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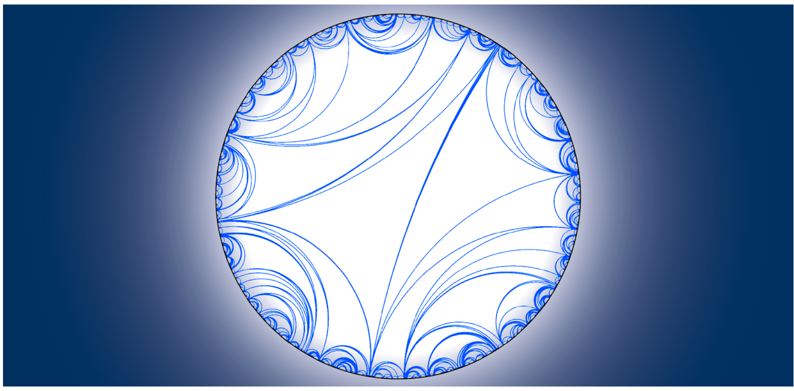

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On the Construction of Point Processes in Statistical Mechanics

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January 25, 2013

Abstract

By means of the cluster expansion method we show that a recent result of Suren Poghosyan and Daniel Ueltschi [21] combined with one of Benjamin Nehring [17] yield a construction of point processes of classical statistical mechanics as well as processes related to the Ginibre Bose gas of Brownian loops and to the dissolution in $\mathbb{R}^d$ of Ginibre's Fermi-Dirac gas of such loops. The latter will be identified as a Gibbs perturbation of the ideal Fermi gas. By generalizing these considerations we will obtain the existence of a large class of Gibbs perturbations of so called KMM-processes as they were introduced in [17]. Moreover, it is shown that certain "limiting Gibbs processes" are Gibbs in the sense of Dobrushin, Lanford and Ruelle if the underlying potential is positive. And finally, Gibbs modifications of infinitely divisible point processes are shown to solve a new integration by parts formula if the underlying potential is positive.

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1 Introductory remarks

We reconsider the problem of construction of interacting point processes which are of importance in statistical physics. They include Gibbs processes of classical statistical mechanics; but also processes which are associated to continuous quantum systems in the sense of Ginibre [4].

Earlier approaches can be found in the work of Kondratiev et al. [8] in the case of Boltzmann statistics, and in the thesis of Kuna [9] as well as in Rebenko [22], where one can find some remarks with respect to Bose-Einstein and Fermi-Dirac statistics. But several questions are left open here.

The method we use is a new version of cluster expansions which had been developed in [18, 17] and which is summarized in Theorem 1. In a first step we then construct in Theorem 2, in the context of statistical mechanics, by means of this method limiting interacting processes by combining a recent result of Poghosyan and Ueltschi [21] with Theorem 1 . As a first application we consider the quantum Bose gas of Ginibre. This yields a point process of interacting winding loops. One of the main assumption of Theorem 2 is the positivity of the reference measure.

But in Ginibre's analysis in [4] of the quantum Fermi-Dirac gas there appears a *signed* reference measure. Therefore we cannot use our construction and cannot proceed as we did in case of the Bose gas. But assuming for a moment, in the case of polygonal loops, the existence of a cluster process for this gas and dissolving its clusters into its particles, we obtain a point process in Euclidean space, which we are able to construct by means of our methods. The resulting process is a Gibbsian modification of a determinantal point process. In a more general setting such processes are then constructed in Theorem 3 . As examples we consider Gibbs modifications of the Poisson respectively determinantal process.

An important question then is what kind of processes are the limiting processes. In Theorem 4 we can show that under natural regularity conditions they are Gibbs in the sense of Dobrushin/Lanford/Ruelle (DLR) if the underlying interaction is positive.

Finally, it is shown that Gibbs modifications with positive pair potential of infinitely divisible point processes solve a new integral equation involving the Campbell measure of the process. This equation generalizes the integration by parts formula of Nguyen X.X., Zessin [20] which is equivalent to the DLR-equation. Examples of such processes are Gibbs modifications of the ideal Bose gas.

2 Random measures and point processes

A point process is a random mechanism realizing configurations of particles in space. Our approach to design such a mechanism uses a generalization of the cluster expansion method, which, in the words of Dobrushin, traces back to the deeps of theoretical physics.

We introduce some basic concepts and standard results from the theory of point processes which we take from the monographs [10, 12, 7]. The basic underlying *phase space* is a Polish space $(X, \mathcal{B}, \mathcal{B}_0)$; i.e. a complete separable metric space (X, d). Our main examples of phase spaces are discrete spaces, the Euclidean space $E = \mathbb{R}^d$, the space $\mathfrak{X} = \mathcal{M}_f^{\cdot\cdot}(E)$ of *finite* configurations in E as defined below, which may have multiple points, and the space of Brownian loops in E. $\mathcal{B}$ denotes the corresponding Borel $\sigma-$field and $\mathcal{B}_0$ the ring of all bounded sets in X.

By $\mathcal{M} = \mathcal{M}(X)$ we denote the set of all measures μ on $\mathcal{B}$ taking only finite values on $\mathcal{B}_0$. We call them *Radon measures* here. This set will be given the following topological and measurable structure: Denote by F the set of all $\mathcal{B}-$measurable mappings $f : X \longrightarrow [0, \infty]$, and F_c the subset of all bounded and continuous $f \in F$ with bounded support supp f. We also need the space F_b of bounded $f \in F$ with bounded support.

Denote then by

$$\zeta_f(\mu) = \mu(f) = \int_X f(x)\, \mu(\mathrm{d}\, x) \qquad , \quad \mu \in \mathcal{M}, f \in F,$$

the integral as a function of the underlying measure. The *vague topology on* $\mathcal{M}$ now is defined as the topology generated by all mappings $\zeta_f, f \in F_c$. $\mathcal{M}$, provided this topology, is a Polish space; the corresponding $\sigma-$algebra of Borel subsets $\mathcal{B}(\mathcal{M})$ is the one generated by all mapping $\zeta_B, B \in \mathcal{B}_0$.

A *random measure on the phase space* X is a random element in $\mathcal{M}(X)$. The collection of their distributions P is denoted by $\mathcal{PM} = \mathcal{PM}(X)$. But we'll consider more generally also other measures on $\mathcal{M}$.

A measure $\mu \in \mathcal{M}$ is called a counting or *point measure* if it takes only integer values on $\mathcal{B}_0$. The set of all point measures is denoted by $\mathcal{M}^{\cdot\cdot} =$

$\mathcal{M}^{\cdot\cdot}(X)$. It is well known that any $\mu \in \mathcal{M}^{\cdot\cdot}$ is of the form

$$\mu = \sum_{x \in \text{supp } \mu} \mu(\{x\}) \cdot \delta_x \quad .$$

$\mathcal{M}^{\cdot\cdot}$ considered as a subspace of $\mathcal{M}$ is vaguely closed and thereby a Borel set in $\mathcal{M}$. Moreover, it is a Polish space; again the corresponding $\sigma-$algebra of Borel subsets is generated by all mapping $\zeta_B, B \in \mathcal{B}_0$. $\mathfrak{X} = \mathcal{M}^{\cdot\cdot}_f(X)$ denotes the subset of finite counting measures on X. If G is a Borel set in X then we denote by $\mathfrak{X}(G)$ the collection of all configurations contained in G.

Now a *point process in* X is a random element in $\mathcal{M}^{\cdot\cdot}(X)$. The collection of their laws P is denoted by $\mathcal{PM}^{\cdot\cdot}(X)$.

The *Laplace transform* of a random measure P is defined by

$$\mathcal{L}_P(f) = \int_{\mathcal{M}} \exp(-\zeta_f) \, \mathrm{d}\, P \quad , f \in F.$$

It determines the process completely. The first moment measure of P is defined by

$$\nu_P(f) = \int_{\mathcal{M}^{\cdot\cdot}} \mu(f) \, P(\mathrm{d}\, \mu) \quad , f \in F.$$

If ν_P is a Radon measure we say that P *is of first order*.

A more general notion containing this one is the Campbell measure of P definied by

$$\mathcal{C}_P(h) = \int_{\mathcal{M}^{\cdot\cdot}} \int_X h(x, \mu) \, \mu(\mathrm{d}\, x) P(\mathrm{d}\, \mu) \quad , h \in F.$$

We'll use these notions also when P is replaced by $\sigma-$finite measures L.

3 A general construction of processes by means of the cluster expansion method

We consider the construction of point processes by means of the cluster expansion method on an abstract level first in the finite case within the setting of [12]. Then we indicate briefly the infinitely extended case. This construction has been developed in [17, 18].

Let $\mathbf{E}$ denote the set of *finite, signed* measures L, M on $\mathcal{M}^{\cdot\cdot}(X)$. Each L can be represented in a unique way as the difference $L = L^+ - L^-$ of measures in $\mathbf{E}^+$, the subspace of positive measures in $\mathbf{E}$, that are purely singular with respect to each others. (*Jordan decomposition of* L) $\mathbf{E}$ is a normed space with respect to $\|L\| = L^+(\mathcal{M}^{\cdot\cdot}) + L^-(\mathcal{M}^{\cdot\cdot})$, the total mass of the *variation* $|L| = L^+ + L^-$ of L. The distance $\|M - L\|$ is called *variation distance.* Denote by $\mathbf{E}^+$ the subspace of positive measures.

With respect to the convolution operation $*$ and $\|.\|$, the vector space $\mathbf{E}$ is a commutative real Banach algebra with unit δ_o, o denoting here the measure zero on $\mathcal{M}^{\cdot\cdot}(X)$. Thus in particular

$$\|L * M\| \leq \|L\| \cdot \|M\| \quad .$$

For all $L \in \mathbf{E}$ the series

$$\exp\, L = \sum_{n=0}^{\infty} \frac{1}{n!} L^{*n}$$

converges absolutely. Here $L^0 = \delta_o$. It has the property

$$\exp(L_1 + L_2) = \exp(L_1) * \exp(L_2) \qquad , L_1, L_2 \in \mathbf{E}. \tag{1}$$

Here $*$ denotes convolution. All this can be found in [12].

Lemma 1 *If $L, M \in \mathbf{E}$ have the same Laplace transform then they coincide.*

This can be seen immediately using the Jordan decomposition of L, F.

The general scheme of the construction

We start with a finite signed measure L on $\mathfrak{X} = \mathcal{M}^{\cdot\cdot}_f(X)$ and consider the finite signed measure $\exp\, L$. Set $\Xi = \exp\left(L(\mathfrak{X})\right)$. Ξ is well defined and strictly positive. Next consider the finite signed measure

$$\Im_L = \frac{1}{\Xi} \cdot \exp\, L \quad . \tag{2}$$

Assume that

$(\mathcal{A}_2')$ $\qquad$ $\exp\, L$ is a positive measure.

This implies that $\Im_L$ is a finite point process in X. This means that the process realizes finite configurations of particles in X which are produced by finitely many independent superpositions of clusters, i.e. configurations generated by the measure L. For this reason we call L a *cluster measure.* And we say that $\Im_L$ *has been constructed by means of the method of cluster expansions.*

We'll see now that this construction of processes is a far reaching generalization of the construction of finite Poisson processes.

Example 1 *([12]) In the case of Poisson processes the cluster measure is given by the positive measure*

$$L(\varphi) = \int_X \varphi(\delta_x)\, \varrho(\mathrm{d}\, x) \quad , \varphi \in F, \tag{3}$$

for some finite measure ϱ on X. In this case one obtains the Poisson process P_ϱ in X with intensity measure ϱ.

But obviously P_ϱ itself can be taken as a cluster measure if ϱ is a Radon measure. The associated process $\Im_{P_\varrho}$ is infinitely divisible.

Example 2 *([27]) We obtain Pólya sum processes for the positive cluster measure*

$$L(\varphi) = \sum_{m=1}^{\infty} \frac{z^m}{m} \int_{X^m} \varphi(m\delta_x)\; \varrho(\mathrm{d}\, x) \quad , \varphi \in F, \tag{4}$$

if we assume that ϱ is a finite measure and $0 < z < 1$*.*

Example 3 *([19]) Pólya difference processes are given by signed cluster measures of the form*

$$L(\varphi) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{z^m}{m} \int_{X^m} \varphi(m\delta_x)\; \varrho(\mathrm{d}\, x) \quad , \varphi \in F, \tag{5}$$

if one assumes that $0 < z < 1$ *and ϱ is a finite point measure on X. It is not evident that the condition $(\mathcal{A}_2')$ is satisfied. We'll see this a bit later.*

Example 4 *The underlying space is now denoted by E, in order to indicate that below the role of E will be taken by the Euclidean space. Determinantal processes are determined by cluster measures which have the following structure:*

$$L(\varphi) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{z^m}{m} \int_{E^m} \varphi(\delta_{a_1} + \cdots + \delta_{a_m})\, b_m^{a_1}(\mathrm{d}\, a_2 \ldots \mathrm{d}\, a_m)\, \lambda(\mathrm{d}\, a_1). \tag{6}$$

Here

$$b_m^a(\mathrm{d}\, a_2 \dots \mathrm{d}\, a_m) = K(a, a_2) \cdots K(a_m, a)\ \lambda(\mathrm{d}\, a_2) \dots \lambda(\mathrm{d}\, a_m) \tag{7}$$

for some nice kernel K, e.g. a centered Gaußian kernel if E is a Euclidean space. λ is some positive finite measure, and K is bounded and satisfies the boundedness condition

$$\sup_{b \in E} \int_E |K(a, b)|\, \lambda(\mathrm{d}\, a) < \infty \quad .$$

In this situation L is finite if $z \in (0, \infty)$ is sufficiently small. Again in this case the positivity of $\Im_L$ is not easy to see.

We first calculate the Laplace transform of $\Im_L$ and obtain immediately that for any $f \in F_b$

$$\begin{aligned} \mathcal{L}_{\Im_L}(f) \quad &= \frac{1}{\Xi} \cdot \exp\, \mathcal{L}_L(f) \\ &= \exp\big(- L(1 - \mathrm{e}^{\zeta_f})\big) \\ &=: \mathcal{K}_L(f) \quad . \end{aligned}$$

Thus the Laplace transform of $\Im_L$ is given by the so called *modified Laplace transform* $\mathcal{K}_L$ of L. This terminology is due to Joseph Mecke [13]. We also say in this case that $\Im_L$ *is the KMM-process with Lévy measure L.*

A special class of L

From now on we consider finite signed measures L on $\mathfrak{X}$ defined by means of signed (finite) symmetric measures Θ_m on X^m as follows:

$$L(\varphi) = \sum_{m=1}^{\infty} \frac{1}{m} \int_{X^m} \varphi(\delta_{x_1} + \cdots + \delta_{x_m})\ \Theta_m(\mathrm{d}\, x_1 \dots \mathrm{d}\, x_m) \quad . \tag{8}$$

We call the Θ_m *cumulant measures* in the sequel. Note that $L\{0\} = 0$ and that all examples given above have this representation. In this case the Laplace transform of L can be written explicitly, on account of the finiteness of L, as an absolutely convergent series:

$$\mathcal{L}_L(f) = \sum_{m \geq 1} \frac{1}{m} \Theta_m(\otimes_m \mathrm{e}^{-f}) \quad , f \in F_b.$$

A well known combinatorial formula, stated explicitly below in (27) and derived in the book of Stanley [25], corollary 5.1.6 , then shows that

$$\exp \mathcal{L}_L(f) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\mathcal{J}\in\pi[m]} \prod_{J\in\mathcal{J}} (|J|-1)! \cdot \Theta_{|J|}(\otimes_J \mathrm{e}^{-f}).$$

Here $\pi[m]$ denotes the set of all partitions of the set $[m] = \{1, \ldots, m\}$. It follows that

$$\mathcal{K}_L(f) = \frac{1}{\Xi} \sum_{k=0}^{\infty} \frac{1}{k!} \int_{X^k} \mathrm{e}^{-f(x_1)} \ldots \mathrm{e}^{-f(x_k)} \;\; \varrho_k(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_k) \quad ,$$

where ϱ_k denotes the signed measure on X^k defined by

$$\varrho_k(\otimes_{j=1}^k f_j) = \sum_{\mathcal{J}\in\pi[k]} \prod_{J\in\mathcal{J}} (|J|-1)! \cdot \Theta_{|J|}(\otimes_{j\in J} f_j) \quad , f_j \in F_c.$$

This can be written equivalently as

$$\varrho_k(\otimes_{j=1}^k f_j) = \sum_{\sigma\in\mathcal{S}_k} \prod_{\omega\in\sigma} \Theta_{\ell(\omega)}(\otimes_{j\in\omega} f_j) \quad , f_j \in F_c. \tag{9}$$

(ϱ_0 is defined by $\varrho(X^0) = 1$.) The sum is taken over all permutations of $[k]$, the product over all cycles of the cycle decomposition; and $\ell(\omega)$ is the length of ω. Following the terminology in [14] we say that the measures ϱ_k have a *cluster representation in terms of the cumulant measures* Θ_m. The measures ϱ_k are called here the (process)*determining measures.*

Thus we have identified $\mathcal{K}_L$, the modified Laplace transform of L, as the Laplace transform of the following finite measure on $\mathfrak{X}$

$$Q(\varphi) = \frac{1}{\Xi} \sum_{k\geq 0} \frac{1}{k!} \int_{X^k} \varphi(\delta_{x_1} + \cdots + \delta_{x_k})\, \varrho_k(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_k) \quad , \varphi \in F. \tag{10}$$

Here the series starts with $\varphi(o)$. Since $\Im_L$ has the same Laplace transform as Q both processes coincide, and thus Q is the finite point process $\Im_L$. Furthermore, since $\Im_L$ is assumed in $(\mathcal{A}_2')$ to be positive, we conclude that all measures ϱ_k satisfy the positivity condition

$$(\mathcal{A}_2) \qquad \text{all measures } \varrho_k \text{ are positive.}$$

Note that in the context considered here the conditions $(\mathcal{A}_2')$ and $(\mathcal{A}_2)$ are even equivalent.

To summarize we have the

Lemma 2 *Given a finite signed measure L on $\mathfrak{X}$, represented by means of finite signed symmetric measures Θ_m via (8) and satisfying the positivity condition $(\mathcal{A}_2')$ resp. $(\mathcal{A}_2)$, then $\Im_L = \frac{1}{\Xi}\exp L$ is a finite point process in X with Laplace transform $\mathcal{K}_L$, which has the cluster representation (10).*

Example 3 (continued) We are now in the position to show the positivity condition for the Pólya difference process of example 3 . We verify $(\mathcal{A}_2)$ by using lemma 4.1.3. of [18]. Given $f_1, \ldots, f_k \in F_b$,

$$\varrho_k(f_1 \otimes \cdots \otimes f_k) = z^k \sum_{\sigma \in \mathcal{S}_k} (-1)^{k-|\sigma|} \prod_{\omega \in \sigma} \varrho(\prod_{j \in \omega} f_j)$$

$$= \quad z^k \int f_1(x_1) \cdot \cdot f_k(x_k)\, \varrho(\mathrm{d}\, x_1)(\varrho - \delta_{x_1})(\mathrm{d}\, x_2) \cdot \cdot (\varrho - \delta_{x_1} - \cdots - \delta_{x_{k-1}})(\mathrm{d}\, x_k).$$

Since ϱ is a point measure this is positive.

Example 4 (continued) The positivity is seen here by the following basic result which is an application of lemma 2 and already foreshadowed in the work of Ginibre [4].

Lemma 3 *The measure* $\exp L$ *coincides with the following determinantal measure $\mathcal{J}_K$ on $\mathfrak{X}$, namely, for $\varphi \in F$,*

$$\mathcal{J}_K(\varphi) = \sum_{\ell=0}^{\infty} \frac{z^\ell}{\ell!} \int_{E^\ell} \varphi(\delta_{a_1} + \cdots + \delta_{a_\ell})\, \det(K(a_i, a_j)_{i,j=1}^{\ell})\, \lambda(\mathrm{d}\, a_1) \ldots \lambda(\mathrm{d}\, a_\ell)$$

As a consequence of this we see that $exp\, L$ is positive if K is non-negative definite.

The *cle*-method in the infinitely extended case

Until now L was assumed to be a finite signed measure. We next present the cle-method *in a locally finite setting as it has been developed in [17, 18].*

We are now given a family of positive, symmetric *Radon, i.e. locally finite,* measures $\Theta_m^{\pm}$ on X^m, $m \geq 1$,. These measures give rise to the cluster measures $L^{\pm}$ by means of (8).

$L^{\pm}$ are positive measures on the space $\mathfrak{X}$ with $L^{\pm}\{0\} = 0$; and we assume the integrability condition

$$(\mathcal{A}_1) \qquad L^{\pm}(1 - \mathrm{e}^{-\zeta_f}) < +\infty \quad , f \in F_b.$$

It is shown in [17] that the condition $(\mathcal{A}_1)$ implies that $|\Theta_m|$ are Radon measures. Then $\Theta_m = \Theta_m^+ - \Theta_m^-$ are signed Radon measures, the cumulant measures.

Given $\mathrm{G} \in \mathcal{B}_0(X)$ we localize $L^{\pm}$ by means of $L_{\mathrm{G}}^{\pm} = 1_{\mathfrak{X}(\mathrm{G})} \cdot L^{\pm}$ and set $L_{\mathrm{G}} = L_{\mathrm{G}}^+ - L_{\mathrm{G}}^-$. Here $\mathfrak{X}(\mathrm{G}) = \mathcal{M}^{\cdot\cdot}(G)$. L_{G} is a finite signed measure on $\mathfrak{X}$ because of the integrability assumption $(\mathcal{A}_1)$. Assuming also condition $(\mathcal{A}_2)$ we are in the situation of lemma 2 . It follows that the local process

$$Q_{\mathrm{G}} = \frac{1}{\Xi_{\mathrm{G}}} \cdot \exp\, L_{\mathrm{G}} \quad , \tag{11}$$

$$\Xi_{\mathrm{G}} = \exp\, L_{\mathrm{G}}(\mathfrak{X}) \quad , \tag{12}$$

has Lévy measure L_{G}, i.e. $Q_G = \Im_{L_{\mathrm{G}}}$.

The convergence of the cle-method has been shown in [17] in the following precise sense: Under the conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ the sequence of processes $Q_{\mathrm{G}}, \mathrm{G} \in \mathcal{B}_0(X)$, converges weakly, as $\mathrm{G} \uparrow X$, to some point process $\Im_L$ having *Lévy-measure* L. Recall that this terminology means that the Laplace transform of $\Im_L$ is of the form $\mathcal{K}_L$. The process $\Im_L$ is called here the *KMM-process with Lévy measure* L. Moreover, the process $\Im_L$ solves the following equation:

$$\Im_{L^+} = \Im_{L^-} * P \quad ;$$

This equation says that P is the *convolution quotient* of the infinitely divisible processes $\Im_{L^+}$ and $\Im_{L^-}$.

The proof of this convergence theorem is based on Mecke's version of Lévy's continuity theorem in [13]. To summarize we have the following construction of point processes by means of the cluster expansion method.

Theorem 1 ([17, 18]) *Let $L^{\pm}$ be measures on $\mathfrak{X}$, given in terms of cumulant measures Θ_m by means of (8), satisfying the integrability condition $(\mathcal{A}_1)$, such that the corresponding process determining measures ϱ_k are all non negative. Then there exists a point process $\Im_L$ in X with Laplace transform $\mathcal{K}_L$.*

In the present context we obtain as examples those from above by replacing the finite measure ϱ by some Radon measure.

Another comment is in order here. The cluster expansion construction of the point process P is based on two assumptions: The integrability condition $(\mathcal{A}_1)$ of L and the positivity condition $(\mathcal{A}_2)$ of the determining measures ϱ_k. We'll see in the next section that in case, where Θ_m are defined by means of the Ursell functions for some underlying pair potential, the verification itself of condition $(\mathcal{A}_1)$ is actually an essential part of the cle-method. For this one has to recall that in this case $L_{\mathrm{G}}(\mathfrak{X})$ has the meaning of the log-partition function, so that the finiteness of L_{G} is in fact equivalent to the absolute convergence of the "traditional" cluster expansion of the log-partition function.

4 Point processes of statistical mechanics

We now consider the case where the cumulant measures Θ_m are determined by Ursell functions defined for some underlying pair potential. In this case Poghosyan and Ueltschi [21] have shown that under natural and fairly general conditions on the potential the integrability condition $(\mathcal{A}_1)$ holds true. Condition $(\mathcal{A}_2)$ is satisfied on account of Ruelle's algebraic method if the reference measure is positive.

Combined with the cle-method *from above this yields a construction of a large class of processes which includes many examples from statistical physics. As main examples we present the Bose process and some polygonal version of the Fermi process of Ginibre [4].*

In the context considered here, the proof of the main result, i.e. lemma 4 below, is itself an important part of the cle-method*.*

The basic estimate of Poghosyan and Ueltschi

The theory of Poghosyan/Ueltschi ([21]) provides sufficient conditions on the underlying potential such that condition $(\mathcal{A}1)$ holds true.

Given a Polish phase space $(X, \mathcal{B}, \mathcal{B}_0)$ together with some (signed) Radon measure $\varrho \in \mathcal{M}(X)$ on it. Moreover, a measurable, symmetric function (a pair potential) $u : X \times X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is given. Set

$$\zeta(x, y) = \exp(-u(x, y)) - 1 \quad , x, y \in X.$$

By convention $\zeta \equiv -1$ on $u \equiv +\infty$. Recall that the corresponding *Ursell function* is defined by

$$U_u(x_1, \ldots, x_m) = \sum_{G \in \mathcal{C}_m} \prod_{\{i,j\} \in G} \zeta(x_i, x_j) \quad , m \geq 2,$$

$U_u(x_1) = 1$, and $u \equiv 1$ if $m = 1$. Here $\mathcal{C}_m$ denotes the set of connected unoriented graphs with m vertices without loops.

We consider now the above point process construction for the cumulant measures of the form

$$\Theta_m^{\pm}(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_m) = \frac{1}{(m-1)!} \cdot U_u^{\pm}(x_1, \ldots, x_m)\, \varrho(\mathrm{d}\, x_1) \ldots \varrho(\mathrm{d}\, x_m) \quad . \tag{13}$$

Here $U_u^{\pm}$ denotes the positive respectively negative part of the Ursell function.

Poghosyan and Ueltschi work under the following conditions:

($\mathcal{B}1$) (weak stability)

There exists $\mathrm{b} \in F$ such that for all n

$$\sum_{1 \leq i < j \leq n} u(x_i, x_j) \geq -\sum_{j=1}^{n} \mathrm{b}(x_j) \quad |\varrho|^n - a.s.[(x_1, \ldots, x_n)].$$

($\mathcal{B}2$) (weak regularity)

There exists $\mathrm{a} \in F$ such that

$$\int_X |\varrho|(\mathrm{d}\, y)\ |\zeta(x,y)| \cdot \mathrm{e}^{(\mathrm{a} + 2\,\mathrm{b})(y)} \leq \mathrm{a}(x) \quad |\varrho| - a.s.[x].$$

We remark that for bounded functions a this implies the regularity of u in the sense of Ruelle [23]. The following condition can replace ($\mathcal{B}2$):

($\mathcal{B}2'$) There exists $\mathrm{a} \in F$ satisfying

$$\int_X |\varrho|(\mathrm{d}\, y)\ |\overline{u}(x,y)| \cdot \mathrm{e}^{(\mathrm{a} + \mathrm{b})(y)} \leq \mathrm{a}(x) \quad |\varrho| - a.s.[x],$$

where

$$\overline{u}(x,y) = \begin{cases} u(x,y), & u(x,y) < \infty \\ 1, & u(x,y) = \infty. \end{cases} \tag{14}$$

($\mathcal{B}3$) (integrability of a, b)

$$\mathrm{e}^{\mathrm{a}+2\,\mathrm{b}} \cdot |\varrho| \in \mathcal{M}(X) \quad .$$

The measure here is $|\varrho|$ having density $\mathrm{e}^{\mathrm{a}+2\,\mathrm{b}}$.

Under condition ($\mathcal{B}2'$) we'll always replace condition ($\mathcal{B}3$) by the integrability condition

($\mathcal{B}3'$)

$$\mathrm{e}^{\mathrm{a}+\mathrm{b}} \cdot |\varrho| \in \mathcal{M}(X) \quad .$$

The following basic theorem will serve as a main lemma in our reasoning:

Lemma 4 *(Poghosyan/Ueltschi [21])*
Assume conditions ($\mathcal{B}1$), ($\mathcal{B}2$) respectively ($\mathcal{B}1$), ($\mathcal{B}2'$). Then the following estimate is true: $|\varrho| - a.s.[x]$

$$\sum_{m\geq 1} \frac{1}{(m-1)!} \int_{X^{m-1}} |U_u(x, x_1, .., x_{m-1})| \qquad |\varrho|(\mathrm{d}\,x_1)...|\varrho|(\mathrm{d}\,x_{m-1})$$
$$\leq \mathrm{e}^{\mathrm{a}(x)+2\,\mathrm{b}(x)} .$$

(Under condition ($\mathcal{B}2'$) this holds true with $\mathrm{e}^{\mathrm{b}(x)}$ instead of $\mathrm{e}^{2\,\mathrm{b}(x)}$.)

This estimate implies that L satisfies condition ($\mathcal{A}_1$). It even implies that $|L|$ is *of first order.* Recall that this means that the intensity measure $\nu^1_{|L|}$ of the variation of the cluster measure L is locally finite. To be more precise, we have for any $f \in F_c$

$$\begin{aligned}
&\nu^1_{|L|}(f) := \\
&= \int_{\mathfrak{X}} \zeta_f \,\mathrm{d}\,|L| = \\
&= \sum_{m\geq 1} \frac{1}{(m-1)!} \int_{X^m} f(x) \cdot |U_u(x, x_1, .., x_{m-1})| \, |\varrho|(\mathrm{d}\,x)|\varrho|(\mathrm{d}\,x_1)..|\varrho|(\mathrm{d}\,x_m) \\
&< \infty.
\end{aligned}$$

Here one uses ($\mathcal{B}3$) resp. ($\mathcal{B}3'$). This is the main consequence, and we are in the situation of Nehring's construction above, if the determining measures are positive. Thus it remains to show the positivity of these measures.

Note first that the measures ϱ_k can be represented as

$$\begin{aligned}\varrho_k(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_k) &= \sum_{\mathcal{J}\in\pi([k])} \prod_{J\in\mathcal{J}} \big(|J|-1\big)! \cdot \Theta_{|J|}(\otimes_{j\in J} f_j) \\ &= \sum_{\mathcal{J}} \prod_{J\in\mathcal{J}} U_u((x_j)_{j\in J}) \;\; \varrho(\mathrm{d}\,x_1)\ldots\varrho(\mathrm{d}\,x_k) \quad .\end{aligned}$$

This follows from the symmetry of product measures.

On the other hand, the density here is given by Ruelle's algebraic exponential (cf. [23])

$$\sum_{\mathcal{J}\in\pi([k])} \prod_{J\in\mathcal{J}} U_u((x_j)_{j\in J}) = \exp(-E_u(\delta_{x_1} + \cdots + \delta_{x_k})), \tag{15}$$

so that

$$\varrho_k(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_k) = \exp(-E_u(\delta_{x_1} + \cdots + \delta_{x_k})) \;\; \varrho(\mathrm{d}\,x_1)\ldots\varrho(\mathrm{d}\,x_k) \tag{16}$$

is a positive measure if ϱ has this property. $E_u(\mu)$ denotes the energy of a finite configuration μ defined by the pair potential u by means of

$$E_u(\mu) = \sum_{1\le i<j\le n} u(x_i, x_j) \quad , \text{ if } \mu = \sum_{k=1}^{n} \delta_{x_k}. \tag{17}$$

To summarize we obtain from the main lemma

Theorem 2 *If the measure ϱ is positive then, under the above conditions on the potential, i.e. under $(\mathcal{B}1)$, $(\mathcal{B}2)$ and $(\mathcal{B}3)$ or $(\mathcal{B}1)$, $(\mathcal{B}2')$ and $(\mathcal{B}3')$; and for the cluster measure L defined by means of the Ursell functions in (13), there exists a unique point process P in X with Lévy measure L.*

The Ginibre Bose gas

An important direct application of this theorem is related to *the Ginibre's Bose gas ([4])*. For precise definitions we refer to [21]. Consider the space X of Brownian loops in $E = (\mathbb{R}^d, \mathrm{d}\,a)$. The measure ϱ is defined by means of some nice pair potential ϕ in E. Given ϕ, define a self-potential υ in X and a pair potential u in X as it is done in [4, 21]. Then for parameters $z, \beta > 0$ let

$$\varrho(f) = \sum_{m \geq 1} \frac{1}{m} \cdot z^m \int_E \int_X f(x)\, \mathrm{e}^{-v(x)}\; P^a_{m\beta}(\mathrm{d}\, x)\, \mathrm{d}\, a \quad , f \in F. \tag{18}$$

Here $P^a_{m\beta}(\mathrm{d}\, x)$ is the non-normalized Brownian bridge measure of loops of length $m\beta$ which start and end at $a \in E$. This defines a positive measure on the loop space X. It is shown in [21] (Section V, B) that for a stable and integrable pair potential ϕ the assumptions ($\mathcal{B}1$), ($\mathcal{B}2'$) and ($\mathcal{B}3'$) holds true for all z from the interval

$$z \leq \exp\Big\{-\beta\Big[\frac{\|\phi\|_1 \zeta(\frac{d}{2})}{(4\pi\beta)^{d/2}} + B\Big]\Big\}. \tag{19}$$

Hence by Theorem 2 there exists a unique point process P in X with Lévy measure L. This process P is the limiting Bose gas of interacting Brownian loops (in the sense of Ginibre). Here, $\zeta(\frac{d}{2}) = \sum_{n \geq 1} n^{-\frac{d}{2}}$ is the Riemann zeta function. When $d = 3$ and if the potential is repulsive, one can rewrite (19) in a more transparent way [21]. Let $a_0 = \frac{1}{8\pi}\|\phi\|_1$ denote the Born approximation to the scattering length. The condition is then

$$z \leq \exp\big\{-\tfrac{\zeta(\frac{3}{2})}{\sqrt{\pi}} \tfrac{a_0}{\sqrt{\beta}}\big\}. \tag{20}$$

In this context we'll consider below another class of examples with a modified ϱ which is even signed.

The Groeneveld process

As an aside we first mention an interesting class of point processes which are even infinitely divisible. Consider a *positive* pair potential u together with the cumulant measures

$$\Theta_m(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_m) = z^m \frac{1}{(m-1)!}\big[(-1)^{m-1} \cdot U_u(x_1, \ldots, x_m)\big]\; \varrho(\mathrm{d}\, x_1) \ldots \varrho(\mathrm{d}\, x_m) \quad . \tag{21}$$

Here again ϱ is a positive measure.

It is well known (see Goeneveld [6]) that in case of a positive potential the Ursell functions have alternating signs, i.e. that the expression in brackets are non-negative. Thus the associated L is positive, so that the process with

Lévy measure L exists and is given by the cluster dissolution of the Poisson process with intensity measure L and as such infinitely divisible. We call this process *Groeneveld process*; we do not know what kind of process this is. Further results in this direction can be found in the interesting paper [26].

Gibbs modifications of determinantal processes

To motivate the main results in section 5 we now present, in the context of the Ginibre Bose gas, a heuristic argument which leads to some new class of interacting non-classical point processes. This argument is based on the hypotheses that a Fermi-Dirac process on the level of clusters exists.

As above for the Bose gas we consider $E = \mathbb{R}^d$ with Lebesgue's measure $\mathrm{d}\, a$. We are given a pair potential on E, i.e. a measurable symmetric function $\phi : E \times E \longrightarrow \mathbb{R} \cup \{+\infty\}$.

We now replace in the definition of ϱ in (18) the term z^m by $(-1)^{m-1} z^m$ and, to be more modest, $P^a_{m\beta}$ by the measure b^a_m from example 4 . The positive measure ϱ is then replaced by

$$\varrho(f) = \sum_{m \geq 1} \frac{1}{m} (-1)^{m-1} z^m \cdot \int_{E^m} f(\delta_{a_1} + .. + \delta_{a_m})\, \mathrm{e}^{-E_\phi(\delta_{a_1} + .. + \delta_{a_m})}\ b^{a_1}_m(\mathrm{d}\, a_2 .. \mathrm{d}\, a_m)\, \mathrm{d}\, a_1.$$

$(f \in F)$ Recall that E_ϕ is defined in (17). This is in general a signed measure on the Polish space $\mathfrak{X}$ of finite configurations in the phase space. Remark that the energy functional E_ϕ on the space $\mathfrak{X}$ is the analog of the self-potential υ on the space X of Brownian loops. The measure ϱ will be the reference measure on $\mathfrak{X}$.

We finally introduce a pair potential Φ on $\mathfrak{X}$, which resembles the pair potential u between brownian loops. An obvious guess is

$$\Phi(\mu, \eta) = \int_E \int_E \phi(a, b)\ \mu(\mathrm{d}\, a) \eta(\mathrm{d}\, b), \qquad \text{for all } \mu, \eta \in \mathfrak{X}.$$

Remark that for any $\mu_1, \ldots, \mu_n \in \mathfrak{X}$ the following identity holds true:

$$E_\Phi(\delta_{\mu_1} + \ldots + \delta_{\mu_n}) + E_\phi(\mu_1) + \ldots + E_\phi(\mu_n) = E_\phi(\mu_1 + \ldots + \mu_n). \tag{22}$$

The main question is: Does there exist a point process having the Lévy measure

$$\Re(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n!} \int\limits_{\mathfrak{X}^n} \varphi(\delta_{\mu_1} + \ldots + \delta_{\mu_n})\, U_\Phi(\mu_1, \ldots, \mu_n)\, \varrho(\mathrm{d}\,\mu_1) \ldots \varrho(\mathrm{d}\,\mu_n)\ . \quad (23)$$

At first we have to check whether the measures ϱ_k, as defined by (16), are positive. But since the first process determining measure ϱ_1 coincides with ϱ, which is a signed measure, this is certainly not the case. So at least our construction does not give a point process corresponding to $\Re$. In case such a process would exist one would obtain a *Fermi process*, which is a process realizing configurations of interacting *polygonal loops* $\delta_{a_1} + .. + \delta_{a_m} \in \mathfrak{X}$.

Now in the sequel let us *assume* that such a process exists. How would the local processes $Q_G = \Im_{\Re_G}$, $G \in \mathcal{B}_0$ look like?

Consider

$$\tau(f) = \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} z^m \int_{E^m} f(\delta_{a_1} + .. + \delta_{a_m})\, b_m^{a_1}(\mathrm{d}\, a_2 .. \mathrm{d}\, a_m)\, \mathrm{d}\, a_1.$$

Here $f \in F$. This is a signed measure on $\mathfrak{X}$. It is the above ϱ without the density e^{-E_ϕ}.

Recall that τ is the Lévy measure of the determinantal point process with interaction kernel K. Using (22) we obtain that

$$\Im_{\Re_{\mathrm{G}}}(\varphi) = \frac{1}{\Xi(\mathrm{G})} \sum_{n=0}^{\infty} \frac{1}{n!} \int\limits_{\mathfrak{X}(\mathrm{G})^n} \varphi(\delta_{\mu_1} + .. + \delta_{\mu_n})\ \mathrm{e}^{-E_\phi(\mu_1 + .. + \mu_n)}\ \tau(\mathrm{d}\,\mu_1) .. \tau(\mathrm{d}\,\mu_n).$$

Consider then the so called *cluster dissolution mapping*

$$\xi : \mathcal{M}_f^{\cdot\cdot}(\mathfrak{X}) \longrightarrow \mathfrak{X}, \quad \delta_{\mu_1} + \delta_{\mu_2} + \ldots \longmapsto \mu_1 + \mu_2 + \ldots\ .$$

The image of $\Im_{\Re_{\mathrm{G}}}$ under ξ, denoted by $\xi\, \Im_{\Re_G}$, becomes an ordinary finite signed measure on the space $\mathfrak{X}(\mathrm{G})$.

If we recall the definition of the exponential of a finite signed measure from section 3 we obtain

$$\xi\, \Im_{\Re_{\mathrm{G}}}(\varphi) = \frac{1}{\Xi(\mathrm{G})} \exp\, \tau_{\mathrm{G}}\, (\varphi\, \mathrm{e}^{-E_\phi}).$$

We saw in lemma 3 that $\exp \tau_{\mathrm{G}}$ coincides with the so called *determinantal measure* $\mathcal{J}_{K_{\mathrm{G}}}$ on $\mathfrak{X}$, where $K_{\mathrm{G}}(x,y) = 1_{\mathrm{G}}(x)K(x,y)1_{\mathrm{G}}(y)$, $x,y \in E$. This is positive if K is a non negative definite kernel. This then implies that $\xi\,\Im_{\Re_{\mathrm{G}}}$ is a finite point process in G. By corollary 6.1.2 in [18] we conclude $\xi\,\Im_{\Re_{\mathrm{G}}} = \Im_{(\xi\Re)_{\mathrm{G}}}$. So we have identified not $\Re$ but $\xi\,\Re$ as a Lévy measure of a point process in E. What remains to be seen, according to Theorem 1, is that $\xi\,|\Re|$ is of first order. This will be established with the help of lemma 4 . Remark that the process determining measures of $\Im_{\xi\,\Re}$ are given by

$$\mathrm{e}^{-E_\phi(\delta_{a_1}+..+\delta_{a_k})}\det(K(a_i,a_j)_{i,j})\,\mathrm{d}\,a_1..\,\mathrm{d}\,a_k \quad .$$

This is why we call $\Im_{\xi\,\Re}$ a *Gibbs modification of the determinantal process with interaction kernel* K.

From now on we consider a general phase space X.

Definition 1 *Let L be a Lévy measure as introduced in section 3, formula (8), with the corresponding point process $\Im_L$ and the family of local processes $\{\Im_{L_{\mathrm{G}}}\}_{\mathrm{G}\in\mathcal{B}_0}$. Furthermore let $\phi : X \times X \to \mathbb{R}\cup\{\infty\}$ be a pair potential such that*

$$0 < \Im_{L_{\mathrm{G}}}(e^{-E_\phi}) < \infty, \quad \mathrm{G}\in\mathcal{B}_0.$$

Now introduce another family of finite point processes

$$\Im^{\phi}_{L_{\mathrm{G}}}(\varphi) := \frac{1}{\Im_{L_{\mathrm{G}}}(e^{-E_\phi})}\,\Im_{L_{\mathrm{G}}}(e^{-E_\phi}\,\varphi), \quad \varphi\in F,\ \mathrm{G}\in\mathcal{B}_0.$$

If a weak limit, denoted $\Im^\phi_L$, of $\Im^\phi_{L_{\mathrm{G}}}$ as $\mathrm{G}\uparrow X$ does exist we call it the Gibbs modification of the KMM process $\Im_L$.

In the next section we will provide sufficient conditions on the pair potential in order for $\Im^\phi_L$ to exist. Remark that the above discussion suggests that the process $\Im^\phi_L$ has a Lévy measure given by

$$\Re^\phi_L(\varphi) = \sum_{n=1}^{\infty}\frac{1}{n!}\int\limits_{\mathfrak{X}^n}\varphi(\mu_1+\ldots+\mu_n)\,U_\Phi(\mu_1,\ldots,\mu_n)\,L^\phi(\mathrm{d}\,\mu_1)\ldots L^\phi(\mathrm{d}\,\mu_n), \quad (24)$$

where

$$L^\phi(\mathrm{d}\,\mu) = \mathrm{e}^{-E_\phi(\mu)}\;L(\mathrm{d}\,\mu).$$

The method of the proof will be to show that $\Re_L^\phi$ satisfies the condition of Theorem 1. Certainly the first question which arises is what kind of family of cumulant measures correspond to the above representation of $\Re_L^\phi$? But a close look at the proof of Theorem 1 yields that we only need to show that the finite signed measures $\Im_{\Re_{L,\mathrm{G}}^\phi}$ as defined by (2), where $\Re_{L,\mathrm{G}}^\phi$ denotes the restriction of $\Re_L^\phi$ to $\mathfrak{X}(\mathrm{G})$, are actually finite point processes in X and we need to establish that $|\Re_L^\phi| = \Re_L^{\phi,+} + \Re_L^{\phi,-}$ is of first order.

But what is the positive $\Re_L^{\phi,+}$ respective negative $\Re_L^{\phi,-}$ part of $\Re_L^\phi$? They are naturally given by the Jordan decomposition of $L = L^+ - L^-$ so that

$$|\Re_L^\phi|(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathfrak{X}^n} \varphi(\mu_1 + \ldots + \mu_n)\, |U_\Phi(\mu_1, \ldots, \mu_n)|\, |L^\phi|(\mathrm{d}\,\mu_1) \ldots |L^\phi|(\mathrm{d}\,\mu_n),$$

where $|L^\phi|(\mathrm{d}\,\mu) = \mathrm{e}^{-E_\phi(\mu)}\, |L|(\mathrm{d}\,\mu)$.

5 Construction of Gibbs modifications of KMM-processes

In the following X is a general phase space. We start with a family of cumulant measures $\{\Theta_m\}_{m\geq 1}$ satisfying the positivity condition $(\mathcal{A}_2)$, i.e. the corresponding family of process determining measures $\{\varrho_k\}_{k\geq 1}$ is non negative.

We introduce a parameter $z \in (0, \infty)$, called the *activity* which will be chosen later small enough. We denote by L_z the Lévy measure corresponding to the family $\{z^m \Theta_m\}_{m\geq 1}$ of cumulant measures. Recall definition (8) here and observe that the process determining measures are now given by $\{z^k \varrho_k\}_{k\geq 1}$. So condition $(\mathcal{A}_2)$ is satisfied for any choice of the activity.

Let $\phi : X \times X \to \mathbb{R} \cup \{\infty\}$ be a stable pair potential in the classical sense, that is there is $B \geq 0$ such that

$$E_\phi(\mu) \geq -B\,|\mu|, \quad \mu \in \mathfrak{X}.$$

Let us denote by ϕ_x for $x \in X$ the function $y \mapsto \phi(x, y)$.

Theorem 3 *Let ϕ be a stable pair potential. Furthermore, assume that there exists $c > 0$ and $z_0 = z_0(c) > 0$ such that*

$$(1) \quad \nu^1_{|L_{z_0 e^{c+B}}|}(f) < \infty, \quad f \in F_b,$$
$$(2) \quad \nu^1_{|L_{z_0 e^{c+B}}|}(|\overline{\phi}_x|) \leq c, \quad \textit{for all } x \in X,$$

where $\overline{\phi}$ is defined as in (14). Then for $z \in (0, z_0]$ the KMM process $\Im_{L_z}$ and its Gibbs modification $\Im^{\phi}_{L_z}$ do exist. Furthermore $\Im^{\phi}_{L_z}$ is a KMM process with Lévy measure $\Re^{\phi}_{L_z}$, that is $\Im^{\phi}_{L_z} = \Im_{\Re^{\phi}_{L_z}}$, where $\Re^{\phi}_{L_z}$ is defined as in (24).

Proof. Let $z \in (0, z_0]$. Due to (i) $|L_z|$ is of first order, which implies condition $(\mathcal{A}_1)$ of Theorem 1. Since $(\mathcal{A}_2)$ was already established we obtain the existence of the KMM process to the Lévy measure L_z. Let us now deduce the existence of the KMM process with Lévy measure $\Re^{\phi}_{L_z}$.

1. Let us start by showing that
$|\Re^{\phi}_{L_z}|$, $z \in (0, z_0]$ is of first order. For any $f \in F_b$

$$\nu^1_{|\Re^{\phi}_{L_z}|}(f) = \int_{\mathfrak{X}} |L^{\phi}_z|(\mathrm{d}\,\mu)\ \zeta_f(\mu) \cdot \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_{\mathfrak{X}^{n-1}} |L^{\phi}_z|(\mathrm{d}\,\mu_2) \ldots |L^{\phi}_z|(\mathrm{d}\,\mu_n)\, |U_{\Phi}(\mu, \mu_2, \ldots, \mu_n)|.$$

Certainly in order to obtain the finiteness of $\nu^1_{|\Re^{\phi}_{L_z}|}(f)$ we want to apply the bound as formulated by lemma 4. So here the underlying space X is now given by the set $\mathfrak{X}$ of finite point configurations in X, the signed reference measure ϱ is given by L^{ϕ}_z and the pair potential u by Φ. The aim is now to show that this triple $(\mathfrak{X}, L^{\phi}_z, \Phi)$ does satisfy the conditions $(\mathcal{B}1)$, $(\mathcal{B}2')$ and $(\mathcal{B}3')$.
Remark that on account of (22) the pair potential Φ on $\mathfrak{X}$ is stable in the weak sense of [21], that is it satisfies assumption $(\mathcal{B}1)$ with

$$b(\mu) = B|\mu| + E_{\phi}(\mu), \quad \mu \in \mathfrak{X}.$$

Let us now establish $(B2')$, that is we have to find $a \in F_+(\mathfrak{X})$ such that

$$\int |L^{\phi}_z|(\mathrm{d}\,\mu)\, |\overline{\Phi}(\eta, \mu)|\, e^{(a+b)(\mu)} \leq a(\eta), \quad \eta \in \mathfrak{X}, \tag{25}$$

where $\overline{\Phi}$ is defined with the help of $\overline{\phi}$ as in (14). Let us try the Ansatz $a(\mu) = c\,|\mu|$, where $c > 0$ is given as in Theorem 3. Introduce $I(x) := \phi_x^{-1}(\{\infty\}) \in \mathcal{B}(X)$ and

$$I(\eta) = \bigcup_{x\in\eta} I(x), \quad \eta \in \mathfrak{X}.$$

Then the integral in (25) can be split up into a hardcore and non-hardcore part

$$\int\limits_{\{\zeta_{I(\eta)}>0\}} |L_z|(\mathrm{d}\,\mu)\, e^{(c+B)|\mu|} + \int\limits_{\mathfrak{X}(E\setminus I(\eta))} |L_z|(\mathrm{d}\,\mu)\, |\Phi(\eta,\mu)|\, e^{(c+B)|\mu|}.$$

Let us call these summands T_1 and T_2. Then

$$T_1 = |L_{ze^{c+B}}|(1_{\{\zeta_{I(\eta)}>0\}}) \le |L_{ze^{c+B}}|(\zeta_{I(\eta)}) \le \sum_{x\in\eta} \nu^1_{|L_{ze^{c+B}}|}(1_{I(x)}).$$

Introduce the non - hard core part of the potential by defining

$$\phi''(x,y) = \begin{cases} 0, & y \in I(x) \\ \phi(x,y), & \text{else.} \end{cases}$$

Certainly we have

$$|\Phi(\eta,\mu)| \le \sum_{x\in\eta} \mu(|\phi''_x|), \quad \mu \in \mathfrak{X}(X \setminus I(\eta)), \tag{26}$$

whence we obtain the following bound

$$T_2 \le \sum_{x\in\eta} \nu^1_{|L_{ze^{c+B}}|}(|\phi''_x|).$$

This finally yields

$$\int |L_z^\phi|(\mathrm{d}\,\mu)\, |\overline{\Phi}(\eta,\mu)|\, e^{(a+b)(\mu)} \le \sum_{x\in\eta} \nu^1_{|L_{ze^{c+B}}|}(1_{I(x)} + |\phi''_x|).$$

So due to the uniform bound (2) and $|\overline{\phi}_x| = 1_{I(x)} + |\phi''_x|$, we have obtained condition $(B2')$. Now we can apply lemma 4, which yields

$$\nu^1_{|\mathfrak{R}^\phi_{L_z}|}(f) \le \int\limits_{\mathfrak{X}} |L_z^\phi|(\mathrm{d}\,\mu)\ \zeta_f(\mu)\, e^{(a+b)(\mu)} = \nu^1_{|L_{ze^{c+B}}|}(f) < \infty,$$

due to condition (1).

2. To finish the proof, it remains to be seen that the process corresponding to the restriction $\Re^{\phi}_{L_z,\mathrm{G}}$ of $\Re^{\phi}_{L_z}$ to $\mathfrak{X}(\mathrm{G}), \mathrm{G} \in \mathcal{B}_0$, is given by the Gibbs modification of the finite point process $\Im_{L_{z,\mathrm{G}}}$ for $z \in (0, z_0]$. Since $|\Re^{\phi}_{L_z}|$ is of first order, $\Re^{\phi}_{L_z,\mathrm{G}}$ is a finite signed measure. As above in section 3 the following combinatorial result will be needed now:

$$\exp(\sum_{k=1}^{\infty} \frac{h_k}{k!}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{J} \in \pi([n])} \prod_{J \in \mathcal{J}} h_{|J|}, \tag{27}$$

where $\sum\limits_{k=1}^{\infty} \frac{h_k}{k!}$ is an absolutely convergent series. Using (27) we obtain

$$\mathcal{K}_{\Re^{\phi}_{L_z,\mathrm{G}}}(f) = \exp(-\Re^{\phi}_{L_z,\mathrm{G}}(1 - e^{-\zeta_f})) =$$
$$\frac{1}{\Xi^{\phi}(\mathrm{G})} \sum_{n=0}^{\infty} \frac{1}{n!} \int\limits_{\mathfrak{X}(\mathrm{G})^n} e^{-(\mu_1 + .. + \mu_n)(f)} \sum_{\mathcal{J} \in \pi([n])} \prod_{J \in \mathcal{J}} U_{\Phi}((\mu_j)_J)\, L^{\phi}_z(\mathrm{d}\,\mu_1)..L^{\phi}_z(\mathrm{d}\,\mu_n),$$

where $\Xi^{\phi}(\mathrm{G}) = \exp(\Re^{\phi}_{L_z,\mathrm{G}}(1))$ and, using Ruelle's algebraic approach, the above expression equals

$$\frac{1}{\Xi^{\phi}(\mathrm{G})} \sum_{n=0}^{\infty} \frac{1}{n!} \int\limits_{\mathfrak{X}(\mathrm{G})^n} e^{-(\mu_1 + \ldots + \mu_n)(f)}\, e^{-E_{\Phi}(\delta_{\mu_1} + \ldots + \delta_{\mu_n})}\, L^{\phi}_z(\mathrm{d}\,\mu_1) \ldots L^{\phi}_z(\mathrm{d}\,\mu_n).$$

Using (22) this can be written as

$$\frac{1}{\Xi^{\phi}(\mathrm{G})} \sum_{n=0}^{\infty} \frac{1}{n!} \int\limits_{\mathfrak{X}(\mathrm{G})^n} e^{-(\mu_1 + \ldots + \mu_n)(f)}\, e^{-E_{\phi}(\mu_1 + \ldots + \mu_n)}\, L_z(\mathrm{d}\,\mu_1) \ldots L_z(\mathrm{d}\,\mu_n).$$

The cle-method then implies that

$$\mathcal{K}_{\Re^{\phi}_{L_z,\mathrm{G}}}(f) = \frac{1}{\Xi^{\phi}(\mathrm{G})} \exp(L_{z,\mathrm{G}})(e^{-E_{\phi}}\, e^{-\zeta_f}) = \frac{\Im_{L_{z,\mathrm{G}}}(e^{-E_{\phi}}\, e^{-\zeta_f})}{\Im_{L_{z,\mathrm{G}}}(e^{-E_{\phi}})}\ .$$

Thus the Laplace transforms of $\Im_{\Re^{\phi}_{L_z,\mathrm{G}}}$ and $\Im^{\phi}_{L_{z,\mathrm{G}}}$ coincide, whence by lemma 1 they are equal. q.e.d.

Corollary 1 *Let L be a Lévy measure defined by a family of cumulant measures $\{\Theta_n\}_{n\geq 1}$. Assume there exist $\alpha, \beta \geq 0$ and $\alpha_f \geq 0$ for $f \in F_b$ such that*

$$\begin{aligned} &(1) \quad |\Theta_n|(f \otimes E^{\otimes(n-1)}) \leq \alpha_f\, \beta^{n-1}, \quad f \in F_b,\, n \geq 1, \\ &(2) \quad |\Theta_n|(|\overline{\phi}_x| \otimes E^{\otimes(n-1)}) \leq \alpha\, \beta^{n-1}, \quad x \in E,\, n \geq 1, \end{aligned}$$

where $0^0 := 1$. Then Theorem 3 holds with $c = 1$ and $z_0 = \frac{e^{-B-1}}{\alpha+\beta}$.

Proof. We verify condition (2) of Theorem 3

$$\begin{aligned} \nu^1_{|L_{z_0 e^{1+B}}|}(|\overline{\phi}_x|) &= \sum_{n=1}^{\infty} (z_0 e^{1+B})^n |\Theta_n|(|\overline{\phi}_x| \otimes E^{\otimes(n-1)}) \\ &\leq \frac{\alpha}{\alpha+\beta} \sum_{n=1}^{\infty} \Big(\frac{\beta}{\alpha+\beta}\Big)^{n-1} = 1. \end{aligned}$$

Condition (1) in Theorem 3 is established in the same way. *q.e.d.*

6 Examples of Gibbs modifications

The underlying general phase space should be thought as a discrete space of the Euclidean space E, or the collection $\mathfrak{X}$ of finite configurations of particles in E or the space of Brownian loops in E.

Definition 2 *Let λ be a non negative reference measure on X and $h : X \times X \to \mathbb{R} \cup \{\infty\}$ some measurable function. Then define*

$$\Upsilon(h) = \sup_{y \in X} \int \lambda(\mathrm{d}\, x)\, |h(x, y)|.$$

Poisson and Gibbs Processes

Let $\lambda \in \mathcal{M}(X)$. Consider the point process whose cumulant measures vanish besides the first one $\Theta_1 = \lambda$. It is called the Poisson point process with intensity measure λ and we denote it by P_λ. Let ϕ be a stable pair potential such that $\Upsilon(\overline{\phi}) < \infty$. As one can easily verify, condition (1) of corollary 1 holds with $\alpha_f = \lambda(f)$, $\beta = 0$; and (2) with $\alpha = \Upsilon(\overline{\phi})$. So corollary 1 respectively Theorem 3 yield that for

$$z \in \Big(0, \frac{e^{-B-1}}{\Upsilon(\overline{\phi})}\Big], \tag{28}$$

the Gibbs modification of $P_{z\lambda}$, that is the weak limit of the finite point processes

$$\frac{1}{\Xi^\phi(\mathrm{G})} \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\mathrm{G}^n} \varphi(\delta_{a_1} + \ldots + \delta_{a_n}) e^{-E_\phi(\delta_{a_1}+\ldots+\delta_{a_n})} \, \lambda(\mathrm{d}\, a_1) \ldots \lambda(\mathrm{d}\, a_n) \quad ,$$

as $\mathcal{B}_0(X) \ni \mathrm{G} \uparrow X$, does exist. It is called classical Gibbs process and its existence is well known [14]. But instead of the classical regularity condition

$$\sup_{y \in X} \int |1 - e^{-\phi(x,y)}| \, \lambda(\mathrm{d}\, x) < \infty$$

we require $\Upsilon(\overline{\phi}) < \infty$. Remark that if ϕ consists only of a hard core part we obtain so called Poisson exclusion processes which have been studied in [16].

Permanental and Determinantal Processes and its modifications

Let $\lambda \in \mathcal{M}(X)$ and $K : X \times X \to \mathbb{R}$ be a bounded non negative definite kernel such that $\Upsilon(K) < \infty$. Furthermore let ϕ be a stable pair potential such that $\Upsilon(\overline{\phi}) < \infty$. Consider the following two families $\epsilon = +1, -1$ of cumulant measures

$$\Theta_n(\epsilon) = \epsilon^{n-1} K(x_1, x_2) K(x_2, x_3) \ldots K(x_n, x_1) \, \lambda(\mathrm{d}\, x_1) \ldots \lambda(\mathrm{d}\, x_n).$$

It is well known (see i.e. [17]) that the corresponding process determining measures are given by

$$\det_\epsilon(K(a_i,a_j)_{i,j})\,\lambda(\mathrm{d}\,a_1)\ldots\lambda(\mathrm{d}\,a_n),$$

where $\det_{+1}$ denotes the permanent and $\det_{-1}$ the determinant, which are non negative due to the non negative definiteness of K. As one can straightforwardly check condition (1) of corollary 1 is satisfied with $\alpha_f = \|K\|_\infty\,\lambda(f)$ and $\beta = \Upsilon(K)$ and condition (2) with $\alpha = \|K\|_\infty\,\Upsilon(\overline{\phi})$. Corollary 1 respectively Theorem 3 now says that for a small activity

$$z \in \Big(0, \frac{e^{-B-1}}{\|K\|_\infty\,\Upsilon(\overline{\phi}) + \Upsilon(K)}\Big],$$

the processes $\Im_{L_z(\epsilon)}$ corresponding to the family of cumulant measures $\{\Theta_n(\epsilon)\}_{n=1}^\infty$ do exist (they can be identified as $\epsilon = +1$ permanantel and $\epsilon = -1$ determinantal processes to the interaction kernel K) and also their Gibbs modifications do exist, that is for any sequence $\mathcal{B}_0(X) \ni \mathrm{G} \uparrow X$ the finite point processes

$$\frac{1}{\Xi^\phi(\mathrm{G})}\sum_{n=0}^\infty \frac{z^n}{n!}\int\limits_{\mathrm{G}^n} \varphi(\delta_{a_1}+ \quad \ldots + \delta_{a_n})e^{-E_\phi(\delta_{a_1}+\ldots+\delta_{a_n})}$$
$$\det_\epsilon(K(a_i,a_j)_{i,j})\,\lambda(\mathrm{d}\,a_1)\ldots\lambda(\mathrm{d}\,a_n) \quad ,$$

do converge weakly to the process $\Im_{\mathfrak{R}^\phi_{L_z(\epsilon)}}$. Again remark that if ϕ consists only of a hard core part we obtain the existence of determinantal respectively permanental exclusion processes.

7 Some integral equations for point processes of statistical mechanics

It is shown that the processes $\Im_L$ of section 4 are Gibbs in the DLR-sense, if the potential u is assumed to be non-negative and satisfies $(\mathcal{B}2')$ and $(\mathcal{B}3')$. As a consequence we obtain a new integration by parts formula for Gibbs modifications of infinitely divisible point processes whichs seems to be a far reaching generalization of the equation $\big(\mathbf{\Sigma}'_{\varrho,\phi}\big)$.

Point processes of statistical mechanics revisited

Let us go back for the moment to the general setting given in section 4. So $(X, \mathcal{B}, \mathcal{B}_0)$ denotes a Polish phase space and $\varrho \in \mathcal{M}(X)$ is a given positive Radon measure on it. Furthermore we let $u : X \times X \to [0, \infty]$ be a non-negative pair potential. In Theorem 2 we have shown that under the conditions $(\mathcal{B}1)$, $(\mathcal{B}2')$ and $(\mathcal{B}3')$ the limiting Gibbs point process $\Im_L$ with Lévy measure

$$L(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n!} \int\limits_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, U_u(x_1, \ldots, x_n)\, \varrho(\mathrm{d}\, x_1) \ldots \varrho(\mathrm{d}\, x_n),$$

does exist.

A more delicate question is whether $\Im_L$ is a Gibbs point process in the DLR sense, that is whether it is a solution to the equation

$$\left(\mathbf{\Sigma}'\right) \quad C_P(h) = \int h(x, \mu + \delta_x)\ \mathrm{e}^{-E_u(x,\mu)}\ \varrho(\mathrm{d}\, x) P(\mathrm{d}\, \mu)\ , h \in F;$$

here the conditional energy $E_u(x, \mu)$ is given by $\mu(u_x)$ for any $x \in X$ and $\mu \in \mathcal{M}^{\cdot\cdot}(X)$, since u is non negative. $\mathcal{C}_P$ denotes the Campbell measure of P. The equivalence of this equation to the DLR-equations in the context of classical statistical mechanics had been shown in [20].

In [17] we saw that, if one assumes classical stability and regularity of u as in [23] and with a reference measure given by $\varrho = z\lambda$, where $z \in (0, \infty)$ and $\lambda \in \mathcal{M}(X)$, then $\Im_L$ is a solution to $\left(\mathbf{\Sigma}'\right)$ for small z. Here we strengthen the stability condition, that is we consider purely repulsive pair potentials, and weaken the regularity condition, that is we only require $(\mathcal{B}2')$ and $(\mathcal{B}3')$.

Theorem 4 *Let u be a non negative pair potential and $\varrho \in \mathcal{M}(X)$. Then under the conditions $(\mathcal{B}2')$ and $(\mathcal{B}3')$ $\Im_L$ solves $\left(\mathbf{\Sigma}'\right)$.*

Proof. We follow the proof in [17]. Due to [27] the finite processes $Q_{\mathrm{G}} = \Im_{L_{\mathrm{G}}}$, $\mathrm{G} \in \mathcal{B}_0$, satisfy

$$C_{Q_{\mathrm{G}}}(h) = \int\limits_{\mathfrak{X}} \int\limits_{\mathrm{G}} h(x, \mu + \delta_x)\ \mathrm{e}^{-E_u(x,\mu)}\ \varrho(\mathrm{d}\, x) Q_{\mathrm{G}}(\mathrm{d}\, \mu)\ , h \in F.$$

Let in the sequel $h = f \otimes e^{-\zeta_g}$, $f, g \in F_b$. In [17] it was shown that $C_{Q_{\mathrm{G}}}(h) \to C_{\Im_L}(h)$ as $\mathrm{G} \uparrow X$ if $|L|$ is of first order. The right hand side of the above equation can be written as

$$\int\limits_X f(x) e^{-g(x)}\, \mathcal{L}_{Q_{\mathrm{G}}}(g + u_x)\, \varrho(\mathrm{d}\, x). \tag{29}$$

The main lemma, which replaces the main lemma in [17] is now given by

Lemma 5 *Let $\Upsilon = \zeta_{g+\overline{u}_x}$. Then $1 - e^{-\zeta_{g+u_x}} \le \Upsilon$ on $\mathfrak{X}$ and there is $c \ge 0$ such that $|L|(\Upsilon) \le a(x) + c$, where a is given as in ($\mathcal{B}2'$).*

Proof. The inequality $1 - e^{-\zeta_{g+u_x}} \le \Upsilon$ is clear. Now we have

$$|L|(\Upsilon) \le \int\limits_X (g(y) + \overline{u}(x,y))\, e^{a(y)} \varrho(\mathrm{d}\, y) \le \int\limits_X g(y)\, e^{a(y)} \varrho(\mathrm{d}\, y) + a(x).$$

The first inequality follows by lemma 4 and the second by definition of ($\mathcal{B}2'$). So we can choose $c := \varrho(g\, e^a)$, which is finite due to ($\mathcal{B}3'$). *q.e.d.*

To finish the proof one can show as in [17] that $\mathcal{L}_{\Im_L}(g+u_x) = \mathcal{K}_L(g+u_x)$, $\mathcal{L}_{Q_{\mathrm{G}}}(g+u_x) = \mathcal{K}_{L_{\mathrm{G}}}(g+u_x)$ and so $\mathcal{L}_{Q_{\mathrm{G}}}(g+u_x) \to \mathcal{L}_{\Im_L}(g+u_x)$ as $\mathrm{G} \uparrow X$ for any $x \in X$. Moreover the bound as given by lemma 5 yields $\mathcal{L}_{Q_{\mathrm{G}}}(g+u_x) \le e^{a(x)+c}$, $\mathrm{G} \in \mathcal{B}_0$, $x \in X$. If we replace $\mathcal{L}_{Q_{\mathrm{G}}}(g + u_x)$ by $e^{a(x)+c}$ we obtain the finiteness of the integral (29) due to condition ($\mathcal{B}3'$). So by Lebesgue's dominated convergence theorem we are allowed to take the limit $\mathrm{G} \uparrow X$ inside the integral of (29) and obtain the assertion as in [17]. *q.e.d.*

The Ginibre Bose gas revisited

Let us again consider the Bose process of Ginibre. If the underlying pair potential ϕ is stable and integrable, then the corresponding pair potential u on the loop space satisfies the conditions ($\mathcal{B}1$), ($\mathcal{B}2'$) and ($\mathcal{B}3'$) for a small value of activity, see (19), and thereby we obtain the existence of the limiting Bose gas by means of Theorem 2. Now if we additionally impose that ϕ is a purely repulsive potential we are able to describe Ginibre's Bose process as a Gibbs process by means of Theorem 4. Formulated a bit more generally we have the

Theorem 5 *Let L_z be a non negative Lévy measure and ϕ be a non negative potential. Then under the conditions of Theorem 3 we have that the Gibbs modification $\Im^{\phi}_{L_z}$ of the infinitely divisible point process $\Im_{L_z}$ is a solution to*

$$\left(\mathbf{\Sigma}^{\phi}_{L_z}\right) \quad C_P(h) = \int h(x, \nu + \mu)\, e^{-(\Phi(\nu,\mu)+E_\phi(\nu))}\, C_{L_z}(\mathrm{d}\, x\, \mathrm{d}\, \nu) P(\mathrm{d}\, \mu).$$

Proof. In the proof to Theorem 3 we have shown that L^{ϕ}_z and Φ satisfy the condition $(\mathcal{B}2')$. The stability condition is satisfied since $\Phi \geq 0$. In the first step we will show that there exists a Gibbs point process in the space $\mathfrak{X}$ to the pair potential Φ with reference measure L^{ϕ}_z. According to Theorem 2 it remains to show that condition $(\mathcal{B}3')$ is valid, but it is well known that the bounded sets in $\mathfrak{X}$ are given by $\{\zeta_{\mathrm{G}} > 0\}$, $\mathrm{G} \in \mathcal{B}_0$ and all measurable subsets of those sets.

Recall that $a(\mu) = c\,|\mu|$, $\mu \in \mathfrak{X}$, so that

$$\int\limits_{\{\zeta_{\mathrm{G}}>0\}} e^{a(\mu)}\, L^{\phi}_z(\mathrm{d}\, \mu) \leq L^{\phi}_{ze^c}(\zeta_{\mathrm{G}}) = \nu^1_{L^{\phi}_{ze^c}}(\mathrm{G}) < \infty.$$

Whence we conclude that $e^a\, L^{\phi}_z$ is a locally finite measure on $\mathfrak{X}$. So we obtain that the Gibbs point process $\Im_{\Re}$ in $\mathfrak{X}$ corresponding to the signed Lévy measure

$$\Re(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n!} \int\limits_{\mathfrak{X}^n} \varphi(\delta_{\mu_1} + \ldots + \delta_{\mu_n})\, U_\Phi(\mu_1, \ldots, \mu_n)\, L^{\phi}_z(\mathrm{d}\, \mu_1) \ldots L^{\phi}_z(\mathrm{d}\, \mu_n)\ .$$

does exist and satisfies $\left(\mathbf{\Sigma}'\right)$ by Theorem 4, that is

$$C_{\Im_{\Re}}(h) = \int h(\nu, \mu + \delta_\nu)\, e^{-\mu(\Phi_\nu)}\, L^{\phi}_z(\mathrm{d}\, \nu) \Im_{\Re}(\mathrm{d}\, \mu), \quad h \in F(\mathfrak{X} \times \mathcal{M}^{\cdot\cdot}(\mathfrak{X})).$$

Corollary 6.1.2 in [18] now gives $\xi \Im_{\Re} = \Im^{\phi}_{L_z}$, that is if we dissolve the clusters realized by the Gibbs process $\Im_{\Re}$ we obtain the Gibbs modification of the infinitely divisible process $\Im_{L_z}$. Therefore

$$\begin{aligned} C_{\Im^\phi_{L_z}}(h) &= \int h(x,\xi(\mu))\,\xi(\mu)(\mathrm{d}\,x)\Im_{\Re}(\mathrm{d}\,\mu) \\ &= \int h(x,\xi(\mu))\,\nu(\mathrm{d}\,x)\mu(\mathrm{d}\,\nu)\Im_{\Re}(\mathrm{d}\,\mu) \\ &= \int h(x,\xi(\mu+\delta_\nu))\,\nu(\mathrm{d}\,x)\,e^{-\mu(\Phi_\nu)}\,L^\phi_z(\mathrm{d}\,\nu)\Im_{\Re}(\mathrm{d}\,\mu) \\ &= \int h(x,\xi(\mu)+\nu)\,e^{-(\mu(\Phi_\nu)+E_\phi(\nu))}\,C_{L_z}(\mathrm{d}\,x\,\mathrm{d}\,\nu)\Im_{\Re}(\mathrm{d}\,\mu). \end{aligned}$$

Now observe that for $\mu = \delta_{\eta_1} + \delta_{\eta_2} + \ldots \in \mathcal{M}^{\cdot\cdot}(\mathfrak{X})$ and $\nu \in \mathfrak{X}$ we have

$$\mu(\Phi_\nu) = \sum_{j=1}^{\infty} \Phi(\nu,\eta_j) = \sum_{j=1}^{\infty}\sum_{x\in\nu}\sum_{y\in\eta_j} \phi(x,y) = \Phi(\nu,\xi(\mu)).$$

Whence we obtain the assertion. *q.e.d.*

Equation $\left(\mathbf{\Sigma}^\phi_{L_z}\right)$ contains equation $\left(\mathbf{\Sigma}'\right)$ for the Lévy measure of the Poisson process. But it contains also, for the case $\phi \equiv 0$, the equation which characterizes infinitely divisible processes. (c.f. [12])

The ideal Bose gas

Here we consider a particular permanental point process the ideal Bose gas. Let $E = \mathbb{R}^d$, where $d \geq 1$, λ the Lebesgue measure on E and let

$$g(x) = \frac{1}{(2\pi\beta)^{d/2}} \exp\Big(-\frac{\parallel x \parallel^2}{2\beta}\Big), \quad x \in E,$$

be the Gaussian density where $\beta > 0$ is a parameter called the inverse temperature. Consider the following interaction kernel $G(x,y) = g(x-y)$, $x, y \in E$. Certainly G is bounded $\|G\|_\infty = (2\pi\beta)^{-d/2}$, non negative definite and satisfies $\Upsilon(G) = 1$. Moreover let ϕ be a non negative potential such that $\Upsilon(\overline{\phi}) < \infty$. So if we recall the above section on permanental processes we obtain for

$$z \in \Big(0, \frac{(2\pi\beta)^{d/2}}{(2\pi\beta)^{d/2} + \Upsilon(\overline{\phi})}\,e^{-1}\Big]$$

the existence of the permanental process $\Im_{L_z(+1)}$, which is called the ideal Bose gas and its Gibbs modification $\Im^\phi_{L_z(+1)}$, which is a solution to $\left(\mathbf{\Sigma}^\phi_{L_z}\right)$ due to Theorem 5.

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