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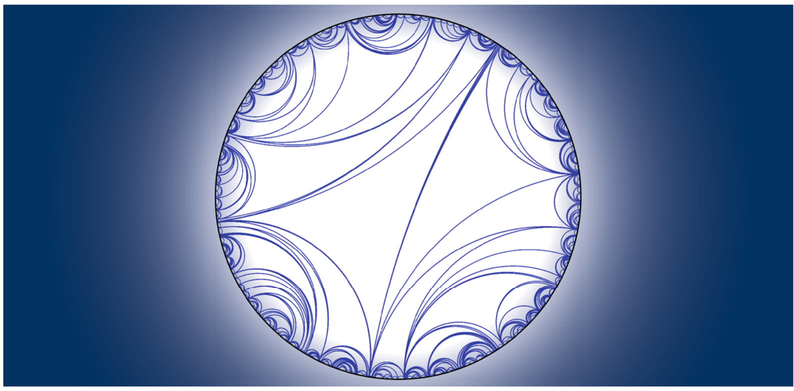

Nikolai Tarkhanov

A Spectral Theorem for Deformation Quantisation

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A SPECTRAL THEOREM FOR DEFORMATION QUANTISATION

NIKOLAI TARKHANOV

This paper is dedicated to B. V. Fedosov

ABSTRACT. This is a short notice on a spectral theorem in deformation quantisation I actually learned from B. V. Fedosov in 2008.

CONTENTS

MOTIVATION

Spectral problems as such are meaningless in the context of deformation quantisation. B. V. Fedosov proposed a reformulation for these problems which was based on the index formula.

The central idea is to look at the unit element of the algebra of quantum observables on a compact symplectic manifold and at its trace, which can be interpreted as the index. The eigenvalue problem can be reformulated as follows in the language of deformation quantization: For fixed λ, find all values of the Planck constant $\hbar > 0$ such that the index $I_\lambda(1/\hbar)$ takes positive integer values. In [FST04], with an example of a 2-dimensional harmonic oscillator in the resonance case, it is shown that the index theorem for symplectic orbifolds gives the correct energy levels. In [Fed06] and [FI07] a construction of an eigenstate for a noncritical level of the Hamiltonian function is elaborated and the contributions of Morse critical points evaluated.

This work is intended to present another approach to the spectral theorem of [Fed06], [FI07] which is closely related to our paper [FST04]. This enables us to complete the study of deformation quantisation on a compact symplectic manifold with boundary, see [FT15].

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Let H be a selfadjoint operator in a Hilbert space U and $u \in U$ an eigenfunction of H corresponding to an eigenvalue λ, i.e., $Hu = \lambda u$. We define a functional $a \mapsto \langle a \rangle_\lambda$ on operators a in U called the eigenstate of H with eigenvalue λ. More precisely, set $\langle a \rangle_\lambda = (au, u)_U$ for $a \in \mathcal{L}(U)$. Using the selfadjointness of H one sees readily that

$$\Big\langle \frac{Ha + aH}{2} \Big\rangle_\lambda = \lambda \, \langle a \rangle_\lambda \qquad (0.1)$$

for all $a \in \mathcal{L}(U)$.

If the spectrum of H is purely discrete and the eigenfunctions are normalised, i.e., $\|u\| = 1$, then

$$\sum_{\lambda \in \operatorname{Spec} H} \langle a \rangle_\lambda = \operatorname{Tr} a \qquad (0.2)$$

for all operators $a \in \mathcal{L}(U)$ of trace class. If the spectrum is absolutely continuous then formula (0.2) takes the form

$$\int_{-\infty}^{\infty} \langle a \rangle_\lambda d\lambda = \operatorname{Tr} a.$$

This equality serves as a normalisation condition and it expresses the completeness of eigenfunctions of H. We are going to set out this spectral formula in the context of deformation quantisation, that is without using the concepts "operator," "eigenfunction," and so on.

1. Deformation quantisation

Suppose $\{\mathcal{X}, \omega\}$ is a symplectic manifold and F a vector bundle over $\mathcal{X}$. The bundle $K = \operatorname{Hom}(F, F)$ will be referred to as the coefficient bundle.

By a formal function with coefficients in K is meant any formal series in powers of a small parameter h

$$a(x, h) = \sum_{k=1}^{\infty} a_k(x) h^k,$$

where $a_k \in C^\infty(\mathcal{X}, K)$ are smooth sections of K over $\mathcal{X}$. We write $A_h = A_h(\mathcal{X}, K)$ for the space of all formal functions with coefficients in K.

Using physical language we call $a_0(x)$ a classical observable, $a(x, h)$ a quantum observable and $a_1(x)h, a_2(x)h^2, \ldots$ quantum corrections of the first, second, etc. order. We regard $a(x, h)$ as a known quantity if we can calculate quantum corrections of any prescribed order.

Deformation quantisation with coefficients in K is an associative algebra structure in A_h with respect to a star product. For $a, b \in A_h$ independent of h, the star product is

$$a * b = \sum_{k=0}^{\infty} c_k(a, b) h^k,$$

where $c_k(a, b)$ are bidifferential operators of order $\le k$ in a and b (locality), such that

1) $c_0(a, b) = ab$ is the pointwise product of matrix-valued functions (a classical product).

The classical product need not be commutative in general, for the matrix product fails to be commutative. However, if F is a vector bundle of rank 1, then K is actually a trivial bundle of rank 1, so the classical product is commutative in this case. From this it follows that the star product is a deformation of the classical product.

2) $c_1(a,b) = (1/2\imath)\omega^{ij}\partial_i a \partial_j b$, where ω^{ij} is the inverse of the matrix ω_{ij} of the symplectic form $\omega = (1/2)\omega_{ij}dx^i \wedge dx^j$ (the Poisson tensor), and ∂ a connection in F and thus in K.

Notice that in the scalar case $\omega^{ij}\partial_i a \partial_j b =: \{a,b\}$ just amounts to a Poisson bracket.

As defined above, the star product extends to general formal functions by linearity. The function $a \equiv 1$ is the unity of the algebra A_h.

By a trace on the algebra $\{A_h, *\}$ is meant any functional $\mathrm{Tr} : (A_h)_{\mathrm{comp}} \to \mathbb{C}[h]]$ of the form

$$a(x,h) \mapsto \sum_{k=0}^{\infty} c_k h^{k-n}$$

with $n = (1/2)\dim \mathcal{X}$, which satisfies $\mathrm{Tr}\, a * b = \mathrm{Tr}\, b * a$.

The simplest and classical example of deformation quantisation with trace is the Weyl algebra on $\mathbb{R}^{2n}$ with a constant symplectic form $\omega = \omega_0$.

Example 1.1. For $a = a(x)$ and $b = b(x)$ one defines

$$\begin{aligned}(a*b)(x,h) &= \frac{1}{(\pi h)^{2n}} \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{2\frac{\imath}{h}\omega(t,\tau)} a(x+t)b(x+\tau)\,dt\,d\tau \\ &= \exp\Big(\frac{1}{2}\frac{h}{\imath}\omega^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\Big) a(x)b(y)\,|_{x=y},\end{aligned}$$

and the trace is

$$\mathrm{Tr}\, a(x,h) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^{2n}} \mathrm{tr}\, a(x,h)\,\frac{\omega^n}{n!}.$$

It is known (see [Fed95]) that

1) Given any symplectic (Poisson) manifold $\mathcal{X}$ and a vector bundle F, there is a star product in A_h.

2) Given any symplectic connection on $\mathcal{X}$ and a connection on F, there is a canonical construction of star product in A_h.

3) There is a unique trace (up to a formal constant factor $c = c(h)$) on the algebra $\{A_h, *\}$. Moreover, there is a natural normalisation of the trace, such that the index formula

$$\mathrm{Tr}\, 1 = \int_{\mathcal{X}} \exp\frac{\omega}{2\pi\imath h}\,\mathrm{ch}F\,\hat{A}(\mathcal{X})$$

holds, where

$$\begin{aligned}\mathrm{ch}F &= \mathrm{tr}\,\exp\left(R^F/2\pi\imath\right), \\ \hat{A}(\mathcal{X}) &= \det{}^{-\frac{1}{2}}\frac{\sinh\left(R^s/4\pi\imath\right)}{R^s/4\pi\imath}.\end{aligned}$$

4) Any algebra $\{A_h, *\}$ is locally equivalent to the Weyl algebra on $\{\mathbb{R}^{2n}, \omega_0\}$ and this isomorphism preserves the traces.

2. Noncritical case

Consider a formal function of the form

$$H(x,h) = H_0(x) + H_1(x)h + \dots,$$

where H_0 is a real-valued function (more precisely, $H_0 \otimes I_K$), and the higher-order terms are any sections of the vector bundle K. Such a function is called a Hamiltonian.

We first discuss the case when H_0 has no critical points in the set of all $x \in \mathcal{X}$, such that $\lambda_1 \le H_0(x) \le \lambda_2$.

Theorem 2.1. *For any $\lambda \in [\lambda_1, \lambda_2]$ there is a unique functional*

$$\langle \cdot \rangle_\lambda : (A_h)_{\mathrm{comp}} \to \mathbb{C}[h]]$$

with the properties

1) $\left\langle \dfrac{H * a + a * H}{2} \right\rangle_\lambda = \lambda \, \langle a \rangle_\lambda$;

2) $\displaystyle\int_{\lambda_1}^{\lambda_2} \langle a \rangle_\lambda d\lambda = \operatorname{Tr} a$ *for all a with support in $\lambda_1 \le H_0 \le \lambda_2$.*

Moreover, for the canonical quantisation defined by the connections ∂^s and ∂^F, this functional has the form

$$\langle a(x) \rangle_\lambda = \frac{1}{(2\pi h)^n} \sum_{k=0}^{\infty} \langle \delta^{(k)}(H_0 - \lambda), \operatorname{tr} S_k(x,h) a(x) \rangle,$$

where $S_k(x,h)$ are formal functions whose coefficients are polynomials in curvatures R^s and R^F and their covariant derivatives and the covariant derivatives of the Hamiltonian. The first two terms are

$$\begin{aligned} \langle a(x) \rangle_\lambda &= \frac{1}{(2\pi h)^n} \Big(\langle \delta(H_0 - \lambda), \operatorname{tr} a \rangle \\ &- \frac{1}{2} \frac{h}{\imath} \langle \delta(H_0 - \lambda), \operatorname{tr} R^F_{ij} \omega^{ij} a(x) \rangle + h \, \langle \delta'(H_0 - \lambda), \operatorname{tr} H_1(x) a(x) \rangle \\ &+ O(h^2) \Big). \end{aligned}$$

The quadratic terms in h also may be written down explicitly, however, this expression is too cumbersome.

Sketch of the proof. Uniqueness We get

$$\begin{aligned} \int \lambda \, \langle a \rangle_\lambda d\lambda &= \int \left\langle \frac{H * a + a * H}{2} \right\rangle_\lambda d\lambda \\ &= \operatorname{Tr} \frac{H * a + a * H}{2} \\ &=: \operatorname{Tr} S_H a \end{aligned}$$

whence

$$\begin{aligned} \int p(\lambda) \, \langle a \rangle_\lambda d\lambda &= \int \langle p(S_H) a \rangle_\lambda d\lambda \\ &= \operatorname{Tr} p(S_H) a \end{aligned}$$

for any polynomial p in λ.

If we have two eigenstates $\langle a \rangle^1_\lambda$ and $\langle a \rangle^2_\lambda$, then for their difference $\Delta \, \langle a \rangle_\lambda$ we obtain

$$\int p(\lambda) \, \Delta \, \langle a \rangle_\lambda d\lambda = 0.$$

Thus, $\Delta \, \langle a \rangle_\lambda = 0$, for $\Delta \, \langle a \rangle_\lambda$ is orthogonal to all polynomials in λ.

Existence Locally the algebra $\{A_h, *\}$ may be reduced to the Weyl algebra on $\{\mathbb{R}^{2n}, \omega_0\}$ and the Hamiltonian may be reduced to x^1. In this particular case the functional

$$\begin{aligned} \langle a \rangle_\lambda &= \frac{1}{(2\pi h)^n} \int_{x^1=\lambda} \operatorname{tr} a(x)\, dx^2 \dots dx^{2n} \\ &= \frac{1}{(2\pi h)^n} \langle \delta(x^1 - \lambda), \operatorname{tr} a \rangle \end{aligned}$$

satisfies all the conditions. Because of uniqueness it does not depend on trivialisations, so the general case can be treated by using a proper partition of unity, as desired. □

Finally, the invariant formulas follow from the canonical construction of deformation quantisation.

3. Morse critical points

Suppose that x_0 is a Morse critical point of $H_0(x)$ in a strip $\lambda_1 \le H_0 \le \lambda_2$. The question is, how does the spectral decomposition change in the presence of the critical point. This problem is purely local, so we can restrict our discussion to the case of the Weyl algebra on $\mathbb{R}^{2n}$ with the critical point of H_0 at the origin $x = 0$. To wit,

$$H_0(x) = \frac{1}{2} g_{ij} x^i x^j + O(x^3).$$

Theorem 3.1. *There is a limit*

$$\lim_{\varepsilon \to 0+} \Big(\operatorname{Tr} a - \Big(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \Big) \langle a \rangle_\lambda \, d\lambda \Big) = \langle f, a \rangle,$$

where f is a distribution supported at $x = 0$. The leading term in h is

$$\frac{\cos\left(\frac{\pi}{2} q\right)}{\sqrt{|\det g|}} \langle \delta(x), \operatorname{tr} a(x) \rangle,$$

q being the inertia index (i.e., the number of negative squares in the canonical form) of g_{ij}.

The theorem shows that the critical values of $H_0(x)$ play actually the role of the discrete spectrum.

Proof. The proof is based on another representation of the functional $\langle a \rangle_\lambda$ for $\lambda \neq 0$ in the Weyl algebra. It is related to a counterpart of the Schrödinger equation in deformation quantisation. □

4. The Schrödinger equation in deformation quantisation

To this end we need the solution of the Schrödinger evolution equation in the Weyl algebra

$$\begin{aligned} \frac{dU}{dt} &= \frac{\imath}{h} H * U, \\ U\,|_{t=0} &= 1. \end{aligned}$$

The first question is what is the meaning of this solution. The point is that U is the star exponential

$$\exp\left(\frac{\imath}{h} H t\right)$$

while this expression makes no sense within the framework of deformation quantisation, for the negative powers of h are not bounded. So, we need more general functions.

By a formal WKB function is meant any expression

$$U(x,h) = e^{\frac{\imath}{h}\varphi(x)}\, v(x,h),$$

where $\varphi(x)$ is a real valued phase function and $v(x,h)$ is any formal function of A_h called the amplitude of U.

Lemma 4.1. *The WKB functions form a module over the algebra A_h with respect to the Weyl star product.*

Proof. Let us define

$$\begin{aligned}(a*U)(x,h) &= \frac{1}{(\pi h)^{2n}} \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{2\frac{\imath}{h}\omega(t,\tau)} a(x+t) e^{\frac{\imath}{h}\varphi(x+\tau)} v(x+\tau) dt d\tau \\ &= \frac{1}{(\pi h)^{2n}} \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{2\frac{\imath}{h}\omega(t,\tau)+\frac{\imath}{h}\varphi(x+\tau)} a(x+t) v(x+\tau) dt d\tau.\end{aligned}$$

We understand the integral on the right-hand side as a formal expansion by the stationary phase method. The phase function is $2\omega(t,\tau)+\varphi(x+\tau)$, and so the critical points are determined from the system

$$\begin{aligned}2\omega_{ij}\tau^j &= 0, \quad \text{for} \quad i = 1,\dots,2n, \\ 2\omega_{ij}t^i + \partial_j\varphi(x+\tau) &= 0, \quad \text{for} \quad j = 1,\dots,2n.\end{aligned}$$

Since the symplectic form ω_0 is nondegenerate, it follows readily that $\tau = 0$ whence $t = (1/2)\,\nabla\varphi(x)$.

Thus, $(a*U)(x,h) = e^{\frac{\imath}{h}\varphi(x)}\, w(x,h)$ holds with some formal function $w(x,h)$, as desired. □

If so, we can solve the Schrödinger evolution equation for sufficiently small interval $|t| < \delta$ using the WKB method. Note that the formal expansion given by the standard WKB method yields an exact solution within the framework of deformation quantisation.

Now, having an evolution operator $U(x,h,t)$, we are in a position to find the spectrum by an inverse Fourier transform. For deformation quantisation the corresponding formula looks like

Theorem 4.2. *For $\lambda \neq 0$, it follows that*

$$\langle a\rangle_\lambda = \frac{1}{2\pi h}\int_{-\infty}^{\infty} e^{-\frac{\imath}{h}\lambda t}\,\chi(t)\,\mathrm{Tr}\,a(x,h) * U(x,h,t)\,dt,$$

where χ is a cut-off function for $0 \in \mathbb{R}$ and the integral is understood as formal stationary phase expansion for

$$\frac{1}{(2\pi h)^{n+1}} \iint_{\mathbb{R}^{2n}\times\mathbb{R}} e^{\frac{\imath}{h}(\varphi(x,t)-\lambda t)}\,\chi(t)\,\mathrm{tr}\,\left(v(x,h,t)a(x,h)\right) dx dt.$$

Recall that by a cut-off function for the origin in $\mathbb{R}$ is meant any C^∞ function on the real axis which is supported in an interval $(-\delta,\delta)$ and equal to 1 in a smaller interval around the origin.

Proof. One can check directly that the functional on the right-hand side satisfies the definition of the eigenstate. By uniqueness, it coincides with $\langle a\rangle_\lambda$ introduced earlier. □

For quadratic Hamiltonians $H(x,h) = (1/2)g_{ij}x^ix^j + ch$ the WKB approximation gives an explicit solution. Then, for a general $H = H_{\mathrm{quad}} + H_{\geq 3}$, the degree of $(\imath/h)H_{\mathrm{quad}}$ just amounts to zero and the degree of $(\imath/h)H_{\geq 3}$ is at least 1, if we prescribe the degree 1 to each variable x^i and the degree 2 to h. So we can develop a perturbation theory using quadratic Hamiltonians as unperturbed ones. It is similar to that of quantum field theory but, of course, much simpler. Like the WKB method it gives a precise formula for a discrete eigenstate within the framework of deformation quantisation.

5. An index formula

There is a link between the eigenstate formula and the index formula. If the Hamiltonian flow is periodic (the free $U(1)$ action) on $\{\mathcal{X}, \omega\}$ and $H_0 = \lambda$ is a noncritical level, then the symplectic reduction to a symplectic base $\{B, \omega_B\}$ is possible (for both classical and deformation quantisation levels). Then, one can easily prove that if $H_0 = \lambda$ is compact then

$$\langle 1 \rangle_\lambda = \mathrm{Tr}\,_B 1$$

which is precisely the index of $\{B, \omega_B\}$.

Remark 5.1. For quadratic Hamiltonians $H(x) = (1/2)g_{ij}x^ix^j$ the eigenstate formula reads

$$\langle a \rangle_\lambda = \frac{1}{(2\pi h)^n} \sum_{k=0}^{\infty} \frac{1}{k!} \Big(\imath h \frac{\partial^2}{\partial t \partial \lambda} \Big)^k \Big\langle \delta(H - \lambda), \det{}^{-\frac{1}{2}} \Big(\frac{\sinh \frac{At}{2}}{\frac{At}{2}} \Big) a \Big(\Big(\frac{\tanh \frac{At}{2}}{\frac{At}{2}} \Big)^{-\frac{1}{2}} x \Big) \Big\rangle |_{t=0},$$

where A is the infinitesimal symplectic matrix $\big(\omega^{ij} g_{jk}\big)$.

The functions

$$\det{}^{-\frac{1}{2}} \Big(\frac{\tanh \frac{At}{2}}{\frac{At}{2}} \Big),$$

$$\det{}^{-\frac{1}{2}} \Big(\frac{\sinh \frac{At}{2}}{\frac{At}{2}} \Big)$$

are familiar from the theory of characteristic classes, see [MS74] and elsewhere. The first function generates the so-called Hirzebruch L-genus and the second one generates the Atiyah-Hirzebruch $\hat{A}$-class. This curious fact should be investigated deeper.

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