-\Omega-A\,Q\sin(\Delta\varphi)|}=\frac{2\pi\mathrm{i}}{AQ}&\left(\frac{\Omega-\omega}{|\Omega-\omega|}-\frac{\Omega-\omega}{\sqrt{(\Omega-\omega)^2-A^2Q^2}}\right)\,.\end{aligned} \tag{3.21}$$

After substitution (3.21) into (3.20), we obtain the final general expression for the main function $F(\Omega,Q)$:

$$\begin{aligned}F(\Omega,Q)=&\int_{|A|>|\Omega-\omega|/Q}g(x)Be^{-\mathrm{i}\psi}\sqrt{1-\frac{(\Omega-\omega)^2}{A^2Q^2}}\;dx-\\&-\mathrm{i}\int g(x)Be^{-\mathrm{i}\psi}\,\frac{\Omega-\omega}{A\,Q}\,dx+\\&+\mathrm{i}\int_{|A|<|\Omega-\omega|/Q}g(x)Be^{-\mathrm{i}\psi}\,\frac{\Omega-\omega}{|\Omega-\omega|}\sqrt{\frac{(\Omega-\omega)^2}{A^2Q^2}-1}\;\;dx\,.\end{aligned} \tag{3.22}$$

Then in new notations the self-consistency condition (3.15) reads

$$Q=\varepsilon e^{-\mathrm{i}\delta}F(\Omega,Q)\;. \tag{3.23}$$

In order to find Q and Ω, it is convenient to consider now Q, Ω not as unknowns but as parameters, and to write explicit equations for the coupling strength constants ε, δ via these parameters:

$$\varepsilon = \frac{Q}{|F(\Omega, Q)|}\,, \quad \delta = \arg(F(\Omega, Q))\,. \tag{3.24}$$

Thus, this solution of the self-consistency problem reduces to finding the solutions of the stationary Liouville equation (3.18) and its integration (3.19) in the parametric form. So, it is quite convenient for a numerical implementation.

3.2.2. Independent parameters

Let us consider the case of independent distributions of the parameters ω, A and B, ψ what means that $g(x) = g_1(\omega, A)\, g_2(B, \psi)$. Since the parameters B and ψ do not enter explicitly the integrals in (3.22), for the case of independent distributions it is convenient to consider ε and δ as scaling parameters of the distribution $\tilde{g}_2(\tilde{B}, \tilde{\psi})$, such that

$$\varepsilon e^{-\mathrm{i}\delta} = \int\int \tilde{g}_2(\tilde{B}, \tilde{\psi}) \tilde{B} e^{-\mathrm{i}\tilde{\psi}} d\tilde{B} d\tilde{\psi}, \tag{3.25}$$

so the parameters $B = \tilde{B}/\varepsilon$ and $\psi = \tilde{\psi} - \delta$ have such a distribution $g_2(B, \psi) = \varepsilon \tilde{g}_2(\tilde{B}, \tilde{\psi})$ that satisfies

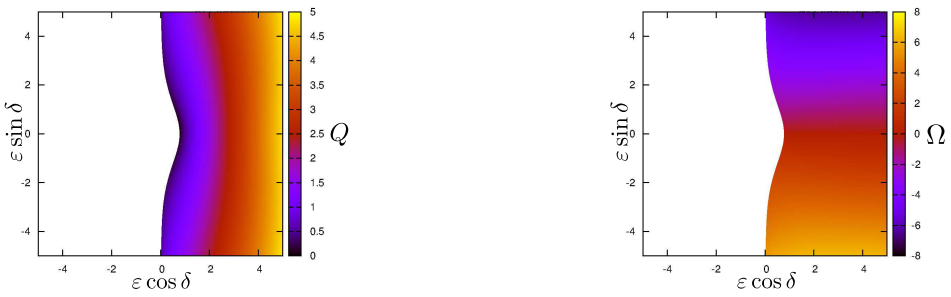
$$\int\int g_2(B, \psi) B e^{-\mathrm{i}\psi} dB d\psi = 1. \tag{3.26}$$

Eq. (3.26) provides that the integration in (3.22) over B and ψ gives 1, and the following expression is obtained :

$$\begin{aligned} F(\Omega, Q) = &\int\int_{|A|>|\Omega-\omega|/Q} g_1(\omega, A) \sqrt{1 - \frac{(\Omega-\omega)^2}{A^2 Q^2}}\; dA d\omega - \\ &- \mathrm{i} \int\int g_1(\omega, A) \frac{\Omega - \omega}{A\, Q}\, dA d\omega + \\ &+ \mathrm{i} \int\int_{|A|<|\Omega-\omega|/Q} g_1(\omega, A)\, \frac{\Omega - \omega}{|\Omega - \omega|} \sqrt{\frac{(\Omega-\omega)^2}{A^2 Q^2} - 1}\; dA d\omega\,. \end{aligned} \tag{3.27}$$

As before the parameters ε and δ can be found from Eqs. (3.24) depending on Ω and Q. Please note that, the distribution of parameters B and ψ is implicitly included in the values of ε and δ, while the distributions of ω, A are explicitly included in the integrals.

As an example of application of our theory, in Fig. 3.2 we present results of the calculation of absolute value Q and the frequency of the global field Ω as function of ε, δ , for $g_1(\omega, A) = g(A)g(\omega)$ where $g(A) = \frac{A}{\theta^2} e^{-A/\theta}$, $g(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2}$.



(a) (b)

Figure 3.2.: (color online) Dependencies of the amplitude Q of the mean field (a) and of its frequency Ω on the parameters ε and δ, for $\theta = 1$. White area corresponds to asynchronous state with vanishing mean field. From [19].

Interestingly, the case of identical individual frequencies of the oscillators, i.e. when $g(\omega) = \delta(\omega - \omega_0)$, gives opportunity to parametrize Eq. (3.27) by one variable. The integration over $d\omega$ can be performed first:

$$F(\Omega, Q) = \int_{|A|>|\Omega-\omega_0|/Q} g(A)\sqrt{1-\frac{(\Omega-\omega_0)^2}{A^2Q^2}}\,dA - \\ -\mathrm{i}\int g(A)\frac{\Omega-\omega_0}{AQ}\,dA + \mathrm{i}\int_{|A|<|\Omega-\omega_0|/Q} g(A)\,\frac{\Omega-\omega_0}{|\Omega-\omega_0|}\sqrt{\frac{(\Omega-\omega_0)^2}{A^2Q^2}-1}\;dA\,. \tag{3.28}$$

Then it is convenient to combine the variables Ω and Q in a new variable $Y = \frac{\Omega-\omega_0}{Q}$ and treat the function $F(\Omega, Q)$ in Eq. (3.28) as a function of Y. Then Eq. (3.28) for $F(\Omega, Q)$ transforms to the following equation for $F(Y)$

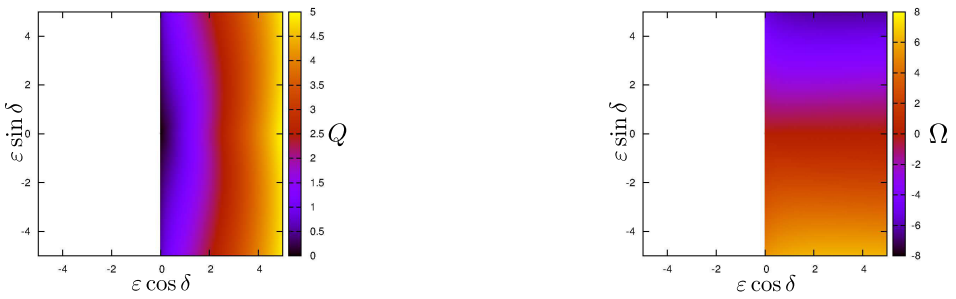
$$F(Y) = \int_{|A|>|Y|} g(A)\sqrt{1-\frac{Y^2}{A^2}}\,dA - \\ -\mathrm{i}\int g(A)\frac{Y}{A}\,dA + \mathrm{i}\int_{|A|<|Y|} g(A)\,\frac{Y}{|Y|}\sqrt{\frac{Y^2}{A^2}-1}\;dA\,, \tag{3.29}$$

where we took into account that $Q \geq 0$.

Eqs (3.24) are still valid for finding ε and δ, but it is more convenient to use Y and ε as a parameters in Eq. (3.23) instead of Q and Ω. Then the final expressions for finding Q, Ω and δ take the following form:

$$Q = \varepsilon\,|F(Y)|\,, \quad \Omega = \omega_0 + \varepsilon Y\,|F(Y)|\,, \quad \delta = \arg(F(Y))\,. \tag{3.30}$$

Fig. 3.3 shows the results of the calculation of $Q(\varepsilon, \delta)$ and $\Omega(\varepsilon, \delta)$ for the identical natural frequencies, where we chose $g_1(\omega, A) = \frac{A}{\theta^2}e^{-A/\theta}\delta(\omega - \omega_0)$.



(a) (b)

Figure 3.3.: (Color online) Dependencies of the amplitude Q of the mean field (a) and of its frequency Ω on the parameters ε and δ, for $\theta = 1$ and $\omega_0 = 0$. White area corresponds to asynchronous state with vanishing mean field. From [19].

3.2.3. Example of a geometric organization of oscillators

Here we present an example of application of the expressions above to the case when the distributions of parameters are determined by geometrical configuration of oscillators. We consider spatially distributed phase oscillators with a common receiver that collects signals from all the oscillators, and a common emitter that receives the summarized signal from the receiver and sends the coupling signal to the oscillators; also we assume that these signals propagate with velocity c. We consider the identical oscillators with the same natural frequencies $\omega_0 = 1$ (cases where the frequencies are distributed (dependent or independent of geometric positions of oscillators) can be straightforwardly treated within the same framework).

We assume that oscillators are distributed uniformly on a circle of radius r. Each oscillator is thus labeled by the angle x_i (Fig. 3.4). The receiver, the emitter, and the center of the circle are supposed to lie on one line. The dynamics of the system is described by Eq. (3.10), where coupling parameters are determined by the place of the oscillator. The phase shifts β_j and α_i are proportional to the distances between the oscillator, the receiver and the emitter, so that

$$\beta_j = \frac{\omega_s}{c}\sqrt{r^2 + b^2 - 2rb\cos x_j}\,, \quad \alpha_i = \frac{\omega_s}{c}\sqrt{r^2 + a^2 - 2ra\cos x_i}\,, \tag{3.31}$$

where ω_s is the frequency of the original signal (since usually phase approximation is made around some common frequency). Coupling strengths B_j and A_i are inversely proportional to the square distances between each oscillator, receiver and emitter:

$$B_j = \frac{1}{r^2 + b^2 - 2rb\cos x_j}, \quad A_i = \frac{1}{r^2 + a^2 - 2ra\cos x_i}\,, \tag{3.32}$$

where a and b are the distances from the center of the circle to the emitter and the receiver respectively (Fig. 3.4). The parameters ε and δ can be interpreted as a coupling coefficient and a phase shift for the signal transfer from the receiver to the emitter.

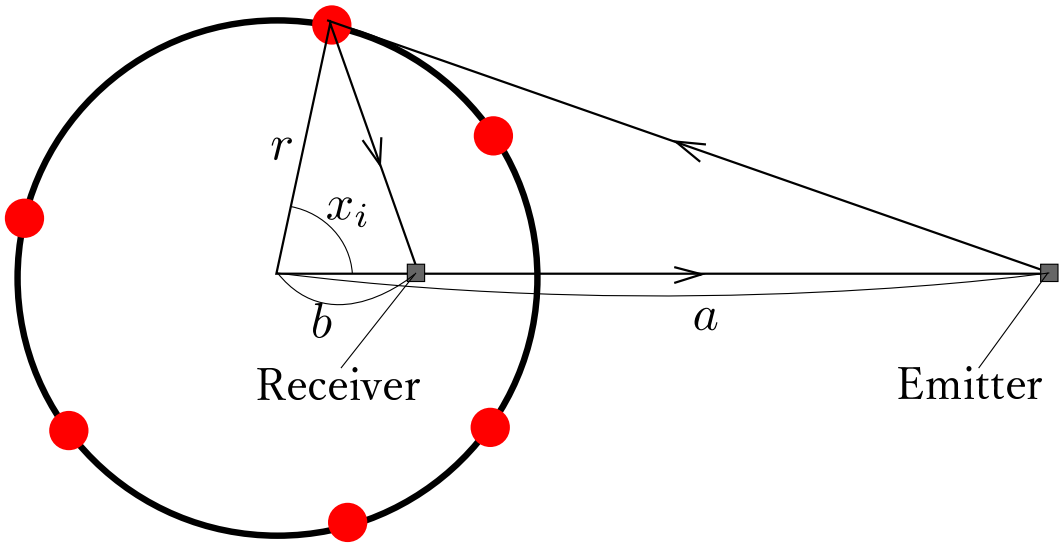


Figure 3.4.: (Color online) The scheme of the system. From [19].

The self-consistent approach gives stable solutions for any given parameters a and b. Since all the oscillators have the same natural frequencies, the variable transformation $Y = (\Omega - \omega_0)/Q$ described above should be performed. Thus, it is suitable to use Eqs. (3.30) in order to find Q, Ω and δ as a functions of ε and Y.

In the numerical example presented in Fig. 3.5, we fixed $b = r/2$ and varied a, finding the absolute value of the global field $Q(a)$ and the frequency of the collective oscillations $\Omega(a)$ for $\varepsilon = 1$ and $\delta = 0$. There are regimes of synchronous motion separated by asynchronous motion. The systems with time delay in the coupling show similar behavior – in our case this delay is due to the separation of the emitter from the community of oscillators, and the finite speed of signal propagation introduced into the phase shifts. The dependencies on the parameter a are not smooth, because as the parameter a varies, some oscillators enter/leave the synchronization domain.

3.3. An ensemble of spatially distributed oscillators with a leader-type coupling

In this section we will present a generalization of the system with leader-type coupling described in previous chapter. The generalization consists of introduction of unequal coupling coefficients and phase shifts. In our setup in the most general case each phase oscillator φ_k has its own frequency ω_k and is coupled to the leader oscillator ϕ with its own coupling strength A_k and phase shift α_k. At the same time the leader ϕ has its own frequency ω_0 and is coupled to every other oscillators φ_j with coupling coefficient

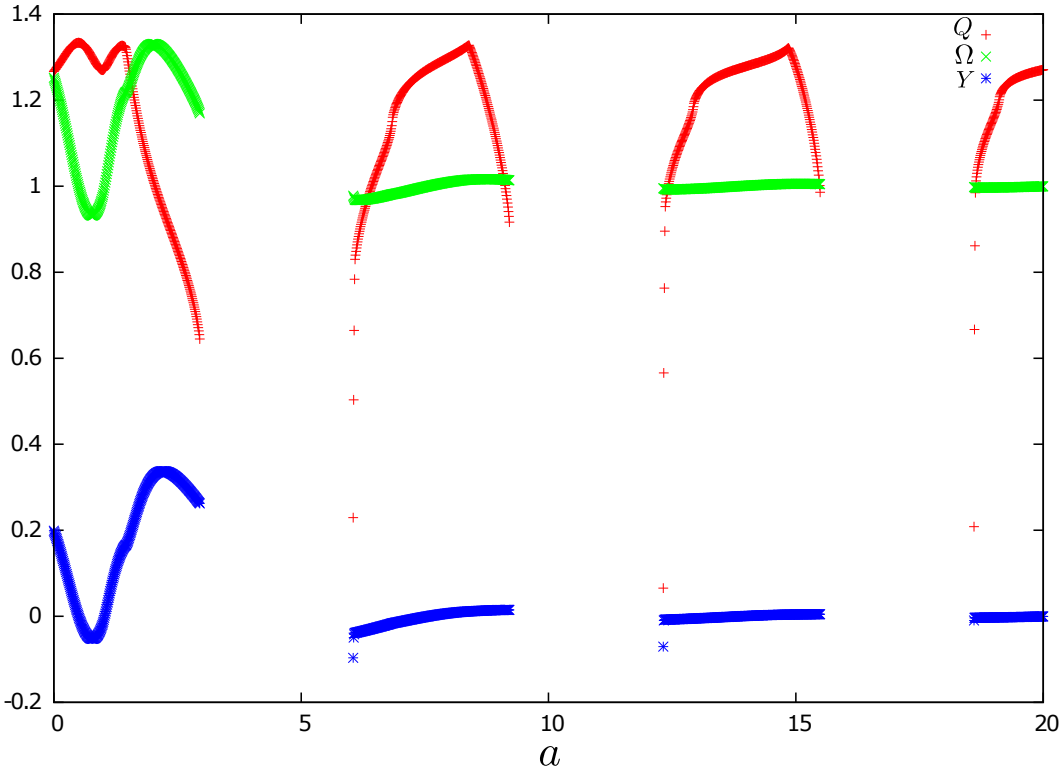


Figure 3.5.: (Color online) The dependence of $Q(a)$, $\Omega(a)$ and $Y(a)$ on the distance from the center of the circle to the emitter a, $b = 0.5$, $r = 1$, $\varepsilon = 1$ and $\delta = 0$. Periodicity in a corresponds to the zones of attractive and repulsive coupling due to delay-induced phase shift. From [19].

B_j and phase shift β_j.

$$\begin{aligned} \dot{\varphi_k} &= \omega_k + A_k \sin(\phi - \varphi_k - \alpha_k), \\ \dot{\phi} &= \omega_0 + \frac{1}{N}\sum_{j=1}^{N} B_j \sin(\varphi_j - \beta_j - \phi). \end{aligned} \tag{3.33}$$

Then we perform the similar procedure as to the homogeneous system. The system (3.33) can be rewritten in terms of the mean-field

$$\begin{aligned} \dot{\varphi_k} &= \omega_k + \mathrm{Im}(A_k e^{\mathrm{i}(\phi - \varphi_k - \alpha_k)}), \\ \dot{\phi} &= \omega_0 + \mathrm{Im}(\tilde{G}(t) e^{-\mathrm{i}\phi}), \\ \tilde{G}(t) &= \frac{1}{N}\sum_{j=1}^{N} B_j e^{\mathrm{i}(\varphi_j - \beta_j)}. \end{aligned} \tag{3.34}$$

It is convenient to make a variable transformation to the phase differences $\Delta\varphi_k$ between the oscillators φ_k and the leader ϕ taking into account the phase shift α_k

$$\Delta\varphi_k = \varphi_k - \phi + \alpha_k. \tag{3.35}$$

Then, the equations for $\Delta\varphi_k$ and ϕ are

$$\begin{aligned} \dot{\Delta\varphi_k} &= -\dot{\phi} + \omega_k + \mathrm{Im}(A_k e^{-\mathrm{i}\Delta\varphi_k}), \\ \dot{\phi} &= \omega_0 + \mathrm{Im}(G(t)), \\ G(t) &= \frac{1}{N}\sum_{j=1}^{N} B_j e^{\mathrm{i}(\Delta\varphi_j - \alpha_j - \beta_j)}. \end{aligned} \tag{3.36}$$

The expression for the leader dynamics can be directly introduced to the equations for $\Delta\varphi_k$ and thus we obtain the closed system

$$\begin{aligned} \dot{\Delta\varphi_k} &= \omega_k - \omega_0 - \mathrm{Im}(G(t)) + \mathrm{Im}(A_k e^{-\mathrm{i}\Delta\varphi_k}), \\ G(t) &= \frac{1}{N}\sum_{j=1}^{N} B_j e^{\mathrm{i}(\Delta\varphi_j - \alpha_j - \beta_j)}. \end{aligned} \tag{3.37}$$

Note that the dynamics of the leader

$$\dot{\phi} = \omega_0 + \mathrm{Im}(G(t)) \tag{3.38}$$

does not enter to the equations for the phase difference.

3.3.1. Self-consistent approach

We present the solution of (3.37) in the thermodynamic limit $N \to \infty$, where in this case the parameters ω, A, B, α and β have a joint distribution density $g(x) = g(\omega, A, B, \alpha, \beta)$, where x is a general vector of parameters. Introducing the conditional probability density function $\rho(\Delta\varphi, t \,|\, x)$, we can rewrite the system (3.37) as

$$\begin{aligned} \dot{\Delta\varphi} &= \omega - \omega_0 - Q\sin\Delta\Theta - A\sin\Delta\varphi, \\ G(t) = Qe^{\mathrm{i}\Delta\Theta} &= \int g(x) B e^{-\mathrm{i}(\alpha+\beta)} \int_0^{2\pi} \rho(\Delta\varphi, t \,|\, x) e^{\mathrm{i}\Delta\varphi} d\Delta\varphi dx, \end{aligned} \tag{3.39}$$

where $\rho(\Delta\varphi, t \,|\, x)$ should be calculated from Liouville equation

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial\Delta\varphi}\left(\left[\omega - \omega_0 - Q\sin\Delta\Theta - A\sin(\Delta\varphi)\right]\rho\right) = 0. \tag{3.40}$$

Then, we look for stationary solution for the phase difference $\Delta\varphi$

$$\dot{\rho}(\Delta\varphi, t \,|\, x) = 0. \tag{3.41}$$

Since we look for the stationary solution it is convenient to denote the frequency of the leader as Ω

$$\Omega = \dot{\phi} = \omega_0 + Q\sin\Delta\Theta, \tag{3.42}$$

and treat the unknowns Q, $\Delta\Theta$ and the parameter ω_0 as functions of Ω.

Thus, we obtain the following solution for the stationary Liouville equation (3.40)

$$\begin{aligned} \sin(\Delta\varphi(A,\omega)) &= \frac{\omega-\Omega}{A}, \quad A \geq |\omega-\Omega|, \\ \rho &= \frac{C(A,\omega)}{|\omega-\Omega-A\sin(\Delta\varphi)|}, \quad A < |\omega-\Omega|. \end{aligned} \tag{3.43}$$

The first equation in (3.43) has two solutions, we take the microscopically stable one

$$e^{\mathrm{i}\Delta\varphi(A,\omega)} = \sqrt{1-\left(\frac{\omega-\Omega}{A}\right)^2} + \mathrm{i}\frac{\omega-\Omega}{A}, \tag{3.44}$$

Also we need to calculate the following integral

$$\begin{aligned} C(A,\omega) = \left(\int_0^{2\pi} \frac{d\Delta\varphi}{|\omega-\Omega-A\sin(\Delta\varphi)|}\right)^{-1} &= \frac{\sqrt{(\Omega-\omega)^2-A^2}}{2\pi}, \\ \int_0^{2\pi} \frac{e^{\mathrm{i}\Delta\varphi}\,d\Delta\varphi}{|\omega-\Omega-A\sin(\Delta\varphi)|} &= \frac{2\pi\mathrm{i}}{A}\left(\frac{\Omega-\omega}{|\Omega-\omega|} - \frac{\Omega-\omega}{\sqrt{(\Omega-\omega)^2-A^2}}\right). \end{aligned} \tag{3.45}$$

Since in the integrals there is no dependence on Q, it is better to denote

$$Qe^{\mathrm{i}\Delta\Theta} = F(\Omega), \tag{3.46}$$

where

$$\begin{aligned} F(\Omega) = &\int_{|A|\geq|\Omega-\omega|} g(x)Be^{-\mathrm{i}(\beta+\alpha)}\sqrt{1-\frac{(\Omega-\omega)^2}{A^2}}\,dx - \\ &-\mathrm{i}\int g(x)Be^{-\mathrm{i}(\beta+\alpha)}\,\frac{\Omega-\omega}{A}\,dx + \\ &+\mathrm{i}\int_{|A|<|\Omega-\omega|} g(x)Be^{-\mathrm{i}(\beta+\alpha)}\,\frac{\Omega-\omega}{|\Omega-\omega|}\sqrt{\frac{(\Omega-\omega)^2}{A^2}-1}\;dx\,. \end{aligned} \tag{3.47}$$

Thus instead of Eq. (3.46) and (3.42) we have

$$Q = |F(\Omega)|, \qquad \Delta\Theta = \arg(F(\Omega)), \qquad \omega_0 = \Omega - \mathrm{Im}(F(\Omega)). \tag{3.48}$$

Contradictionary to the previous case of the Kuramoto-type model with generic coupling, the solution here is parameterized only by the frequency of the leader Ω and thereby we have only one non-distributed parameter of the original system that is found implicitly, namely the natural frequency of the leader ω_0. So hereinafter in this section we will represent the solutions in the form of the dependence of Q and Ω on the ω_0. Also the phase $\Delta\Theta$ is not indicative, so we will not find it in the examples below.

In this model, the amplitude of the global field Q that determines the forcing acting on the oscillators is not normalized and can be larger than unity. Besides it does not vanish for asynchronous regime. Thus it is not convenient to use it as an order parameter. As an order parameter it is much more convenient to use the relative number of locked oscillators, or in the thermodynamic limit the parameter R (3.49).

$$R = \int_{|A| \geq |\Omega - \omega|} g(x) dx. \tag{3.49}$$

3.3.2. Drums with a leader

Here, as an example of the application of this method, we will consider the system (3.37) as a model for the drum orchestra or any other 2D organized ensemble of oscillators. We assume that the drum orchestra is a manifold of oscillators equally distributed on a unit square located at the origin (Fig. 3.6).

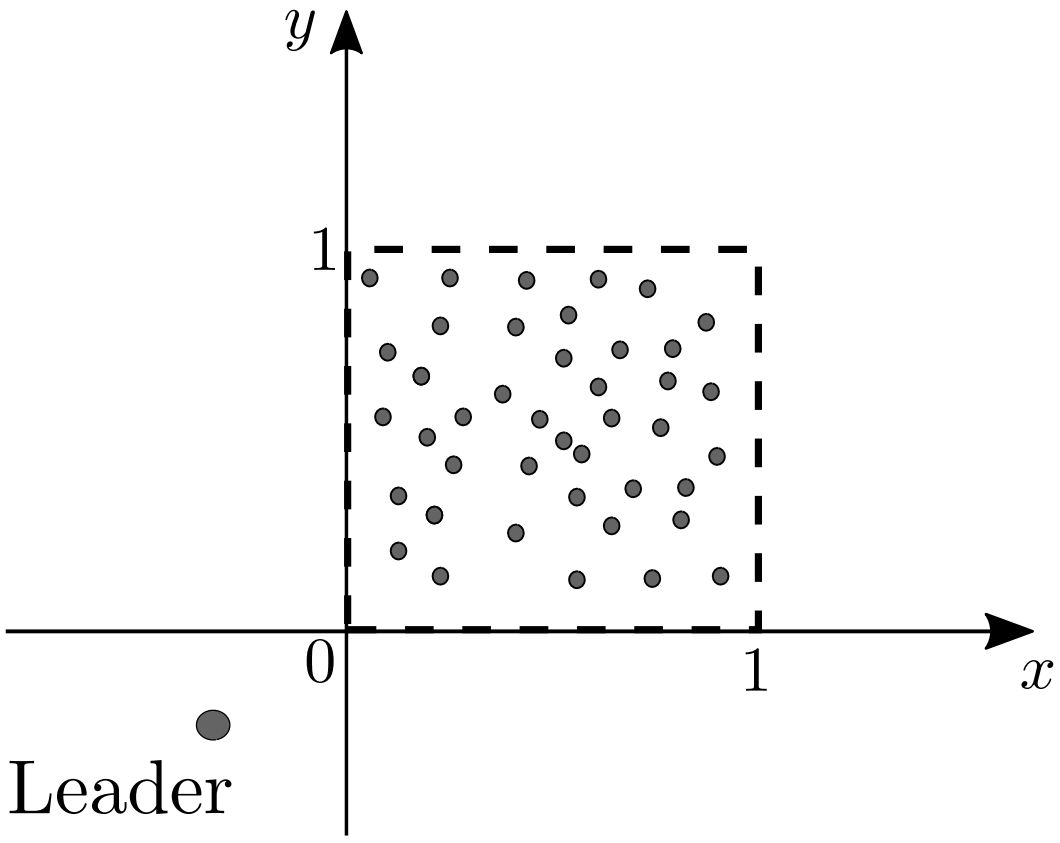


Figure 3.6.: The scheme of the organization of the drum orchestra.

As in the example of geometric organization of oscillators, we assume that the phase shifts β_j and α_i are proportional to the distances between the oscillator and the leader,

so

$$\beta_j = \frac{\omega_s}{c}\sqrt{(x_j - x_l)^2 + (y_j - y_l)^2}\,, \quad \alpha_i = \frac{\omega_s}{c}\sqrt{(x_i - x_l)^2 + (y_i - y_l)^2}\,, \tag{3.50}$$

where as before ω_s is the frequency of the original signal (around what the phase approximation was made). Coupling strengths B_j and A_i are inversely proportional to the square distances between each oscillator and the leader:

$$B_j = \frac{W_B}{(x_j - x_l)^2 + (y_j - y_l)^2}, \quad A_i = \frac{W_A}{(x_i - x_l)^2 + (y_i - y_l)^2}\,, \tag{3.51}$$

here additional initial intensities of the signals W_A and W_B were added in order to have coupling coefficients of the order 1 for any distant position of the leader.

Then in the thermodynamic limit the parameter distribution $g(A, B, \alpha, \beta) = g(x, y)$, where all the parameters are the functions (3.50,3.51) of the coordinates (x, y) of the 2D plane, except for, perhaps, ω that can be independently distributed. In our numerical simulations all the oscillators have identical frequencies. Self-consistent approach gives solutions for any given position of the leader outside the manifold of the oscillators and its own individual frequency. As a measure of synchrony, we will use previously introduced order parameter R (3.49) (if R is close to unity the synchronous regime is observed and if R is small we call this regime asynchronous). The terms "synchronous" and "asynchronous" are used here in order to show the resemblance between the solutions of homogenous and non-homogenous systems. For the latter, however, the usage of these terms is not entirely correct as can be seen on Fig. (3.7), where it is impossible to distinguish between the partial synchrony and asynchrony because there is no abrupt transitions and, except for a small region when all the phases are locked ($R = 1$), there is a fraction of locked phases and rotating phases with stationary distribution, that can be named both as partial synchrony and asynchrony in this case.

We present self-consistent solutions together with numerical simulations for different values of the parameters in order to show different possible regimes. While we cannot exclude all the complicated regimes, next we will present the solutions that qualitatively coincide with the solutions (2.48, 2.50, 2.52) of the homogenous system.

First, we show the regime that represents the case when there are two stable fixed points (one asynchronous and one synchronous) with hysteretic transition between them. The dependences of the amplitude Q and the frequency Ω of the global field on the frequency mismatch $\Delta\omega = \omega - \omega_0$ for this case together with the order parameter R are expressed on the Fig. (3.8) and Fig. (3.9) respectively. On these figures we show both the results obtained by the self-consistent method and direct numerical simulations that converge (slight differences are due to the finite-size effects and the fact that we stop calculations at finite time) everywhere except for the area of the hysteresis that can be observed in the neighborhood of the maximum amplitude of the global field and when the values of the order parameter R is near to unity. For the large absolute values of the frequency mismatch $|\Delta\omega|$ the order parameter R is small, what means that the number of locked phases is low and tends to zero with further increasing of $|\Delta\omega|$.

The second regime represents the case when there are one stable synchronous fixed point and one unstable asynchronous fixed point. The results of numerical simulations and the self-consistent method for this case are expressed on the Fig. (3.10). For the negative and small positive values of $\Delta\omega$ there is a steady solution. Asynchronous for large negative $\Delta\omega$, it gradually becomes partly synchronous for small negative $\Delta\omega$ transforming to synchronous solution ($R = 1$) for small positive $\Delta\omega$. As can be seen with the help of numerical simulations, with further increasing of $\Delta\omega$ steady solution becomes unstable and we observe the oscillating regime (on Fig. (3.11) it is shown that this oscillating regime is a limit cycle).

3.4. Summary

In this chapter we have described the self-consistent approach and its application to noise-free systems. We applied this approach to the systems with distributed parameters and obtain a unified description of the frequency and the amplitude of the global field in a parametric form. The application to the Kuramoto-type model with generic coupling gives the possibility to find regions of parameters when the behavior of the system is synchronous and when asynchronous. As one of the examples we considered a situation, where contributions to the global field and its action on oscillators are prescribed by a geometric configuration of the oscillators; phase shifts and the contribution factors result from the propagation of the signals as waves having certain velocity. The application to ensembles with leader-type coupling leads to analogous results, but due to the fact that in case of asynchrony the global field does not vanish, an additional order parameter, namely the amount of locked phases, have to be introduced. As an example we considered a simplified model of the drum orchestra with a leader, where we used the same assumption of finite propagation velocity of the signals. Finally we compared obtained solutions with the results from the analysis of the analogous identical system from the previous chapter.

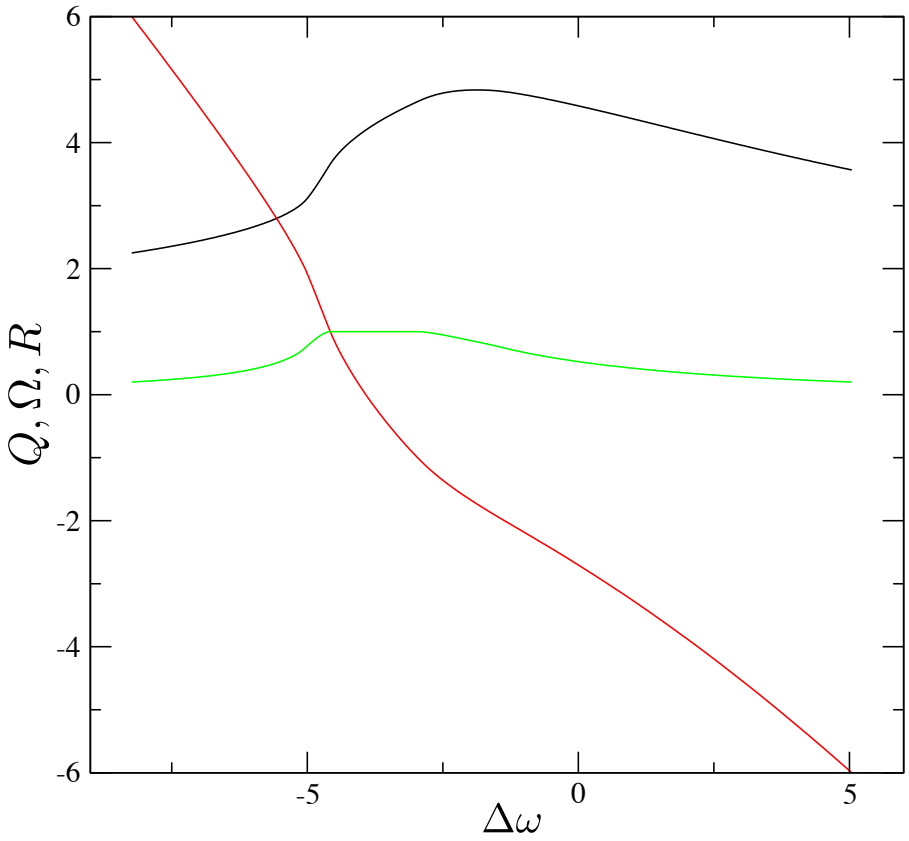


Figure 3.7.: The dependences of the amplitude Q (black curve) and the frequency Ω (red curve) of the global field together with the order parameter R (green curve) on the frequency mismatch $\Delta\omega = \omega - \omega_0$, obtained self-consistently for the following values of the parameters $x_l = -0.1$, $y_l = -0.1$ and $W_A = W_B$, $\omega_s/c = 1$.

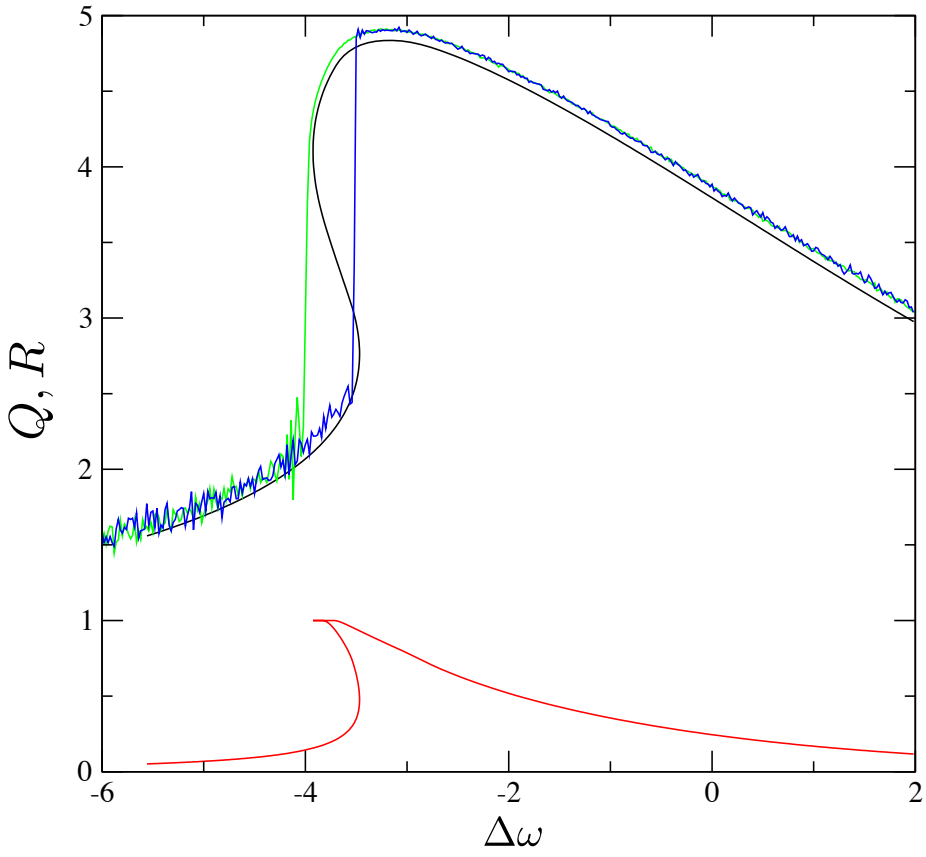


Figure 3.8.: The dependences of the amplitude Q of the global field (black curve represents self-consistent solution, blue and green curves — numeric) and the order parameter R (red curve obtained from self-consistent method) as functions of the frequency mismatch $\Delta\omega = \omega - \omega_0$, for the case $x_l = -0.1$, $y_l = -0.1$ and $W_A = 0.25W_B$, $\omega_s/c = 1$.

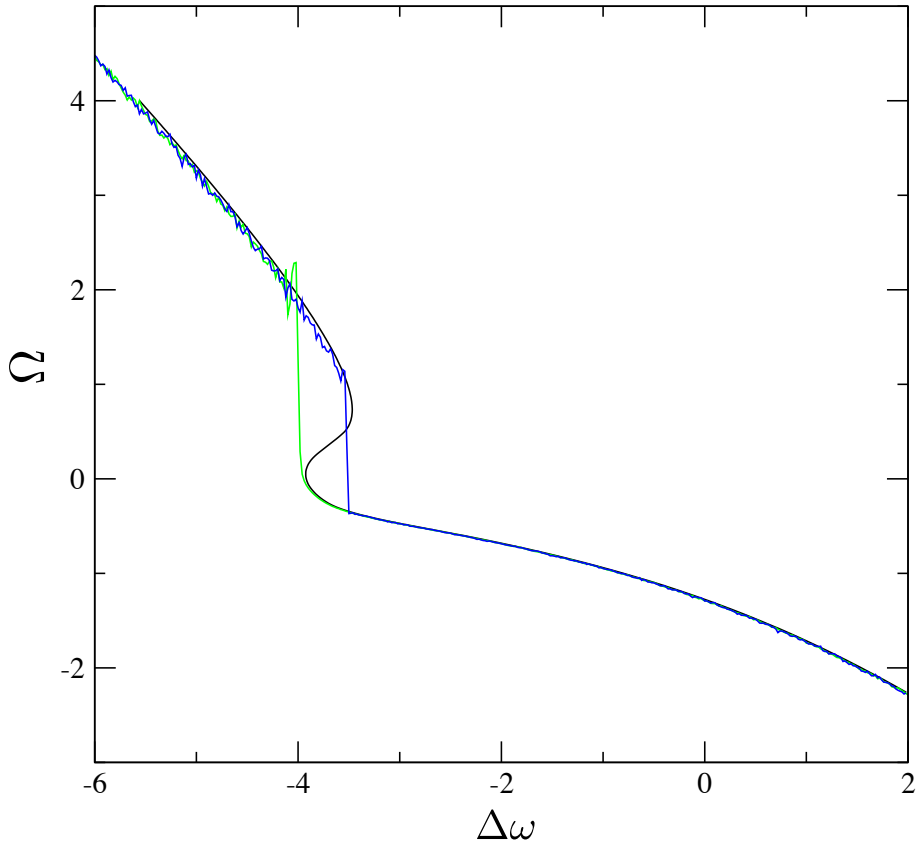


Figure 3.9.: The dependences of the frequency Ω of the global field (black curve from self-consistent approach, blue and green curves from numerics) as functions of the frequency mismatch $\Delta\omega = \omega - \omega_0$, for the case $x_l = -0.1$, $y_l = -0.1$ and $W_A = 0.25 W_B$, $\omega_s/c = 1$.

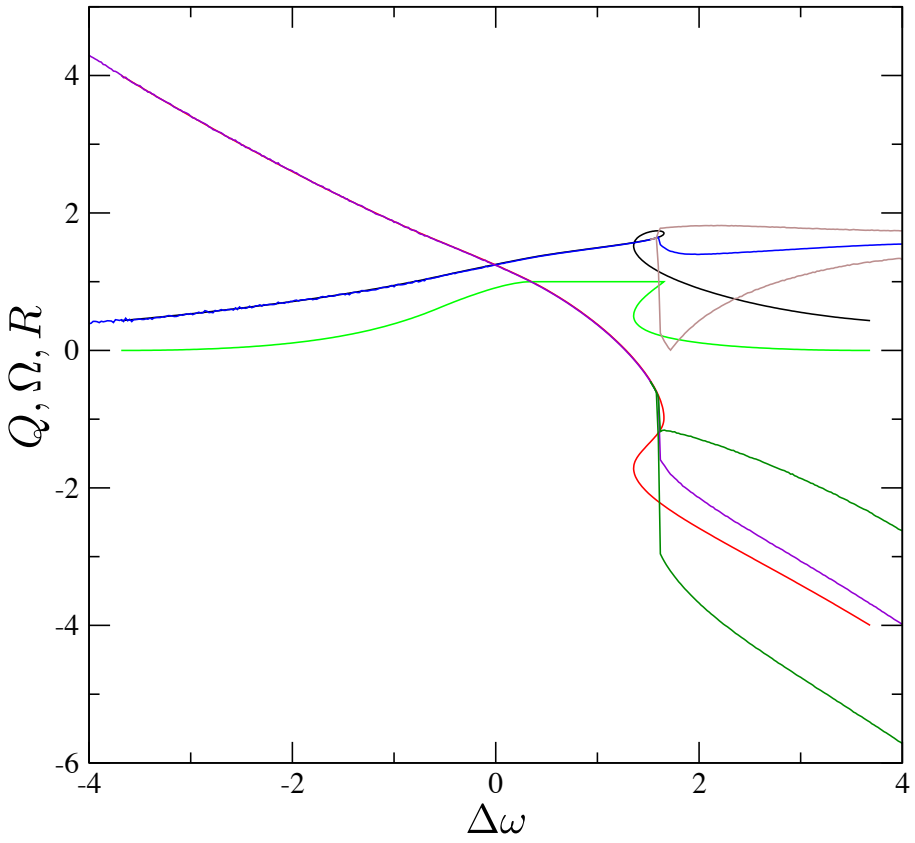


Figure 3.10.: The dependences of the amplitude Q (black curve is obtained self-consistently, blue curve is an average value on the limit cycle obtained numericly and brown curves are minimum and maximum values on the limit cycle) and the frequency Ω (red curve — self-consistent solution, violet curve is numerical average over the limit cycle, dark green curves are the minimum and the maximum on the limit cycle) of the global field on the frequency mismatch $\Delta\omega = \omega - \omega_0$ together with the order parameter R (light green curve obtained from self-consistent method). The following values of the parameters were used $x_l = -1$, $y_l = -1$ and $W_A = W_B$, $\omega_s/c = 1$.

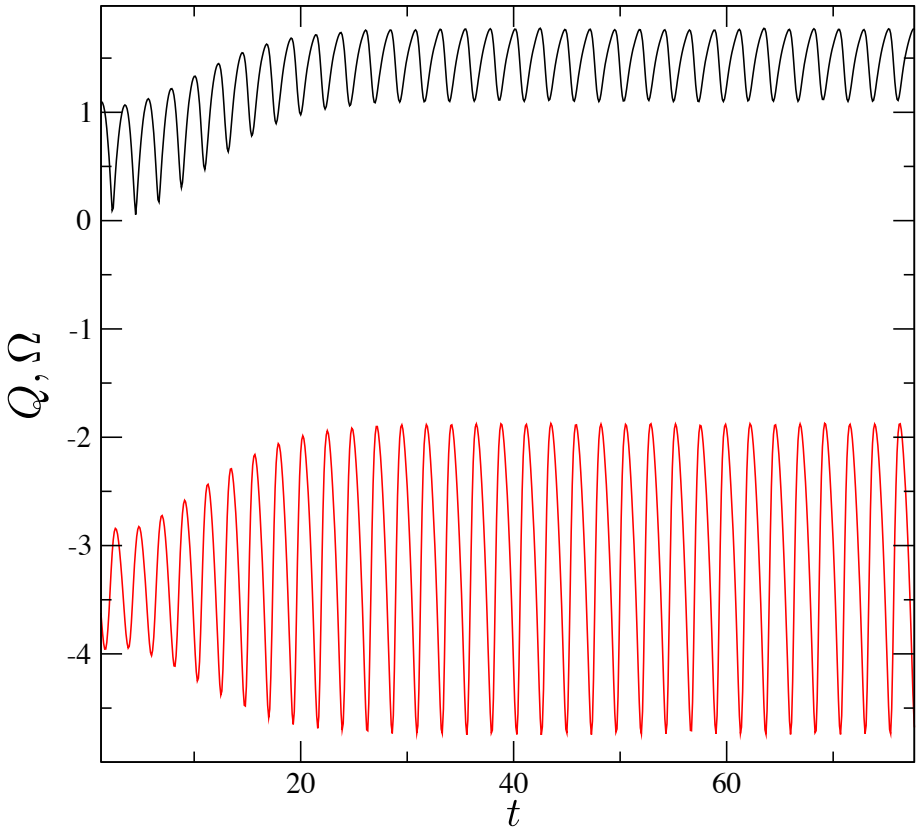


Figure 3.11.: Numerical simulations for the amplitude Q (black curve) and the frequency Ω (red curve) of the global field as functions of time for $\Delta\omega = \omega - \omega_0 = 3$, for the same values of the parameters as in Fig. (3.10).

4. Self-consistent approach in the presence of noise

4.1. Schematic description of the method

The noise is another kind of disorder beside diversity of frequencies and coupling parameters, and the Watanabe-Strogatz approach cannot be applied for that case. However it is still possible to obtain the solutions in terms of global variables with the help of the self-consistent approach.

As in the noise-free case let us consider the system (3.1) with all the notations and the condition (3.3) subject to independent Gaussian white noise forces ($\langle \xi_i(t)\xi_j(t')\rangle = 2\delta_{ij}\delta(t-t')$) with intensity D

$$\dot{\varphi_k} = f(H(p,t), \varphi_k, x_k, p) + \sqrt{D}\xi_k(t). \tag{4.1}$$

The global field $H(p,t)$ is described by (3.2)

$$H(p,t) = \frac{1}{N}\sum_{j=1}^{N} h(\varphi_j(t), x_j, p),$$

In the thermodynamic limit instead of the continuity equation (3.4), the conditional probability density function $\rho(\varphi, t\,|\,x)$ should satisfy the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \varphi}\Big(f(H(p,t),\varphi,x,p)\rho\Big) = D\frac{\partial^2 \rho}{\partial \varphi^2}, \tag{4.2}$$

where the global field $H(t)$ should be calculated from (3.5)

$$H(p,t) = \int g(x)\int_0^{2\pi} \rho(\varphi, t\,|\,x)h(\varphi,x,p)d\varphi\, dx.$$

Analogous to the noise-free case, we perform the variable transformation $\Delta\varphi = \varphi - \Phi(t)$, where $\dot{\Phi} = \text{const} = \Omega$. We are looking for traveling wave solutions with constant frequency Ω and stationary distribution of phase difference $\Delta\varphi$. The equation for $\Delta\varphi$ is the equation (3.6) with noise

$$\dot{\Delta\varphi} = f(H'(p), \Delta\varphi, x, p) - \Omega + \sqrt{D}\xi(t). \tag{4.3}$$

Stationary global field $H'(p)$ is determined by the expression (3.7), with substituted stationary density function $\rho(\Delta\varphi, t \,|\, x) = \rho(\Delta\varphi \,|\, x)$

$$H'(p) = \int g(x) \int_0^{2\pi} \rho(\Delta\varphi \,|\, x) h(\Delta\varphi, x, p) d\Delta\varphi \, dx. \tag{4.4}$$

Where stationary density $\rho(\Delta\varphi \,|\, x)$ can be found from stationary Fokker-Planck equation

$$\frac{\partial}{\partial \Delta\varphi} \left(\Big[f(H'(p), \Delta\varphi, x, p) - \Omega \Big] \rho \right) = D \frac{\partial^2 \rho}{\partial \Delta\varphi^2}. \tag{4.5}$$

A solution of (4.5) can be represented as double integrals, but it does not give any practical results. For further analysis it is more convenient to solve it in the Fourier modes representation

$$\rho(\Delta\varphi \,|\, x) = \frac{1}{2\pi} \sum_n C_n(x) e^{\mathrm{i}n\Delta\varphi} \qquad C_n(x) = \int_0^{2\pi} \rho e^{-\mathrm{i}n\Delta\varphi} d\Delta\varphi \qquad C_0(x) = 1. \tag{4.6}$$

Substitution of (4.6) to Eq. (4.5) gives the equation for the modes

$$\begin{aligned} &\int_0^{2\pi} d\Delta\varphi \left[-\frac{\partial}{\partial \Delta\varphi} \Big(\Big[f(H'(p), \Delta\varphi, x, p) - \Omega \Big] \rho \Big) + D \frac{\partial^2 \rho}{\partial \Delta\varphi^2} \right] e^{-\mathrm{i}k\Delta\varphi} = \\ &= -k^2 D C_k + \mathrm{i}k\Omega C_k - \mathrm{i}k \sum_n f_n^f C_{k-n} = 0, \end{aligned} \tag{4.7}$$

where f_n^f are Fourier modes (with different normalization) of $f(H'(p), \Delta\varphi, x, p)$ determined by

$$f(H'(p), \Delta\varphi, x, p) = \sum_n f_n^f(H'(p), x, p) e^{\mathrm{i}n\Delta\varphi} \qquad f_n^f(H'(p), x, p) = \frac{1}{2\pi} \int_0^{2\pi} f e^{-\mathrm{i}n\Delta\varphi} d\Delta\varphi. \tag{4.8}$$

From the equation (4.7) it follows that Fourier coefficients can be found as a function of system's parameters and variables $C_n = C_n(H'(p), \Omega, x, p)$. Also it is convenient to represent the coupling function $h(\Delta\varphi, x, p)$ in Fourier modes (with different normalization)

$$h(\Delta\varphi, x, p) = \sum_n h_n^f(x, p) e^{\mathrm{i}n\Delta\varphi} \qquad h_n^f(x, p) = \frac{1}{2\pi} \int_0^{2\pi} h e^{-\mathrm{i}n\Delta\varphi} d\Delta\varphi. \tag{4.9}$$

Then the expression for the global field $H'(p)$ can be rewritten as

$$\begin{aligned} H'(p) = Q(p) e^{\mathrm{i}\Delta\Theta(p)} &= \int g(x) \int_0^{2\pi} \rho(\Delta\varphi \,|\, x) h(\Delta\varphi, x, p) d\Delta\varphi \, dx = \\ &= \int g(x) \sum_n h_n^f(x, p) \int_0^{2\pi} \rho(\Delta\varphi \,|\, x) e^{\mathrm{i}n\Delta\varphi} d\Delta\varphi \, dx = \\ &= \int g(x) \sum_n h_n^f(x, p) C_n^*(H'(p), \Omega, x, p) \, dx. \end{aligned} \tag{4.10}$$

Thus we have obtained the self-consistent equation (4.10) for $H'(p)$ analogous to (3.9) that can be solved by the same method by treating parameter family p as unknowns and finding them as a function $p = F(Q, \Delta\Theta, \Omega)$.

This is the scheme of the finding self-consistent solutions for a globally coupled systems of noisy oscillators. The application of this scheme strictly depends on a particular setup of a system, so we will present this method based on two examples of noisy systems: Kuramoto-type model with generic coupling and noise (published in [19]) and an ensemble of noisy oscillators with bi-harmonic coupling (published in [25]).

4.2. Kuramoto-type model with generic coupling and noise

We will consider the same system of Kuramoto-type model with generic coupling in the form (4.11) with additional independent Gaussian white noise forces ($\langle \xi_i(t)\xi_j(t')\rangle = 2\delta_{ij}\delta(t-t')$) with intensity D:

$$\begin{aligned} \dot{\varphi}_i &= \omega_i + A_i \operatorname{Im}\left(H(t)e^{-\mathrm{i}\varphi_i}\right) + \sqrt{D}\xi_i(t), \\ H(t) &= \frac{\varepsilon e^{-\mathrm{i}\delta}}{N}\sum_{j=1}^{N} B_j e^{\mathrm{i}(\varphi_j-\psi_j)}. \end{aligned} \tag{4.11}$$

As in the case of self-consistent approach we consider thermodynamic limit and describe the system in terms of conditional probability function $\rho(\varphi, t \,|\, x)$, where $x = \{\omega, A, B, \psi\}$ and $g(x) = g(\omega, A, B, \psi)$.

Furthermore similar to the noise-free case, we perform the same variable transformation $\Delta\varphi = \varphi - \Theta$, where $\Theta = \Theta(t)$ is the phase of the common mean field $H(t)$. A conditional probability density function of the variable $\Delta\varphi$ is $\rho(\Delta\varphi, t \,|\, x) = \rho(\varphi - \Theta, t \,|\, x)$.

$$\dot{\Delta\varphi} = \omega - \dot{\Theta} - A\,Q\sin(\Delta\varphi) + \sqrt{D}\xi(t), \tag{4.12}$$

$$Q = \varepsilon e^{\mathrm{i}\delta}\int g(x)Be^{-\mathrm{i}\psi}\int_0^{2\pi}\rho(\Delta\varphi, t \,|\, x)e^{\mathrm{i}\Delta\varphi}d\Delta\varphi\,dx. \tag{4.13}$$

The Fokker-Plank equation for $\rho(\Delta\varphi, t \,|\, x)$ reads

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial\Delta\varphi}\left(\left[\omega - \dot{\Theta} - A\,Q\sin(\Delta\varphi)\right]\rho\right) = D\frac{\partial^2\rho}{\partial\Delta\varphi^2}. \tag{4.14}$$

As before, we are looking for special solutions when the mean field rotates uniformly which satisfy the conditions (3.17).

$$\dot{\Theta} = \Omega, \quad \dot{\rho}(\Delta\varphi, t \,|\, x) = 0. \tag{4.15}$$

Thus, in order to find $\rho(\Delta\varphi, t \,|\, x) = \rho(\Delta\varphi \,|\, x)$ we need to solve Eq. (4.16).

$$\frac{\partial}{\partial \Delta\varphi}\left([\omega - \Omega - A\,Q\sin(\Delta\varphi)]\,\rho\right) = D\frac{\partial^2 \rho}{\partial \Delta\varphi^2}. \tag{4.16}$$

Eq. (4.16) can be solved with introduction of Fourier modes (4.17).

$$\rho(\Delta\varphi \,|\, x) = \frac{1}{2\pi}\sum_n C_n(x)e^{\mathrm{i}n\Delta\varphi} \qquad C_n(x) = \int_0^{2\pi} \rho e^{-\mathrm{i}n\Delta\varphi} d\Delta\varphi \qquad C_0(x) = 1. \tag{4.17}$$

After the substitution of (4.17) to Eq. (4.16) the equation for the modes (4.18) is obtained.

$$\begin{aligned}&\int_0^{2\pi} d\Delta\varphi \left[-\frac{\partial}{\partial \Delta\varphi}([\omega - \Omega - AQ\sin(\Delta\varphi)]\rho) + D\frac{\partial^2 \rho}{\partial \Delta\varphi^2}\right] e^{-\mathrm{i}k\Delta\varphi} = \\ &= -k^2 D C_k + \mathrm{i}k(\Omega - \omega)C_k + \mathrm{i}kAQ\frac{C_{k-1} - C_{k+1}}{2\mathrm{i}} = 0.\end{aligned} \tag{4.18}$$

As a consequence, we get a system of algebraic equations with a tridiagonal matrix:

$$[2kD - \mathrm{i}2(\Omega - \omega)]C_k + AQ(C_{k+1} - C_{k-1}) = 0. \tag{4.19}$$

The system (4.19) can be solved by truncation of wavenumbers k at some large K using the following recursion:

$$C_k = \alpha_k C_{k-1}, \quad a_k = 2kD - \mathrm{i}2(\Omega - \omega). \tag{4.20}$$

$$\begin{aligned}\alpha_K &= \frac{AQ}{a_K}, \\ \alpha_k &= \frac{AQ}{a_k + AQ\alpha_{k+1}}.\end{aligned} \tag{4.21}$$

As a result, C_1 can be found from the application of the recursion

$$C_1 = \alpha_1 = \frac{AQ}{a_1 + AQ\alpha_2}. \tag{4.22}$$

From Eq. (4.22) it is easy to see that in general C_1 is a function of Ω, Q, ω and A .

$$C_1 = C_1(\Omega, Q, \omega, A). \tag{4.23}$$

The integral over $\Delta\varphi$ in (4.13) can be calculated with the help of Fourier modes (4.17).

$$\int_0^{2\pi} \rho(\Delta\varphi \,|\, x)e^{\mathrm{i}\Delta\varphi} d\Delta\varphi = C_1^*(\Omega, Q, \omega, A). \tag{4.24}$$

Thus the expression for Q reads

$$Q = \varepsilon e^{\mathrm{i}\delta} \int g(x) B e^{-\mathrm{i}\psi} C_1^*(\Omega, Q, \omega, A) dx. \tag{4.25}$$

Denoting the integral in (4.25) as a function $F(\Omega, Q)$ of unknown variables Ω and Q

$$F(\Omega, Q) = \int g(x) B e^{-\mathrm{i}\psi} C_1^*(\Omega, Q, \omega, A) dx. \tag{4.26}$$

we obtain analogous to the noise free case equation (3.23)

$$Q = \varepsilon e^{-\mathrm{i}\delta} F(\Omega, Q)$$

As in the noise-free case the parameters ε and δ should be treated as unknowns and can be found from Eqs. (3.24) by using another expression (4.26) for $F(\Omega, Q)$.

$$\varepsilon = \frac{Q}{|F(\Omega, Q)|}, \quad \delta = \arg(F(\Omega, Q)).$$

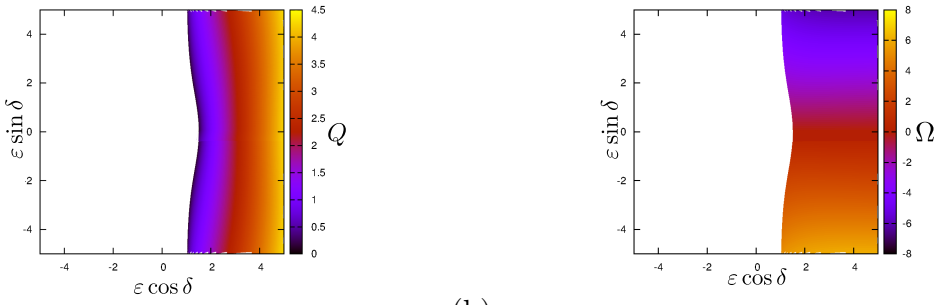
4.2.1. Independent parameters

Similar to the noise-free case, because B and ψ do not enter to the function $F(\Omega, Q)$ explicitly it is clear that the integral in (4.26) simplifies in the same case of independent distribution of the parameters $g(x) = g_1(\omega, A)\, g_2(B, \psi)$. Here the same notations as before are used, including condition (3.26), that yields the meaning for ε and δ.

The parameters ε and δ can be found from Eqs. (3.24), where $F(\Omega, Q)$ is determined from

$$F(\Omega, Q) = \int g_1(\omega, A) C_1^*(\Omega, Q, \omega, A) dA d\omega. \tag{4.27}$$

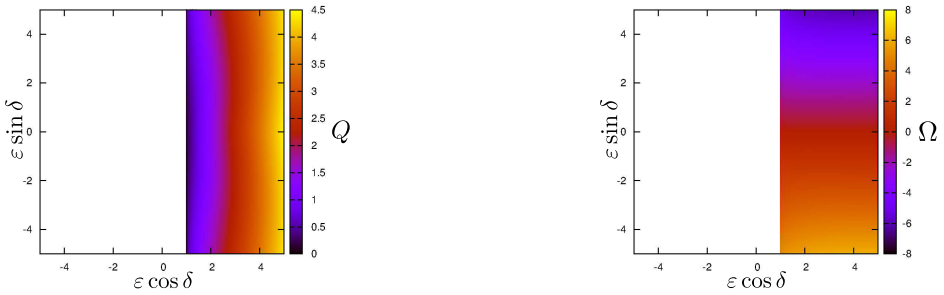
In this way we obtain $Q(\varepsilon, \delta)$ and $\Omega(\varepsilon, \delta)$ (Fig. 4.1). For calculations we used the same distribution $g_1(\omega, A)$ as in the noise-free case.



(a) (b)

Figure 4.1.: (Color online) The same as Fig. 3.2, but with noise $D = 1$. From [19].

Contrary to the noise-free case, there is no further simplification of $F(\Omega, Q)$ possible when oscillator's individual frequencies are identical (delta-function distribution). In Fig. 4.2 we report the results for the same parameters as in Fig. 3.3, but with noise $D = 1$.



(a) (b)

Figure 4.2.: (Color online)The same as Fig. 3.3, but with noise $D = 1$. From [19].

In the considered model the main effect of noise consists of moving the synchronization threshold to larger values of the coupling strength ε. The influence of noise is quite similar to the disorder caused by the distribution of natural frequencies. In the case of identical oscillator's individual frequencies noise leads to a non-zero threshold in the coupling.

4.3. An ensemble of noisy oscillators with bi-harmonic coupling

Here we will present how noisy system with bi-harmonic coupling function can be treated self-consistently. We study an ensemble of phase oscillators (phase variables ϕ_k), coupled through a mean-field bi-harmonic function and noise [we start with formulation in "primed" variables, which will be transformed to dimensionless ones]:

$$\frac{d\phi_k}{dt'} = \omega_k \Delta' + \frac{\varepsilon}{N}\sum_{j=1}^{N}\sin(\phi_j - \phi_k) + \frac{\gamma}{N}\sum_{j=1}^{N}\sin(2\phi_j - 2\phi_k) + \sqrt{D'}\,\xi_k(t')\ . \qquad (4.28)$$

Here ω_k are normalized natural frequencies of oscillators that have a symmetrical unimodal distribution $g(\omega)$ with unit width (the parameter Δ is responsible for the width of the distribution) and zero mean value (what can be always achieved by transforming to a properly rotating reference frame). Gaussian white noise is defined according to $\langle \xi_k(t'_1)\xi_j(t'_2)\rangle = 2\delta(t'_1 - t'_2)\delta_{kj}$. Parameters ε and γ define the coupling strengths of the first and the second harmonics, respectively.

Eq. (4.28) can be rewritten as

$$\frac{d\phi_k}{dt'} = \omega_k \Delta' + \varepsilon R_1 \sin(\Theta_1 - \phi_k) + \gamma R_2 \sin(\Theta_2 - 2\phi_k) + \sqrt{D'}\,\xi_k(t'), \qquad (4.29)$$

where $R_m e^{\mathrm{i}\Theta_m} = N^{-1}\sum_j e^{\mathrm{i}m\phi_j}$, $m = 1, 2$, are the two relevant order parameters [26].

Eq. (4.29) has 4 parameters, all of them of dimension $1/t'$: $\Delta', \varepsilon, \gamma, D'$. It is possible to reduce the number of parameters by rescaling time. We choose the following rescaling $t = (\varepsilon + \gamma)t'$ and get

$$\dot{\phi}_k = \omega_k \Delta + qR_1 \sin(\Theta_1 - \phi_k) + (1-q)R_2 \sin(\Theta_2 - 2\phi_k) + \sqrt{D}\,\xi_k(t), \tag{4.30}$$

where $\Delta = \Delta'/(\varepsilon + \gamma)$, $q = \varepsilon/(\varepsilon + \gamma)$ and $D = D'/(\varepsilon + \gamma)$. The new parameter q describes the relation between coupling coefficients ε and γ, so that the case $q = 0$ corresponds to a pure second harmonic coupling with $\varepsilon = 0$, and $q = 1$ corresponds to a pure Kuramoto-type first harmonic coupling with $\gamma = 0$. In this new normalization, changing of coupling strength is equivalent to changing of the disorder parameters Δ (spread of frequencies) and D (noise), while keeping a constant relation Δ/D between them. Such a feature suggests to introduce new parameters T, s in a way that $\Delta = (1-s)T$, $D = sT$. Therefore, the overall disorder (noise and spread of frequencies) is measured by the parameter T, normalized by the overall coupling strength $\varepsilon + \gamma$. The share between two disorders is measured by the parameter s: for $s = 0$ the system is purely deterministic, and for $s = 1$ it describes an ensemble of identical noisy oscillators. Then Eq. (4.30) with the new parameters q, T, s becomes

$$\dot{\phi}_k = \omega_k(1-s)T + qR_1 \sin(\Theta_1 - \phi_k) + (1-q)R_2 \sin(\Theta_2 - 2\phi_k) + \sqrt{sT}\,\xi_k(t). \tag{4.31}$$

As in the other sections in this work we consider the thermodynamic limit $N \to \infty$, where the order parameters are just ensemble averages $R_m e^{\mathrm{i}\Theta_m} = \langle e^{\mathrm{i}m\phi} \rangle$. They can be represented through the conditional probability density function of the phases $\rho(\varphi, t \,|\, \omega)$, as

$$R_m e^{\mathrm{i}\Theta_m} = \langle e^{\mathrm{i}m\varphi} \rangle = \int\int g(\omega)\, \rho(\varphi, t \,|\, \omega) e^{\mathrm{i}m\varphi} d\varphi\, d\omega \tag{4.32}$$

In the thermodynamic-limit formulation we use the continuous (without indices) variable φ to describe the phase, therefore, according to (4.31) the equation for the phase variable φ at given ω reads

$$\frac{d\varphi}{dt} = \omega(1-s)T + qR_1 \sin(\Theta_1 - \varphi) + (1-q)R_2 \sin(\Theta_2 - 2\varphi) + \sqrt{sT}\,\xi(t)\ . \tag{4.33}$$

The Fokker-Planck equation for $\rho(\varphi, t \,|\, \omega)$ follows from Eq. (4.33):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \varphi}\Big[\Big(\omega(1-s)T + qR_1 \sin(\Theta_1 - \varphi) + (1-q)R_2 \sin(\Theta_2 - 2\varphi)\Big)\rho\Big] = sT \frac{\partial^2 \rho}{\partial \varphi^2}\,. \tag{4.34}$$

The limiting noise-free case when $s = 0$ has been described in details in [17, 18]. Here we will present a general analysis for systems with noise and a finite distribution of frequencies. The limit $s \ll 1$ appears to be singular and goes beyond the scope of these work (see [25] for detailed consideration of this case). The other limiting case $s = 1$ is the case of the identical natural frequencies and in terms of the analysis presented below is not special but, due to the fact that an additional stability analysis for $s = 1$ can be performed, this case will be also considered in details separately.

4.3.1. Stationary solutions in a parametric form

The first type and the most trivial solution of the system (4.34, 4.32) is a disordered state with a uniform distribution of phases $\rho = (2\pi)^{-1}$, where the order parameters vanish $R_1 = R_2 = 0$. The second type is an ordered state with two nontrivial synchronized regimes: (i) all order parameters are non-zero, and (ii) a symmetric 2-cluster distribution where all odd order parameters vanish $R_{2m+1} = 0$ and $R_{2m} \neq 0$, $m \in \mathbb{N}_0$.

As mentioned above we restrict ourselves to the symmetric distribution of natural frequencies $g(\omega)$, so due to the symmetry of the coupling function the frequency of the mean fields for the nontrivial solutions is exactly the average oscillator frequency (that we have considered to be zero). Thus these solutions could be found by setting $\Theta_1 = \Theta_2 = 0$ (in fact, one of the phases of the order parameters is arbitrary, the condition above means that the phase shift between the two order parameters is zero). For details please see [25], where this setting is discussed in full detail. For this work we will focus on the analysis of the symmetric setup.

In the symmetric setting, the stationary conditional probability density function $\rho(\varphi \,|\, \omega)$ satisfies the stationary Fokker-Planck equation

$$\frac{\partial}{\partial\varphi}\left[\left(\omega(1-s) - \frac{qR_1}{T}\sin(\varphi) - \frac{(1-q)R_2}{T}\sin(2\varphi)\right)\rho\right] = s\frac{\partial^2\rho}{\partial\varphi^2}\,, \tag{4.35}$$

where because of symmetry

$$R_m = \int\int g(\omega)\,\rho(\varphi \,|\, \omega)\cos(m\varphi)d\varphi\,d\omega\ . \tag{4.36}$$

The explicit solution of this self-consistent system can be found with the help of the introduction of two new auxiliary variables R and α (together with definitions u, v, and x) according to

$$\begin{aligned} R &= \sqrt{\Big(qR_1/T\Big)^2 + \Big((1-q)R_2/T\Big)^2}, \\ u &= \cos\alpha = \frac{qR_1}{TR}, \\ v &= \sin\alpha = \frac{(1-q)R_2}{TR}, \\ x &= \frac{\omega}{R}\ . \end{aligned} \tag{4.37}$$

Then the stationary Fokker-Plank equation (4.35) for the stationary distribution density $\rho(\varphi \,|\, x)$ (which depends on R, α, s as parameters) should be rewritten as

$$\frac{\partial}{\partial\varphi}\Big[R\Big(x(1-s) - u\sin(\varphi) - v\sin(2\varphi)\Big)\rho\Big] = s\frac{\partial^2\rho}{\partial\varphi^2}\,. \tag{4.38}$$

A formal solution of (4.38) is a double integral, but practically it is more convenient to solve it in the Fourier modes representation

$$\rho(\varphi\,|\,x)=\frac{1}{2\pi}\sum_n C_n(\alpha,R,s,x)e^{\mathrm{i}n\varphi},\quad C_n(\alpha,R,s,x)=\int_0^{2\pi}\rho e^{-\mathrm{i}n\varphi}d\varphi,\quad C_0=1. \tag{4.39}$$

Substituting (4.39) in Eq. (4.38) we obtain

$$\begin{aligned}0&=\int_0^{2\pi}\left[-\frac{\partial}{\partial\varphi}\Big[R\Big(x(1-s)-u\sin(\varphi)-v\sin(2\varphi)\Big)\rho\Big]+s\frac{\partial^2\rho}{\partial\varphi^2}\right]e^{-\mathrm{i}k\varphi}\,d\varphi=\\&=R\left[\Big(-\mathrm{i}x(1-s)k-k^2s/R\Big)C_k+\mathrm{i}ku\frac{C_{k-1}-C_{k+1}}{2\mathrm{i}}+\mathrm{i}kv\frac{C_{k-2}-C_{k+2}}{2\mathrm{i}}\right].\end{aligned} \tag{4.40}$$

Thus, from (4.40) a system of linear algebraic equations for the mode amplitudes is obtained:

$$2\Big(\mathrm{i}x(1-s)+ks/R\Big)C_k+u(C_{k+1}-C_{k-1})+v(C_{k+2}-C_{k-2})=0\,. \tag{4.41}$$

This equation should be truncated to a finite number of Fourier modes (which controls accuracy of the solution) and then solved by standard methods. After finding $C_{1,2}(\alpha,R,s,x)$, we have to calculate integrals

$$F_{1,2}(\alpha,R,s)=\int g(Rx)\,\mathrm{Re}\,[C_{1,2}(\alpha,R,s,x)]\,dx. \tag{4.42}$$

This allows us to represent the order parameters as

$$R_{1,2}(\alpha,R,s)=R\int g(Rx)\,\mathrm{Re}\,[C_{1,2}(\alpha,R,s,x)]\,dx=RF_{1,2}(\alpha,R,s). \tag{4.43}$$

Substituting this in Eq. (4.37), we obtain our parameters T,q as functions of the auxiliary variables

$$\begin{aligned}T&=\frac{1}{\frac{\cos\alpha}{F_1}+\frac{\sin\alpha}{F_2}},\\q&=\frac{\frac{\cos\alpha}{F_1}}{\frac{\cos\alpha}{F_1}+\frac{\sin\alpha}{F_2}}=\frac{1}{1+\frac{F_1}{F_2}\tan\alpha}.\end{aligned} \tag{4.44}$$

Thus for each fixed s, by varying $\alpha\in[0,\pi/2]$ and $R\in[0,\infty)$, we obtain the explicit solution of the self-consistent problem in a parametric form: $R_{1,2}=R_{1,2}(\alpha,R)$ according to (4.43), $T=T(\alpha,R)$ and $q=q(\alpha,R)$ according to (4.44).

The case $\alpha=\pi/2$ is singular in (4.43, 4.44) and corresponds to purely two-cluster state with $R_1=0$. Here the solution is represented as

$$R_2=RF_2,\qquad T=(1-q)F_2. \tag{4.45}$$

So the method presented above provides stationary solutions of the Eq. (4.34) for any given q, T and s. Note that in the general case of $s < 1$, Eq. (4.34) is integro-differential equation and the analysis of the stability of all solutions is quite difficult, except for the simplest incoherent solution $\rho = (2\pi)^{-1}$. However, in the limiting case of identical natural frequencies $s = 1$, density ρ is ω-independent, and integration in (4.36) over the frequencies always gives unity. What means that the real values of the Fourier modes (4.39) are in fact the order parameters: $R_m(q, T, 1) = \mathrm{Re}\,[C_m(q, T, 1, 0)]$. So, in the case of identical natural frequencies $s = 1$ the full time-dependent problem can be written as a system of nonlinear ordinary differential equations for time-depended Fourier modes of the density, which can be analyzed for stability after a proper truncation.

First we will present the method for obtaining stability borders for the simplest incoherent solution $\rho = (2\pi)^{-1}$. And second we will show how to analyze the stability of the nontrivial solution in the case of identical natural frequencies $s = 1$.

4.3.2. Stability analysis of the incoherent solution

The stability analysis of the incoherent solution has been performed in [27, 28] for the system of phase equations in the case of multi-harmonic coupling function. Here we will restrict ourselves to the case of bi-harmonic coupling function and present the analysis in the new parameter plane (q, T, s).

Let us return the Fokker-Planck equation (4.34) for conditional probability density function $\rho(\varphi, t\,|\,\omega)$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \varphi}\Big[\Big(\omega(1-s)T + q\,\mathrm{Im}\left(Z_1 e^{-\mathrm{i}\varphi}\right) + (1-q)\,\mathrm{Im}\left(Z_2 e^{-2\mathrm{i}\varphi}\right)\Big)\rho\Big] = sT\frac{\partial^2 \rho}{\partial \varphi^2}\,, \quad (4.46)$$

where

$$Z_m(t) = \int\int g(\omega)\,\rho(\varphi, t\,|\,\omega) e^{\mathrm{i}m\varphi} d\varphi\, d\omega \quad (4.47)$$

are the order parameters [26]. Then we perform the same procedure, namely go in the Fourier modes representation

$$\rho(\varphi, t\,|\,\omega) = \frac{1}{2\pi}\sum_n C_n(t,\omega) e^{\mathrm{i}n\varphi} \qquad C_n(t,\omega) = \int_0^{2\pi} \rho e^{-\mathrm{i}n\varphi} d\varphi, \qquad C_0(t,\omega) = 1 \quad (4.48)$$

and introduce it to the Fokker-Planck equation (4.46)

$$\begin{aligned}\frac{dC_k}{dt} &= \int_0^{2\pi}\left[-\frac{\partial}{\partial \varphi}\Big[\Big(\omega(1-s)T + q\,\mathrm{Im}\left(Z_1 e^{-\mathrm{i}\varphi}\right) + (1-q)\,\mathrm{Im}\left(Z_2 e^{-2\mathrm{i}\varphi}\right)\Big)\rho\Big] + sT\frac{\partial^2 \rho}{\partial \varphi^2}\right] e^{-\mathrm{i}k\varphi}\, d\varphi = \\ &= \Big(-\mathrm{i}k\omega(1-s)T - k^2 sT\Big)C_k + \mathrm{i}kq\,\frac{Z_1^* C_{k-1} - Z_1 C_{k+1}}{2\mathrm{i}} + \mathrm{i}k(1-q)\frac{Z_2^* C_{k-2} - Z_2 C_{k+2}}{2\mathrm{i}}.\end{aligned} \quad (4.49)$$

Thus the system of differential equations for time-dependent Fourier modes $C_k(t,\omega)$ is obtained

$$\frac{dC_k}{dt} = k\left[-\Big(\mathrm{i}\omega(1-s)T + ksT\Big)C_k + q\,\frac{Z_1^* C_{k-1} - Z_1 C_{k+1}}{2} + (1-q)\frac{Z_2^* C_{k-2} - Z_2 C_{k+2}}{2}\right], \tag{4.50}$$

where

$$Z_{1,2}(t) = \int g(\omega)\, C_{1,2}^*(t,\omega)\, d\omega. \tag{4.51}$$

Incoherent solution $\rho(\varphi, t\,|\,\omega) = (2\pi)^{-1}$ means that $C_k = 0$ and in order to analyze the stability of this solution, a small perturbation should be added to it. So we can take $C_k << 1$ for $k \neq 0$ and thus $Z_m << 1$, so in linearized system we should neglect all the terms such as $C_m C_n$ and $Z_m C_n$ and higher. Then for $k > 0$

$$\begin{aligned}
\frac{dC_1}{dt} &= -T\Big(\mathrm{i}\omega(1-s) + s\Big)C_1 + q\,\frac{Z_1^*}{2},\\
\frac{dC_2}{dt} &= -2T\Big(\mathrm{i}\omega(1-s) + 2s\Big)C_2 + (1-q)Z_2^*,\\
\frac{dC_k}{dt} &= -kT\Big(\mathrm{i}\omega(1-s) + ks\Big)C_k, \quad \text{for } k = 3,4,...
\end{aligned} \tag{4.52}$$

and complex conjugate equations for $k < 0$ because $C_{-k} = C_k^*$. Next, the expressions (4.51) for $Z_{1,2}$ should be introduced to (4.52). Then, only for $C_{1,2}$ the integro-differential equations are obtained

$$\begin{aligned}
\frac{dC_1}{dt} &= -T\Big(\mathrm{i}\omega(1-s) + s\Big)C_1 + q\,\frac{1}{2}\int g(\omega)\, C_1\, d\omega,\\
\frac{dC_2}{dt} &= -2T\Big(\mathrm{i}\omega(1-s) + 2s\Big)C_2 + (1-q)\int g(\omega)\, C_2\, d\omega,\\
\frac{dC_k}{dt} &= -kT\Big(\mathrm{i}\omega(1-s) + ks\Big)C_k, \quad \text{for } k = 3,4,...
\end{aligned} \tag{4.53}$$

In the linearized system (4.53) the equations for the harmonics split and all the high harmonics with $k \geq 3$ and their complex conjugates are stable because $T \geq 0$ and $0 \leq s \leq 1$, whereas instability appears in first and second harmonics. And because the harmonics split it appears independently, depending on q, T.

Since the equations for modes (4.53) are decoupled, boundary conditions when the first and the second harmonics become unstable can be found independently. If one put in (4.53) $dC_1/dt = 0$ and $dC_2/dt = 0$ one can self-consistently obtain two conditions on the parameters q, T. Then, by using expression (4.51) we obtain

$$\begin{aligned}
T\Big(\mathrm{i}\omega(1-s) + s\Big)C_1 &= q\,\frac{1}{2}\,Z_1^*,\\
2T\Big(\mathrm{i}\omega(1-s) + 2s\Big)C_2 &= (1-q)\,Z_2^*.
\end{aligned} \tag{4.54}$$

Introducing (4.54) to (4.51)

$$\begin{aligned} Z_1^* &= \frac{q}{2T}\int \frac{g(\omega)\, Z_1^*}{\mathrm{i}\omega(1-s)+s}\, d\omega, \\ Z_2^* &= \frac{1-q}{2T}\int \frac{g(\omega)\, Z_2^*}{\mathrm{i}\omega(1-s)+2s}\, d\omega, \end{aligned} \tag{4.55}$$

we obtain two lines on the (q, T) parameter plane for any given s:

$$T = q\,\frac{1}{2}\int \frac{g(\omega)s}{\omega^2(1-s)^2+s^2}\, d\omega, \tag{4.56}$$

and

$$T = (1-q)\int \frac{g(\omega)s}{\omega^2(1-s)^2+4s^2}\, d\omega. \tag{4.57}$$

Here the line (4.56) on the (q, T) plane corresponds to the linear stability boundary for $R_1 = 0$ and another line (4.57) corresponds to the linear stability boundary for $R_2 = 0$.

4.3.3. Limiting case of identical oscillators

As already have been mentioned, the stability analysis of the nontrivial solution can be performed only in the case of identical natural frequencies of the oscillators, what means that $\Delta = (1-s)T = 0$ or $s = 1$.

As before we will rewrite the Fokker-Planck equation (4.34) in terms of time-dependent Fourier modes $C_m(t)$ (4.50), but taking into account that $s = 1$, so that the complex order parameters $Z_m = C_m^*$:

$$\frac{dC_k}{dt} = -k^2 T C_k + kq\frac{C_1 C_{k-1} - C_1^* C_{k+1}}{2} + k(1-q)\frac{C_2 C_{k-2} - C_2^* C_{k+2}}{2}. \tag{4.58}$$

A stationary solution can be still represented through the parametric formulas (4.43, 4.44) or (4.45). Here we are interested in the stability analysis of these solutions.

So we add a small perturbation around stationary solution $\tilde{C}_k$ so we set $C_k = \tilde{C}_k + c_k$ in Eq. (4.58), and in the first order in c_k obtain

$$\begin{aligned} \frac{dc_k}{dt} =& -k^2 T c_k + \frac{kq}{2}\Big(c_1\tilde{C}_{k-1} - c_1^*\tilde{C}_{k+1} + \tilde{C}_1 c_{k-1} - \tilde{C}_1^* c_{k+1}\Big) + \\ &+ \frac{k(1-q)}{2}\Big(c_2\tilde{C}_{k-2} - c_2^*\tilde{C}_{k+2} + \tilde{C}_2 c_{k-2} - \tilde{C}_2^* c_{k+2}\Big). \end{aligned} \tag{4.59}$$

The system of equations (4.59) is an infinite system. But the amplitudes of the modes with large k tend to zero,so it is appropriate to truncate it at some large K, and to write a finite system of Eqs. (4.59), with k varying from 1 to K, where K is large enough. Since we obtain the stationary solution $\tilde{C}_k$ as a function of q, T, for any given

pair of this parameters we can find a maximum eigenvalue of the corresponding matrix. In this way we find stability properties of the solution $\tilde{C}_k$. Based on that it is possible to build the boundary $q = q(T)$ where the solution $\tilde{C}_k$ changes its stability. This can be done both for general solutions (4.43, 4.44) and for the two-cluster states (4.45).

Fig. 4.3 represents the diagram of synchronous states in the parameter plane (q, T), for the case of identical oscillators $s = 1$. The stability lines of the disordered state $\rho = (2\pi)^{-1}$ are shown with dashed lines. As a result, there are three major states: a disordered one, one with all non-zero order parameters, and a two-cluster one where all odd order parameters vanish. For the values of the parameter q where one of the coupling modes dominates (the first harmonics coupling for $0 << q \leq 1$ or the second harmonics coupling for $1 >> q \geq 0$), the transitions are supercritical, it is illustrated in Fig. 4.4 (panels (a),(d)), showing dependencies of the order parameters on T for constant values of q (marked on the Fig. 4.3 (panel (a)). Whereas in the middle part of the phase diagram (between the points marked p_1, p_2 in Fig. 4.3), for q close to 0.3, the transitions are subcritical, so that a bistability occurs. These regimes are illustrated in Fig. 4.4 (panels (b),(c)). Note that the transition from the disordered to the two-cluster state is always supercritical, what can be seen in panels (c,d) of Fig. 4.4 (dashed red line).

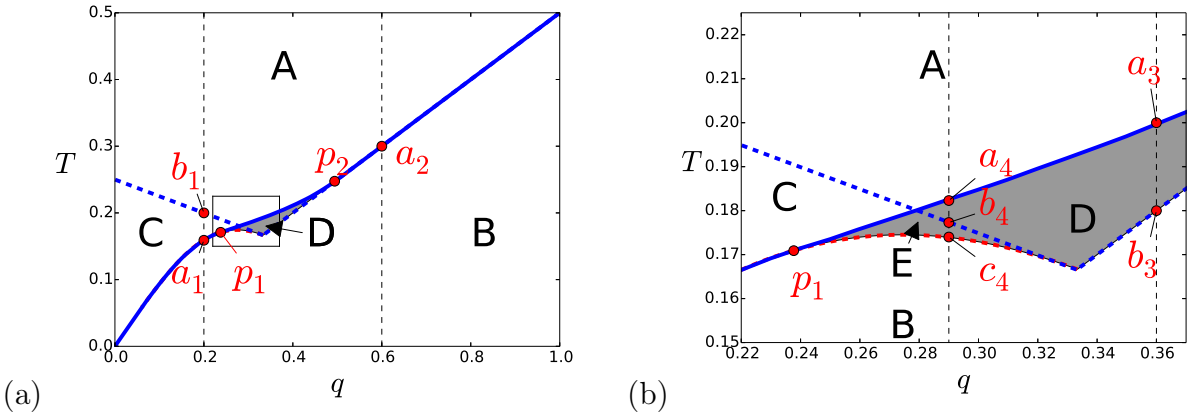


Figure 4.3.: (a) Different regimes in the parameter plane (q, T) are shown for $s = 1$. Area A: asynchronous solution. Area B: coherent regime with $R_{1,2} \neq 0$. Area C: two-cluster coherent regime with only order parameter $R_2 \neq 0$, $R_1 = 0$. Area D: region of bistability of incoherent and synchronous solutions. Area E: bistability of the two-cluster state and a state with $R_{1,2} \neq 0$. Dashed blue lines are stability lines of the disordered state, obtained from (4.56) and (4.57). Between the points p_1 and p_2 the transition is hysteretic; dashed red line is the stability line of Area C, obtained from (4.59), it coincides with the line where on the branch existing for small T the first order parameter tends to zero. (b) Enlarged central region of panel (a). Vertical dashed lines are cuts of the diagram illustrated in Fig. 4.4. From [25].

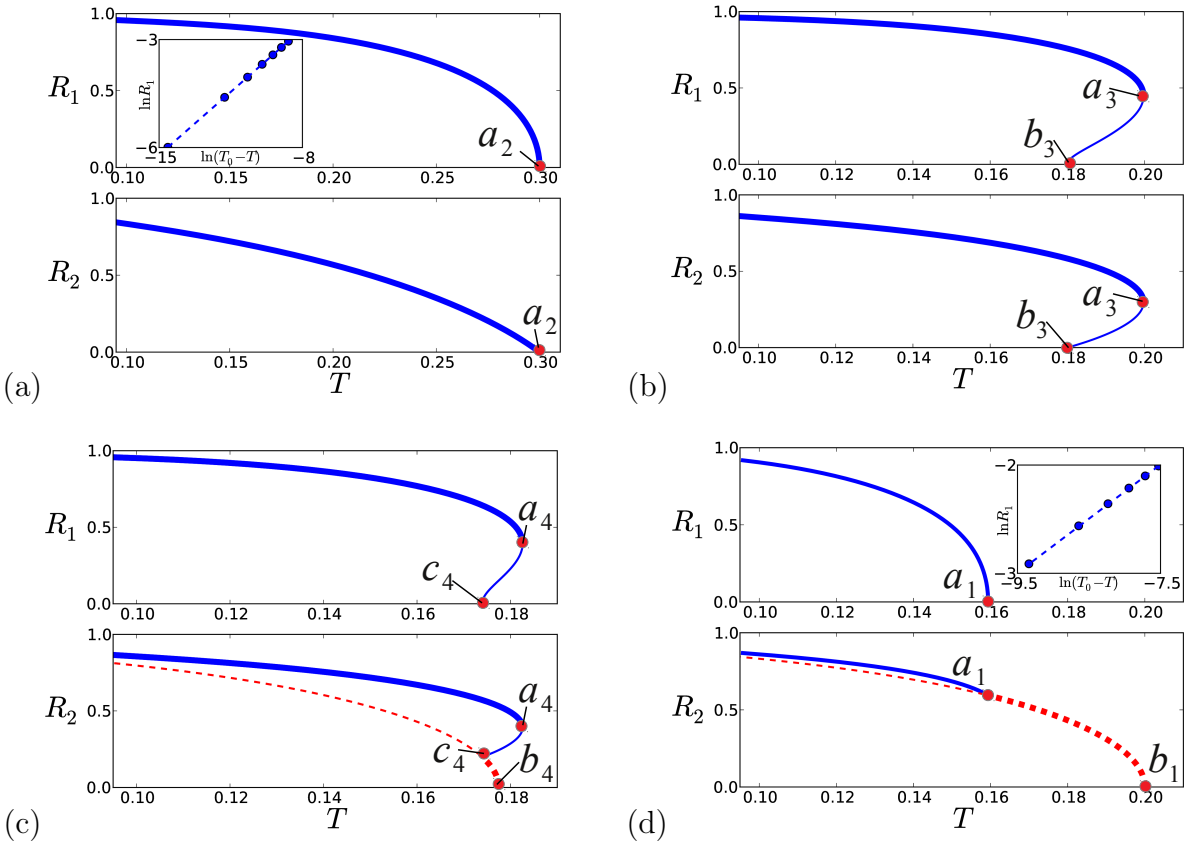


Figure 4.4.: Dependencies of order parameters R_{12} on the disorder parameter T, for $s = 1$ and different values of q: (a) $q = 0.6$, (b) $q = 0.36$, (c) $q = 0.29$, (d) $q = 0.2$. Solid blue line: branch of general solution $R_1 \neq 0$; dashed red line: branch of the two-cluster state with $R_1 = 0$, $R_2 \neq 0$. Thin curves depict unstable solutions, whereas the bold curves depict stable ones. Insets in the panels (a) and (d) present logarithmic plots indicating square-root dependencies of the order parameter R_1 on the criticality $T_0 - T$ (T_0 is a bifurcation point). Here markers denote solution of the self-consistent equations, the dashed line has a slope 1/2. From [25].

4.3.4. General phase diagram

In this section we present the phase diagram for the Gaussian distribution of natural frequencies ($g(\omega) = (2\pi)^{-1} \exp(-\omega^2/2)$). The lines of stability of the disordered state (bold dashed lines) (4.56,4.57) are obtained according to stability analysis presented above. Fig. 4.5 shows two phase diagrams on the plane of basic parameters (q, T) for $s = 0.1$ (relatively weak noise, wide distribution of frequencies) and $s = 0.5$. The picture for general values of s is qualitatively the same as Fig. 4.3, and although it cannot be checked, the stability properties of different solutions are expected to be like in Fig. 4.4.

The size of the region of the first-order transition changes for different s. In the case of weak noise (s is small) (Fig. 4.5a) it is rather large. On the other hand with increasing of s (for stronger noise and narrower distribution of frequencies) it becomes smaller

(Figs. 4.3,4.5b).

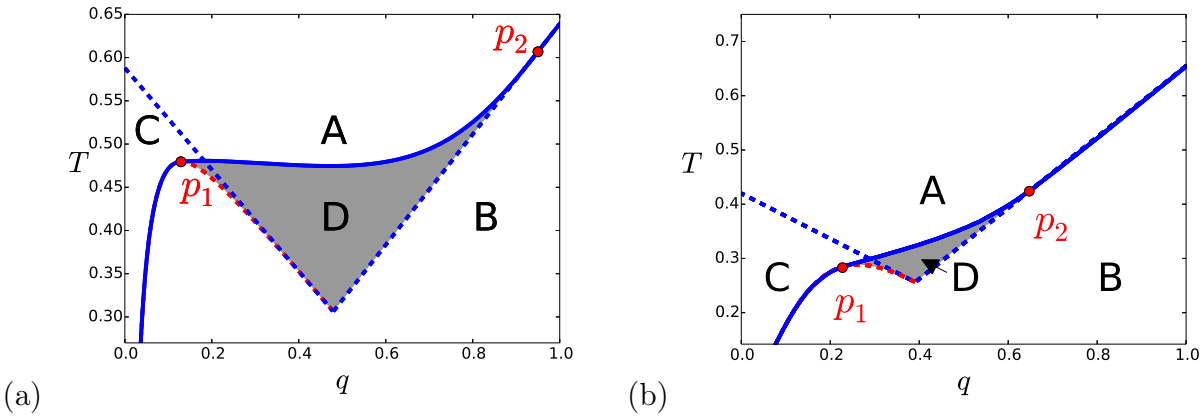


Figure 4.5.: The same as Fig. 4.3, but for $s = 0.1$ (a) and $s = 0.5$ (b). Region E is not denoted because it is very tiny on these plots. From [25].

4.4. Summary

In this chapter we have considered the application of the self-consistent approach to noisy systems. The main difference from the noise-free case is the way of finding of self-consistent solutions, where numerical evaluation of a continued fraction is needed in order to solve the corresponding stationary Fokker-Planck equation in the Fourier space. The application to the Kuramoto-type model with generic coupling and noise leads to the similar results as in the noise-free system, except for the fact that the synchronization threshold is shifted to larger values of the coupling strength. The application to an ensemble of noisy oscillators with a bi-harmonic coupling gives a possibility to calculate a general bifurcation diagram and to describe all possible stationary regimes of the model, for different values of noise strength, spread of frequencies distribution, and coupling constants. Quite remarkable is a synchronous two-cluster state with zero first order parameter (so-called "nematic phase") which can be observed for a large area of parameters. Also this method reveals all possible scenarios of the transitions to synchrony in this model.

5. Conclusions

In this work we have studied two methods that provide a global description of networks of globally coupled phase oscillators. Namely, the Watanabe-Strogatz (WS) and the self-consistent approaches. These are completely different methods that have an overlapping area of application. For both of these methods the global coupling is necessary condition and the self-consistent approach is applicable for the large variety of such networks. However the application of the WS approach is restricted to the particular type of phase equation and inhomogeneity. Strictly speaking, in its original form, it is applicable only to the homogenous systems of identical oscillators but can be used for inhomogeneous systems that can be divided into the groups of identical elements. For more general nonidentical systems and noisy systems the WS method is not applicable. Nevertheless, the expressions for the solutions of a special type, namely traveling wave solutions, in the thermodynamic limit can be obtained with the help of the self-consistent approach, generally without a possibility to obtain equations for global variables. In the thermodynamic limit a system can be described in terms of the probability density function of the phases. The idea behind this approach is to find self-consistent solutions for the probability density function of the phases of corresponding continuity equation for a noise-free case or of the Fokker-Planck equation for noisy systems. While it is not possible to analyze stability of obtained solutions for general case, in some particular cases (like it is shown based on the example of biharmonic coupling) the stability analysis of some solutions (simple incoherent solution in the considered example) can be done.

We have applied these methods to different systems. With the help of the WS approach we have analyzed arrays of Josephson junctions and networks of phase oscillators with leader-type coupling. In both of these systems we have obtained that the transition from asynchrony to synchrony is hysteretic. With the help of the WS approach the stability analysis appeared to be possible and proved the existence of bistability for some regions of the parameters. The self-consistent approach was applied for nonidentical systems with and without noise. The key idea is to write the equation for a conditional probability density function of the phases at every given value of the distributed parameters and to perform the integration over their distribution. Then the traveling wave solutions can be obtained as functions of non-distributed parameters. As a particular example of systems with a distribution of parameters we took a geometric organization of oscillators with common receiver and emitter and an ensemble of spatially distributed oscillators with a leader-type coupling. This method gives the regions of synchrony and asynchrony for any given parameters.

All the systems mentioned above have one-harmonic (sine) coupling function, but the self-consistent approach can be applied to a coupling function with several harmonics. As an example we have studied an ensemble of noisy phase oscillators with bi-harmonic coupling. The application of the self-consistent theory appeared to be very productive as it showed the existence of an unexpected region of bistability and of a large area of parameters when a solution is represented by a synchronous two-cluster state with zero first order parameter and non-zero second order parameter (so-called "nematic phase"). Also linear stability borders of the asynchronous state with zero order parameters could be obtained analytically with the help of the self-consistent approach. Remarkably, further stability analysis could be performed for the case of the delta distribution of natural frequencies (identical oscillators) in the presence of noise. A limiting case of small noise appeared to be singular because there is a problem in finding a stationary probability density function. This case goes beyond the scope of this work dedicated to the methods of acquiring global description of different systems.

In essence, we have built the theory of synchronization for different systems that together with direct numerical simulation provides significant results in obtaining solutions for different values of the system's parameters.

Acknowledgments

First and foremost I wish to express sincere gratitude to my supervisor, Prof. Dr. Arkady Pikovsky, who has supported me throughout my work with his patience and knowledge. I am also thankful for his careful reading and giving suggestions to the manuscript. Prof. Dr. Michael Rosenblum and Dr. Maxim Komarov are warmly acknowledged for giving me useful advices and suggestions during my work on this thesis.

I am grateful to Prof. Dr. Elbert E. N. Macau, my second supervisor, for his guidance throughout all my work and, specially, for the warm welcome in Brazil. Furthermore, I wish to thank David Hansmann for organizational support.

I acknowledge the IRTG 1740/TRP 2011/50151-0, funded by the DFG /FAPESP for financial support.

Finally, I would like to thank my parents, grandparents and my brother. Also special thanks to Tetiana Tsenova, Kesha and Charlie. They are always supporting me and encouraging me with their best wishes.

A. Watanabe-Strogatz transformation as a Möbius transformation

The connection between the Watanabe-Strogatz transformation and a Möbius transformation was shown in [8], where the authors give mathematically rigorous proof of the equivalence of the time dependent Möbius transformation applied to $e^{\mathrm{i}\varphi_k(t)}$ and phase equation (2.1) for a phase $\varphi_k(t)$. Here we will present less rigorous, but in our opinion quite demonstrative explanation of such equivalence from a different point of view.

As it was shown, the phase equation (2.1) can be rewritten as the equation (2.3) for $e^{\mathrm{i}\varphi_k}$. This equation, at any given moment of time, can be considered as a transformation of the complex variable $w(0) = r(0)e^{\mathrm{i}\varphi(0)}$ with $|w(0)| = r(0) = 1$ to another complex variable $w(t) = r(t)e^{\mathrm{i}\varphi(t)}$ with $|w(t)| = r(t) = 1$. Or in the other words the transformation from the border of the unit disk to the border of the unit disk on the complex plane. The border of the unit disk is subset of the closed unit disk. So this transformation (2.3) can be treated as a part of more general transformation of the unit disk to the unit disk such that a point on the border transforms to another point on the border. Let us consider the following equation:

$$\dot{w} = q_t(w(w_0, t), t) = \mathrm{i}f(t)w + \frac{1}{2}F(t) - \frac{w^2}{2}F^*(t). \tag{A.1}$$

In the integral form the transformation (A.1) can be formally written as

$$w(w_0, t) = Q_t(w_0, t) = w_0 + \int_0^t q_t(w(w_0, t), t)\, dt, \tag{A.2}$$

where it is assumed that the time dependence of $w(w_0, t)$ can be explicitly introduced to $q_t(w(w_0, t), t)$ in order to obtain function $Q_t(w_0, t)$.

First, let us prove that $Q_t(w_0, t)$ is holomorphic on the unit disk. Then by the definition of holomorphic functions, we need to prove that $Q_t(w_0, t)$ is complex differentiable in a neighborhood of every point on the unit disk. If $F(t)$ and $f(t)$ are holomorphic functions then the functions $q_t(w(w_0, t), t)$ and $\dfrac{\partial q_t}{\partial w}$ are holomorphic, thus they are complex differentiable in a neighborhood of every point on the unit disk. In the function

$Q_t(w_0,t)$ the time t is included as a parameter and $\dfrac{\partial Q_t}{\partial t} = q_t$, so we can calculate the derivative of $Q_t(w_0,t)$ over w_0 from the following equation:

$$\frac{\partial}{\partial t}\frac{\partial Q_t}{\partial w_0} = \frac{\partial q_t}{\partial w_0} = \frac{\partial q_t}{\partial w}\frac{\partial w}{\partial w_0} = \frac{\partial q_t}{\partial w}\frac{\partial Q_t}{\partial w_0}. \tag{A.3}$$

The equation (A.3) can be rewritten in the integral form:

$$\frac{\partial Q_t}{\partial w_0} = \exp\int_0^t \frac{\partial q_t}{\partial w}dt = \exp\int_0^t \left[\mathrm{i}f(t) - Q_t(w_0,t)F^*(t)\right]dt, \tag{A.4}$$

where it is taken into account that $Q_0 = Q_t(w_0,0) = w(w_0,0) = w_0$ and thus $\dfrac{\partial Q_0}{\partial w_0} = 1$.

It is well-known that the exponent function is holomorphic on a complex plane. So, we have proved that $Q_t(w_0,t)$ is complex differentiable and thus it is a holomorphic function of w_0 on the same area where the functions $f(t)$ and $F(t)$ are holomorphic, so if they are holomorphic on the unit disk, then $Q_t(w_0,t)$ is holomorphic on the unit disk.

Second, from the Cauchy's integral formula it follows that any holomorphic on the disk function is completely determined by its values on the disk's boundary. This means that the transformation (A.1) is fully determined by the transformation (2.1).

Third, let us show that, if $|w_0| \le 1$ then $|w(t)| = |Q_t(w_0,t)| \le 1$. In order to do that let us return to the polar representation and substitute $w = re^{\mathrm{i}\varphi}$ to the eq. (A.1).

$$\begin{aligned} \dot r &= \frac{1-r^2}{2}\mathrm{Re}(Fe^{\mathrm{i}\varphi}),\\ \dot\varphi &= f - \frac{1+r^2}{2r}\mathrm{Im}(Fe^{\mathrm{i}\varphi}). \end{aligned} \tag{A.5}$$

From the system (A.5) follows that if $r(0) = 1$ then $r(t) = 1$ and if $r(0) < 1$ then $r(t) \le 1$. That means that $|Q_t(w_0,t)| \le 1$ inside the unit disk, and if $|w_0| = 1$ then $|Q_t(w_0,t)| = 1$. Also, it is clear that $Q_t(w_0,t)$ is continuous on the boundary of the unit disk $|w_0| = 1$. Then, from the maximum modulus principle follows that holomorphic function within a bounded domain D, continuous up to the boundary of D, takes its maximum value on the boundary of D, or equal to constant on the domain D. Thus, either $|w| = |Q_t(w_0,t)| = 1$ if and only if $|w_0| = 1$ and $|w| = |Q_t(w_0,t)| < 1$ if and only if $|w_0| < 1$, or $|w| = |Q_t(w_0,t)| = 1$ and $w = Q_t(w_0,t)$ is a constant for any time and for any $|w_0| \le 1$.

Forth, any map defined by the holomorphic function with non-zero derivative is a conformal map. And from the eq. (A.4) it follows that $\dfrac{\partial Q_t}{\partial w_0} \neq 0$ for any w_0 and $t < \infty$. However, if $\mathrm{Re}\,\langle\dfrac{\partial q_t}{\partial w}\rangle_t < 0$ then $\lim\limits_{t\to\infty}\dfrac{\partial Q_t}{\partial w_0} = 0$.

Eventually, from the four properties mentioned above follows that the transformation (A.2) $w = Q_t(w_0, t)$ is an automorphism of the unit disk for any finite moment of time $t < \infty$.

From the theory of functions of a complex variable, it is well-known that any automorphism of the main areas of the complex plane is a fractional linear transformation or a Möbius transformation. And any Möbius transformation is fully characterized by three complex variables. The unit disk is one of the main areas of the complex plane and its automorphism has specific form:

$$w = e^{\mathrm{i}\phi}\frac{b + w_0}{1 + b^* w_0}\,, \tag{A.6}$$

where since we consider time dependent transformation $b = b(t)$ and $\phi = \phi(t)$. The transformation (A.6) has the same form as the Watanabe-Strogatz transformation (2.2) in the case, when $\phi(t) = \Psi(t)$ and $b(t) = z(t)e^{-\mathrm{i}\Psi(t)}$:

$$w = e^{\mathrm{i}\Psi}\frac{ze^{-\mathrm{i}\Psi} + w_0}{1 + z^* e^{\mathrm{i}\Psi} w_0} = \frac{z + w_0 e^{\mathrm{i}\Psi}}{1 + z^* w_0 e^{\mathrm{i}\Psi}}\,, \tag{A.7}$$

where if $w_0 = e^{\mathrm{i}\varphi_0}$ then $w = e^{\mathrm{i}\varphi}$. The transformation (A.7) is the transformation of a value w_0 to $w = w(t)$, such that the latter is a solution of (A.1) with initial condition $w(0) = w_0$, to satisfy that the transformation parameters $z = z(t)$ and $\Psi = \Psi(t)$ should be time dependent variables with initial conditions $z(0) = \Psi(0) = 0$. If $w_0 = 0$ then $w = z$ for any moment of time and it means that the dynamics of the transformation parameter $z = z(t)$ is determined by the equation (A.1) with zero initial condition $z(0) = 0$.

Next we will explain how the WS constant ψ is related to initial variable $\varphi(0)$. Let us introduce constant ψ as an inverse to (A.7) transformation

$$e^{\mathrm{i}\psi} = e^{-\mathrm{i}\Psi_0}\frac{-z_0 + e^{\mathrm{i}\varphi(0)}}{1 - z_0^* e^{\mathrm{i}\varphi(0)}}, \tag{A.8}$$

with some arbitrary z_0 such that $|z_0| < 1$ and Ψ_0. So that $\varphi(0)$ is the result of time-independent transformation (A.7) with the transformation parameters z_0 and Ψ_0 applied to some constant ψ

$$e^{\mathrm{i}\varphi(0)} = \frac{z_0 + e^{\mathrm{i}(\psi+\Psi_0)}}{1 + z_0^* e^{\mathrm{i}(\psi+\Psi_0)}}. \tag{A.9}$$

The transformation (A.7) applied on the border of the unit disk reads

$$e^{\mathrm{i}\varphi} = \frac{z + e^{\mathrm{i}(\varphi(0)+\Psi)}}{1 + z^* e^{\mathrm{i}(\varphi(0)+\Psi)}}. \tag{A.10}$$

Introducing (A.9) into (A.10) we obtain

$$e^{\mathrm{i}\varphi} = \frac{z + e^{\mathrm{i}\Psi}\frac{z_0 + e^{\mathrm{i}(\psi+\Psi_0)}}{1 + z_0^* e^{\mathrm{i}(\psi+\Psi_0)}}}{1 + z^* e^{\mathrm{i}\Psi}\frac{z_0 + e^{\mathrm{i}(\psi+\Psi_0)}}{1 + z_0^* e^{\mathrm{i}(\psi+\Psi_0)}}} = \frac{z + z_0 e^{\mathrm{i}\Psi} + e^{\mathrm{i}\psi}e^{\mathrm{i}(\Psi_0+\Psi)}(1 + z z_0^* e^{-\mathrm{i}\Psi})}{1 + z^* z_0 e^{\mathrm{i}\Psi} + (z^* + z_0^* e^{-\mathrm{i}\Psi})e^{\mathrm{i}\psi}e^{\mathrm{i}(\Psi_0+\Psi)}}. \tag{A.11}$$

Let us denote z_{WS} as the transformation (A.7) applied to z_0

$$z_{WS} = \frac{z + z_0 e^{\mathrm{i}\Psi}}{1 + z^* z_0 e^{\mathrm{i}\Psi}}\,. \tag{A.12}$$

Then (A.11) can be rewritten as

$$e^{\mathrm{i}\varphi} = \frac{z_{WS}(1 + z^* z_0 e^{\mathrm{i}\Psi}) + e^{\mathrm{i}\psi} e^{\mathrm{i}(\Psi_0+\Psi)}(1 + z z_0^* e^{-\mathrm{i}\Psi})}{1 + z^* z_0 e^{\mathrm{i}\Psi} + z_{WS}^* e^{\mathrm{i}\psi} e^{\mathrm{i}(\Psi_0+\Psi)}(1 + z z_0^* e^{-\mathrm{i}\Psi})}. \tag{A.13}$$

By dividing both the numerator and denominator in (A.13) by $1 + z^* z_0 e^{\mathrm{i}\Psi}$ we obtain

$$e^{\mathrm{i}\varphi} = \frac{z_{WS} + e^{\mathrm{i}(\psi+\Psi_{WS})}}{1 + z_{WS}^* e^{\mathrm{i}(\psi+\Psi_{WS})}}, \tag{A.14}$$

where

$$e^{\mathrm{i}\Psi_{WS}} = e^{\mathrm{i}(\Psi_0+\Psi)} \frac{1 + z z_0^* e^{-\mathrm{i}\Psi}}{1 + z^* z_0 e^{\mathrm{i}\Psi}} = e^{\mathrm{i}\Psi_0} \frac{z z_0^* + e^{\mathrm{i}\Psi}}{1 + z^* z_0 e^{\mathrm{i}\Psi}}. \tag{A.15}$$

Finally we have obtained the transformation (A.14) that coincide with WS transformation (2.2). Also from (A.12) follows that the dynamics for $z_{WS} = z_{WS}(t)$ is determined by the equation (A.1) with initial condition $z_{WS}(0) = z_0$.

Note that we have chosen z_0 such that $|z_0| < 1$. The case $z_0 = e^{\mathrm{i}\Phi_0}$ is singular, because after substitution to (A.9) we obtain $e^{\mathrm{i}\varphi(0)} = e^{\mathrm{i}\Phi_0}$ and there is no link between $\varphi(0)$ and ψ. Basically that is exactly what happens when the synchrony in an ensemble of identical oscillators is established. And if initially an ensemble is synchronized it stays synchronized. The condition $|z_0| < 1$ means that we consider non synchronous initial conditions.

B. Obtaining the set of WS constants for a given set of initial conditions

We will start with the formulation of the equations in the spirit of the original approach of Watanabe and Strogatz [6] in our notations [7].

The WS variable transformation (2.2) introduces N constants ψ_k and the complex time-dependent variable $z(t)$ and real $\Psi(t)$. But initially there are N original phases $\varphi_k(t)$. Thus in order to get unique set of constants ψ_k we need to determine constraints how we choose $z(0)$ and $\Psi(0)$.

The simplest case would be to choose $z(0) = \Psi(0) = 0$ and then $\psi_k = \varphi_k(0)$ but it is not convenient for the general analysis. As shown in the Appendix A, by introducing yet arbitrary initial values $z(0) = z_0$ and $\Psi(0) = \Psi_0$ it is always possible to obtain the WS transformation in the form (2.2), where constants ψ_k are determined by inverse WS transform from the initial values of the phases $\varphi_k(0)$

$$e^{\mathrm{i}\psi_k} = e^{-\mathrm{i}\Psi_0}\frac{-z_0 + e^{\mathrm{i}\varphi_k(0)}}{1 - z_0^* e^{\mathrm{i}\varphi_k(0)}}. \tag{B.1}$$

Since the transformation (B.1) is also a Möbius transformation, for each $\varphi_k(0)$ it uniquely determines the only ψ_k for any given z_0 and Ψ_0.

Generally, we can take any z_0 and Ψ_0 but, as it argued in [6], it is more convenient to choose z_0 such that

$$\frac{1}{N}\sum_{k=1}^{N} e^{\mathrm{i}\psi_k} = 0. \tag{B.2}$$

Then the corresponding $z(0)$ can be found by introduction of (B.1) into (B.2)

$$\frac{1}{N}\sum_{k=1}^{N} e^{-\mathrm{i}\Psi_0}\frac{-z_0 + e^{\mathrm{i}\varphi_k(0)}}{1 - z_0^* e^{\mathrm{i}\varphi_k(0)}} = 0. \tag{B.3}$$

Note that $z_0 = 0$ is the solution if and only if $\sum_{k=1}^{N} e^{\mathrm{i}\varphi_k(0)} = 0$. Below we assume that this is not the case and $z_0 \neq 0$ and as mentioned in Appendix A $|z_0| < 1$.

By taking into account the identity

$$\left(1 - z_0^* e^{\mathrm{i}\varphi_k(0)}\right)^{-1} = \sum_{l=0}^{\infty} z_0^{*l} e^{\mathrm{i}l\varphi_k(0)} \tag{B.4}$$

and the fact that $e^{\mathrm{i}\Psi_0} \neq 0$ the expression (B.3) can be rewritten as

$$\begin{aligned}
0 &= \frac{1}{N}\sum_{k=1}^{N} \frac{-z_0 + e^{\mathrm{i}\varphi_k(0)}}{1 - z_0^* e^{\mathrm{i}\varphi_k(0)}} = \sum_{k=1}^{N}\left(-z_0 + e^{\mathrm{i}\varphi_k(0)}\right)\sum_{l=0}^{\infty} z_0^{*l} e^{\mathrm{i}l\varphi_k(0)} = \\
&= \frac{1}{N}\sum_{k=1}^{N}\left(-z_0 - z_0\sum_{l=1}^{\infty} z_0^{*l} e^{\mathrm{i}l\varphi_k(0)} + \frac{z_0}{|z_0|^2}\sum_{l=0}^{\infty} z_0^{*l+1} e^{\mathrm{i}(l+1)\varphi_k(0)}\right) = \\
&= \frac{1}{N}\sum_{k=1}^{N}\left(-z_0 - z_0\left(1 - \frac{1}{|z_0|^2}\right)\sum_{l=1}^{\infty} z_0^{*l} e^{\mathrm{i}l\varphi_k(0)}\right) = \\
&= \frac{1}{N}\sum_{k=1}^{N}\left(-z_0 - z_0\left(1 - \frac{1}{|z_0|^2}\right)\left(\sum_{l=0}^{\infty} z_0^{*l} e^{\mathrm{i}l\varphi_k(0)} - 1\right)\right) = \\
&= \frac{z_0}{|z_0|^2}\left(-1 + \left(1 - |z_0|^2\right)\frac{1}{N}\sum_{k=1}^{N}\sum_{l=0}^{\infty} z_0^{*l} e^{\mathrm{i}l\varphi_k(0)}\right).
\end{aligned} \tag{B.5}$$

In general the identity (B.4) can be used again and we obtain

$$\frac{1}{N}\sum_{k=1}^{N} \frac{1}{1 - z_0^* e^{\mathrm{i}\varphi_k(0)}} = \frac{1}{1 - |z_0|^2}, \tag{B.6}$$

where we took into account that we consider the case when $z_0 \neq 0$. But this expression is not much simpler than (B.3). But we can change the order of summation and the expression (B.5) can be rewritten as

$$\frac{z_0}{|z_0|^2}\left(-1 + \left(1 - |z_0|^2\right)\sum_{l=0}^{\infty} z_0^{*l} C_l(0)\right) = 0, \tag{B.7}$$

where

$$C_l(t) = \frac{1}{N}\sum_{k=1}^{N} e^{\mathrm{i}l\varphi_k(t)}. \tag{B.8}$$

In the thermodynamic limit there is a special manifold of initial conditions $\varphi_k(0)$ when the expression (B.7) can be greatly simplified. In the thermodynamic limit the phase can be described as a continuous variable with some time-dependent distribution. If initially this distribution is a Poisson kernel (see [8] for detail analysis) then

$$C_l(0) = C_1^l(0) \tag{B.9}$$

and in this case the expression (B.7) becomes

$$\frac{1}{1 - z_0^* C_1(0)} = \frac{1}{1 - |z_0|^2}. \tag{B.10}$$

What gives the only solution

$$z_0 = z(0) = C_1(0) = \frac{1}{N} \sum_{k=1}^{N} e^{\mathrm{i}\varphi_k(0)}. \tag{B.11}$$

With the calculations similar to (B.5) in the case of initial conditions satisfying (B.9) and (B.11) it is easy to show that

$$\frac{1}{N} \sum_{k=1}^{N} e^{\mathrm{i}n\psi_k} = 0, \tag{B.12}$$

for any $n = 0, 1, 2, ...$ and $N \to \infty$. What gives uniform distribution of constants ψ. And the fact that WS constants ψ have the uniform distribution gives (see expressions (2.13–2.16))

$$C_l(t) = C_1^l(t) = z^l(t). \tag{B.13}$$

What means the well-known fact that Poisson kernel is invariant manifold of Möbius transformation.

The third constraint to obtain $\Psi(0) = \Psi_0$ is somewhat arbitrary, for example in [6] it was chosen to be $\sum_{k=1}^{N} \psi_k = 0$ and in [7] it was $\sum_{k=1}^{N} \cos 2\psi_k = 0$.

C. Special case when $\cos\delta = -1$ and $B = A$

If $\cos\delta = -1$ and $B = A$ then Eqs. (2.33) transform to

$$\begin{aligned} \dot{\rho} &= A\frac{1-\rho^2}{2}\mathrm{Re}(e^{\mathrm{i}\Delta\Phi}), \\ \dot{\Delta\Phi} &= \Delta\omega - A\frac{1-\rho^2}{2\rho}\,\mathrm{Im}(e^{\mathrm{i}\Delta\Phi}). \end{aligned} \tag{C.1}$$

Thus in this special case, there are no synchronous steady states, only limit cycle with $\rho = 1$ and $\Delta\Phi(t) = \Delta\omega t$. The asynchronous steady states could be found from the equations analogous to Eqs. (2.40), they read

$$\begin{aligned} \Delta\Phi &= \pm\,\pi/2, \\ 0 &= \Delta\omega \mp A\frac{1-\rho^2}{2\rho}. \end{aligned} \tag{C.2}$$

Then the steady asynchronous solutions are

$$z_{a1,2} = \mathrm{sign}(\Delta\omega)\,\mathrm{i}\frac{-|\Delta\omega| \mp \sqrt{\Delta\omega^2 + A^2}}{A}. \tag{C.3}$$

From (C.3) follows that $|z_{a1}| > 1$ and $|z_{a2}| < 1$ if $\Delta\omega \neq 0$. And thus there is only one asynchronous steady solution z_{a2}. After linearization around z_{a2} the following linear system is obtained

$$\begin{aligned} \dot{a} &= -\,\mathrm{sign}(\Delta\omega)\,\sqrt{\Delta\omega^2 + A^2}\;b, \\ \dot{b} &= \mathrm{sign}(\Delta\omega)\,|\Delta\omega|a, \end{aligned} \tag{C.4}$$

where $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z) - \mathrm{Im}(z_{a2})$. Linear system (C.4) has two eigenvalues:

$$\lambda_{a2}^{1,2} = \pm\sqrt{-|\Delta\omega|\sqrt{\Delta\omega^2 + A^2}}\,, \tag{C.5}$$

what means that z_{a2} is neutrally stable as in the general case when $\sin\delta = 0$.

Bibliography

[1] A. Pikovsky, M. Rosenblum and J. Kurths. *Synchronization. A Universal Concept in Nonlinear Sciences.* Cambridge University Press, Cambridge, 2001.

[2] Y. Kuramoto. *Chemical Oscillations, Waves and Turbulence.* Springer, Berlin, 1984.

[3] H. Sakaguchi and Y. Kuramoto. A soluble active rotator model showing phase transition via mutual entrainment. *Prog. Theor. Phys.*, 76(3):576–581, 1986.

[4] J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort and R. Spigler. The Kuramoto model: A simple paradigm for synchronization phenomena. *Rev. Mod. Phys.*, 77(1):137–175, 2005.

[5] S. Watanabe and S. H. Strogatz. Integrability of a globally coupled oscillator array. *Phys. Rev. Lett.*, 70(16):2391–2394, 1993.

[6] S. Watanabe and S. H. Strogatz. Constants of motion for superconducting Josephson arrays. *Physica D*, 74:197–253, 1994.

[7] A. Pikovsky and M. Rosenblum. Dynamics of heterogeneous oscillator ensembles in terms of collective variables. *Physica D*, 240:872–881, 2011.

[8] S. A. Marvel, R. E. Mirollo and S. H. Strogatz. Phase oscillators with global sinusoidal coupling evolve by Mobius group action. *Chaos*, 19:043104., 2009.

[9] V. Vlasov and A. Pikovsky. Synchronization of a Josephson junction array in terms of global variables. *Phys. Rev. E*, 88:022908, 2013.

[10] K. Wiesenfeld and J. W. Swift. Averaged equations for Josephson junction series arrays. *Phys. Rev. E*, 51(2):1020–1025, 1995.

[11] K. Wiesenfeld, P. Colet and S. H. Strogatz. Synchronization transition in a disordered Josephson series array. *Phys. Rev. Lett.*, 76(3):404–407, 1996.

[12] K. Wiesenfeld, P. Colet and S. H. Strogatz. Frequency locking in Josephson arrays: Connection with the Kuramoto model. *Physical Review E*, 57(2):1563–1569, February 1998.

[13] M. Rosenblum and A. Pikovsky. Self-organized quasiperiodicity in oscillator ensembles with global nonlinear coupling. *Phys. Rev. Lett.*, 98:064101, 2007.

[14] T. Heath and K. Wiesenfeld. Synchronization transitions in Josephson arrays: a puzzle and its resolution. *Ann. Phys. (Leipzig)*, 9(9-10):689–696, 2000.

[15] E. Ott and Th. M. Antonsen Low dimensional behavior of large systems of globally coupled oscillators *CHAOS*, 18(3):037113, 2008

[16] V. Vlasov, Y. Zou and T. Pereira. Explosive Synchronization is Discontinuous. *Phys. Rev. E*, 92:012904, 2015

[17] M. Komarov and A. Pikovsky. Multiplicity of singular synchronous states in the Kuramoto model of coupled oscillators. *Phys. Rev. Lett.*, 111:204101, 2013.

[18] M. Komarov and A. Pikovsky. The Kuramoto model of coupled oscillators with a bi-harmonic coupling function. *Physica D*, 289:18–31, 2014.

[19] V. Vlasov, E. E. N. Macau and A. Pikovsky. Synchronization of oscillators in a Kuramoto-type model with generic coupling. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 24(2):023120, 2014.

[20] E. Montbrió and D. Pazó. Shear diversity prevents collective synchronization. *Phys. Rev. Lett.*, 106:254101, 2011.

[21] E. Montbrió and D. Pazó. Collective synchronization in the presence of reactive coupling and shear diversity. *Phys. Rev. E*, 84:046206, 2011.

[22] H. Hong and S. H. Strogatz. Kuramoto model of coupled oscillators with positive and negative coupling parameters: An example of conformist and contrarian oscillators. *Phys. Rev. Lett.*, 106:054102, Feb 2011.

[23] G. H. Paissan and D. H. Zanette. Synchronization of phase oscillators with heterogeneous coupling: A solvable case. *Physica D*, 237(6):818 – 828, 2008.

[24] D. Iatsenko, P. V. E. McClintock and A. Stefanovska. Glassy states and superrelaxation in populations of coupled phase oscillators. Nature Comm. 5, 4118

[25] V. Vlasov, M. Komarov and A. Pikovsky. Synchronization transitions in ensembles of noisy oscillators with bi-harmonic coupling. *J. Phys. A: Mathematical and Theoretical*, 48(10):105101, 2015.

[26] H. Daido. Onset of cooperative entrainment in limit-cycle oscillators with uniform all-to-all interactions: Bifurcation of the order function. *Physica D*, 91:24–66, 1996.

[27] J. D. Crawford. Scaling and singularities in the entrainment of globally coupled oscillators. *Phys. Rev. Lett.*, 74(21):4341–4344, 1995.

[28] J. D. Crawford and K. T. R. Davies. Synchronization of globally coupled phase oscillators: Singularities and scaling for general couplings. *Physica D*, 125(1-2):1–46, 1999.