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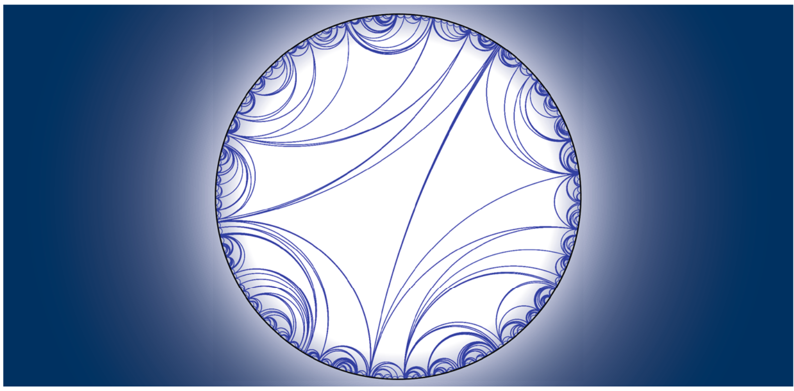

Mark Elin | David Shoikhet | Nikolai Tarkhanov

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ANALYTIC SEMIGROUPS OF HOLOMORPHIC MAPPINGS AND COMPOSITION OPERATORS

MARK ELIN, DAVID SHOIKHET, AND NIKOLAI TARKHANOV

Abstract. In this paper we study the problem of analytic extension in parameter for a semigroup of holomorphic self-mappings of the unit ball in a complex Banach space and its relation to the linear continuous semigroup of composition operators. We also provide a brief review around this topic.

Contents

Introduction	1
1. One-parameter semigroups of holomorphic self-mappings	2
1.1. Analytic semigroups	2
1.2. The one-dimesional case	4
1.3. General Banach spaces	6
1.4. The Schwarz-Pick type estimates	7
2. Semigroups of composition operators	9
2.1. Analytic extension of nonlinear resolvents and semigroups	9
2.2. C_0-semigroups	18
2.3. Composition operators and semigroups	19
2.4. Analytic semigroups of composition operators	23
References	28

Introduction

For a continuous semigroup of bounded linear operators on a complex Banach space the problem of analytic continuation in the parameter goes back to the pioneer works [HP57] and [Yos65]. In these works some criteria of analytic continuation were established along with estimates of those sector in the complex plane to which the analytic continuation is possible. This study was developed by many mathematician (see for instance [BB67, Paz79, Kan10]). The recent paper [ACP15] presents

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also specific criteria for analytic continuation of a continuous semigroup of composition operators.

On the other hand, the linear semigroup of composition operators is induced by a nonlinear semigroup of holomorphic self-mappings of a domain in an underlying complex space. The relations between those semigroups and the topological structure of their domains play a crucial role in the study of analytic continuation.

Although the nonlinear semigroup theory of holomorphic self-mappings of a bounded convex domain in a complex Banach space was developed very intensively in the last thirty years, a little has been known of analytic continuation of such semigroups in a complex parameter. This problem is very well motivated by diverse applications in geometric function theory, such as the radii problem for starlike and spirallike mappings, Bloch constants for locally biholomorphic mappings, etc., and the continuous Newton method for solving algebraical or more general nonlinear equations, see [BLRS15]. Specifically for a generated nonlinear semigroup of holomorphic mappings there arises an additional problem of studying local analytic continuation in a complex parameter. More precisely, if the semigroup generator has a singular point inside the domain, then there exists a family of subdomains which are invariant under semigroup action, e.g., hyperbolic balls around the critical point. It turns out that although there are semigroups which have no analytical continuation in parameter, for an invariant subdomain, all semigroups can be continued analytically into a sector in the complex plane with vertex at the origin. The paper is aimed in studying this phenomenon.

It should also be mentioned that our approach is based mostly on analytic continuation in parameter of the so-called nonlinear resolvent function which is of independent interest by itself. We show that the existence domain of the resolvent which preserves some subdomain is much wider than the domain of analyticity of the semigroup induced by using the nonlinear exponential formula.

Returning to linear continuous semigroups of composition operators, we apply our results to study the problem of analytic continuation for suitable Banach spaces of holomorphic self-mappings on shrinking invariant subdomains.

1. One-parameter semigroups of holomorphic self-mappings

1.1. **Analytic semigroups.** Let D be a domain in a complex Banach space X. A mapping $f : D \to X$ is said to be holomorphic in D if the

Frechét derivative $f'(x)$ exists at each point $x \in D$ as a bounded linear operator on X.

The set of all holomorphic mappings of D to X whose values belong to a set $V \subset X$ is denoted by $\mathrm{Hol}(D, V)$. For $V = D$ we will write it simply $\mathrm{Hol}(D)$. Note that $\mathrm{Hol}(D)$ is a unital algebra with respect to the composition operation.

Definition 1.1. A family $\mathcal{S} = \{F(t)\}_{t\geq 0}$ in $\mathrm{Hol}(D)$ is called a one-parameter semigroup on D if

1) $F(t+s) = F(t) \circ F(s)$ for all $s, t \geq 0$;
2) $F(0) = I$, where $I(x) := x$ for each $x \in D$.

Such a semigroup is said to be continuous at the point $t = 0$ if

$$\lim_{t\to 0+} F(t) = I$$

pointwise in D. If this limit is achieved uniformly in a neighbourhood of each point of D, then one says that the semigroup is locally uniformly continuous in D.

Given a family $\mathcal{S} = \{F(t)\}_{t\in\mathbb{R}}$ in $\mathrm{Hol}(D)$ satisfying both 1) and 2) for all real s and t, one sees easily that each F_t is a biholomorphic mapping (automorphism) of D whose inverse is F_{-t}. In this case $\mathcal{S}$ bears actually the structure of a continuous one-parameter group of automorphisms of D.

From the composition rule it follows immediately that if a semigroup is continuous at $t = 0$ then it is continuous at each point $t > 0$, i.e., $F(t+s) \to F(s)$ pointwise in D, as $t \to 0+$. Moreover, it is known that if D is a bounded convex domain in X and $\{F(t)\}_{t\geq 0}$ a locally uniformly continuous semigroup in $\mathrm{Hol}(D)$ then it is also differentiable in $t \geq 0$ and satisfies the Cauchy problem

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}\,(F(t)x) & = & f(F(t)x), \\ F(0) & = & I, \end{array} \right. \tag{1.1}$$

where

$$f(x) = \lim_{t\to 0+} \frac{1}{t}\,(F(t) - I)x$$

exists and belongs to $\mathrm{Hol}(D, X)$ (see [BP78] for the one-dimensional case and [Aba92], [RS96, RS05] for the case of arbitrary Banach spaces). This mapping $f : D \to X$ is called the (infinitesimal) generator of the semigroup.

Denote by $\mathcal{G}(D)$ the set of all holomorphic semigroup generators on D. If D is a convex domain, the set $\mathcal{G}(D)$ is a real cone in $\mathrm{Hol}(D, X)$, while the set of all holomorphic group generators on D is a real Banach algebra (see [RS96]).

Various characterizations and parametric representation of the class $\mathcal{G}(D)$ are established in [BP78, Aba92, AERS99] and [RS97a] (see also the results presented below).

However, even continuous semigroups of bounded linear operators in X need not be locally uniformly continuous on X. Hence, they may fail to be differentiable at each point of X. In particular, this is mostly the case for the one-parameter semigroups of composition operators to be considered below.

Definition 1.2. A set $\mathcal{S} = \{F(t)\}_{t\in\Lambda}$ indexed by a parameter t in a (nonempty) sector $\Lambda = \{|\arg t| < \alpha\}$ of the complex plane is said to be a one-parameter analytic semigroup if the composition rule holds for all $s, t \in \Lambda$ and the mapping $(x, t) \mapsto F(t)x$ is jointly holomorphic in $D \times \Lambda$.

For a family $\{F(t)\}$ of nonlinear mappings of the domain D parametrised by real or complex parameter $t \in \Lambda$, we set

$$F_t(x) := F(t)x$$

whenever $(x, t) \in D \times \Lambda$.

1.2. **The one-dimesional case.** First we consider the one-dimensional case and quote a familiar formula of [BP78] which gave a great push in the development of semigroup theory [Aba92, Gor93, RS96, CD05, CDP06, ES10], Loewner's evolution equation theory [BCD10] and the theory of semigroups of composition operators. Write $\mathbb{D}$ for the unit (open) disk in $\mathbb{C}$.

A natural question is whether, given a holomorphic function f in $\mathbb{D}$, there are conditions which insure that $f \in \mathcal{G}(\mathbb{D})$. The paper [RS96] provided a simple criterion for a special case. Namely, if f has a continuous extension to the closure of $\mathbb{D}$, then f is a semigroup generator in $\mathbb{D}$ if and only if $\Re(f(z)\overline{z}) \le 0$ for all $z \in \partial\mathbb{D}$. However, there are holomorphic functions on the disk which have no continuous extension to $\overline{\mathbb{D}}$.

Theorem 1.3. *For a holomorphic function f in $\mathbb{D}$, the following are equivalent:*

1) $f \in \mathcal{G}(\mathbb{D})$.

2) There exists a unique point $a \in \overline{\mathbb{D}}$ such that

$$f(z) = (z - a)(\overline{a}z - 1)p(z), \tag{1.2}$$

where p is a holomorphic function in $\mathbb{D}$ with nonnegative real part.

3) f admits the representation

$$f(z) = c - \overline{c}z^2 - zq(z), \tag{1.3}$$

where $c \in \mathbb{C}$ and q is a holomorphic function in $\mathbb{D}$ with nonnegative real part. Moreover, f generates a group of automorphisms of $\mathbb{D}$ if and only if $\Re q(z) \equiv 0$.

As mentioned, formula (1.2) is due to [BP78]. Representation (1.3) was established in [AERS99].

The equivalence of 1) and 3) implies that the set $\mathcal{G}(\mathbb{D})$ is a real cone, i.e., it survives under multiplication by nonnegative real numbers. Furthermore, any group generator in D is actually a polynomial of order at most 2 which has the form $f(z) = c - \overline{c}z^2 - \imath rz$ with complex c and real r.

The following assertion combines and generalizes those criteria which are established in [Aba92, ARS99] and [RS97a].

Theorem 1.4. *Suppose that f is a holomorphic function in the disk $\mathbb{D}$. Then $f \in \mathcal{G}(\mathbb{D})$ if and only if there is a constant $\lambda \in [0,1]$ with the property that*

$$\Re(f(z)\overline{z}) \leq (1 - |z|^2) \left(\lambda \Re(f(0)\overline{z}) - (1-\lambda)\frac{1}{2}\Re f'(z)\right)$$

for all $z \in \mathbb{D}$.

To prove the theorem we use the following result of [AERS99].

Lemma 1.5. *Let q be a holomorphic function in $\mathbb{D}$. Then $\Re q(z) \geq 0$ if and only if there is a nonnegative function χ on $[0,1)$ satisfying*

$$\Re\left(zq'(z) + \chi(|z|)q(z)\right) \geq 0$$

for all $z \in \mathbb{D}$.

Proof of Theorem 1.4. Write f in the form $f(z) = c - \overline{c}z^2 - zq(z)$ for $z \in \mathbb{D}$, where q is a holomorphic function in $\mathbb{D}$. By Theorem 1.3, we shall have established the lemma if we prove that the inequality in our assertion just amount to saying that $\Re q(z) \geq 0$. To this end, we note that it is equivalent to

$$\Re\left((1-\lambda)(1-|z|^2)\, zq'(z)\right) + \Re q(z)\left((1-\lambda)(1-|z|^2) + |z|^2\right) \geq 0$$

for some $\lambda \in [0,1]$. If $\lambda = 1$, then the required inequality $\Re q(z) \geq 0$ follows immediately. If $\lambda < 1$, then we obtain the same inequality by setting

$$\chi(m) = \frac{m^2}{(1-\lambda)(1-m^2)} + 1$$

in Lemma 1.5. $\square$

1.3. **General Banach spaces.** Appropriate nonlinear analogues of the familiar Lumer-Phillips and Hille-Yosida theorems for holomorphic mappings were first studied in [HRS00]. This paper shows also several applications.

In this section we assume that D is a bounded convex domain in a complex Banach space X, such that $0 \in D$. Let X^* stand for the dual space of X. For a boundary point $x \in \partial D$, we denote by $s(x)$ the set of all linear functionals l on X which are tangent to D at x. In other words, $s(x)$ consists of those $l \in X^*$ which satisfy $l(x) = 1$ and $\Re l(y) \leq 1$ for all $y \in D$.

The concept of a numerical range was first introduced in [Har71] and developed later in [HRS00].

Definition 1.6. Let f be a continuous mapping of $\overline{D}$ to X. By the numerical range of f with respect to D is meant the subset of $\mathbb{C}$ given by

$$V(f, D) = \{l(f(x)) : \, l \in s(x), \, x \in \partial D\}.$$

By the Hahn-Banach theorem, for each $x \in X$ there is a functional $l_x \in X^*$ with the property that $l_x(x) = \|l_x\| \, \|x\|$. On normalising l_x one obtains a functional whose norm is $\|x\|$. Write x^* for any functional $l \in X^*$ satisfying $\Re l(x) = \|x\|^2 = \|l\|^2$. Such a functional x^* is in general not unique. However, if X is a Hilbert space, then the element x^* is unique and it can be identified with x, which is due to the Riesz theorem.

Roughly speaking, by the dissipative mappings $f : \overline{D} \to X$ are meant those which satisfy $\Re \langle f(x), x^* \rangle \leq 0$ for all $x \in \partial D$.

Definition 1.7. Let $f : B \to X$ be a continuous mapping of an open ball with centre at the origin in X. We say that f is dissipative on B if

$$\limsup_{r \to 1-} \Big(\sup_{\lambda \in V(f_r, B)} \Re \lambda \Big) \leq 0,$$

where $f_r(x) := f(rx)$ for $x \in \overline{B}$ and $0 \leq r < 1$.

In view of their numerous applications, dissipative mappings constitute an important class of mappings in complex Banach spaces. We are now in a position to formulate nonlinear versions of the theorems mentioned at the beginning. For more details we refer the reader to [RS05] (see also [RS97a]).

Theorem 1.8. *Assume that D is a bounded convex domain in a complex Banach space X, and let $f : D \to X$ be a holomorphic mapping. Then there exists a real number λ_0 with the property that the mapping*

$f-\lambda_0 I$ is dissipative on D if and only if, for each $\lambda > \lambda_0$, the resolvent equation $(\lambda I - f)(x) = (\lambda - \lambda_0)y$ has a unique solution $x \in D$ for all $y \in D$.

Thus, the inverse $R(\lambda, f) := (\lambda I - f)^{-1}\,(\lambda - \lambda_0)I$ is a well-defined self-mapping of D for each $\lambda > \lambda_0$. We call it the associate resolvent mapping for $f - \lambda_0 I$. The factor $(\lambda - \lambda_0)I$ is really required, for otherwise the associate resolvent mapping fails to be a self-mapping of D. The latter property is needed to define its iterates. In the case of linear operators λ can be placed before the operator, but not for nonlinear mappings f.

For $n = 1, 2, \ldots$, we write $R(\lambda, f)^n$ for the n-fold iterate of the self-mapping $R(\lambda, f)$ of D.

Theorem 1.9. *Suppose D is a bounded convex domain in a complex Banach space X, and let $f : D \to X$ be a holomorphic mapping on D. Then, for f to be dissipative on D, it is necessary and sufficient that its associate resolvent mapping $R(\lambda, f) = (\lambda I - f)^{-1}\,\lambda I$ would exist for all $\lambda > 0$ and be a holomorphic self-mapping of D (we take $\lambda_0 = 0$). Moreover, if $\mathcal{S} = \{F_t\}_{t \geq 0}$ is the semigroup generated by f, then the exponential formula*

$$F_t(x) = \lim_{n\to\infty} R\Big(\frac{n}{t}, f\Big)^n (x)$$

holds for all $x \in D$, where the limit is achieved uniformly in a neighbourhood of each point $x \in D$.

From Theorems 1.8 and 1.9 we conclude immediately that for holomorphic mappings of an open ball with centre at the origin in X the concepts of a generator and a dissipative mapping are actually the same.

1.4. **The Schwarz-Pick type estimates.** Let $\mathbb{B}$ be the open unit ball with centre at the origin in a complex Banach space X, and let f be the generator of a one-parameter continuous semigroup $\mathcal{S} = \{F_t\}_{t\geq 0}$ on $\mathbb{B}$. Suppose that $a \in \mathbb{B}$ is a null point of f, and assume that the derivative $A := f'(a)$ is strongly dissipative in the sense that $\Re\langle Ax, x^*\rangle \leq k\,\|x\|^2$ for all $x \in X$, with a negative constant k. It follows that the spectrum of the linear operator A lies strictly inside the left half-plane. In this case, a is an attractive fixed point of the semigroup $\mathcal{S}$ and the linear semigroup

$$F_t'(a) = \exp(tA)$$

converges to 0 uniformly on bounded subsets of X, as $t \to \infty$ (see [Sho01] and [RS96]). If assuming $f(0) = 0$, one establishes, by the familiar Schwarz lemma, the invariance condition $\|F_t(x)\| \leq \|x\|$ for all $x \in \mathbb{B}$ and $t \geq 0$. The fact that $\mathcal{S}$ is a continuous semigroup generated by f leads actually to a more qualified estimate (see for instance [Sho01]).

Theorem 1.10. *Let $f \in \mathcal{G}(\mathbb{B})$ and $\mathcal{S} = \{F_t\}_{t\geq 0}$ be the semigroup generated by f. Assume that $f(0) = 0$ and*

$$k = \sup_{\|x\|=1} \Re\langle Ax, x^*\rangle \leq 0.$$

Then there is $c \in [0,1]$, such that

$$\begin{array}{rcccl} \exp\left(kt\,\dfrac{1+c\|x\|}{1-c\|x\|}\right)\|x\| & \leq & \|F_t(x)\| & \leq & \exp\left(kt\,\dfrac{1-c\|x\|}{1+c\|x\|}\right)\|x\|, \\ \exp(kt)\,\dfrac{\|x\|}{(1+c\|x\|)^2} & \leq & \dfrac{\|F_t(x)\|}{(1-c\|F_t(x)\|)^2} & \leq & \exp(kt)\,\dfrac{\|x\|}{(1-c\|x\|)^2} \end{array} \tag{1.4}$$

for all $x \in \mathbb{B}$ and $t \geq 0$.

Theorem 1.10 refines not only the upper bound estimate for $\|F_t(x)\|$ in $\mathbb{B}$ but also the lower bound estimate. In fact, the inequlities in the first line of (1.4) imply that, for each $x \in \mathbb{B}$, the rate of convergence of the semigroup to its interior Denjoy-Wolff point is exponential. The estimates in the first line with $c = 1$ are established in [Gur75] and those in the second line in [Por91].

The following consequence of Theorem 1.10 is useful in the theory of starlike and spirallike mappings of complex Banach spaces (see for instance [Por91, Gur75, ERS04, GK03]).

Theorem 1.11. *Suppose that $f \in \mathcal{G}(\mathbb{B})$ and $\mathcal{S} = \{F_t\}_{t\geq 0}$ is the semigroup in $\mathrm{Hol}(\mathbb{B})$ generated by f. If $f(0) = 0$ and $f'(0) = -I$, then the limit*

$$h(x) := \lim_{t\to\infty} e^t F_t(x)$$

is achieved uniformly in a neighbourhood of each point $x \in \mathbb{B}$ and it satisfies the so-called Shröder functional equation

$$h(F_t(x)) = e^{-t} h(x). \tag{1.5}$$

The Shröder equation shows readily that the mapping h is starlike with respect to the origin. For a discrete type semigroup this mapping was introduced in [Kœn84], see also [ES10] and references therein. Nowadays it plays a crucial role in the study of asymptotic behavior of semigroups and applies to various problems of geometric function theory. This mapping is often called the Kœnigs function associated with the semigroup $\mathcal{S}$.

2. Semigroups of composition operators

2.1. **Analytic extension of nonlinear resolvents and semigroups.**

Our approach to analytic extension of semigroups of holomorphic self-mappings is based on the following two results.

Theorem 2.1. *Let $\mathcal{S} = \{F_t\}_{t\geq 0}$ be a semigroup of holomorphic self-mappings of $\mathbb{B}$ generated by $f : \mathbb{B} \to X$ with $f(0) = 0$ and $f'(0) = -I$. Then $\mathcal{S}$ extends holomorphically to a sector $\Lambda = \{|\arg\lambda| < \alpha \leq \pi/2\}$ in $\mathbb{C}$ if and only if, for each φ satisfying $|\varphi| < \alpha$, the mapping $e^{\imath\varphi} f$ is dissipative on $\mathbb{B}$.*

As a matter of fact it is sufficient to verify if the mappings $e^{\pm\imath\alpha} f$ are dissipative on $\mathbb{B}$.

Proof. Assume that, for any φ satisfying $|\varphi| < \alpha$, the holomorphic mapping $e^{\imath\varphi} f$ is dissipative on $\mathbb{B}$. By Theorem 1.9, for each $t \geq 0$ one can define the resolvent

$$R = R(1/t, e^{\imath\varphi} f) = (I - te^{\imath\varphi} f)^{-1}$$

which is a holomorphic self-mapping of $\mathbb{B}$. In other words, for each $\lambda = te^{\imath\varphi}$ in Λ, the equation $x - \lambda f(x) = y$ has a unique solution $x = x(y, \lambda) \in \mathbb{B}$ for any $y \in \mathbb{B}$. One defines $R(y) := x(y, \lambda)$ for $y \in \mathbb{B}$. For each $t > 0$, the fixed point set of the resolvent R is known to coincide with the null point set of the generator $e^{\imath\varphi} f$ (see for instance [RS05]). Therefore, in our situation the value $x(0, \lambda)$ just amounts to zero for all $\lambda \in \Lambda$. On the other hand, the mapping $g(\cdot, y, \lambda) : \mathbb{B} \to X$ defined by $g(x, y, \lambda) = y - (x - \lambda f(x))$ for fixed $(y, \lambda) \in \mathbb{B} \times \Lambda$ is dissipative on $\mathbb{B}$, since

$$\Re\langle g(x, y, \lambda), x^*\rangle = \Re\langle y, x^*\rangle - \|x\|^2 + \Re\langle \lambda f(x), x^*\rangle < 0$$

on each sphere $\|x\| = r$ with $\|y\| < r < 1$. For each $\lambda \in \Lambda$, we get $g(0, 0, \lambda) = 0$ and the operator $g'_x(0, 0, \lambda, 0) = -I + \lambda f'(0) = -(1 + \lambda)I$ is invertible. Hence it follows by a global version of the implicit function theorem for holomorphic generators (see for instance [RS05]) that the (unique) solution $x(y, \lambda) = (I - \lambda f)^{-1}$ of the equation $g(x, y, \lambda) = 0$ is holomorphic in $(y, \lambda) \in \mathbb{B} \times \Lambda$. On applying Theorem 1.9 we deduce that the limit

$$\lim_{n\to\infty} \Big(I - \frac{\lambda}{n} f\Big)^{-n} (y) =: F_\lambda(y)$$

exists for all $(y, \lambda) \in \mathbb{B} \times \Lambda$ and thus defines an operator family which is analytic in $\lambda \in \Lambda$ and satisfies

$$\begin{aligned} \lim_{t\to 0+} F_{te^{\imath\varphi}}(y) &= y, \\ F_{(t+s)e^{\imath\varphi}} &= F_{te^{\imath\varphi}} \circ F_{se^{\imath\varphi}} \end{aligned}$$

for all $y \in \mathbb{B}$ and φ with $|\varphi| < \alpha$. The last equality means that F_λ preserves the semigroup property on any ray $\lambda = te^{\imath\varphi}$ in Λ. Since a ray is a uniqueness set for analytic functions we conclude readily that $F_{\lambda+\kappa} = F_\lambda \circ F_\kappa$ is actually valid for all κ and λ in Λ. Moreover, if $|\varphi| < \alpha$, then

$$e^{\imath\varphi} f(x) = \lim_{t\to 0+} \frac{F_{te^{\imath\varphi}}(x) - x}{t}$$

or, what is the same,

$$f(x) = \lim_{\lambda\to 0} \frac{F_\lambda(x) - x}{\lambda},$$

λ converging to zero along the ray $\lambda = te^{\imath\varphi}$. The converse considerations by using the last two formulas complete the proof. □

To formulate our next result we need the definition of a starlike mapping. For mappings of the complex plane it was first introduced in [Sta66, BK69].

A biholomorphic mapping $h \in \mathrm{Hol}(\mathbb{B}, X)$ satisfying $h(0) = 0$ and $h'(0) = I$ is said to be strongly starlike of order α, where $0 \le \alpha < 1$, if it fulfils

$$|\arg\langle (h'(x))^{-1} h(x), x^*\rangle| \le (1-\alpha)\,\frac{\pi}{2} \tag{2.1}$$

for all $x \in \mathbb{B} \setminus \{0\}$.

Theorem 2.2. *Let $\mathcal{S} = \{F_t\}_{t\ge 0}$ be a semigroup of holomorphic self-mappings of $\mathbb{B}$ generated by a mapping $f : \mathbb{B} \to X$ satisfying $f(0) = 0$ and $f'(0) = -I$. Then $\mathcal{S}$ can be holomorphically continued to a sector $\Lambda = \{|\arg\lambda| < \alpha(\pi/2)\}$ in the complex plane with $0 < \alpha < 1$ if and only if the Kœnigs function h associated with $\mathcal{S}$ is strongly starlike of order α.*

Proof. Differentiating (1.5) at $t = 0$, one sees that h and f are related by the equation

$$h'(x) f(x) = -h(x).$$

So we get by (2.1)

$$|\arg\langle -f(x), x^*\rangle| \le (1-\alpha)\,\frac{\pi}{2}$$

whence

$$|\arg\langle -e^{\imath\varphi} f(x), x^*\rangle| = |\varphi + \arg\langle -f(x), x^*\rangle| \le \frac{\pi}{2}.$$

Therefore, $\Re\langle e^{\imath\varphi} f(x), x^*\rangle \le 0$, which means that the mapping $e^{\imath\varphi} f$ is dissipative. The converse arguments along with Theorem 2.1 complete our proof. □

By using the same techniques as in [AERS99] one proves the following simple assertion.

Theorem 2.3. *Let $f : \mathbb{B} \to X$ be a holomorphic mapping. Then f is dissipative on $\mathbb{B}$ if and only if, for any point $x \in \partial\mathbb{B}$, the holomorphic mapping $g : \mathbb{D} \to \mathbb{C}$ given by $g(z) = \langle f(zx), x^* \rangle$ is dissipative on the disk $\mathbb{D}$.*

In the case of strongly convex domains in $\mathbb{C}^n$, Theorem 2.3 holds for the restriction to any complex geodesic (see Proposition 4.5 in [BCD10]).

Lemma 2.4. *Suppose $f : \mathbb{B} \to X$ is a dissipative holomorphic mapping satisfying $f(0) = 0$ and $f'(0) = -I$. Then, for each $\varphi \in [0, 2\pi)$, it follows that*

$$\cos\varphi\, r^2 \frac{1+r^2}{1-r^2} - \frac{2r^3}{1-r^2} \le \Re\langle -e^{\imath\varphi} f(x), x^* \rangle \le \cos\varphi\, r^2 \frac{1+r^2}{1-r^2} + \frac{2r^3}{1-r^2}$$

whenever $\|x\| = r < 1$.

Notice that these estimates generalise in certain sense the classical Harnack inequalities.

Proof. Fix $x' \in X$ with $\|x'\| = 1$ and define a holomorphic function p on $\mathbb{D}$ by

$$p(z) = -\frac{1}{|z|^2} \langle f(zx'), (zx')^* \rangle$$

for $z \in \mathbb{D}$. Since p is a Carathéodory's function with $p(0) = 1$ and $\Re p(z) \ge 0$ for $z \in \mathbb{D}$, it follows from the Schwarz lemma applied to the function

$$z \mapsto \frac{p(z) - 1}{p(z) + 1},$$

which maps $\mathbb{D}$ into $\mathbb{D}$, that $\left| \frac{p(z) - 1}{p(z) + 1} \right|^2 \le |z|^2$. Hence,

$$|p(z)|^2 - 2\Re p(z) + 1 \le |z|^2 \left(|p(z)|^2 + 2\Re p(z) + 1 \right)$$

or

$$|p(z)|^2 (1 - |z|^2) - 2\Re p(z)\, (1 + |z|^2) \le |z|^2 - 1,$$

implying

$$|p(z)|^2 - 2\Re p(z)\, \frac{1 + |z|^2}{1 - |z|^2} \le -1$$

and

$$\left| p(z) - \frac{1 + |z|^2}{1 - |z|^2} \right|^2 \le \left(\frac{2|z|}{1 - |z|^2} \right)^2.$$

Given any $\varphi \in [0, 2\pi)$, we thus obtain

$$\Big| e^{\imath\varphi} p(z) - e^{\imath\varphi}\, \frac{1+|z|^2}{1-|z|^2} \Big| \le \frac{2|z|}{1-|z|^2},$$

showing that

$$-\frac{2|z|}{1-|z|^2} + \cos\varphi\, \frac{1+|z|^2}{1-|z|^2} \le \Re e^{\imath\varphi} p(z) \le \frac{2|z|}{1-|z|^2} + \cos\varphi\, \frac{1+|z|^2}{1-|z|^2}.$$

If $x \in \mathbb{B}$ is arbitrary, then we write $x = zx'$ with $|z| = \|x\| = r$ to get

$$\Big| \langle -e^{\imath\varphi} f(x), x^* \rangle - e^{\imath\varphi}\, r^2 \frac{1+r^2}{1-r^2} \Big| \le \frac{2r^3}{1-r^2},$$

as desired. □

We now consider the following problem. Let $f : \mathbb{B} \to X$ be a dissipative holomorphic mapping of the ball $\mathbb{B}$ with $f(0) = 0$ and $f'(0) = -I$. For a given $r \in (0, 1)$, find the set of all complex numbers λ, such that the associated resolvent $(\lambda I - f)^{-1}(\lambda I)$ is a well-defined holomorphic self-mapping of B_r. (Here, B_r stands the open ball with centre 0 and radius r in X.)

To this end, fix $y \in B_r$ and consider the resolvent equation

$$\lambda x - f(x) = \lambda y.$$

This equation has a unique solution in B_r if and only if the mapping g defined by $g(x) = \lambda y - (\lambda x - f(x))$ has a unique null point in B_r. (By abuse of notation, we write $g(x)$ instead of $g(x, y, \lambda)$.) By [HRS00], the latter condition is satisfied if there is $\varphi \in [0, 2\pi)$ depending on λ, such that the inequality

$$\Re \langle e^{\imath\varphi} g(x), x^* \rangle < 0$$

holds whenever $\|x\| = \|x^*\| = r$.

We compute

$$\begin{aligned}
\Re \langle e^{\imath\varphi} g(x), x^* \rangle &= \Re \langle e^{\imath\varphi} (\lambda y - \lambda x + f(x)), x^* \rangle \\
&= \Re \langle e^{\imath\varphi} \lambda y, x^* \rangle - \Re \langle e^{\imath\varphi} \lambda x, x^* \rangle + \Re \langle e^{\imath\varphi} f(x), x^* \rangle \\
&\le |\lambda \langle y, x^* \rangle| - \Re \langle e^{\imath\varphi} \lambda x, x^* \rangle + \Re \langle e^{\imath\varphi} f(x), x^* \rangle.
\end{aligned}$$

Using Lemma 2.4 yields

$$\Re \langle e^{\imath\varphi} g(x), x^* \rangle \le r^2 \Big(|\lambda| - \Re (e^{\imath\varphi} \lambda) - \cos\varphi\, \frac{1+r^2}{1-r^2} + \frac{2r}{1-r^2} \Big).$$

So, the inequality $\Re \langle e^{\imath\varphi} g(x), x^* \rangle < 0$ is fulfilled provided that

$$\cos\varphi\, \Re\lambda - \sin\varphi\, \Im\lambda + \cos\varphi\, \frac{1+r^2}{1-r^2} > |\lambda| + \frac{2r}{1-r^2}. \tag{2.2}$$

Set

$$\begin{aligned} a &:= \Re\lambda + \frac{1+r^2}{1-r^2}, \\ b &:= \Im\lambda. \end{aligned}$$

Now, for some fixed $\lambda \in \mathbb{C}$, we wish to verify whether there is φ, such that the inequality (2.2) holds. Since

$$a\cos\varphi - b\sin\varphi = \sqrt{a^2+b^2}\Big(\frac{a}{\sqrt{a^2+b^2}}\cos\varphi - \frac{b}{\sqrt{a^2+b^2}}\sin\varphi\Big),$$

then, on denoting

$$\begin{aligned} \cos\alpha &:= \frac{a}{\sqrt{a^2+b^2}}, \\ \sin\alpha &:= \frac{b}{\sqrt{a^2+b^2}}, \end{aligned}$$

we get

$$a\cos\varphi - b\sin\varphi = \sqrt{a^2+b^2}\cos(\varphi+\alpha).$$

Choosing $\varphi = -\alpha$ we conclude that the inequality $\Re\langle e^{\imath\varphi}g(x), x^*\rangle < 0$ is fulfilled if

$$\sqrt{a^2+b^2} > |\lambda| + \frac{2r}{1-r^2}.$$

Substituting the formulas for a and b we get

$$2\,\Re\lambda\,\frac{1+r^2}{1-r^2} + 1 > \frac{4r}{1-r^2}\,|\lambda|$$

or

$$\frac{\Big(\Re\lambda + \dfrac{1+r^2}{2(1-r^2)}\Big)^2}{\dfrac{r^2}{(1-r^2)^2}} - \frac{(\Im\lambda)^2}{\dfrac{1}{4}} > 1. \tag{2.3}$$

The domain Ω in the plane of the complex variable λ, which is bounded by the hyperbola (2.3), is illustrated on Fig. 1. Summarising we arrive at the following theorem.

Theorem 2.5. *Let $f : \mathbb{B} \to X$ be a dissipative holomorphic mapping on $\mathbb{B}$ satisfying $f(0) = 0$ and $f'(0) = -I$. Then, for each $r \in (0,1)$ and λ in the domain Ω given by (2.3), the nonlinear associated resolvent $R(\lambda, f) := (\lambda I - f)^{-1}\lambda I$ is a well-defined holomorphic self-mapping of the ball B_r.*

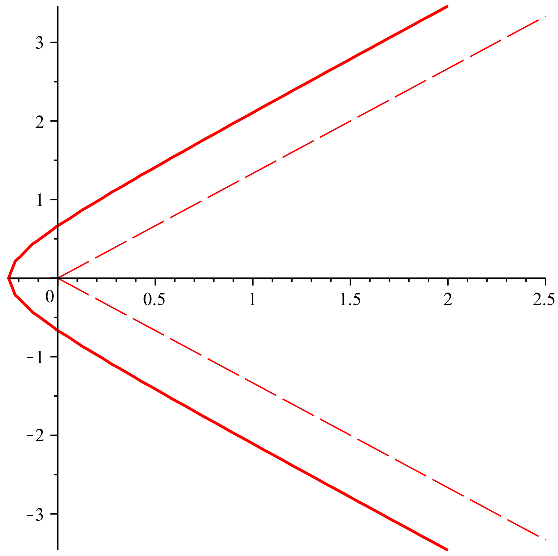

FIG. 1. The domain Ω for $r = 1/3$.

It is easy to see that Ω contains a keyhole domain of the form $\Omega' \cup \Omega''$ where

$$\begin{array}{rcl}\Omega' & = & \Big\{ \lambda \in \mathbb{C} : |\lambda| < \dfrac{1}{2}\, \dfrac{1-r}{1+r} \Big\}, \\ \Omega'' & = & \Big\{ \lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \arcsin \dfrac{1-r^2}{1+r^2} \Big\}\end{array}$$

are a disk around the origin and a sector around the nonnegative semi-axis in the complex plane, respectively.

Theorem 2.6. *Under the hypotheses of Theorem 2.5, the associated resolvent mapping $R(\lambda, f)$ is holomorphic in $\lambda \in \Omega' \cup \Omega''$.*

Proof. The mapping g defined by $g(x, y, \lambda) = \lambda y - (\lambda x - f(x))$ depends holomorphically on $(x, y, \lambda) \in B_r \times B_r \times \Omega$. Since $x = R(\lambda, f)(y)$ is a solution of $g(x, y, \lambda) = 0$, we want to show that the mapping $g(\cdot, y, \lambda)$ is dissipative on B_r, i.e.,

$$\sup_{\|x\|=r} \Re \langle g(x, y, \lambda), x^* \rangle < 0,$$

for each fixed pair $(y, \lambda) \in B_r \times (\Omega' \cup \Omega'')$.

Indeed, if $\lambda \in \Omega'$, then on applying Lemma 2.4 with $\varphi = 0$ we get

$$\begin{aligned}\sup_{\|x\|=r} \Re\langle g(x,y,\lambda), x^*\rangle &\le \sup_{\|x\|=r} |\lambda|\|y-x\|r - r^2\frac{1-r}{1+r}\\ &< \frac{1}{2}\frac{1-r}{1+r}2r^2 - r^2\frac{1-r}{1+r}\\ &= 0.\end{aligned}$$

The point $x = 0$ is a regular point of the mapping $g(x,y,\lambda)$, for $g'_x(0,y,\lambda) = -(\lambda+1)I$. Hence it follows by a global version of the implicit function theorem (see [KRS01] and [RS05]) that $R(\lambda, f)$ is a well defined holomorphic self-mapping of B_r, which depends holomorphically on $(\lambda, y) \in \Omega' \times B_r$.

If $\lambda \in \Omega''$ (and so $\lambda \neq 0$), then we rewrite the equation $g(x,y,\lambda) = 0$ in the form

$$x - \kappa f(x) = y,$$

where $\kappa = \lambda^{-1}$. Consider the mapping $h(x,y,\kappa) = y - x + \kappa f(x)$. Since the sector $\Lambda = \Omega''$ is invariant under the inversion $\lambda \mapsto \lambda^{-1}$, we conclude that the equation $x - \kappa f(x) = y$ has a unique solution $x \in B_r$, for each $y \in B_r$ and $\kappa \in \Omega''$, if the mapping $e^{-\imath\varphi} f$ is dissipative on B_r, where $\varphi = \arg\lambda$. This is the case if

$$\sup_{\|x\|=r} \Re\langle e^{-\imath\varphi}|\kappa| f(x), x^*\rangle \le 0.$$

From Lemma 2.4 it follows that

$$\begin{aligned}\sup_{\|x\|=r} |\kappa|\, \Re\langle e^{-\imath\varphi} f(x), x^*\rangle &\le |\kappa|\, r^2\Big(\frac{2r(1-r\cos\varphi)}{1-r^2} - \cos\varphi\Big)\\ &\le 0,\end{aligned}$$

for

$$|\varphi| = |\arg\lambda| < \arcsin\frac{1-r^2}{1+r^2}.$$

To finish the proof we just notice that, for each $y \in B_r$ and $\kappa \in \Omega''$, the closure of the numerical range of the mapping $h(\cdot, y, \kappa)$ lies in the open right half-plane. Hence it follows that the closure of the numerical range of the linear operator $h'_x(R(1/\kappa, f), y, \kappa)$ and its spectrum lie in the left half-plane, which means that this operator is continuously invertible. By the implicit function theorem, $R(1/\kappa, f)$ is holomorphic in $\kappa \in \Omega''$, as desired. □

Now, by using the exponential formula, Vitali's theorem and the uniqueness property of analytic functions, we get the following assertion.

Corollary 2.7. *Let $f : \mathbb{B} \to X$ be a dissipative holomorphic mapping satisfying $f(0) = 0$ and $f'(0) = -I$. Then, for any $r \in (0,1)$, the semigroup $\{F_t\}_{t\geq 0}$ in $\mathrm{Hol}(B_r)$ generated by f can be analytically continued to the sector*

$$\Lambda = \Big\{\lambda \in \ C \setminus \{0\} : |\arg\lambda| < \arcsin\frac{1-r^2}{1+r^2}\Big\}.$$

Moreover, the family $\{F_\lambda\}_{\lambda\in\Lambda}$ forms a one-parameter analytic semigroup in $\lambda \in \Lambda$ which satisfies

$$\lim_{\substack{\lambda\in\Lambda\\ \lambda\to 0}} F_\lambda(x) = x$$

uniformly in x on each smaller ball $\mathbb{B}_{r'}$ with $r' < r$.

Example 2.8. Let $\mathbb{D}$ be the unit open disk in the complex plane $\mathbb{C}$. Consider the mapping $f \in \mathrm{Hol}(\mathbb{D},\mathbb{C})$ given by $f(z) = zp(z)$, where

$$p(z) = \frac{z+1}{z-1}.$$

Since $\Re f(z)\overline{z} < 0$ for all $z \in \mathbb{D}$, this mapping is dissipative. Solving the Cauchy problem (1.1), we see that the semigroup (with real or complex parameter λ) generated by f satisfies the equation

$$\frac{F_\lambda(z)}{(1-F_\lambda(z))^2} = e^{-\lambda}\frac{z}{(1-z)^2}$$

for all $z \in \mathbb{D}$. The problem reads as follows. Given any $r \in (0,1)$, find a sector $\Lambda := \{\zeta \in \mathbb{C}\setminus\{0\} : |\arg\lambda| < \alpha\}$ depending on r, such that $|F_\lambda(z)| < r$ whenever $|z| < r$ and $\lambda \in \Lambda$. To solve the problem we need computations which seem to be fairly complicated. However, by Corollary 2.7, we get an universal sector Λ with

$$\alpha = \arcsin\frac{1-r^2}{1+r^2}.$$

Moreover, this estimate is sharp, i.e., if $\alpha < \alpha' < \pi/2$, then the semigroup F_λ cannot be analytically continued into the whole sector $\{\lambda \in \mathbb{C}\setminus\{0\} : |\arg\lambda| < \alpha'\}$. Because of Theorem 2.1, to prove this it suffices to show that there is a point z_0 with $|z_0| = r$, such that

$$\max\Re(e^{\pm\imath\alpha'}f(z_0)\overline{z}_0) = r^2\max\Re(e^{\pm\imath\alpha'}p(z_0)) > 0.$$

Since the function p maps the circle $|z| = r$ onto the circle

$$w = -\frac{1+r^2}{1-r^2} + e^{\imath\varphi}\frac{2r}{1-r^2},$$

with $\varphi \in [0, 2\pi)$, and $\cos\alpha' < \dfrac{2r}{1-r^2}$, we get

$$\begin{aligned}\sup_{\varphi\in[0,2\pi)} e^{-\imath\alpha'} w &= -\frac{1+r^2}{1-r^2}\cos\alpha' + \sup_{\varphi\in[0,2\pi)} \frac{2r}{1-r^2}\cos(\varphi-\alpha') \\ &= -\frac{1+r^2}{1-r^2}\cos\alpha' + \frac{2r}{1-r^2} \\ &> 0.\end{aligned}$$

This gives the desired conclusion.

As a matter of fact, on using Theorem 2.2 one can prove the following more general assertion.

Theorem 2.9. *Let $f : \mathbb{B} \to X$ be a dissipative holomorphic mapping with $f(0) = 0$ and $f'(0) = -I$, and let $\{F_t\}_{t\geq 0}$ be the semigroup in* $\mathrm{Hol}(\mathbb{B})$ *generated by f. Assume that the Kœnigs function h associated with the semigroup is strongly starlike of order $\alpha \in [0, 1)$. Then, for each $r \in (0, 1)$, the restriction of $\{F_t\}_{t\geq 0}$ to the ball B_r can be analytically continued to the sector $\Lambda = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| < (\alpha+\beta)\,\pi/2\}$, where*

$$0 \leq \beta \leq \beta(r, \alpha) := \frac{2}{\pi}\,\arcsin\frac{(1-r^2)\cos\left(\alpha\frac{\pi}{2}\right)}{1+2r\sin\left(\alpha\frac{\pi}{2}\right)+r^2}.$$

Proof. Given any $x \in \partial\mathbb{B}$, consider the holomorphic functions on $\mathbb{D}$ defined by

$$g(z) = \langle f(zx), x^*\rangle$$

and $e^{\pm\imath\alpha(\pi/2)}g(z)$. By Theorems 2.3 and 2.2, these three mappings are dissipative on $\mathbb{D}$. Therefore, by Theorem 2.1, it is sufficient to show that, for each $r \in (0, 1)$, the mappings $e^{\pm\imath(\alpha+\beta)\,\pi/2}g(z)$ are still dissipative on $r\mathbb{D}$.

Consider

$$p(z) = e^{\imath\alpha(\pi/2)}\,\frac{g(z)}{z}$$

for $z \in \mathbb{D}$. Since $e^{\imath\alpha(\pi/2)}g(z)$ is dissipative, it follows that $\Re p(z) \leq 0$. Moreover,

$$p(0) = e^{\imath\alpha(\pi/2)}\langle f'(0)x, x^*\rangle = -e^{\imath\alpha(\pi/2)}.$$

For each z with $|z| = r$, the same calculations as in Lemma 2.4 show that

$$\left| e^{\imath\beta(\pi/2)}p(z) + e^{\imath(\alpha+\beta)\,\pi/2} + e^{\imath\beta(\pi/2)}\,\frac{2r^2\cos\left(\alpha\frac{\pi}{2}\right)}{1-r^2}\right| \leq \frac{2r\cos\left(\alpha\frac{\pi}{2}\right)}{1-r^2}.$$

Therefore,

$$\begin{aligned}\Re\Big(e^{\imath(\alpha+\beta)\,\pi/2}\,\frac{g(z)}{z}\Big) &= \Re\Big(e^{\imath\beta(\pi/2)}p(z)\Big)\\ &\leq -\cos\big((\alpha+\beta)\frac{\pi}{2}\big)+\frac{2r\cos\big(\alpha\frac{\pi}{2}\big)}{1-r^2}\Big(1-r\cos\big(\beta\frac{\pi}{2}\big)\Big).\end{aligned}$$

Denote the right-hand side of this inequality by $s(r,\alpha,\beta)$. Suppose temporarily that we have already solved the inequality $s(r,\alpha,\beta)\leq 0$ relatively to $0\leq\beta\leq 1-\alpha$. Since the semigroup preserves each smaller ball invariant, we conclude that the inequality $s(r',\alpha,\beta)\leq 0$ is satisfied for each $r'\leq r$. On solving it in r' we conclude that r' (and hence r) does not exceed

$$r(\alpha,\beta):=\frac{\cos\alpha\frac{\pi}{2}-\sin\beta\frac{\pi}{2}}{\cos(\alpha-\beta)\frac{\pi}{2}}.$$

Solving the inequality $r\leq r(\alpha,\beta)$ relative to β we get our assertion. The function $e^{-\imath(\alpha+\beta)\,\pi/2}g(z)$ can be considered similarly. □

If $\alpha=0$, then $\beta(r,0)=\dfrac{2}{\pi}\arcsin\dfrac{1-r^2}{1+r^2}$ and we arrive again at Corollary 2.7 above.

2.2. C_0**-semigroups.** Assume that X is a complex Banach space. As usual, we denote by $\mathcal{L}(X)$ the space of all bounded linear operators on X. It is clear that each bounded linear operator $T\in\mathcal{L}(X)$ is holomorphic in X by the very definition. Therefore, setting $D=X$ in Definition 1.1, one can consider a semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ in $\mathcal{L}(X)\subset \mathrm{Hol}(X)$.

Definition 2.10. A semigroup $\mathcal{S}=\{T(t)\}_{t\geq 0}$ in $\mathcal{L}(X)$ is said to be strongly continuous at the point $t=0$ (shortly, a C_0-semigroup) if

$$\lim_{t\to 0+}T(t)x=x$$

pointwise at each point $x\in X$.

The infinitesimal generator A of $\{T(t)\}_{t\geq 0}$ is defined by

$$Ax=\lim_{t\to 0+}\frac{1}{t}\,(T(t)-I)x$$

with domain $\mathcal{D}_A$ consisting of those points $x\in X$ for which the limit exists. The following assertion characterises the main properties of the infinitesimal generator A and its domain $\mathcal{D}_A$ (see for instance [BB67, Paz79]).

Theorem 2.11. *Assume that A is an infinitesimal generator of a one-parameter semigroup $\{T(t)\}_{t\geq 0}$ of class C_0. Then*

1) $\mathcal{D}_A$ is a dense subspace of X;

2) A is a closed linear operator on $\mathcal{D}_A$;

3) For each $x \in X$, the orbit $\{T(t)x\}_{t\geq 0}$ belongs to $\mathcal{D}_A$ and satisfies the Candy problem

$$\begin{cases} \dfrac{d}{dt}T(t)x &= AT(t)x, \qquad t > 0, \\ T(0)x &= x. \end{cases}$$

2.3. **Composition operators and semigroups.** Suppose D is a domain in a complex Banach space X. For $F \in \mathrm{Hol}(D)$, one can define a linear composition operator $T_F)$: $\mathrm{Hol}(D, X) \to \mathrm{Hol}(D, X)$ by the formula

$$T_F h = h \circ F$$

for $h \in \mathrm{Hol}(D, X)$, which is certainly specified within the general concept of adjoint operators F^* in mathematics.

Let Σ be a Banach space of mappings in $\mathrm{Hol}(D, X)$. We write $\|\cdot\|_\Sigma$ and assume that the norm topology in Σ is stronger than the topology of uniform convergence on compact subsets of D. Here are some classical Banach spaces of holomorphic functions in the unit disk $\mathbb{D}$ of the complex plane.

Example 2.12. For any real number $1 \leq p \leq \infty$, the Hardy space $H^p(\mathbb{D})$ is defined to consist of all holomorphic functions h in $\mathbb{D}$ with finite norm

$$\|h\|_p = \sup_{0<r<1} \Big(\frac{1}{2\pi}\int_0^{2\pi} |h(re^{\imath\varphi})|^p d\varphi\Big)^{1/p}.$$

Example 2.13. For a real number $1 \leq p \leq \infty$, the Bergman space $B^p(\mathbb{D})$ is defined to consist of all holomorphic functions h in $\mathbb{D}$ with finite norm

$$\|h\|_{B^p} = \Big(\frac{1}{\pi}\int_{\mathbb{D}} |h(z)|^p dv(z)\Big)^{1/p},$$

where $dv(z)$ is the Lebesgue measure on $\mathbb{D}$.

Example 2.14. By the Dirichlet space $D(\mathbb{D})$ is meant the Banach space consisting of all holomorphic functions in $\mathbb{D}$ with finite (squared) norm

$$\|h\|_D^2 = \frac{1}{\pi}\int_{\mathbb{D}} |h'(z)|^2 dv(z) + |h(0)|^2.$$

In particular, if $\Sigma = H^p(\mathbb{D})$ is a Hardy space on the disk, then we get

$$\Big(\frac{1-|F(0)|}{1+|F(0)|}\Big)^{1/p}\|h\|_p \le \|T_F h\|_p \le \Big(\frac{1+|F(0)|}{1-|F(0)|}\Big)^{1/p}\|h\|_p$$

for all $h \in \Sigma$ (see for instance [CM95]).

In the case, where Σ is the Dirichlet space on $\mathbb{D}$, there are composition operators which fail to preserve Σ. However, if F is a univalent function on $\mathbb{D}$, then the composition operator T_F maps Σ continuously into Σ.

If $\{F_t\}_{t\geq 0}$ is a one-parameter semigroup in $\mathrm{Hol}(D)$, then the family of composition operators $\mathcal{S} = \{T(t)\}_{t\geq 0}$ defined by

$$T(t)h := h \circ F_t, \tag{2.4}$$

for $h \in \mathrm{Hol}(D)$, is also a one-parameter semigroup on $\mathrm{Hol}(D)$. If $\{F_t\}_{t\geq 0}$ is a continuous semigroup on D, then the semigroup $\mathcal{S}$ given by (2.4) is strongly continuous on each of the spaces discussed above (see [Sis98]).

We now focus on the Hilbert space case which corresponds to $p = 2$. To this end, we consider a more general construction based on the control of Taylor coefficients. To wit, pick a sequence of positive real numbers $s = \{s_n\}_{n=0,1,\ldots}$. A holomorphic function $f : \mathbb{D} \to \mathbb{C}$ with power series expansion

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

is said to belong to $H^2(s)$ if

$$\|f\|_{H^2(s)} := \Big(\sum_{n=0}^{\infty} s_n^2\,|c_n|^2\Big)^{1/2} < \infty.$$

It is easy to see that $H^2(s) \subset \mathrm{Hol}(\mathbb{D},\mathbb{C})$ is a Hilbert space endowed with the norm $f \mapsto \|f\|_{H^2(s)}$.

If $s_n = 1$ for all $n = 0, 1, \ldots$, then $H^2(s)$ is the Hardy space $H^2(\mathbb{D})$ and the corresponding norms coincide. If $s_0 = 1$ and $s_n = 1/\sqrt{n+1}$, then $H^2(s)$ is the Bergman space $B^2(\mathbb{D})$ and the norms in these spaces are equal. If $s_0 = 1$ and $s_n = \sqrt{n}$ for all $n > 0$, then $H^2(s)$ just amounts to the Dirichlet space $D(\mathbb{D})$ and the corresponding norms coincide. Observe that

$$D(\mathbb{D}) \hookrightarrow H^2(\mathbb{D}) \hookrightarrow B^2(\mathbb{D}),$$

all the embeddings being continuous.

Remark 2.15. On using the Cauchy integral formula one sees that the convergence of a sequence in $B^1(\mathbb{D})$ implies its uniform convergence on compact subsets of $\mathbb{D}$. The converse assertion is more delicate (cf. [Bea13, Sis98]).

From [CP14] one easily derives a condition for the strong continuity of the semigroup of composition operators on $H^2(s)$.

Theorem 2.16. *Let $s = \{s_n\}_{n\in\mathbb{Z}_{\geq 0}}$ be any sequence of positive numbers, such that $D(\mathbb{D}) \subset H^2(s)$. Then $\mathcal{S}$ is a strongly continuous semigroup on $H^2(s)$.*

Returning to the general case, we fix a Banach space Σ in $\mathrm{Hol}(D,X)$, and let $\{F_t\}_{t\geq 0}$ be a locally uniformly continuous one-parameter semigroup. Assume that the semigroup $\mathcal{S}$ of composition operators defined by (2.4) is of class C_0 on Σ. In this case one can define the infinitesimal generator

$$Ah = \lim_{t\to 0+} \frac{T(t)h - h}{t}$$

with a domain $\mathcal{D}_A$ which is dense in Σ and A is a closed linear operator on Σ.

On the other hand, by the chain rule for each $h \in \mathrm{D}(A)$ we have formally that

$$\begin{aligned} Ah\,(x) &= \frac{d}{dt}\,(T(t)h)(x)\big|_{t=0+} \\ &= \frac{d}{dt}\,(h\circ F_t)(x)\big|_{t=0+} \\ &= h'(x)\,\frac{d}{dt}\,F_t(x)\big|_{t=0+} \\ &= h'(x)f(x), \end{aligned} \tag{2.5}$$

where $f \in \mathcal{G}(D)$ is the infinitesimal generator of the semigroup $\{F_t\}_{t\geq 0}$ on D.

Repeating the proof of Theorem 2 in [Bea13] yields

Theorem 2.17. *Let Σ be a complex Banach space in* $\mathrm{Hol}(D,X)$*. Suppose that A is a closely defined linear operator on Σ which is the generator of a C_0-semigroup $\mathcal{S} = \{T(t)\}_{t\geq 0}$ of composition operators on Σ defined by equality* (2.4)*, where $\{F_t\}_{t\geq 0}$ is a locally uniformly continuous semigroup on D. Then $Ah = h'f$, where f is the generator of $\{F_t\}_{t\geq 0}$ and the domain of A consists of all $h \in$* $\mathrm{Hol}(D,X)$*, such that $h'f \in \Sigma$.*

We also notice that if A is a linear operator with domain $\mathcal{D}_A$ given by $Ah = h'f$, where $f \in \mathrm{Hol}(D,X)$, then the identity operator on Σ

belongs to $\mathcal{D}_A$ if and only if $f \in \Sigma$. In this case it follows from (2.5) that f is dissipative on D. In fact, this fact holds true in a more general setting.

Theorem 2.18. *Let $\Sigma \subset \mathrm{Hol}(D, X)$ be a complex Banach space. Assume that the linear operator A defined by $Ah = h'f$ is the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ of composition operators on Σ defined by (2.4), where $\{F_t\}_{t\geq 0}$ is a locally uniformly continuous semigroup on D. If there is a mapping $h \in \mathcal{D}_A$ with the property that $h'(x_0)$ is invertible for at least one $x_0 \in D$, then the mapping f is dissipative on D and it generates $\{F_t\}_{t\geq 0}$.*

Proof. Under our assumptions we get

$$\lim_{t\to 0+} \|T(t)h - h\|_\Sigma = 0$$

for all $h \in \Sigma$. Taking here h to be the identity mapping of D we deduce that

$$\lim_{t\to 0+} \|F_t(x) - x\| = 0$$

uniformly in x on small balls in D. In other words, $\{F_t\}_{t\geq 0}$ is a locally uniformly continuous semigroup in $\mathrm{Hol}(D)$. Therefore, it possesses a generator $g \in \mathrm{Hol}(D, X)$ which has to satisfy $h'(x)g(x) = h'(x)f(x)$ for all $x \in D$. Since one can choose here h which is locally biholomorphic in a neighborhood U of the point $x_0 \in D$, we conclude that $g = f$ in U. By the familiar uniqueness theorem it follows that $g = f$ in all of D, as desired. □

Using the Lumer-Phillips theorem (see for instance [Yos65]) one shows the following assertion.

Theorem 2.19. *Let Σ be a complex Banach space in* $\mathrm{Hol}(D, X)$ *and $f \in \mathrm{Hol}(D, X)$ a dissipative mapping of D. Then the linear operator A defined by $Ah = h'f$ is a generator of C_0-semigroup of composition operators on Σ if and only if*

1) The set $\mathcal{D}_A$ consisting of those $h \in \Sigma$, which satisfy $h'f \in \Sigma$, is dense in Σ.

2) There is a number $\lambda_0 \geq 0$ such that, for each $\lambda > \lambda_0$, the equation $\lambda h(x) - h'(x)f(x) = g(x)$ has a unique solution $h(x) = (\lambda I - A)^{-1}g(x)$ in Σ whenever $g \in \Sigma$.

3) There is a positive number C such that, for each $n = 0, 1, \ldots$ and $\lambda > \lambda_0$, it follows that

$$\|(\lambda I - A)^{-n}\| \leq C\,(\lambda - \lambda_0)^{-n}.$$

In particular, suppose $\{F_t\}_{t\in(0,t_0)}$ is a family of holomorphic self-mappings of D, such that the limit

$$f(x) := \lim_{t\to 0+} \frac{1}{t}\left(F_t(x) - x\right)$$

exists for all $x \in D$. If the operator $A : \mathcal{D}_A \to \Sigma$ given by $Ah = h'f$ is closely defined in Σ, then A is a generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ in $\mathcal{L}(\Sigma)$. Moreover,

$$T(t)h = h \circ \lim_{n\to\infty} \left(I - \frac{t}{n} f\right)^{-n}.$$

2.4. **Analytic semigroups of composition operators.** We start this section with a general approach to analytic continuation of semigroups of linear operators (see [BB67]).

Theorem 2.20. *Let $\mathcal{S} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators in Σ and A its infinitesimal generator with domain $\mathcal{D}_A \subset \Sigma$. The following are equivalent:*

1) There is a constant $C > 0$ such that $t\,\|AT(t)\| \leq C$ holds true for all $t \in [0, 1]$.

2) The semigroup $\mathcal{S}$ admits an analytic continuation $\{T(\lambda)\}_{\lambda\in\Lambda}$ to the sector Λ in the complex plane consisting of all $\lambda \in \Lambda$ with $\Re\lambda > 0$ and $|\arg \lambda| < (eC)^{-1}$.

Another way of stating 2) is to say that $T(\lambda + \kappa) = T(\lambda) \circ T(\kappa)$ for all $\kappa, \lambda \in \Lambda$ and $T(\lambda)x \to x$ for each $x \in \Sigma$, as $\lambda \to 0$ within a smaller sector $|\arg \lambda| < \vartheta(eC)^{-1}$ with $\vartheta < 1$.

Theorem 2.20 is traced back to a result in [Yos65] for equicontinuous semigroups on locally convex spaces X. The theorem can be also used in the study of analytic continuation of composition operators. However, to this end one can apply the results of previous sections along with an obvious fact.

Lemma 2.21. *Let $\Sigma \subset \mathrm{Hol}(D, X)$ be a complex Banach space containing I, and let $\mathcal{S} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup of composition operators on Σ defined by (2.4), where $\{F_t\}_{t\geq 0}$ is a locally uniformly continuous semigroup on D. Then $\mathcal{S}$ extends to an analytic semigroup into a sector Λ if and only if the semigroup $\{F_t\}_{t\geq 0}$ can be analytically continued into Λ.*

The lemma implies immediately that there are semigroups of composition operators which possess no analytic extension to a sector in the right half-plane. As but one example we mention the semigroup of composition operators in a Hardy space $H^p(\mathbb{D})$ which is induced by the semigroup $\{F_t\}_{t\geq 0}$ generated by the function

$$f(z) = -z\,\frac{1+z}{1-z}$$

of $\mathcal{G}(\mathbb{D})$. However, if one confines himself to the shrinking disks $r\mathbb{D}$ with $r \in (0,1)$, then for each r one can find a sector into which any continuous semigroup has a holomorphic extension. To develop this approach we need certain notation.

Pick a scale of Banach spaces Σ^r depending continuously on the parameter $r \in (0,1)$, such that $\Sigma^r \hookrightarrow \mathrm{Hol}(B_r, X)$ and $\Sigma^{r_2} \hookrightarrow \Sigma^{r_1}$, if $r_1 < r_2$. Consider a locally uniformly continuous semigroup $\{F_t\}_{t\geq 0}$ in $\mathrm{Hol}(\mathbb{B})$, such that $F_t(0) = 0$ and $F_t'(0) = e^{\lambda t} I$ for some $\lambda \in \mathbb{C}$, whenever $t \geq 0$. Note that if the semigroup is generated by a mapping $f \in \mathrm{Hol}(\mathbb{B}, X)$ then, by the Schwarz lemma, it preserves each smaller ball B_r invariant. In other words, the restrictions of F_t to B_r form a semigroup in $\mathrm{Hol}(B_r)$, for any $r \in (0,1]$, which is obviously locally uniformly continuous. Conversely, if $\{F_t\}_{t\geq 0}$ is a semigroup in $\mathrm{Hol}(\mathbb{B})$ and it is locally uniformly continuous on some B_r, with $r \in (0,1)$, then it is locally uniformly continuous on all of $\mathbb{B}$, which is due to the Vitali theorem. The scale Σ^r is said to be consistent with the semigroup if each space Σ^r survives under the action of the composition operators $T(t) = F_t^*$ defined by (2.4), and the norm of any $T(t)$ in $\mathcal{L}(\Sigma^r)$ is bounded uniformly in $r \in (0,1]$.

As mentioned, for semigroups of composition operators one uses classical methods to find a sector in the complex plane into which they can be continued analytically. We illustrate such a method for the scale of Banach spaces $\Sigma_r = H^p(D_r)$, where $D_r = r\mathbb{D}$ is the disk of radius r around $z = 0$.

Let $f \in \mathcal{G}(\mathbb{D})$ and $\{F_t\}_{t\geq 0}$ be the semigroup in $\mathrm{Hol}(\mathbb{D})$ generated by f. By Theorem 1.10,

$$|F_t(re^{\imath\varphi})| \leq r\,\exp\Big(-t\,\frac{1-r}{1+r}\Big) \tag{2.6}$$

holds for all $r \in (0,1)$ and $\varphi \in [0, 2\pi)$. Therefore, for each $r \in (0,1)$, the semigroup elements preserve the disk D_r invariant, and so $T(t)$ maps $H^p(D_r)$ into itself.

Theorem 2.22. *Let $\mathcal{S} = \{T(t)\}_{t\geq 0}$ be the semigroup of composition operators defined by $T(t)h = h \circ F_t$ for $h \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$. When acting on the Banach space $H^p(D_r)$ with $r \in (0,1)$ and $p \geq 1$, the semigroup can* *be continued analytically in the parameter t into the sector*

$$\Lambda := \Big\{ \lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \frac{2}{e} \Big(\frac{1-r}{1+r} \Big)^2 \Big\}.$$

Proof. First we recall that the infinitesimal generator A of the semigroup $\{T(t)\}_{t\ge 0}$ is given by $Ah = h'f$, where f is the infinitesimal generator of the semigroup $\{F_t\}_{t\ge 0}$. Further, we wish to use Theorem 2.20 above. For this purpose, we estimate $\|(h'f)(F_t)\|_{H^p(D_r)}$. By definition, we get

$$\|(h'f)(F_t)\|^p_{H^p(D_r)} = \frac{1}{2\pi} \int_0^{2\pi} |f \circ F_t\,(re^{\imath\varphi})|^p |h' \circ F_t\,(re^{\imath\varphi})|^p \, d\varphi.$$

By Theorem 1.3, the generator has the form $f(z) = -zq(z)$, where q is an analytic function in $\mathbb{D}$ satisfying $\Re q(z) > 0$ and $q(0) = 1$. Hence it follows that

$$|q(z)| \le \frac{1+|z|}{1-|z|}$$

for all $z \in \mathbb{D}$, and so $\|(h'f)(F_t)\|^p_{H^p(D_r)}$ is majorised by

$$\frac{1}{2\pi} \int_0^{2\pi} |F_t(re^{\imath\varphi})|^p \Big(\frac{1+|F_t(re^{\imath\varphi})|}{1-|F_t(re^{\imath\varphi})|} \Big)^p |h' \circ F_t\,(re^{\imath\varphi})|^p \, d\varphi.$$

Applying inequality (2.6) and the subordination principle of Littlewood (see instance Theorem 6.1 of [Dur83]) yields

$$\begin{aligned} \|(h'f)(F_t)\|_{H^p(D_r)} &\le \rho\, \frac{1+\rho}{1-\rho} \Big(\int_0^{2\pi} |h' \circ F_t\,(re^{\imath\varphi})|^p \, \frac{d\varphi}{2\pi} \Big)^{1/p} \\ &\le \rho\, \frac{1+\rho}{1-\rho} \Big(\int_0^{2\pi} |h'(\rho e^{\imath\varphi})|^p \, \frac{d\varphi}{2\pi} \Big)^{1/p}, \end{aligned} \tag{2.7}$$

where $\rho = r\, \exp\Big(-t \dfrac{1-r}{1+r} \Big)$.

In order to estimate the last integral we recall that each $h \in H^p(D_r)$ possesses nontangential boundary values h^* of the class $L^p(\partial D_r)$ and the Cauchy integral formula for the derivative can be written in the form

$$h'(\rho e^{\imath\varphi}) = \int_0^{2\pi} h^*(re^{\imath(\psi+\varphi)}) \frac{re^{\imath(\psi-\varphi)}}{(re^{\imath\psi} - \rho)^2} \, \frac{d\psi}{2\pi}$$

for $\rho < r$. On using an integral version of the Minkowski inequality we get

$$\begin{aligned}
\Big(\int_0^{2\pi} |h'(\rho e^{\imath\varphi})|^p \, \frac{d\varphi}{2\pi} \Big)^{1/p} & \leq & \int_0^{2\pi} \Big\| h^*(r e^{\imath(\psi+\varphi)}) \frac{r e^{\imath(\psi-\varphi)}}{(r e^{\imath\psi} - \rho)^2} \Big\|_{L^p[0,2\pi]} \frac{d\psi}{2\pi} \\
& = & \|h^*\|_{L^p(\partial D_r)} \int_0^{2\pi} \frac{r}{r^2 - 2r\rho\cos\psi + \rho^2} \, \frac{d\psi}{2\pi} \\
& = & \frac{r}{r^2-\rho^2} \, \|h\|_{H^p(D_r)}.
\end{aligned}$$

Combining this estimate with (2.7) yields

$$\|(h'f)(F_t)\|_{H^p(D_r)} \leq \rho \, \frac{1+\rho}{1-\rho} \, \frac{r}{r^2-\rho^2} \, \|h\|_{H^p(D_r)}$$

or, equivalently,

$$\frac{\|(h'f)(F_t)\|_{H^p(D_r)}}{\|h\|_{H^p(D_r)}} \leq \frac{C(t)}{t},$$

where

$$C(t) = \frac{t}{2\sinh\Big(t\dfrac{1-r}{1+r}\Big)} \, \frac{1 + r\exp\Big(-t\dfrac{1-r}{1+r}\Big)}{1 - r\exp\Big(-t\dfrac{1-r}{1+r}\Big)}.$$

Now, for a fixed $r \in (0,1)$, we consider the function $C = C(t)$ of $t \in (0,1]$. It is easy to see that both factors in presentation of C are decreasing functions in t, hence,

$$\sup_{t\in(0,1]} C(t) = \lim_{t\to 0+} C(t) = \frac{1}{2} \Big(\frac{1+r}{1-r} \Big)^2.$$

By Theorem 2.20, the semigroup of composition operators can be analytically continued into the sector

$$\varLambda = \Big\{ \lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| < \frac{2}{e} \Big(\frac{1-r}{1+r} \Big)^2 \Big\},$$

as desired. $\square$

Even in a more general situation, on applying the results concerning analytic continuation of semigroups of holomorphic self-mappings of the unit ball and Theorem 2.17 we establish new theorems on analytic continuation in parameter of semigroups of (linear) composition operators.

Theorem 2.23. *Let $\{F_t\}_{t\geq 0}$ be a locally uniformly continuous semigroup in* Hol$(\mathbb{B})$*, such that $F_t(0) = 0$ and $F_t'(0) = e^{\lambda t} I$ for some $\lambda \in \mathbb{C}$, whenever $t \geq 0$. Assume that $\varSigma^r$ is a scale of Banach spaces consistent with the semigroup. If for some $\alpha \in [0,1)$ the semigroup $\mathcal{S} = \{T(t)\}_{t\geq 0}$*

of composition operators defined by equality (2.4) *can be analytically continued into the sector* $|\arg\lambda| < \alpha\,\pi/2$*, then for each* $r \in (0,1)$*, this semigroup acting on* Σ_r *can be analytically continued in the parameter* t *into the sector*

$$\Lambda = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| < (\alpha+\beta)\,\pi/2\},$$

where

$$0 \le \beta \le \beta(r,\alpha) := \frac{2}{\pi}\,\arcsin\frac{(1-r^2)\cos\left(\alpha\frac{\pi}{2}\right)}{1+2r\sin\left(\alpha\frac{\pi}{2}\right)+r^2}.$$

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