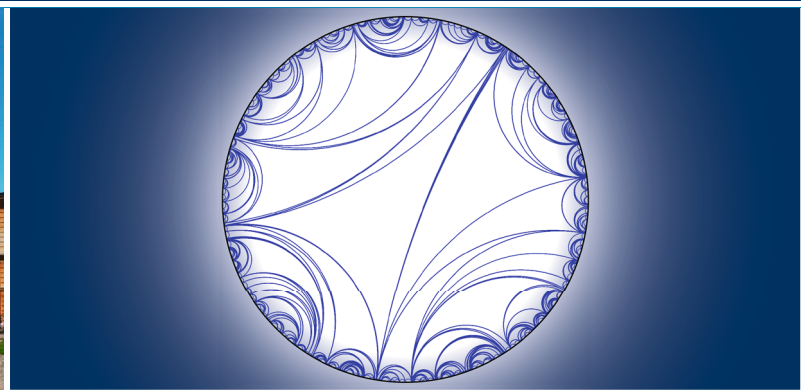




Universität Potsdam



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Deformation Quantisation and Boundary Value Problems

Preprints des Instituts für Mathematik der Universität Potsdam
4 (2015) 5

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Bibliografische Information der Deutschen Nationalbibliothek

Die Deutsche Nationalbibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliografie; detaillierte bibliografische Daten sind im Internet über <http://dnb.dnb.de> abrufbar.

Universitätsverlag Potsdam 2015

<http://verlag.ub.uni-potsdam.de/>

Am Neuen Palais 10, 14469 Potsdam
Tel.: +49 (0)331 977 2533 / Fax: 2292
E-Mail: verlag@uni-potsdam.de

Die Schriftenreihe **Preprints des Instituts für Mathematik der Universität Potsdam** wird herausgegeben vom Institut für Mathematik der Universität Potsdam.

ISSN (online) 2193-6943

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Titelabbildungen:

1. Karla Fritze | Institutsgebäude auf dem Campus Neues Palais
 2. Nicolas Curien, Wendelin Werner | Random hyperbolic triangulation
- Published at: <http://arxiv.org/abs/1105.5089>
Das Manuskript ist urheberrechtlich geschützt.

Online veröffentlicht auf dem Publikationsserver der Universität Potsdam
URN <urn:nbn:de:kobv:517-opus4-77150>
<http://nbn-resolving.de/urn:nbn:de:kobv:517-opus4-77150>

DEFORMATION QUANTISATION AND BOUNDARY VALUE PROBLEMS

B. FEDOSOV AND N. TARKHANOV

Dedicated to S. Grudsky on the occasion of his 60th birthday

ABSTRACT. We describe a natural construction of deformation quantisation on a compact symplectic manifold with boundary. On the algebra of quantum observables a trace functional is defined which as usual annihilates the commutators. This gives rise to an index as the trace of the unity element. We formulate the index theorem as a conjecture and examine it by the classical harmonic oscillator.

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Part 1. Deformation quantisation on manifolds with boundary

1. INTRODUCTION

In this paper a deformation quantisation on a symplectic manifold with boundary is elaborated. A prototype for our construction is the Boutet de Monvel algebra of boundary value problems in much the same way as the usual algebra of pseudo-differential operators is a prototype for deformation quantisation on a symplectic manifold without boundary. Our quantum observables are thus pairs consisting

Date: May 24, 2015.

1991 Mathematics Subject Classification. Primary 53D55; Secondary 37J10.

Key words and phrases. Symplectic manifold, star product, trace, index.

of interior and boundary components, and the composition (star product) gives a nontrivial contribution from interior components to the boundary ones.

We also define a trace functional on our algebra of quantum observables which vanishes on commutators. When having an algebra with a trace, we introduce in a familiar way the index as the trace of the unity element and we formulate as conjecture an index theorem. This latter expresses the trace of unity through topological invariants of the manifold with boundary, bundle connections and the symplectic structure. Unfortunately, we have not been able to prove our index theorem in full generality and we address this to our forthcoming investigations in this direction.

Deformation quantisation on a symplectic manifold with boundary has first been considered in the paper of Nest and Tsygan [NT96]. The prototype of their construction is however the b -calculus of Melrose. The corresponding algebra of quantum observables does not contain any boundary components. Moreover, there is no trace on the algebra which vanishes on commutators, and hence no index with standard properties is available. Our construction has the advantage of being free of these drawbacks, which is due to stronger assumptions on the geometry and symplectic structure.

By quantum mechanical systems in a mathematical model of quantum mechanics are meant couples $\{\hat{H}, S\}$, where \hat{H} is the so-called Hamilton operator of the system and S the eigenstate of the system. In the deformation quantisation model \hat{H} is an element of an associative algebra of quantum observables corresponding to a classical observable H which is a smooth real-valued function on a manifold M . In order to treat a concrete physical problem within the framework of deformation quantisation one needs the corresponding “operator” \hat{H} . To this end, one uses the rules of classical mechanics to determine the Hamilton function H on a phase space. Under symplectic structure, one exploits a simple quantisation procedure leading from H to \hat{H} , see [Fed94].

The idempotent elements in the quantum algebra can be thought of as projectors, and so their traces are the dimensions of the corresponding “eigenspaces”, if there are any. If $\lambda \in \mathbb{R}$ is a noncritical value of H , then the level surface $\{H = \lambda\}$ is often a circle bundle over a compact symplectic manifold B . It bears a canonical quantum algebra and the trace of the unity element is expected to contain encoded spectral data. Deformation quantisation on the sublevel manifold $\{H \leq \lambda\}$ leads to spectral asymptotics.

In our opinion, the spectral interpretation of the index formula is of especial interest, and it was actually a crucial motivation of our treatment here. When directly understood, the spectral problems do not make sense in the framework of deformation quantisation. One may ask whether this is still possible to show such a reformulation of spectral problems which would have meaning within deformation quantisation. For this purpose we suggest to make use of index theorems. Thus, in our preceding paper [FST04] we treated several examples of evaluating spectra with the help of index theorem for symplectic orbifolds. As we recently learned, similar methods had been regularly used in physics and chemistry for qualitative description of spectra of molecules with many atoms [FZ02]. While the proofs are not of crucial importance for physicists, the punctual setting of a spectral problem in the framework of deformation quantisation is as much important for us as its solution.

In the present paper we deal with spectral problems which are slightly different from those in [FST04]. Their prototype in the theory of pseudodifferential operators are spectral asymptotics for $\hbar \rightarrow 0$. Concerning this we mention the method of approximate spectral projection of [Shu87], which consists of replacing the step-function of an operator by smoothed step-like data followed by applying pseudodifferential techniques. The approach we take is quite different. Instead of smoothing we construct an algebra with trace, in which the step-function is a perfect element. It is actually the unity element in this algebra. Then the trace of this unity element can be thought of as the trace of spectral projection, the latter being exact, not approximate. On the other hand, the trace of the unity element is an index, and the index theorem evaluates it explicitly. The quantisation condition consists of the requirement that this index had to be a positive integer number, namely, be equal to the dimension of the spectral subspace. While being still incomplete, these observations demonstrate rather strikingly that there exists a spectral theory in deformation quantisation rich in content and based on the index theorem and its variants. For a fuller treatment we refer the reader to [Fed06], see also [Tar15].

2. DEFORMATION QUANTISATION

Here we give a brief summary of results concerning deformation quantisation on a symplectic manifold $\{M, \omega\}$. More details and proofs may be found in [Fed94] or in the book [Fed96].

In the sequel the letter W refers to (Hermann) Weyl and D stands for a special connection. The notation W_D means that our quantum objects are flat sections of a bundle W (the Weyl algebras bundle) with respect to the connection D .

Let E be a complex vector bundle over M and K the bundle $\text{Hom}(E, E)$. This latter will be referred to as coefficient bundle. A connection ∂^E on E defines a connection on K (with the same notation) given in local frames by

$$(\partial^E a)(h) = \partial^E(a(h)) - a(\partial^E h)$$

for $h \in C^\infty(M, E)$ (the Leibniz rule). Thus,

$$\partial^E a = da + [\Gamma^E, a]$$

for all sections $a \in C^\infty(M, K)$, where $\Gamma^E = \Gamma_i dx^i$ is the local connection one-form of ∂^E .

We introduce the Weyl algebras bundle $W = W(M, K)$ by describing the space $C^\infty(M, W)$ of its sections over M . A section a of the bundle W is a function of $x \in M$ with values in formal power series in a small parameter \hbar (Planck constant) whose coefficients are formal power series in $y \in T_x M$ with coefficients in K_x . To wit

$$(2.1) \quad a = a(x, y, \hbar) = \sum_{2k+|\alpha| \geq 0} \hbar^k a_{k,\alpha}(x) y^\alpha,$$

where $y = (y^1, \dots, y^{2n})$ is a tangent vector in $T_x M$, $y^\alpha = (y^1)^{\alpha_1} \dots (y^{2n})^{\alpha_{2n}}$, and $a_{k,\alpha}(x)$ are symmetric tensors on M with values in K_x . We prescribe the degrees 2 and 1 to \hbar and y^i , respectively, and order the terms of (2.1) by the total degree $2k + |\alpha|$ in each tangent space $T_x M$.

The latter carries a linear symplectic structure given by the form ω at the point $x \in M$. Using this structure, we define a fibrewise product, called the Weyl (or Moyal) product, by

$$(2.2) \quad \begin{aligned} a \circ b &= \exp\left(-\frac{i\hbar}{2}\omega^{ij}(x)\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right)a(x, y, \hbar)b(x, z, \hbar)|_{z=y} \\ &= \sum_{k=0}^{\infty}\left(-\frac{i\hbar}{2}\right)^k\frac{1}{k!}\omega^{i_1j_1}(x)\dots\omega^{i_kj_k}(x)\frac{\partial^k a}{\partial y^{i_1}\dots\partial y^{i_k}}\frac{\partial^k b}{\partial y^{j_1}\dots\partial y^{j_k}}. \end{aligned}$$

We will also need differential forms on M with values in W . These are sections of the bundle $W \otimes \Lambda$, where $\Lambda = \bigoplus \Lambda^q T^*M$ means the bundle of exterior forms. In local coordinates such a section looks like

$$a = \sum h^k a_{kpq},$$

where

$$(2.3) \quad a_{kpq} = a_{ki_1\dots i_p j_1\dots j_q}(x) y^{i_1}\dots y^{i_p} dx^{j_1} \wedge \dots \wedge dx^{j_p}$$

with the total degree of terms $2k+p$. The coefficients in (2.3) are tensors symmetric in i_1, \dots, i_p and skewsymmetric in j_1, \dots, j_q .

Let ∂^s be a symplectic connection on M , i.e., a torsion-free connection preserving the symplectic tensor $\omega_{ij}(x)$. Given connections ∂^s and ∂^E , we define a connection ∂ in the bundle W (or $W \otimes \Lambda$) by taking covariant differentials of coefficients in (2.1) which are tensors on M with values in K . In local Darboux coordinates we have

$$\partial a = d_x a + \frac{i}{\hbar} [G, a],$$

where

$$(2.4) \quad G = \frac{1}{2}\Gamma_{ijk}(x)y^i y^j dx^k - i\hbar \Gamma_i(x) dx^i$$

and $\Gamma_{ijk} = \omega_{il}\Gamma_{jk}^l$ are the coefficients of the symplectic connection ∂^s (Christoffel symbols). They are completely symmetric in i, j, k .

Introduce an important derivation

$$\delta a = dx^i \wedge \frac{\partial a}{\partial y^i} = -\frac{i}{\hbar} [\omega_{ij} y^i dx^j, a]$$

for $a \in C^\infty(M, W)$. It acts on summands in (2.3) by replacing in turn y^{i_1} by dx^{i_1} , etc., y^{i_p} by dx^{i_p} , and summing up the results. A direct verification actually shows that

$$\begin{aligned} \delta^2 &= 0, \\ \delta\partial + \partial\delta &= 0. \end{aligned}$$

Consider the operator

$$\delta^* a = y^k i\left(\frac{\partial}{\partial x^k}\right)a$$

for $a \in C^\infty(M, W)$. It acts on summands in (2.3) by replacing in turn dx^{j_1} by y^{j_1} , etc., dx^{j_q} by $(-1)^{q-1}y^{j_q}$, and summing up the results. It is clear that $(\delta^*)^2 = 0$, however, δ^* fails to be a derivation.

Finally, we introduce the fibrewise Laplace operator

$$\delta^* \delta + \delta \delta^* = (\delta + \delta^*)^2$$

which acts on the coefficients (2.3) by $(\delta^* \delta + \delta \delta^*) a_{kpq} = (p+q) a_{kpq}$. Put

$$\delta^{-1} a_{kpq} = \frac{1}{p+q} \delta^* a_{kpq}$$

if $p+q > 0$, and $\delta^{-1} a_{k00} := 0$. This leads immediately to the fibrewise Hodge-de Rham decomposition

$$(2.5) \quad a = a_{00} + \delta^{-1} \delta a + \delta \delta^{-1} a$$

for $a \in C^\infty(M, W)$.

We will consider more general connections on the bundle W which are of the form

$$(2.6) \quad Da = \partial a - \delta a + \frac{\imath}{\hbar} [r, a],$$

where $r \in C^\infty(M, W \otimes \Lambda^1)$ is a globally defined one-form with $\deg r \geq 3$. A simple calculation shows that

$$\partial^2 a = \frac{\imath}{\hbar} [R, a],$$

where

$$R = d\Gamma + \frac{\imath}{\hbar} \Gamma \circ \Gamma = \frac{1}{4} R_{ijkl} y^i y^j dx^k \wedge dx^l - \frac{\imath \hbar}{2} R_{kl}^E dx^k \wedge dx^l$$

is the curvature of ∂ , R_{kl}^E is the curvature tensor of ∂^E and $R_{ijkl} = \omega_{im} R_{jkl}^m$ is the curvature tensor of the symplectic connection ∂^s .

In local Darboux coordinates we may rewrite (2.6) in the form

$$(2.7) \quad Da = d_x a + \frac{\imath}{\hbar} [\omega_{ij} y^i dx^j + \Gamma + r, a]$$

for $a \in C^\infty(M, W)$. Then, an easy calculation yields

$$D^2 a = \frac{\imath}{\hbar} \left[-\frac{1}{2} \omega_{ij} dx^i \wedge dx^j + R + \partial r - \delta r + \frac{\imath}{\hbar} r^2, a \right].$$

The two-form $\Omega = -\omega + R + \partial r - \delta r + \frac{\imath}{\hbar} r^2$ appearing in this formula is called the Weyl curvature of D .

We now look for a special connection D whose Weyl curvature just amounts to $-\omega$. The form ω is central, so we would have

$$D^2 a = -\frac{\imath}{\hbar} [\omega, a] \equiv 0$$

for all $a \in C^\infty(M, W)$. A connection D with this property is called Abelian. From the formula for the Weyl curvature Ω we obtain an equality for the one-form r , namely

$$(2.8) \quad \delta r = R + \partial r + \frac{\imath}{\hbar} r^2.$$

If we impose an additional condition $\delta^{-1} r = 0$ on r (this just amounts to saying that r is fibrewise coclosed), then by virtue of (2.5) we obtain an equivalent equation

$$(2.9) \quad r = \delta^{-1} R + \delta^{-1} \left(\partial r + \frac{\imath}{\hbar} r^2 \right)$$

with $\delta^{-1} R = \frac{1}{8} R_{ijkl} y^i y^j y^k dx^l - \frac{\imath \hbar}{2} R_{ij} y^i dx^j$.

The basic theorem of deformation quantisation reads

Theorem 2.1. *There exists a unique solution r of (2.8) satisfying $\delta^{-1} r = 0$. It may be found by iterations of (2.9).*

Proof. See [Fed94]. □

We define quantum objects to be flat sections of the bundle W with respect to the special connection D . In other words, W_D is the space of all $a \in C^\infty(M, W)$ satisfying $Da = 0$. The property of being flat may be written by (2.6) as

$$(2.10) \quad \delta a = \partial a + \frac{\imath}{\hbar} [r, a]$$

or, using (2.5), as

$$(2.11) \quad a = a_0 + \delta^{-1} \left(\partial a + \frac{\imath}{\hbar} [r, a] \right),$$

where $a_0(x, \hbar) = a(x, 0, \hbar)$ is a function on M with values in formal power series in \hbar with coefficients in K .

Theorem 2.2. *Equation (2.10) has a unique solution satisfying $a(x, 0, \hbar) = a_0$. It may be found by iterations of (2.11).*

Proof. See [Fed94]. □

We will use the notation $a = \hat{a}_0 = Q(a_0)$ for the solution of Theorem 2.2. The operator $Q : C^\infty(M, K)[[\hbar]] \rightarrow W_D$ is called the quantisation procedure or simply quantisation map. Its inverse is the restriction to $y = 0$. On using Q and Q^{-1} we may transport the product \circ in W_D directly to the functions on M , defining the so-called star-product $a(x, \hbar) * b(x, \hbar) = Q^{-1}(Q(a) \circ Q(b))$. However, it is more convenient to work with the algebra W_D and the fibrewise product \circ than with the algebra of functions equipped with the star-product.

3. LOCAL ISOMORPHISMS

The simplest example of the above construction is the standard symplectic space \mathbb{R}^{2n} with a constant symplectic form

$$\omega = \frac{1}{2} w_{ij} dx^i \wedge dx^j$$

and a trivial bundle K . We set $\partial = d_x$ and

$$(3.1) \quad D_0 = d - \delta = d + \frac{\imath}{\hbar} [w_{ij} y^i \wedge dx^j, \cdot],$$

so that the flat sections in $W_{D_0}(\mathbb{R}^{2n})$ are those of the form

$$\hat{a} = Q(a(x, \hbar)) = a(x + y, \hbar) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{2n}} \frac{1}{\alpha!} a^{(\alpha)}(x, \hbar) y^\alpha.$$

Choose a Darboux chart O in M and a local frame of the bundle E and consider the restriction $W|_O$ of the Weyl algebras bundle to O . We have two Abelian connections on $W|_O$. One of them D has been constructed in the previous section globally on M . The other connection D_0 is defined locally on $W|_O$ by (3.1). Thus, we have two different algebras W_D and W_{D_0} over O and two different quantisation maps Q and Q_0 corresponding to D and D_0 , respectively. It turns out that the algebras W_D and W_{D_0} are isomorphic. Moreover, the isomorphism can be taken in the special form

$$(3.2) \quad a_0 = (\text{ad } U)a := U \circ a \circ U^{-1}$$

with

$$U = \exp\left(\frac{\imath}{\hbar}H\right),$$

where $H \in C^\infty(O, W)$ is a section of the bundle W over O whose degree is at least 3. Although U is not defined as a section of W , the expression (3.2) is defined correctly, for

$$\begin{aligned} U \circ a \circ U^{-1} &= \exp\left(\frac{\imath}{\hbar} \operatorname{ad} H\right)a \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\imath}{\hbar} \underbrace{\left[H, \frac{\imath}{\hbar} \left[H, \dots \frac{\imath}{\hbar} [H, a] \dots \right] \right]}_{k \text{ times}}. \end{aligned}$$

We may also define the exponential U in an extension of the bundle W which admits negative powers of \hbar . To this end, consider an algebra bundle W^+ whose sections are sums (2.1) where k may be negative, provided the total degree $2k + |\alpha|$ is positive, with a finite number of terms at each total degree entering into the sum (2.1). Then

$$U = 1 + \frac{\imath}{\hbar}H + \frac{1}{2}\left(\frac{\imath}{\hbar}\right)^2 H \circ H + \dots$$

is an invertible section of the bundle W^+ since $\deg H \geq 3$.

If we require a_0 given by (3.2) to belong to W_{D_0} for any $a \in W_D$, then we arrive at the condition

$$[U^{-1} \circ D_0 U - \frac{\imath}{\hbar} \Delta \Gamma, a] = 0,$$

which is obtained by differentiating (3.2) with respect to the connection D_0 and using

$$D_0 a = Da - \frac{\imath}{\hbar} [\Delta \Gamma, a]$$

with $\Delta \Gamma = \Gamma + r$, as is easily seen from (3.1) and (2.7). To satisfy this condition, we set

$$(3.3) \quad U^{-1} \circ D_0 U = \frac{\imath}{\hbar} \Delta \Gamma.$$

First of all we observe that the solution of (3.3) is not unique, for multiplying U by an invertible section $V \in C^\infty(O, W_{D_0})$ from the left yields another solution $V \circ U$. To find a particular solution of (3.3), we look for an exponential solution and make use of the well-known formula

$$D_0 \exp w = \exp w \circ \frac{1 - \exp(-\operatorname{ad} w)}{\operatorname{ad} w} D_0 w.$$

Thus, since $D_0 = d - \delta$, we would have

$$\delta H = dH - \frac{\operatorname{ad} \frac{\imath}{\hbar} H}{1 - \exp\left(-\operatorname{ad} \frac{\imath}{\hbar} H\right)} (\Gamma + r).$$

On applying δ^{-1} and setting $H|_{y=0} = 0$, we come to the equation

$$H = \delta^{-1} \left(dH - \frac{\operatorname{ad} \frac{\imath}{\hbar} H}{1 - \exp\left(-\operatorname{ad} \frac{\imath}{\hbar} H\right)} (\Gamma + r) \right)$$

which can be solved by iterations starting with $H_0 \equiv 0$. This gives

$$H = \frac{1}{6} \Gamma_{ijk} y^i y^j y^k - \imath \hbar \Gamma_i y^i + \dots$$

and we arrive at the following theorem.

Theorem 3.1. *Any algebra W_D is locally isomorphic to the standard algebra W_{D_0} and the isomorphism can be taken in the form (3.2).*

This theorem allows one to define a trace functional on elements $a \in W_D$ having compact support in M . So, let a be supported in a local Darboux coordinate chart $O \subset M$. We choose an isomorphism of the form (3.2) to the algebra $W_{D_0}(\mathbb{R}^n)$ and set

$$\mathrm{Tr} a = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} \mathrm{tr}(\mathrm{ad} U) a|_{y=0} \frac{\omega^n}{n!}.$$

For an arbitrary section $a \in W_D$ with compact support we take a locally finite covering of M by Darboux charts O_i and trivialisations of the bundle E over O_i . Next, taking a partition of unity $\rho_i(x) \in C^\infty(M)$ subordinate to the covering O_i , we construct flat sections $\hat{\rho}_i = Q(\rho_i)$ which give a partition of unity in the algebra W_D . Now, choosing isomorphisms U_i for each chart O_i , we set

$$(3.4) \quad \mathrm{Tr} a = \sum_i \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} \mathrm{tr}(\mathrm{ad} U_i) \hat{\rho}_i \circ a|_{y=0} \frac{\omega^n}{n!}.$$

Theorem 3.2. *The functional Tr is correctly defined (that is, independent of the particular choices of O_i , ρ_i , U_i) and satisfies the trace property $\mathrm{Tr} a \circ b = \mathrm{Tr} b \circ a$ for any $a, b \in W_D$ with compactly supported product. This is a unique (up to a normalisation factor) functional with this property.*

The factor $(2\pi\hbar)^{-n}$ in (3.4) is taken by analogy with pseudodifferential operators. There are deeper reasons for this choice. If M is a compact symplectic manifold, then any $a \in W_D$ possesses a trace.

Take a scalar-valued function $a \in C^\infty(M)$ with compact support, consider the section $a(x) \otimes 1 \in C_{\mathrm{comp}}^\infty(M, K)$, where 1 means the unit in the algebra $K = \mathrm{Hom}(E, E)$, quantise this section obtaining $\hat{a} = Q(a \otimes 1)$ in $W_D(M, K)$, and calculate its trace. The result will be

$$\mathrm{Tr} \hat{a} = \frac{1}{(2\pi\hbar)^n} \int_M t(x, \hbar) a(x) \frac{\omega^n}{n!},$$

where $t(x, \hbar) = \mathrm{rank} E + \hbar t_1(x) + \hbar^2 t_2(x) + \dots$ is an element of $C^\infty(M)[[\hbar]]$ called trace density. The explicit formulas of Section 3 show that the coefficients $t_k(x)$ may be expressed as polynomials in connection coefficients Γ_i , Γ_{ijk} and their derivatives in local coordinates. One can do it explicitly for lower dimensions 2 and 4, but even for $\dim M = 4$ the calculations become extremely tiresome and lead to the formula which at first glance is quite different from the formula for the number of quantum states (index formula)

$$(3.5) \quad \mathrm{Tr} \hat{1} = \int_M \mathrm{ch} E \hat{A}(M) \exp\left(\frac{\omega}{2\pi\hbar}\right),$$

where $\mathrm{ch} E$ is the Chern character of the connection ∂^E and $\hat{A}(M)$ the Atiyah-Hirzebruch form of the connection ∂^s , see [Fed91], [Fed96], etc.

4. SYMPLECTIC MANIFOLDS WITH BOUNDARY

Let M be a smooth symplectic manifolds, and ω_M a symplectic form on M . Let moreover $H(x)$ be a smooth real-valued function on M , called Hamiltonian. Consider the level surface $M_0 = \{H = 0\}$ and the sublevel set $M_- = \{H \leq 0\}$. Suppose that M_0 is a smooth compact submanifold of M , i.e., 0 is not a critical value of H . Moreover, M_- is assumed to be a compact manifold whose boundary is $\partial M_- = M_0$. The Hamiltonian vector field V_H induced by H defines an one-dimensional foliation of M_0 . We assume that this foliation is actually a circle bundle. Thus, the trajectories of the vector field are orbits of free action of the group $U(1)$, hence the space of trajectories

$$B = M_0/U(1)$$

is a smooth compact manifold.

In other words, we consider a smooth reduction of M under the action of the group $U(1)$. The base B is automatically a symplectic manifold with the form ω_B that is uniquely determined by the condition

$$i^*\omega_M = p^*\omega_B,$$

where $i : M_0 \rightarrow M$ is the embedding and $p : M_0 \rightarrow B$ the projection. This symplectic manifold is called the reduced manifold (or simply reduction) and it is going to bear the boundary conditions in the sequel. In the classical theory of boundary value problems on a compact manifold X with boundary ∂X the role of M_- is played by T^*X . The submanifold M_0 is defined by the equation $H(x, \xi) \equiv x_n = 0$, i.e., M_0 just amounts to the restriction of T^*X to ∂X . Since $V_H = \partial/\partial \xi_n$, the trajectories of V_H are straight lines along conormal vectors, and the space of trajectories is $B = T^*(\partial X)$. The analogy to smooth reduction is evident, and what is still lacking is the compactness, which is however impossible in the case of cotangent bundles.

Generally speaking, as is in the theory of boundary value problems, the function H need not be defined on all of M . It may be given merely in a neighbourhood of M_0 , i.e., M_- need not be a sublevel manifold of H on the whole.

The neighbourhood U of the level manifold M_0 in M given by the inequality $|H| < \varepsilon$ admits the following standard model, cf. for instance [Fed96]. Topologically U is the direct product $M_0 \times I$, where I is the interval $-\varepsilon < H < \varepsilon$, and the symplectic form has the form

$$\begin{aligned} \omega_U &= \omega_M|_U \\ &= \omega_B + d(H\gamma). \end{aligned}$$

Here ω_B means the symplectic form on the base B pulled back to $M_0 \times I$, and γ is the 1-form of a connection on M_0 pulled back to $M_0 \times I$. Recall that on M_0 , which is a principal $U(1)$ -bundle, there exists a connection form, i.e., a 1-form γ , satisfying

$$(4.1) \quad \begin{aligned} i(V_H)\gamma &= 1, \\ L_{V_H}\gamma &= 0. \end{aligned}$$

Here the vector field V_H is thought of as a vector field on $M_0 \times I$ with zero projection on I . From (4.1) it easily follows that

$$i(V_H)\omega_U = -dH,$$

i.e., V_H is a Hamiltonian vector field with Hamiltonian H .

Sometimes it is convenient to pass from the principal bundle M_0 to the associated one-dimensional complex bundle

$$\begin{aligned} E &= M_0 \times_{U(1)} \mathbb{C} \\ &= M_0 \times \mathbb{C}/U(1) \end{aligned}$$

with associated connection ∂^E and Hermitian metric (\cdot, \cdot) . We regard the Hermitian metric as being conjugate linear in the first variable and linear in the second one. Then we get

$$\omega_U = \Omega_B + \Im d(z, \partial z)$$

and

$$\begin{aligned} H &= (z, z) - 1 = |z|^2 - 1, \\ V_H &= iz \frac{\partial}{\partial z} - i\bar{z} \frac{\partial}{\partial \bar{z}}. \end{aligned}$$

Here z designates a vector of the fibre of E , and $\partial z = dz + i\gamma z$ is the covariant differential of the “tautological” section z at the point $(b, z) \in E$, the 1-form γ satisfying (4.1).

5. THE TOEPLITZ-BOUTET DE MONVEL ALGEBRA

We begin with an algebra of pseudodifferential operators on the circle S which depend on a small positive parameter \hbar . The dependence on \hbar is understood in an asymptotic sense. We restrict our discussion to merely indicating formal expansions which stem from formal Taylor series, and we skip the proofs of the fact that the formal expansions are actually asymptotic series. Such assertions are well known in the theory of pseudodifferential operators, cf. for instance [Shu87]. For the Boutet de Monvel algebra of boundary value problems similar assertions are discussed in detail in [Gru86]. Furthermore, when passing to deformation quantisation, we will regard the asymptotic series as formal ones. Note that both algebraic and analytical relations between asymptotic series imply analogous relations between formal series which can be obvious by no means.

Functions on a circle are given by the Fourier series or Laurent series, in the latter case the circle is specified as the unit circle in the complex plane. For calculations it is convenient to assume that these Laurent series actually converge in a small annular neighbourhood of the unit circle. Such functions are dense in L^2 . There is a projection Π^+ onto the functions holomorphic in the unit disk, and a projection Π^- onto the functions holomorphic in the complement and vanishing at infinity. These projections are given by the Cauchy-type integrals

$$(\Pi^\pm u)(z) = \frac{1}{2\pi i} \int_{S^\pm} \frac{u(\zeta)}{\zeta - z} d\zeta,$$

where S^\pm is the contour consisting of the circle $|\zeta| = 1$ with going around the point z over a small outer or inner semicircle, respectively. The ranges of these projections in L^2 will be denoted by H^\pm . For functions u on S , we also introduce their means over the circle

$$\begin{aligned} \langle u \rangle &= \frac{1}{2\pi i} \int_S \frac{u(\zeta)}{\zeta} d\zeta \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(x) dx. \end{aligned}$$

We want to study the operator algebra generated by the \hbar -pseudodifferential operators on S and the projection Π^+ . We first consider \hbar -pseudodifferential operators. Let $a(x, \xi)$ be a smooth function with compact support on the cylinder $T^*S = S \times \mathbb{R}$, called symbol. Define a pseudodifferential operator $A_\hbar = \text{Op}(a)$ by the formula

$$(A_\hbar u)(x) = \sum_{k=-\infty}^{\infty} e^{ikx} a(x, \hbar k) \hat{u}(k),$$

where $\hat{u}(k)$ is a Fourier coefficient of u . An equivalent definition of A_\hbar in Fourier images is

$$(5.1) \quad \hat{v}(l) = \sum_{k=-\infty}^{\infty} \hat{a}(l-k, \hbar k) \hat{u}(k),$$

$\hat{a}(k, \xi)$ being a Fourier coefficient of $a(x, \xi)$ as function of $x \in S$. For the product of two operators we get

$$\hat{w}(m) = \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \hat{b}(m-l, \hbar l) \hat{a}(l-k, \hbar k) \hat{u}(k).$$

Expanding $\hat{b}(m-l, \hbar l)$ in the formal Taylor series at the point $\xi = \hbar k$

$$\hat{b}(m-l, \hbar l) \sim \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} \partial_\xi^\alpha \hat{b}(m-l, \hbar k) (\hbar l - \hbar k)^\alpha$$

we arrive at a formula for the composition of symbols

$$(5.2) \quad (b \circ a)(x, \xi) = \sum_{\alpha=0}^{\infty} \frac{(-i\hbar)^\alpha}{\alpha!} \partial_\xi^\alpha b(x, \xi) \partial_x^\alpha a(x, \xi)$$

in the form of a formal series in powers of \hbar . As is known from the theory of pseudodifferential operators, this series is asymptotic.

It follows that if one introduces symbols which formally depend on the parameter \hbar by

$$(5.3) \quad a(x, \xi, \hbar) = \sum_{k=0}^{\infty} a_k(x, \xi) \hbar^k$$

then the composition (5.2) makes the set of formal symbols an associative algebra called the algebra of quantum observables.

We next consider a more complicated algebra generated by the operators of the form

$$(5.4) \quad A_\hbar^+ = \Pi^+ \text{Op}(a(x, \xi)) \Pi^+.$$

Such operators are called Toeplitz operators. In the book [BdMG81] they are treated in detail even for the higher dimensional case. Similarly to pseudodifferential operators they form an algebra. However, in contrast to pseudodifferential operators, the transition from operators depending on a small parameter \hbar to formal series in \hbar is not possible within this algebra. In the framework of deformation quantisation there appear the so-called Green operators.

Definition 5.1. By a Green kernel $g(z, \zeta)$ is meant a smooth function on $S \times S$ which belongs to H^+ in z and H^- in ζ . The integral operator $G : H^+ \rightarrow H^+$ given by

$$(5.5) \quad (Gu)(z) = \int_S g(z, \zeta)u(\zeta) d\zeta$$

for $u \in H^+$ is said to be a Green operator.

At first sight, when rewriting the operator (5.5) as an integral operator over the interval $[0, 2\pi]$, one can pass from the kernel to a symbol and thus represent a Green operator in the form $\text{Op}(a(x, \xi))$. However, it is easily seen that the symbol $a(x, \xi)$ will have the form $\tilde{a}(x, \xi/\hbar)$, where $\tilde{a}(x, \tilde{\xi})$ is a smooth function vanishing for $\tilde{\xi} < 0$. This shows that a is a function of boundary layer type, i.e., it is rapidly decreasing for $\xi < 0$. Such functions are not defined in deformation quantisation. On the contrary, an operator of the form (5.5), where the kernel $g(z, \zeta, \hbar)$ is a formal series in \hbar , makes sense in deformation theory. We thus conclude that introducing Green operators just amounts to legitimating boundary layers in the deformation approach.

Definition 5.2. An operator of the form

$$(5.6) \quad A_{\hbar}^+ + G : H^+ \rightarrow H^+$$

is said to be a Toeplitz-Boutet de Monvel operator.

The operators (5.6) form already an algebra even in the deformation approach, as we will see soon. This algebra is very similar to the usual Boutet de Monvel algebra, where the role of conormal variable ξ_n is played by the variable of the circle. Since x varies over a compact set (unlike ξ_n), the situation gets considerably simplified. Namely, one needs no transmission property (it is automatically fulfilled), there does not arise the condition of invertibility of the symbol outside of a compact set, and hence no trace and potential operators are required. They can be added if one wishes. However, in the simplest setting one can do without them as well. Note that if one maps the circle onto the real axis by a linear fractional map then one arrives at the genuine Boutet de Monvel algebra. This allows one to invoke the results of [Gru86], when one needs to verify the asymptotic character of formal series.

We will give an operator (5.6) as a pair

$$(5.7) \quad t = \{a(x, \xi), g(z, \zeta)\}$$

where $a \in C_c^\infty(T^*S)$ is the symbol of the pseudodifferential operator $A_{\hbar} = \text{Op}(a)$ that enters into the Toeplitz operator A_{\hbar}^+ , and g is a Green kernel. Note that the symbol a is uniquely determined but up to symbols $\Delta a \in C_c^\infty((T^*S)^-)$, i.e., those vanishing on $(T^*S)^+$, where $(T^*S)^\pm$ stands for the semicylinder $\pm\xi > 0$. To prove this we observe that the operator A_{\hbar}^+ acts by the formula similar to (5.1), however, the indices k and l now vary over nonnegative integers. Hence it follows that for the symbols $\Delta a \in C_c^\infty((T^*S)^-)$ the corresponding Toeplitz operator $\Pi^+ \text{Op}(\Delta a) \Pi^+$ is equal to 0. Thus, it would be more precise to say that a in (5.4) belongs to the quotient space

$$C_c^\infty(T^*S)/C_c^\infty((T^*S)^-).$$

However, we will not be so pedantic and we will work with representatives of equivalence classes, i.e., with symbols on the whole cylinder T^*S .

We will also consider the formal series

$$t(h) = \sum_{k=0}^{\infty} t_k \hbar^k,$$

the coefficients t_k being pairs (5.7). We call such series the formal Toeplitz-Boutet de Monvel symbols.

Theorem 5.3. *The operators (5.6) form an algebra, which survives under passing to deformation quantisation.*

A bit vague phrase that the algebra survives under passing to deformation quantisation means in fact that the composition rule extends to formal symbols of Toeplitz-Boutet de Monvel.

Proof. We have

$$(B_h^+ + G_2)(A_h^+ + G_1) = (B_h A_h)^+ + (B_h^+ A_h^+ - (B_h A_h)^+) + B_h^+ G_1 + G_2 A_h^+ + G_2 G_1.$$

Here the first summand is a Toeplitz-Boutet de Monvel operator defined by the symbol $b \circ a$. To complete the proof it remains to show that the other summands are Green operators. We give the proof only for the most difficult case of the operator $B_h^+ A_h^+ - (B_h A_h)^+$.

For a composition $w = \Pi^+ \text{Op}(b) \Pi^+ \text{Op}(a) u$ with $u \in H^+$, we get similarly to formula (5.1)

$$\hat{w}(m) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \hat{b}(m-l, \hbar l) \hat{a}(l-k, \hbar k) \hat{u}(k),$$

in contrast to (5.1) the sums being over nonnegative indices k and l . For the operator $B_h^+ A_h^+ - (B_h A_h)^+$ we get an analogous formula but with summation over negative integers l

$$(5.8) \quad \hat{w}(m) = - \sum_{l=-\infty}^{-1} \sum_{k=0}^{\infty} \hat{b}(m-l, \hbar l) \hat{a}(l-k, \hbar k) \hat{u}(k),$$

the index m takes nonnegative values. Clearly, the difference $l-k$ is always negative and the difference $m-l$ is always positive for all admissible values k and l . Moreover, both $l-k$ and $m-l$ tend to $\mp\infty$, respectively. Therefore, one can replace the symbol a by $a^-(x, \xi) = \Pi_x^- a(x, \xi)$ and the symbol b by $b^+(x, \xi) = \Pi_x^+ b(x, \xi)$. Since the Fourier coefficients are rapidly decreasing, the double series in (5.8) converges rapidly.

We first consider the case where a and b do not depend on ξ , i.e., $\text{Op}(a)$ is the multiplication operator by a function $a(z)$, and similarly for $\text{Op}(b)$.

Lemma 5.4. *If a and b are independent of ξ then (5.8) is a Green operator with kernel*

$$(5.9) \quad g(z, \zeta) = \frac{1}{(2\pi i)^2} \int_S \frac{b^+(v) - b^+(z)}{v - z} \frac{a^-(\zeta) - a^-(v)}{\zeta - v} dv.$$

Proof. It is easily verified that

$$\begin{aligned} (B_h^+ A_h^+ - (B_h A_h)^+) u &= \Pi^+ b(z) \Pi^+ (a(z)u(z)) - \Pi^+ (b(z)a(z)u(z)) \\ &= \Pi^+ b(z) \Pi^- (a(z)u(z)). \end{aligned}$$

The function $a(z)$ can be replaced by $a^-(z)$, for $a^+(z)u(z) \in H^+$ and so Π^- annihilates such a function. In the same manner we can see that b can be replaced by b^+ . Thus, the operator takes the form

$$(B_{\hbar}^+ A_{\hbar}^+ - (B_{\hbar} A_{\hbar})^+)u(z) = \frac{1}{(2\pi i)^2} \int_{S^+} \frac{b^+(v)}{v-z} dv \int_{S^-} \frac{a^-(\zeta)u(\zeta)}{\zeta-v} d\zeta.$$

Clearly,

$$\begin{aligned} \int_{S^-} \frac{a^-(v)u(\zeta)}{\zeta-v} d\zeta &= a^-(v) \int_{S^-} \frac{u(\zeta)}{\zeta-v} d\zeta \\ &= 0 \end{aligned}$$

whence

$$\begin{aligned} v(v) &= \int_{S^-} \frac{a^-(\zeta)u(\zeta)}{\zeta-v} d\zeta \\ &= \int_S \frac{a^-(\zeta) - a^-(v)}{\zeta-v} u(\zeta) d\zeta, \end{aligned}$$

the contour S^- can be replaced by S , for the singularity at $\zeta = v$ disappears. Similar arguments apply to the external integral. Taking into account that $v \in H^-$ we get

$$\int_{S^+} \frac{b^+(v)v(v)}{v-z} dv = \int_S \frac{b^+(v) - b^-(z)}{v-z} v(v) dv,$$

which proves the lemma. \square

In the sequel, for symbols $a(z)$ and $b(z)$ independent of ξ , we use the designation $G(b, a)$ for the Green operator with kernel (5.9). The kernel itself will be denoted by $g(b, a)$. When passing to the real variables $z = e^{ix}$ and $\zeta = e^{iy}$, one should attach the factor $i\zeta$ to the kernel $g(b, a)$, for $d\zeta = i e^{iy} dy$. We will use both the real and the complex form of the kernel without further comments.

Returning now to the general case, we expand the symbols $a(x, \xi)$ and $b(x, \xi)$ in the equality (5.8) in a formal Taylor series in ξ at $\xi = 0$, obtaining

$$\hat{a}(l-k, \hbar k) = \sum_{\alpha=0}^{\infty} \partial_{\xi}^{\alpha} \hat{a}(l-k, 0) \frac{(\hbar k)^{\alpha}}{\alpha!}$$

as well as

$$\begin{aligned} \hat{b}(m-l, \hbar l) &= \sum_{\alpha=0}^{\infty} \partial_{\xi}^{\alpha} \hat{b}(m-l, 0) \frac{(\hbar l)^{\alpha}}{\alpha!} \\ &= \sum_{\beta, \gamma=0}^{\infty} \partial_{\xi}^{\beta+\gamma} \hat{b}(m-l, 0) \frac{(\hbar m)^{\beta} (\hbar l - \hbar m)^{\gamma}}{\beta! \gamma!} \\ &= \sum_{\beta, \gamma=0}^{\infty} \partial_{\xi}^{\beta+\gamma} \widehat{\partial_x^{\gamma} b}(m-l, 0) \frac{i^{\gamma} \hbar^{\beta+\gamma} m^{\beta}}{\beta! \gamma!}. \end{aligned}$$

Substituting these into formula (5.8) yields

$$\hat{w}(m) = \sum_{\alpha, \beta, \gamma=0}^{\infty} m^{\beta} \sum_{l=-\infty}^{-1} \sum_{k=0}^{\infty} \frac{i^{\gamma} \hbar^{\alpha+\beta+\gamma}}{\alpha! \beta! \gamma!} \partial_{\xi}^{\beta+\gamma} \widehat{\partial_x^{\gamma} b}(m-l, 0) \partial_{\xi}^{\alpha} \hat{a}(l-k, 0) k^{\alpha} \hat{u}(k),$$

which in turn can be written, by Lemma 5.4, in the form

$$w(x) = \sum_{\alpha, \beta, \gamma=0}^{\infty} \frac{h^{\alpha+\beta+\gamma}}{\alpha! \beta! \gamma!} (-i\partial_x)^\beta G \left(\partial_\xi^{\beta+\gamma} (i\partial_x)^\gamma b(x, 0), \partial_\xi^\alpha a(x, 0) \right) (-i\partial_x)^\alpha u(x).$$

For the kernel of this operator we readily obtain a formal series in powers of \hbar , namely

$$(5.10) \quad \sum_{\alpha, \beta, \gamma=0}^{\infty} \frac{h^{\alpha+\beta+\gamma}}{\alpha! \beta! \gamma!} (-i\partial_x)^\beta (i\partial_y)^\alpha g \left(\partial_\xi^{\beta+\gamma} (i\partial_x)^\gamma b(x, 0), \partial_\xi^\alpha a(x, 0) \right) (x, y),$$

which is asymptotic by [Gru86].

The compositions $G_2 A_h^+$ and $B_h^+ G_1$ can be handled in much the same way. We first assume that the symbol a does not depend on ξ , i.e., A_h is the operator of multiplication by a function $a(z)$. Then as in Lemma 5.4 we deduce that $G_2 A_h^+$ is a Green operator. In the general case we expand the symbol $a(x, \xi)$ in a formal Taylor series in ξ at $\xi = 0$, which leads to a formal power series in h , the coefficients being Green operators. The asymptotic character of these series is a consequence of results of [Gru86]. The same reasoning applies to the case of $B_h^+ G_1$. It is not difficult to write down explicit formulas of the type (5.10), however, we need not them in the sequel. \square

Part 2. Index theorem for manifolds with boundary

6. TRACE IN THE TOEPLITZ-BOUTET DE MONVEL ALGEBRA

Once again we start with the algebra of pseudodifferential operators on a circle. For symbols with compact support, the operator $A_\hbar = \text{Op}(a(x, \xi))$ is of trace class in the space $L^2(S)$, and the trace of this operator is equal to the sum of diagonal elements in any orthonormal basis. Taking as a basis the functions e^{ikx} , with $k \in \mathbb{Z}$, and using (5.1), we get

$$\begin{aligned} \text{Tr } A_\hbar &= \sum_{k=-\infty}^{\infty} \hat{a}(0, \hbar k) \\ &= \sum_{k=-\infty}^{\infty} \langle a(x, \hbar k) \rangle \\ (6.1) \quad &= \frac{1}{2\pi\hbar} \int_{T^*S} a(x, \xi) dx d\xi + O(\hbar^\infty). \end{aligned}$$

Here we used the summation formula of Euler-Maclaurin to replace the sum over k by integration over ξ . The expression (6.1) makes also sense for formal series (5.3). It follows that one can introduce a trace on the algebra of quantum observables as termwise integral

$$(6.2) \quad \text{Tr } a(x, \xi, \hbar) = \frac{1}{2\pi\hbar} \int_{T^*S} a(x, \xi, \hbar) dx d\xi.$$

Thus, the trace is a linear functional on the algebra of quantum observables with values in $\mathbb{C}[[\hbar]]$ (formal series with complex coefficients), which vanishes on the commutators. This latter property can be directly verified by using the composition formula. However, it is not necessary because the operator origin of the trace formula (6.2) guarantees this property.

We now turn to Toeplitz operators (5.4) in the space H^+ . As an orthonormal basis we take the same functions e^{ikx} with nonnegative frequencies $k = 0, 1, \dots$. Then we get similarly to (6.1)

$$\begin{aligned}
\mathrm{Tr} A_{\hbar}^+ &= \sum_{k=0}^{\infty} \hat{a}(0, \hbar k) \\
&= \sum_{k=0}^{\infty} \langle a(x, \hbar k) \rangle \\
(6.3) \quad &\sim \frac{1}{2\pi\hbar} \int_S \int_0^{\infty} a(x, \xi) dx d\xi - \sum_{j=1}^{\infty} \frac{B_j}{j!} \langle \partial_{\xi}^{j-1} a(x, 0) \rangle \hbar^{j-1}.
\end{aligned}$$

We have once again used the Euler-Maclaurin formula for the interval $0 \leq k\hbar \leq N$, where N is the width of the symbol support, i.e., $a(x, \xi) = 0$ for $\xi \geq N$. Along with the integral over the semicylinder there are also boundary terms at $\xi = 0$, and B_j are Bernoulli numbers. The series is asymptotic if $\hbar \rightarrow 0$.

The trace of a Green operator G with kernel

$$g(z, \zeta) = \sum_{m,n=0}^{\infty} g_{m,n} z^m \zeta^{-n-1}$$

is also defined as the sum of diagonal elements in the basis $z^m = e^{imx}$. This sum can be rewritten as the integral of the kernel over the diagonal, i.e.,

$$\begin{aligned}
\mathrm{Tr} G &= \sum_{n=0}^{\infty} g_{n,n} \\
(6.4) \quad &= \int_S g(z, z) dz.
\end{aligned}$$

The formulas (6.3) and (6.4) allow one to introduce a trace functional for the formal algebra of Toeplitz-Boutet de Monvel by linearity. It splits into an interior and boundary parts in a natural way. Namely, let $t = \{a, g\}$ where

$$\begin{aligned}
a &= a(x, \xi, \hbar) = \sum_{k=0}^{\infty} \hbar^k a_k(x, \xi), \\
g &= g(z, \zeta, \hbar) = \sum_{k=0}^{\infty} \hbar^k g_k(z, \zeta).
\end{aligned}$$

Then we set

$$\mathrm{Tr} t = \mathrm{Tr}_i t + \mathrm{Tr}_b t,$$

where

$$\begin{aligned}
\mathrm{Tr}_i t &= \frac{1}{2\pi\hbar} \int_{(T^*S)^+} a(x, \xi, \hbar) dx d\xi, \\
(6.5) \quad \mathrm{Tr}_b t &= \int_S g(z, z, \hbar) dz - \frac{1}{2\pi} \int_S \sum_{j=1}^{\infty} \frac{B_j}{j!} \hbar^{j-1} \partial_{\xi}^{j-1} a(x, 0, \hbar) dx,
\end{aligned}$$

all the integrals being understood as termwise ones. The very operator nature of these formulas ensures that this way defined trace vanishes on commutators. Indeed, for the Toeplitz-Boutet de Monvel operators $t_1 = A_{\hbar}^+ + G_1$ and $t_2 = B_{\hbar}^+ + G_2$ the equality $\mathrm{Tr} [t_1, t_2] = 0$ is fulfilled for all positive values \hbar , since for operators in

a Hilbert space it always holds. Hence, this equality remains valid if by the trace is meant the asymptotic expansion in \hbar given by (6.3) and (6.4). From this it in turn follows that the equality still holds if the asymptotic expansion is replaced by the formal one, and then it extends to formal linear combinations. It is worth pointing out, however, that the interior and the boundary traces do not vanish on commutators separately.

Note that the direct verification of the equality $\text{Tr}[t_1, t_2] = 0$ is rather lavish similarly to cumbersome calculations that are done in [FGLS96] for the noncommutative residue of Wodzicki. Here we can prove this equality in a roundabout way without invoking cumbersome calculations.

7. GLOBAL CONSTRUCTION

In the section we construct an algebra of quantum observables $\mathcal{A} = \{\mathcal{A}_i, \mathcal{A}_b\}$ consisting of interior and boundary components. Along with symplectic structures described in Section 4, we need symplectic connections ∂_M^s and ∂_B^s and vector bundles E_M and E_B with connections ∂^{E_M} and ∂^{E_B} over M and B , respectively. In a boundary neighbourhood U we require

$$E_M|_U = \pi^* E_B,$$

where $\pi : U \rightarrow B$ is a projection. The connections ∂_M^s and ∂^{E_M} in the neighbourhood U are assumed to be invariant under the action of the group $U(1)$.

We will use the following notation. By x is meant a point of M and by $y \in TM$ a tangent vector. The variables x' and y' have the same meaning with respect to the base B . In the neighbourhood U we have

$$\begin{aligned} x &= (x', \varphi, H), \\ y &= (y', \Delta\varphi, \Delta H) \end{aligned}$$

where φ, H are the coordinates ‘angle’ and ‘action’ in the fibre $S \times I$ over x' , or $x = (x', z, \bar{z})$ if one uses complex coordinates in the fibres of the bundle E (cf. Section 4). By $K_M = \text{Hom}(E_M, E_M)$ and $K_B = \text{Hom}(E_B, E_B)$ we denote the coefficient bundles with connections induced by ∂^{E_M} and ∂^{E_B} .

The algebra \mathcal{A}_i is the standard algebra of quantum observables on M restricted to M_- , cf. [Fed96]. To construct it one introduces the bundle of Weyl algebras $W(M)$ on tangent spaces, and the bundle of Weyl algebras with coefficients in K_M , i.e., $W(M, K_M) = W(M) \otimes K_M$ equipped with the connection $\partial_M = \partial_M^s \otimes 1 + 1 \otimes \partial^{E_M}$. Furthermore, one constructs an Abelian connection $D_M = \partial_M + [\gamma_M, \cdot]$ where γ_M is an 1-form with values in W_M of the form

$$\gamma_M = \frac{i}{\hbar} (\omega_M)_{ij} y^i dx^j + \frac{i}{\hbar} r_M.$$

The principal term of γ_M is of degree -1 in y and \hbar under the convention $\deg y = 1$ and $\deg \hbar = 2$, and the remainder has degree ≥ 1 . The Abelian property of D_M means that the Weyl curvature of D_M

$$\Omega_M := \partial_M \gamma_M + \gamma_M^2 + \Omega_M^s + \Omega^{E_M}$$

coincides with $-i\omega_M/\hbar$, where Ω_M^s and Ω^{E_M} are curvatures of the connections ∂_M^s and ∂^{E_M} . In particular, Ω_M is a form with values in the centre. The form γ_M is defined uniquely modulo a scalar (central) summand. The Weyl normalisation assumes that

$$\gamma_M|_{y=0} = 0.$$

As a consequence of the Abelian property we readily deduce $D_M^2 = 0$, and \mathcal{A}_i is actually the algebra of all flat sections $a = a(x, y, \hbar)$ of the bundle $W(M, K_M)$.

The algebra \mathcal{A}_b is constructed in much the same way on the manifold B , the only difference being in more involved coefficients. More precisely, the coefficients are operator-valued. In the theory of pseudodifferential operators the corresponding objects are called operator-valued symbols. Recall (cf. Section 4) that the neighbourhood U of the boundary is a bundle with fibre $S \times I$, and the transition functions are angle shifts $\varphi \rightarrow \varphi + \varphi_0(x')$. In the fibres we have the Toeplitz-Boutet de Monvel algebra, on which the group $U(1)$ acts by angle shifts, i.e., through multiplication with complex numbers of modulus 1. Thus, over B there live the bundle H^+ and the bundle T of Toeplitz algebras that can be thought of as $\text{Hom}(H^+, H^+)$. The sections of the bundle T are pairs

$$t = \{a(x', \varphi, H, \hbar), g(x', z, \zeta, \hbar)\}.$$

When comparing to Section 4, we have slightly changed the notation: instead of $(x, \xi) \in T^*S$ we now write $(\varphi, H) \in S \times I$. Moreover, there has appeared a dependence of $x' \in B$ as a parameter. On the bundle T one can introduce the connection

$$(7.1) \quad \partial^T = d_{x'} + \left[\frac{\imath}{\hbar} H \gamma, \cdot \right].$$

Here $d_{x'}$ is the de Rham differential in $x' \in B$, γ is a connection form in the principal bundle M_0 , and H is regarded as an element of the Toeplitz-Boutet de Monvel algebra with $\{a, g\} = \{H, 0\}$. Clearly, the commutator with H yields the differentiation in the angle, hence the connection (7.1) in fact corresponds to infinitesimal angle shifts, see Chapter 8 in [Fed96] for more details.

We can now introduce the coefficient bundle $K' = K_B \otimes T$ with connection $\partial' = \partial^{E_B} \otimes 1 + 1 \otimes \partial^T$, and the bundle of Weyl algebras $W(B, K') = W(B) \otimes K'$, the tensor product of the latter expression is understood over $\mathbb{C}[[\hbar]]$. The rest runs in the same manner as for M . Namely, we introduce the connection $\partial_B = \partial_B^s \otimes 1 + 1 \otimes \partial'$ and then construct an Abelian connection

$$(7.2) \quad D_B = \partial_B + [\gamma_B, \cdot]$$

whose Weyl connection is

$$\begin{aligned} \Omega_B &:= \partial_B \gamma_B + \gamma_B^2 + \Omega_B^s + \Omega^{E_B} + \Omega^T \\ &= -\frac{\imath}{\hbar} \omega_B. \end{aligned}$$

Then \mathcal{A}_b is the algebra of all flat sections of $W(B, K')$ relative to the connection D_B . Technically this construction is much more complicated than that for M , and we refer the reader to Chapter 8 of [Fed96] for more details.

The interior and boundary components in a pair $\{a_i, a_b\} \in \mathcal{A}$ should be compatible in an appropriate manner. The boundary component is also a pair consisting of a Toeplitz operator and of a Green operator, both depending on parameters x', y' and the formal parameter \hbar . Omitting the projection Π^+ in the designation of a Toeplitz operator, we write a flat section of $W(B, K')$ in the form

$$(7.3) \quad a_b(x', y', \hbar) = \{a(x', y', \hbar, \varphi, H), g(x', y', \hbar, z, \zeta)\}.$$

Loosely speaking, the compatibility condition means that the symbol a in the pair (7.3) and a flat section $a_i = a_i(x, y, \hbar)$ of the bundle $W(M, K_M)$ are diverse forms of the same object. In other words, we must be able to rewrite any symbol a of

(7.3) in the form of a flat section of the bundle $W(M, K_M)$. This is done in three steps.

1. *Formalisation:* Omitting the arguments x', y' and \hbar , we consider a function $a(\varphi, H)$ of (7.3) on the fibre $F = S \times I$ of the bundle $U \rightarrow B$. We assign to $a(\varphi, H)$ the formal Taylor series $\mathcal{F}a = a(\varphi + \Delta\varphi, H + \Delta H)$ in powers of $\Delta\varphi$ and ΔH . Such expansions form an algebra with respect to the Leibniz product

$$(7.4) \quad a \circ b = \sum_{\alpha=0}^{\infty} \frac{(-i\hbar)^\alpha}{\alpha!} \frac{\partial^\alpha a}{\partial(\Delta\varphi)^\alpha} \frac{\partial^\alpha b}{\partial(\Delta H)^\alpha}.$$

This algebra is isomorphic to the algebra of left symbols of pseudodifferential operators on a circle. The isomorphism is given by the formalisation map \mathcal{F} , and the inverse isomorphism reduces to the substitution $\Delta\varphi = \Delta H = 0$. Obviously, $\mathcal{F}a$ satisfies the equation

$$(7.5) \quad \begin{aligned} D_F(\mathcal{F}a) &:= d_{\varphi, H}\mathcal{F}a + \left[\frac{i}{\hbar} (\Delta\varphi dH - \Delta H d\varphi), \mathcal{F}a \right] \\ &= 0 \end{aligned}$$

where the commutator is taken with respect to multiplication (7.4). This equality expresses the fact that $\mathcal{F}a$ depends only on the combinations $\varphi + \Delta\varphi$ and $H + \Delta H$. On the other hand, the equality just amounts to saying that $\mathcal{F}a$ is a flat section of the bundle of Leibniz algebras in a fibre $F = S \times I$ relative to the Abelian connection D_F . Here $d_{\varphi, H}$ stands for the de Rham differential in the fibre F , and the connection form is

$$\gamma_F = \frac{i}{\hbar} (\Delta\varphi dH - \Delta H d\varphi).$$

2. *Transition to Weyl symbols:* From the Leibniz algebra with product (7.4) one can pass to the Weyl algebra with multiplication

$$a \circ b = \sum_{\alpha, \beta=0}^{\infty} \left(-\frac{i\hbar}{2} \right)^{\alpha+\beta} \frac{(-1)^\alpha}{\alpha!\beta!} \frac{\partial^{\alpha+\beta} a}{\partial(\Delta\varphi)^\alpha \partial(\Delta H)^\beta} \frac{\partial^{\alpha+\beta} b}{\partial(\Delta H)^\alpha \partial(\Delta\varphi)^\beta}.$$

These two products are equivalent, for the left symbols are transferred to Weyl symbols by the formal differential operator

$$P = \exp \left(\frac{i\hbar}{2} \frac{\partial^2}{\partial(\Delta\varphi)\partial(\Delta H)} \right).$$

The “function” $P\mathcal{F}a$ can be thus thought of as a section of the bundle of Weyl algebras over $U \subset M$. However, it is not in general a flat section of the bundle with respect to the Abelian connection D_M .

3. *Transition to another Abelian connection:* The original section a in the pair $t = \{a, g\}$ is flat with respect to the Abelian connection D_B , cf. (7.2). Hence the section $P\mathcal{F}a$ is flat relative to the connection \tilde{D}_B that is the image of the connection D_B under the isomorphism $P\mathcal{F}$. Moreover, it is flat in each fibre $S \times I$ relative to D_F , cf. (7.5). Therefore, $P\mathcal{F}a$ is a flat section of the bundle of Weyl algebras over $U \subset M$ with respect to the connection $\tilde{D}_B + D_F$. In [Fed96, Chapter 8] it is shown that the Weyl curvatures of the connections $\tilde{D}_B + D_F$ and D_M coincide in a small neighbourhood $U \subset M$ of the boundary M_0 . Hence the corresponding algebras of flat sections of the Weyl bundle over U are isomorphic, cf. Chapter 5 in [Fed96].

Denoting this isomorphism by J , we conclude that the composition $\mathcal{I} = JPF$ gives an epimorphism of the algebras

$$W_{D_B} \ni \{a, g\} \mapsto \mathcal{I}a \in W_{D_M}.$$

Given a pair $\{a_i, a_b\}$, with $a_i \in W_{D_M}$ and $a_b = \{a, g\} \in W_{D_B}$, we say that a_i and a_b are compatible if $a_i = \mathcal{I}a$ in a neighbourhood of M_0 . The component-wise product does not destroy the compatibility condition, for $\mathcal{I} = JPF$ is an isomorphism of algebras. Thus, we introduce the algebra of quantum observables \mathcal{A} as the set of all compatible pairs with component-wise product.

8. TRACE AND INDEX

Recall that we consider a compact symplectic manifold $M_- = \{H \leq 0\}$ with boundary embedded into a larger symplectic manifold M . Our main assumption is that the boundary of M_- is a circle bundle over a compact closed manifold B whose dimension is thus two less than that of M_- . We use the construction of smooth reduction with respect to the action of the group $U(1)$ elaborated in [Fed98]. This yields deformation quantisation on B with coefficients in $K' = K_B \otimes T$, where $K_B = \text{Hom}(E_B, E_B)$ and $T = \text{Hom}(H^+, H^+)$ is the bundle of Toeplitz algebras over B . The algebra of quantum observables on M_- is defined to consist of interior and boundary components, i.e., $\mathcal{A} = \{\mathcal{A}_i, \mathcal{A}_b\}$, where \mathcal{A}_i is the restriction of W_{D_M} to M_- and \mathcal{A}_b just amounts to W_{D_B} . The trace functional on W_{D_M} restricted to the elements of \mathcal{A}_i does not vanish on commutators, for M_- bears a boundary. On the other hand, the Toeplitz operators $t = \{a, g\}$ need not be of trace class unless $a(x, \xi, \hbar)$ is $o(|\xi|^{-1})$ for $\xi \rightarrow \infty$. Hence, the fibres of T are no longer endowed with operator trace. To get rid of these difficulties and construct a trace functional on \mathcal{A} , we require the interior and boundary components of $\{a_i, a_b\}$ with $a_b = (a, g)$ to be compatible in the sense clarified in the preceding section. Then a_i and a are diverse forms of the same objects in W_{D_M} and we introduce

$$\text{Tr} \{a_i, a_b\} = \text{Tr}_i \{a_i, a_b\} + \text{Tr}_b \{a_i, a_b\}$$

where

$$(8.1) \quad \begin{aligned} \text{Tr}_i \{a_i, a_b\} &= \int_{M_-} \exp \frac{\omega_M}{2\pi\hbar} \text{tr} a_i |_{y=0}, \\ \text{Tr}_b \{a_i, a_b\} &= \int_B \exp \frac{\omega_B}{2\pi\hbar} \text{tr}_b a_b |_{y'=0} \end{aligned}$$

for $\{a_i, a_b\} \in \mathcal{A}$. Here, we have slightly changed the notation and write tr_b for the trace in the fibres of T given by the second formula of (6.5).

Lemma 8.1. *As defined above, the functional $\text{Tr} \{a_i, a_b\}$ is a trace on the quantum algebra \mathcal{A} , i.e., it vanishes on commutators.*

Proof. For those couples $\{a_i, 0\}$ whose components a_i are compactly supported in the interior of M_- the assertion follows immediately from what has been said in Section 3. Using a partition of unity in the quantum algebra \mathcal{A}_i , one can write an arbitrary $\{a_i, a_b\}$ as the sum of two couples $\{a'_i, 0\}$ and $\{a''_i, a_b\}$, where a'_i is compactly supported in the interior of M_- and a''_i has compact support in a neighbourhood $\{-\varepsilon < H \leq 0\}$ of ∂M_- . Since a_i and a_b are compatible, it follows that

the trace of $\{a_i, a_b\}$ with a_i compactly supported in $\{-\varepsilon < H \leq 0\}$ is given by the integral

$$\int_B \exp \frac{\omega_B}{2\pi\hbar} (\mathrm{tr}_i a_b + \mathrm{tr}_b a_b) |_{y'=0}.$$

From the construction of the trace in the Toeplitz-Boutet de Monvel algebra given in Section 6 we conclude readily that the functional vanishes on commutators, as desired. \square

It is clear that Lemma 8.1 can be proved immediately by integration by parts. However, the direct verification is rather cumbersome.

In [FGLS96] it is shown that any continuous trace on Boutet de Monvel's algebra on a compact manifold with boundary is a scalar multiple of the noncommutative residue constructed there.

Note that for the couples of the form $\{a_i, 0\}$, where $a_i \in W_{D_M}$ is compactly supported in the interior of M_- , the trace functional coincides with that in the quantum algebra W_{D_M} . By Theorem 3.2, the trace in this algebra is defined uniquely up to a normalisation factor. On the other hand, if a_i is supported in a sufficiently small neighbourhood of the boundary ∂M_- , then the trace of $\{a_i, a_b\} \mathcal{A}$ coincides with the trace of a_b in the quantum algebra W_{D_B} endowed with the operator trace in the fibres, as is shown in the proof of Lemma 8.1. Theorem 3.2 still applies to this algebra thus proving the uniqueness of the trace up to a normalisation factor. Using the localisation procedure based on the partition of unity in the quantum algebra W_{D_M} , we can represent any $\{a_i, a_b\} \in \mathcal{A}$ as the sum of two elements $\{a'_i, 0\}$ and $\{a''_i, a_b\}$ as above. Since the components a_i and a_b are compatible, it follows that the trace functionals in W_{D_M} and W_{D_B} should agree. Hence, the trace functional on \mathcal{A} is defined uniquely up to a normalisation factor, if one assumes some continuity of the trace.

Take a scalar-valued function $a \in C^\infty(M_-)$ and consider the section $a \otimes 1$ in $C^\infty(M, K_M)$, where 1 stands for the unit in the algebra K_M . Similarly to the calculus of pseudodifferential operators on a compact manifold with boundary, there is no canonical way to quantise this section in the quantum algebra \mathcal{A} . Write $a = a' + a''$, where a' is a C^∞ function with compact support in the interior of M_- and $a'' \in C^\infty(M_-)$ is supported in a sufficiently small neighbourhood of the boundary ∂M_- . Then $a' \otimes 1$ can be quantised in the quantum algebra \mathcal{A}_i while $a'' \otimes 1$ can be quantised in the quantum algebra \mathcal{A}_b . In this way we obtain an element

$$\hat{a} = \{Q(a' \otimes 1), Q(a'' \otimes 1)\}$$

in \mathcal{A} , where we use the same letter Q to designate the different quantisation maps in \mathcal{A}_i and \mathcal{A}_b . Nevertheless the trace $\mathrm{Tr} \hat{a}$ does not depend on the particular splitting $a = a' + a''$, as is easy to see. Proceeding by analogy with the index theorem for deformation quantisations on a compact closed manifold, cf. [Fed96], we come to the conjecture

$$(8.2) \quad \mathrm{Tr} \hat{1} = \int_{M_-} \mathrm{ch} E_M \exp \frac{\omega_M}{2\pi\hbar} \hat{A}(M) + \int_B \mathrm{ch}' E_B \exp \frac{\omega_B}{2\pi\hbar} \hat{A}(B),$$

where $\mathrm{ch}' E_B$ means the Chern character of the bundle E_B with respect to the "trace" tr_b .

Formula (8.2) is very similar to the formula for the index of an elliptic boundary value problem, see [Fed91, (2.24)]. The arguments of [Fed01] show that the index

theorem for elliptic boundary value problems may be obtained as a consequence of the index theorem for deformation quantisation. Formula (8.2) has also intimate relations to the index formula for symplectic orbifolds of [FST04, (6.3)]. It should be noted that the techniques for establishing formulas [FST04, (6.3)] and (8.2) has been understood well. A challenging problem consists in finding a direct proof of the index theorem.

9. SPECTRAL ASYMPTOTICS

We use the designation Hamiltonian for any smooth real-valued function $H(x)$ on M used an element of a quantum algebra. An eigenstate $\langle \cdot \rangle_\lambda$ with eigenvalue $\lambda \in \mathbb{R}$ for the Hamiltonian $H(x)$ is a functional on the compactly supported elements of the quantum algebra

$$\langle a \rangle_\lambda = \sum_{k \gg -\infty} \hbar^k c_k(\hbar),$$

such that $\langle \hat{H} * a \rangle_\lambda = \langle a * \hat{H} \rangle_\lambda = \lambda \langle a \rangle_\lambda$ for all a , where $k \gg -\infty$ means that k is bounded away from $-\infty$. See [Fed06], [Tar15].

The property $\langle \hat{H} * a \rangle_\lambda = \langle a * \hat{H} \rangle_\lambda$ means that the functional $\langle \cdot \rangle_\lambda$ vanishes on commutators with \hat{H} . We say that the spectral theorem holds if

$$(9.1) \quad \int_{-\infty}^{\infty} \langle a \rangle_\lambda d\lambda = \text{Tr } a$$

for any a in the quantum algebra with compact support. It is clear that the eigenstate may be multiplied with any number, and the equality (9.1) gives a proper normalisation.

Let $\lambda \in \mathbb{R}$ be a noncritical value of $H(x)$. Then the construction of Section 8 applies to yield a quantum algebra $\mathcal{A} = \mathcal{A}_{\{H \leq \lambda\}}$ with trivial coefficients in K_M , where $E_M = M \times \mathbb{C}$, on the compact manifold with boundary $M_- = \{H \leq \lambda\}$. The trace functional $\text{Tr} = \text{Tr}_{\{H \leq \lambda\}}$ gives rise to the construction of an eigenstate with eigenvalue λ for H .

Namely, assuming that the derivative exists we set

$$\langle a \rangle_\lambda = \frac{d}{d\lambda} \text{Tr } a$$

for $a \in \mathcal{A}$. By the very construction, $\langle a \rangle_\lambda$ is a formal Laurent series in \hbar with a finite number of negative powers of \hbar , whose coefficients depend on λ . Write $\hat{H} = Q_{\{H \leq \lambda\}}(H)$ for a quantisation of $H(x)$ in the quantum algebra \mathcal{A} . By the trace property, we obtain

$$\langle \hat{H} * a \rangle_\lambda = \frac{d}{d\lambda} \text{Tr } \hat{H} * a = \frac{d}{d\lambda} \text{Tr } a * \hat{H} = \langle a * \hat{H} \rangle_\lambda$$

for all $a \in \mathcal{A}$. Moreover,

$$\begin{aligned} \langle \hat{H} * a \rangle_\lambda &= \lim_{\Delta\lambda \rightarrow 0} \frac{\text{Tr}_{\{H \leq \lambda + \Delta\lambda\}} \hat{H} * a - \text{Tr}_{\{H \leq \lambda\}} \hat{H} * a}{\Delta\lambda} \\ &= \lambda \lim_{\Delta\lambda \rightarrow 0} \frac{\text{Tr}_{\{H \leq \lambda + \Delta\lambda\}} a - \text{Tr}_{\{H \leq \lambda\}} a}{\Delta\lambda} \\ &= \lambda \langle a \rangle_\lambda, \end{aligned}$$

as desired. The spectral theorem holds for $H(x)$ in the sense that

$$\begin{aligned} \int_{-\infty}^{\infty} \langle a \rangle_{\lambda} d\lambda &= \int_{-\infty}^{\infty} \frac{d}{d\lambda} (\mathrm{Tr}_{\{H \leq \lambda\}} a) d\lambda \\ &= \mathrm{Tr}_{\{H < \infty\}} a \end{aligned}$$

whenever a is supported away from the critical levels of H .

Having eigenstates at his disposal, one may introduce further important spectral notions purely within the framework of deformation quantisation. Let Δ be a closed interval in the real λ -axis which does not contain any critical value of the Hamiltonian $H(x)$. A formal spectral projector $E(\Delta)$ is the functional on the compactly supported elements of the quantum algebra with values in $\mathbb{C}[\hbar^{-1}, \hbar]$ given by

$$\langle E(\Delta), a \rangle = \int_{\Delta} \langle a \rangle_{\lambda} d\lambda.$$

If an interval $\Delta = [\lambda_1, \lambda_2]$ is free of the critical values of the Hamiltonian $H(x)$, then

$$\langle E(\Delta), a \rangle = \mathrm{Tr}_{\{H \leq \lambda_2\}} a - \mathrm{Tr}_{\{H \leq \lambda_1\}} a,$$

showing the relation of $E(\Delta)$ to the index formula for compact symplectic manifolds with boundary.

Definition 9.1. In the case where $E(\Delta)$ is a distribution with compact support, the expression $\langle E(\Delta), \hat{1} \rangle \in \mathbb{C}[\hbar^{-1}, \hbar]$ is called a formal spectral asymptotics and denoted by $N(\Delta)$.

This definition is motivated by analogy with operator formulas but in fact it has nothing to do with spectral theory. However, it is instructive to compare these formal spectral objects with genuine ones defined for \hbar -pseudodifferential operators, see [Ivr98].

For a treatment of the contribution of Morse critical points to the spectral decomposition we refer the reader to [Fed06], see also [Tar15].

10. AN EXAMPLE

The simplest example for which all the quantities in question may be calculated explicitly is an n -dimensional harmonic oscillator in \mathbb{R}^{2n} . Let $M = \mathbb{C}^n$ be endowed with symplectic form

$$\omega_M = \frac{1}{2i} \sum_{k=1}^n dz^k \wedge d\bar{z}^k.$$

Given any $\lambda > 0$, consider the Hamiltonian

$$H = \frac{1}{2} |z|^2 - \lambda$$

which has the only critical point at the origin. Then $M_- = \{|z| \leq \sqrt{2\lambda}\}$ is a compact manifold with boundary. The Hamiltonian vector field V_H on the boundary $M_0 = \{|z| = \sqrt{2\lambda}\}$ induced by H is

$$V_H = \sum_{k=1}^n iz^k \frac{\partial}{\partial z^k} - i\bar{z}^k \frac{\partial}{\partial \bar{z}^k},$$

i.e. $i(V_H)\omega_M = -dH$. The vector field V_H defines the structure of a circle bundle on M_0 whose base is $B = \mathbb{C}\mathbb{P}^{n-1}$.

The base B is a compact symplectic manifold with symplectic form $\omega_B = 2\pi\lambda c$ where

$$c = \frac{1}{2\pi i} \sum_{k=1}^n d \frac{\bar{z}^k}{|z|} \wedge d \frac{z^k}{|z|}$$

is the standard Fubini-Study form on $\mathbb{C}\mathbb{P}^{n-1}$.

The connection 1-form on $M_0 \times I$ is

$$\gamma = \frac{1}{2i} \sum_{k=1}^n \frac{\bar{z}^k}{|z|} d \frac{z^k}{|z|} - \frac{z^k}{|z|} d \frac{\bar{z}^k}{|z|} = \frac{1}{2i} \sum_{k=1}^n \frac{\bar{z}^k}{|z|^2} dz^k - \frac{z^k}{|z|^2} d\bar{z}^k$$

whence

$$\begin{aligned} i(V_H)\gamma &= 1, \\ d\gamma &= \frac{2}{2i} \sum_{k=1}^n d \frac{\bar{z}^k}{|z|} \wedge d \frac{z^k}{|z|} = 2\pi c. \end{aligned}$$

It is easy to verify that

$$\omega_M = \omega_B + d(H\gamma).$$

The index formula (8.2) for M_- reads

$$\mathrm{Tr} \hat{1} = \int_{M_-} \exp \frac{\omega_M}{2\pi\hbar} + \int_B \mathrm{tr}_b \exp \left(- \frac{d(H\gamma)}{2\pi\hbar} \right) \exp \frac{\omega_B}{2\pi\hbar} \hat{A}(\mathbb{C}\mathbb{P}^{n-1})$$

Since

$$\begin{aligned} \mathrm{tr}_b \exp \left(- \frac{Hd\gamma}{2\pi\hbar} \right) &= - \sum_{j=1}^{\infty} \frac{B_j}{j!} \hbar^{j-1} \partial_H^{j-1} \exp \left(- \frac{Hd\gamma}{2\pi\hbar} \right) \Big|_{H=0} \\ &= - \frac{1}{z} \left(\frac{z}{e^z - 1} - 1 \right) \Big|_{z = -\frac{d\gamma}{2\pi} = -c} \\ &= \frac{1}{1 - e^{-c}} - \frac{1}{c} \end{aligned}$$

and the Atiyah-Hirzebruch class is represented by the form

$$\begin{aligned} \hat{A}(\mathbb{C}\mathbb{P}^{n-1}) &= \left(\frac{c}{e^{c/2} - e^{-c/2}} \right)^n \\ &= e^{(n/2)c} \left(\frac{c}{e^c - 1} \right)^n \end{aligned}$$

(see [Fed96]), we obtain

$$(10.1) \quad \mathrm{Tr} \hat{1} = \int_{M_-} \exp \frac{\omega_M}{2\pi\hbar} + \int_B \left(\frac{1}{1 - e^{-c}} - \frac{1}{c} \right) \exp \frac{\lambda}{\hbar} c \left(\frac{c}{e^{c/2} - e^{-c/2}} \right)^n.$$

The integral over M_- of (10.1) is easily evaluated, namely

$$\int_{|z|^2 \leq 2\lambda} \exp \frac{\omega_M}{2\pi\hbar} = \frac{1}{n!} \left(\frac{\lambda}{\hbar} \right)^n,$$

and it remains to calculate the integral over B . An easy computation shows that the integrand of the boundary integral just amounts to

$$\begin{aligned} \left(\frac{1}{1 - e^{-c}} - \frac{1}{c} \right) e^{\frac{\lambda}{\hbar} c} \left(\frac{c}{e^{c/2} - e^{-c/2}} \right)^n &= \left(\frac{e^c}{e^c - 1} - \frac{1}{c} \right) e^{\left(\frac{\lambda}{\hbar} + \frac{n}{2} \right) c} \frac{c^n}{(e^c - 1)^n} \\ &= \frac{e^{\left(\frac{\lambda}{\hbar} + \frac{n}{2} + 1 \right) c}}{(e^c - 1)^{n+1}} c^n - \frac{e^{\left(\frac{\lambda}{\hbar} + \frac{n}{2} \right) c}}{(e^c - 1)^n} c^{n-1}. \end{aligned}$$

Write $f(c)$ for the function on the right-hand side of this equality. Since the integral

$$\int_{\mathbb{C}\mathbb{P}^{n-1}} c^{n-1}$$

just amounts to 1, it follows that the boundary integral in (10.1) is equal to the Taylor coefficient of $f(z)$ at z^{n-1} . This coefficient can be evaluated as the residue

$$\text{Res}_0 \frac{f(z)}{z^n} = \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{f(z)}{z^n} dz,$$

where ε is a small positive number. We now calculate

$$\text{Res}_0 \frac{f(z)}{z^n} = \text{Res}_0 \frac{e^{\left(\frac{\lambda}{\hbar} + \frac{n}{2} + 1\right)z}}{(e^z - 1)^{n+1}} - \text{Res}_0 \frac{e^{\left(\frac{\lambda}{\hbar} + \frac{n}{2}\right)z}}{z(e^z - 1)^n}.$$

The first residue is easily seen to be $C_{\frac{\lambda}{\hbar} + \frac{n}{2}}^n$, where

$$C_Q^n = \binom{Q}{n} = \frac{Q(Q-1)\dots(Q-n+1)}{n!},$$

see Lemma 3.5 of [Fed06]. To calculate the more difficult second residue, we make the change of variables $\zeta = e^z - 1$, thus obtaining

$$\begin{aligned} z &= \log(1 + \zeta), \\ dz &= \frac{d\zeta}{1 + \zeta}. \end{aligned}$$

Then

$$(10.2) \quad \text{Res}_0 \frac{e^{\left(\frac{\lambda}{\hbar} + \frac{n}{2}\right)z}}{z(e^z - 1)^n} = \text{Res}_0 \frac{(1 + \zeta)^{\frac{\lambda}{\hbar} + \frac{n}{2} - 1}}{\zeta^n \log(1 + \zeta)},$$

the function $\log(1 + \zeta)$ being holomorphic in the disk of radius 1 with centre at the origin and it has a simple zero at $\zeta = 0$. Write

$$\frac{\zeta}{\log(1 + \zeta)} = \sum_{k=0}^{\infty} c_k \zeta^k,$$

where c_k are explicit real numbers. For instance, we have

$$c_0 = 1, c_1 = \frac{1}{2}, c_2 = -\frac{1}{12}, c_3 = \frac{1}{24},$$

etc. Substituting this expansion into (10.2) yields

$$\begin{aligned} \text{Res}_0 \frac{e^{\left(\frac{\lambda}{\hbar} + \frac{n}{2}\right)z}}{z(e^z - 1)^n} &= \text{Res}_0 \frac{\sum_{j=0}^{\infty} \left(\sum_{k=0}^j \binom{Q}{k} c_{j-k} \right) \zeta^j}{\zeta^{n+1}} \\ &= \sum_{k=0}^n \binom{Q}{k} c_{n-k}, \end{aligned}$$

where $Q = \frac{\lambda}{\hbar} + \frac{n}{2} - 1$.

Summarising we arrive at the following index formula for the harmonic oscillator at the sublevel $\{|z| \leq \sqrt{2\lambda}\}$ in \mathbb{C}^n .

Theorem 10.1. *For each $\lambda > 0$, the index of deformation quantisation on the manifold $M_- = \{|z| \leq \sqrt{2\lambda}\}$ is given by*

$$\mathrm{Tr} \hat{1} = \frac{1}{n!} \left(\frac{\lambda}{\hbar} \right)^n + C_{\frac{\lambda}{\hbar} + \frac{n}{2}}^n - \sum_{k=0}^n C_{\frac{\lambda}{\hbar} + \frac{n}{2} - 1}^k C_{n-k}.$$

We complete the section by a short discussion of the index formula for the particular cases $n = 1, 2, 3$. For $n = 1$, it takes the form

$$\mathrm{Tr} \hat{1} = \frac{\lambda}{\hbar} + \left(\frac{\lambda}{\hbar} + \frac{1}{2} \right) - \left(\frac{1}{2} + \frac{\lambda}{\hbar} - \frac{1}{2} \right) = \frac{\lambda}{\hbar} + \frac{1}{2},$$

which is precisely $C_{\frac{\lambda}{\hbar} + \frac{1}{2}}^1$.

When starting this research the authors believed that the index formula would look like

$$\mathrm{Tr} \hat{1} = C_{\frac{\lambda}{\hbar} + \frac{n}{2}}^n.$$

However, this equality holds merely in an asymptotic sense when $\lambda \rightarrow \infty$. Namely, for $n = 2$ the formula reads

$$\mathrm{Tr} \hat{1} = \frac{1}{2} \left(\frac{\lambda}{\hbar} \right)^2 + \frac{1}{2} \left(\frac{\lambda}{\hbar} + 1 \right) \frac{\lambda}{\hbar} - \left(-\frac{1}{12} + \frac{1}{2} \frac{\lambda}{\hbar} + \frac{1}{2} \frac{\lambda}{\hbar} \left(\frac{\lambda}{\hbar} - 1 \right) \right),$$

which reduces to $C_{\frac{\lambda}{\hbar} + 1}^2 + \frac{1}{12}$.

Finally, in the case $n = 3$ we get

$$\begin{aligned} \mathrm{Tr} \hat{1} &= \frac{1}{6} \left(\frac{\lambda}{\hbar} \right)^3 + \frac{1}{6} \left(\frac{\lambda}{\hbar} + \frac{3}{2} \right) \left(\frac{\lambda}{\hbar} + \frac{1}{2} \right) \left(\frac{\lambda}{\hbar} - \frac{1}{2} \right) \\ &\quad - \left(\frac{1}{24} - \frac{1}{12} \left(\frac{\lambda}{\hbar} + \frac{1}{2} \right) + \frac{1}{4} \left(\frac{\lambda}{\hbar} + \frac{1}{2} \right) \left(\frac{\lambda}{\hbar} - \frac{1}{2} \right) + \frac{1}{6} \left(\frac{\lambda}{\hbar} + \frac{1}{2} \right) \left(\frac{\lambda}{\hbar} - \frac{1}{2} \right) \left(\frac{\lambda}{\hbar} - \frac{3}{2} \right) \right), \end{aligned}$$

which reduces to $C_{\frac{\lambda}{\hbar} + \frac{3}{2}}^3 + \frac{1}{8} \frac{\lambda}{\hbar}$.

REFERENCES

- [APS75] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Camb. Phil. Soc. **77** (1975), 43–69.
- [BFFLS78] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerovitz, and D. Sternheimer, *Deformation theory and quantization*, Ann. Phys. **111** (1978), 61–151.
- [BdM71] L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math. **126** (1971), no. 1–2, 11–51.
- [BdMG81] L. Boutet de Monvel and V. Guillemin, *The Spectral Theory of Toeplitz Operators*, Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1981.
- [CMR14] A. S. Cattaneo, P. Mnev, and N. Reshetikhin, *Classical BV theories on manifolds with boundary*, Commun. Math. Phys. **332** (2014), 535–603.
- [Fed74] B. V. Fedosov, *An analytic formula for the index of an elliptic boundary value problem*, I, Math. USSR Sbornik **22** (1974), No. 1, 61–90; II, Math. USSR Sbornik **24** (1974), No. 4, 511–535; III, Math. USSR Sbornik **30** (1976), No. 3, 341–359.
- [Fed91] B. V. Fedosov, *Index Theorems*, Current Problems of Mathematics. Fundamental Directions. Vol. 65, VINITI, Moscow, 1991, pp. 165–268.
- [Fed94] B. V. Fedosov, *A simple geometrical construction of deformation quantization*, J. Differential Geom. **40** (1994), 213–238.
- [Fed96] B. V. Fedosov, *Deformation Quantization and Index Theory*, Akademie-Verlag, Berlin, 1996.
- [Fed98] B. Fedosov, *Non-Abelian reduction in deformation quantization*, Lett. Math. Phys. **43** (1998), 137–154.
- [Fed00] B. Fedosov, *On G-trace and G-index in deformation quantization*, Lett. Math. Phys. **52** (2000), 29–49.
- [Fed01] B. Fedosov, *Pseudodifferential operators and deformation quantization*, Quantization of Singular Symplectic Quotients, Vol. 198, Birkhäuser Verlag, Basel/Switzerland, 2001, pp. 95–118.
- [FGLS96] B. V. Fedosov, F. Golse, E. Leichtmann, and E. Schrohe, *The non-commutative residue for manifolds with boundary*, J. Funct. Anal. **142** (1996), 1–31.
- [FST04] B. Fedosov, B.-W. Schulze, and N. Tarkhanov, *On index theorem for symplectic orbifolds*, Ann. Inst. Fourier (Grenoble) **54** (2004), No. 5, 1601–1639.
- [Fed06] B. Fedosov, *On a spectral theorem for deformation quantisation*, Int. J. Geom. Methods Mod. Phys. **3** (2006), No. 8, 1609–1626.
- [FZ02] F. Faure and B. Zhilinskii, *Qualitative features of intra-molecular dynamics. What can be learned from symmetry and topology?* Acta Applicandae Mathematicae **70** (2002), 265–282.
- [Gru86] G. Grubb, *Functional Calculus of Pseudodifferential Boundary Problems*, Birkhäuser Verlag, Basel et al., 1986.
- [Ivr98] V. Ivrii, *Microlocal Analysis and Precise Spectral Asymptotics*, Springer Verlag, Berlin et al., 1998.
- [NT96] R. Nest and B. Tsygan, *Formal deformations on symplectic manifolds with boundary*, J. Reine Angew. Math. **481** (1996), 27–54.
- [Shu87] M. A. Shubin, *Pseudodifferential Operators and Spectral Theory*, Springer-Verlag, Berlin et al., 1987.
- [Tar15] N. Tarkhanov, *A Spectral Theorem for Deformation Quantisation*, Preprints of Inst. of Math., Univ. of Potsdam **4** (2015), No. 4, 7 pp.

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