Ph.D Thesis

The Milnor-Moore and Poincaré-Birkhoff-Witt theorems in the locality set up and the polar structure of Shintani zeta functions

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Introduction

This thesis bridges two areas of mathematics, algebra on the one hand with the Milnor-Moore theorem (also called Cartier-Quillen-Milnor-Moore theorem) as well as the Poincaré-Birkhoff-Witt theorem, and analysis on the other hand with Shintani zeta functions which generalise multiple zeta functions. This thesis indeed consists of two main parts: the first one is devoted to an algebraic formulation of the locality principle in physics and generalisations of classification theorems such as Milnor-Moore and Poincaré-Birkhooff-Witt theorems to the locality framework. The locality principle roughly says that events that take place far apart in spacetime do not influence each other. The algebraic formulation of this principle discussed here is useful when analysing singularities which arise from events located far apart in space, in order to renormalise them while keeping a memory of the fact that they do not influence each other. This idea will be developed in further detail in the sequel. A multivariable renormalisation approach à la Speer, enables a "separation of singularities" compatible with the locality principle, which can be applied to several mathematical objects presenting singularities. This includes generalisations of multiple zeta functions, namely Shintani zeta functions which are the object of study of the second part of the thesis. We describe their polar structure in relation with the geometry of Newton polytopes, an essential step towards their renormalisation yet to be investigated. Let us now proceed to describe the state of the art in both the areas of mathematics we touch upon in the thesis.

State of the art

The locality setup

The notion of locality lies at the intersection of physics and mathematics. It can be roughly summarised in the requirement that an object can only be directly affected by its surroundings. It is then necessary to specify how objects relate, and how close must they be to interact with each other, or equivalently, when are two objects distant enough so that they do not interact. In the latter formulation, objects which are far away from each other form a symmetric relation, and observables measured on pairs of events lying in such relation should behave "nicely".

Locality naturally arises in several fields. To name a few occurrences of the notion of locality in mathematics, let us mention local operators [6], localised geometry [81], localised rings [19], and locality in index theory [95]. On the physics side, in classical field theory for instance, locality is sometimes understood as the disjointness of supports of test functions on which fields are evaluated. More precisely, for a classical action $A(f) := B(f, f)$ defined through a bilinear form $B : \mathcal{D}(U, \mathbb{C}^k) \times \mathcal{D}(U, \mathbb{C}^k) \to \mathbb{C}$, where $\mathcal{D}(U, \mathbb{C}^k)$ is the space of smooth, compactly supported functions from an open set $U \subset \mathbb{R}^n$ to $\mathbb{C}^k$, the locality of A reads:

$$\forall (f_1, f_2) \in (\mathcal{D}(U, \mathbb{C}^k))^2 : \operatorname{supp}(f_1) \cap \operatorname{supp}(f_2) = \emptyset \Longrightarrow B(f_1, f_2) = 0.$$

The previous implication can be understood in terms of a symmetric relation $\top$ on elements of $\mathcal{D}(U, \mathbb{C}^k)$, namely $f_1 \top f_2 :\Leftrightarrow B(f_1, f_2) = 0$, were we have set $f_1 \top f :\Leftrightarrow \operatorname{supp}(f_1) \cap \operatorname{supp}(f_2) = \emptyset$, which indeed defines a symmetric binary relation.

Locality serves as a guiding thread when dealing with singularities, in the sense that singularities that arise from non-interacting events, should be considered and measured separately in accordance with the locality principle. Such observations led P. Clavier, L. Guo, S. Paycha, and B. Zhang to develop an algebraic formulation of the locality principle [22], which we shall refer to as the "locality framework". It was tailored to keep track of the locality principle; its relation with the causality principle in perturbative algebraic quantum field theory was studied in [83]. P. Clavier, L. Guo, S. Paycha, and B. Zhang enhanced to the locality framework, commonly used algebraic tools, such as sets, monoids, vector spaces, algebras, coalgebras and Hopf algebras. A set X is endowed with a symmetric relation $\top$, called the locality relation — in which case the pair $(X, \top)$ is called a locality set — whose graph corresponds to the set of pairs of elements that are locality independent. When the set carries some algebraic structure e.g. if it is a monoid, resp. a vector space, resp. an algebra, resp. a coalgebra, resp. a Hopf algebra, the locality relation is required to fulfill some compatibility conditions with the underlying algebraic operations. In particular, partial products are defined only on those pairs that lie in $\top$, and for every subset U of X, the polar set $U^\top$, namely those elements x in X such that $u\top x$ for any u in U, should carry the same algebraic structure as X.

In this locality framework, the authors proved a locality version of the celebrated algebraic Birkhoff factorisation implemented by A. Connes and D. Kreimer in the context of renormalisation. It yields a factorisation of a map φ from a commutative, graded, connected, Hopf algebra to the space of meromorphic germs as a convolution product of a term involving the holomorphic part φ_+ and one involving the polar part φ_- [22, Theorem 5.8]. P. Clavier, L. Guo, S. Paycha, and B. Zhang proved that in the locality framework, and under an algebraic assumption on the polar part of the meromorphic germs under consideration which should be a locality ideal, the algebraic Birkhoff factorisation is equivalent to a minimal subtraction scheme in several variables, thereby showing the relevance of the locality framework in the context of renormalisation.

Various tools entering the algebraic Birkhoff factorisation were then generalised to the locality setup. A prototype is the locality tensor product, namely the tensor product of two locality vector spaces which reflects their locality relations, was introduced and used in [22]. This opened the path to the study of the properties of the locality tensor product, starting with its universal property. It plays an important role in two fundamental theorems discussed in this thesis in the locality framework, namely the Cartier-Milnor-Moore and the Poincaré-Birkhoff-Witt theorems. Interestingly, some of the steps of their proofs in the locality framework turn out to be simpler than in the usual setup.

Shintani zeta functions

Another central protagonist of this thesis is the Riemann zeta function, also called Euler-Riemann zeta function, which was first introduced and studied by Euler for real values, and later revisited by Riemann as a function of one complex variable. It is defined as $\zeta(s) = \sum_{n\geq 1} n^{-s}$ and is known to be absolutely convergent whenever $\Re(s) > 1$. Riemann proved [84] that it admits a meromorphic continuation to the whole complex plane with a simple pole at $s = 1$. Also in [84], he proved the functional equation of the zeta function, the relation between its zeros and the distribution of prime numbers, and stated what is probably the most famous open problem in mathematics: the Riemann hypothesis. Several generalisations of the famous Riemann zeta function have been proposed and studied, such as the Hurwitz zeta function [4] defined as $\zeta_H(s, a) = \sum_{n\geq 1}(n + a)^{-s}$ whenever a is not a negative integer.

Multivariable generalisations have also been considered starting from the multiple zeta function (see (1.27)), also called Euler-Riemann-Zagier zeta function [98],[100] or polyzeta functions [15]. They were first introduced by Euler in the eighteenth century for two complex variables [31] and much later by Ecalle [30] in 1981 for n complex variables. Hoffman [49] and Zagier [98] revisited them later during the last decade of the 20th century, which revived the interest on such objects. Their domain of absolute convergence was identified, and it was also proven that they admit a meromorphic continuation to the whole complex space with linear poles in certain hyperplanes [100], [2]. Multiple zeta functions connect with different areas of mathematics, such as arithmetic geometry, quantum groups, mathematical physics, renormalisation theory, etc., see for instance [10], [13], [15], [35], [46], [50], [51], [62], [70], [72], [67], [96]. An interesting property of the multiple zeta functions is that they satisfy polynomial relations known as shuffle and stuffle relations (also known as double-shuffle relations) [54]. Interestingly, some renormalisation schemes have been suggested in which the renormalised values of the multiple zeta functions on its divergences also satisfy the shuffle and stuffle relations, see for instance [46], [67].

Other common and useful multivariable generalisations of the Riemann zeta functions are the conical zeta functions [42], [99], Mordell-Tornheim zeta functions [69],[68], branched or arborified zeta functions [23], [25], [26] [66], Schur multiple zeta functions [71], [76], multiple Hurwitz-Lerch zeta functions [58], [55], some of which will be described in Paragraph 3.2. Meromorphic continuations of such generalisations have called the attention of numerous mathematicians, as can be seen in [2], [58], [69], [68], and [100].

Shintani zeta functions which are parametrised by matrices, form a class of multivariable generalisations of the Riemann zeta function, which encompasses multiple zeta functions, conical zeta functions, Mordell-Tornheim zeta functions, and many others. These functions where first introduced by Shintani in a series of papers in the 1970s [86], [87], [88], [89], [90], [91] motivated by problems of number theory. Shintani first built a function of only one complex variable, and determined, among several other things, the precise domain of absolute convergence of such functions, proving that they admit a meromorphic continuation to the whole complex plane $\mathbb{C}$ with the same linear poles as the function $\Gamma(rs-n)/\Gamma(s)$, where s is the complex variable, r is a positive real coefficient and n takes values in $\mathbb{Z}_{\geq 0}$. His work inspired some authors (see for instance [3], [18]) to study a multivariable (or multidimensional) version the Shintani zeta function. We focus on the formulation of Shintani zeta functions presented in the following series.

$$\sum_{m_1 \geq 1} \cdots \sum_{m_r \geq 1} (a_{11}m_1 + \cdots + a_{1r}m_r)^{-s_1} \times \cdots \times (a_{n1}m_1 + \cdots + a_{nr}m_r)^{-s_n},$$

where the a_{ij} are the real, non-negative coefficients of the parametrising matrix, and the s_j are the complex variables. The precise definition is provided in Definition 3.12. In 2003 Matsumoto proved [69] that the Shintani zeta functions admit a meromorphic continuation to $\mathbb{C}^n$ with possible linear poles, and determined a class of hyperplanes which might carry the poles. An interesting question remains open, namely as to whether these Shintani zeta functions satisfy some type of polynomial relations which generalise the double-shuffle relations of the multiple zeta functions. If so, the next step would be to renormalise these functions at their poles in such a way that the renormalised values still satisfy the polynomial relations as in the case of the multiple zetas. A first step in this direction is to refine the description of the polar structure of the Shintani zeta functions.

Objectives

As mentioned previously, this thesis focuses on two different areas geared around precise objectives, which we now briefly present. Concerning the algebraic formulation of the locality principle, the main objective is to provide a rigorous algebraic construction and to study the usual algebraic structures used in renormalisation such as tensor products, tensor algebras, symmetric algebras, Hopf algebras, and universal enveloping algebras, in the context of locality. A further aim of this work is to prove a locality version of the Milnor-Moore and Poincaré-Birkhoff-Witt theorems, which are often used in renormalisation. This extends the results of joint work with P. Clavier, L. Foissy and S. Paycha [21].

For this purpose, we introduce the concepts of locality symmetric algebras, locality Lie algebras, and locality universal enveloping algebras, which are new with respect to former work from [22], and moreover endow the existing locality structures, namely locality tensor product, locality tensor algebra, locality coalgebra, locality bialgebra and locality Hopf algebra with a natural locality relation induced by the locality relation of the original locality vector space.

A definition of bilinearity in the context of locality is presented (Definition 2.12), different from the one used in [22], and which is compatible with the construction of the locality tensor product as is demonstrated in its universal property (Theorem 5.37). In the Appendix A we discussed why the definition of locality bilinearity presented in [22] is not compatible with the locality tensor product and propose an alternative locality tensor product for which it works. The universal properties of the locality tensor algebra, locality symmetric algebra and locality universal enveloping algebra are also proved in Theorems 5.38, 5.41 and 5.45. They are later used to endow the locality symmetric algebra and the locality universal enveloping algebra with a structure of locality Hopf algebras (Propositions 6.16 and 6.17). Finally the locality version of the Milnor-Moore theorem (Theorem 6.22) (resp. the Poincaré-Birkhoff-Witt theorem (Theorem 6.39)) build an isomorphism of locality Hopf algebras between a graded, connected, cocommutative locality Hopf algebra and the universal enveloping algebra of its primitive elements (resp. an isomorphism of locality coalgebras between the locality symmetric algebra and the locality universal enveloping algebra of a locality Lie algebra).

Turning to Shintani zeta functions, our main objective is to refine the description of the polar structure of the Shintani zeta functions given in [69] to relate it with the matrix underlying the Shintani function under consideration. It is shown in Theorem 8.7 that the polar structure of the Shintani zeta functions is determined by normal vectors to the facets of the Newton polytopes corresponding to the product of the linear forms given by the columns of the parametrising matrix A. This result is an enhancement of that in [69], in that it indicates the coefficients of the hyperplanes that might carry the poles$<$ and provides a geometric interpretation of the polar structure. We moreover prove in Theorem 8.18 that the coefficients in the canonical basis of those vectors, and therefore the coefficients of the hyperplanes carrying the poles, are either 0 or 1. The latter implies that the poles of the Shintani zeta functions generalise those of the generic Feynman amplitudes via analytic regularisation, using what mathematicians call Riesz regularisation in each variable. More precisely, it was shown in [92], [28] that the poles of the generic Feynman integrals using analytic regularisation are of the form

$$s_{j_1}(s_{j_1} + s_{j_2}) \cdots (s_{j_1} + \cdots + s_{j_r}) = 0 \tag{1}$$

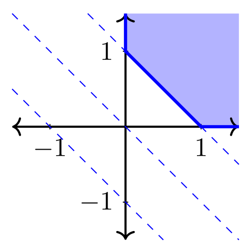
where the s_{j_i} are vectors in the canonical basis of $\mathbb{C}^n$. Since the coefficients of the hyperplanes carrying the poles are 0 or 1, such poles also correspond to those of the Shintani zeta functions. However, the ones of the Shintani zeta functions are more general in the sense that they do not require the vectors to be nested as in (1).

Let us illustrate the result of Theorem 8.7 with an example.

Example 0.1. *Consider the Shintani zeta associated to the matrix*

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The columns C_i of the matrix A induce linear forms in $(\mathbb{R}^2)^*$ given by $C_i(\epsilon) = \langle C_i, \epsilon \rangle$ which we denote also by C_i making use of the identification of $\mathbb{R}^2$ with its dual using the canonical basis. In this case the linear forms induced by the columns are $C_1(\epsilon_1, \epsilon_2) = \epsilon_1 + \epsilon_2$ and $C_2(\epsilon_1 + \epsilon_2) = \epsilon_2$. It follows from Theorem 8.7 that in this case the poles of ζ_A are parallel to the facets of the Newton polytopes of the polynomials C_1, C_2 and C_1C_2 when adding the set R_+^2 with the Minkowski sum. In the following figure, the blue area represents the polyhedron obtained by adding the Newton polynomial of C_1 with the positive orthant $\mathbb{R}_+^2$ and the dashed lines are the hyperplanes parallel to its facets which carry the poles.



Our method consists of three main steps: The first step is to express the Shintani zeta function in its domain of convergence as a multiple integral, namely as the multivariable Mellin transform of a Schwartz function on $\mathbb{R}_+^n$ which, in a neighborhood of zero, can be extended to a meromorphic function with linear poles at zero. More precisely the integrand is a product of a Schwartz function on $\mathbb{R}_{\geq 0}^n$ and the inverse $1/C_J$ of a polynomial where $C_J(\boldsymbol{\epsilon}) = \prod_{j \in J} \langle C_j, \boldsymbol{\epsilon} \rangle$, the C_j are the columns of the matrix A, $\langle , \rangle$ is the canonical inner product in $\mathbb{R}^n$, and $J \subset [r]$. The second step borrows ideas from Nilsson and Passare [77] who determined the domain of convergence and the analytic continuation of the Mellin transform of a rational function. We adapt their results to the case of a product of a Schwartz function times a rational function, which we then apply to the Mellin transform obtained in step one. The final step is to prove that the vectors $\boldsymbol{\mu}_k$ on the inward normal direction of the polyhedra obtained in step 2 are either zero or one and to provide an easy way to derive them from the columns of the matrix A. For this purpose, we borrow some tools from graph theory. More precisely, we provide an algorithm to distribute weight over a graph, such that the weight at each vertex is never lower than an imposed bound.

Structure

This thesis is divided in three chapters and two appendices.

Chapter 1

This first chapter presents the necessary prerequisites for what follows. In Section 1, we recall the algebraic building blocks in the usual (non-locality) context, for the later study of the universal properties and the proofs of the Milnor-Moore and Poincaré-Birkhoff-Witt theorems in the locality setup. For that purpose, we review in the Paragraph 1.1 Zorn's Lemma together with some of its consequences in linear algebra for infinite dimensional vector spaces. In Paragraph 1.2 we introduce the tensor product and tensor algebra of a vector space together with their respective universal properties. Paragraph 1.3 deals with the definition of coalgebras, bialgeras and Hopf algebras with some related concepts as the reduced coproduct, coideals, the convolution product on linear endomorphisms of a bialgebra and the primitive elements of a connected Hopf algebra. In paragraph 1.4 we build two quotient algebras of the tensor algebra, namely the symmetric algebra of a vector space, and the universal enveloping algebra of a Lie algebra. We also prove their universal properties and use them to endow them with a Hopf algebra structure which is essential for the formulations of the Milnor-Moore and Poincaré-Birkhoff-Witt theorems. Finally, in paragraph 1.5 we present the Milnor-Moore theorem and two versions of the Poincaré-Birkhoff-Witt theorem, together with some of their consequences.

In Section 2 we present the basic concepts underlying the locality structures, mostly borrowed from [22], necessary for our further formulation of the locality versions of the Milnor-Moore and Poincaré-Birkhoff-Witt theorems in Chapter 2. We stress out that some concepts presented in this thesis are new such as the concepts of pre-locality vector spaces and (pre-)locality subsets (resp. subspaces, resp. subalgebras). Those concepts are not introduced in a later chapter since their early introduction provides a better structure of the topic. Section 3 is devoted to the necessary background tools for our study of the polar structure of the Shintani zeta functions. We first recall some well known concepts and results from complex analysis such as meromorphic functions, analytic continuation, and Morera's theorem. We also introduce the Gamma function and the multivariable Mellin transform together with some of their properties. In the second paragraph we review some notions of number theory, more precisely the Riemann zeta function and some of some of its generalisations, such as the multiple zeta functions (or Euler-Riemann-Zagier zeta functions) also called poly zeta functions, and conical zeta functions. We present well known results about their convergence and meromorphic continuation. We also introduce the mathematical object of our study, namely the Shintani zeta functions. Finally, in the third paragraph of this section, we review some geometric concepts regarding Newton polytopes and their relation with the multivariable Mellin transform following the results of Nilsson and Passare [77].

Chapter 2

This chapter, based on [21], enhances to the context of locality the universal properties in the pre-locality context of the tensor product (Theorem 5.37), the tensor algebra (Theorem 5.38), the universal enveloping algebra (Theorem 5.38), and the symmetric algebra (Theorem 5.45), together with the locality versions of the Milnor-Moore and Poincaré-Birkhoff-Witt theorems (Theorems 6.22 and 6.39). For that purpose, we study in Section 4 the construction and universal properties in the context of pre-locality of the locality tensor product, locality tensor algebra, locality symmetric algebra, and locality universal enveloping algebra. The locality tensor product of subspaces of a pre-locality vector space presented here differs from the one in Section 2 in so far as the former is naturally endowed with a locality relation which turns it into a pre-locality vector space. Since all of the constructions presented in Section 4 are quotients of pre-locality vector spaces or of pre-locality algebras, a question which naturally arises is when the quotient of locality vector spaces is again a locality vector space and not only a pre-locality vector space (Question 2.15). We devote Section 5 to the study of this question. We provide in Paragraph 5.1 examples for which the quotient is again a locality vector space and cases in which it is not. In Paragraphs 5.2 and 5.3 we provide two sufficient conditions to have a positive answer. Finally, in Paragraphs 5.4 and 5.5 the universal properties introduced in Section 4 in the context of pre-locality are upgraded to the context of locality provided some sufficient conditions are satisfied. In Section 7.2 we introduce locality coalgebras, locality bialgebras, and locality Hopf algebras together with some techical lemmas which we then use to state and prove the locality versions of the Milnor-Moore and Poincaré-Birkhoff-Witt theorems. Most of this chapter is based on [21], and some parts are identical to those in [21], this in agreement with the other authors of the paper. Nonetheless, to our knowledge, the locality Poincaré-Birkhoff-Witt theorem, together with all its necessary previous constructions and Lemmas presented in this Ph.D. thesis are. More specifically, the contents of Paragraphs 4.4 and 6.6 and the parts of Paragraphs 5.5 and 6.3 regarding the locality symmetric algebra are original results.

Chapter 3

This third and last chapter based on [64] is dedicated to the meromorphic continuation of the Shintani zeta functions. In Theorems 7.10 and 7.11 from Section 7, we provide a domain of absolute convergence and meromorphic continuation of the Mellin transform of some class of functions, Theorems 8.7 and 8.18 in Section 8, describe the polar structure of the Shintani zeta functions which refines the result from Matsumoto [69] as we mentioned before. In Theorem 9.1 of Section 9, an algorithm is given to distribute a multidimensional weight over the vertices of a graph such that the weight on each vertex is larger than a given bound. Although at first glance, the topic of Section 9 might look unrelated to the rest of the chapter, the results provided there are essential to prove Theorem 8.18. It says that the possible hyperplanes carrying the Shintani zeta functions have normal vectors with coefficients 0 or 1 when written in terms of the canonical basis with integer and mutually coprime coefficients (Theorem 8.18). This implies that the poles at zero are similar to the ones of generic Feynman amplitudes studied in [92], [28]. Interestingly, Theorem 9.1 in Section 9 is related to Hall's marriage theorem [47] and the theory of optimal transport.

Openings

The first part of the thesis dedicated to enhancing to the locality setup classical algebraic and coalgebraic results, triggers many open questions, starting with the proof of the conjectures formulated along the way. They are interesting for their own sake and we saw how they relate to open questions in group theory. Having proved the locality Milnore-Moore theorem for cocommutative graded connected algebras, now arises the question how to classify commutative graded connected algebras in the locality setup. This further raises the question of how to handle the concept of duality in the locality framework.

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Notations

- For $a \in \mathbb{R}$, we define

$$\begin{aligned}\mathbb{R}_{\geq a} &:= \{x \in \mathbb{R} : x \geq a\},\\ \mathbb{R}_{\leq a} &:= \{x \in \mathbb{R} : x \geq a\},\\ \mathbb{R}_{>a} &:= \{x \in \mathbb{R} : x > a\},\\ \mathbb{R}_{<a} &:= \{x \in \mathbb{R} : x < a\},\\ \mathbb{R}_{+} &:= \{x \in \mathbb{R} : x > 0\},\\ \mathbb{R}_{-} &:= \{x \in \mathbb{R} : x < 0\}.\end{aligned}$$

  Analogous for $\mathbb{Z}$ instead of $\mathbb{R}$.

- $\mathbb{R}^\infty := \cup_{n\geq 1}\mathbb{R}^n$, where we assume for every $n \geq 1$ that $\mathbb{R}^{n-1}$ is embedded in $\mathbb{R}^n$ by adding one rightmost supplementary coordinate and setting it to zero. Analogous for $\mathbb{Z}^\infty$.

- For $n \in \mathbb{Z}_{>0}$, $[n] := \{1, 2, \ldots, n\}$.

- We assume that the field underlying every algebraic structure (vector space, algebra, coalgebra, bialgebra, etc), is the same, and it is denoted by $\mathbb{K}$. We assume it has characteristic zero unless it is otherwise specified.

- $\{e_i\}_{I=1}^n$ is the canonical basis of $\mathbb{R}^n$ or $\mathbb{C}^n$, depending on the context.

- For a set S, we use $\mathbb{K}(S)$ or $\langle S\rangle$ in distinctively for the free vector space spanned by S, namely $\bigoplus_{s\in S}\mathbb{K}(s)$. In the case where S is a subset of a vector space V, we use $\mathrm{span}(S)$ to denote the span of S with respect to the space V. Notice that $\mathbb{K}(S)$ and $\mathrm{span}(S)$ are isomorphic only when S is linearly independent (see Definition 1.2).

- For $n \in \mathbb{Z}_{>0}$, $\mathfrak{S}_n$ is the symmetric group of order n.

- The symbol $\subset$ means contained, not strictly contained.

- For any finite set S, we denote its cardinal as $|S|$.

- The identity map on a set S is denoted by Id_S.

- The unit element of a unital (resp. locality) algebra is denoted by 1_A. It must not be confused with the identity map on A denoted by Id_A.

- The symbol $\sim$ denotes isomorphic. The type of isomorphism is specified each time.

- Given a set S in $\mathbb{R}^n$, we denote by $\mathrm{int}(S)$ the interior of S in the usual topology of $\mathbb{R}^n$.

Chapter 1

Prerequisites

1 Algebraic prerequisites

The main objective of this first introductory section is to recall the algebraic prerequisites necessary for our study of the universal properties and the Milnor-Moore and Poincaré-Birkhoff-Witt theorems in the locality setup. For that purpose, we review in the Paragraph 1.1 Zorn's Lemma together with some of its consequences in linear algebra for infinite dimensional vector spaces. In Paragraph 1.2 we introduce the tensor product and tensor algebra of a vector space together with their respective universal properties. Paragraph 1.3 deals with the definition of coalgebras, bialgebras and Hopf algebras with some related concepts as the reduced coproduct, coideals, the convolution product on linear endomorphisms of a bialgebra and the primitive elements of a connected Hopf algebra. In Paragraph 1.4 we build two quotient algebras of the tensor algebra, namely the symmetric algebra of a vector space, and the universal enveloping algebra of a Lie algebra. We also prove their universal properties and use them to endow them with a Hopf algebra structure which is essential for the formulations of the Milnor-Moore and Poincaré-Birkhoff-Witt theorems. Finally, in Paragraph 1.5 we present the Milnor-Moore theorem and two versions of the Poincaré-Birkhoff-Witt theorem, together with some of their consequences. Contrarily to Paragraphs 1.1 to 1.4, in Paragraph 1.5 we do not provide the proofs of all theorems there presented since they are particular cases of the ones in the locality setup. However, we provide the reader with a reference when the proof is omitted. The presentation of Paragraphs 1.2, 1.3, 1.4 and 1.5 are based in [33], [65], and [17], where a complete presentation of such topics can be found. The results of Paragraph 1.1 can be found in any book about the axiom of choice and linear algebra, for instance [7].

1.1 Vector spaces, Hamel basis and bilinearity

In this first paragraph we recall some basic concepts and results of linear algebra and set theory which will be of use in all the document. We start by recalling Zorn's Lemma since it is a very useful tool when dealing with vector spaces of arbitrary dimension. Zorn's lemma is equivalent to the axiom of choice, we refer the reader to [7] for a complete discussion about this topic. The other results in this paragraph can be found in most books about linear algebra.

Recall that a partially ordered set (poset) $(P, \leq)$ is a set P together with a reflexive, transitive and antisymmetric relation $\leq \subset P \times P$. Two elements x and y in P are said to be comparable if $x \leq y$ or $y \leq x$. A totally ordered set $(S, \leq)$ is a partially ordered set where every pair of elements are comparable. A chain C of a partially ordered set $(P \leq)$ is a subset of P in which every pair of elements are comparable, this means that $(C, \leq|_C)$ is a totally ordered set, where $\leq_C := \leq \cap (C \times C)$.

Lemma 1.1 (Zorn's Lemma)**.** Let $(P, \leq)$ be a partially ordered set with the property that every chain C has an upper bound in P. Then the set P contains at least one maximal element.

We now recall the definition of an algebraic or Hamel basis of a vector space.

Definition 1.2. Let V be a $\mathbb{K}$ vector space and B a subset of V.

1. We say that B **generates** V as a vector space if every element in V is a finite linear combination of elements in B. This is $\text{span}(B) = V$.

2. We say that B is **linearly independent** or **free** if

$$\sum_{b\in B} \alpha_b b = 0$$

implies that $\alpha_b = 0$ for all $b \in B$. Moreover, the **dimension** of a vector space V is

$$\max\{|B| \in \mathbb{Z}_{\geq}0 : \text{B is linearly independent and finite}\}.$$

If the maximum does not exist, we say that the dimension of V is infinite.

3. We say that B is a **Hamel** or **algebraic basis** or simply a **basis** of V, if it generates V and is linearly independent.

Notice that this is the usual definition of a basis of a vector space when V is of finite dimension. In the infinite dimension however, it should not be confused with a Hilbert or Schauder basis since we do not consider any topology, and therefore, convergence of infinite sums is not defined.

Two folklore results from linear algebra are the following.

Lemma 1.3. Let $B_1 \subset \cdots \subset B_n \subset \cdots$ be an infinite nested family of linearly independent sets. Then $B := \bigcup_{n\geq 1} B_n$ is a linearly independent set.

Proof. The proof is by contradiction. Assume that there is a possible choice of coefficients α_b such that

$$\sum_{b\in B} \alpha_b b = 0$$

where only finitely many of them are not zero. Set $\mathsf{B} := \{b \in B : \alpha_b \neq 0\}$, by assumption B is finite and thus, there is $n \in \mathbb{Z}_{\geq 1}$ such that $\mathsf{B} \subset B_n$. However, this contradicts the linearly independence of B_n which yields the result. □

Lemma 1.4. Let S and S' be subsets of a vector space V. Then

- $\text{span}(\text{span}(S)) = \text{span}(S)$.
- $S \subset S' \Longrightarrow \text{span}(S) \subset \text{span}(S')$.

Proof. The proof follows directly from the definition of span of a subset of a vector space. □

The following useful lemma about bases of vector spaces follows from Zorn's lemma in the case where the dimension of the vector space is infinite, as it will be shown in the proof.

Lemma 1.5. Let G be a generating subset of a vector space V. A linearly independent set $A \subset G$ can be extended to a Hamel basis $B \subset G$ of V.

Proof. We assume that G is not linearly independent, since otherwise G is the expected basis. If $\text{span}(A) \neq V$, there is an element $x \in G \setminus \text{span}(A)$, otherwise Lemma 1.4 implies $V = \text{span}(G) \subset \text{span}(\text{span}(A)) = \text{span}(A)$. Thus $\mathcal{B}_1 := A \cup \{x\}$ is also a linearly independent set. If V is of finite dimension, repeating the process n times, for some n big enough, yields the existence of a set $\mathcal{B}_n$ which is linearly independent and generates V. Therefore $\mathcal{B}_n$ is a basis of V satisfying $A \subset \mathcal{B}_n \subset G$.

Consider now the case where V is of infinite dimension. Set $\mathcal{O} := \{B \in \mathcal{P}(V) : A \subset B \subset G \,\text{and}\, B \,\text{is linearly independent}\}$, and endow it with the relation $B \preceq B' :\Leftrightarrow B \subset B'$. Then $(\mathcal{O}, \preceq)$ is a partially ordered set. Moreover, by means of Lemma 1.3, every chain $B_1 \preceq \cdots B_n \preceq \cdots$ of $(\mathcal{O}, \preceq)$ has an upper bound in $\mathcal{O}$, namely $\bigcup_{n>0} B_n$. By means of Zorn's lemma (Lemma 1.1) there exist a maximal element $\mathcal{B} \in \mathcal{O}$ with respect to the order $\preceq$. Since $\mathcal{B} \in \mathcal{O}$ it is linearly independent and such that $A \subset B \subset G$, and since $\mathcal{B}$ is maximal, it generates V. Otherwise there would be an element x in $G \setminus \text{span}(\mathcal{B})$ implying that $\mathcal{B} \prec \mathcal{B} \cup \{x\}$, which contradicts the maximality of $\mathcal{B}$. □

A simple consequence of the previous lemma is the following.

Lemma 1.6. If $V \neq \{0_V\}$ is a vector space, then it has a basis $\mathcal{B}$.

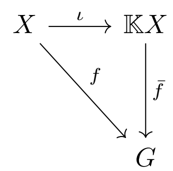
Proof. Let $0 \neq x \in V$, then Lemma 1.5 implies the result for $A = \{x\}$ and $G = V$. $\square$

We proceed to recall a universal property of the freely generated vector space $\mathbb{K}(X)$ of a set X and its relation with bilinear maps.

Lemma 1.7. Given a set X and a vector space G, any map $f : X \to G$ uniquely extends to a linear map $\bar{f} : \mathbb{K}(X) \to G$ as follows

$$\bar{f}\left(\sum_{x\in X} \alpha_x\, x\right) = \sum_{x\in X} \alpha_x\, f(x). \tag{1.1}$$

This can be seen as the commutativity of the following diagram, where ι stands for the canonical injection.



Proof. It follows from (1.1) that $\bar{f}$ is linear and extends f. The uniqueness is a consequence of $\iota(X)$ being a Hamel basis of $\mathbb{K}(X)$, and thus $\bar{f}$ is completely determined by its values in $\iota(X)$. $\square$

For V and W two vector spaces over the same field $\mathbb{K}$, we consider the linear subspace $I_{\text{bil}}(V,W)$ of the free linear span $\mathbb{K}(V \times W)$, generated by all elements of the form

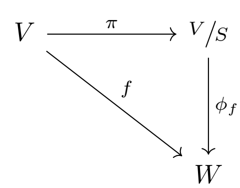
$$(a+b,x) - (a,x) - (b,x) \tag{1.2}$$
$$(a,x+y) - (a,x) - (a,y) \tag{1.3}$$
$$(ka,x) - k(a,x) \tag{1.4}$$
$$(a,kx) - k(a,x) \tag{1.5}$$

with $a,b \in V$, $x,y \in W$ and $k \in \mathbb{K}$. The following lemma follows directly from the definition of bilinearity, and will be useful for the proof of Theorem 1.11.

Lemma 1.8. Let V, W, and G be vector spaces. A map $f : X \times Y \to G$ is bilinear if, and only if $\bar{f}(I_{\text{bil}}(V,W)) = \{0_G\}$.

We conclude this first paragraph with a well known result of linear algebra which will be central to many theorems in the sequel.

Proposition 1.9. *Let $f : V \to W$ be a linear map and S a subset of V such that $S \subset \ker(f)$. Then there exists a unique linear map $\phi_f : {}^{V}\!/_{S} \to W$ such that $f = \phi_f \circ \pi$ where $\pi : V \to {}^{V}\!/_{S}$ is the canonical quotient map. This can be seen as the commutativity of the following diagram.*



Proof. Define $\phi_f([x]) := f(x)$ where $[x] \in {}^{V}\!/_{S}$ is the equivalence class of x. Let $s \in S$, then

$$\phi_f([x+s]) = f(x+s) = f(x) + f(s) = f(x) = \phi_f([x])$$

and thus ϕ_f is well defined. To prove that the uniqueness of ϕ_f, assume there is another linear map $\phi' : {}^{V}\!/_{S} \to W$ such that $f = \phi' \circ \pi$, then for $x \in V$ by assumption $\phi_f([x]) = f(x) = \phi'([x])$ which yields the result. $\square$

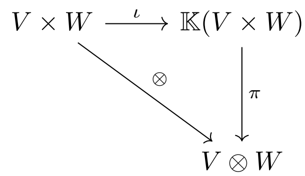
1.2 Tensor product and tensor algebra

In this paragraph we focus on the construction of the tensor product between vector spaces and of the tensor algebra of a vector space. We also present some of its well known properties such as associativity (Corollary 1.15), distributivity with respect to sums, direct sums and intersections (Proposition 1.17), and their universal properties (Theorems 1.11, 1.16, and 1.25). This paragraph is based on [33], [65] where the reader can find a good, complete and understandable presentation of these topics. We recall the construction of the tensor product of two vector spaces via quotients of vector spaces.

Definition 1.10. The **tensor product** of V and W is a vector space defined as the following quotient space:

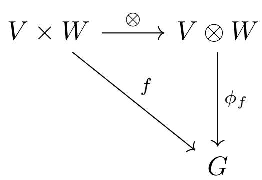
$$V \otimes W := {}^{\mathbb{K}(V \times W)}\!/_{I_{\text{bil}}(V,W)}.$$

Notice that the map $\otimes := \pi \circ \iota : V \times W \to V \otimes W$, built from the canonical inclusion map $\iota : V \times W \to \mathbb{K}(V \times W)$ and the canonical quotient map $\pi : \mathbb{K}(V \times W) \to V \otimes W$ makes the following diagram commute.



It follows from its construction that the map $\otimes$ is an injective bilinear map, and that $\otimes(V \times W)$ generates $V \otimes W$ as a vector space. Then the tensor product satisfies the following universal property.

Theorem 1.11. Let V, W, and G be vector spaces, and $f : V \times W \to G$ a bilinear map. Then, there exists a unique linear map $\phi_f : V \otimes W \to G$ which makes the following diagram commute.



Moreover, the pair $(V \otimes W, \otimes)$ is the only one (up to isomorphism) which satisfies this property.

Proof. Existence and uniqueness of ϕ_f: By means of Lemma 1.7 there exists a unique linear map $\bar{f}$: $\mathbb{K}(V \times W) \to G$ extending f. Since f is bilinear, it follows from Lemma 1.8 that $\bar{f}(I_{\text{bil}}(V,W)) = \{0_G\}$, and thus, Proposition 1.9 yields the existence of a unique linear map $\phi_f : V \otimes W \to G$ satisfying $\bar{f} = \phi_f \circ \pi$. Precomposing both sides with the canonical inclusion ι implies that

$$f = \bar{f} \circ \iota = \phi_f \circ \pi \circ \iota = \phi_f \circ \otimes$$

as expected. The uniqueness of ϕ_f follows from the facts that $\otimes(V \times W)$ generates $V \otimes W$ as a vector space, and that the values of $\otimes(V \times W)$ are completely determined by f.

Uniqueness of $(V \otimes W, \otimes)$: Assume that there is another space $\overline{V \otimes W}$ and another bilinear map $\bar{\otimes} : V \times W \to \overline{V \otimes W}$ which satisfies the property in the first part of the theorem. Let us fix the notations $v \otimes w := \otimes(v,w)$ and $\overline{v \otimes w} := \bar{\otimes}(v,w)$. Since $V \otimes W$ satisfies the property above, for $f = \bar{\otimes}$ there exist a unique linear map $\phi : V \otimes W \to \overline{V \otimes W}$ such that $\phi(v \otimes w) = \overline{v \otimes w}$. Conversely, since $\overline{V \otimes W}$ also satisfies the condition above, for $f = \otimes$ there exists a unique linear map $\bar{\phi} : \overline{V \otimes W} \to V \otimes W$ such that $\bar{\phi}(\overline{v \otimes w}) = v \otimes w$, and thus $\phi \circ \bar{\phi}\big|_{\otimes(V \times W)} = \text{Id}_{\overline{V \otimes W}}\big|_{\bar{\otimes}(V \times W)}$. We apply once again the property of the first part of the theorem to the map $\otimes$ and the space $V \otimes W$. It follows that $\phi_{\otimes} = \text{Id}_{V \otimes W}$ is the only linear map from $V \otimes W$ to itself that stabilizes $\otimes(V \times W)$. Thus $\phi \circ \bar{\phi} = \text{Id}_{V \otimes W}$. Analogously $\bar{\phi} \circ \phi = \text{Id}_{\overline{V \otimes W}}$ implying that ϕ is an isomorphism of vector spaces with inverse $\bar{\phi}$. □

Given two linear maps $f_i : V_i \to W_i$ for $i \in \{1,2\}$, the map $f_1 \times f_2 : V_1 \times V_2 \to W_1 \otimes W_2$ defined as $(f_1 \times f_2)(v_1, v_2) = f_1(v_1) \otimes f_2(v_2)$ is a bilinear map. We will denote by $f_1 \otimes f_2$ the map $\phi_{f_1 \times f_2}$ given in the universal property of the tensor product (Theorem 1.11). Another notation which will be useful in the sequel is the map $\tau_{12} : V \otimes V \to V \otimes V$ which is the only linear map such that

$$\tau_{12}(v_1 \otimes v_2) = v_2 \otimes v_1. \tag{1.6}$$

Its existence and uniqueness can be easily deduced by applying Theorem 1.11 to the bilinear map $f(v_1, v_2) = v_2 \otimes v_1$.

We proceed to prove the associativity of the tensor product. For that purpose, we relate first a basis of V and W with a basis of $V \otimes W$. This same process will not work on the locality context since bases do not always behave well with the locality relation as it is mentioned in Chapter 2.

Lemma 1.12. Let $\mathcal{B}_V = \{v_i\}_{i\in I}$ be an algebraic basis of V and $\mathcal{B}_W = \{w_j\}_{j\in J}$ be an algebraic basis of W, then $\mathcal{B}_V \otimes \mathcal{B}_W := \{v_i \otimes w_j : v_i \in \mathcal{B}_V \text{ and } w_j \in \mathcal{B}_W\}$ is an algebraic basis of $V \otimes W$.

Proof. Since the map $\otimes$ is bilinear, every tensor $v \otimes w$ can be expressed as a sum of tensors of the form $v_i \otimes w_j$, and thus $\mathcal{B}_V \otimes \mathcal{B}_W$ generates $V \otimes W$. Consider now a linear combination $\sum_{(i,j)\in I\times J} k_{i,j} v_i \otimes w_j = 0$ where $k_{i,j} \in \mathbb{K}$ and only a finite number of them is different from zero. Fix $i_o \in I$ and $j_o \in J$ and consider the bilinear map $f : V \times W \to \mathbb{K}$ defined as $f(v,w) = e^*_{i_o}(v) f^*_{j_o}(w)$, where $e^*_{i_o} \in V^*$ (resp. $f^*_{j_o} \in W^*$) sends a vector to the coefficient of v_{i_o} (resp. w_{j_o}) when expressed in the basis $\mathcal{B}_V$ (resp. $\mathcal{B}_W$). Let $\phi_f : V \otimes W \to K$ be the unique linear map obtained by means of the universal property of the tensor product (Theorem 1.11), then

$$0 = \phi_f\Big(\sum_{(i,j)\in I\times J} k_{i,j} v_i \otimes w_j\Big) = \sum_{(i,j)\in I\times J} k_{i,j} e^*_{i_o}(v_i) f^*_{j_o}(w_j) = k_{i_o,j_o}.$$

We conclude that $\mathcal{B}_V \otimes \mathcal{B}_W$ is free. □

We present some simple consequences of the previous lemma.

Corollary 1.13. Let V be a $\mathbb{K}$-vector space. Then $\mathbb{K} \otimes V$ and $V \otimes \mathbb{K}$ are isomorphic to V as vector spaces.

Proof. Let $\{v_i\}_{i\in I}$ be a basis of V, and consider the map $f : V \to \mathbb{K} \otimes V$ defined by $f(v_i) = 1 \otimes v_i$. By means of Lemma 1.12, it is an isomorphism of vector spaces. A similar argument proves that $\mathbb{K} \sim V \otimes \mathbb{K}$ as vector spaces. □

Corollary 1.14. Let $f_i : V_i \to W_i$ be isomorphisms of vector spaces for $i \in \{1,2\}$. Then $f_1 \otimes f_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ is also an isomorphism of vector spaces.

Proof. Consider a basis $\mathcal{B}_{V_i}$ of V_i for $i \in \{1,2\}$. Since f_i is an isomorphism, then $f_i(\mathcal{B}_{V_i})$ is also a basis of W_i. The result follows from Lemma 1.12. □

We proceed to prove the associativity of the tensor product.

Corollary 1.15. Let V_1, V_2, and V_3 be three vector spaces. The map

$$\begin{aligned} f :&(V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3) \\ &(v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3) \end{aligned}$$

is an isomorphism of vector spaces.

Proof. Considering basis of the three vector spaces, by means of Lemma 1.12 f sends a basis injectively into a basis, and therefore it is an isomorphism of vector spaces. □

Notice that Corollary 1.15 allows us to extend the tensor product to n vector spaces. Indeed the tensor product $V_1 \otimes \cdots \otimes V_n$ of n vector spaces can be defined iteratively. Also the universal property of the tensor product (Theorem 1.11) can be extended to n-linear maps.

Theorem 1.16. Let $f : V_1 \times \cdots \times V_n \to G$ be an n-linear map. There exists a unique linear map $\phi_f : V_1 \otimes \cdots \otimes V_n \to G$ such that $\phi_f \circ \otimes = f$.

Proof. The statement follows from Theorem 1.11 applied $n-1$ times to the maps $f|_{V_1\times\cdots\times V_i\times\{0_{V_{i+1}}\}\times\cdots\times\{0_{V_n}\}}$ for $3\leq i\leq n+1$. This yields the existence of a unique linear map $\phi_f : V_1\otimes\cdots\otimes V_n\to G$ as expected. □

It follows from Theorem 1.16, similar as before, that given n linear maps $f_i : V_i\to W_i$, the map $f_1\times\cdots\times f_n : V_1\times\cdots\times V_n\to W_1\otimes\cdots\otimes W_n$ defined as $(f_1\times\cdots\times f_n)(v_1,\ldots,v_n) = f_1(v_1)\otimes\cdots\otimes f_n(v_n)$ is an n-linear map. We will denote by $f_\otimes\cdots\otimes f_n$ the map $\phi_{f_1\times\cdots\times f_n}$ obtained from the universal property of the tensor product.

The following proposition presents the behavior of the tensor product with respect to sums, direct sums and intersections. Such properties rely on the existence of algebraic complements of any subspace of a vector space, which can be easily shown by completing a basis. On the locality context, however, bases do not always behave well with the locality relation as it will be discussed in Chapter 2, thus the locality counterpart of the following proposition and its consequences need some extra assumptions (see Proposition 4.21 and Corollary 4.22).

Proposition 1.17. *Let V_1 and V_2, be subspaces of a vector space V, and W_1 and W_2 subspaces of a vector space W, then:*

1. $(V_1+V_2)\otimes(W_1+W_2) = (V_1\otimes W_1)+(V_1\otimes W_2)+(V_2\otimes W_1)+(V_2\otimes W_2)$.
2. $(V_1\cap V_2)\otimes(W_1\cap W_2) = (V_1\otimes W_1)\cap(V_2\otimes W_2)$.
3. $(V_1\oplus V_2)\otimes(W_1\oplus W_2) = (V_1\otimes W_1)\oplus(V_1\otimes W_2)\oplus(V_2\otimes W_1)\oplus(V_2\otimes W_2)$.

Proof. [33, Proposition 9]

1. Both sides of the equality are subspaces of $V\otimes W$ generated by the elements $v\otimes w$ with $v\in V_1\cup V_2$ and $w\in W_1\cup W_2$. Therefore they are equal.

2. The inclusion from left to right follows directly from the observation $(V_1\cap V_2)\otimes(W_1\cap W_2)\subset(V_i\otimes W_i)$ for $i\in\{1,2\}$. For the second inclusion, consider a basis $\{v_i\}_{i\in I'}$ of $V_1\cap V_2$ (resp. a basis $\{w_j\}_{j\in J'}$ of $W_1\cap W_2$). Extend it to a basis $\{v_i\}_{i\in I_1}$ of V_1 (resp. $\{w_j\}_{j\in J_1}$ of W_1) and to a basis $\{v_i\}_{i\in I_2}$ of V_2 (resp. $\{w_j\}_{j\in J_2}$ of W_2), such that $I' = I_1\cap I_2$ (resp. $J' = J_1\cap J_2$). Finally, extend the linearly independent set $\{v_i\}_{i\in I'}\cup\{v_i\}_{i\in I_1\setminus I'}\cup\{v_i\}_{i\in I_2\setminus I'}$ (resp. $\{w_j\}_{j\in J'}\cup\{w_j\}_{j\in J_1\setminus J'}\cup\{w_j\}_{j\in J_2\setminus J'}$) to a basis $\{v_i\}_{i\in I}$ of V (resp. $\{w_j\}_{j\in J}$ of W). Let $x\in(V_1\otimes W_1)\cap(V_2\otimes W_2)$. It can then be written in a unique way as
$$x = \sum_{(i,j)\in I\times J} a_{ij}v_i\otimes w_j.$$
where the a_{ij} lie in $\mathbb{K}$. Consider $i_o\in I\setminus I_1$ and $e^*_{i_o}$ the map on V^* which sends an element to the coefficient of v_{i_o} when expressed in the basis $\{v_i\}_{i\in I}$. Since $x\in V_1\otimes W_1$, then
$$0 = (e^*_{i_o}\otimes \mathrm{Id}_W)(x) = \sum_{j\in J} a_{i_o,j}w_j.$$
Since the w_js are linearly independent, then $a_{i_o,j}=0$ for every j. A similar argument proves that $a_{i,j}=0$ for every $(i,j)\notin I'\times J'$. Thus $x\in(V_1\cap V_2)\otimes(W_1\cap W_2)$ as expected.

3. From the first item, it follows that $(V_1\oplus V_2)\otimes(W_1\oplus W_2) = (V_1\otimes W_1)+(V_1\otimes W_2)+(V_2\otimes W_1)+(V_2\otimes W_2)$. From the second item $(V_1\otimes W_1)\cap(V_1\otimes W_2)=\{0\}$ and similarly for all the other intersections, which implies the result.

□

A useful consequence of the previous proposition is the following.

Lemma 1.18. For $i\in\{1,2\}$, let $f_i : V_i\to W_i$ be linear maps from a vector space V_i to a vector space W_i. We have
$$\ker(f_1\otimes f_2) = \ker f_1\otimes V_2 + V_1\otimes\ker f_2.$$

Proof. Let $K_i := \ker(f_i) \subset V_i$ and let $X_i \subset V_i$ be any direct complement space in V_i so that $V_i = K_i \oplus X_i$. It follows from Proposition 1.17 item (3) that

$$V_1 \otimes V_2 = (K_1 \otimes K_2) \oplus (X_1 \otimes K_2) \oplus (K_1 \otimes X_2) \oplus (X_1 \otimes X_2).$$

As a consequence of the first isomorphism theorem for vector spaces there are isomorphisms $\phi_i : X_i \to \mathrm{Im}(f_i)$ from which we build a linear map $(\phi_1^{-1} \otimes \phi_2^{-1}) \circ (f_1 \otimes f_2)$ from $V_1 \otimes V_2$ onto $X_1 \otimes X_2$ which does not vanish on $X_1 \otimes X_2$ outside the null tensor.

Since $(K_1 \otimes K_2) \oplus (X_1 \otimes K_2) \oplus (K_1 \otimes X_2) \subset \ker((\phi_1^{-1} \otimes \phi_2^{-1}) \circ (f_1 \otimes f_2))$ and since by construction $(\phi_1^{-1} \otimes \phi_2^{-1}) \circ (f_1 \otimes f_2)$ does not vanish on $X_1 \otimes X_2 \setminus \{0\}$ (where 0 is the null tensor), we have $\ker((\phi_1^{-1} \otimes \phi_2^{-1}) \circ (f_1 \otimes f_2)) = (K_1 \otimes K_2) \oplus (X_1 \otimes K_2) \oplus (K_1 \otimes X_2) = \ker f_1 \otimes V_2 + V_1 \otimes \ker f_2$. By means of Corollary 1.14, $\phi_1 \otimes \phi_2$ is an isomorphism of vector spaces which yields the result. $\square$

Algebras and the tensor algebra

Throughout this document, unless otherwise stated, algebras refer to associative unitary algebras.

Definition 1.19.

1. An algebra is a triple (A, m, u) where A is a vector space, $m : A \otimes A \to A$ and $u : \mathbb{K} \to A$ are linear maps satisfying:

   - *Associativity:* The following diagram commutes.

   $$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes \mathrm{Id}_A} & A \otimes A \\ \downarrow{\scriptstyle \mathrm{Id}_A \otimes m} & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{\quad m \quad} & A \end{array}$$

   or equivalently $m \circ (\mathrm{Id}_m \otimes m) = m \circ (m \otimes \mathrm{Id}_A)$.

   - *Unit:* The following diagram commutes.

   $$\begin{array}{ccccc} A \otimes \mathbb{K} & \xrightarrow{\mathrm{Id}_A \otimes u} & A \otimes A & \xleftarrow[u \otimes \mathrm{Id}_A]{} & \mathbb{K} \otimes A \\ & {\scriptstyle \sim} \searrow & \downarrow{\scriptstyle m} & \swarrow {\scriptstyle \sim} & \\ & & A & & \end{array}$$

   or equivalently $m \circ (\mathrm{Id}_A \otimes u) = \mathrm{Id}_A = m \circ (u \otimes \mathrm{Id}_A)$.

   We moreover call an algebra commutative if $m = \tau_{12} \circ m$ (see (1.6)). We sometimes refer to an algebra simply as A when the product and unit are clear from the context.

2. A subspace S of A is a subalgebra if $m(S \otimes S) \subset S$ and $u(\mathbb{K}) \subset S$.

3. We say that an algebra (A, m, u) is **graded** if there is a sequence $\{A_i\}_{i \in \mathbb{Z}_{\geq 0}}$ of subspaces of A such that
$$A = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_n, \quad m(A_p \otimes A_q) \subset A_{p+q}, \quad u(\mathbb{K}) \subset A_0.$$

4. A **filtered algebra** is an algebra (A, m, u) together with a sequence of nested vector spaces $A^0 \subset A^1 \subset \cdots \subset A^n \subset \cdots$ called the filtration, such that
$$A = \bigcup_{n \in \mathbb{Z}_{\geq 0}} A^n, \quad m(A^p \otimes A^q) \subset A^{p+q}, \quad u(\mathbb{K}) \subset A^0.$$

5. A graded (resp. filtered) algebra is said to be **connected** if $\dim(A_0) = 1$ (resp. $\dim(A^0) = 1$).

6. We say a linear map $f : A \to A'$, where (A, m, u) and (A', m', u') are algebras, is an **algebra morphism** if $m' \circ (f \otimes f) = f \circ m$.

Notice that a grading $\{A_i\}_{i\in\mathbb{Z}_{\geq 0}}$ in an algebra induces naturally a filtration $\{A^n\}_{n\in\mathbb{Z}\geq 0}$ by setting $A^n := \bigoplus_{j=0}^n A_j$. Conversely, a filtration $\{A^n\}_{n\in\mathbb{Z}_{\geq 0}}$ induces a grading by setting $A_0 := A^0$ and $A_i := {}^{A^n}/_{A_{n-1}}$ for $i > 0$. It follows that A and $\bigoplus_{n\in\mathbb{Z}_{\geq 0}} A_n$ are isomorphic as vector spaces but not necessarily as algebras. However, the last identification depends on the choice of isomorphism since, in general, there is no canonical way of building one.

Remark 1.20. With some abuse of notation we write 1 instead of $u(1)$ or 1_A when there is no ambiguity.

The following proposition will be used implicitly in the sequel.

Proposition 1.21. *Let A and B be two algebras, then $A\otimes B$ has the structure of an algebra with product*

$$m_{A\otimes B}\big((a \otimes b) \otimes (a' \otimes b')\big) := m_A(a \otimes a') \otimes m_B(b \otimes b'),$$

and unit $u_{A\otimes B} := u_A \otimes u_B$.

Proof. Notice that $m_{A\otimes B} = (m_A \otimes m_B) \circ \tau_{23}$, where $\tau_{23} : A \otimes B \otimes A \otimes B \to A \otimes A \otimes B \otimes B$ switches the 2nd and 3rd component of the vector. Then $m_{A\otimes B}$ is linear since τ_{23}, m_A and m_B are linear. The associativity follows directly from the associativity of m_A and m_B.

For the unit map, consider $a \otimes b \in A \otimes B$ and $k \in \mathbb{K}$. Then

$$m_{A\otimes B} \circ (\mathrm{Id}_{A\otimes B} \otimes u_{A\otimes B})(a \otimes b \otimes k) = m_{A\otimes B}\big((a \otimes b) \otimes k(1_A \otimes 1_B)\big) = k(a \otimes b).$$

An analogous calculation proves the same result using $(u_{A\otimes B} \otimes \mathrm{Id}_{A\otimes B})$ instead of $(\mathrm{Id}_{A\otimes B} \otimes u_{A\otimes B})$, thus $u_{A\otimes B}$ satisfies the unit axiom. □

We proceed to recall the concept of ideal and its relation with quotients of algebras.

Definition 1.22. Let A be an algebra and $I \subset A$ a subspace of A.

- We say that I is a **left ideal** (resp. **right ideal**) of A if, and only if

$$m(A \otimes I) \subset I \quad (\text{resp. } m(I \otimes A) \subset I).$$

- We say that I is an **ideal** of A if, and only if it is both a right and a left ideal of A.

Proposition 1.23. *Let A be an algebra and I and ideal of A, then $({}^{A}/_{I}, \bar{m}, \bar{u})$ is an algebra, where $\bar{m}([x] \otimes [y]) := [m(x \otimes y)]$, and $\bar{u}(k) = [u(k)]$.*

Proof. By construction $\bar{u} = \pi \circ u$ where π is the canonical quotient map, thus it is well defined and linear. The fact that it satisfies the unity axiom follows from the same property of u. To show that $\bar{m}$ is well defined, consider w and w' elements of I, then

$$\bar{m}([x+w]\otimes[y+w']) = [m((x+w)\otimes(y+w'))] = [m(x\otimes y)]+[m(x\otimes w')]+[m(w\otimes y)]+[m(w\otimes w')] = \bar{m}([x]\otimes[y])$$

thus $\bar{m}$ is well defined. Linearity and associativity of $\bar{m}$ follow from linearity and associativity of m. □

A folklore result which will be useful in the sequel is the following. We omit the proof, since it follows from its locality version (Lemma 2.29) when considering the trivial locality $\top = A \times A$.

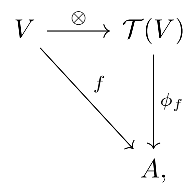
Proposition 1.24. *Let $f : A \to A'$ be a morphism of algebras, then $\ker(f)$ is an ideal of A, and $\mathrm{Im}(f)$ is a subalgebra of A'.*

We proceed to recall the construction and properties of the tensor algebra of a vector space. Recall that throughout the document we assume that all algebras are unital and associative unless stated otherwise. Let V be a vector space, its tensor algebra is the vector space defined as

$$\mathcal{T}(V) := \bigoplus_{n\geq 0} V^{\otimes n}, \tag{1.7}$$

where $V^{\otimes n} := \underbrace{V \otimes \cdots \otimes V}_{n-\text{times}}$ for $n \geq 2$, $V^{\otimes 1} = V$, and $V^{\otimes 0} = \mathbb{K}$. Notice that as a consequence of Lemma 1.12 and Corollary 1.15, given a basis $\{v_i\}_{i \in I}$ of V, the set $\{v_{i_1} \otimes \cdots \otimes v_{i_k} : k \in \mathbb{Z}_{\geq 0} \wedge i_j \in I \text{ for } 1 \leq j \leq k\}$ is a basis of $\mathcal{T}(V)$. We define the concatenation product $m_\otimes$ on $\mathcal{T}(V)$ as $m_\otimes(v_1 \otimes \cdots \otimes v_k, v_{k+1} \otimes \cdots \otimes v_n) := v_1 \otimes \cdots \otimes v_k \otimes v_{k+1} \otimes \cdots \otimes v_n$. It is associative as a consequence of Corollary 1.15. Moreover the canonical injection $u : \mathbb{K} \to V^{\otimes 0} \subset \mathcal{T}(V)$ is a unit for the concatenation product. Thus $(\mathcal{T}(V), m_\otimes, u)$ is an associative, unital algebra which satisfies the following universal property.

Theorem 1.25. Let V be a vector space, A an algebra, and $f : V \to A$ a linear map. There is a unique morphism of algebras $\phi_f : \mathcal{T}(V) \to A$ such that the following diagram commutes



where $\otimes : V \to \mathcal{T}(V)$ is the canonical injection map.

Proof. It is clear that the maps $(v_1, \ldots, v_n) \mapsto f(v_1) \cdots f(v_n)$ are n-linear. Thus, by means of the universal property of the tensor product (Theorem 1.16) there exist linear maps $F_n : V^{\otimes n} \to A$ such that $F_n(v_1, \ldots, v_n) = f(v_1) \cdots f(v_n)$. Consider the linear the map $\phi_f : \mathcal{T}(V) \to A$ defined as the direct sum of the maps F_n. It is clearly an algebra morphism which satisfies $f = \phi_f \circ \otimes$.

The uniqueness follows from the fact that ϕ_f is completely determined by the values it takes on the set $\otimes(V)$ since it generates $\mathcal{T}(V)$ as an algebra. □

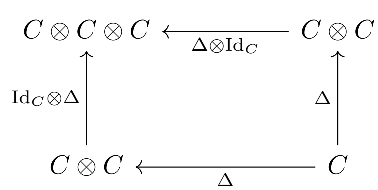
Analogous to the proof of Theorem 1.11, it can be shown that the pair $(\mathcal{T}(V), \otimes)$ is the only pair (up to isomorphism of algebras) which satisfies the conditions of Theorem 1.25.

1.3 Coalgebras, bialgebras, and Hopf algebras

Other algebraic concepts essential to our study are those of coalgebras, bialgebras and Hopf algebras. We introduce those concepts, together with some examples and well known properties in this paragraph. We base this paragraph in [33], [65], and also in the introductory chapters of the Ph.D. thesis of Pierre Clavier[20]. Another complete introduction to this topic can be found in [17, page 3] for a better introduction.

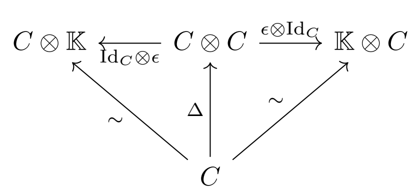
Definition 1.26. 1. A **coalgebra** is a triplet (C, Δ, ϵ) where C is a vector space, $\Delta : C \to C \otimes C$ and $\epsilon : C \to \mathbb{K}$ are linear maps satisfying the following conditions:

- *Coassociativity:* The following diagram commutes.



or equivalently $(\mathrm{Id}_C \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{Id}_C) \circ \Delta$.

- *Counit:* The following diagram commutes.



or equivalently $(\mathrm{Id}_C \otimes \epsilon) \circ \Delta = \mathrm{Id}_C = (\epsilon \circ \mathrm{Id}_C) \circ \Delta$.

We say moreover that a coalgebra is cocommutative if $\Delta = \tau_{12} \circ \Delta$ (see (1.6)). We sometimes write only C to a coalgebra (C, Δ, ϵ) whenever there is no risk of ambiguity.

2. We say a coalgebra (C, Δ, ϵ) is **graded** if there is a sequence $\{C_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of subspaces of C such that

$$C = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} C_n, \quad \Delta(C_n) \subset \sum_{p+q=n} C_p \otimes C_q, \quad \epsilon(C_n) = \{0\}\ \forall n \geq 1.$$

3. A **filtered coalgebra** is a coalgebra (C, Δ, ϵ) together with a sequence of nested vector spaces $C^0 \subset C^1 \subset \cdots \subset C^n \subset \cdots$ called the filtration, such that

$$C = \bigcup_{n\in\mathbb{Z}_{\geq 0}} C^n, \quad \Delta(C^n) \subset \sum_{p+q=n} C^p \otimes C^q, \quad (\forall x \notin C^0) : \epsilon(x) = 0.$$

4. A graded (resp. filtered) coalgebra is said to be **connected** if $\dim(C_0) = 1$ (resp. $\dim(C^0) = 1$).

5. A map $f : C \to C'$, where (C, Δ, ϵ) and (C', Δ', ϵ') are coalgebras, is a **coalgebra morphism** if $(f \otimes f) \circ \Delta = \Delta' \circ f$, and $\epsilon' \circ f = \epsilon$.

A notation which is commonly used to simplify the calculations with the coproduct is the Sweedler notation [94]:

$$\Delta(x) =: \sum_x x^{(1)} \otimes x^{(2)} \in C \otimes C.$$

Using Sweedler notation, the coassociativity axiom reads

$$\sum_x \sum_{x^{(1)}} \left(x^{(1)}\right)^{(1)} \otimes \left(x^{(1)}\right)^{(2)} \otimes x^{(2)} = \sum_{x^{(2)}} x^{(1)} \otimes \left(x^{(2)}\right)^{(1)} \otimes \left(x^{(2)}\right)^{(2)} =: \sum_x x^{(1)} \otimes x^{(2)} \otimes x^{(3)},$$

the counit axiom reads

$$\sum_x \epsilon\left(x^{(1)}\right) x^{(2)} = x = \sum_x x^{(1)} \epsilon\left(x^{(2)}\right),$$

and the cocommutativity reads

$$\sum_x x^{(1)} \otimes x^{(2)} = \sum_x x^{(2)} \otimes x^{(1)}.$$

Analogous to the concept of ideal of an algebra, we can define the concept of coideal in order to preserve the coalgebraic structure under quotients.

Definition 1.27. Let (C, Δ, ϵ) be a coalgebra, and $J \subset C$ a subspace. We say J is:

- a **sub-coalgebra** of C if $\Delta(J) \subset J \otimes J$.
- a **right coideal** of C if $\epsilon(J) = \{0\}$, and

$$\Delta(J) \subset C \otimes J.$$

- a **left coideal** of C if $\epsilon(J) = \{0\}$, and

$$\Delta(J) \subset J \otimes C.$$

- a **coideal** of C if $\epsilon(J) = \{0\}$, and

$$\Delta(J) \subset J \otimes C + C \otimes J.$$

Proposition 1.28. *Let C be a coalgebra and J a coideal, then the quotient space ${}^{C}/_{J}$ inherits a coalgebra structure with counit $\bar{\epsilon}([x]) = \epsilon(x)$, and coproduct $\bar{\Delta}([x]) = \sum_x [x^{(1)}] \otimes [x^{(2)}]$.*

Proof. The coproduct $\bar{\Delta}$ and counit $\bar{\epsilon}$ are well defined as a consequence of Proposition 1.9. Indeed, the map $(\pi\otimes\pi)\circ\Delta$ (resp. ϵ) vanishes in J, thus the map $\bar{\Delta}$ satisfying $(\pi\otimes\pi)\circ\Delta=\bar{\Delta}\circ\pi$ (resp. $\bar{\epsilon}$ satisfying $\epsilon=\bar{\epsilon}\circ\pi$) is well defined, linear and unique. It is then straightforward to see that they inherit the coassociativity and counit properties from Δ and ϵ respectively. □

The following result corresponds to the coalgebraic counterpart of Proposition 1.24.

Proposition 1.29. *Let $f: C\to C'$ be a coalgebra morphism, then* $\ker C$ *is a coideal of C and* $\mathrm{Im}(f)$ *is a sub-coalgebra of C'.*

Proof. We prove first that $\mathrm{Im}(f)$ is a sub-coalgebra of C': for any $c\in C$, since f is a coalgebra morphism, $\Delta' f(c)=(f\otimes f)\circ\Delta(c)=\sum_{(c)} f(c^{(1)})\otimes f(c^{(2)})\in\mathrm{Im}(f)\otimes\mathrm{Im}(f)$, showing that $\mathrm{Im}(f)$ is a sub-coalgebra of C'.

We prove that $\ker(f)$ is a coideal. For $c\in\ker(f)$, $\epsilon(c)=\epsilon'(f(c))=0$. Also, since f is a coalgebra morphism, then $0=\Delta'(f(c))=(f\otimes f)\Delta(f)$, and thus $\ker(f)\subset\ker(f\otimes f)$. The result follows from Lemma 1.18. □

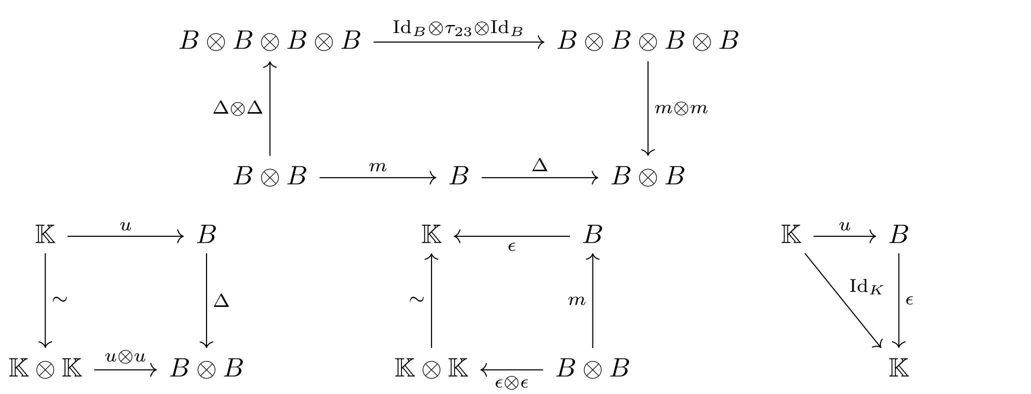
Notice that the proof of $\ker(f)$ being a coideal relies on Lemma 1.18 which ultimately relies on the fact that a linearly independent set can be completed to a basis as discussed before Proposition 1.17. Therefore, the locality counterpart of Proposition 1.29, namely Lemma 6.7 uses some extra assumptions.

The following algebraic structure links the concept of algebra and coalgebra in a same structure.

Definition 1.30. 1. A **bialgebra** is a quintuple (B,m,u,Δ,ϵ) where (B,m,u) is an algebra and (B,Δ,ϵ) is a coalgebra compatible in the sense that ϵ is an algebra morphism, m is a coalgebra morphism, and Δ is an algebra morphism (equivalently m is a coalgebra morphism). This means that

$$\Delta\circ m|_{B^{\otimes 2}}=\underbrace{(m\otimes m)}_{\text{domain } B^{\otimes 4}}\circ(\mathrm{Id}_B\otimes\tau_{23}\otimes\mathrm{Id}_B)\circ\underbrace{(\Delta\otimes\Delta)|_{B^{\otimes 2}}}_{\text{range } B^{\otimes 4}};\quad \epsilon\circ m=\epsilon\otimes\epsilon;\quad \Delta\circ u=u\otimes u;\quad \epsilon\circ u=\mathrm{Id}_{\mathbb{K}},$$

where $\tau_{23}: B^{\otimes^4}\to B^{\otimes 4}$ is the map that switches the terms on the second and third position of the tensor. This can be seen as the commutativity of the following diagrams:



2. We say that a bialgebra (B,m,u,Δ,ϵ) is graded (resp. filtered) if there is a grading $\{B_n\}_{n\in\mathbb{Z}_{\geq 0}}$ (resp. a filtration $\{B^n\}_{n\in\mathbb{Z}_{\geq 0}}$) which makes it graded (resp. filtrated) both as an algebra and as a coalgebra. We moreover say that it is connected if $\dim(B_0)=1$ (resp. $\dim(B^0)=1$).

3. Let $(B_i,m_i,u_i,\Delta_i,\epsilon_i)$ $(i\in\{1,2\})$ be two bialgebras. A **bialgebra morphism** from B_1 to B_2 is a locality map $f: B_1\longrightarrow B_2$ that is a morphism of algebras and of coalgebras.

4. A subspace V of a bialgebra B is a sub-bialgebra if it is both a subalgebra and a sub-coalgebra.

5. We say that an element x of a bialgebra B is a **primitive element** if $\Delta(x) = x \otimes 1 + 1 \otimes x$. We denote the set of primitive elements of B as $\mathrm{Prim}(B)$ which is actually a subspace of B. If $\Delta(x) = x \otimes x$, we say that x is a group like element.

Remark 1.31. In a filtered connected bialgebra B, $\Delta(1) \in B^0 \otimes B^0$ which is a space of dimension 1 (see Lemma 1.12). Then $\Delta(1) = a(1 \otimes 1)$. It follows from $\Delta \circ u = u \otimes u$ that $a = 1$ and thus 1 is a group like element.

Example 1.32. *For a vector space V, its tensor algebra $\mathcal{T}(V)$ can be endowed with a bialgebra structure using its universal property (Theorem 1.25). Let $\delta : V \to \mathcal{T}(V) \otimes \mathcal{T}(V)$ be the linear map defined by $\delta(v) := v \otimes 1 + 1 \otimes v$. Theorem 1.25 yields the existence of a unique algebra morphism $\Delta : \mathcal{T}(V) \to \mathcal{T}(V) \otimes \mathcal{T}(V)$ extending δ. This is the so called deshuffle coproduct, namely for $v_i \in V$ for $1 \leq i \leq n$,*

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{S \subset [n]} v_S \otimes v_{[n] \setminus S}.$$

Here $v_S := v_{i_1} \otimes \cdots \otimes v_{i_{|S|}}$ where $S = \{i_1 < \cdots < i_{|S|}\}$, and $v_\emptyset = 1$. The counit on the other hand is the unique algebra morphism $\epsilon : \mathcal{T}(V) \to \mathbb{K}$, the kernel of which is $\ker(\epsilon) = \bigoplus_{n=1}^{\infty} V^{\otimes n}$. It is straightforward to check that $\mathcal{T}(V)$ together with the deshuffle coproduct and the counit above mentioned is a cocommutative, graded, connected bialgebra.

Notice that if a graded bialgebra B is connected, then $\ker(\epsilon) = \bigoplus_{n \geq 1} B_n$. If the bialgebra is connected and filtered, then $B = B^0 \oplus \ker(\epsilon)$. A well known result which will be of use later is the following.

Lemma 1.33. Let B be a graded (resp. filtered) connected bialgebra, then for every $x \in B_n$ (resp. $x \in B^n$)

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_x x' \otimes x''$$

where x' and x'' are of degree less than n and greater than 0.

Proof. Since B is graded (resp. filtered), then

$$\Delta(B_n) \subset \sum_{p+q=n} B_p \otimes B_q = B_0 \otimes B_n + B_n \otimes B_0 + \sum_{\substack{p+q=n \\ p \neq 0 \neq q}} B_p \otimes B_q$$

$$\Big(\text{resp.} \quad \Delta(B^n) \subset \sum_{p+q=n} B^p \otimes B^q = B^0 \otimes B^n + B^n \otimes B^0 + \sum_{\substack{p+q=n \\ p \neq 0 \neq q}} B^p \otimes B^q\Big).$$

Thus $\Delta(x) = k_1(x_1 \otimes 1) + k_2(1 \otimes x_2) + \sum_x x' \otimes x''$ where all the x' and x'' are of degree lower than n and greater than 0, x_1, x_2 lie in B_n (resp. B^n), and $k_1, k_2 \in \mathbb{K}$. The elements 1 appearing i the previous expression follow from the connectedness of B since every element of B_0 (resp. B^0) is a multiple of 1. By means of the counit axiom, and since for every $y \notin B_0$ (resp. $y \notin B^0$) $\epsilon(y) = 0$, then $x = (\mathrm{Id}_C \otimes \epsilon)(\Delta(x)) = k_1 x_1$. A similar argument shows that $k_2 x_2 = x$ as expected. □

In lights of the previous lemma, given a connected graded or filtered bialgebra, we may define the **reduced coproduct** as a map

$$\tilde{\Delta} : C \longrightarrow \ker(\epsilon) \otimes \ker(\epsilon) \tag{1.8}$$

$$x \mapsto \tilde{\Delta}(x) := \Delta(x) - 1 \otimes x - x \otimes 1. \tag{1.9}$$

Notice that $\tilde{\Delta}$ is coassociative, as a consequence of the coassociativity of Δ, and its kernel is $\mathbb{K} \oplus \mathrm{Prim}(B)$. Moreover, if Δ is cocommutative, then $\tilde{\Delta}$ is also cocommutative. We may define recursively $\tilde{\Delta}^{(n)}$ as $\tilde{\Delta}^{(1)} := \tilde{\Delta}$ and

$$\tilde{\Delta}^{(n)} := (\underbrace{\mathrm{Id}_B \otimes \cdots \otimes \mathrm{Id}_B}_{n-1 \text{ times}} \otimes \tilde{\Delta}) \circ \tilde{\Delta}^{(n-1)}.$$

Lemma 1.34. Let B be a filtered connected bialgebra and x and element in B^k with $k \geq 1$, then

$$\tilde{\Delta}^{(n)}(x) = 0$$

for all $n \geq k$.

Proof. For any $y \in B$ set $|y| := \min\{n :\in \mathbb{Z}_{\geq 0} : y \in B^n\}$. Consider $x \in B^k$, then

$$\tilde{\Delta}(x)^{(n)} = \sum_x x^{(1)} \otimes \cdots \otimes x^{(n+1)}$$

where $\sum_{j=1}^{n+1} |x^{(j)}| = |x| \leq k$. Since $n+1 > k$, this imposes that at least a $|x^{(j)}| = 0$ and thus $\widetilde{\Delta}^{(n)}(x) = 0$. □

Apart from the intrinsic beauty of the bialgebras relating the dual concepts of algebra and coalgebra in the same object, it also leads to a convolution product among linear maps from the bialgebra to itself.

Definition-Proposition 1.35. Let $(B, m, u, \Delta, \epsilon)$ be a bialgebra, and $\phi, \psi : B \to B$ two linear maps. The convolution product of ϕ and ψ is a linear map $B \to B$ defined by

$$\phi \star \psi = m(\phi \otimes \psi)\Delta.$$

Moreover, if ϕ and ψ are algebra morphisms and B is commutative, then $\phi \star \psi$ is also an algebra morphism.

Proof. The linearity of $\phi \star \psi$ follows from the linearity of Δ, m, ϕ and ψ. The multiplicativity of $\phi \star \psi$ follows also directly from the multiplicativity of Δ, ϕ and ψ. □

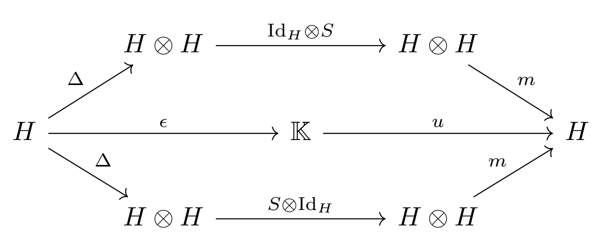
One can easily deduce from the associativity of m and coassociativity of Δ, that the convolution product is associative, and that it has a unit $e = u \circ \epsilon$. Thus $(\mathcal{L}(B, B), \star)$ is a monoid. It follows from the linearity of m and Δ that the convolution product is bilinear. The existence of an inverse however, is in general not guaranteed, which leads us naturally to the concept of Hopf algebra.

Definition 1.36.

1. A **Hopf algebra** is a bialgebra $(H, m, u, \Delta, \epsilon)$ together with a linear map $S : H \to H$ such that

$$S \star Id_H = Id_H \star S = u \circ \epsilon.$$

This can be seen as the commutativity of the following diagram.



We say that H is commutative if it is commutative as an algebra. We say that H is cocommutative if it is cocommutative as a coalgebra.

2. A **graded (resp. filtered) Hopf algebra** is a Hopf algebra together with a grading (resp. filtration) which makes it a graded (resp. filtered) algebra, and a graded (resp. filtered) coalgebra and such that

$$S(H_n) \subset H_n \text{ (resp. } S(H^n) \subset H^n).$$

A **connected Hopf algebra** is a graded (resp. filtered) algebra such that H_0 (resp. H^0) has dimension 1.

3. Let $(H_i, m_i, u_i, \Delta_i, \epsilon_i, S_i)$ for $i \in \{1, 2\}$ be two Hopf algebras. A **Hopf algebra morphism** between H_1 and H_2 is a morphism of locality bialgebras $f : H_1 \longrightarrow H_2$ which also satisfies $f \circ S_1 = S_2 \circ f$.

4. A subspace H' of Hopf algebra H is a **sub-Hopf algebra** of H if, and only if it is a sub-bialgebra and $S(H') \subset H'$.

Proposition 1.37. *[20, Proposition 1.1.3] Let H be a Hopf algebra and G the set of algebra morphisms from H to itself, then $(G, \star)$ is a group. Moreover, the inverse of an element $\phi \in G$ is $\phi \circ S$ where S is the antipode of H.*

Proof. The associativity and existence of a unit were discussed after Definition-Proposition 1.35. We are left to prove that $(\phi \circ S) \star \phi = \phi \star (\phi \circ S) = u \circ \epsilon$. Notice first that for every algebra morphism $\phi \in G$, $\phi \circ u \circ \epsilon = u \circ \epsilon$. Indeed, from the bialgebra axioms $u \circ \epsilon(H) = \mathbb{K}(1_H)$ and $\phi(1_H) = 1_H$. Then

$$\begin{aligned}
\phi \star (\phi \circ S)(x) =& m \circ (\phi \otimes (\phi \circ S)) \circ \Delta(x) \\
=& m \circ (\phi \otimes \phi) \circ (\mathrm{Id}_H \circ S) \circ \Delta(x) \\
=& \phi \circ m \circ (\mathrm{Id}_H \circ S) \circ \Delta(x) && \text{since } \phi \text{ is an algebra morphism,} \\
=& \phi \circ (\mathrm{Id}_H \star S)(x) \\
=& \phi \circ u \circ \epsilon(x) && \text{from the Hopf algebra axioms,} \\
=& u \circ \epsilon(x) && \text{from the previous argument.}
\end{aligned}$$

A similar argument proves that $(\phi \circ S) \star \phi = u \circ \epsilon$ which yields the result. ☐

Notice that the previous proposition also works in the more general case that G is the set of algebra morphisms from a Hopf algebra H to an algebra A. If the algebra is commutative, such group is commonly known as the character group, [20, Section 1.1.4] and it plays a central role in the Connes and Kreimer's formalism of renormalisation. There is also a similar result for a bigger monoid containing G whenever H is filtered and connected. We use here the common notation $\tilde{\Delta}(x) = \sum_x x' \otimes x''$.

Proposition 1.38. *Let H be a filtered connected Hopf algebra, then the set $G = \{\phi \in \mathcal{L}(H, H) : \phi(1) = 1\}$ is a group with respect to the convolution product.*

Proof. For ϕ and ψ in G, $\phi \star \psi(1) = \phi(1)\psi(1) = 1$ and thus $\star$ stabilizes G. We are only left to prove the existence of an inverse $\phi^{\star -1} \in G$ for every $\phi \in G$. We claim that

$$\psi := \sum_{k \in \mathbb{Z}_{\geq 0}} (u \circ \epsilon - \phi)^{\star k}$$

lies in G and is the desired inverse of ϕ. Here $(u \circ \epsilon - \phi)^{\star 0} = u \circ \epsilon$. We prove that the sum is finite when evaluated in any $x \in H$. Notice first that $(u \circ \epsilon - \phi)(1) = 0$ and thus $(u \circ \epsilon - \phi)^{\star k}(1) = 0$ for $k > 0$ and $\psi(1) = u \circ \epsilon(1) = 1$. For $x \in \ker(\epsilon)$ fix $n > 0$ such that $x \in H^n$,

$$(u \circ \epsilon - \phi)^{\star k}(x) = m^{(k-1)} \circ (\underbrace{\phi \otimes \cdots \otimes \phi}_{k-\text{times}}) \circ \tilde{\Delta}^{(k-1)}(x).$$

By means of Lemma 1.34 $\tilde{\Delta}^{(k-1)}(x) = 0$ for $k - 1 \geq n$ and thus, when evaluated in some x, the sum is finite and ψ well defined.

We now show that $\phi^{\star -1} = \psi$. For $x = 1$ it follows from the previous computations that $\psi(1) = 1$ and thus $\psi \star \phi(1) = 1$. For $x \in \ker(\epsilon)$ such that $x \in H^n$, it follows that

$$\begin{aligned}
\psi \star \phi(x) =& \psi \star (u \circ \epsilon - (u \circ \epsilon - \phi))(x) \\
=& \psi(x) - \psi \star (u \circ \epsilon - \phi)(x) \\
=& \sum_{k=0}^{n} (u \circ \epsilon - \phi)^{\star k}(x) - \sum_{k=0}^{n} (u \circ \epsilon - \phi)^{\star k+1}(x) \\
=& (u \circ \epsilon - \phi)^{\star 0}(x) - (u \circ \epsilon - \phi)^{\star n+1}(x) \\
=& (u \circ \epsilon)(x) \\
=& 0,
\end{aligned}$$

and thus ψ is a left inverse for ϕ. Here we used the bilinearity of $\star$, and the fact that for $x \in H^n$, the terms of order higher than n in the sum of ψ vanish. A similar computation proves that ψ is also a right inverse for ϕ which yields the result. □

In the same spirit of the previous result, the following proposition states that in a filtered, connected bialgebra, the antipode comes for free.

Proposition 1.39. *Let H be a filtered, connected bialgebra, then H is a Hopf algebra and the antipode is defined recursively as $S(1) = 1$, and for $x \in \ker(\epsilon)$*

$$S(x) = -x - \sum_x S(x')x'' = -x - \sum_x x' S(x'').$$

Proof. The antipode is the inverse of the identity map with respect to the convolution product, then for $x \in \ker(\epsilon)$, the antipode must satisfy $m(S \otimes \mathrm{Id}_H)\Delta(x) = m(\mathrm{Id}_H \otimes S)\Delta(x) = 0$. It is easy to see that the recurrent formulas from the antipode satisfy this condition. Notice moreover that if Id_H has a left inverse S and a right inverse S' for the convolution product, then $S = S \star u \circ \epsilon = S \star \mathrm{Id}_H \star S' = u \circ \epsilon \star S' = S'$, and thus the two formulas for the antipode coincide. □

As a consequence of Proposition 1.39, the tensor algebra together with the deshuffle coproduct (see Example 1.32) is a graded, connected, cocommutative Hopf algebra. We introduce other examples of Hopf algebras on rooted trees.

Definition 1.40.
- A **rooted tree** T is a finite, non-empty, loopless, connected, oriented graph which has a minimum vertex called the root. A **rooted forest** is a commutative concatenation of trees. We admit the existence of the empty forest $\emptyset$ which is the only forest without trees in it. The free $\mathbb{K}$ span of the set of rooted forests is denoted by $\mathcal{F}$. Setting $\mathcal{F}_n$ as the free $\mathbb{K}$ span of forests with n vertices, $\mathcal{F}$ becomes a graded vector space, namely $\mathcal{F} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{F}_n$.
- An **admissible cut** of a rooted forest F is a subset c of edges of F such that any path from one of the roots of F to a leaf of F meet c at most once. We write $\mathrm{Adm}(F)$ the set of admissible cuts of F. For $c \in \mathrm{Adm}(F)$, we write $R_c(F)$ the subforest of F below the c and $T_c(F)$ the subforest of F above the cut c. Notice that we have $|V(F)| = |V(R_c(F))| + |V(T_c(F))|$ where $|\bullet|$ represents the number of vertices of a tree.
- For F_1, F_2 and F three rooted forests, we set

  $$n(F_1, F_2, F) := |\{c \in \mathrm{Adm}(F) | R_c(F) = F_1 \ \wedge \ T_c(F) = F_2\}|.$$

  Notice that for F_1 and F_2 two given rooted forests, $n(F_1, F_2, F) = 0$ except for a finite number of rooted forests F.

Example 1.41. *The Connes-Kreimer [32, 61] Hopf algebra is defined on the free span of rooted forests $\mathcal{F}$.*

- *The product of two forests is the commutative concatenation of them, namely*

  $$m_{CK}(T_1 \otimes T_2) := T_1 T_2.$$

  It is clear that this product respects the grading of $\mathcal{F}$.
- *The unit u is the linear application $u : \mathbb{K} \to \mathcal{F}$ which sends 1 to the empty tree $\emptyset$.*
- *The coproduct is defined in a forest as*

  $$\Delta_{CK}(F) := \sum_{c \in \mathrm{Adm}(F)} T_c(F) \otimes R_c(F),$$

  and extends linearly to all of $\mathcal{F}$. Notice that Δ_{CK} also respects the grading of $\mathcal{F}$.

- *The counit $\epsilon : \mathcal{F} \to \mathbb{K}$ sends $\epsilon(\emptyset) = 1$ and all other forests to zero.*

One can then show that $(\mathcal{F}, m_{CK}, u, \Delta_{CK}, \epsilon)$ is a graded, connected, commutative bialgebra, and by means of Proposition 1.39, it is a connected, graded, commutative Hopf algebra.

Example 1.42. *We now describe the so called Grossman-Larson Hopf algebra which is isomorphic to the dual of the Connes-Kreimer Hopf algebra (see [79] with corrections in [49],[32]). The Grossman-Larson algebra, whose primitive elements are rooted trees, was introduced in [38, 39, 40]. The unit and counit are the same as in the Connes-Kreimer Hopf algebra.*

- *We define the product of two rooted forests F_1 and F_2 by*

$$F_1 * F_2 = \sum_{F \in \mathcal{F}} n(F_1, F_2, F) F.$$

  It is well-defined since the sum contains only finitely many non-zero terms. It moreover respects the grading of $\mathcal{F}$.

- *The coproduct Δ_* is defined by its action on rooted forests $F = T_1 \cdots T_n$, namely*

$$\Delta_*(F) = \sum_{I \subseteq [n]} T_I \otimes T_{[n] \setminus I}$$

  with, for $I \subseteq [n]$, we have $T_I := \prod_{i \in I} T_i$.

$(\mathcal{F}, *, u, \Delta_*, \epsilon)$ is a graded, connected, cocommutative bialgebra, and once again by means of Proposition 1.39 it is a Hopf algebra.

1.4 Lie algebras, universal enveloping algebra, and symmetric algebra

The purpose of this paragraph is to recall the construction and universal properties of the symmetric algebra of a vector space and the universal enveloping algebra of a Lie algebra. Making use of such universal properties, we endow each of these algebras with a Hopf algebra structure. This paragraph is based in [33], [17] where a complete presentation of such topics can be found.

The following result from basic algebra states under which conditions is the quotient of an algebra over a subspace again an algebra. Since both the symmetric algebra and the universal enveloping algebra are quotients of the tensor algebra, it is particularly useful in this paragraph.

Lemma 1.43. Let A and B be two algebras, I an ideal of A and $\psi : A \to B$ an algebra morphism such that $\psi(I) = \{0_B\}$. Then there exists a unique algebra morphism $\phi : A/I \to B$ which satisfies $\psi = \phi \circ \pi$, where $\pi : A \to A/I$ is the canonical quotient map. Here, the algebra structure of A/I is the one introduced in Proposition 1.23.

Proof. By means of Proposition 1.9, there exists a unique linear map $\phi : A/I \to B$ satisfying $\psi = \phi \circ \pi$. We check that it is an algebra morphism: let x and y be elements of A, and w and w' elements in the ideal I. Then

$$\begin{aligned}
\phi([xy]) =& \phi([x][y]) && \text{from the definition of the product on } A/I, \\
=& \psi((x+w)(y+w')) && \text{from the definition of } \phi, \\
=& \psi(x)\psi(y) + \psi(x)\psi(w') + \psi(w)\psi(y) + \psi(w)\psi(w') && \text{since } \psi \text{ is an algebra morphism,} \\
=& \phi([x])\psi([y]).
\end{aligned}$$

The last line follows from the definition of ϕ and from the fact that $\psi(I) = \{0_B\}$. □

Symmetric, antisymmetric tensors and symmetric algebra

The following folklore result about $\mathfrak{S}_n$-modules, where $\mathfrak{S}_n$ is the symmetric group of degree n, will be useful in the sequel. We provide a proof for completeness.

Proposition 1.44. *Let V be a $\mathfrak{S}_n$-module and set*

$$SV := \{v \in V \mid \forall \sigma \in \mathfrak{S}_n,\ \sigma.v = v\},$$
$$AV := \langle \{v - \sigma.v \mid \sigma \in \mathfrak{S}_n,\ v \in V\}\rangle.$$

Then

1. *SV and AV are submodules of V such that $V = SV \oplus AV$.*
2. *If W is a submodule of V, then $SW = SV \cap W$ and $AW = AV \cap W$, where AW and SW are defined in a similar manner as AV and SV.*
3. *The quotient V/AV is an $\mathfrak{S}_n$-module, and the following map is a well-defined $\mathfrak{S}_n$-module morphism:*

$$\begin{array}{ccc} V/AV & \longrightarrow & SV \\ [v] & \longmapsto & \dfrac{1}{n!}\displaystyle\sum_{\sigma \in \mathfrak{S}_n} \sigma.v. \end{array}$$

The elements in SV are often referred to as the invariants of the $\mathfrak{S}_n$-module, and the elements in V/AV as the coinvariants.

Proof. Consider the map

$$\begin{array}{rcl} \pi_V : V & \longrightarrow & V \\ v & \longmapsto & \dfrac{1}{n!}\displaystyle\sum_{\sigma \in \mathfrak{S}_n} \sigma.v. \end{array}$$

Notice that for any $v \in V$, and for any $\tau \in \mathfrak{S}_n$,

$$\pi_V(\tau.v) = \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} (\sigma\tau).v = \frac{1}{n!}\sum_{\sigma' \in \mathfrak{S}_n} \sigma'.v = \pi_V(v),$$

whereas

$$\tau.\pi_V(v) = \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} (\tau\sigma).v = \frac{1}{n!}\sum_{\sigma' \in \mathfrak{S}_n} \sigma'.v = \pi_V(v).$$

Therefore $\pi_V(\tau.v) = \tau.\pi_V(v) = \pi_V(v)$, so π_V is a $\mathfrak{S}_n$-module endomorphism with values in SV. Moreover, if $v \in SV$,

$$\pi_V(v) = \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \sigma.v = v,$$

so π_V is a projection on SV. Consequently, $SV = Im(\pi_V)$, as well as $\ker(\pi_V)$, are submodules of V, and $V = SV \oplus \ker(\pi_V)$.

Let us now prove that $AV = \ker(\pi_V)$. If $v \in V$ and $\sigma \in \mathfrak{S}_n$,

$$\pi_V(v - \sigma.v) = \pi_V(v) - \pi_V(\sigma.v) = \pi_V(v) - \pi_V(v) = 0,$$

so $AV \subseteq \ker(\pi_V)$. On the other hand, for $v \in \ker(\pi_V)$

$$v = v - \pi(v) = v - \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \sigma.v = \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} (\underbrace{v - \sigma.v}_{\in AV}) \in AV,$$

which proves the first item.

For the second item, consider W a submodule of V. It is clear from the construction that $SW = SV \cap W$, since SW are those elements in W invariant under the action of $\mathfrak{S}_n$. Moreover, $\pi_W = \pi_V\big|_W$, so

$$AW = \ker(\pi_W) = \ker(\pi_V\big|_W) = \ker(\pi_V) \cap W = AV \cap W,$$

which completes the proof of the second item.

For the third item, notice that V/AV is still a $\mathfrak{S}_n$-module where $\sigma.[v] := [\sigma.v]$. Indeed, for any v and v' in V, and σ and τ in $\mathfrak{S}_n$,

$$[\tau.(v + v' - \sigma.v')] = [\tau.v + \tau.v' - (\tau\sigma\tau^{-1}).\tau.v'] = [\tau.v]$$

and thus the action of S_n is well defined. Finally, it is straightforward to check that the following map is an isomorphism of $\mathfrak{S}_n$-modules:

$$\begin{array}{rcl} V/AV = V/\ker(\pi_V) & \longrightarrow & SV = Im(\pi_V) \\ \overline{v} & \longmapsto & \pi_V(v). \end{array}$$

□

We proceed to recall some known properties of the symmetric and antisymmetric tensors. The results provided here can be found in [17], [29] where the reader can found a deeper presentation of these subjects.

Let V be a vector space, we denote by Ξ the canonical linear action $\Xi : \mathfrak{S}_n \times \mathcal{T}(V) \to \mathcal{T}(V)$ of the elements in the symmetric group $\mathfrak{S}_n$ in the tensor algebra $\mathcal{T}(V)$ given by

$$\mathfrak{S}_n \times V^{\otimes^m} \ni (\sigma, v_1 \otimes \cdots \otimes v_m) \mapsto \Xi(\sigma, v_1 \otimes \cdots \otimes v_m) := \begin{cases} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases} \tag{1.10}$$

The **symmetric tensors** of V are the elements of $\mathcal{T}(V)$ invariant under the symmetric action Ξ defined in (1.10). More precisely, for $n \geq 1$, $(\mathcal{ST}(V))_n$ is the subspace of $V^{\otimes n}$ invariant under the action of the symmetric group $\mathfrak{S}_n$ as defined in (1.10), and the set of symmetric tensors of V is the direct sum $\mathcal{ST}(V) := \bigoplus_{n \in \mathbb{Z}_{\geq 0}} (\mathcal{ST}(V))_n$, where we have set $(\mathcal{ST}(V))_0 := V^{\otimes 0} = \mathbb{K}$. Similarly, we denote by $\mathcal{AT}(V)$ the ideal of $\mathcal{T}(V)$ generated by all elements of the form

$$v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

where $n \in \mathbb{Z}_{\geq 1}$ and σ is any permutation in S_n. It is clear that the set of antisymmetric tensors is a graded ideal with $(\mathcal{AT}(V)_n = \mathcal{AT}(V) \cap V^{\otimes n}$.

It is well known (see for instance [17, Section 4.2] or [29, Section 2.4]) that the direct sum of symmetric and antisymmetric tensors is the whole tensor algebra.

Lemma 1.45. Let V be a vector space, then

$$\mathcal{T}(V) = \mathcal{AT}(V) \oplus \mathcal{ST}(V).$$

The proof follows directly from Proposition 1.44. The decomposition of a tensor $v_1 \otimes \cdots \otimes v_n \in \mathcal{T}(V)$ in symmetric and antisymmetric part is as follows:

$$v_1 \otimes \cdots \otimes v_n = \frac{1}{n!}\Big(\sum_{\Sigma \in \mathfrak{S}_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}\Big) - \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n}\Big(v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}\Big).$$

Definition 1.46. Let V be a vector space, its **symmetric algebra** is the quotient algebra

$$S(V) := {}^{\mathcal{T}(V)}\!/\!_{\mathcal{AT}(V)}.$$

The product $m_S : S(V) \otimes S(V) \to S(V)$ is defined as the equivalent class of the product of two representatives (See Proposition 1.23). We denote by $\pi_S : \mathcal{T}(V) \to S(V)$ the canonical quotient map, and $\iota_S := \pi_S \circ \otimes : V \to S(V)$ is the canonical map from V to $S(V)$.

Some properties of the symmetric algebra which follow directly from the previous definition are the following:

1. By means of Lemma 1.45, $S(V)$ is isomorphic as vector space to $\mathcal{ST}(V)$.

2. The symmetric algebra $S(V)$ inherits a grading from that on $\mathcal{T}(V)$ because $\mathcal{AT}(V)$ is a graded ideal.

3. Since $(\mathcal{ST}(V))_0 = \mathbb{K}$ and $(\mathcal{AT}(V)_0 = \{0\}$, then $(S(V))_0 = \mathbb{K}$, thus $S(V) = \bigoplus_{n\in\mathbb{Z}_{\geq 0}}(S(V))_n$ is a graded, connected algebra.

4. Similarly, the facts that $(\mathcal{ST}(V))_1 = V$, and $(\mathcal{AT}(V)_1 = \{0\}$, imply that $(S(V))_1 = \iota_S(V) = V$, and therefore the canonical map $\iota_S : V \to S(V)$ is injective. This justifies the following abuse of notation: for $v \in V$, we denote also by v the element $\iota_S(v) \in S(V)$. We denote by $\odot$ the product on $S(V)$, i.e., $v_1 \odot v_2 := m_S(v_1 \otimes v_2) = m_S(\iota_S(v_1) \otimes \iota_S(v_2))$.

5. The symmetric algebra $S(V)$ is commutative. Indeed, for any $\sigma \in \mathfrak{S}_n$,

$$\begin{aligned} v_1 \odot \cdots \odot v_n =& \pi_S(v_1 \otimes \cdots \otimes v_n) \\ =& \pi_S(v_1 \otimes \cdots \otimes v_n - v_1 \otimes \cdots \otimes v_n + v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}) \\ =& \pi_S(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}) \\ =& v_{\sigma(1)} \odot \cdots \odot v_{\sigma(n)}. \end{aligned}$$

6. Given an ordered basis $\{v_i\}_{i\in\mathbb{Z}_{\geq 1}}$ of V, the set of ordered monomials $\{v_{i_1} \odot \cdots \odot v_{i_n} : i_1 \leq \cdots \leq i_n\}$ is a basis of $(S(V))_n$. This follows from the fact that tensor products of basis elements form a basis of $\mathcal{T}(V)$ and from item 1.

The symmetric algebra of a vector space satisfies the following universal property.

Theorem 1.47 (Universal property of the symmetric algebra)**.** Let V be a vector space, A a commutative algebra, and $f : V \to A$ a linear map. There is a unique morphism of commutative algebras $\phi_f : S(V) \to A$ such that $f = \phi_f \circ \iota_S$. This can be seen as the commutativity of the following diagram.

$$\begin{array}{ccc} V & \xrightarrow{\iota_S} & S(V) \\ & \searrow^{f} & \downarrow{\scriptstyle \phi_f} \\ & & A \end{array}$$

Proof. By means of the universal property of the tensor algebra (Theorem 1.25), since f is a linear map, there exists a unique algebra morphism $\psi : \mathcal{T}(V) \to A$ such that $f = \psi \circ \otimes$ where $\otimes$ is the canonical map from V to $\mathcal{T}(V)$. Since A is a commutative algebra, for every $n \in \mathbb{Z}_{\geq 1}$, every $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$ and σ in $\mathfrak{S}_n$

$$\psi(v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}) = f(v_1) \cdots f(v_n) - f(v_{\sigma(1)}) \cdots f(v_{\sigma(n)}) = 0,$$

and thus $\psi(\mathcal{AT}(V)) = \{0_A\}$. Lemma 1.43 yields the existence of a unique algebra morphism $\phi : U(V) \to A$ which satisfies $\psi = \phi \circ \pi_S$. From $\iota_S = \pi_S \circ \otimes$, it follows that

$$f = \psi \circ \otimes = \phi \circ \pi_S \circ \otimes = \phi \circ \iota_S$$

as expected. The uniqueness of ϕ is granted from the fact that $\iota_S(V)$ generates $S(V)$ as an algebra, and thus ϕ is completely determined by its values in $\iota_S(V)$. ☐

The symmetric algebra $S(V)$ of a vector space V can be endowed naturally with the structure of a graded, cocommutative, connected Hopf algebra. So far we have only described it as a graded, connected algebra. In order to equip it with a coproduct, consider the linear map $\delta : V \to S(V) \otimes S(V)$ defined by $\delta(x) = \iota_S(x) \otimes 1 + 1 \otimes \iota_S(x)$. By means of Theorem 1.47 there exists a unique morphism of commutative algebras $\Delta : S(V) \to S(V) \otimes S(V)$ which extends δ, this is $\delta = \Delta \circ \iota_S$. Notice that by construction the elements in $\iota_S(V)$ are primitive and the coproduct Δ is cocommutative. For the counit we consider the zero map from V to $\mathbb{K}$. This is again a linear map and once again by means of Theorem 1.47, there is a unique algebra morphism $\epsilon : S(V) \to \mathbb{K}$ which vanishes identically on $\iota_S(V)$. Therefore $S(V)$ together with this coproduct and counit is a graded connected bialgebra over $\mathbb{K}$, and thus, by means of Proposition 1.39 $S(V)$ is a Hopf algbera.

Lie algebras and universal enveloping algebras

The second part of this paragraph is devoted to Lie algebras, the construction of its universal enveloping algebra, and how it is naturally endowed with a Hopf algebraic structure by means of its universal property.

Definition 1.48. 1. A **Lie algebra** is a pair $(\mathfrak{g}, [,])$ where $\mathfrak{g}$ is a vector space, and $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is an antisymmetric bilinear map which satisfies the Jacobi identity for every $(a, b, c) \in V^3$:

$$[[a, b], c] + [[c, a], b] + [[b, c], a] = 0.$$

The map $[,]$ is called the **Lie bracket**.

2. Let $(\mathfrak{g}_1, [,]_1)$ and $(\mathfrak{g}_2, [,]_2)$ be two Lie algebras. A linear map $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ is called a **Lie algebra morphism** if $f([x, y]_1) = [f(x), f(y)]_2$, for every pair $(x, y) \in \mathfrak{g}^2$.

3. Let $(\mathfrak{g}, [,])$ be a Lie algebra and $V \subseteq \mathfrak{g}$ a linear subspace. Then V is a **Lie subalgebra** of $\mathfrak{g}$ if the Lie bracket stabilises V, namely $[,](V \times V) \subset V$. In that case, it follows directly that the inclusion map $\iota : V \to \mathfrak{g}$ is a Lie algebra morphism.

Since a Lie algebra is in particular a vector space, we may consider its tensor algebra. The universal enveloping algebra is a quotient of the tensor algebra as described in the following definition.

Definition 1.49. Let $(\mathfrak{g}, [,])$ be a Lie algebra. Consider the ideal $J(\mathfrak{g})$ of $\mathcal{T}(\mathfrak{g})$ generated by all terms of the form $a \otimes b - b \otimes a - [a, b]$ for $(a, b) \in \mathfrak{g}^2$. The **universal enveloping algebra** of $\mathfrak{g}$ is defined as

$$U(\mathfrak{g}) := \mathcal{T}(\mathfrak{g})/J(\mathfrak{g}). \tag{1.11}$$

The product $m_U : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g})$ is defined by the product of its representatives of each equivalent class (see Proposition 1.23). The canonical map $\iota_\mathfrak{g} : \mathfrak{g} \to U(\mathfrak{g})$ is defined as $\iota_\mathfrak{g} := \pi_U \circ \otimes$, where $\otimes : \mathfrak{g} \to \mathcal{T}(\mathfrak{g})$ is the canonical inclusion of $\mathfrak{g}$ into its tensor algebra, and $\pi_U : \mathcal{T}(\mathfrak{g}) \to U(\mathfrak{g})$ is the canonical quotient map.

The following classical and useful example of Lie algebras can be found for instance in [1, Theorem 2.1.3].

Example 1.50. *Let B be a bialgebra, the set of primitive elements of B define a Lie algebra where the Lie bracket is given by the commutator $[x, y] := xy - yx$. It is easy to check that the commutator is antisymmetric and satisfies the Jacobi identity. Moreover, for x and y primitive elements of B*

$$\begin{aligned}
\Delta[x, y] &= \Delta(xy - yx) \\
&= \Delta(x)\Delta(y) - \Delta(y)\Delta(x) \\
&= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) - (y \otimes 1 + 1 \otimes y)(x \otimes 1 + 1 \otimes x) \\
&= xy \otimes 1 + x \otimes y + y \otimes x + 1 \otimes xy - yx \otimes 1 - x \otimes y - y \otimes x - 1 \otimes yx \\
&= (xy - yx) \otimes 1 + 1 \otimes (xy - yx) \\
&= [x, y] \otimes 1 + 1 \otimes [x, y],
\end{aligned}$$

and thus $\mathrm{Prim}(B)$ *is closed under the commutator making* $(\mathrm{Prim}(B), [,])$ *a Lie algebra.*

Remark 1.51. Since the elements that generate the ideal $J(\mathfrak{g})$ are not homogeneous with respect to the natural grading of $\mathcal{T}(\mathfrak{g})$, J is not a homogeneous ideal. Indeed, a generator of $\mathfrak{g}$ of the form $a \otimes b - b \otimes$

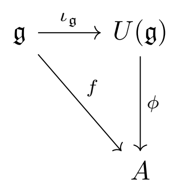
$a - [a,b]$ is composed by elements of degree 1 and 2. This implies that the quotient $U(\mathfrak{g}) = \mathcal{T}/J(\mathfrak{g})$ does not inherit a natural grading from that on the tensor algebra. Yet it does inherit a filtration

$$(U(\mathfrak{g}))^n = {}^{(\mathcal{T}(\mathfrak{g}))^n}/_{J(\mathfrak{g}) \cap (\mathcal{T}(\mathfrak{g}))^n},$$

where $(\mathcal{T}(\mathfrak{g}))^n = \bigoplus_{i=0}^{n} \mathfrak{g}^{\otimes i}$. It is easy to check that this indeed endows $U(\mathfrak{g})$ with a filtered algebra structure which is moreover connected as $\mathcal{T}(\mathfrak{g})$ is connected.

The reason behind the name of the universal enveloping algebra is that it satisfies the following universal property.

Theorem 1.52. Let $(\mathfrak{g}, [,])$ be a Lie algebra, A an algebra, and $f : \mathfrak{g} \to A$ a Lie algebra morphism where the Lie bracket on A is the commutator defined by the product. Then, there is a unique algebra morphism $\phi : U(\mathfrak{g}) \to A$ such that $f = \phi \circ \iota_{\mathfrak{g}}$ where $\iota_{\mathfrak{g}}$ is the canonical map from $\mathfrak{g}$ to $U(\mathfrak{g})$. This can be seen as the commutativity of the following diagram.



Proof. By means of the universal property of the tensor algebra (Theorem 1.25), since f is a linear map, there exists a unique algebra morphism $\psi : \mathcal{T}(\mathfrak{g}) \to A$ such that $f = \psi \circ \otimes$ where $\otimes$ is the canonical map from $\mathfrak{g}$ to $\mathcal{T}(\mathfrak{g})$. Since f is a Lie algebra morphism, for every a and b in $\mathfrak{g}$

$$\psi(a \otimes b - b \otimes a - [a,b]) = \psi(a \otimes b) - \psi(b \otimes a) - \psi([a,b]) = f(a)f(b) - f(b)f(a) - f([a,b]) = 0,$$

and thus $\psi(J(\mathfrak{g})) = \{0_A\}$. Lemma 1.43 yields the existence of a unique algebra morphism $\phi : U(\mathfrak{g}) \to A$ which satisfies $\psi = \phi \circ \pi_U$ where π_U is the canonical quotient map from $\mathcal{T}(\mathfrak{g}) \to U(\mathfrak{g})$. From $\iota_{\mathfrak{g}} = \pi_U \circ \otimes$, it follows that

$$f = \psi \circ \otimes = \phi \circ \pi_U \circ \otimes = \phi \circ \iota_{\mathfrak{g}}$$

as expected. The uniqueness of ϕ is granted from the fact that $\iota_{\mathfrak{g}}(\mathfrak{g})$ generates $U(\mathfrak{g})$ as an algebra, and thus ϕ is completely determined by its values in $\iota_{\mathfrak{g}}(\mathfrak{g})$. ☐

Analogous to the proof of Theorem 1.11, it can be shown that the pair $(U(\mathfrak{g}), \iota_{\mathfrak{g}})$ it the only pair (up to isomorphism of algebras) which satisfies the conditions of Theorem 1.52.

The universal enveloping algebra of a Lie algebra can be endowed naturally with the structure of a filtered, cocommutative, connected Hopf algebra. So far we have only described it as a filtered connected algebra (see Remark 1.51). In order to equip it with a coproduct, consider the Lie algebra morphism $\delta : \mathfrak{g} \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ defined by $\delta(x) := \iota_{\mathfrak{g}}(x) \otimes 1 + 1 \otimes \iota_{\mathfrak{g}}(x)$. By means of Theorem 1.52 there exists a unique algebra morphism $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ which extends δ, this is $\delta = \Delta \circ \iota_{\mathfrak{g}}$. Notice that by construction, the elements in $\iota_{\mathfrak{g}}(\mathfrak{g})$ are primitive and Δ is cocommutative. For the counit we consider the zero map from $\mathfrak{g}$ to $\mathbb{K}$. This is again a Lie algebra morphism and once again by means of Theorem 1.52, there is a unique algebra morphism $\epsilon : U(\mathfrak{g}) \to \mathbb{K}$ which vanishes identically on $\iota_{\mathfrak{g}}(\mathfrak{g})$. Therefore $U(\mathfrak{g})$ together with this coproduct and counit is a filtered connected bialgebra over $\mathbb{K}$, and thus, by means of Proposition 1.39 $U(\mathfrak{g})$ is a Hopf algebra. The antipode in this case could be alternatively built using once again the universal property (Theorem 1.52) as it will be done in Proposition 6.17 for the locality case.

1.5 Milnor-Moore theorem and Poincaré Birkhoff Witt theorem

Since the appearance in 1960 of the book *"Lie groups and Lie algebras"* of Bourbaki [11], the following theorem which provides some information about the structure of the universal enveloping algebra of a Lie algebra is called the Poincaré-Birkhoff-Witt theorem (from now on PBW theorem). It has been largely studied since then with several generalisations to other contexts of mathematics (see for instance [8], [12],

[56], [57], and [85]) . We present two equivalent formulations which will be of use in the sequel together with a simple, yet useful, corollary which is necessary for the proof of the Milnor Moore theorem. For a historical exposition of the PBW theorem we refer the reader to [37]. For the purpose of not extending unnecessarily this first chapter, we do not provide in this paragraph the proofs of the aforementioned theorems but rather refer the reader to one of the many classical references. Also some of the results here presented are particular cases of the locality counterpart which will be fully demonstrated in the sequel (Theorem 6.39).

Theorem 1.53 (Poincaré-Birkhoff-Witt theorem)**.** Let $\{g_i\}_{i\in\mathbb{Z}_{>0}}$ be a totally ordered basis of a Lie algebra $\mathfrak{g}$. For every $n \in \mathbb{Z}_{\geq 0}$, the set of ordered monomials of the form $\iota_{\mathfrak{g}}(g_{i_1})\cdots\iota_{\mathfrak{g}}(g_{i_k})$ where $k \leq n$ and $1 \leq i_1 \leq \cdots \leq i_k$ is a basis of $U^n(\mathfrak{g})$ (see Remark 1.51). In particular the set

$$B := \{g_{\iota_1}\cdots g_{\iota_k} : 1 \leq i_1 \leq \cdots \leq i_k \wedge k \in \mathbb{Z}_{\geq 0}\}$$

is a basis of $U(\mathfrak{g})$.

It follows from construction that the set of ordered monomials generates $U(\mathfrak{g})$, the difficulty lies in proving that it is a linearly independent set. A complete proof of this theorem can be found in [53]. Some of the consequences of Theorem 1.53 which will be of use are the following.

Corollary 1.54. Let $\mathfrak{g}$ be a Lie algebra, then the canonical map $\iota_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})$ is a linear injection.

Proof. The linearity is by construction. By means of Theorem 1.53, any basis of g is maps by $\iota_{\mathfrak{g}}$ to a linearly independent set which yields the result. □

On the basis of Corollary 1.54, we write g instead of $\iota_{\mathfrak{g}}(g)$ for any $g \in \mathfrak{g}$.

Corollary 1.55. Let $\mathfrak{g}$ be a Lie algebra and $U(\mathfrak{g})$ its universal enveloping algebra. The set of primitive elements of $U(\mathfrak{g})$ is exactly $\mathfrak{g}$.

Proof. The inclusion $\mathfrak{g} \subset \mathrm{Prim}(U(\mathfrak{g}))$ is trivial by the way the coproduct on $U(\mathfrak{g})$ is defined. For the other inclusion, consider an ordered basis $\{g_i\}_{i=1}^N$ of $\mathfrak{g}$ and let B be the corresponging basis of $U(\mathfrak{g})$ described in Theorem 1.53. Then y can be expressed in terms of the basis B as

$$y = \sum_{\vec{g}^{\vec{k}}\in B} \alpha_{\vec{g}^{\vec{k}}}\, \vec{g}^{\vec{k}}$$

where only finitely many $\alpha_{\vec{g}^{\vec{k}}}$ are non zero. Here we are using the compact notation $\vec{g}^{\vec{k}} := g_{i_1}^{k_1}\cdots g_{i_n}^{k_n}$ where $i_1 < \cdots < i_n$. Let $N := \max\{|\vec{k}| : \alpha_{\vec{g}^{\vec{k}}} \neq 0\}$. If $N = 1$ we have $y \in \mathfrak{g}$ as required. Let us now assume that $N > 1$. Since y is primitive, then $\tilde{\Delta}^{(N-1)}(y) = 0$. On the other hand it can be seen that

$$0 = \tilde{\Delta}^{(N-1)}(y) = \sum_{|\vec{k}|=N} \alpha_{\vec{g}^{\vec{k}}} \sum_{\sigma\in S_n} g_{\sigma(1)} \otimes \cdots \otimes g_{\sigma(N)},$$

with $g_{\sigma(1)} \otimes \cdots \otimes g_{\sigma(N)} \in \mathrm{Prim}(U(\mathfrak{g}))^{\otimes N}$.

Applying the product $N-1$ times yields

$$0 = m^{(N-1)}(\tilde{\Delta}^{(N-1)}(y)) = \sum_{|\vec{k}|=N} \alpha_{\vec{g}^{\vec{k}}} \sum_{\sigma\in S_n} g_{\sigma(1)}\cdots g_{\sigma(N)} \in U(\mathfrak{g}).$$

Since $\mathfrak{g} = (U(\mathfrak{g}))^1 \ni [g_i, g_j] = g_i g_j - g_j g_i$ for every i and j, we may reorder the g_i's to get the original elements of B at the cost of adding some lower order terms (l.o.t.) with respect to the natural filtration of $U(\mathfrak{g})$ (see Remark 1.51). The resulting products arising in the new linear combination are linearly independent of the leading term as a consequence of Theorem 1.53. Hence, we have

$$0 = \sum_{|\vec{k}|=N} \frac{\alpha_{\vec{g}^{\vec{k}}}}{N!}\vec{g}^{\vec{k}} + \text{l.o.t.}$$

Since the elements of the basis B are linearly independent, we may conclude that all $\alpha_{\vec{g}^{\vec{k}}} = 0$ except if $N = 1$. Therefore $\mathrm{Prim}(U(\mathfrak{g})) \subset \mathfrak{g}$. Thus $\mathrm{Prim}(U(\mathfrak{g})) = \mathfrak{g}$. □

The formulation of the Poincaré-Birkhoff-Witt theorem presented in Theorem 1.53 is not adequate when looking for an extension to the locality setup since, as mentioned before, vector space basis do not always behave well with locality as we will discuss it in Chapter 2. For that purpose, we introduce a slightly stronger formulation presented by Quillen [82, Appendix B] which not only provides a basis for $U(\mathfrak{g})$ from a basis of $\mathfrak{g}$, it moreover provides a coalgebra isomorphism. Another proof for this theorem can be found in [29].

Theorem 1.56 (Poincaré-Birkhoff-Witt theorem Quillen's version)**.** Let $\mathfrak{g}$ be a Lie algebra, the map

$$\begin{aligned}\Phi : S(\mathfrak{g}) &\longrightarrow U(\mathfrak{g})\\ g_1 \odot \cdots \odot g_n &\mapsto \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}(g_{\sigma(1)}\cdots g_{\sigma(n)})\end{aligned}$$

is an isomorphism of coalgebras.

We proceed to state one of the most known structural theorems of Hopf algebras, namely the Milnor-Moore theorem also known as Cartier-Quillen-Milnor-Moore theorem. This theorem first appeared in Cartier's seminar lectures [14] in 1956 and was later popularized by Milnor and Moore [73] in 1965. Quillen provided a simple proof of this theorem in [82, Appendix B]. Recall that the set of primitive elements of any connected bialgebra forms a Lie algebra (see Example 1.50).

Theorem 1.57 (Cartier-Quillen-Milnor-Moore theorem)**.** Let H be a cocommutative, graded, connected Hopf algebra over a field $\mathbb{K}$ of characteristic zero. Then H is isomorphic as a Hopf algebra to the universal enveloping algebra of its primitive elements, this is

$$H \sim U(\mathrm{Prim}(H)).$$

A self contained proof of the Milnoor-Moore Theorem can be found in [33]. That proof is adapted and generalised in Section 6.4 to the locality case, so fixing the trivial locality relation $\top = H\times H$, the original result is recovered.

We generalise the theorem of Milnor-Moore to the locality context in Theorem 6.22. The proof provided there is an adaption to the locality setup of that one in [33, Section 5.4]. When Theorem 6.22 is applied to the trivial locality relation $\top = H\times H$ Theorem 1.57 is recovered.

Cartier also proved a more general version of this theorem [16], [14] involving unipotent bialgebras, i.e., bialgebras in which for every b in the bialgebra, there is a $K := K(x) > 0$ such that $m^{(K)}\circ\tilde{\Delta}^{(K)}(x) = 0$, where $\tilde{\Delta}^{(K)}$ is the reduced coproduct iterated K times, and $m^{(K)}$ is the product iterated K times. A complete modern proof of this theorem can be found in [17, Section 4.3].

Theorem 1.58 (Cartier's theorem)**.** Let H be a cocommutative unipotent Hopf algebra over a field $\mathbb{K}$ of characteristic zero. Then its primitive elements $\mathrm{Prim}(H)$ form a Lie algebra, the universal enveloping algebra of which is isomorphic as Hopf algebra to H:

$$H \sim U(\mathrm{Prim}(H)).$$

Finally, the following theorem from Loday [63, Theorem 4.1.3] summarizes the relation between the Poincaré-Birkhoff-Witt theorem and the Milnor-Moore theorem.

Theorem 1.59 (Confront [63] Theorem 4.1.3)**.** For any graded, cocommutative Hopf algebra H over a filed $\mathbb{K}$ of characteristic zero, the following are equivalent.

1. H is connected.

2. There is an isomorphism of Hopf algebras $H \sim U(\mathrm{Prim}(H))$.

3. There is an isomorphism of connected coalgebras $H \sim S(\mathrm{Prim}(H))$.

The implication 1. $\Rightarrow$ 2. is the Milnor-Moore theorem (Theorem 1.57). Implication 2. $\Rightarrow$ 3. is the Quillen version of Poincaré-Birkhoff-Witt theorem (Theorem 1.56), and 3. $\Rightarrow$ 1. is straightforward.

2 Locality

The algebraic formulation of the locality principle in renormalisation was first introduced by Pierre Clavier, Lie Guo, Sylvie Paycha, and Bin Zhang in [22] in 2018. Since then, various authors have shown interest in such formulation and continued the development of such theory for instance [23],[24],[83], and [21]. In this paragraph we present the introductory concepts of such locality structures, mostly from [22], necessary for our further formulation of the locality versions of the Milnor-Moore and Poincaré-Birkhoff-Witt theorems in Chapter 2. We point out that some concepts presented here are original from this Ph.D. thesis as the ideas of pre-locality and (pre-)locality subsets (resp. subspaces, resp. subalgebras). Those concepts are not introduced in a latter chapter since we think their early introduction provides a better structure to the topic.

2.1 Locality sets and pre-locality spaces

We begin this section recalling some definitions from the locality context introduced first by P. Clavier, L. Guo, S. Paycha and B. Zhang in [22], like those of locality sets, locality maps, and locality semigroups. We also introduce the concept of pre-locality vector spaces which is first appeared during my Ph.D. and we first wrote it in [21] . Also the concepts we present of locality subset and pre-locality subspace are original of my Ph.D. since they are more general than the ones used in [22]. Despite being original ideas of my thesis, they are introduced in the chapter of prerequisites for the purpose of presenting a well structured introduction to the algebraic formulation of locality.

Definition 2.1.

- A **locality set** is a pair $(S,\top)$ where S is a set and $\top \subset S\times S$ is a symmetric relation on S called the **locality relation**. We sometimes denote $(x,y)\in\top$ as $x\top y$ and say that x and y are **locality independent**.
- Given a locality set $(S,\top)$, and a subset $U\subset S$, the **polar set** of U, denoted by $U^\top$, is defined as
$$U^\top := \{x\in S : (\forall u\in U)\, x\top u\}.$$
- A **locality map** is a map $f:(X,\top_X)\longrightarrow(Y,\top_Y)$ between locality sets which preserves locality in the following sense $(f\times f)(\top_X)\subset\top_Y$. In other words, f is a locality map if for every pair $(x_1,x_2)\in X^2$, $x_1\top_X x_2$ implies that $f(x_1)\top_Y f(x_2)$.
- Two locality sets $(X,\top_X)$ and $(Y,\top_Y)$ are said to be isomorphic as locality sets if there exists a locality bijection $f:(X,\top_X)\longrightarrow(Y,\top_Y)$ such that its inverse function $f^{-1}:(Y,\top_Y)\longrightarrow(X,\top_X)$ is also a locality map.
- Two maps $f,g:(X,\top_X)\longrightarrow(Y,\top_Y)$ between locality sets are called locality independent (or simply independent if there is no risk of ambiguity) if
$$(f\times g)(\top_X)\subset\top_Y.$$
When $X=Y$ and $\top_X=\top_Y=\top$, we sometimes denote f and g being locality independent as $f\top g$.
- Let $(S,\top)$ and $(Q,\top_Q)$ be two locality sets with $Q\subset S$. We say that $(Q,\top_Q)$ is a **locality subset** of $(S,\top)$ if the injection $\iota: Q\hookrightarrow S$ is a locality map.

A type of locality relations that will be particularly useful in the sequel is the **subset locality** relation [22, Lema 2.3] built from subsets of a locality set. More precisely, for a locality set $(S,\top)$ and a subset $U\subset S$, the subset locality relation on U, denoted by $\top\big|_U$, is the locality relation inherited from that on S, namely
$$\top\big|_U := \top\cap(U\times U). \tag{1.12}$$
Notice that $(U,\top\big|_U)$ is always a locality subset of $(S,\top)$ since the injection map $\iota:(U,\top\big|_U)\hookrightarrow(S,\top)$ is always a locality map.

Let us illustrate the previous definition with some examples.

Example 2.2. *1. For a set S, consider the following two locality relations on the power set of S.*

- $\forall (X,Y) \in \mathcal{P}(S), \quad X \top_D Y :\Leftrightarrow X \cap Y = \emptyset.$
- $\forall (X,Y) \in \mathcal{P}(S), \quad X \top_C Y :\Leftrightarrow X \cup Y = S.$

Then $(\mathcal{P}(S), \top_D)$ and $(\mathcal{P}(S), \top_C)$ are locality sets. The application $c : (\mathcal{P}(S), \top_D) \to (\mathcal{P}(S), \top_C)$ which maps a set X to its complement $X^c := S \setminus X$ is a locality map. Indeed, it follows from $(X \cap Y)^c = X^c \cup Y^c$ that $X \cap Y = \emptyset$ implies $X^c \cup Y^c = S$, and conversely c^{-1} is also a locality map, thus c is an isomorphism of locality sets. The polar sets of any $\mathcal{U} \subset \mathcal{P}(X)$ are

$$\mathcal{U}^{\top_D} = \mathcal{P}(\mathcal{U}^c) \subset \mathcal{P}(S), \quad \textit{and} \quad \mathcal{U}^{\top_C} = \{X \in \mathcal{P}(S) : \mathcal{U} \subset X\}.$$

Consider moreover a subset $\emptyset \neq A \subset S$ and define the function $\phi_A : (\mathcal{P}(A^c), \top_D\big|_{\mathcal{P}(A^c)}) \to (\mathcal{S}, \top_D)$ as $\phi_A(X) := X \cup A$. It is easy to see that it is not a locality map since in particular

$$\phi_A(\emptyset) \cap \phi_A(\emptyset) = A \cap A = A.$$

However it is locality independent to the canonical injection $\iota : (\mathcal{P}(A^c), \top_D\big|_{\mathcal{P}(A^c)}) \to (\mathcal{S}, \top_D)$.

2. Consider the set of maps $M(S,S)$ from a locality set $(S, \top)$ to itself. Then the locality independence of maps is a locality relation on $M(S,S)$. In particular, a locality map is a map which is locality independent to itself.

The following proposition states that the composition of locality maps remains locality

Proposition 2.3. *Let $(S, \top_S)$, $(P, \top_P)$, and $(U, \top_U)$ be locality sets, $f : S \to P$, and $g : P \to U$ locality maps. Then $g \circ f$ is a locality map.*

Proof. The statement follows from the following inclusions

$$\big((g \circ f) \times (g \circ f)\big)(\top_S) = (g \times g)(f \times f)(\top_S) \subset (g \times g)(\top_P) \subset \top_U.$$

□

Definition 2.4. [22, Section 3.1] Let $S_1, \ldots, S_n$ be subsets of a locality set $(S, \top)$. We define the locality Cartesian product of S_1 to S_n as

$$S_1 \times_\top \cdots \times_\top S_n := \{(x_1, \ldots, x_n) \in S_1 \times \cdots \times S_n : (\forall (i,j) \in [n]^2) \ x_i \top x_j\}.$$

In the case where $S_i = S$ for every $i \in [n]$, we write

$$S^{\times_\top^n} := \underbrace{S \times_\top \cdots \times_\top S}_{n-\text{times}}.$$

We also take the convention $S^{\times_\top^1} = S$.

Remark 2.5. In particular, for a locality set $(S, \top)$

$$S \times_\top S = \top.$$

We use the two notations indistinctly depending on whether we emphasize the Cartesian product or the locality relation.

We extend the locality setup to semigroups, monoids, and groups following [22].

Definition 2.6.
- A **locality semigroup** is a triple $(G, \top, m_\top)$ where $(G, \top)$ is a locality set, and $m_\top$ is a **partial product** defined only on $G \times_\top G$, i.e.

$$m_\top : G \times_\top G \to G \tag{1.13}$$

compatible with the locality relation in the following sense

$$\forall (U \subset G) \quad m_\top(U^\top \times_\top U^\top) \subset U^\top,$$

and satisfying the **locality associativity** condition, namely

$$\forall (x,y,z) \in G^{\times_\top^3} \qquad m_\top(m_\top(x,y),z) = m_\top(x, m_\top(y,z)). \tag{1.14}$$

Notice that Condition (1.13) ensures that both sides of (1.14) are well defined whenever $(x,y,z) \in G^{\times_\top^3}$.

- A **locality monoid** is a locality semigroup $(G, \top, m_\top)$ together with a unit element 1_G for the partial product which satisfies

$$1_G \in G^\top, \qquad \forall x \in G \quad m_\top(x, 1_G) = m_\top(1_G, x) = x.$$

- A **locality group** is a locality monoid $(G, \top, m_\top, 1_G)$ together with a locality map $\iota : G \to G$ called the inverse, which is locality independent of the identity map Id_G, and such that

$$\forall x \in G \qquad m_\top(\iota(x), x) = m_\top(x, \iota(x)) = 1_G.$$

Example 2.7. *An example of a locality monoid is given by $\mathbb{Z}_{>0}$, with the locality relation $\top_{\mathrm{cop}}$ defined by*

$$x \top_{\mathrm{cop}} y \Leftrightarrow x \text{ and } y \text{ are coprime},$$

and the multiplication on coprime elements is the usual multiplication in $\mathbb{R}$.

We proceed to define pre-locality vector spaces. This concept was defined in [21] and is less restrictive than the concept of locality vector spaces (Definition 2.21). The main reason why it is introduced is because in general the locality tensor product of locality vector spaces is at least a pre-locality vector space (see Remark 4.10). However it is not enough to define locality coalgebras, and therefore locality Hopf algebras as it is discussed in Section 6.1.

Definition 2.8.

- A **pre-locality $\mathbb{K}$-vector space** is a locality set $(V, \top)$ such that V has the structure of a $\mathbb{K}$-vector space, and $(0_V, 0_V) \in \top$.
- Let $(V, \top_V)$ and $(W, \top_W)$ be two pre-locality vector spaces. We call a linear map $f : V \to W$ a **locality linear morphism** if it is also a locality map.

  We sometimes write $f : (V, \top_V) \to (W, \top_W)$ instead of $f : V \to W$ to emphasize the locality relation in each space.
- Let $(W, \top_W)$ and $(V, \top_V)$ be two pre-locality vector spaces where $W \subset V$. We call $(W, \top_W)$ a **pre-locality subspace** of $(V, \top_V)$ if the injection $\iota : W \hookrightarrow V$ is a locality linear morphism.
- Let $(V, \top_V)$ and $(W, \top_W)$ be two pre-locality $\mathbb{K}$-vector spaces. We say that they are isomorphic as pre-locality vector spaces if there is a bijective locality linear morphism $f : V \to W$ such that f^{-1} is also a locality linear morphism.

Remark 2.9. The previous definitions of pre-locality vector space, ensures that for $(E, \top)$, the locality Cartesian product of two subspaces V and W of E is never empty since in particular $0 \top 0$.

Remark 2.10. Pre-locality is an hereditary property. We mean by this that for $(V, \top)$ a pre-locality vector space and $W \subset V$ a subspace, $(W, \top\big|_W)$ is a pre-locality subspace of $(V, \top)$, where $\top\big|_W$ is the subset locality (see (1.12)). Indeed, $0_W \top\big|_W 0_W$ and the injection $\iota : (W, \top\big|_W) \to (V, \top)$ is a locality linear morphism.

The following proposition is a generalization of a known result of linear algebra to the pre-locality setup.

Proposition 2.11. *Let $(V, \top_V)$ and $(W, \top_W)$ be two pre-locality subspaces, and $f : V \to W$ a locality linear morphism. Then the image of f is a pre-locality subspace of W.*

Proof. Since f is linear, $f(V)$ is a subspace of W. The result follows from Remark 2.10. □

2.2 Bilinearity and the locality tensor product of pre-locality vector spaces

We extend the definition of bilinearity of a map to the pre-locality setup. Namely for $(E,\top)$ and $(G,\top_G)$ pre-locality vector spaces and V and W subspaces of $(E,\top)$, we need a consistent definition of bilinearity for maps of the form $f: V\times_\top W\to G$. The usual definition of a map being bilinear if it is linear in each of its components, is not useful in the pre-locality case. Indeed, it makes no sense to require

$$f(v_1+v_2,w)=f(v_1,w)+f(v_2,w) \tag{1.15}$$

for every $(v_1,v_2)\in V^2$ and every $w\in W$ since not all terms in (1.15) are necessarily defined. A first attempt to solve this problem is to require f to satisfy (1.15) whenever all terms are defined in the locality relation. That is the definition used in [22, Paragraph 3.3]. We provide a new definition which is compatible with the locality tensor product proposed in [22] and which we use. This definition of bilinearity in the context of (pre-)locality is original of this Ph.D. thesis. However, it is presented in the chapter of prerequisites for the purpose of a well structured presentation. The reasons why we do not follow the definition used in [22] are discussed in Appendix A. The path we follow can be found in [21, Definition 1.7], and is inspired by Lemma 1.8.

Definition 2.12. Let V and W be subspaces of a pre-locality vector space $(E,\top)$ and G any vector space. We call $\top_\times$**-bilinear** a map $f: V\times_\top W\to G$ which satisfies the $\top_\times$-bilinearity condition:

$$\bar{f}(I_{\rm bil}^{\top_\times})=\{0_G\}, \tag{1.16}$$

where we have set $I_{\rm bil}^{\top_\times}:=\mathbb{K}(V\times_\top W)\cap I_{\rm bil}(V,W)$, $I_{\rm bil}(V,W)$ is defined in Equations (1.2) to (1.5), and the map $\bar{f}$ is given by Lemma 1.7.

The following result will be useful in the sequel.

Proposition 2.13. *Let $f: V\times_\top W\to G$ be a $\top_\times$-bilinear map where V and W are subspaces of a pre-locality vector space $(E,\top)$, and G is any vector space, and consider $(E',\top')$ a pre-locality subspace of $(E,\top)$. Then the restriction $f':=f\big|_{(V\times_\top W)\cap(E'\times_{\top'}E')}:(V\cap E')\times_{\top'}(W\cap E')\to G$ is also a $\top_\times$-bilinear map.*

Proof. Let us set $V':=V\cap E'$ and $W':=W\cap E'$ for simplicity. As a consequence of the inclusions $V'\subset V$, $W'\subset W$, and $\top'\subset\top$, it follows that $\bar{f'}=\bar{f}\big|_{\mathbb{K}(V'\times_{\top'}W')}$, and $I_{\rm bil}(V',W')\subset I_{\rm bil}(V,W)$. Therefore, for any $x\in I_{\rm bil}(V',W')\cap\mathbb{K}(V'\times_{\top'}W')\subset I_{\rm bil}(V,W)\cap\mathbb{K}(V\times_\top W)$ it follows that

$$\bar{f'}(x)=\bar{f}(x)=0,$$

and thus f' is $\top_\times$-bilinear. □

The locality tensor product was defined in [22, Section 4.1] for relative locality vector spaces. Following the same idea, it was defined in [21, Definition 1.8] for pre-locality vector spaces. We follow here the definition in [21] which is equivalent to the one in [22] for the case of locality vector spaces. Following Definition 1.10, the tensor product on the in the pre-locality setup is defined as follows.

Definition 2.14. Given V and W subspaces of a pre-locality vector space $(E,\top)$, the locality tensor product is the vector space

$$V\otimes_\top W:={}^{\mathbb{K}(V\times_\top W)}/_{I_{\rm bil}^{\top_\times}}, \tag{1.17}$$

where $I_{\rm bil}^{\top_\times}:=\mathbb{K}(V\times_\top W)\cap I_{\rm bil}(V,W)$ as in (1.16).

Remark 2.15. Since $V\times_\top W\subset V\times W$ and $I_{\rm bil}^{\top_\times}:=\mathbb{K}(V\times_\top W)\cap I_{\rm bil}$, we have an inclusion of vector spaces $V\otimes_\top W\subset V\otimes W$. If $V\times_\top W=V\times W$, then $V\otimes_\top W=V\otimes W$.

$V\otimes_\top W$ comes with a map

$$\otimes_\top:=\pi_\top\circ\iota_\top: V\times_\top W\to V\otimes_\top W \tag{1.18}$$

built from the canonical inclusion $\iota_\top: V\times_\top W\to\mathbb{K}(V\times_\top W)$ and the canonical quotient map $\pi_\top:\mathbb{K}(V\times_\top W)\to V\otimes_\top W$, which makes the following diagram commute:

$$\begin{array}{ccc} V \times_\top W & \xrightarrow{\iota_\top} & \mathbb{K}(V \times_\top W) \\ & \searrow^{\otimes_\top} & \downarrow \pi_\top \\ & & V \otimes_\top W \end{array}$$

Proposition 2.16. *Given V and W subspaces of a pre-locality vector space $(E, \top)$, the map*

$$\otimes_\top : V \times_\top W \to V \otimes_\top W$$

is a $\top_\times$-bilinear map.

Proof. Let $\overline{\otimes_\top}$ be the linear extension of (1.18) to a map $\mathbb{K}(V \times_\top W) \to V \otimes_\top W$. By construction, $\overline{\otimes_\top}(I_{\text{bil}}^{\top_\times}) = \pi_\top \left(I_{\text{bil}}^{\top_\times} \right)$ coincides with $\{0_{V \otimes_\top W}\}$. The map $\otimes_\top$ therefore satisfies (1.16) and defines a $\top_\times$-bilinear map . $\square$

We now extend Definition 2.14 for several subspaces of a pre-locality vector space. In the rest of this paragraph, $(E, \top)$ is a pre-locality vector space over $\mathbb{K}$, and $V_1, \ldots, V_n$ are linear subspaces of E. Recall that in the usual (non-locality) tensor product, $V_1 \otimes \cdots \otimes V_n$ is the quotient of $\mathbb{K}(V_1 \times \cdots \times V_n)$ and its subspace $I_{\text{mult}}(V_1, \ldots, V_n)$ generated by all elements of the form

$$(x_1, ..., x_{i-1}, a_i + b_i, x_{i+1}, ..., x_n) - (x_1, ..., x_{i-1}, a_i, x_{i+1}, ..., x_n) - (x_1, ..., x_{i-1}, b_i, x_{i+1}, ..., x_n) \tag{1.19}$$

$$(x_1, ..., kx_i, ..., x_n) - k(x_1, ..., x_i, ..., x_n) \tag{1.20}$$

for every $i \in [n]$, $k \in \mathbb{K}$ and $a_i, b_i, x_i \in V_i$. If $V_1 = \cdots = V_n = V$, we write $I_{\text{mult},n}(V)$.

Definition 2.17. [22, Section 4.1] We define the **locality tensor product**

$$V_1 \otimes_\top \cdots \otimes_\top V_n \; := \; \mathbb{K}(V_1 \times_\top \cdots \times_\top V_n) / (I_{\text{mult}}(V_1, \ldots, V_n) \cap \mathbb{K}(V_1 \times_\top \cdots \times_\top V_n)). \tag{1.21}$$

If $V_i = V$ for any $i \in [n]$, we set $V^{\otimes_\top^n} := V_1 \otimes_\top \cdots \otimes_\top V_n$.

Remark 2.18. For $n = 2$ we recover Definition 2.14.

The size of $V^{\otimes_\top^n} \subset V^{\otimes n}$ depends on the locality relation, namely on how many mutually independent elements it allows as the following example illustrates.

Example 2.19. *Consider the pre-locality vector space $(\mathbb{R}^n, \perp)$ where $\perp$ stands for the canonical orthogonality relation: $u \perp v :\Longleftrightarrow \langle u, v \rangle = 0$ (see [22, Subsetcion 2.2.1]). Then $V^{\times_\top m} = \{0\}$ for all $m > n$ since there are no $n+1$ pairwise orthogonal non zero elements in $\mathbb{R}^n$.*

On the other hand, if we consider the vector space $V := \mathbb{R}^\infty$ again with the canonical orthogonality relation $\perp$ as locality relation, one easily checks that there is no integer n in $\mathbb{N}$ such that $V^{\otimes_\perp n} = \{0\}$.

The following result is a characterization of the locality tensor product

Proposition 2.20. *Let $V_1, \ldots, V_n$ be subspaces of a pre-locality vector space $(E, \top)$, then the locality tensor product $V_1 \otimes_\top \cdots \otimes_\top V_n$ is the subspace of $V_1 \otimes \cdots \otimes V_n$ generated by terms of the form $v_1 \otimes \cdots \otimes v_n$ such that $(v_1, \ldots, v_n) \in V_1 \times_\top \cdots \times_\top V_n$.*

Proof. Since every element $x \in V_1 \otimes_\top \cdots \otimes_\top V_n$ is an equivalence class $x = [v]$ of some $v \in \mathbb{K}(V_1 \times_\top \cdots \times_\top V_n)$, by means of the canonical quotient map $\pi_\top$, $x = \pi_\top(v)$ is a linear combination of elements of the form $v_1 \otimes \cdots \otimes v_n$ such that $(v_1, \ldots, v_n) \in V_1 \times_\top \cdots \times_\top V_n$. Conversely, an element $v_1 \otimes \cdots \otimes v_n$ such that $(v_1, \ldots, v_n) \in V_1 \times_\top \cdots \times_\top V_n$ is clearly the image under $\pi_\top$ of $(v_1, \ldots, v_n)$ which completes the proof. $\square$

2.3 Locality vector spaces and (pre-) locality algebras

In this paragraph we present the notion of locality vector spaces first introduced by [22, Definition 3.8] and recalled in [21, Definition 1.1]. This notion requires the compatibility of the locality relation with the linear structure of the vector space, which is sometimes referred to as linear locality. We also present the definition of pre-locality algebra presented in [21]. Even though the definition of locality algebra was first introduced by [22], we delay its introduction to a later chapter since it requires some results presented there for a complete definition.

Definition 2.21. A **locality vector space** is a pre-locality vector space $(V, \top)$ such that the polar set $U^\top$ of any subset $U \subset V$ is a linear subspace of V. Equivalently, the following condition should be fulfilled

$$\forall(\lambda,\lambda') \in \mathbb{K}^2, \wedge \forall(u,u',v) \in V^3 \quad u\top v \,\text{and}\, u'\top v \Longrightarrow (\lambda u + \lambda' u')\top v. \tag{1.22}$$

We sometimes refer to the previous property as **linear locality.**

The linear locality condition can be motivated by the following example, and is necessary for the definition of locality coalgebras and for the proof of the locality Milnor-Moore theorem, more precisely in Definition 6.1 and Lemma 6.12.

Example 2.22. *Let $(V, \langle,\rangle)$ be a Hilbert space and $\top$ the locality relation given by orthogonality, i.e., $v\top w \Leftrightarrow \langle v,w\rangle = 0$. Then $(V,\top)$ is a locality vector space since the polar set $U^\top$ of any subset $U \subset V$ corresponds to the orthogonal of U which is always a subspace of V.*

A more pedestrian example is the one that follows. However, this type of examples should not be underestimated, since they are the source of many counter-examples, for instance, Counter-examples 4.51 and 5.24.

Example 2.23. *Let $V = \mathbb{R}^2$, $\{e_1, e_2\}$ the canonical basis of $\mathbb{R}^2$, and define*

$$\top := \mathbb{R}^2 \times \{0\} \cup \{0\} \times \mathbb{R}^2 \cup \langle e_1\rangle \times \langle e_2\rangle \cup \langle e_2\rangle \times \langle e_1\rangle \cup \langle e_1+e_2\rangle \times \langle e_1+e_2\rangle.$$

Then $(V,\top)$ is a locality vector space.

Since a locality vector space is in particular a pre-locality vector space, the definitions of locality linear morphism, locality subspace and isomorphism of locality vector spaces are the same as in Definition 2.8 but with V and W locality vector spaces.

Remark 2.24. Similar to pre-locality, locality is a hereditary property. Indeed, any linear subspace W of a locality vector space $(V,\top)$ endowed with the subset locality $\top\big|_W = \top \cap (W \times W)$ is a locality vector space. Indeed, for any $U \subset W$,

$$U^{\top|_W} = U^\top \cap W$$

is the intersection of two linear subspaces, and therefore a subspace itself.

Moreover, it is a locality subspace of $(V,\top)$ since the canonical injection $\iota : (W, \top\big|_W) \to (V,\top)$ is a locality linear morphism.

Free locality vector spaces

A special type of locality vector spaces are those generated by a locality set, namely the free locality vector spaces. We recall their construction following [21, Section 2.1].

A locality relation $\top$ on a set S further induces a locality relation (denoted with some abuse of notation by the same symbol $\top$) on the vector space $\mathbb{K}S$ generated by S given by the linear extension of the locality relation $\top$ on S. Explicitly, two elements a and b in $\mathbb{K}S$ are independent if the basis elements from S appearing in a are independent of the basis elements arising in b. More precisely, the linear span $(\mathbb{K}S,\top)$ of a locality set $(S,\top)$ is a pre-locality vector space when equipped with the symmetric binary relation

$$\left(a := \sum_{x\in X_a\subset S} \alpha_x\, x\right) \top \left(b := \sum_{y\in Y_b\subset S} \beta_y\, y\right) \Longleftrightarrow \forall(x,y) \in X_a \times Y_b : x\top y, \tag{1.23}$$

where the coefficients α_x and β_y are all different from zero, and X_a and Y_b finite subsets of S.

Lemma 2.25. The linear span $(\mathbb{K}\,X,\top)$ of a locality set $(X,\top)$ is a locality (and hence also a pre-locality) vector space.

Proof. By definition we have

$$\left(\sum_{i=1}^{n}\lambda_i\,u_i\right)\top\left(\sum_{j=1}^{n'}\lambda'_j\,u'_j\right)\Longleftrightarrow u_i\top u'_j\ \forall(i,j)\in\{1,\ldots,n\}\times\{1,\ldots,n'\}.$$

In order to check Condition (1.22), we take $v:=\sum_{k=1}^{m}\mu_k\,v_k$, $u=\sum_{i=1}^{n}\lambda_i\,u_i$ in $\mathbb{K}X$, and $u'=\sum_{j=1}^{n'}\lambda'_j\,u'_j$ with $u\top v$ and $u'\top v$. For any $(\lambda,\lambda')\in\mathbb{K}^2$, the element $\lambda u+\lambda' u'=\sum_{i=1}^{n}\lambda\,\lambda_i\,u_i+\sum_{j=1}^{n'}\lambda'\,\lambda'_j\,u'_j$ is locality independent of v. Indeed, it follows from the definition of the linearly extended relation $\top$, and from the u_i's and u'_j's being locality independent of v for all $(i,j)\in\{1,\ldots,n\}\times\{1,\ldots,n'\}$. ☐

Notice also that as a consequence of the last remark, Definitions 2.14 and 2.17 of the locality tensor product of two and of several subspaces of a pre-locality vector space extend naturally to locality vector spaces.

We must say at this point that there are two different definitions of the locality Cartesian product of locality vector spaces in [21, Definition 1.4] and in [22, Sect. 4.1]. Ours coincide with the former one when applying Definition 2.4 to subspaces V and W of a locality vector space $(E,\top)$. Indeed $V\times_\top W=(V\times W)\cap\top$. In the definition from [22], V and W are not required to be subspaces of a bigger locality vector space $(E,\top)$. Instead, it is introduced the concept of relative locality vector space. That is a triple $(V,W,V\times_\top W)$ where V and W are vector spaces and $V\times_\top W$ is a subset of $V\times W$, such that for every $X\subset V$ (resp. $Y\subset W$) its relative polar set

$$X^\top:=\{w\in W:(\forall x\in X)\,(x,w)\in V\times_\top W\}$$

$$(\text{resp. }\ {}^\top Y:=\{v\in V:(\forall y\in Y)\,(v,y)\in V\times_\top W\})$$

is a subspace of W (resp. of V).

Notice however that the two definitions are equivalent up to isomorphism of the vector spaces. Indeed, if V and W are subspaces of a locality vector space $(E,\top)$, then $(V,W,\top\cap(V\times W))$ is a relative locality vector space. On the other hand, given a relative locality vector space $(V,W,V\times_\top W)$, one can endow the vector space $V\oplus W$ with the locality relation

$$\top=\{((v,0),(0,w))\in(V\oplus W)^2|(v,w)\in V\times_\top W\}.$$

Then $(V\oplus W,\top)$ is a locality vector space, the canonical injections $\iota_V:V\to V\oplus W$ and $\iota_W:W\to V\oplus W$ are such that $V\sim\iota_V(V)$ and $W\sim\iota_W(W)$ are isomorphic as locality vector spaces, and $(v,w)\in V\times_\top W\Leftrightarrow(v,0)\top(0,w)$. We stick to Definition 2.4 which is equivalent to the one in [21] since it simplifies the construction of Cartesian products of several locality vector spaces and therefore the locality tensor product of higher degrees as it was done in Definition 2.17.

Pre-locality and locality algebras

We finish this section with the concept of pre-locality and locality algebras taken from [22, Definition 3.16] and [21, Definition 4.1].

Definition 2.26.

- A **non-unital pre-locality algebra** is a triple $(A,\top,m)$, where $(A,\top)$ is a pre-locality vector space, equipped with a partially defined product, namely a $\top_\times-$bilinear map $m:A\times_\top A\to A$, which is associative in the following sense

  $$m(m(x,y),z)=m(x,m(y,z))\quad\forall(x,y,z)\in A^{\times_\top 3},$$

  whenever $m(m(x,y),z)$ and $m(x,m(y,z))$ are defined.

- [22, Definition 3.16 (ii)] We call **non-unital locality algebra** a non-unital pre-locality algebra $(A, \top, m)$, whose underlying vector space is a locality vector space (so that in particular $U^\top$ is a vector space for any $U \subset A$), and such that $(A, \top, m)$ is a locality semigroup. This means that the partial product $m : A \times_\top A \to A$ is compatible with the locality relation in the following sense

$$m\left(U^\top \times_\top U^\top\right) \subset U^\top \quad \forall U \subset A. \tag{1.24}$$

- A subspace I of a non-unital pre-locality algebra $(A, \top, m)$ is called a left, resp. right **pre-locality ideal** of A, if

$$m\left(I \times I^\top\right) \subset I; \text{resp.}\, m\left(I^\top \times I\right) \subset I. \tag{1.25}$$

  If it is both a left and a right pre-locality ideal, we call it a pre-locality ideal.

  If $(A, \top, m)$ is moreover a non-unital locality algebra, then $I^\top$ is a linear subspace of A, and we call I a locality ideal.

- [22, Definition 3.16 (ii)] Given two non-unital (resp. pre-) locality algebras $(A_i, \top_i, m_i, u_i), i = 1, 2$, a locality linear morphism $f : A_1 \to A_2$ is called a **(resp. pre-) locality algebra morphism** if

$$f \circ m_1|_{\top_1} = m_2 \circ (f \times f)|_{\top_1}. \tag{1.26}$$

- We call $(A_1, \top_{A_1}, m_1, u_1)$ a non-unital **(pre-)locality subalgebra** of $(A_2, \top_{A_2}, m_2, u_2)$ if $A_1 \subset A_2$, and the inclusion map $\iota : A_1 \hookrightarrow A_2$ is also a (pre-)locality algebra morphism.

Similar as Definition 2.8, the concept of (pre)-locality subalgebra here is more general than the one given in [22] in that, the locality relation on the (pre-)locality subalgebra can be smaller than the one in the bigger (pre-)locality algebra. This degree of generality is needed for the locality version of the Milnor-Moore theorem. A case of particular importance is when $A_1 = A_2$ and $\top_2 \subsetneq \top_1$.

Example 2.27. *Let $(A, \top, m)$ be a non-unital locality algebra, the polar set $U^\top$ of any non-empty subset U of A gives rise to a non-unital locality subalgebra $(U^\top, \top\big|_{U^\top}, m)$ of $(A, \top, m)$. Here $\top\big|_{U^\top}$ stands for the subset locality (see* (1.12)*).*

Remark 2.28. Notice that for a non-unital locality algebra $(A, \top, m)$, Condition (1.24) is equivalent to the product $m : (A \times_\top A, \top_{A\times_\top A}) \to (A, \top)$ being a locality map, and thus a locality $\top_\times$-bilinear map.

Lemma 2.29. Let $f : A_1 \longrightarrow A_2$ be a locality linear morphism between two non-unital (resp. pre-) locality algebras $(A_i, m_i, \top_i), i \in \{1, 2\}$. Its kernel is a (resp. pre-) locality ideal of A_1 and its range is a (resp. pre-) locality subalgebra of A_2.

Proof. We prove that the kernel $\mathrm{Ker}(f)$ is a (resp. pre-)locality ideal. Take $a \in \mathrm{Ker}(f)$ and $b \in \mathrm{Ker}(f)^{\top_1}$, then $f(m_1(a, b)) = m_2(f(a), f(b)) = m_2(0, f(b)) = 0$, hence $m_1\left(\mathrm{Ker}(f) \times \mathrm{Ker}(f)^{\top_1}\right) \subset \mathrm{Ker}(f)$. Similarly we check that $m_1\left(\mathrm{Ker}(f)^{\top_1} \times \mathrm{Ker}(f)\right) \subset \mathrm{Ker}(f)$.

If A_1 is a locality algebra, then $\mathrm{Ker}(f)^\top$ is a linear subspace of A_1 and $\mathrm{Ker}(f)$ a locality ideal in A_1.

We prove that the range $\mathrm{Im}(f)$ is a (resp. pre-)locality algebra. Given $(f(a), f(b)) \in (\mathrm{Im}(f)\times\mathrm{Im}(f))\cap \top_2$, by (1.26) we have $m_2(f(a), f(b)) = f \circ m_1(a, b) \in \mathrm{Im}(f)$.

If A_2 is a locality algebra, then $\mathrm{Im}(f)^\top$ is a linear subspace of A_2. Moreover, setting $\top_{\mathrm{Im}(f)} := \top_2\cap(\mathrm{Im}(f)\times\mathrm{Im}(f))$, and given $U \subset \mathrm{Im}(f)$, we write $U^{\top_{\mathrm{Im}(f)}} \times_{\top_{\mathrm{Im}(f)}} U^{\top_{\mathrm{Im}(f)}} := \top_{\mathrm{Im}(f)}\cap(U^{\top_{\mathrm{Im}(f)}}\times U^{\top_{\mathrm{Im}(f)}})$ and thus

$$\begin{aligned} m(U^{\top_{\mathrm{Im}(f)}} \times_{\top_{\mathrm{Im}(f)}} U^{\top_{\mathrm{Im}(f)}}) &= m\left((U^{\top_2} \cap \mathrm{Im}(f)) \times_{\top_2} (U^{\top_2} \cap \mathrm{Im}(f))\right) \\ &= m\left((U^{\top_2} \times_{\top_2} U^{\top_2}) \cap \mathrm{Im}(f)^{\times_{\top_2}2}\right) \\ &\subset U^{\top_2} \cap \mathrm{Im}(f) = U^{\top_{\mathrm{Im}(f)}}. \end{aligned}$$

The last inclusion is a consequence of Condition (1.24) for A_2 and $\mathrm{Im}(f)$ being closed under the product m. Therefore Condition (1.24) is satisfied for $\mathrm{Im}(f)$, so that $\mathrm{Im}(f)$ a locality subalgebra of A_2.

□

3 Complex analytic, geometric and number theoretical prerequisites

We devote the following section to set the necessary background tools for our study of the polar structure of the Shintani zeta functions. We first recall some well known concepts and results from complex analysis such as meromorphic functions, analytic continuation, and Morera's theorem. We also introduce the Gamma function and the multivariable Mellin transform together with some of their properties. In the second paragraph we review some elements of number theory, more precisely the Riemann zeta function and some of its generalisations, such as the multiple zeta functions (or Euler-Riemann-Zagier zeta functions) also called poly zeta functions, and conical zeta functions. We present well known results about their convergence and meromorphic continuation. We also introduce the mathematical object of our study, namely the Shintani zeta functions. Finally, in this section's third paragraph, we review some geometric concepts regarding Newton polytopes and their relation with the multivariable Mellin transform following the results of Nilsson and Passare [77].

3.1 The Gamma function and the Mellin transform.

The objective of this paragraph is to recall **without proof** some well known tools of complex analysis in one and several variables like Cauchy's theorem (Theorem 3.2), Morera's theorem (Theorem 3.3), and Riemann's theorem of removable singularities (Theorem 3.5) with some of their consequences. For a complete introduction to such topics, we refer the reader to one of the many books in complex analysis, for instance, [93]. We further present the Mellin transform and Gamma functions together with some of their properties.

Definition 3.1. A function $f : \mathbb{C} \supset \mathcal{O} \to \mathbb{C}$ is said to be **holomorphic** at some interior point s of $\mathcal{O}$ if the limit

$$\lim_{h\to 0,\, h\in\mathbb{C}\setminus\{0\}} \frac{f(s+h)-f(h)}{h}$$

exists. We say a function is holomorphic in an open $O \subset \mathcal{O}$ if it is holomorphic at every point of O. Moreover a function is called **entire** if it is holomorphic on $\mathbb{C}$.

It is well known that holomorphic functions are analytic and analytic functions are holomorphic. For a complete introduction to the beautiful world of complex analysis we refer the reader to [93]. We make use of some very well known results about holomorphic functions which we now recall. Since a complete demonstration of them would require a lengthier introduction, we omit their proofs. However, they can be found in almost every book of complex analysis, for instance [93]. The following is one version of the famous Cauchy's Theorem.

Theorem 3.2 (Cauchy's theorem)**.** Let $f : \mathcal{O} \to \mathbb{C}$ be a holomorphic function and $\mathcal{O}$ a simply connected open set. Then

$$\int_\gamma f(s)ds = 0$$

for any closed curve $\gamma \subset \mathcal{O}$.

A converse of Cauchy's theorem (Theorem 3.2) is also true.

Theorem 3.3 (Morera's theorem [74])**.** Let $f : \mathcal{O} \to \mathbb{C}$ be a continuous function with $\mathcal{O} \subset \mathbb{C}$ open. If $\int_\gamma f(s)ds = 0$ for any closed path $\gamma \subset \mathcal{O}$, then f is holomorphic in $\mathcal{O}$.

The power of Morera's theorem lies in the fact that none of its hypotheses involves differentiability, yet its conclusion is that f is holomorphic (complex differentiable). The two previous theorems have a simple, yet useful Corollary which contrasts with the usual smooth dominated convergence theorem in that it requires less assumptions.

Corollary 3.4. Let $\mathcal{O}$ be a simply connected open subset of $\mathbb{C}$, and $f : \mathbb{R}_{\geq 0}\times\mathcal{O} \to \mathbb{C}$ a continuous function such that for every $\epsilon \in \mathbb{R}_{\geq 0}$ the function $s \mapsto f(\epsilon, s)$ is holomorphic on $\mathcal{O}$. If $s \mapsto F(s) := \int_0^\infty f(\epsilon, s)d\epsilon$ is convergent for every $s \in \mathcal{O}$, then F is holomorphic on $\mathcal{O}$.

Proof. By means of Morera's theorem (Theorem 3.3) it is enough to prove that $\int_\gamma F(s)ds = 0$ for every closed path $\gamma \in \mathcal{O}$. By means of Fubini's theorem

$$\int_\gamma F(s)ds = \int_\gamma \int_0^\infty f(\epsilon, s)d\epsilon ds = \int_0^\infty \int_\gamma f(\epsilon, s)dsd\epsilon = 0.$$

The last equality follows from Cauchy's theorem (Theorem 3.2) since the function $s \mapsto f(\epsilon, s)$ is holomorphic, and $\mathcal{O}$ is simply connected. □

We now state another famous result of Complex analysis which we will use in the sequel.

Theorem 3.5. [Riemann's theorem on removable singularities] Let $f : \mathcal{O} \setminus \{s_0\} \to \mathbb{C}$ be a holomorphic function bounded on $O_{s_0} \setminus \{s_0\}$, where O_{s_0} is an open set containing s_0. Then there is a holomorphic function $g : \mathcal{O} \to \mathbb{C}$ such that $g\big|_{\mathcal{O}\setminus\{s_0\}} = f$.

We now recall some notions of complex analysis in several variables. Our object of study are multivariable complex functions of the type $f : \mathbb{C}^n \supset \mathcal{O} \to \mathbb{C}$ where $n \in \mathbb{Z}_{\geq 0}$.

Definition 3.6.
1. A function $f : \mathbb{C}^n \supset \mathcal{O} \to \mathbb{C}$ is said to be **holomorphic** at an interior point $\boldsymbol{z} = (z_1, \ldots, z_n)$ of $\mathcal{O} \subset \mathbb{C}^n$ if it is holomorphic on each of its variables. This means that the map $s \mapsto f(z_1, \ldots, z_{j-1}, s, z_{j+1}, \ldots, z_n)$ is holomorphic at $s = z_j$ for all $j \in [n]$. We say that a function is holomorphic in an open $O \subset \mathcal{O}$ if it is holomorphic at every point of O. Moreover a function is called **entire** if it is holomorphic in $\mathbb{C}^n$.

2. Let $\mathcal{O} \subset \mathbb{C}^n$ be an open set and S a set with Lebesgue measure equal to 0. A **meromorphic** function on $\mathcal{O}$, is a function $f : \mathcal{O} \setminus S \to \mathbb{C}$ which locally is the quotient of two holomorphic functions. More precisely, for every $\boldsymbol{s} \in \mathcal{O} \setminus S$, there is an open neighborhood $O_{\boldsymbol{s}}$ of $\boldsymbol{s}$ where $f\big|_{O_{\boldsymbol{s}}\setminus S} = \frac{h_{\boldsymbol{s}}}{g_{\boldsymbol{s}}}$, where $h_{\boldsymbol{s}}, g_{\boldsymbol{s}} : O_{\boldsymbol{s}} \to \mathbb{C}$ are two holomorphic functions such that the zeros of $g_{\boldsymbol{s}}$ lie inside S. A meromorphic function $f = \frac{h}{g}$ is said to have linear poles if g is a product of linear polynomials in n complex variables. This is

$$g = \prod_{i=1}^m L_i,$$

where $L_i(\boldsymbol{s}) = \sum_{j=1}^n \alpha_{i,j} s_j + b_i$ for some coefficients $\alpha_{i,j}$ and b_i in $\mathbb{C}$.

3. The **polar locus** of a meromorphic function $f : \mathcal{O} \setminus S \to \mathbb{C}$ is the set of $\boldsymbol{z} \in S$ satisfying that for every open set O containing $\boldsymbol{z}$, and every pair of holomorphic functions $h, g : O \to \mathbb{C}$ such that $f\big|_{O\setminus S} = \frac{h}{g}$, then $g(\boldsymbol{z}) = 0$.

Notice that in the definition of holomorphic function at $\boldsymbol{s} \in \mathbb{C}^n$, we only require the function f to be "separately" holomorphic on each variable at s_i. It follows from Hartogs theorem [48] that f is actually continuous as a function from $\mathcal{O} \subset \mathbb{C}^n$ to $\mathbb{C}$ at $\boldsymbol{s}$, and then from Osgood's Lemma [78] that f is analytic in n complex variables as one would expect from a holomorphic function in several complex variables. On this ground, we use the terms analytic and holomorphic interchangeably similar to the one complex variable case. For a complete discussion of this topic we refer the reader to one of the following books [41, 60].

Definition 3.7. Let $f : \mathbb{C}^n \supset \mathcal{O} \to \mathbb{C}^n$ be a holomorphic function and $O \supset \mathcal{O}$ an open set. A function $g : O \to \mathbb{C}^n$ is called an **analytic continuation** of f to O if it is holomorphic and $g\big|_{\mathcal{O}} = f$.

Analytic continuations are unique also in several complex variables as we recall in the following proposition. The proof is totally analogous to that in one complex variable. However, several new phenomena appear in the multivariable approach. We do not deal which such phenomena in this document and therefore they are not introduced. We refer the reader to [41], [60] for information in this regard.

Proposition 3.8. *[59, Theorem 1.5.4] Let $\mathcal{O}$ be a connected open set, and $f, g : \mathcal{O} \to \mathbb{C}^n$ two holomorphic functions such that $f - g$ is identically zero in a non empty open set $O \subset \mathcal{O}$. Then $f = g$ in all $\mathcal{O}$.*

Proof. Let $h := f - g$ and define the set

$$E := \{\boldsymbol{s} \in \mathcal{O} : (\forall \boldsymbol{\alpha} \in \mathbb{Z}^n_{\geq 0})\, \partial^{(\boldsymbol{\alpha})} h(\boldsymbol{s}) = 0\}.$$

Notice that $O \subset E$. The proof consists of showing that both E and $\mathcal{O} \setminus E$ are open, and since $\mathcal{O}$ is connected and $E \neq \emptyset$, then $E = \mathcal{O}$. Finally since h is analytic, it is determined by its derivatives so h is identically zero in $\mathcal{O}$.

E is open: Let $\boldsymbol{a}$ in E, since h is analytic, there is a small open set O_a containing $\boldsymbol{a}$ where $h(s)$ is equal to the Taylor series $\sum \partial^{(\boldsymbol{\alpha})} h(\boldsymbol{a}) \frac{(\boldsymbol{s}-\boldsymbol{a})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}$. Hence $h = 0$ throughout O_a.

$\mathcal{O} \setminus E$ *is open:* Let $\boldsymbol{b}$ in $\mathcal{O} \setminus E$, then, there is an element $\boldsymbol{\beta}$ in $\mathbb{Z}^n_{\geq 0}$ such that $\partial^{(\boldsymbol{\beta})}(\boldsymbol{b})$ is non zero. By continuity of the derivatives, there is an open set O_b of $\boldsymbol{b}$ where $\partial^{(\boldsymbol{\beta})}(\boldsymbol{b})$ is never zero. Thus O_b is a subset of $\mathcal{O} \setminus E$. □

A function which will be used several times in the sequel is the Gamma function which we now briefly recall. Consider the integral

$$\Gamma(s) = \int_0^\infty \epsilon^{s-1} e^{-\epsilon} d\epsilon.$$

It is easy to see that it converges absolutely if $\Re(s) > 0$ and it is nowhere zero. Moreover, it follows from integration by parts that

$$\begin{aligned}\Gamma(s+1) &= \int_0^\infty \epsilon^{s} e^{-\epsilon} d\epsilon \\ &= \big[-\epsilon^s e^{-\epsilon}\big]_{\epsilon=0}^{\infty} + s\int_0^\infty \epsilon^{s-1} e^{-\epsilon} d\epsilon \\ &= s\Gamma(s),\end{aligned}$$

which yields the recursive formula

$$\Gamma(s) = \frac{\Gamma(s+n)}{s(s+1)\cdots(s+n-1)}$$

for $n \in \mathbb{Z}_{\geq 0}$. Using this recursive formula, we may define an analytic continuation $\overline{\Gamma}$ of Γ for any value of $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. Such analytic continuation is the famous **Gamma function**. Notice that $\overline{\Gamma}$ is also nowhere zero, and it can be written as the quotient of two holomorphic functions, thus it is meromorphic with linear poles and $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ is its polar locus. With some abuse of notation, and following the usual convention we denote the Gamma function $\overline{\Gamma}$ also by Γ.

We proceed to recall a transformation which will be our main tool for finding meromorphic continuations, namely the multivariable Mellin transform.

Definition 3.9. Let $g : \mathbb{R}^n_+ \to \mathbb{R}$ be a measurable function. The **Mellin tranform** of g is defined as

$$\mathbb{C}^n \subset \mathcal{O} \ni \boldsymbol{s} \mapsto \mathcal{M}_g(\boldsymbol{s}) := \int_{\mathbb{R}^n_+} \boldsymbol{\epsilon}^{\boldsymbol{s}-1} g(\boldsymbol{\epsilon}) d\boldsymbol{\epsilon},$$

where $\boldsymbol{\epsilon}^{\boldsymbol{s}-1} = \epsilon_1^{s_1-1}\cdots\epsilon_n^{s_n-1}$.

The maximal set $\mathcal{O} \subset \mathbb{C}^n$ for which the above integral is defined depends on the function g. The following lemma will be of use in the sequel, more precisely in Proposition 7.8 and Theorem 7.10.

Lemma 3.10. Let $g : \mathbb{R}^n_+ \to \mathbb{R}$ be a measurable function. If the integral defining $\mathcal{M}_g$ is convergent for every $\boldsymbol{s}$ in an open set $O \subset \mathbb{C}^n$, then $\mathcal{M}_g$ is holomorphic on O.

Proof. Let $\boldsymbol{s} = (s_1, \ldots, s_n) \in O$. Since $z \mapsto \epsilon_1^{z-1}\epsilon_2^{s_2-1}\cdots\epsilon_n^{s_n-1} g(z, s_2, \ldots, s_n)$ is holomorphic, then $z \mapsto \int_0^\infty \epsilon_1^{z-1}\epsilon_2^{s_2-1}\cdots\epsilon_n^{s_n-1} g(z, s_2, \ldots, s_n) dz$ is holomorphic at s_1 as a consequence of Corollary 3.4. The same argument used recursively in each coordinate implies that $\mathcal{M}_g$ is holomorphic in each coordinate and thus holomorphic as a multivariable complex function.

□

3.2 Zeta functions

The Riemann zeta function, also called Euler-Riemann zeta function is defined as $\zeta(s) := \sum_{n\geq 1} n^{-s}$ and is known to be absolutely convergent whenever $\Re(s) > 1$. It was first introduced and studied by Euler for real values of s. Later, Riemann studied the function for complex values of s and proved in his famous article "Über die Anzahl der Primzahlen unter einer gegebenen Grösse" [84] that ζ admits a meromorphic continuation to the whole complex plane with a simple pole at $s = 1$. In the same article, he proved the functional equation of the zeta function, the relation between its zeros and the distribution of prime numbers, and stated probably the most famous open problem in mathematics: the Riemann hypothesis. Very few functions have been so extensively studied in the history of mathematics, and thus, it is practically impossible to give a complete introduction to the Riemann zeta function. However, [52] is a very good starting point for the curious reader. Instead, we focus on some of the multivariable generalisations of ζ, the most famous probably being the **multiple zeta functions** (or Euler-Riemann-Zagier zeta function) [98],[100] also called polyzeta functions [15], which are defined as follows.

$$\mathbb{C}^n \ni (s_1, \dots, s_n) \mapsto \zeta(s_1, \dots, s_n) := \sum_{0<\mathfrak{m}_1<\mathfrak{m}_2<\cdots<\mathfrak{m}_n} \mathfrak{m}_1^{-s_1}\mathfrak{m}_2^{-s_2}\cdots\mathfrak{m}_n^{-s_n}. \tag{1.27}$$

The values of the multiple zeta functions at positive integers are known as multiple zeta values. Notice that with the change of variables $\mathfrak{m}_i = m_1 + \cdots + m_i$, the right hand side of (1.27) can be rewritten as

$$\mathbb{C}^n \ni (s_1, \dots, s_n) \mapsto \zeta(s_1, \dots, s_n) = \sum_{m_1\geq 1}\cdots\sum_{m_n\geq 1} m_1^{-s_1}(m_1+m_2)^{-s_2}\cdots(m_1+\cdots+m_n)^{-s_n}, \tag{1.28}$$

an expression that will be useful in the sequel. Multiple zeta values were first introduced by Euler in the eighteenth century for $n = 2$ [31], and in 1981 by Ecalle [30] for any $n > 2$. There was a revival of their study during the last decade of the 20th century with the works of Hoffman [49] and Zagier [98]. It is known that the sum on the right hand side of (1.28) converges absolutely whenever $\Re(s_n) > 1$ and $\sum_{i=1}^{n} \Re(s_i) > n$ (see for instance [100, Proposition 1].) We skip for the moment the proof of the last statement since it is a particular case of Corollary 8.6. It was proved by Zhao [100] and later more precisely by Akiyama, Egami, and Tanigawa [2] that multiple zeta functions admit a meromorphic continuation to the whole space $\mathbb{C}^n$ with simple poles on the hyperplanes $s_n = 1$, $s_{n-1} + s_n = 2, 0, -2, -4, \cdots$, and for $3 \leq k \leq n$

$$\sum_{i=1}^{k} s_{n-i+1} = k - l \quad \text{where}\, l \in \mathbb{Z}_{\geq 0}.$$

During the last three decades multiple zeta functions have been the object of study of several authors, linking them with other areas of mathematics such as arithmetic geometry, quantum groups, mathematical physics, renormalisation theory, etc. See for instance [10], [13], [15], [35], [46], [50], [51], [62], [70], [72], [67], [96].

A geometric generalisation of multiple zeta functions introduced by Guo, Paycha and Zhang [42] are the conical zeta functions. We recall from [34], and [101] some basic concepts about polyhedral cones.

Definition 3.11.

- An **open convex polyhedral cone** in $\mathbb{R}^n$ with vertex at zero (or simply cone) is an open subset C of $\mathbb{R}^n$ such that for every $\lambda \in \mathbb{R}_{>0}$

$$\boldsymbol{x} \in C \Longrightarrow \lambda\boldsymbol{x} \in C.$$

- A **closed convex polyhedral cone** in $\mathbb{R}^n$ with vertex at zero (or simply closed cone) is the closure of an open cone.
- A **generating set** $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ of a cone C is a set of vectors in $\mathbb{R}^n$ such that $C = \sum_{i=0}^{k} \mathbb{R}_{>0}\boldsymbol{v}_i$. The **dimension of the cone** C is the dimension of the subspace spanned by a generating set of it.
- If a cone has a generating set, the vectors of which lie in $\mathbb{Q}^n$, it is called a **rational cone**. Thus a rational cone is spanned by vectors in $\mathbb{Q}^n$ and equivalently by vectors in $\mathbb{Z}^n$.

- A **smooth cone** C is a rational cone together with a generating set which is a basis of $\mathbb{Z}^n$. In this case the generating set is unique and is called the **primary generating set** of C.
- A **face** of a cone C is a set of the form $\overline{C} \cap \{u = 0\}$ where $\overline{C}$ is the closure of C, and $u : \mathbb{R}^n \to \mathbb{R}$ is a linear function with coefficients in $\mathbb{R}$ which is non-negative on $\overline{C}$. Notice that a face of a cone is itself a cone.
- If C is a cone of dimension m, a **facet** of C is a face of dimension $m-1$.
- [42, Definition 2.1] A **subdivision** of a cone C is a set of cones $\{C_1, \ldots, C_r\}$ such that
  1. $\overline{C} = \bigcup_{i=1}^r \overline{C}_i$,
  2. $C_1, \ldots, C_r$ have the same dimension as C and
  3. intersect along the faces, i.e., $\overline{C}_i \cap \overline{C}_j$ is a face of both C_i and C_j.

In Chapter 3, a cone is always an open polyhedral convex cone with vertex at zero unless stated otherwise. Given a cone $C \subset \mathbb{R}^n_{\geq 0}$ and $\boldsymbol{s} \in \mathbb{C}^n$, Guo, Paycha and Zhang [42, Definition 2.4] defined the conical zeta function associated to C as

$$\zeta(C; \boldsymbol{s}) := \sum_{\mathfrak{m} \in C \cap \mathbb{Z}^n_{\geq 0}} \mathfrak{m}_1^{-s_1} \cdots \mathfrak{m}_n^{-s_n}, \tag{1.29}$$

whenever the sum is convergent. When $\boldsymbol{s}$ is a vector with integer components, $\zeta(C, \boldsymbol{s})$ is called a conical zeta value. The sum on the right hand side of (1.29) is convergent for $\boldsymbol{s} \in \mathbb{Z}^n$ whenever $s_i \geq 2$ [42, Lemma 2.5]. Consider a smooth cone C, and $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ its primary generating set with $\boldsymbol{v}_j = \sum_{i=1}^n v_{ij} e_i$. Since $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is a basis of $\mathbb{Z}^n$, using the change of coordinates $\mathfrak{m}_i = v_{i1} m_1 + \cdots + v_{in} m_n$ we can rewrite the right hand side of (1.29) as

$$\zeta(C; \boldsymbol{s}) = \sum_{m_1 \geq 1} \cdots \sum_{m_n \geq 1} (v_{11} m_1 + \cdots + v_{1n} m_n)^{-s_1} \cdots (v_{n1} m_1 + \cdots + v_{nn} m_n)^{-s_n}. \tag{1.30}$$

Moreover, any cone of maximal dimension can be subdivided in smooth cones [42][Proposition 2.2] and given subdivision $\{C_1, \ldots, C_r\}$ of a cone C

$$\zeta(C; \boldsymbol{s}) = \sum_{i=1}^r \zeta(C_i, \boldsymbol{s}),$$

whenever both sides of the equation are well defined [42, Lemma 2.7]. This implies that every conical zeta function can be seen as a linear combination of functions with the form on the right hand side of (1.30). It was shown in [43] that conical zeta functions admit a meromorphic continuation to the whole space $\mathbb{C}^n$ with linear poles. Renormalised values of conical zeta functions were further studied in [44] and [22].

Several other multivariable generalisations of the Riemann zeta functions have been made, for instance Mordell-Tornheim zeta functions [69],[68], branched or arborified zeta functions [23], [25], [26] [66], Schur multiple zeta functions [71], [76], multiple Hurwitz-Lerch zeta functions [58], [55], etc. Meromorphic continuations of such generalisations have called the attention of numerous mathematicians [2], [58], [69], [68], [100].

In Chapter 3, we focus on a particular generalisation which is parametrised by matrices, namely the Shintani zeta functions. These functions where introduced by Shintani in a series of papers in the 1970s [86, 87, 88, 89, 90, 91] motivated by some problems of number theory. He considered the one complex variable function

$$s \mapsto \sum_{m_1 \geq 0} \cdots \sum_{m_r \geq 0} (a_{11} m_1 + \cdots + a_{1r} m_r + b_1)^{-s} \times \cdots \times (a_{n1} m_1 + \cdots + a_{nr} m_r + b_n)^{-s} \tag{1.31}$$

where $a_{ij} > 0$ and $b_i > 0$ for all $1 \leq i \leq n$ and all $1 \leq j \leq r$. Shintani proved, among several other things, that the sum on the right hand side of (1.31) converges whenever $\Re(s) > \frac{r}{n}$, and that this function admits a meromorphic continuation to the whole complex plane $\mathbb{C}$ with the same linear poles as the function $\Gamma(rs-n)/\Gamma(s)$. In particular, the poles do not depend on the values of the b_i's and of the a_{ij}'s as long as they are positive. His work inspired some authors (see for instance [3], [18]) to study a multivariable (or multidimensional) version of (1.31), namely

$$\boldsymbol{s} \mapsto \sum_{m_1 \geq 0} \cdots \sum_{m_r \geq 0} (a_{11}m_1 + \cdots + a_{1r}m_r + b_1)^{-s_1} \times \cdots \times (a_{n1}m_1 + \cdots + a_{nr}m_r + b_n)^{-s_n}. \tag{1.32}$$

where once again all b_i's and a_{ij}'s are positive. We consider in Chapter 3 a slight generalisation of (1.32), which was also considered in [42] and [69], and that we introduce now.

Definition 3.12.

- Let $\Sigma_{n\times r}(\mathbb{R}_{\geq 0})$ be the set of $n \times r$ matrices with real non-negative arguments, and with at least one positive argument in each row and in each column.
- Given a matrix $A = \{a_{ij}\}_{1\leq i\leq n, 1\leq j\leq r} \in \Sigma_{n\times r}(\mathbb{R}_{\geq 0})$ the **Shintani zeta function** associated to A is given by

$$\zeta_A(\boldsymbol{s}) := \sum_{m_1 \geq 1} \cdots \sum_{m_r \geq 1} (a_{11}m_1 + \cdots + a_{1r}m_r)^{-s_1} \times \cdots \times (a_{n1}m_1 + \cdots + a_{nr}m_r)^{-s_n}. \tag{1.33}$$

Our objective is to give a precise description of the polar loci of the Shintani zeta functions. We point out in Remark 8.10 that the polar structure of ζ_A remains the same if we start the sums at zero and instead add a $b_i > 0$ inside of each parenthesis. Thus, for the sake of simplicity in the notations, we omit the terms b_i and start the sums at 1. The condition that there is at least one non zero element in each column of the matrix guarantees that every m_j has at least one non-zero coefficient, otherwise the sum on the right hand side of (1.33) would never converge. On the other hand, if there is a row full of zeros, the term 0^{-s_i} would cause problems for $\Re(s_i) > 0$. The sum on the right hand side of (1.33) is absolutely convergent whenever $\Re(s_i) > r$ for every $1 \leq i \leq n$ as we recall in Corollary 8.6.

Notice that Shintani zeta functions as we are considering them generalise some of the zeta functions we previously described. Indeed, the Riemann zeta function corresponds to the case when A is a 1×1 matrix, multiple zeta functions correspond to the case when the matrix A is a square lower triangular matrix with ones on and under the diagonal as it can be seen in (1.28), also the conical zeta function of a smooth (open) cone is in particular a Shintani zeta function when written in the form of (1.30). It was moreover shown in [42, Proposition 5.16] that Shintani zeta values, i.e. when $\boldsymbol{s} \in \mathbb{Z}^n_{>r}$, span the space of conical zeta values. Shintani zeta functions also generalize Mordell-Tornheim zeta functions and arborified of tree-like zeta functions (see [26]).

Matsumoto proved in [69, Theorem 3] that (1.33) admits a meromorphic continuation to $\mathbb{C}^n$ with possible linear poles located on the hyperplanes

$$c_1 s_1 + \cdots + c_n s_n = u(c_1, \ldots, c_n) - l, \tag{1.34}$$

where the c_i lie in $\mathbb{Z}_{\geq 0}$, $u(c_1, \ldots, c_n)$ in $\mathbb{Z}$, and l in $\mathbb{Z}_{\geq 0}$. His proof uses the Mellin-Barnes integration formula which involves a contour integral of the product of Gamma functions (See for instance [68]). The main objective of this chapter is to refine Matsumoto's result in specifying the coefficients c_i in the equations of the hyperplanes describing the poles (Theorem 8.7). For this purpose we make use of another ingredient which we introoduce in the next paragraph.

3.3 Newton polytopes and the Mellin transform of rational functions

Newton polytopes provide a geometric tool to study the polar structure of Shintani zeta functions. In this chapter we briefly recall their definition together with Theorems 3.20 and 3.21 of Nilsson and Passare regarding the Mellin transform of rational functions using Newton polytopes. We do not provide their proofs since they are very similar in spirit to those of Theorems 7.10 and 7.11. This paragraph is based on [77].

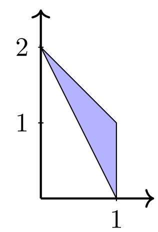
Throughout the rest of this chapter, we consider complex Laurent polynomials in n variables $p : \mathbb{C}^n \to \mathbb{C}$ of the form

$$p(\boldsymbol{\epsilon}) = \sum_{\alpha \in \mathcal{A}} a_\alpha \boldsymbol{\epsilon}^\alpha, \tag{1.35}$$

where $\mathcal{A}$ is a finite subset of $\mathbb{Z}^n$, and we have set $\boldsymbol{\epsilon}^\alpha := \boldsymbol{\epsilon}_1^{\alpha_1} \cdots \boldsymbol{\epsilon}_n^{\alpha_n}$. One of the main tools we use to study Shintani zeta functions are Newton polytopes, the definition of which we recall now.

Definition 3.13. Consider a Laurent polynomial $p : \mathbb{C}^n \to \mathbb{C}$, the **Newton polytope** of p, denoted by Δ_p, is the convex hull generated by $\mathcal{A}$. Notice that Δ_p is a compact subset of $\mathbb{R}^n$ since $\mathcal{A}$ is finite.

Example 3.14. *For $p(\boldsymbol{\epsilon}) = \epsilon_1 + \epsilon_2^2 + \epsilon_1\epsilon_2$. $\mathcal{A}$ is given by $\{(1,0),(0,2),(1,1)\}$ and Δ_p is pictured below in blue color.*



We recall from [5, Definition 2.1] that a polyhedron in $\mathbb{R}^n$ is a set $P \subset \mathbb{R}^n$ defined by finitely many linear inequalitites of the form

$$P := \{x \in \mathbb{R}^n : \ell_i(x) \geq \alpha_i,\ i \in I\}.$$

Here I is finite, $\alpha_i \in \mathbb{R}$ and $\ell_i : \mathbb{R}^n \to \mathbb{R}$ are linear functions. It is sometimes useful to see a polytope as a polyhedron. This is always possible as a consequence of the Weyl-Minkowski theorem which we now recall. We omit the proof, but refer the reader to [5, Theorems 4.4 & 4.7] for a complete proof.

Theorem 3.15 (Weyl-Minkowski theorem)**.** Let P be a subset of $\mathbb{R}^n$. Then P is a polytope if, and only if, P is a bounded polyhedron.

In particular, Theorem 3.15 implies that any convex polytope in $\mathbb{R}^n$, the vertices of which lie in $\mathbb{Z}^n$, as well as being the convex hull of its vertices, can also be described as the intersection of half spaces determined by its facets:

$$\bigcap_{k=1}^{N} \{\boldsymbol{\sigma} \in \mathbb{R}^n ; \langle \boldsymbol{\mu}_k, \boldsymbol{\sigma} \rangle \geq \nu_k\}. \tag{1.36}$$

Here N is the number of facets of the polytope, the ν_k's are integers, and the $\boldsymbol{\mu}_k$'s are vectors in the lattice $\mathbb{Z}^n$ on the inward normal direction of the facets with mutually coprime coordinates. Since the vertices of the polytope are in the lattice $\mathbb{Z}^n$, the choice of the vectors $\boldsymbol{\mu}_k$ and $\boldsymbol{\nu}$ in (1.36) is unique.

Continuation of Example 3.14: *For $p(\boldsymbol{\epsilon}) = \epsilon_1 + \epsilon_2^2 + \epsilon_1\epsilon_2$, Δ_p can be described as the intersection of half spaces as follows.*

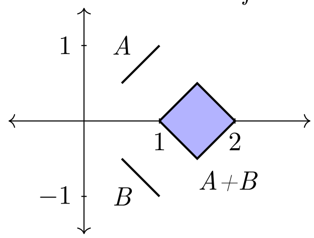
$$\Delta_p = \{\boldsymbol{\sigma} \in \mathbb{R}^2 : \langle \boldsymbol{\sigma}, (2,1) \rangle \geq 2\} \cap \{\boldsymbol{\sigma} \in \mathbb{R}^2 : \langle \boldsymbol{\sigma}, (-1,0) \rangle \geq -1\} \cap \{\boldsymbol{\sigma} \in \mathbb{R}^2 : \langle \boldsymbol{\sigma}, (-1,-1) \rangle \geq -2\}. \tag{1.37}$$

We now recall the definition of the Minkowski sum of two subsets of a vector space.

Definition 3.16. [5, Definition 3.4] Let V be a vector space and $A, B \subset V$ not empty. The Minkowski sum $A + B$ is defined as

$$A + B := \{a + b \in V : a \in A \wedge b \in B\}.$$

Example 3.17. *For $V = \mathbb{R}^2$, the Minkowski sum of the line segments A and B in the following figure is*



the blue rhombus on the right.

The Minkowski sum inherits the associativity and commutativity of those in the sum of the vector space. A type of sets which will be used in the sequel is of the form $\Delta + \mathbb{R}^n_+$ where Δ is a polytope. Notice that $\Delta + \mathbb{R}^n_+$ is not a polytope since it is not bounded, however it is a polyhedron as a consequence of the following theorem.

Theorem 3.18. [5, Theorem 3.5 (1)] For P_1 and P_2 two non empty polyhedra in $\mathbb{R}^n$, their Minkowski sum $P_1 + P_2 \subset \mathbb{R}^n$ is also a polyhedron.

We introduce Newton polytopes in order to adapt some work of Nilsson and Passare [77] regarding the convergence and meromorphic continuation of the Mellin transform of rational functions. We proceed to recall their results. The following definition is taken from [77], and will be of use in the sequel.

Definition 3.19.

- Let p be a complex Laurent polynomial on n variables and Γ a face of its Newton polytope Δ_p. The **truncated polynomial** p_Γ associated to the face Γ is the sum of the monomials of p whose exponents lie in Γ.
- We say that a polynomial is **completely non-vanishing** on a subset $\mathcal{O} \subset \mathbb{C}^n$ if neither it, nor its truncated polynomials, vanish on $\mathcal{O}$.

Continuation of Example 3.14: *Consider $p(\boldsymbol{\epsilon}) = \epsilon_1 + \epsilon_2^2 + \epsilon_1\epsilon_2$, and Γ the facet of Δ_p lying on the hyperplane $\{\boldsymbol{\sigma} \in \mathbb{R}^2 : \langle \boldsymbol{\sigma}, (2,1)\rangle = 2\}$ (see (1.37)), then the truncated polynomial p_Γ is*

$$p_\Gamma(\boldsymbol{\epsilon}) = \epsilon_1 + \epsilon_2^2.$$

The following result by Nilsson and Passare relates the domain of convergence of the Mellin transform of a function of the type $1/p$ where p is a Laurent polynomial, with the Newton polytope of p. Since every rational function can be expressed as a linear combination of such functions, this theorem also determines the domain of convergence of the Mellin transform of any rational function. The original proof is very similar to that of Theorem 7.10 and thus, we do not write it here.

Theorem 3.20. [77, Theorem 1] If a polynomial p is completely non-vanishing on $\mathbb{R}^n_+$, then the integral

$$\mathcal{M}_{1/p}(\boldsymbol{s}) = \int_{\mathbb{R}^n_+} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}-1}}{p(\boldsymbol{\epsilon})} d\boldsymbol{\epsilon}$$

converges absolutely and defines an analytic function $\boldsymbol{s} \mapsto \mathcal{M}_{1/p}(\boldsymbol{s})$ on the tube domain

$$D_{1/p} := \{\boldsymbol{s} \in \mathbb{C}^n : \Re(\boldsymbol{s}) = \boldsymbol{\sigma} \in \mathrm{int}(\Delta_p)\}.$$

Furthermore, Nilsson and Passare proved that the Mellin transform of a function of the type $1/p$ admits a meromorphic continuation to the whole space $\mathbb{C}^n$ with linear poles as stated in the following theorem. Once again, the proof is very similar to that of Theorem 7.11 and therefore we omit it here.

Theorem 3.21. [77, Theorem 2] Let p be a completely non-vanishing polynomial on the positive orthant $\mathbb{R}^n_+$ such that its Newton polytope Δ_p is of full dimension. Then the Mellin transform of $1/p$

$$\mathcal{M}_{1/p}(\boldsymbol{s}) := \int_{\mathbb{R}^n_+} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}-1}}{p(\boldsymbol{\epsilon})} d\boldsymbol{\epsilon}$$

admits a meromorphic continuation of the form

$$\mathcal{M}_{1/p}(\boldsymbol{s}) = \Phi(\boldsymbol{s}) \prod_{k=i}^{N} \Gamma(\langle \boldsymbol{\mu}_k, \boldsymbol{s} \rangle - \nu_k),$$

where Φ is an entire function, and N, $\boldsymbol{\mu}_k$ and ν_k are as in (1.36).

Chapter 2

Locality tensor products and locality Milnor-Moore and Poincaré-Birkhoff-Witt theorems

In this chapter, we present the principal results on the context of locality, namely the universal properties of the locality tensor product (Theorem 5.37), the locality tensor algebra (Theorem 5.38), the locality universal enveloping algebra (Theorem 5.38), and the locality symmetric algebra (Theorem 5.45), together with the locality versions of the Milnor-Moore and Poincaré-Birkhoff-Witt theorems (Theorems 6.22 and 6.39). For that purpose, we study in Section 4 the construction and universal properties on the context of pre-locality of the locality tensor product, locality tensor algebra, locality symmetric algebra, and locality universal enveloping algebra. The main difference between the locality tensor product of subspaces of a pre-locality vector space presented here and the one in Section 2 is that the one presented here is naturally endowed with a locality relation which makes it a pre-locality vector space. Since all of the constructions presented in Section 4 are quotients of pre-locality vector spaces or of pre-locality algebras, a question that naturally arises is when the quotient of locality vector spaces is again a locality vector space and not only a pre-locality vector space (Question 2.15). We devote Section 5 to study such question and provide sufficient conditions to have a positive answer. Also, the universal properties introduced in Section 4 in the context of pre-locality are upgraded to the context of locality provided some sufficient conditions. In Section 7.2 we introduce locality coalgebras, locality bialgebras, and locality Hopf algebras together with some technical lemmas which we then use to state and prove the locality versions of the Milnor-Moore and Poincaré-Birkhoff-Witt theorems. Most of this chapter is based on [21], and some parts are written exactly the same with the permission of the other authors of the paper. We stress that the locality Poincaré-Birkhoff-Witt theorem, together with all its necessary previous constructions and lemmas, are original from this thesis. Therefore the contents of Paragraphs 4.4 and 6.6 and the parts of Paragraphs 5.5 and 6.3 regarding the locality symmetric algebra are original of this document.

4 Locality relations and universal properties in the context of pre-locality

In this section we present the complete construction of the locality tensor product, locality algebra, locality symmetric algebra and locality universal enveloping algebra endowed with natural locality relations which turn them into pre-locality vector spaces (resp. pre-locality algebras). The main results of this section are the universal properties of such objects (Theorems 4.14, 4.33, 4.41, and 4.48).

4.1 Locality relations

Similar to topological spaces, a set, a vector space, or an algebra can be endowed with different locality relations which will have an impact on its properties in the context of (pre-)locality. In this paragraph, we study locality relations induced by maps and the possible inclusions among them. Such locality relations are essential to endow the locality tensor product and locality algebra with a locality relation in order to have a full fledged (pre-)locality theory, as it will be discussed in 4.2.

Recall that given two topologies τ_1, τ_2 on some set X, τ_1 is said to be **coarser (weaker or smaller)** than τ_2, or equivalently τ_2 **finer (stronger or larger)** than τ_1 if, and only if $\tau_1 \subset \tau_2$. Also, given a set X and $(X_i, \tau_i)_{i\in I}$ a family of topological spaces together with a family of maps $f_i : X_i \to X$, the **final topology (or strong, colimit, coinduced, or inductive topology)** $\overline{\tau}$ is the finest topology on X such that all maps f_i are continuous. With a small abuse of language, one says that the topology $\overline{\tau}$ is final with respect to the maps f_i.

A typical example is the quotient topology on X/I where I is a subset of a set $(X, \top)$, defined as the final topology for the projection map $\pi : X \to X/I$.

We now transpose this terminology to the locality setup.

Definition 4.1. Let $\top_1$ and $\top_2$ be two locality relations over a set A. We say $\top_1$ is **coarser** than $\top_2$ or equivalently, that $\top_2$ is **finer** than $\top_1$ if, and only if $\top_1 \subset \top_2$.

The following example provides a justification of the terminology in our transposition from a topological to a locality context.

Example 4.2. *Let X be a set and $\mathcal{P}(X)$ its powerset. Disjointness of sets:*

$$A \top B \Longleftrightarrow A \cap B = \emptyset$$

defines a locality relation on any subset $\mathcal{O}$ of $\mathcal{P}(X)$. If $(X, \mathcal{O})$ is a topological space with topology $\mathcal{O} \subset \mathcal{P}(X)$, this disjointness relation gives rise to another locality relation (which with some abuse of notation, we denote by the same notation) given by the separation of points:

$$x \top y \Longleftrightarrow \exists U, V \in \mathcal{O}, \quad (U \top V) \wedge (x \in U \wedge y \in V).$$

The finer (resp. coarser) the topology $\mathcal{O}$, the larger (resp. smaller) the graph $\{(x, y),\, x \top y\}$ of the locality relation, hence the terminology we have chosen.

Definition 4.3. Let X be a set, $(X_i, \top_i)_{i\in I}$ a family of locality sets, and $f_i : X_i \to X$ a family of maps. The **final locality relation** $\overline{\top}$ on X is the coarsest locality relation among the locality relations $\top$ on X for which

$$f_i : (X_i, \top_i) \longrightarrow (X, \top), \quad i \in I$$

are locality maps.

As before, with a slight abuse of language, we shall say that $\overline{\top}$ is a **final locality relation** on X for the maps f_i.

Proposition 4.4. *Given a locality set $(A; \top)$ and a surjective map $\phi : A \to B$, the locality relation $\top$ on A induces a locality relation $\overline{\top}$ on B defined by*

$$b_1 \overline{\top} b_2 \Longleftrightarrow (\exists (a_1, a_2) \in A \times A : \phi(a_i) = b_i \text{ and } a_1 \top a_2),$$

which is the final locality relation for the map ϕ.

Proof. It is clear from the definition of $\overline{\top}$, that $\phi : (A, \top) \longrightarrow (B, \overline{\top})$ is a locality map.

Let $\top_B$ be a locality relation on B such that $\phi : (A, \top) \longrightarrow (B, \top_B)$ is a locality map. For any $(b_1, b_2) \in B^2$ we have

$$\begin{aligned} b_1 \overline{\top} b_2 &\Longrightarrow \big(\exists (a_1, a_2) \in A^2 | \phi(a_i) = b_i \ \wedge \ a_1 \top a_2\big) \quad \text{for } i \in \{1, 2\} \\ &\Longrightarrow \big(\exists (a_1, a_2) \in A^2 | \phi(a_i) = b_i \ \wedge \ \phi(a_1) \top_B \phi(a_2)\big) \quad \text{since } \phi \text{ is a locality map} \\ &\Longrightarrow b_1 \top_B b_2. \end{aligned}$$

Therefore $\overline{\top} \subseteq \top_B$.

□

Example 4.5. *The map*

$$\begin{array}{rcl} \phi : \mathbb{N} & \longrightarrow & 2\,\mathbb{N} \\ m & \longmapsto & 2m \end{array}$$

is surjective. We equip $A := \mathbb{N}$ *with the locality relation* $m_1 \top m_2 \iff |m_1 - m_2| = 3$. *Then* $n_1 \overline{\top} n_2$ *if and only if* $|n_1 - n_2| = 6$.

Applying Proposition 4.4 to the canonical projection map $\pi : V \to V/W$ of a pre-locality vector space $(V, \top)$ to its quotient V/W by a linear subspace W, we equip the quotient with the quotient locality relation.

Definition-Proposition 4.6. *For a subspace* W *of a pre-locality vector space* $(V, \top)$*, we call* **quotient locality** *on the quotient* V/W*, the final locality relation*

$$\left([u] \overline{\top} [v] \iff \exists (u', v') \in [u] \times [v] : \ u' \top v'\right) \qquad \forall ([u], [v]) \in (V/W)^2 \tag{2.1}$$

for the canonical projection map $\pi : V \to V/W$. *This way, the pre-locality space* $(V, \top)$ *gives rise to a pre-locality vector quotient space* $(V/W, \overline{\top})$ *and the projection map* $\pi : (V, \top) \to (V/W, \overline{\top})$ *is a morphism of pre-locality vector spaces.*

Proof. The facts that $(V/W, \overline{\top})$ is a pre-locality space and that $\pi : (V, \top) \to (V/W, \overline{\top})$ is a morphism of pre-locality spaces hold by definition of $\overline{\top}$, since it is the coarsest locality relation such that π is a locality map. $\square$

The following simple examples illustrate this last concept.

Example 4.7. *Consider the pre-locality vector space* $(\mathbb{R}^3, \top)$ *where* $\top$ *is the orthogonality relation, namely* $v \top w \Leftrightarrow v \perp w$. *Let* $W = \mathbb{K} e_1 \subset \mathbb{R}^3$ *be the span of* e_1 *where* $\{e_i\}_{i=1}^3$ *is the canonical basis of* $\mathbb{R}^3$. *The quotient locality on* $\mathbb{R}^3/W$ *is* $\overline{\top} = (\mathbb{R}^3/W) \times (\mathbb{R}^3/W)$ *since for any pair* $([q_2 e_2 + q_3 e_3], [k_2 e_2 + k_3 e_3]) \in (\mathbb{R}^3/W)^2$ *there are scalars* q_1 *and* k_1 *in* $\mathbb{K}$ *such that* $(q_1 e_1 + q_2 e_2 + q_3 e_3) \perp (k_1 e_1 + k_2 e_2 + k_3 e_3)$*, so that* $[q_2 e_2 + q_3 e_3] \overline{\top} [k_2 e_2 + k_3 e_3]$.

Example 4.8. *Consider the pre-locality vector space* $(V, \top)$ *where* $V = \mathbb{R}^4$ *and* $\top = \mathbb{R}^4 \times \{0\} \cup \{0\} \times \mathbb{R}^4 \cup (\langle \{e_1, e_3\} \rangle \times \langle e_2 + e_4 \rangle) \cup (\langle e_2 + e_4 \rangle \times \langle \{e_1, e_3\} \rangle)$. *For* $W = \mathbb{K}(e_4)$*, the quotient locality on* V/W *is given by* $\overline{\top} = (V/W \times \{[0]\}) \cup (\{[0]\} \times V/W) \cup (\langle [e_1 + e_3] \rangle \times \langle [e_2] \rangle) \cup (\langle [e_2] \rangle \times \langle [e_1 + e_3] \rangle)$.

4.2 Universal property of the locality tensor product

Up to this point, the locality tensor product of subspaces $V_1, \ldots, V_n$ of a (pre-) locality vector space $(E, \top)$, is only a vector space $V \otimes_\top \cdots \otimes_\top V_n$. We have not endowed it with any locality relation induced by $\top$. This paragraph has two objectives: the first one is to define a locality relation $\top_{\otimes n}$ in the locality tensor product naturally induced by that on the original space. The second objective is to extend the universal property of the usual (non-locality) tensor product to the pre-locality set up and to present some of its direct consequences. For this purpose, and for the rest of the paragraph, $V_1, \ldots, V_n$ are subspaces of a pre-locality vector space $(E, \top)$, where $n \geq 2$.

Definition 4.9.

- We define the locality relation $\top_{V_1 \times_\top \cdots \times_\top V_n}$ on $V_1 \times_\top \cdots \times_\top V_n$ as

$$\top_{V_1 \times_\top \cdots \times_\top V_n} := \left\{ \left((v_1, \ldots, v_n), (v_1', \ldots, v_n') \right) \in (V_1 \times_\top \cdots \times_\top V_n)^2 : \forall (i,j) \in [n]^2, v_i \top v_j' \right\}. \tag{2.2}$$

  It extends linearly to a locality relation on $\mathbb{K}(V_1 \times_\top \cdots \times_\top V_n)$ as in (1.23). In the case $V_1 = \cdots = V_n = V$ we write $\top_{\times n}$ instead of $\top_{V \times_\top \cdots \times_\top V}$.

- The locality relation $\top_\otimes(V_1, \ldots, V_n)$ on $V_1 \otimes_\top \cdots \otimes_\top V_n$ is defined as the quotient relation (see Definition 4.6) for the quotient map $\pi : \left(\mathbb{K}\left(V_1 \times_\top \cdots \times_\top V_n\right), \top_{V_1 \times_\top \cdots \times_\top V_n}\right) \to V_1 \otimes_\top \cdots \otimes_\top V_n$. For the case $V_1 = \cdots = V_n = V$ we write $\top_{\otimes n}$ instead of $\top_\otimes(V_1, \ldots, V_n)$.

Remark 4.10. Observe that the locality tensor product $V_1 \otimes_\top \cdots \otimes_\top V_n$ turns into a pre-locality vector space when endowed with the locality relation $\top_\otimes(V_1, \ldots, V_n)$.

The first item of the previous definition provides a canonical locality relation on the locality Cartesian product of two spaces. This suggests an enhancement of the definition of $\top_\times$-bilinearity (Definition 2.12).

Definition 4.11. Let $V_1, \ldots, V_n$ be subspaces of a pre-locality vector space $(E, \top)$, and $(G, \top_G)$ a pre-locality vector space.

- We call a $\top_\times$ **n-linear** a map $f : V_1 \times_\top \cdots \times_\top V_n \to G$ which satisfies the $\top_\times$ n-linearity condition:

$$\bar{f}(I_{\text{mult,n}}^{\top_\times}) = \{0_G\}, \tag{2.3}$$

  where we have set $I_{\text{mult,n}}^{\top_\times} := \mathbb{K}(V_1 \times_\top \cdots \times_\top V_n) \cap I_{\text{mult,n}}(V_1, \ldots, V_n)$. When $n = 2$, we say that f is $\top_\times$-bilinear as in Definition 2.12.

- We call a map $f : (V_1 \times_\top \cdots \times_\top V_n, \top_{V_1 \times_\top \cdots \times_\top V_n}) \to (G, \top_G)$ **locality** $\top_\times$ **n-linear** or **locality multilinear**, if it is $\top_\times$ n-linear, and is moreover a locality map, namely

$$(f \times f)(\top_{V_1 \times_\top \cdots \times_\top V_n}) \subset \top_G, \tag{2.4}$$

  or equivalently

$$(v_1, \ldots, v_n) \top_{V_1 \times_\top \cdots \times_\top V_n} (v_1', \ldots, v_n') \implies f(v_1, \ldots, v_n) \top_G f(v_1', \ldots, v_n').$$

  When $n = 2$ we say that f is locality $\top_\times$-bilinear, instead of locality $\top_\times$ 2-linear.

Proposition 4.12. *The map* $\otimes_\top : (V_1 \times_\top \cdots, \times_\top V_n, \top_{V_1 \times_\top \cdots \times_\top V_n}) \to (V_1 \otimes_\top \cdots \otimes_\top V_n, \top_\otimes(V_1, \ldots, V_n))$ *is a locality* $\top_\times$ *n-linear map.*

Proof. For $n = 2$, it was shown in Proposition 2.16 that $\otimes_\top$ is $\top_\times$-bilinear. A similar argument proves that $\otimes_\top$ is $\top_\times$ n-linear for any $n > 2$. We therefore only need to show that it is a locality map. Recall that $\otimes_\top = \pi_\top \circ \iota_\top$, where $\iota_\top : V_1 \times_\top \cdots \times_\top V_n \to \mathbb{K}(V_1 \times_\top \cdots \times_\top V_n)$ is the canonical inclusion map. The latter is a locality map since the locality relation$\top_{V_1 \times_\top \cdots \times_\top V_n}$ on $\mathbb{K}(V_1 \times_\top \cdots \times_\top V_n)$ is a linear extension of the locality relation in $V_1 \times_\top \cdots \times_\top V_n$. The map $\pi_\top : \mathbb{K}(V_1 \times_\top \cdots \times_\top V_n) \to V_1 \otimes_\top \cdots \otimes_\top V_n$ is also a locality map by construction of the locality relation $\top_\otimes(V_1, \ldots, V_n)$. The statement then follows from the fact that the composition of locality maps is again a locality map (see Proposition 2.3). $\square$

Before stating and proving the universal property of the locality tensor product, we recall that a subset B of a vector space V is a (Hamel or algebraic) basis if it satisfies:

1. the linear independence property, i.e., for every finite subset $\{b_1, \ldots, b_n\}$ of B and every $\alpha_1, \ldots, \alpha_n$ in $\mathbb{K}$ we have $\sum_{i=1}^n \alpha_i b_i = 0 \Longrightarrow \alpha_1 = \cdots = \alpha_n = 0$, and
2. the spanning property, i.e., every vector v in V can be written as a finite linear combination $v = \sum_{k=1}^n \alpha_i b_i$ in which case there is an isomorphism of vector spaces $\mathbb{K}B \simeq V$.

Recall that, since we do not have a topology, a basis always refer to a Hamel basis (Definition 1.2) and not a Hilbert basis, even if the space is infinite dimensional.

The following result states that any $\top_\times$ n-linear map can be extended (non uniquely in general) to an n-linear map. The arguments in the proof follow those in [21, Proposition 1.15] and make use of Zorn's lemma 1.1, more precisely Lemma 1.5.

Proposition 4.13. *Let* $V_1, \ldots, V_n$ *be subspaces of a pre-locality vector space* $(E, \top)$ *and* G *a linear space. Any* $\top_\times$ *n-linear map* $f : V_1 \times_\top \cdots \times_\top V_n \to G$ *extends to an n-linear map* $g : V_1 \times \cdots \times V_n \to G$ *i.e.,* $g|_{V_1 \times_\top \cdots \times_\top V_n} = f$.

Proof. Consider the set $\mathcal{O} := \{B \subset V_1 \times_\top \cdots \times_\top V_n | \otimes(B)$ is a linearly independent subset of $V_1 \otimes \cdots \otimes V_n\}$, where $\otimes(B)$ is the image of B under the canonical map $\otimes : V_1 \times \cdots \times V_n \to V_1 \otimes \cdots \otimes V_n$. From $(\underbrace{0, \ldots, 0}_{n-\text{times}}) \in \mathcal{O}$ follows that $\mathcal{O}$ is not empty. We equip $\mathcal{O}$ with the partial inclusion order $B_1 \subset B_2$ and consider a chain $\mathcal{C}$ of $\mathcal{O}$. We observe that $\bigcup_{B \in \mathcal{C}} B \in \mathcal{O}$ since $\otimes\left(\bigcup_{B \in \mathcal{C}} B\right) = \bigcup_{B \in \mathcal{C}} \otimes(B)$, and the union of nested linearly independent sets is a linearly independent set (see Lemma 1.3). Thus, $\mathcal{O}$ satisfies the assumption of Zorn's Lemma (Lemma 1.1) which ensures the existence of a maximal element $\mathcal{B} \in \mathcal{O}$ and correspondingly a linearly independent set $\otimes(\mathcal{B}) \subset V_1 \otimes \cdots \otimes V_n$. Since $\otimes(\mathcal{B}) \subset \otimes(V_1 \times \cdots \times V_n)$ and $\otimes(V_1 \times \cdots \times V_n)$ generates $V_1 \otimes \cdots \otimes V_n$, by Lemma 1.5, we can complete $\otimes(\mathcal{B})$ to a basis $\overline{\otimes(\mathcal{B})} \subset \otimes(V_1 \times \cdots \times V_n)$ of $V_1 \otimes \cdots \otimes V_n$.

Since by construction, $\overline{\otimes(\mathcal{B})} \subset \otimes(V_1 \times \cdots \times V_n)$, any element $y \in \overline{\otimes(\mathcal{B})} \setminus \otimes(\mathcal{B})$ can be written $y = \otimes(x_y)$ for some $x_y \in V_1 \times \cdots \times V_n$. We claim that the set $\overline{\mathcal{B}} := \{x_y : y \in \overline{\otimes(\mathcal{B})} \setminus \otimes(\mathcal{B})\} \cup \mathcal{B}$ fulfills the relation

$$\overline{\otimes(\mathcal{B})} = \otimes(\overline{\mathcal{B}}).$$

Indeed, if $x \in \overline{\mathcal{B}}$, either $x \in \mathcal{B}$, in which case $y := \otimes(x) \in \otimes(\mathcal{B}) \subset \overline{\otimes(\mathcal{B})}$, or $y := \otimes(x) \in \overline{\otimes(\mathcal{B})} \setminus \otimes(\mathcal{B}) \subset \overline{\otimes(\mathcal{B})}$ and therefore $\otimes(\overline{\mathcal{B}}) \subset \overline{\otimes(\mathcal{B})}$. Conversely, for $y \in \overline{\otimes(\mathcal{B})}$ either $y \in \otimes(\mathcal{B}) \subset \otimes(\overline{\mathcal{B}})$ or $y \in \overline{\otimes(\mathcal{B})} \setminus \otimes(\mathcal{B})$ in which case $x_y \in \overline{\mathcal{B}}$ and therefore $\overline{\otimes(\mathcal{B})} \subset \otimes(\overline{\mathcal{B}})$.

Let $g : V_1 \times \cdots \times V_n \to G$ be the unique n-linear map defined on $\overline{\mathcal{B}}$ by

$$g(x_1, \ldots, x_n) := \begin{cases} f(x_1, \ldots, x_n) & \text{if } (x_1, \ldots, x_n) \in \mathcal{B} \\ 0 & \text{if } (x_1, \ldots, x_n) \notin \mathcal{B} \end{cases}$$

The existence and uniqueness of this map is granted by the universal property of the tensor product (Theorem 1.16). It remains to show that $g|_{V_1 \times_\top \cdots \times_\top V_n} = f$. Given $p \in V_1 \times_\top \cdots \times_\top V_n$, the maximality of $\mathcal{B}$ yields the existence of $(x_{1,i}, \ldots, x_{n,i}) \in \mathcal{B}$ with $1 \le i \le N$ for some $N \in \mathbb{N}$ such that $\sum_{i=1}^N \alpha_i x_{1,i} \otimes \cdots x_{n,i} = \otimes(p)$, and thus $\sum_{i=1}^N \alpha_i (x_{1,i}, \ldots, x_{n,i}) = p + \omega$ which amounts to $\omega \in I_{\text{mult}}(V_1, \ldots, V_n) \cap \mathbb{K}(V_1 \times_\top \cdots \times_\top V_n)$. Using the extensions of f and g to $\bar{f}$ and $\bar{g}$ (see (1.1)), and the fact that $\bar{f}(\omega) = \bar{g}(\omega) = 0$, this implies that

$$f(p) = \bar{f}\Big(\sum_{i=1}^N \alpha_i (x_{1,i}, \ldots, x_{n,i}) - \omega\Big) = \sum_{i=1}^N \alpha_i f(x_{1,i}, \ldots, x_{n,i}) = \sum_{i=1}^N \alpha_i g(x_{1,i}, \ldots, x_{n,i}) = g(p).$$

It follows that $g|_{V_1 \times_\top \cdots \times_\top V_n} = f$ as expected. □

We deduce from the universal property of the usual tensor product (Theorem 1.11), a universal property for the locality tensor product of subspaces of a pre-locality vector space.

Theorem 4.14 (Universal property of the locality tensor product)**.** Given $V_1, \ldots, V_n$ subspaces of a pre-locality vector space $(E, \top)$, G a linear space and $f_\top : V_1 \times_\top \cdots \times_\top V_n \to G$ a $\top_\times$ n-linear map, there is a unique linear map $\phi_{f_\top} : V_1 \otimes_\top \cdots \otimes_\top V_n \to G$ such that the following diagram commutes:

$$\begin{array}{ccc} V_1 \times_\top \cdots \times_\top V_n & \xrightarrow{\otimes_\top} & V_1 \otimes_\top \cdots \otimes_\top V_n \\ & {\scriptstyle f_\top} \searrow \quad \swarrow {\scriptstyle \phi_{f_\top}} & \\ & G & \end{array} \tag{2.5}$$

If moreover $(G, \top_G)$ is a pre-locality vector space and $f_\top$ is a locality $\top_\times$ n-linear map, then the map $\phi_{f_\top} : (V_1 \otimes_\top \cdots \otimes_\top V_n, \top_\otimes(V_1, \ldots, V_n)) \to (G, \top_G)$ is a locality linear morphism. In particular the following diagram commutes, and all maps in it are locality maps.

$$\begin{array}{ccc} (V_1 \times_\top \cdots \times_\top V_n, \top_{V_1 \times_\top \cdots \times_\top V_n}) & \xrightarrow{\otimes_\top} & (V_1 \otimes_\top \cdots \otimes_\top V_n, \top_\otimes(V_1, \ldots, V_n)) \\ & {\scriptstyle f_\top} \searrow \quad \swarrow {\scriptstyle \phi_{f_\top}} & \\ & (G, \top_G) & \end{array}$$

Remark 4.15. The statement holds for the trivial locality relation $\top = \underbrace{E \times \cdots \times E}_{n-\text{times}}$ as a consequence of the universal property of the usual (non-locality) tensor product (Theorem 1.16). This fact is used in the proof to build the map $\phi_{f_\top}$ for any locality relation $\top$ on E.

Proof. To build $\phi_{f_\top}$, we use Proposition 4.13, to extend $f_\top$ to a multilinear map $g : V_1 \times \cdots \times V_n \to G$ such that $g|_{V_1 \times_\top \cdots \times_\top V_n} = f_\top$. The universal property of the usual tensor product yields the existence of a unique linear map $\phi_g : V_1 \otimes \cdots \otimes V_n \to G$ such that

$$g = \phi_g \circ \otimes. \tag{2.6}$$

Since $V_1 \otimes_\top \cdots \otimes_\top V_n \subset V_1 \otimes \cdots \otimes V_n$, we can restrict ϕ_g to $V_1 \otimes_\top \cdots \otimes_\top V_n$ and set $\phi_{f_\top} := \phi_g|_{V_1 \otimes_\top \cdots \otimes_\top V_n}$. As a consequence of $\otimes|_{V_1 \times_\top \cdots \times_\top V_n} = \otimes_\top$, we can further restrict (2.6) to $V \times_\top \cdots \times_\top V_n$ and thus

$$f_\top = \phi_{f_\top} \circ \otimes_\top$$

as expected.

For the uniqueness it is useful to make the following observation. Thanks to the universal property of the usual tensor product (Theorem 1.16), any given map g induces a uniquely defined map ϕ_g. So is its restriction $\phi_{f_\top}$ is uniquely defined. However the construction of the extension g of the map $f_\top$ by means of Zorn's lemma, does not ensure uniqueness. Nonetheless, given g_1, $g_2 : V_1 \times \top \times V_n \to G$ multilinear maps such that $g_1|_{V_1 \times_\top \cdots \times_\top V_n} = g_2|_{V_1 \times_\top \cdots \times_\top V_n} = f_\top$, we have that $\phi_{g_1}|_{V_1 \otimes_\top \cdots \otimes_\top V_n} = \phi_{g_2}|_{V_1 \otimes_\top \cdots \otimes_\top V_n}$. Indeed, given an equivalence class $[x] \in V_1 \otimes_\top \cdots \otimes_\top V_n$, a representative of such class $x' \in [x]$ can be written as $x' = \sum_{i \in I} \alpha_i (v_{1,i}, \ldots, v_{n,i})$ where I is a finite set, $\alpha_i \in \mathbb{K}$, and $(v_{1,i}, \ldots, v_{n,i}) \in V_1 \times_\top \cdots \times_\top V_n$ for every $i \in I$. Thus, for $j \in \{1, 2\}$

$$\begin{aligned}
\phi_{g_j}([x]) &= \sum_{i \in I} \alpha_i \phi_{g_j}\left([(v_{1,i}, \ldots, v_{n,i})]\right) \\
&= \sum_{i \in I} \alpha_i \phi_{g_j}\left(v_{1,i} \otimes \cdots \otimes v_{n,i}\right) \\
&= \sum_{i \in I} \alpha_i g_j\left(v_{1,i}, \ldots, w_{n,i}\right) \\
&= \sum_{i \in I} \alpha_i f_\top\left(v_{1,i}, \ldots, v_{n,i}\right).
\end{aligned}$$

It follows that the restriction of ϕ_{g_j} to $V_1 \otimes_\top \cdots \otimes_\top V_n$ is completely determined by $f_\top$. Hence $\phi_{g_1}|_{V_1 \otimes_\top \cdots \otimes_\top V_n} = \phi_{g_2}|_{V_1 \otimes_\top \cdots \otimes_\top V_n} = \phi_{f_\top}$, implying the uniqueness of $\phi_{f_\top}$.

We are only left to show that ϕ is a locality map if $f_\top$ is locality $\top_\times$ n-linear. Recall that two equivalence classes $[a]$ and $[b]$ in $V_1 \otimes_\top \cdots \otimes_\top V_n$ verify $[a] \top_\otimes (V_1, \ldots, V_n) [b]$ if there are $\sum_{i=1}^n \alpha_i (x_{1,i}, \ldots, x_{n,i}) \in [a]$ and $\sum_{j=1}^m \beta_j (y_{1,j}, \ldots, y_{n,j}) \in [b]$ such that for every $(k, i, q, j) \in [n] \times I \times [n] \times J$, $x_{k,i} \top y_{q,j}$. Since $f_\top$ is locality $\top_\times$ n-linear, then $f_\top(\sum_{i=1}^n \alpha_i (x_{1,i}, \ldots, x_{n,i})) \top_G f(\sum_{j=1}^m \beta_j (y_{1,j}, \ldots, y_{n,j}))$ which amounts to $\phi([a]) \top_G \phi([b])$. Therefore ϕ is local as expected. □

Corollary 4.16. The universal property of the locality tensor product of subspaces of a pre-locality vector space (Theorem 4.14) is equivalent to the universal property of the usual (non locality) tensor product of subspaces of an ordinary vector space (Theorem 1.16).

Proof. It was proven in Theorem 4.14 that the universal property of the locality tensor product of subspaces of a pre-locality vector space E is implied by Theorem 1.16 (the universal product of the usual (non-locality) tensor product of subspaces of E).

Conversely, assuming that the universal property of the locality tensor product of subspaces of any pre-locality vector space (Theorem 4.14) holds, we want to show that it holds for ordinary tensor products. Given subspaces $V_1, \ldots, V_n$ of a vector space E. Consider the trivial locality relation $\top = E \times E$. Then Theorem 1.16 follows from Theorem 4.14. □

As a consequence of the universal property of the locality tensor product, we may now define the tensor product of locality independent linear maps. Recall that for $(E,\top_E)$ and $(F,\top_F)$ pre-locality vector spaces, V_1 and V_2 subspaces of E, and W_1 and W_2 subspaces of F, two maps $f_i : V_i \to W_i$ with $i \in \{1,2\}$ are locality independent to each other if, and only if (see Definition 2.1)

$$(f_1 \times f_2)(V_1 \times_{\top_E} V_2) \subset W_1 \times_{\top_F} W_2.$$

Lemma 4.17. Let $(E,\top_E)$ and $(F,\top_F)$ be pre-locality vector spaces, $V_1,\dots,V_n$ (resp. $W_1,\dots,W_n$) subspaces of E (resp. F), and $\otimes_\top^W : W_1 \times_\top, \dots, \times_\top W_n \to W_1 \otimes_\top \cdots \otimes_\top W_n$ the canonical injection map. If there are linear maps $f_i : V_i \to W_i$ with $i \in [n]$, locality independent two by two, then

$$\otimes_\top^W \circ (f_1 \times \cdots \times f_n)\big|_{V_1 \times_\top \cdots \times_\top V_n} : V_1 \times_\top \cdots \times_\top V_n \to W_1 \otimes_\top \cdots \otimes_\top W_n$$

is a $\top_\times$ n-linear map. If moreover every f_i is a locality map, then $\otimes_\top^W \circ (f_1 \times \cdots \times f_n)\big|_{V_1 \times_\top \cdots \times_\top V_n}$ is a locality $\top_\times$ n-linear map.

Proof. Observe first that $\otimes_\top^W \circ (f_1 \times \cdots \times f_n)\big|_{V_1 \times_\top \cdots \times_\top V_n}$ is well defined since the f_i are locality independent two by two, and therefore $(f_1 \times \cdots \times f_n)\big|_{V_1 \times_\top \cdots \times_\top V_n} : V_1 \times_\top \cdots \times_\top V_n \to W_1 \times_\top \cdots \times_\top W_n$. Define

$$\overline{(f_1 \times \cdots \times f_n)} : \mathbb{K}(V_1 \times \cdots \times V_n) \to \mathbb{K}(W_1 \times \cdots \times W_n)$$

as the only linear map whose restriction to $V_1 \times \cdots \times V_n$ is equal to $(f_1 \times \cdots \times f_n)$. We denote with some abuse of notation $\overline{(f_1 \times \cdots \times f_n)}|_\top$ the restriction of $\overline{(f_1 \times \cdots \times f_n)}$ to $\mathbb{K}(V_1 \times_\top \cdots \times_\top V_n)$. It follows from the assertion before that

$$\overline{(f_1 \times \cdots \times f_n)}|_\top : \mathbb{K}(V_1 \times_\top \cdots \times_\top V_n) \to \mathbb{K}(W_1 \times_\top \cdots \times_\top W_n). \tag{2.7}$$

We proceed to prove that $\overline{\otimes_\top^W \circ (f_1 \times \cdots \times f_n)\big|_{V_1 \times_\top \cdots \times_\top V_n}} = \overline{\otimes_\top^W} \circ \overline{(f_1 \times \cdots \times f_n)}|_\top$. Indeed, for any $\sum_{i\in I} \alpha_i (v_{1,i},\dots,v_{n,i}) \in \mathbb{K}(V_1 \times_\top \cdots \times_\top V_n)$, with I a finite set,

$$\begin{aligned}\overline{\otimes_\top^W \circ (f_1 \times \cdots \times f_n)\big|_{V_1 \times_\top \cdots \times_\top V_n}}\Big(\sum_{i\in I} \alpha_i (v_{1,i},\dots,v_{n,i})\Big) &= \sum_{i\in I} \alpha_i \otimes_\top^W (f_1 \times \cdots \times f_n)(v_{1,i},\dots,v_{n,i}) \\ &= \overline{\otimes_\top^W}\Big(\sum_{i\in I} \alpha_i (f_1 \times \cdots \times f_n)(v_{1,i},\dots,v_{n,i})\Big) \\ &= \overline{\otimes_\top^W} \circ \overline{(f_1 \times \cdots \times f_n)}|_\top\Big(\sum_{i\in I} \alpha_i (v_{1,i},\dots,v_{n,i})\Big).\end{aligned}$$

The first and second lines follow from the definition of the maps $\overline{\otimes_\top^W \circ (f_1 \times \cdots \times f_n)\big|_{V_1 \times_\top \cdots \times_\top V_n}}$ and $\overline{\otimes_\top^W}$ (see (1.1)), and the third one is a consequence of the definition of $\overline{(f_1 \times \cdots \times f_n)}|_\top$.

In order to prove that $\otimes_\top^W \circ (f_1 \times \cdots \times f_n)\big|_{V_1 \times_\top \cdots \times_\top V_n}$ is $\top_\times$ n-linear, we must show (see Definition 2.12) that $\overline{\otimes_\top^W} \circ \overline{(f_1 \times \cdots \times f_n)}|_\top \big(\mathbb{K}(V_1 \times_\top \cdots \times_\top V_n) \cap I_{\text{mult}}(V_1,\dots,V_n)\big) = \{0_{W_1 \otimes_\top \cdots \otimes_\top W_n}\}$. It follows from the linearity of every f_i that $\overline{(f_1 \times \cdots \times f_n)}(I_{\text{mult}}(V_1,\dots,V_n)) \subset I_{\text{mult}}(W_1,\dots,W_n)$. This assertion together with (2.7) imply that $\overline{(f_1 \times \cdots \times f_n)}\big(\mathbb{K}(V_1 \times_\top \cdots \times_\top V_n) \cap I_{\text{mult}}(V_1,\dots,V_n)\big) \subset \big(\mathbb{K}(W_1 \times_\top \cdots \times_\top W_n) \cap I_{\text{mult}}(W_1,\dots,W_n)\big)$. As a consequence of the $\top_\times$ n-linearity of $\otimes_\top^W$ (see Proposition 2.16), that $\overline{\otimes_\top^W} \circ \overline{(f_1 \times \cdots \times f_n)}|_\top \big(\mathbb{K}(V_1 \times_\top \cdots \times_\top V_n) \cap I_{\text{mult}}(V_1,\dots,V_n)\big)$ is equal to $\{0_{W_1 \otimes_\top \cdots \otimes_\top W_n}\}$ as expected, and thus $\otimes_\top^W \circ (f \times \cdots \times f_n)|_{V_1 \times_\top \cdots \times_\top V_n} : V \times_\top \cdots \times_\top V_n \to W_1 \otimes_\top \cdots \otimes_\top W_n$ is a $\top_\times$ n-linear map.

If moreover the f_i are locality linear maps, it is easy to see that $(f_1 \times \cdots \times f_n)(\top_{V_1 \times_\top \cdots \times_\top V_n}) \subset (\top_{W_1 \times_\top \cdots \times_\top W_n})$. Since $\otimes_\top^W$ is a locality $\top_\times$ n-linear map (see Proposition 4.12) and the composition of locality maps is again a locality map (see Proposition 2.3), then $\otimes_\top^W \circ (f_1 \times \cdots \times f_n)\big|_{V_1 \times_\top \cdots \times_\top V_n}$ is a locality $\top_\times$ n-linear map. □

Definition-Proposition 4.18. With the hypothesis of Lemma 4.17, the tensor product of the maps f_i for $1 \le i \le n$, is the only linear map $f_1 \otimes \cdots \otimes f_n : V_1 \otimes_\top \cdots \otimes_\top V_n \to W_1 \otimes_\top \cdots \otimes_\top W_n$ such that

$$f_1 \otimes \cdots \otimes f_n \circ \otimes_\top^V = \otimes_\top^W \circ (f_1 \times \cdots \times f_n)\big|_{V_1 \times_\top \cdots \times_\top V_n}.$$

Here $\otimes_\top^V$ is the canonical map from $V_1 \times_\top \cdots \times_\top V_n$ to $V_1 \otimes_\top \cdots \otimes_\top V_n$.

This can be visualized in the commutativity of the following diagram.

$$\begin{array}{ccc} V_1 \times_\top \cdots \times_\top V_n & \xrightarrow{\otimes_\top^V} & V_1 \otimes_\top \cdots \otimes_\top V_n \\ \downarrow{\scriptstyle f_1 \times \cdots \times f_n} & & \downarrow{\scriptstyle f_1 \otimes \cdots \otimes f_n} \\ W_1 \times_\top \cdots \times_\top W_n & \xrightarrow{\otimes_\top^W} & W_1 \otimes_\top \cdots \otimes_\top W_n \end{array}$$

If moreover the f_i are locality linear maps, then $f_1 \otimes \cdots \otimes f_n$ is also a locality linear map.

Proof. The existence and uniqueness of the map $f_1 \otimes \cdots \otimes f_n$ in the previous definition is a consequence of Lemma 4.17 and of the universal property of the locality tensor product (Theorem 4.14). These two results also imply that if the f_i are locality maps, then $f_1 \otimes \cdots \otimes f_n$ is also a locality map. □

Remark 4.19. Notice that with some abuse of notation, we write $f_1 \otimes \cdots \otimes f_n$ instead of $f_1 \otimes_\top \cdots \otimes_\top f_n$. This is because the map in Definition-Proposition 4.18 is the restriction of the usual tensor product of the f_i maps to $V \otimes_\top W \subset V \otimes W$, and therefore there is no risk of ambiguity.

Distributivity of the locality tensor product

Two properties of the non-locality tensor product commonly used are the distributivity with respect to direct sums and with respect to intersections (see Proposition 1.17). These two properties rely on the existence of a direct complement to every subspace $V_1 \subset V$, i.e, the existence of a subspace $V_2 \subset V$ such that $V_1 \oplus V_2 = V$. However, for the locality tensor product, distributivity does not always hold as the following example illustrates.

Counter-example 4.20. Consider $\mathbb{R}^2$ with the orthogonality locality. Then

$$(\langle e_1 \rangle \otimes_\top \mathbb{R}^2) \oplus (\langle e_2 \rangle \otimes_\top \mathbb{R}^2) = \langle e_1 \otimes e_2 \rangle \oplus \langle e_2 \otimes e_1 \rangle, \tag{2.8}$$

is different from $(\langle e_1 \rangle \oplus \langle e_2 \rangle) \otimes_\top \mathbb{R}^2$ since it does not contain $(e_1 + e_2) \otimes (e_1 - e_2) \in \mathbb{R}^2 \otimes_\top \mathbb{R}^2$.

It is therefore necessary to ensure some compatibility of the splitting $V = V_1 \oplus V_2$ with the locality relation $\top$, in order to accommodate for distributivity properties of the tensor product in the locality set up. Such properties will be useful in the sequel (Proposition 6.6). The following proposition gives sufficient conditions to ensure the distributivity of the locality tensor product with respect to the direct sum.

Proposition 4.21. *Let V and W be linear subspaces of a pre-locality vector space $(E, \top)$, and let V_1 and V_2 be subspaces of V such that $V_1 \oplus V_2 = V$.*

1. *If for $\{i, j\} = \{1, 2\}$ the projection maps $\pi_i : V \to V_i$ onto V_i along V_j are locally independent of the identity map Id_W on W (i.e., $(\pi_i \times \mathrm{Id}_W)(\top|_{V \times W}) \subset \top|_{V_i \times W}$), then*

$$(V_1 \otimes_\top W) \oplus (V_2 \otimes_\top W) = V \otimes_\top W.$$

2. *If $(E, \top)$ is a locality vector space, and one of the projections π_i is locally independent of Id_W, then the other projection is also locally independent of Id_W.*

Proof. 1. We first prove that $(V_1 \otimes_\top W) \oplus (V_2 \otimes_\top W) \subset V \otimes_\top W$. Since $V_i \subset V$ then $V_i \otimes_\top W \subset V \otimes_\top W$ and the expected inclusion follows. To prove the other direction, without lost of generality consider $a \otimes b$ in $V \otimes_\top W$ such that (a, b) lies in $\top|_{V \times W}$. Since π_i and Id_W are locally independent $(\pi_i(a), b) \in \top|_{V_i \times W}$ thus $\pi_i(a) \otimes b \in V_i \otimes_\top W$, which implies that

$$a \otimes b = (\pi_1(a) + \pi_2(a)) \otimes b = \pi_1(a) \otimes b + \pi_2(a) \otimes b \in (V_1 \otimes_\top W) \oplus (V_2 \otimes_\top W).$$

2. Moreover, if $(E, \top)$ is a locality vector space and if π_1 is independent of Id_W, then for any pair $(a, b) \in V \times W$ we have $a \top b \Longrightarrow \pi_1(a) \top b$. Using $\pi_2(a) = a - \pi_1(a)$, it follows that $\pi_2(a) \top b$ since $\{b\}^\top$ is a vector subspace of E. Thus π_2 is independent of Id_W as claimed. □

We prove a (weak) locality version of the distributivity property w.r. to the intersection $(V_1 \cap V_2) \otimes W = (V_1 \otimes W) \cap (V_2 \otimes W)$.

Corollary 4.22. Let V and W be linear subspaces of a pre-locality vector space $(E, \top)$, and let V_1, V_2 be subspaces of V. Let V_2' be a direct complement of the intersection $V_1 \cap V_2$ in V_2 i.e. $(V_1 \cap V_2) \oplus V_2' = V_2$.

1. If the projection maps $\pi : V_2 \to V_1 \cap V_2$ onto $V_1 \cap V_2$ along V_2' and $\mathrm{Id}_{V_2} - \pi$ are locally independent of the identity map Id_W on W, then

$$(V_1 \cap V_2) \otimes_\top W = (V_1 \otimes W) \cap (V_2 \otimes_\top W).$$

2. In particular, if $V_1 \subset V$, and if the projection maps $\pi : V \to V_1$ onto V_1 along V_2' and $\mathrm{Id}_{V_2} - \pi$ are locally independent of the identity map Id_W on W, then

$$V_1 \otimes_\top W = (V_1 \otimes W) \cap (V \otimes_\top W).$$

Proof. 1. Using the distributivity property of the locality tensor product $((V_1 \cap V_2) \otimes_\top W) \oplus (V_2' \otimes_\top W) = V_2 \otimes_\top W$ which follows from Proposition 4.21, we have

$$(V_1 \otimes W) \cap (V_2 \otimes_\top W) = (V_1 \otimes W) \bigcap \left(((V_1 \cap V_2) \otimes_\top W) \oplus (V_2' \otimes_\top W)\right). \tag{2.9}$$

We now make use of a result of elementary linear algebra, namely $A \cap (B \oplus C) = B$ whenever A, B and C are linear subspaces of a linear space E with $B \subset A$ and $A \cap C = \{0\}$. Since $(V_1 \cap V_2) \otimes_\top W \subset V_1 \otimes W$ and $(V_1 \otimes W) \cap (V_2' \otimes_\top W) = \{0\}$, the right hand side of (2.9) is equal to $(V_1 \cap V_2) \otimes_\top W$, which yields the result.

2. Setting $V_2 =: V$ in the previous item, yields the result since $V_1' = \{0\}$. □

4.3 Locality tensor algebra and its universal property

In the present paragraph we extend the results of Paragraph 4.2 to the locality tensor algebra of a pre-locality vector spaces. For that purpose, we build a natural locality relation in the tensor algebra induced from the one in the pre-locality vector space and then state and prove its universal property (Theorem 4.33), together with the equivalence with the usual universal property (Corollary 4.34).

Recall that the tensor algebra of a vector space V is a unital algebra. It is therefore necessary to extend Definition 2.26 to include unital (pre-) locality algebras (or simply (pre-) locality algebras). This can be done as a consequence of Definition-Proposition 4.18. The following definition is taken from [22, Definition 3.16] and [21, Definition 4.1].

Definition 4.23. A non-unital (pre-) locality algebra $(A, \top, m)$ is called **unital** (or simply (pre-) locality algebra) if there is a map $u : \mathbb{K} \to A$ such that $u(\mathbb{K}) \subseteq A^\top$, which makes the following diagram commute

$$\begin{array}{ccccc} \mathbb{K} \otimes A & \xrightarrow{u \otimes \mathrm{Id}} & A \otimes_\top A & \xleftarrow{\mathrm{Id} \otimes u} & A \otimes \mathbb{K} \\ & \searrow^{\simeq} & \downarrow m & \swarrow_{\simeq} & \\ & & A & & \end{array} \tag{2.10}$$

We set $\mathbb{K} \otimes_\top A := \mathbb{K} \otimes A$ and $A \otimes_\top \mathbb{K} := A \otimes \mathbb{K}$, meaning that every element in A is locality independent of the unit element.

Recall also from (1.7) that the tensor algebra of a vector space V is defined as

$$\mathcal{T}(V) := \bigoplus_{n \geq 0} V^{\otimes n},$$

where we keep the conventions $V^{\otimes 0} = \mathbb{K}$, and $V^{\otimes 1} = V$. This motivates the following definition

Definition 4.24. [21, Definition 4.5] The **locality tensor algebra** over a pre-locality vector space $(V, \top)$ is defined as

$$\mathcal{T}_\top(V) := \bigoplus_{n\geq 0} V^{\otimes_\top n}.$$

So far, the locality tensor algebra of a pre-locality vector space is only a vector space. The following proposition lays the foundations to build a locality relation on it by expressing $\mathcal{T}_\top(V)$ as a quotient space. We follow the convention $V^{\times_\top^0} = \{0\}$.

Proposition 4.25. *Given a pre-locality vector space* $(V, \top)$*, then*

$$\mathcal{T}_\top(V) = {}^{\mathbb{K}\left(\bigcup_{k=0}^\infty V^{\times_\top k}\right)}\Big/_{\left(I_{\text{mult}}^\infty(V)\cap\mathbb{K}\left(\bigcup_{k=0}^\infty V^{\times_\top k}\right)\right)}$$

where we have set $I_{\text{mult}}^\infty(V) := \bigoplus_{k=1}^\infty I_{\text{mult},k}(V)$.

Proof. Given a direct sum of vector spaces $V = \bigoplus_{k=0}^\infty V_k$ and I_k a linear subspace of V_k for each k in $\mathbb{Z}_{\geq 0}$, it is a standard result of linear algebra that the quotient space $\mathbb{V} = {}^{(\bigoplus_{k=0}^\infty V_k)}/_{\bigoplus_{k=0}^\infty I_k}$ inherits the structure of a direct sum of vector space $\mathbb{V} = \oplus_{k=0}^\infty \mathbb{V}_k$ with $\mathbb{V}_k = {}^{V_k}/_{I_k}$. Applying this to $V_k := \mathbb{K}(V^{\times_\top k})$ and $I_k := I_{\text{mult,k}} \cap \mathbb{K}(V^{\times_\top k})$ (we write $I_{\text{mult},k}$ instead of $I_{\text{mult}}(\underbrace{V,\ldots,V}_{k-\text{times}})$ to simplify the notation), and noticing that

$$\bigoplus_{k=0}^\infty I_k = \bigoplus_{k=0}^\infty \left(I_{\text{mult,k}} \cap \mathbb{K}(V^{\times_\top k})\right) = \bigoplus_{k=0}^\infty I_{\text{mult,k}} \cap \bigoplus_{k=0}^\infty \mathbb{K}(V^{\times_\top k}) = I_{\text{mult}}^\infty(V) \cap \mathbb{K}\Big(\bigcup_{k=0}^\infty V^{\times_\top k}\Big).$$

then yields the result, where we have set $I_{\text{mult},0} := \{0\}$. □

We proceed to define a locality relation in the locality tensor algebra in a similar manner as it was done for locality tensor products.

Definition 4.26. For a pre-locality vector space $(V, \top)$ we define

- the locality relation $\top_\times$ on $\bigcup_{k=0}^\infty V^{\times_\top k}$ as

  $$(v_1, \ldots, v_n)\top_\times(w_1, \ldots, w_m) \Longleftrightarrow \forall (i, j) \in [n] \times [m],\ v_i \top w_j,$$

  where n and m lie in $\mathbb{Z}_{\geq 1}$. The relation $\top_\times$ linearly extend to $\mathbb{K}(\bigcup_{k=0}^\infty V^{\times_\top k})$ as in (1.23).
- The locality relation $\top_\otimes$ on $\mathcal{T}_\top(V)$ is defined as the quotient relation for the canonical map $\otimes$: $\left(\mathbb{K}(\bigcup_{k=0}^\infty V^{\times_\top k}), \top_\times\right) \to \mathcal{T}_\top(V)$.

Remark 4.27. For any $(V, \top)$ pre-locality vector space, the pair $(\mathcal{T}_\top(V), \top_\otimes)$ is trivially a pre-locality vector space, $\top_\otimes$ is symmetric by construction and $0\top_\otimes 0$.

Proposition 4.28. *The locality relation* $\top_\times$ *(resp.* $\top_\otimes$*) when restricted to* $V^{\times_\top^n} \times V^{\times_\top^n}$ *(resp.* $V^{\otimes_\top^n} \times V^{\otimes_\top^n}$*) is equal to the locality relation* $\top_{\times n}$ *(resp.* $\top_{\otimes n}$*) from Definition 4.9, whenever* $n > 2$.

Proof. The first statement, namely that $\top_\times\big|_{V^{\times_\top^n}\times V^{\times_\top^n}} = \top_{\times n}$, follows directly from (2.2) and Definition 4.26. The second statement follows from the previous one and from the equalities $\otimes(V^{\times_\top^n}) = V^{\otimes_\top^n}$, and $\otimes^{-1}(V^{\otimes_\top^n}) = V^{\times_\top^n}$. □

Remark 4.29. Notice that $(\mathbb{K} = V^{\otimes_\top^0})\top_\otimes\mathcal{T}_\top(V)$. Indeed, since for every $k \in \mathbb{K}$ and every $(v_1, \ldots, v_n) \in V^{\times_\top^n}$, we have that $k\top_\times(v_1, \ldots, v_n)$, then $k\top_\otimes\ (v_1 \otimes \cdots \otimes v_n)$ for every $v_1 \otimes \cdots \otimes v_n \in V^{\otimes_\top^n}$, and the claim follows.

The following proves the associativity of the locality tensor product of spaces. Notice that for $V_1, \ldots, V_m, \ldots, V_n$ subspaces of a pre-locality vector space $(E, \top)$, the tensor products $V_1 \otimes_\top \cdots \otimes_\top V_m$ and $V_{m+1} \otimes_\top \cdots \otimes_\top V_n$ are subspaces of the pre-locality vector space $(\mathcal{T}_\top(E), \top_\otimes)$, and therefore its locality tensor product is well defined.

Proposition 4.30. *Let $V_1, \ldots, V_m, \ldots, V_n$ be subspaces of a pre-locality vector space $(E, \top)$, where $m < n$ are positive integers. Then, there is an isomorphism of pre-locality vector spaces*

$$(V_1 \otimes_\top \cdots \otimes_\top V_m) \otimes_{\top_\otimes} (V_{m+1} \otimes_\top \cdots \otimes_\top V_n) \sim V_1 \otimes_\top \cdots \otimes_\top V_n.$$

Proof. By means of Proposition 2.20, $V_1 \otimes_\top \cdots \otimes_\top V_m$ (resp. $V_{m+1} \otimes_\top \cdots \otimes_\top V_n$, resp. $V_1 \otimes_\top \cdots \otimes_\top V_n$) is the subspace of $V_1 \otimes \cdots \otimes V_m$ (resp. $V_{m+1} \otimes \cdots \otimes V_n$, resp. $V_1 \otimes \cdots \otimes V_n$) generated by all elements of the form $v_1 \otimes \cdots \otimes v_m$ (resp. $v_{m+1} \otimes \cdots \otimes v_n$, resp. $v_1 \otimes \cdots \otimes v_n$) such that $(v_1, \ldots, v_m)$ lies in $V_1 \times_\top \cdots \times_\top V_m$ (resp. $(v_{m+1}, \ldots, v_n)$ lies in $V_{m+1} \times_\top \cdots \times_\top V_n$, resp. $(v_1, \ldots, v_n)$ lies in $V_1 \times_\top \cdots \times_\top V_n$). Moreover $(V_1 \otimes_\top \cdots \otimes_\top V_m) \otimes_{\top_\otimes} (V_{m+1} \otimes_\top \cdots \otimes_\top V_n)$ is the subspace of $(V_1 \otimes \cdots \otimes V_m) \otimes (V_{m+1} \otimes \cdots \otimes V_n)$ generated by all elements of the form $(v_1 \otimes \cdots \otimes v_m) \otimes (v_{m+1} \otimes \cdots \otimes v_n)$ such that $(v_1, \ldots, v_n)$ lies in $V_1 \times_\top \cdots \times_\top V_n$. The isomorphism as vector spaces then follows from the associativity of the usual tensor product (Corollary 1.15). We are left to check that the locality relations are isomorphic. This follows directly from the definition of the localities $\top_\otimes$ in the tensor product as quotient localities (Definition 4.6). Indeed, two elements in either space are locality independent if there is a way of writing them where each element on the first tensor is locality independent to each element on the second tensor. Thus, the isomorphism is of pre-locality vector spaces. □

So far the locality tensor algebra is only a pre-locality vector space. In order to make it a pre-locality algebra, we have to endow it with a partial product. For that purpose, notice that for any pre-locality vector space $(V, \top)$ the locality tensor algebra $\mathcal{T}_\top(V)$ is a subspace of the usual tensor algebra $\mathcal{T}(V)$. Indeed, it follows from $V^{\otimes_\top^n}$ being a subspace of $V^{\otimes n}$ for any n in $\mathbb{Z}_{\geq 0}$.

Proposition 4.31. *[21, Proposition 4.12] Let $(V, \top)$ be a pre-locality vector space. The usual concatenation product $m_\otimes : \mathcal{T}(V) \times \mathcal{T}(V) \to \mathcal{T}(V)$ on the (non-locality) tensor algebra restricts to $\top_\otimes = \mathcal{T}_\top(V) \times_{\top_\otimes} \mathcal{T}_\top(V)$ where it defines a $\top_\times$-bilinear map (see Definition 2.12) and*

$$(\mathcal{T}_\top(V), \top_\otimes, m_\otimes, u)$$

defines a pre-locality algebra, where u is the canonical injection $u : \mathbb{K} \to V^{\otimes_\top^0}$.

Proof.

- Let us first check that the restriction is $\mathcal{T}_\top(V)$-valued, namely that $m_\otimes(\top_\otimes) \subset \mathcal{T}_\top(V)$.
  For $([a], [b]) \in \top_\otimes$, we may assume without loss of generality that $a = (a_1, \ldots, a_m) \in V^{\times_\top m}$, $b = (b_1, \ldots, b_n) \in V^{\times_\top n}$ and $a \top_\times b$. Therefore $a_i \top b_j$ for every i and j implying that $ab := (a_1, \ldots, a_m, b_1, \ldots, b_n) \in V^{\times_\top (m+n)}$ so that $m_\otimes([a], [b]) = [ab]$ lies in $\mathcal{T}_\top(V)$ as expected.
- The $\top_\times$-bilinearity follows from the fact that $m_\otimes$ is a restriction of the usual concatenation product on the tensor algebra which is bilinear, and from Proposition 2.13.
- The associativity of the usual concatenation product is preserved when we restrict to $\top_\otimes$ whenever it is well defined. Therefore $(\mathcal{T}_\top(V), \top_\otimes, m_\otimes, u)$ is indeed a pre-locality algebra.

□

The following Proposition states the relation between (pre-)locality subspaces and their respective locality tensor algebras.

Proposition 4.32. *[21, Proposition 4.13] Let $(W, \top')$ be a (pre-)locality subspace of $(V, \top)$ a (pre-)locality vector space. Then $\mathcal{T}_{\top'}(W)$ is a (pre-)locality subalgebra of $\mathcal{T}_\top(V)$.*

Proof. We prove first that for every $n \in \mathbb{Z}_{\geq 0}$, $W^{\otimes_{\top'}^n}$ is a subspace of $V^{\otimes_\top^n}$. This is trivially true for $n \in \{0, 1\}$. For $n \geq 2$, notice that

$$I_{\text{mult}}(\underbrace{W, \ldots, W}_{n-\text{times}}) = I_{\text{mult}}(\underbrace{V, \ldots, V}_{n-\text{times}}) \cap \mathbb{K}(\underbrace{W \times \cdots \times W}_{n-\text{times}}).$$

Intersecting both sides with $\mathbb{K}(\underbrace{W \times_{\top'} \cdots \times_{\top'} W}_{n-\text{times}})$ it follows that

$$\Big(I_{\text{mult}}(\underbrace{V, \ldots, V}_{n-\text{times}}) \cap \mathbb{K}(\underbrace{W \times \cdots \times W}_{n-\text{times}})\Big) \cap \mathbb{K}(\underbrace{W \times_{\top'} \cdots \times_{\top'} W}_{n-\text{times}}) = I_{\text{mult}}(\underbrace{W, \ldots, W}_{n-\text{times}}) \cap \mathbb{K}(\underbrace{W \times_{\top'} \cdots \times_{\top'} W}_{n-\text{times}}).$$

Moreover, since $\mathbb{K}(W \times_{\top'} \cdots \times_{\top'} W) \subset \mathbb{K}(W \times \cdots \times W)$ and $\mathbb{K}(W \times_{\top'} \cdots \times_{\top'} W) \subset \mathbb{K}(V \times_\top \cdots \times_\top V)$ then

$$\Big(I_{\text{mult}}(\underbrace{V, \ldots, V}_{n-\text{times}}) \cap \mathbb{K}(\underbrace{V \times_\top \cdots \times_\top V}_{n-\text{times}})\Big) \cap \mathbb{K}(\underbrace{W \times_{\top'} \cdots \times_{\top'} W}_{n-\text{times}}) = I_{\text{mult}}(\underbrace{W, \ldots, W}_{n-\text{times}}) \cap \mathbb{K}(\underbrace{W \times_{\top'} \cdots \times_{\top'} W}_{n-\text{times}})$$

and hence, using the identity $\mathbb{K}(W \times_{\top'} \cdots \times_{\top'} W) \cap \mathbb{K}(V \times_\top \cdots \times_\top V) = \mathbb{K}(W \times_{\top'} \cdots \times_{\top'} W)$ we have

$$\begin{aligned} W^{\otimes^n_{\top'}} &= \mathbb{K}(W \times_{\top'} \cdots \times_{\top'} W) \,/\, I_{\text{mult}}(W, \ldots, W) \cap \mathbb{K}(W \times_{\top'} \cdots \times_{\top'} W) \\ &= \mathbb{K}(V \times_\top \cdots \times_\top V) \cap \mathbb{K}(W \times_{\top'} \cdots \times_{\top'} W) \,/\, I_{\text{mult}}(W, \ldots, W) \cap \mathbb{K}(W \times_{\top'} \cdots \times_{\top'} W) \\ &\subset \mathbb{K}(V \times_\top \cdots \times_\top V) \,/\, I_{\text{mult}}(V, \ldots, V) \cap \mathbb{K}(V \times_\top \cdots \times_\top V) \\ &= V^{\otimes^n_\top}, \end{aligned}$$

(since $(A \cap C)/(B \cap C) \subseteq A/B$ for $B \subseteq A$). In particular $\mathcal{T}_{\top'}(W)$ is a subspace of $\mathcal{T}_\top(V)$. We are only left to prove that the injection map $\iota : \mathcal{T}_{\top'}(W) \to \mathcal{T}_\top(V)$ is a locality map. The inclusion $\top' \subset \top$ implies that $\top'_\times \subset \top_\times$ (see Definition 4.26), and therefore

$$\begin{aligned} [w_1]\top'_\otimes[w_2] &\Rightarrow (\exists w'_1 \in [w_1])(\exists w'_2 \in [w_2]) : \; w'_1 \top'_\times w'_2 \\ &\Rightarrow (\exists w'_1 \in [w_1])(\exists w'_2 \in [w_2]) : \; w'_1 \top_\times w'_2 \\ &\Rightarrow [w_1]\top_\otimes[w_2]. \end{aligned}$$

Thus ι is a morphism of (pre-)locality vector spaces. One easily checks that it is moreover a morphism of (pre-)locality algebras which proves the statement of the proposition. □

As promised in the title of this paragraph, we proceed to prove the universal property on the locality tensor algebra of a pre-locality vector space.

Theorem 4.33 (Universal property of locality tensor algebra over a pre-locality vector space)**.** [21, Theorem 4.14] Let $(V, \top)$ be a pre-locality vector space, $(A, \top_A)$ a pre-locality algebra whose product $m_A : (A \times_\top A, \top_{A \times_\top A}) \to (A, \top_A)$ is a locality map, and $f : V \to A$ a locality linear map. There is a unique pre-locality algebra morphism $\phi : \mathcal{T}_\top(V) \to A$ such that the following diagram commutes

$$\begin{array}{ccc} (V, \top) & \xrightarrow{\otimes_\top} & (\mathcal{T}_\top(V), \top_\otimes) \\ & \searrow^{f} & \downarrow{\scriptstyle \phi} \\ & & (A, \top_A) \end{array}$$

where $\otimes_\top : V \to \mathcal{T}_\top(V)$ is the canonical (locality) injection map.

Proof. Let $f : V \to A$ be a locality linear map. We define for every $n \in \mathbb{Z}_{\geq 2}$, the map $f_n : V^{\times_\top n} \to A$ as $f_n(x_1, \ldots, x_n) := m_A^{n-1}(f(x_1), \ldots, f(x_n))$. Notice that since the maps f and m_A are locality maps, it follows that f_n is well defined. Moreover, from the fact that composition of locality maps is again locality (Proposition 2.3), it follows that f_n is also a locality map. Furthermore, since f is linear and m_A is a $\top_\times$-bilinear map, then f_n is $\top_\times$ n-linear, and thus a locality $\top_\times$ n-linear map.

By means of the universal property of the locality tensor product (Theorem 4.14), there are locality linear maps $\phi_n : V^{\otimes_\top n} \to A$ such that $f_n = \phi_n \circ \otimes_{\top n}$ where $\otimes_{\top n}$ is the canonical map from $V^{\times_\top n} \to V^{\otimes_\top n}$. Set $\phi_1 : V^{\otimes^1_\top} \to A$ as the only map such that

$$f = \phi_1 \circ \otimes_{\top 1},$$

where $\otimes_{\top 1} : V \to V^{\otimes^1_\top}$ is the natural isomorphism. Finally set $\phi_0 : V^{\otimes^0_\top} \sim \mathbb{K} \to A$ the only linear map such that $\phi_0(1_{\mathcal{T}_\top(V)}) = 1_A$. We define the map $\phi : \mathcal{T}_\top(V) \to A$ as the direct sum of the maps ϕ_n for $n \geq 0$, and show that it is a locality algebra morphism such that $f = \phi \circ \otimes_\top$. The latter follows from $\otimes_\top(V) = V^{\otimes^1_\top}$ and from the construction of ϕ_1. In order to prove that ϕ is a locality algebra morphism consider $v_1 \otimes \cdots \otimes v_n$ (resp. $w_1 \otimes \cdots \otimes w_m$) an element of $\mathcal{T}_\top(V)$ with $(v_1, \ldots, v_n)$ lying in $V^{\times_\top^n}$ (resp. $(w_1, \ldots, w_m)$ lying in $V^{\times_\top^m}$), and such that $(v_1, \ldots, v_n) \top_\times (w_1, \ldots, w_m)$. Since f and m_A are locality maps, it follows that

$$\phi(v_1 \otimes \cdots \otimes v_n) = f_n(v_1, \ldots, v_n) \top_A f_m(w_1, \ldots, w_m) = \phi(w_1 \otimes \cdots \otimes w_m),$$

and thus ϕ is a locality map. Furthermore

$$\begin{aligned}
\phi(m_\otimes(v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_m)) &= \phi_{n+m}(v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m) \\
&= f_{n+m}(v_1, \ldots, v_n, w_1, \ldots, w_m) \\
&= m_A^{n+m-1}(f(v_1), \ldots, f(v_n), f(w_1), \ldots, f(w_m)) \\
&= m_A(m_A^{n-1}(f(v_1), \ldots, f(v_n)), m_A^{m-1}(f(w_1), \ldots, f(w_m))) \\
&= m_A(\phi_n(v_1 \otimes \cdots \otimes v_n), \phi_m(w_1 \otimes \cdots \otimes w_m)).
\end{aligned}$$

Therefore ϕ is indeed a locality algebra morphism as announced. □

Similarly to the universal property of the locality tensor product, this last statement is equivalent to the universal property of the usual (non-locality) tensor algebra.

Corollary 4.34. The universal property of the locality tensor algebra (Theorem 4.33) is equivalent to the universal property of the usual (non-locality) tensor algebra (Theorem 1.25).

Proof. Theorem 1.25 is a particular case of Theorem 4.33 when the locality relation is the trivial one $\top = V \times V$.

For the converse: Assuming the universal property of the usual tensor algebra and given a locality linear map $f : V \to A$, it is in particular a linear map. Applying the usual universal property, we get an algebra morphism $\phi : \mathcal{T}(V) \to A$ such that $f = \phi \circ \otimes$ where $\otimes$ is the canonical injection of V into $\mathcal{T}(V)$. Recall that $\mathcal{T}_\top(V)$ is a linear subspace of $\mathcal{T}(V)$ which contains V, so $\otimes$ is also the canonical injection of V into $\mathcal{T}_\top(V)$. The fact that the restriction $\phi|_{\mathcal{T}_\top(V)}$ is a locality map follows from the locality of f, from $f = \phi \circ \otimes$, and from the fact that V generates $\mathcal{T}_\top(V)$ as a locality algebra. Thus, the restriction $\phi|_{\mathcal{T}_\top(V)}$ is a locality algebra morphism such that $f = \phi|_{\mathcal{T}_\top(V)} \circ \otimes$. We conclude that the usual universal property implies the locality one. □

4.4 Locality symmetric algebra and its universal property

In this brief paragraph we introduce the construction of the locality symmetric algebra and its universal property on the pre-locality context. For that purpose, we transpose the idea of symmetric and antisymmetric tensors to the (pre-)locality setup. For a pre-locality vector space $(V, \top)$, we denote by $\Xi_\top$ the restriction to the locality tensor algebra $\mathcal{T}_\top(V)$ of the canonical linear action $\Xi : \mathfrak{S}_n \times \mathcal{T}(V) \to \mathcal{T}(V)$ described in (1.10) of the elements in the symmetric group $\mathfrak{S}_n$ in $\mathcal{T}_\top(V)$, i.e.,

$$\mathfrak{S}_n \times V^{\otimes_\top^m} \ni (\sigma, v_1 \otimes \cdots \otimes v_m) \mapsto \Xi(\sigma, v_1 \otimes \cdots \otimes v_m) := \begin{cases} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \tag{2.11}$$

As a direct consequence of the symmetry of the locality relation $\top$, if $v \in \mathcal{T}_\top(V)$ then $\Xi(\sigma, v) \in \mathcal{T}_\top(V)$, and thus the action $\Xi_\top$ stabilises the locality tensor algebra. In particular, for any fixed $\sigma \in \mathfrak{S}_n$ we denote $\Xi_\top^\sigma(v) := \Xi_\top(\sigma, v)$. The following lemma is a consequence of the symmetry of $\top$.

Lemma 4.35. The locality relation $\top_\otimes$ on $\mathcal{T}_\top(V)$ is invariant under the action $\Xi_\top$. This means that for any $\sigma \in \mathfrak{S}_n$, the map $\Xi_\top^\sigma : (V^{\otimes_\top^n}, \top_\otimes\big|_{V^{\otimes_\top^n}}) \to (V^{\otimes_\top^n}, \top_\otimes\big|_{V^{\otimes_\top^n}})$ is a locality linear map. Moreover it is locality independent to the identity map in $\mathcal{T}_\top(V)$.

Proof. By definition of the locality relation $\top_{\times n}$ (Definition 4.9), it is invariant under the canonical action of $\mathfrak{S}_n$ in $\mathbb{K}(V^{\times_\top^n})$. Moreover, for any $\sigma \in \mathfrak{S}_n$, the action of σ in $\mathbb{K}(V^{\times_\top^n})$ is locality independent of the identity map in $\mathbb{K}(V^{\times_\top^n})$. Then, the fact that $\top_\otimes$ is the quotient locality for $\top_\times$ (Definition 4.9) yields the result. □

The decomposition of the usual (non-locality) tensor algebra in symmetric and antisymmetric vectors (Lemma 1.45) has a direct counterpart in the locality setup.

Definition-Proposition 4.36. Let $(V,\top)$ be a pre-locality vector space, and define the space $\mathcal{AT}_\top(V)$ as the ideal of $\mathcal{T}_\top(V)$ generated by all elements of the form

$$v_1\otimes\cdots\otimes v_n - v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)}$$

where $(v_1,\ldots,v_n)\in V^{\times_\top^n}$ and $\sigma\in\mathfrak{S}_n$. Define also $\mathcal{ST}_\top(V)$ the subspace of $\mathcal{T}_\top(V)$ of symmetric tensors. Then

1.
$$\mathcal{ST}_\top(V)=\mathcal{ST}(V)\cap\mathcal{T}_\top(V),\quad\text{and}\quad \mathcal{AT}_\top(V)=\mathcal{AT}(V)\cap\mathcal{T}_\top(V).$$

2.
$$\mathcal{T}_\top(V)=\mathcal{AT}_\top(V)\oplus\mathcal{ST}_\top(V).$$

   Moreover, the projection $\pi_A:\mathcal{T}_\top(V)\to\mathcal{AT}_\top(V)$ parallel to $\mathcal{ST}_\top(V)$ is locality independent of the identity map $\mathrm{Id}_{\mathcal{T}_\top(V)}$.

Proof. The first item follows from the second property of the S_n-modules in Proposition 1.44, since $\mathcal{T}_\top(V)$ is a subspace of $\mathcal{T}(V)$. The direct sum on the second item follows from the decomposition in Lemma 1.45 and from the distributivity of intersections and direct sums. The fact that $\pi_A\top\mathrm{Id}_{\mathcal{T}_\top(V)}$ follows from $\Xi_\top^\sigma\top\mathrm{Id}_{\mathcal{T}_\top(V)}$ in Lemma 4.35. □

For the purpose of building the locality symmetric algebra as a quotient of the locality tensor algebra over a locality ideal, we prove first the pre-locality version of Proposition 1.23.

Lemma 4.37. Let $(A,\top,m)$ be a pre-locality algebra and $I\subset A$ a locality ideal (see Definition 2.26). Then $(A/I,\overline{\top},m')$ is also a pre-locality algebra where $\overline{\top}$ is the quotient locality relation (see Definition 4.6), and m' is the product as expected, namely

$$m'([x],[y])=[m(x,y)]\quad\text{for } x\top y.$$

Proof. We prove only what is particular to the locality context, since the rest of the arguments are the same as in the non-locality case. We see first that m' is well defined. Let $\pi:A\to A/I$ be the canonical quotient map. By the definition of quotient locality it is clear that

$$(\pi\times\pi)(A\times_\top A)=A/I\times_{\overline{\top}}A/I, \tag{2.12}$$

and thus m' is well defined.

We prove that m' is a $\top_\times$-bilinear map. Consider $\overline{\pi\times\pi}:\mathbb{K}(A\times A)\to\mathbb{K}(A/I\times A/I)$ the only linear map which extends $\pi\times\pi$. It is easy to check that $\overline{\pi\times\pi}(I_{\mathrm{bil}}(A,A))=I_{\mathrm{bill}}(A/I,A/I)$. Also from (2.12), it follows that $\overline{\pi\times\pi}(\mathbb{K}(A\times_\top A))=\mathbb{K}(A/I\times_{\overline{\top}}A/I)$. Therefore

$$\overline{m'}\big(I_{\mathrm{bil}}(A/I,A/I)\cap\mathbb{K}(A/I\times_{\overline{\top}}A/I)\big)=\overline{m'}\big(\overline{\pi\times\pi}(I_{\mathrm{bil}}(A,A)\cap\mathbb{K}(A\times_\top A))\big)=\overline{\pi\times\pi}\big(\overline{m}(I_{\mathrm{bil}}(A,A)\cap\mathbb{K}(A\times_\top A))\big)$$

where the last equality follows from the definition of m'. The fact that m is a $\top_\times$-bilinear map yields the result. □

We proceed to build the locality symmetric algebra in a similar manner as the usual (non-locality) symmetric algebra (Definition 1.46).

Definition 4.38. Let $(V,\top)$ be a pre-locality vector space. We define the **locality symmetric algebra** of V as the quotient

$$S_\top(V):={}^{\mathcal{T}_\top(V)}\!/_{\mathcal{AT}_\top(V)}.$$

The locality relation $\top_S$ on $S_\top(V)$ is the quotient locality relation (Definition 4.6) given by the canonical quotient map $\pi_S:\mathcal{T}_\top(V)\to S_\top(V)$, and the product $m_S:S_\top(V)\otimes_\top S_\top(V)\to S_\top(V)$ is defined as the product of the representatives of the equivalent classes whenever it is defined (see Lemma 4.37). With some abuse of notation, we denote by ι_S (as in the non-locality case) the canonical map $\iota_S:=\pi_S\circ\otimes_\top:V\to S_\top(V)$.

Some direct properties of the locality symmetric algebra are summarized in the following proposition.

Proposition 4.39. *Let $(V, \top)$ be a pre-locality vector space. Then*

1. *The locality symmetric algebra is isomorphic as a vector space to $\mathcal{ST}_\top(V)$.*
2. *The locality symmetric algebra $S_\top(V)$ is a subspace of the usual symmetric algebra. Moreover, the product m_S on $S_\top(V)$ is a restriction of the usual product on the usual symmetric algebra to $S_\top(V)$.*
3. *The locality symmmetric algebra is commutative.*
4. *The map $\iota_S : V \to S_\top(V)$ is an injective locality linear map.*

Proof. 1. This follows directly from the second item in Definition-Proposition 4.36.

2. The fact that $S_\top(V) = S(V)$ follows from its definition and the fact that $\mathcal{AT}_\top(V) = \mathcal{AT}(V) \cap \mathcal{T}_\top(V)$. The fact that the product m_S is a restriction of the product on the usual symmetric algebra follows from the fact that the product on the locality tensor algebra $m_\otimes$ is also a restriction of the usual product on the usual tensor algebra (Proposition 4.31).

3. This follows from the last item and the commutativity of $S(V)$.

4. The locality and linearity of $\iota_S : \pi_S \circ \otimes_\top$ follow from the locality and linearity of both $\otimes_\top : V \to \mathcal{T}_\top(V)$ and $\pi_S : \mathcal{T}_\top(V) \to S_\top(V)$. The injectivity follows from the injectivity of $\otimes_\top$ and from the fact that $(\mathcal{AT}_\top(V))_1 = \{0\}$, thus $(S_\top(V))_1 = V^{\otimes_\top^1}/\{0\} \sim V$ and therefore $\pi_S\big|_V$ is injective.

□

By means of the Proposition 4.39 item 4, we identify the elements in V with their images through ι_S. For $(v_1, \ldots, v_n) \in V^{\times_\top^n}$ we use the notation $v_1 \odot \cdots \odot v_n = \pi_S(v_1 \odot \cdots \odot v_n)$ to distinguish an element in $S_\top(V)$ from that on $\mathcal{T}_\top(V)$.

Much like its non-locality counterpart, the symmetric locality algebra satisfies a universal property which we now present.

Lemma 4.40. Let $(A, \top_A, m_A)$ and $(B, \top_B, m_B)$ be two pre-locality algebras, $I \subset A$ a pre-locality ideal and $\psi : A \to B$ a pre-locality algebra morphism such that $\psi(I) = \{0_B\}$. Then the map $\phi : A/I \to B$ defined by $\phi([x]) = \psi(x)$ is also a pre-locality algebra morphism where the pre-locality algebra structure of A/I is the same as in Lemma 4.37.

Proof. The proof is very similar to the non-locality case, namely the existence and uniqueness of ϕ is granted by the fact that $\psi(I) = \{0_B\}$, and the fact that it is multiplicative follows from I being a pre-locality ideal and ψ also being a pre-locality algebra morphism. The locality of the map ϕ follows from the locality of ψ and the definition of quotient locality (Definition 4.6). □

Theorem 4.41 (Universal Property of the locality symmetric algebra over a pre-locality vector space)**.** Let $(V, \top)$ be a pre-locality vector space, $(A, \top_A)$ a commutative pre-locality algebra, and $f : V \to A$ a locality linear map. There is a unique morphism of commutative pre-locality algebras $\phi_f : S_\top(V) \to A$ such that $f = \phi_f \circ \iota_S$, i.e., such that the following diagram commutes:

$$\begin{array}{ccc} (V,\top) & \xrightarrow{\iota_S} & (S_\top(V), \top_S) \\ & \searrow^{f} & \downarrow{\scriptstyle \phi_f} \\ & & (A, \top_A) \end{array}$$

Figure 2.1: Universal property of the locality symmetric tensor algebra.

Proof. By means of the universal property of the locality tensor algebra (Theorem 4.33) there is a unique pre-locality algebra morphism $\psi : \mathcal{T}_\top(V) \to A$ satisfying $f = \psi \circ \otimes_\top$. Since A is commutative, then $\psi(\mathcal{AT}(V)) = \{0_A\}$ and thus, by means of Lemma 4.40, the linear map $\phi : S_\top(V) \to A$ defined by $\phi([x]) = \psi(x)$ is a well defined pre-locality algebra morphism such that $\psi = \phi \circ \pi_S$. Therefore

$$f = \psi \circ \otimes_\top = \phi \circ \pi_S \circ \otimes_\top = \phi \circ \iota_S.$$

The uniqueness of ϕ is granted by the fact that it is completely defined by its values on V, and V generates $S_\top(V)$ as a locality algebra. □

Similar to the tensor algebra, there is an equivalence of the universal properties of the locality and non-locality symmetric algebras.

Corollary 4.42. The universal property of the locality symmetric algebra (Theorem 4.41) is equivalent to the universal property of the usual (non-locality) symmetric algebra (Theorem 1.47).

Proof. The fact that the universal property of the locality symmetric algebra implies the usual one follows from choosing the trivial locality relation $\top = V \times V$.

On the other hand, assuming the universal property of the usual symmetric algebra and given a locality linear map $f : V \to A$, it is in particular a linear map. Applying Theorem 1.47, we get an algebra morphism $\phi : S(V) \to A$ such that $f = \phi \circ \iota_S$ where ι_S is the canonical injection of V into $S(V)$. Since $S_\top(V)$ is a linear subspace of $S(V)$ which contains V, then ι_S is also the canonical injection of V into $S_\top(V)$. The fact that the restriction $\phi|_{S_\top(V)}$ is a locality map follows from the locality of f, from $f = \phi \circ \iota_S$, and from the fact that V generates $S_\top(V)$ as a locality algebra. Thus, the restriction $\phi|_{S_\top(V)}$ is the desired morphism of commutative pre-locality algebras such that $f = \phi|_{S_\top(V)} \circ \iota_S$. We conclude that the usual universal property implies the locality one. □

The following corollary of Theorem 4.41 shows how the locality symmetric algebra is completely determined by the structure of the pre-locality vector space.

Corollary 4.43. Let $(V, \top)$ and $(V', \top')$ be two isomorphic pre-locality vector spaces, then their locality symmetric algebras $S_\top(V)$ and $S_{\top'}(V')$ are isomorphic as pre-locality algebras.

Proof. The proof is very similar to the non-locality case. Let $\psi : V \to V'$ be an isomorphism of pre-locality vector spaces, then $\iota_{V'} \circ \psi : V \to S_{V'}(V')$ is a locality linear map. By means of Theorem 4.41, there is a unique pre-locality algebra morphism $\phi : S_\top(V) \to S_{\top'}(V')$ which makes the following diagram commute.

$$\begin{array}{ccc} (V, \top) & \xrightarrow{\iota_S} & (S_\top(V), \top_S) \\ \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \phi} \\ (V', \top') & \xrightarrow{\iota_{S'}} & (S_{\top'}(V'), \top'_S) \end{array}$$

On the other hand, since ψ is an isomorphism, again Theorem 4.41 applied to the locality linear map $\iota_S \circ \psi^{-1} : V' \to S_\top(V)$ yields the existence of a pre-locality algebra morphism $\phi' : S_{\top'}(V') \to S_\top(V)$ such that $\iota_S \circ \psi^{-1} = \phi' \circ \iota_{S'}$. It follows from the bijectivity of ψ that $\phi'\big|_{\iota_{S'}(V')} = \phi^{-1}\big|_{\iota_{S'}(V')}$, and since $\iota_S(V)$ (resp. $\iota_{S'}(V')$) generates $S_\top(V)$ (resp. $S_{\top'}(V')$) as a pre-locality algebra, then $\phi' = \phi^{-1}$. Moreover, ϕ and ϕ' are locality maps as a consequence of Theorem 4.41, then ϕ is an isomorphism of commutative pre-locality algebras. □

4.5 Pre-locality Lie algebras and their locality universal enveloping algebra

In this paragraph we introduce locality Lie algebras and build their locality universal enveloping algebra as a quotient of the locality tensor algebra. We also prove the universal property of the locality universal enveloping algebra (Theorem 4.33) which happens to be essential for the upcoming Locality Milnor-Moore and Poincaré-Birkhoff-Witt theorems. Finally we prove in Counter-example 4.51 that, contrary to pre-locality vector spaces and $\top_\times$-bilinear maps, not all locality Lie algebras can be extended to a usual

Lie algebra since the extended Lie bracket might not always satisfies the Jacobi identity. We proceed to introduce the concept of (pre-) locality Lie algebra.

Definition 4.44. [21, Definition 4.16]

- A **pre-locality Lie algebra** is a triple $(\mathfrak{g}, \top_{\mathfrak{g}}, [,])$ where $(\mathfrak{g}, \top_{\mathfrak{g}})$ is a pre-locality vector space, and $[,] : (\top_{\mathfrak{g}} = \mathfrak{g} \times_\top \mathfrak{g}) \to \mathfrak{g}$ is a $\top_\times$-bilinear map antisymmetric which satisfies the following properties:
  - For every $U \subset \mathfrak{g}$, the partial Lie bracket stabilises polar sets, i.e. it maps $(U^\top \times U^\top) \cap \top_{\mathfrak{g}}$ into $U^\top$.
  - For $(a, b, c) \in \mathfrak{g}^{\times_\top 3}$ we have that $[[a, b], c] + [[c, a], b] + [[b, c], a] = 0$.
- A **locality Lie algebra** is a pre-locality Lie algebra $(\mathfrak{g}, \top_{\mathfrak{g}}, [,])$ such that $(\mathfrak{g}, \top_{\mathfrak{g}})$ is also a locality vector space.
- Let $(\mathfrak{g}_1, \top_{\mathfrak{g}_1}, [,]_1)$ and $(\mathfrak{g}_2, \top_{\mathfrak{g}_2}, [,]_2)$ be two (resp. pre-)locality Lie algebras. A locality linear map $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ is called a **(resp. pre-)locality Lie algebra morphism** if $f([x, y]_1) = [f(x), f(y)]_2$, for every independent pair $x \top_1 y$.
- Let $(\mathfrak{g}_2, \top_{\mathfrak{g}_2}, [,]_2)$ be a (pre-)locality Lie algebra and $\mathfrak{g}_1 \subseteq \mathfrak{g}_2$. We call $(\mathfrak{g}_1, \top_{\mathfrak{g}_1}, [,]_1)$ a **(pre-) locality Lie subalgebra** of $(\mathfrak{g}_2, \top_{\mathfrak{g}_2}, [,]_2)$ if the inclusion map $\iota : \mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2$ is a (pre-) locality Lie algebra morphism.

Remark 4.45. Notice that a the partial Lie bracket $[,] : (\mathfrak{g} \times_\top \mathfrak{g}, \top_{\mathfrak{g} \times_\top \mathfrak{g}}) \to (\mathfrak{g}, \top_{\mathfrak{g}})$ is a locality map, and thus a locality $\top_\times$-bilinear map. Indeed, for $(g_1, g_2) \top_{\mathfrak{g} \times_\top \mathfrak{g}} (g_1', g_2')$, or equivalently $(g_1, g_2, g_1', g_2') \in \mathfrak{g}^{\times_\top^4}$, since the bracket $[,]$ stabilises polar sets, then $[g_1, g_2] \top_{\mathfrak{g}} [g_1', g_2']$ as expected.

Similar to usual (non-locality) Lie algebras, it is possible to build a universal enveloping algebra from a locality Lie algebra which satisfies a similar property than the one in Theorem 1.52.

Definition 4.46. Let $(\mathfrak{g}, \top_{\mathfrak{g}}, [,])$ be a pre-locality Lie algebra. Consider the pre-locality ideal $J_\top(\mathfrak{g})$ of $\mathcal{T}_\top(\mathfrak{g})$ generated by all terms of the form $a \otimes b - b \otimes a - [a, b]$ for $(a, b) \in \top_{\mathfrak{g}}$. The **locality universal enveloping algebra** of $\mathfrak{g}$ is defined as

$$U_\top(\mathfrak{g}) := \mathcal{T}_\top(\mathfrak{g}) / J_\top(\mathfrak{g}). \tag{2.13}$$

The locality relation $\top_U$ on $U_\top(\mathfrak{g})$ is the quotient locality relation for the quotient map $\pi_U : (\mathcal{T}_\top(\mathfrak{g}), \top_\otimes) \to U_\top(\mathfrak{g})$ (see Definition 4.6), and the product m_U of the equivalence classes is defined by the product of its representatives whenever defined (see Lemma 4.37). Finally, we denote by $\iota_{\mathfrak{g}}$ the canonical map from $\mathfrak{g}$ to $U_\top(\mathfrak{g})$ defined as $\iota_{\mathfrak{g}} := \pi_U \circ \otimes$.

Notice that the map $\iota_{\mathfrak{g}}$ is a locality map since it is the composition of two locality maps (see Proposition 2.3). Also in the usual (non-locality) case, the injectivity of the map $\iota_{\mathfrak{g}}$ follows from the Poincaré-Birkhoff-Witt theorem (see Corollary1.54). However, in the locality setup, we cannot assume the injectivity. We will discuss this fact in more detail in Section 6.6

Similarly to Proposition 4.32, the following proposition compares the universal enveloping algebras of two locality Lie algebras.

Proposition 4.47. *Let $(\mathfrak{g}', \top')$ be a (pre-)locality Lie subalgebra of $(\mathfrak{g}, \top)$ a (pre-)locality Lie algebra. Then $(U_{\top'}(\mathfrak{g}'), \top_U')$ is a (pre-)locality subalgebra of $(U_\top(\mathfrak{g}), \top_U)$.*

Proof. By Proposition 4.32 $\mathcal{T}_{\top'}(\mathfrak{g}') \subset \mathcal{T}_\top(\mathfrak{g})$, and by construction $J_{\top'}(\mathfrak{g}') = J_\top(\mathfrak{g}) \cap \mathcal{T}_{\top'}(\mathfrak{g}')$. It follows that $U_{\top'}(\mathfrak{g}')$ is a subspace of $U_\top(\mathfrak{g})$. Proposition 4.32 also states that $\top_\otimes' \subset \top_\otimes$ and we can show the inclusion $\top_U' \subset \top_U$ in a similar manner. Therefore $U_{\top'}(\mathfrak{g}')$ is (pre-)locality subspace of $U_\top(\mathfrak{g})$. It is straightforward to see that it is moreover a (pre-)locality subalgebra as expected. □

We proceed to state and prove the so announced universal property of the locality universal enveloping algebra.

Theorem 4.48 (Universal property of the locality universal enveloping algebra)**.** Let $(\mathfrak{g}, \top_{\mathfrak{g}}, [,])$ be a pre-locality Lie algebra, $(A, \top_A)$ a pre-locality algebra whose product $m_A : (A \times_\top A, \top_{A\times_\top A}) \to (A, \top_A)$ is a locality map, and $f : \mathfrak{g} \to A$ a pre-locality Lie algebra morphism where the Lie bracket on A is the commutator defined by the product. There is a unique pre-locality algebra morphism $\phi : U_\top(\mathfrak{g}) \to A$ such that $f = \phi \circ \iota_{\mathfrak{g}}$ where $\iota_{\mathfrak{g}}$ is the canonical (locality) map from $\mathfrak{g}$ to $U_\top(\mathfrak{g})$.

$$\begin{array}{ccc} (\mathfrak{g}, \top_{\mathfrak{g}}) & \xrightarrow{\iota_{\mathfrak{g}}} & (U_\top(\mathfrak{g}), \top_U) \\ & \searrow^{f} & \downarrow \phi \\ & & (A, \top_A) \end{array}$$

Proof. By means of the universal property of the locality tensor algebra (Theorem 4.33), since f is a locality linear map and m_A is a locality map, there exists a unique pre-locality algebra morphism $\psi : \mathcal{T}_\top(\mathfrak{g}) \to A$ such that $f = \psi \circ \otimes_\top$ where $\otimes_\top$ is the canonical map from $\mathfrak{g}$ to $\mathcal{T}_\top(\mathfrak{g})$. Since f is a locality Lie algebra morphism, for every $a \top_{\mathfrak{g}} b$

$$\psi(a \otimes b - b \otimes a - [a, b]) = \psi(a \otimes b) - \psi(b \otimes a) - \psi([a, b]) = f(a)f(b) - f(b)f(a) - f([a, b]) = 0,$$

and thus $\psi(J_\top(\mathfrak{g})) = \{0_A\}$. By means of Proposition 1.9, the map $\phi : U_\top(\mathfrak{g}) \to A$ defined as $\phi([x]) := \psi(x)$ for every $x \in \mathcal{T}_\top(\mathfrak{g})$ is the only map which satisfies $\psi = \phi \circ \pi_U$ where π_U is the canonical quotient map from $\mathcal{T}_\top(\mathfrak{g}) \to U_\top(\mathfrak{g})$. From $\iota_{\mathfrak{g}} = \pi_U \circ \otimes_\top$, it follows that

$$f = \psi \circ \otimes_\top = \phi \circ \pi_U \circ \otimes_\top = \phi \circ \iota_{\mathfrak{g}}$$

as expected. Finally ϕ is a pre-locality algebra morphism as a consequence of Lemma 4.40. □

Corollary 4.49. The universal property of the locality universal enveloping algebra $U_\top(\mathfrak{g})$ (Theorem 4.48) implies the universal property of the usual universal enveloping algebra $U(\mathfrak{g})$ (Theorem 1.52).

Proof. Given a Lie algebra $\mathfrak{g}$, an algebra A and a Lie algebra morphism $f : \mathfrak{g} \to A$, it is enough to consider the trivial locality relation $\top = \mathfrak{g} \times \mathfrak{g}$ and this yields the existence and uniqueness of the algebra morphism $\phi : U(\mathfrak{g}) \to A$ required for the universal property of the universal enveloping algebra. □

Remark 4.50. Contrarily to Corollaries 4.16 and 4.34, in Corollary 4.49 only universal property of the locality universal enveloping algebra implies the universal property of the usual universal enveloping algebra. The reason for this is that a locality Lie algebra does not in general extend to a Lie algebra. More precisely, one would need to extend the Lie bracket of a pre-locality Lie algebra to the whole Lie algebra. However this is not always possible as can be seen from Counterexample 4.51 below.

Proposition 4.13 yields a bilinear map which is antisymmetric on $\mathfrak{g} \times_\top \mathfrak{g}$ by construction since $\top_{\mathfrak{g}}$ is symmetric. Outside of the span of the image of the original (non-extended map), the extended map vanishes identically so that the extended map is antisymmetric. However, as the next counter-example shows, it does not in general satisfy Jacobi identity.

Counter-example 4.51. Lie algebras with three generators are known, and there are finitely many (see Mubarakzyanov's Classification [75]). We build an infinite family of distinct locality Lie algebras, which as a consequence, does not correspond to an ordinary Lie algebra equipped with a locality algebra structure.

Take $\mathfrak{g} = \mathbb{R}^3$, (e_1, e_2, e_3) its canonical basis and the locality relation $\top_{\mathfrak{g}}$ defined by the subset of $\mathbb{R}^3 \times \mathbb{R}^3$ obtained by symmetrising the following set

$$\langle e_1 \rangle \times \langle e_1 \rangle \bigcup \langle e_2 \rangle \times \langle e_2 \rangle \bigcup \langle e_3 \rangle \times \langle e_3 \rangle \bigcup \langle e_1, e_2 \rangle \times \langle e_3 \rangle.$$

Let $[\cdot, \cdot] : \top_{\mathfrak{g}} \longrightarrow \mathfrak{g}$ be the bilinear antisymmetric map defined by

$$[e_1, e_3] = \lambda e_1 + \mu e_3, \quad [e_2, e_3] = \mu' e_3 \tag{2.14}$$

with $(\lambda, \mu, \mu') \in (\mathbb{R}^*)^3$. Notice that since $(e_1, e_2) \notin \top_{\mathfrak{g}}$, $[e_1, e_2]$ does not need to be defined. We argue that it *cannot* be defined such that $(\mathfrak{g}, [\cdot, \cdot])$ is a usual (non-locality) Lie algebra. Indeed, let us write $[e_1, e_2] = xe_1 + ye_2 + ze_3$. Then computing $[[e_1, e_2], e_3]$ and its permutations, we find that the Jacobi identity is satisfied if, and only if:

$$-\mu'\lambda e_1 - \lambda y e_2 + (x\mu + y\mu' - \lambda z)e_3 = 0.$$

This equation has no solutions since $\mu'\lambda \neq 0$.

However, it is easy to see that for any $(\lambda, \mu, \mu') \in (\mathbb{R}^*)^3$, $(\mathfrak{g}, \top_{\mathfrak{g}}, [\cdot, \cdot])$ is a locality Lie algebra. This follows from the fact that since $(e_1, e_2) \notin \top_{\mathfrak{g}}$, (e_i, e_j, e_k) cannot lie in $\mathfrak{g}^{\times_\top^3}$ if i, j and k are all different, $[\cdot, \cdot]$ trivially satisfies the locality Jacobi identity.

We conclude this section with a simple corollary of the universal property of the locality universal enveloping algebra (Theorem 4.48) which is the pre-locality counterpart of a classical result.

Corollary 4.52. Let $(\mathfrak{g}, [,]_{\mathfrak{g}}, \top)$ and $(\mathfrak{g}', [,]_{\mathfrak{g}'}, \top')$ be two isomorphic pre-locality Lie algebras, then their locality universal enveloping algebras $U_\top(\mathfrak{g})$ and $U_{\top'}(\mathfrak{g}')$ are isomorphic as pre-locality algebras.

Proof. The proof is very similar to the non-locality case. Let $\psi : \mathfrak{g} \to \mathfrak{g}'$ be an isomorphism of pre-locality Lie algebras, then $\iota_{\mathfrak{g}'} \circ \psi : \mathfrak{g} \to U_{\top'}(\mathfrak{g}')$ is a morphism of pre-locality Lie algebras (where the bracket in $U_{\top'}(\mathfrak{g}')$ is the commutator as usual). By means of Theorem 4.48, there is a unique pre-locality algebra morphism $\phi : U_\top(\mathfrak{g}) \to U_{\top'}(\mathfrak{g}')$ which makes the following diagram commute.

$$\begin{array}{ccc}
(\mathfrak{g}, \top) & \xrightarrow{\iota_{\mathfrak{g}}} & (U_\top(\mathfrak{g}), \top_U) \\
\Big\downarrow \psi & & \Big\downarrow \phi \\
(\mathfrak{g}', \top') & \xrightarrow{\iota_{\mathfrak{g}'}} & (U_{\top'}(\mathfrak{g}'), \top'_U)
\end{array}$$

On the other hand, since ψ is an isomorphism, again Theorem 4.33 applied to the pre-locality Lie algebra morphism $\iota_{\mathfrak{g}} \circ \psi^{-1} : \mathfrak{g}' \to U_\top(\mathfrak{g})$ yields the existence of a pre-locality algebra morphism $\phi' : U_{\top'}(\mathfrak{g}') \to U_\top(\mathfrak{g})$ such that $\iota_{\mathfrak{g}} \circ \psi^{-1} = \phi' \circ \iota_{\mathfrak{g}'}$. It follows from the bijectivity of ψ that $\phi'\big|_{\iota_{\mathfrak{g}'}(\mathfrak{g}')} = \phi^{-1}\big|_{\iota_{\mathfrak{g}'}(\mathfrak{g}')}$, and since $\iota_{\mathfrak{g}}(\mathfrak{g})$ (resp. $\iota_{\mathfrak{g}'}(\mathfrak{g}')$) generates $U_\top(\mathfrak{g})$ (resp. $U_{\top'}(\mathfrak{g}')$) as a pre-locality algebra, then $\phi' = \phi^{-1}$. Moreover, ϕ and ϕ' are locality maps as a consequence of Theorem 4.33, then ϕ is an isomorphism of pre-locality algebras. □

5 Quotient of locality vector spaces

Up to this point, the main objects studied in the past section are only in the pre-locality setting, namely the locality tensor product, locality tensor algebra, locality symmetric algebra and locality universal enveloping algebra. However, it is necessary to enhance those constructions to the locality set up in order to obtain a locality version of the Milnor-Moore and Poincaré-Birkhoff-Witt theorems, more precisely for the coalgebraic construction as it is discussed in Section 6.1. Since all the previously named objects are obtained via quotients, the following natural question arises:

When is the quotient of a locality vector space by a linear subspace, a locality vector space, if equipped with the quotient locality relation of Definition 4.6 ? (2.15)

Given a locality vector space $(V, \top)$ and a subspace W, the question can be reformulated as follows: *When does the following implication*

$$\forall (v_1, v_2.v_3) \in V^3 \wedge \forall w \in W, \quad \big(v_1 \top v_2 \wedge (v_1 + w) \top v_3\big) \Longrightarrow \big(\exists (w', w'') \in W^2,\, (v_1 + w') \top (v_2 + v_3 + w'')\big), \tag{2.16}$$

hold?

This section is devoted to the study and better understanding of Question (2.15). Such study leads to conjectural statements 5.30 and 5.33 as it will be introduced later. Also, some important consequences of those conjectural statements are presented in Paragraph 5.5.

5.1 Examples of locality quotient vector spaces

The answer of Question 2.15 is not absolute, namely there are quotient spaces that, when equipped with the quotient locality, are locality vector spaces, and there are other which are only pre-locality vector spaces. We present in this paragraph examples of both types.

The following proposition provides a first class of examples of locality quotient vector spaces for the quotient locality relation $\overline{\top}$ induced by the orthogonality relation.

Proposition 5.1. *Take V be any Hilbert space with scalar product $\langle , \rangle$, and consider the locality relation $\top$ on V given by the orthogonality relation: $v \top v' :\Longleftrightarrow \langle v, v' \rangle = 0$. Then for any closed linear subspace W of V different from $\{0\}$, the quotient locality $\overline{\top}$ on V/W induced by $\top$ is the complete locality relation: $\overline{\top} = (V/W) \times (V/W)$.*

In particular, for any closed linear subspace W of V, $(V/W, \overline{\top})$ is a locality vector space.

Proof. We want to show that if $W \neq \{0\}$ then for any $(v_1, v_2) \in V^2$, there exists $(w_1, w_2) \in W^2$ such that $\langle v_1 + w_1, v_2 + w_2 \rangle = 0$. Write $v_i = u_i + \tilde{w}_i$ for $i \in \{1, 2\}$ with $\tilde{w}_i \in W$ and $u_i \in W^\perp$. Then:

$$\langle v_1 + w_1, v_2 + w_2 \rangle = 0 \iff \langle v_1, v_2 \rangle + \langle \tilde{w}_1, w_2 \rangle + \langle w_1, \tilde{w}_2 \rangle + \langle w_1, w_2 \rangle = 0, \tag{2.17}$$

and we want to find w_1, w_2 that solve (2.17). Let us consider three different cases:

- If $\tilde{w}_1 \neq 0$, then
$$(w_1, w_2) = \left(0, -\frac{\langle v_1, v_2 \rangle}{||\tilde{w}_1||^2} \tilde{w}_1 \right) \in W^2$$
(with $||w|| := \sqrt{\langle w, w \rangle}$) solves (2.17).
- If $\tilde{w}_1 = 0$ and $\tilde{w}_2 \neq 0$, then we find a solution to (2.17) as in the first item by exchanging the roles of w_1 and w_2.
- If $\tilde{w}_1 = \tilde{w}_2 = 0$, we pick $w \in W \neq \{0\}$ and set
$$(w_1, w_2) = \left(\frac{w}{||w||}, -\frac{\langle v_1, v_2 \rangle}{||w||} w \right) \in W^2$$
which solves (2.17).

□

We proceed to present an example of a locality vector space with a linear subspace whose quotient is not a locality vector space for the locality relation $\overline{\top}$ (see Definition 4.6). It shows that the answer to Question (2.15) cannot be always positive.

Example 5.2. *Consider the vector space $V = M(\mathbb{R})$ of real valued maps on $\mathbb{R}$ together with the locality relation $\top$ given by disjoint supports: $f \top g \Leftrightarrow \mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$. It is easy to check that $(V, \top)$ is a locality vector space. Let W denote the linear subspace of constant functions and consider the functions*

$$x \mapsto v(x) := \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}, \quad x \mapsto u(x) := \begin{cases} 0 & \text{if } x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}, \quad x \mapsto w(x) := \begin{cases} 0 & \text{if } x \leq 2 \\ 1 & \text{if } x > 2 \end{cases}$$

It follows that $[u] \overline{\top} [v]$ since $u \top (v - 1)$, and $[u] \overline{\top} [w]$ since $(u - 1) \top w$. However, $[u] \overline{\not\top} [v + w]$. Thus V/W is not a locality vector space for $\overline{\top}$.

We proceed to give an example which is related to the locality tensor algebra. For this purpose, we first recall a folklore result which provides a justification for the convention $V^{\otimes_\top^1} = V$. We give a constructive proof for it since it will be of use in Proposition 5.5.

Lemma 5.3. Consider the subspace $I_{\mathrm{lin}}(V)$ of $\mathbb{K}(V)$ generated by all elements of the form

$$(a + b) - (a) - (b), \quad \text{and} \quad k(a) - (ka), \tag{2.18}$$

for a and b in V and k in $\mathbb{K}$. Then $V \sim \mathbb{K}(V)/I_{\mathrm{lin}}(V)$ are isomorphic as vector spaces. Moreover, the map $\phi : V \to \mathbb{K}(V)/I_{\mathrm{lin}}(V)$ defined by $\phi(v) = [(v)]$ is an isomorphism.

Remark 5.4. We use the notation (a) with rounded brackets to denote elements of $\mathbb{K}(V)$ and to distinguish them from elements of V. These brackets should not be confused with the squared one $[a]$ used to denote equivalence classes.

Notice that by construction of $I_{\text{lin}}(V)$, the only element a in V such that $(a) \in I_{\text{lin}}(V)$ is $a = 0$.

Proof. Set $\iota : V \to \mathbb{K}(V)$ the canonical injection, and $\pi : \mathbb{K}(V) \to {}^{\mathbb{K}(V)}/_{I_{\text{lin}}(V)}$ the canonical quotient map. We prove that the linear map $\pi \circ \iota : V \to {}^{\mathbb{K}(V)}/_{I_{\text{lin}}(V)}$, which takes $x \mapsto [x]$ is an isomorphism of vector spaces. The injectivity follows from the fact that $\iota(V) \cap I_{\text{lin}}(V) = \{(0_V)\}$, then the kernel of $\pi \circ \iota$ is $\{0_V\}$. For the surjectivity, notice that $\iota(V)$ generates $\mathbb{K}(V)$, and since π is surjective, then $(\pi\circ\iota)(V)$ generates ${}^{\mathbb{K}(V)}/_{I_{\text{lin}}(V)}$. In other words, ${}^{\mathbb{K}(V)}/_{I_{\text{lin}}(V)}$ is the smaller vector space containing $(\pi\circ\iota)(V)$. However, since V is a vector space itself, and $\pi \circ \iota$ is linear, then $(\pi \circ \iota)(V)$ is a vector space, implying that $(\pi \circ \iota)(V) = {}^{\mathbb{K}(V)}/_{I_{\text{lin}}(V)}$, and the result follows. $\square$

If $(V, \top)$ is a locality vector space, $\mathbb{K}(V)$ can be endowed with the locality relation $\top_\times \cap (\mathbb{K}(V) \times \mathbb{K}(V))$ (see Definition 4.26), which induces a quotient locality on ${}^{\mathbb{K}(V)}/_{I_{\text{lin}}(V)}$. The following statement is an enhancement of the isomorphism of Lemma 5.3, to an isomorphism of locality vector spaces.

Proposition 5.5. *Let $(V, \top)$ be a locality vector space, then $(V, \top)$ and $\left({}^{\mathbb{K}(V)}/_{I_{\text{lin}}(V)}, \overline{\top}\right)$ are isomorphic as locality vector spaces:*

$$(V, \top) \simeq \left({}^{\mathbb{K}(V)}/_{I_{\text{lin}}(V)}, \overline{\top}\right), \tag{2.19}$$

where $\overline{\top}$ is the quotient locality (see Definition 4.6).

Proof. We need to show that the isomorphism $\pi \circ \iota : V \to {}^{\mathbb{K}(V)}/_{I_{\text{lin}}(V)}$ in the proof of Lemma 5.3 is a locality map as well as its inverse. For $x \top y$, it follows that $(x) \top_\times (y)$ and thus $[x] \overline{\top} [y]$. This proves that $\pi \circ \iota$ is a locality map. Conversely if $[x] \overline{\top} [y]$, then there exist $\sum_{i\in I} \alpha_i (x_i) \in [x]$ and $\sum_{j \in J} \beta_j (y_j) \in [y]$, for I and J finite sets, such that $\left(\sum_{i\in I} \alpha_i(x_i), \sum_{j\in J} \beta_j(y_j)\right) \in \top_\times$ or equivalently $x_i \top y_j$ for every $(i, j) \in I \times J$. Since $(V, \top)$ is a locality vector space the latter implies that $\left(\sum_{i\in I} \alpha_i x_i, \sum_{j\in J} \beta_j y_j\right) \in \top$. However, by means of Lemma 5.3, $\left[\sum_{i\in I} \alpha_i x_i\right] = [x]$ and $\left[\sum_{j\in J} \beta_j y_j\right] = [y]$ imply that $\sum_{i\in I} \alpha_i x_i = x$ and $\sum_{j\in J} \beta_j y_j = y$. Thus $x \top y$ as announced. $\square$

Corollary 5.6. For $(V, \top)$ a locality vector space, the quotient space ${}^{\mathbb{K}(V)}/_{I_{\text{lin}}(V)}$ endowed with the quotient locality $\overline{\top}$ (see Definition 4.6) is a locality vector space.

Proof. The proof follows from Proposition 5.5 and from the fact that $(V, \top)$ is a locality vector space. $\square$

In the proof of Corollary 5.6, the linear locality property of $(V, \top)$ was used. The following counterexample shows that the isomorphism of vector spaces is not necessarily a locality isomorphism when V is only a pre-locality vector space.

Counter-example 5.7. Let $V = \mathbb{R}^2$ and $\top = \{(e_1, e_2), (e_1, 3e_2), (e_2, e_1), (3e_2, e_1)\}$. Notice that $(3e_2) - (e_2) \top_\times (e_1)$ and thus $([3e_2 - e_2] = [2e_2]) \overline{\top} [e_1]$. However $2\, e_2 \not\top e_1$ proving that in this case $(V, \top)$ and $({}^{\mathbb{K}(V)}/_{I_{\text{lin}}(V)}, \overline{\top})$ are not isomorphic as pre-locality vector spaces.

5.2 Split locality exact sequences

The natural relation between quotient spaces and short exact sequences suggests an approach to tackle Question (2.15). In this paragraph we study such relation in the context of locality and provide a sufficient condition so have a positive answer to Question (2.15). Let us first recall a result from linear algebra which makes precise the aforementioned relation.

Proposition 5.8. *For V_1, V_2 and V three vector spaces, the following statements are equivalent:*

1. *There is a short exact sequence $0 \to V_1 \overset{\iota_1}{\to} V \overset{\pi_2}{\to} V_2 \to 0$ i.e., ι_1 is an injective linear map, π_2 is a surjective linear map, and $\mathrm{Im}(\iota_1) = \mathrm{Ker}(\pi_2)$.*

2. *There is an injective linear map $\iota_1 : V_1 \to V$ such that $V/V_1 \simeq V_2$ are isomorphic as vector spaces.*

3. The are injective linear maps $\iota_i : V_i \to V$ for $i \in \{1, 2\}$ such that V splits as $V = \iota_1(V_1) \oplus \iota_2(V_2)$.

In order to give a locality counterpart of the previous result, we must define the locality counterpart of short exact sequences and split exact sequences first.

Definition 5.9.

- A **locality short exact sequence** (resp. a **pre-locality short exact sequence**) is a sequence $0 \to (V_1, \top_1) \stackrel{\iota_1}{\to} (V, \top) \stackrel{\pi_2}{\to} (V_2, \top_2) \to 0$ such that
  1. $(V_1, \top_1)$, $(V_2, \top_2)$ and $(V, \top)$ are locality vector spaces (resp. pre-locality vector spaces),
  2. ι_1 and π_2 are locality maps,
  3. $0 \to V_1 \stackrel{\iota_1}{\to} V \stackrel{\pi_2}{\to} V_2 \to 0$ is a short exact sequence.
- A locality short exact sequence $0 \to (V_1, \top_1) \stackrel{\iota_1}{\to} (V, \top) \stackrel{\pi_2}{\to} (V_2, \top_2) \to 0$ is a **split locality exact sequence** if there exists a locality linear map $\iota_2 : V_2 \to V$ which is a right inverse for π_2, i.e., $\pi_2 \circ \iota_2 = \mathrm{Id}_{V_2}$, and such that $V = \iota_1(V_1) \oplus \iota_2(V_2)$.

Notice that items 2. and 3. of the definition of locality short exact sequence imply that ι_1 and π_2 are locality linear maps. The following statement is the locality counterpart of items 1. and 2. of Proposition 5.8.

Lemma 5.10. Let $(V_1, \top_1), (V_2, \top_2), (V, \top)$ be three locality (resp. pre-locality) linear spaces. The following properties are equivalent:

1. There is a short locality (resp. pre-locality) exact sequence $0 \to (V_1, \top_1) \stackrel{\iota_1}{\to} (V, \top) \stackrel{\pi_2}{\to} (V_2, \top_2) \to 0$, where $\top_2$ is the final locality relation for the map π_2 (see Definition 4.3).

2. There is an injective locality morphism $\iota_1 : V_1 \to V$ such that the canonical isomorphism $(V/\iota_1(V_1)) \stackrel{\phi}{\to} V_2$ is a locality (resp. pre-locality) isomorphism for the locality relations $\overline{\top}$ and $\top_2$.

Proof. By means of Proposition 5.8, the map π_2 in item 1. induces a unique isomorphism of vector spaces $\phi : (V/\iota_1(V_1)) \to V_2$. Conversely, an isomorphism $\phi : (V/\iota_1(V_1)) \to V_2$ as in item 2. induces a unique surjective linear map $\pi_2 : V \to V_2$. The relation between these two maps is represented in the following commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{h} & V/\iota_1(V_1) \\ & \searrow^{\pi_2} & \downarrow{\scriptstyle \phi} \\ & & V_2 \end{array}$$

where h denotes the quotient map.

- 1$\Rightarrow$ 2: We are only left to check the locality of the maps ϕ and ϕ^{-1}. If $[v]\overline{\top}[v']$, there is some $w \in [v]$ and some $w' \in [v']$ such that $w \top w'$. The locality of π_2 then implies that $\pi_2(w) \top_2 \pi_2(w')$ so that $\phi([v]) \top_2 \phi([v'])$ which proves the locality of ϕ.

  The inverse map ϕ^{-1} is defined by $\phi^{-1}(v) = [w]$ for any $w \in \pi_2^{-1}(v)$. Since $\top_2$ is the final locality relation for π_2, $v_1 \top_2 v_2$ implies that there are $w_1 \in \pi_2^{-1}(\{v_1\})$ and $w_2 \in \pi_2^{-1}(\{v_2\})$ such that $w_1 \top w_2$. By definition of the locality $\overline{\top}$ on the quotient space, $[w_1]\overline{\top}[w_2]$ or equivalently $\phi^{-1}(v_1)\overline{\top}\phi^{-1}(v_2)$.

- 2$\Rightarrow$ 1: We only have to show that π_2 is a locality map, and that $\top_2$ is the final locality relation for π_2. We show the locality of π_2. Recall that the surjective map $\pi_2 : V \to V_2$ is given by $\pi_2(v) := \phi([v])$. From $v \top v'$, it follows that $[v]\overline{\top}[v']$. The locality of ϕ implies that $\pi_2(v) = \phi([v]) \top_2 \pi_2(v') = \phi([v'])$, and thus π_2 is a locality map. We prove that $\top_2$ is the final locality for π_2. Let $\top_2'$ be another locality relation on V_2 such that $\pi_2 : (V, \top) \to (V_2, \top_2')$ is a locality map, and consider two elements v and v' in V_2 such that $v \top_2 v'$. Since ϕ^{-1} is a locality map $\phi^{-1}(v)\overline{\top}\phi^{-1}(v')$. By definition of $\overline{\top}$, there are elements $w \in \phi^{-1}(v)$ and $w' \in \phi^{-1}(v')$ such that $w \top w'$. Notice that in particular $\pi_2(w) = v$ and $\pi_2(w') = v'$. Since $\top_2'$ makes π_2 a locality map, then $v \top_2' v'$ which implies that $\top_2 \subset \top_2'$ as expected.

□

From the preceding lemma, a simple corollary follows which allows us to rephrase Question (2.15) in terms of locality exact sequences.

Corollary 5.11. Given two locality vector spaces $(V,\top)$ and $(V_1,\top_1)$, and an injective locality linear map $\iota_1 : V_1 \longrightarrow V$, The quotient space $(V/\iota_1(V_1),\overline{\top})$ is a locality vector space if, and only if, there is a locality vector space $(V_2,\top_2)$ and a map $\pi_2 : V \longrightarrow V_2$ such that $\top_2$ is the final locality relation for π_2, and $0 \to (V_1,\top_1) \overset{\iota_1}{\to} (V,\top) \overset{\pi_2}{\to} (V_2,\top_2) \to 0$ is a locality short exact sequence.

Proof. We prove both implications

- Assuming that there exists such locality vector space $(V_2,\top_2)$, by Lemma 5.10, there is an isomorphism of locality vector spaces $(V/\iota_1(V_1),\overline{\top}) \simeq (V_2,\top_2)$. It follows that $(V/\iota_1(V_1),\overline{\top})$ is a locality vector space.
- On the other hand, if $(V/\iota_1(V_1),\overline{\top})$ is a locality vector space, we set $(V_2,\top_2) := (V/\iota_1(V_1),\overline{\top})$ and $\pi_2(v) := [v]$. By construction $0 \to (V_1,\top_1) \overset{\iota_1}{\to} (V,\top) \overset{\pi_2}{\to} (V_2,\top_2) \to 0$ is a locality short exact sequence, and by definition of the quotient locality $\overline{\top}$, $\top_2$ is the final locality relation for π_2.

□

This corollary implies that Question (2.15) can be rephrased as follows:

Given a locality vector space $(V,\top)$ *and* V_1 *a subspace of* V*, when is the pre-locality short exact sequence* $0 \to (V_1,\top|_{V_1\times V_1}) \overset{\iota_1}{\to} (V,\top) \overset{\pi_2}{\to} ({}^{V}\!/_{V_1},\overline{\top}) \to 0$ *a locality short exact sequence?*

However, this new formulation does not provide any new hint for an answer. Notice however, that we have not yet developed the locality counterpart of item 3. from Proposition 5.8. This is nonetheless an implication and not an equivalence as in the non-locality case.

Proposition 5.12. *Let* $0 \to (V_1,\top_1) \overset{\iota_1}{\to} (V,\top) \overset{\pi_2}{\to} (V_2,\top_2) \to 0$ *be a split locality exact sequence, then* $\top_2$ *is the final locality relation for the map* π_2*.*

Proof. Let $\top'$ be another locality relation in V_2 such that the map $\pi_2 : (V,\top) \to (V_2,\top')$ is a locality map and consider x and y in V_2. The statement follows form the subsequent implications:

$$\begin{aligned} x\top_2 y &\Rightarrow \iota_1(x)\top\iota_1(y) \\ &\Rightarrow (\pi_2\circ\iota_1)(x)\top'(\pi_2\circ\iota_1)(y) \\ &\Leftrightarrow x\top' y. \end{aligned}$$

□

This proposition leads us to a first sufficient condition to answer question (2.15) positively.

Definition 5.13. Let W_1 be a subspace of a locality vector space $(V,\top)$. We say that W_1 admits a **locality complement** with respect to $(V,\top)$ if there is another subspace W_2 such that $V = W_1 \oplus W_2$ and the projection $\pi_2 : V \to W_2$ parallel to W_1 is a locality map and hence a locality morphism.

In such case we say that W_2 is a locality complement of W_1.

Example 5.14. *Let* $V = \mathbb{R}^2$ *and consider the locality relation* $\top = \mathbb{R}^2\times\{0\}\cup\{0\}\times\mathbb{R}^2\cup\langle e_1\rangle\times\langle e_1\rangle\cup\langle e_2\rangle\times\langle e_2\rangle$*. The subspace* $W_1 = \langle e_1+e_2\rangle$ *admits a locality complement, namely* $W_2 = \langle e_1\rangle$ *(also* $W_2' = \langle e_2\rangle$*). Indeed, the projection* $\pi_2 : \mathbb{R}^2 \to W_2$ *parallel to* W_1 *is a locality map since, in particular,* $T\big|_{W_2} = W_2\times W_2$*.*

Notice however that W_1 *is not a locality complement of* W_2 *since the projection* $\pi_1 : \mathbb{R}^2 \to W_1$ *parallel to* W_2 *is not a locality map.*

Corollary 5.15. Let W be a subspace of a locality vector space $(V,\top)$. If W admits a locality complement with respect to $(V,\top)$, then $(V/W,\overline{\top})$ is a locality vector space.

Proof. Let W_2 be a locality complement of W, then $0 \to (W,\top\big|_W) \overset{\iota}{\to} (V,\top) \overset{\pi_2}{\to} (W_2,\top\big|_{W_2}) \to 0$ is a split locality exact sequence where ι is the canonical injection and $\pi : V \to W_2$ is the projection parallel to W. The result follows from Proposition 5.12 and from Corollary 5.11. □

The previous corollary gives a sufficient but not necessary condition to answer Question (2.15) positively as the following example illustrates.

Example 5.16. *Let $(V, \langle , \rangle)$ be a Hilbert space and consider the locality relation given by the orthogonality relation. It was shown in Example 5.1 that for any closed subspace W_1 of V different from $\{0_V\}$, the quotient $(V/W_1, \overline{\top})$ is a locality vector space. However, if $W_1 \neq V$, it does not admit a locality complement. Indeed, let W_2 be another subspace of V such that $V = W_1 \oplus W_2$, and consider the projection $\pi_2 : V \to W_2$ parallel to W_1. Consider also w_1 (resp. w_2) a non zero element of W_1 (resp. W_2), and define the elements $x_1 := w_2 + \frac{||w_2||}{||w_1||} w_1$ and $x_2 := w_2 - \frac{||w_2||}{||w_1||} w_1$. Then x_1 and x_2 are locality independent since*

$$\langle x_1, x_2 \rangle = ||w_2||^2 - \left(\frac{||w_2||}{||w_1||} \right)^2 ||w_1||^2 = 0$$

but their projections are not:

$$\langle \pi_2(x_1), \pi_2(x_2) \rangle = \langle w_2, w_2 \rangle = ||w_2||^2 \neq 0.$$

Therefore W_2 is not a locality complement of W_1. Since W_2 is arbitrary, then W_1 does not admit a locality complement.

Notice that, contrary to the non locality case, the property of "being a locality complement of" is not symmetric as shown in Example 5.14. To obtain symmetry, we must ask for a slightly stronger condition.

We proceed to define the concept of split locality exact sequences, which indeed provides a first sufficient condition over the spaces $(V, \top)$ and V_1 in order to give a positive answer to Question (2.15).

Definition 5.17. Let $(V, \top)$ be a locality vector space, W_1 a subspace of it, and W_2 a locality complement of W_1. We say that W_2 is a **strong locality complement** of W_1 with respect to $(V, \top)$ if, and only if, the projection $\pi_2 : W \to W_2$ parallel to W_1 is locality independent to the identity map on V, i.e., $\pi_2 \top \mathrm{Id}_V$. Here again the locality relation in W_2 is the subset locality $\top\big|_{W_2}$ as defined in (1.12).

If W_1 is a strong locality complement of W_2 with respect to V, we write

$$V = W_1 \oplus_\top W_2.$$

This notation is justified by the following proposition since the property of being a strong locality complement is symmetric.

Proposition 5.18. *Let $(V, \top)$ be a locality vector space, and W_1 and W_2 two subspaces of it such that $V = W_1 \oplus W_2$. The following properties are equivalent*

1. *$\pi_1 \top \pi_1$ and $\pi_1 \top \pi_2$;*
2. *$\pi_1 \top \mathrm{Id}_V$ (W_1 is a strong locality complement of W_2);*
3. *$\pi_2 \top \pi_2$ and $\pi_1 \top \pi_2$;*
4. *$\pi_2 \top \mathrm{Id}_V$ (W_2 is a strong locality complement of W_1);*

where π_1 (resp. π_2) is the projection onto W_1 (resp. onto W_2) parallel to W_2 (resp. parallel to W_1).

Proof. We prove first two implications which will be useful: If φ, φ_1, and φ_2 are linear maps from V to itself, then

$$(\varphi \top \varphi_1 \,\text{and}\, \varphi \top \varphi_2) \Longrightarrow (\varphi \top (\varphi_1 \pm \varphi_2)), \quad \text{and} \tag{2.20}$$

$$\varphi \top \mathrm{Id}_V \Longrightarrow \varphi \top \varphi. \tag{2.21}$$

Indeed, for $x, y \in V$, by assumption $x \top y \Rightarrow \varphi(x) \top \varphi_i(y)$ for $i = 1, 2$ and the locality of the vector space $(V, \top)$ implies that $\varphi(x) \top \varphi_1(y) \pm \varphi_2(y)$, which shows (2.20). Furthermore, if $\varphi \top \mathrm{Id}_V$ then $x \top y \Rightarrow \varphi(x) \top y$ which, using the symmetry of $\top$ implies that $\varphi(y) \top \mathrm{Id}_V(\varphi(x))$ which is equivalent to $\varphi(x) \top \varphi(y)$ proving (2.21).

- 1) $\Rightarrow$ 2) follows from (2.20) applied to $\varphi = \pi_1$ and $\varphi_1 = \pi_2, \varphi_2 = \pi_1$, which implies $\pi_1 \top \mathrm{Id}_V$ since $\mathrm{Id}_V = \pi_1 + \pi_2$.
- 2) $\Rightarrow$ 1) : Follows from (2.21) applied to $\varphi = \pi_1$ followed by (2.20) applied to $\varphi = \pi_1$ and $\varphi_1 = \mathrm{Id}_V$, $\varphi_2 = \pi_1$ using the fact that $\pi_2 = \mathrm{Id}_V - \pi_1$.
- 3) $\Leftrightarrow$ 4) : Analogous to the last two points exchanging the roles of π_1 and π_2.
- 2) $\Rightarrow$ 4) : Since $\mathrm{Id}_V \top \mathrm{Id}_V$, it follows from (2.20) applied to $\varphi = \mathrm{Id}_V$, $\varphi_1 = \mathrm{Id}_V$, and $\varphi_2 = \pi_1$.
- 4) $\Rightarrow$ 2) : Analogous to the last point exchanging the roles of π_1 and π_2.

□

Notice that any two of the four points of Proposition 5.18 imply that

$$\pi_1 \top \pi_1 \ \wedge \ \pi_2 \top \pi_2.$$

However, $\pi_1 \top \pi_1 \ \wedge \ \pi_2 \top \pi_2$ is not the fifth point of Proposition 5.18 since it does not imply any of the four points of the aforementioned proposition as the following enhancement of Example 5.14 shows.

Counter-example 5.19. Let $V = \mathbb{R}^2$ and consider the locality relation $\top$ on V defined by

$$x \top y \Leftrightarrow (\exists \lambda \in \mathbb{R}) : x = \lambda y.$$

Set $W_1 = \langle e_1 \rangle$ and $W_2 = \langle e_2 \rangle$, then the projection $\pi_1 : V \to W_1$ (resp. $\pi_2 : V \to W_2$) parallel to W_2 (resp. W_1) are indeed locality maps. However none of them are independent of the identity map on V since $(e_1 + e_2) \top (e_1 + e_2)$ but $\pi_i(e_1 + e_2) \not\top (e_1 + e_2)$ for $i \in \{1, 2\}$.

The following proposition is related to the degree 1 component in the locality tensor algebra of a locality vector space. It provides an example of a subspace of a vector space freely generated by a locality basis, which admits a strong locality complement.

Proposition 5.20. *Let $(V, \top)$ be a locality vector space, then $I_{\mathrm{lin}}(V) \subseteq \mathbb{K}(V)$ (see Lemma 5.3) admits a strong locality complement with respect to $(\mathbb{K}(V), \top_\times)$ (see Definition 4.26).*

Proof. We prove that the subspace $\iota(V) \subset \mathbb{K}(V)$ is a strong locality complement of $I_{\mathrm{lin}}(V)$, where $\iota : V \to \mathbb{K}(V)$ is the canonical inclusion. Since Lemma 5.3 implies that $\mathbb{K}(V) = \iota(V) \oplus I_{\mathrm{lin}}(V)$, we are only left to prove that the projection $\pi : \mathbb{K}(V) \to \iota(V)$ parallel to $I_{\mathrm{lin}}(V)$ is locality independent to the identity map on $\mathbb{K}(V)$. Let $x = \sum_{i \in I} \alpha_i (x_i)$, and $y = \sum_{j \in J} \beta_j (y_j)$ be elements of $\mathbb{K}(V)$ (where, as before, we use brackets to distinguish elements of V from elements of $\mathbb{K}(V)$) with $x \top_\times y$, i.e., $x_i \top y_j$ for every $(i, j) \in I \times J$. By means of Lemma 5.3 there is $x_V \in V$ such that $[(x_V)] = [x] = \sum_{i \in I} \alpha_i [(x_i)]$ (or equivalently $(x_V) = \pi(x)$) where moreover $x_V = \sum_{i \in I} \alpha_i x_i$. Since $(V, \top)$ is a locality vector space and for every $j \in J$ we have that $x_i \top y_j$ for every $i \in I$, we may conclude that $x_V \top y_j$ for every $j \in J$. It follows from definition of $\top_\times$ that $(x_V) \top_\times y$ or equivalently $\pi(x) \top \mathrm{Id}_{\mathbb{K}(V)}(y)$, which concludes the proof. □

5.3 Locality compatibility

Despite of being rather natural to look into split exact sequences in order to get an answer for Question (2.15), the condition obtained in the last paragraph, namely for a subspace to have a strong locality complement, is very strong as Counter-example 5.24 illustrates. In this paragraph we present a weaker sufficient condition to answer Question (2.15) positively which, even though doesn't seem very natural at first sight, it is more computational friendly than the condition given in the previous paragraph.

Definition 5.21. Let $(V, \top)$ be a locality vector space and $W \subset V$ a linear subspace. We say that W **is locality compatible with** $\top$ if $\forall (x, y, z) \in V^3, \ \forall w \in W$,

$$x \top y \wedge (x + w) \top z \Longrightarrow (\exists w' \in W) : (x + w') \top y \wedge (x + w') \top z. \tag{2.22}$$

As we will prove later (Theorem 5.26), if a subspace W is locality compatible with $\top$ then the quotient V/W is a locality vector space. Let us first illustrate the previous definition with an example.

Example 5.22. *Let $(V,\top)$ be a locality vector space. Then V and $\{0\}$ are locality compatible with $\top$. Indeed, to see that V is locality compatible with $\top$ it is enough to consider $w' = -x$ using the notations of Definition 5.21. To see that $\{0\}$ is locality compatible with $\top$, it is enough to notice that $w = w' = 0$ and thus $x\top y$ and $x\top z$.*

Proposition 5.23. *If a subspace W of $(V,\top)$ admits a strong locality complement, then it is locality compatible with $\top$.*

Proof. Let W_2 be a strong locality complement of W and let π_2 be the projection onto W_2 parallel to W. With the notations of (2.22), since π_2 is locality independent of Id_V,

$$x\top y \wedge (x+w)\top z \Longrightarrow \pi_2(x)\top y \wedge \pi_2(x)\top z.$$

Setting $-w' := x - \pi_2(x)$ which lies in W, we have $(x + w' = \pi_2(x))\top y$ and $(x + w' = \pi_2(x))\top z$ as expected. $\square$

The converse of Proposition 5.23 is not true in general. The following counterexample shows that for a locality vector space $(V,\top)$ and a subspace W, the condition of W having a strong locality complement is stronger than the condition of W being locality compatible with $\top$.

Counter-example 5.24. Consider the vector space $\mathbb{R}^7$, its subspace $W = \langle\{e_1, e_2, e_3\}\rangle$ where $\{e_i\}_{i=1}^7$ are the elements of the canonical basis, and

$$\begin{aligned}\top =& \mathbb{R}^7 \times \{0\} \bigcup \langle e_1 + e_7\rangle \times \langle\{e_4, e_5\}\rangle \bigcup \langle e_2 + e_7\rangle \times \langle\{e_5, e_6\}\rangle \bigcup \langle e_3 + e_7\rangle \times \langle\{e_4, e_6\}\rangle \\ &\bigcup \langle\{e_1 + e_7, e_3 + e_7\}\rangle \times \langle e_4\rangle \bigcup \langle\{e_1 + e_7, e_2 + e_7\}\rangle \times \langle e_5\rangle \bigcup \langle\{e_2 + e_7, e_3 + e_7\}\rangle \times \langle e_6\rangle \bigcup \text{Sym. terms.}\end{aligned}$$

Notice that $\top$ is invariant under the natural action of the subgroup $\Omega := \langle\sigma_1, \sigma_2\rangle$ of the symmetric group $\mathfrak{S}_7$ generated by $\sigma_1 := (1,2)(4,6)$ and $\sigma_2 := (1,3)(5,6)$, where (i,j) stands for the transposition of i and j. Ω has six elements:

$$\Omega = \{\sigma_1 := (1,2)(4,6),\ \sigma_2 := (1,3)(5,6),\ \sigma_3 = (2,3)(4,5),\ \sigma_4 = (1,2,3)(4,5,6),\ \sigma_5 = (3,2,1)(6,5,4),\ \mathrm{Id}_7\}.$$

This follows from the relations $\sigma_3 = \sigma_1 \circ \sigma_2 \circ \sigma_1$, $\sigma_4 = \sigma_1 \circ \sigma_2$, $\sigma_5 = \sigma_2 \circ \sigma_1$, $\sigma_5^2 = \sigma_4$ and $\sigma_4^2 = \sigma_5$, combined with the involutivity of the transpositions.

One can see that W is locality compatible with $\top$ by checking all cases. For instance, if $k \in \mathbb{R}$, $k(e_1 + e_7)\top e_4$ and $k(e_1 + e_7) + k(e_2 - e_1) = k(e_2 + e_7)\top e_6$, there is $k(e_3 + e_7) = k(e_1 + e_7) + k(e_3 - e_1)$ such that $k(e_3 + e_7)\top e_4$ and $k(e_3 + e_7)\top e_6$. In terms of Equation (2.22), $x = k(e_1 + e_7)$, $w = k(e_2 - e_1)$, $y = e_4$, $z = e_6$, and $w' = k(e_3 - e_1)$. Another possible case is when $x = k(e_1 + e_7) + q(e_3 + e_7)$ for $k, q \in \mathbb{K}$, then $x\top e_4$. If $w = q(-e_3 + e_2)$, then $x + w = k(e_1 + e_7) + q(e_2 + e_7)\top e_5$. In this case $w' = q(-e_3 + e_1)$ makes $x + w' = (k + q)(e_1 + e_7)$ locality independent to both e_4 and e_5. All other possible cases are analogous, in the sense that they are obtained from the previous two via the action of a permutation in Ω on the subindices of the e_i's.

We show that W has no strong locality complement by proving that there is no projection $\pi : \mathbb{R}^7 \to W$ such that $\pi\top\mathrm{Id}_{\mathbb{R}^7}$. Indeed, if there were such projection, then $\pi(e_1 + e_7)\top e_4$, but $\pi(e_1 + e_7) = e_1 + \pi(e_7)$ where $\pi(e_7) \in W$. From the construction of $\top$, the only option is $\pi(e_7) = -e_1$. On the other hand, $(e_3 + e_7)\top e_4$ but $\pi(e_3 + e_7) = e_3 - e_1$ is not locality independent to e_4 which yields the contradiction.

Remark 5.25. Notice that the proof of Proposition 5.23 relies on the fact that the locality complement is strong. The subspace W_1 from Counter-example 5.19 is an example of a subspace admitting a locality complement without being locality compatible with the locality relation.

As a result of the above constructions we get that the quotient by a locality compatible subspace yields a locality quotient. Moreover, the quotient of a locality algebra by a locality ideal is again a locality algebra if the ideal is locality compatible. On these two facts will rely most of the forthcoming constructions.

Theorem 5.26.

1. Let $(V, \top)$ be a locality vector space and $W \subset V$ a subspace locality compatible with $\top$, then $(V/W, \overline{\top})$ is a locality vector space, where $\overline{\top}$ is the quotient locality (see Definition 4.6).

2. Let $(A, \top_A, m)$ be a non-unital locality algebra and $I \subset A$ a locality ideal of A which is locality compatible with $\top_A$. Then $(A/I, \overline{\top}, \bar{m})$ is a non-unital locality algebra where $\overline{\top}$ is the quotient locality.

Proof. We prove the first item: Given any subset U of V/W, we want to prove that $U^{\overline{\top}}$ is a linear subspace. For any $[x] \in U^{\overline{\top}}$, for every $[u] \in U$, there is an element u' of $[u]$ and an element x' of $[x]$, such that $(u', x') \in \top$. The fact that V is a linear vector space implies that for $\lambda \in \mathbb{K}$, $u' \top \lambda x'$, therefore $\lambda[x] \in U^{\overline{\top}}$.

On the other hand, for $[x]$ and $[y]$ equivalence classes in $U^{\overline{\top}}$ and for every $[u] \in U$, there are $u' \in [u]$ and $w \in W$ such that $x' \top u'$ and $y' \top u' + w$ for some $x' \in [x]$ and some $y' \in [y]$. Since W is locality compatible with $\top$, there is $w' \in W$ such that $x' \top u' + w'$ and $y' \top u' + w'$. Since $(V, \top)$ is a locality vector space, it follows that $x' + y' \top u' + w'$ and therefore $[x] + [y] \in U^{\overline{\top}}$. Hence V/W is a locality vector space.

We prove the second item which is similar in spirit as the last one: By means of item 1. $(A/I, \overline{\top})$ has the structure of a locality vector space. Analogous to the usual (non locality) set up, the induced product $\bar{m} : A/I \times_{\overline{\top}} A/I \to A/I$ is an associative $\top_\times$−bilinear map. We are left to prove that for any $U \subset A/I$, $\bar{m}(U^{\overline{\top}} \times_{\overline{\top}} U^{\overline{\top}}) \subset U^{\overline{\top}}$. Given $[x] \overline{\top} [y]$ such that both $[x]$ and $[y]$ are in $U^{\overline{\top}}$, consider also $[u] \in U$. Then there are $x' \in [x]$, $y' \in [y]$, $u' \in [u]$, and $(w, w_1, w_2) \in I^3$ such that

$$x' \top_A y', \tag{2.23}$$

$$(x' + w_1) \top_A u', \tag{2.24}$$

$$(y' + w_2) \top_A (u' + w). \tag{2.25}$$

Since I is locality compatible with $\top_A$, from (2.23) and (2.24) we conclude that there is $w_1' \in I$ such that

$$(x' + w_1') \top_A y', \quad \text{and} \tag{2.26}$$

$$(x' + w_1') \top_A u'. \tag{2.27}$$

From (2.25) and (2.26), there is $w_2' \in I$ such that

$$(y' + w_2') \top_A (x' + w_1'), \quad \text{and} \tag{2.28}$$

$$(y' + w_2') \top_A (u' + w). \tag{2.29}$$

And finally from (2.27) and (2.29), there is $w' \in I$ such that

$$(x' + w_1') \top_A (u' + w'), \quad \text{and} \tag{2.30}$$

$$(y' + w_2') \top_A (u' + w'). \tag{2.31}$$

By (2.28) $m(x' + w_1', y' + w_2')$ is well defined and the fact that A is a locality algebra together with (2.30) and (2.31) imply that $m(x' + w_1', y' + w_2') \top_A (u' + w')$. Hence $\bar{m}([x], [y]) \in U^{\overline{\top}}$. □

The following counter-example shows that locality compatibility is not necessary to have a local quotient space (compare with Example 5.1).

Counter-example 5.27. Take V be any Hilbert space with scalar product $\langle , \rangle$. We equip V with the locality relation $\top$ given by the orthogonality relation: $v \top v' :\Longleftrightarrow \langle v, v' \rangle = 0$. Then a closed linear subspace $\{0\} \subsetneq W \subsetneq V$ is not locality compatible with $\top$.

Proof. Since $W \neq V$ and $W \neq \{0\}$, we can choose vectors $w^\perp \in W^\perp \setminus \{0\}$, $w \in W \setminus \{0\}$ and we set

$$x := \frac{w^\perp}{||w^\perp||}, \quad y := \frac{w}{||w||} =: w_0 \in W, \quad z := \frac{w}{||w||} - \frac{w^\perp}{||w^\perp||}.$$

We have $\langle x, y\rangle = \langle x + w_0, z\rangle = 0$. However there is no $w_1 \in W$ such that $\langle x + w_1, y\rangle = \langle x + w_1, z\rangle = 0$. Indeed $\langle x + w_1, y\rangle = 0$ implies $\langle w_1, y\rangle = 0$. But since $z = y - x$, then $\langle x + w_1, z\rangle = 0$ which implies $\langle w^\perp, w^\perp\rangle = 0$ leading to a contradiction. □

In the case of a vector space freely generated by a locality set, the locality compatibility property actually implies a stronger version with $N \geq 2$ elements instead of two:

Proposition 5.28. *Let $(S, \top)$ be a locality set, $(\mathbb{K}(S), \top)$ the locality vector space generated by extending linearly the relation $\top$, and $W \subset \mathbb{K}(S)$ a subspace locality compatible with $\top$. Let $x \in \mathbb{K}(S)$, $y_i \in \mathbb{K}(S)$ such that for every $1 \leq i \leq N$ there exists $w_i \in W$ such that $x + w_i \top y_i$. Then there is an element w' of W such that $x + w' \top y_i$ for every $1 \leq i \leq N$.*

Proof. We prove the statement by induction on N. The case $N = 2$ is immediate since W is locality compatible with $\top$. Assume it is true for $N - 1$, and let $x \in \mathbb{K}(S)$, $y_i \in \mathbb{K}(S)$ and $w_i \in W$ for $1 \leq i \leq N$, such that $x + w_i \top y_i$ for every i. By induction there is $w_0' \in W$ such that $x + w_0' \top y_i$ for every $1 \leq i \leq N-1$. We can write every y_i in terms of the basis elements of S as $y_i = \sum_{s\in S} \alpha_{i,s} s$ where only finitely many $\alpha_{i,s} \neq 0$, and define $\bar{y} = \sum_{s\in S} M_s s$ where

$$M_s = \begin{cases} 0 & \text{if} \quad (\forall i \in [N-1]) : \alpha_{i,s} = 0, \\ 1 & \text{if} \quad (\exists i \in [N-1]) : \alpha_{i,s} \neq 0. \end{cases}$$

Notice that only finitely many $M_s \neq 0$ and thus $\bar{y}$ is well defined. Moreover $x + w_0' \top \bar{y}$ and, as a consequence of the definition of $\bar{y}$, for every $z \in \mathbb{K}(S)$ such that $z \top \bar{y}$ then $z \top y_i$ for every $i < N$.

Since W is locality compatible with $\top$, there is an element w' in W such that $x + w' \top \bar{y}$ and $x + w' \top y_N$ which implies the expected result. □

5.4 Two conjectural statements

We now formulate two conjectural statements which will play an important role in the sequel. The first one is the conjectural statement 5.30 which gives a sufficient condition for the tensor product of n subspaces of a locality vector space to be again a locality vector space. The second one is the conjectural statement 5.33 which enhances the tensor algebra to a locality algebra by giving sufficient conditions for the filtered components from Definition 4.24 to become locality vector spaces. In the following V_1 and V_2 are subspaces of a pre-locality vector space $(E, \top)$. We equip $V_1 \times_\top V_2$ with a locality relation $\top_{V_1\times_\top V_2}$ as in (2.2). We discuss why those conjectural statements are very difficult to prove together with some reasons and examples why we believe them to be true in Appendix B.

Remark 5.29. If $V_1 = V_2 = V$, we have $\top_{V_1\times_\top V_2} = V^{\times_\top 4}$ with the notations of Definition 2.4.

Recall that the locality tensor product of V_1 and V_2 was defined in Definition 2.14 as the quotient space

$$V_1 \otimes_\top V_2 = \mathbb{K}(V_1 \times_\top V_2)/I_{\text{bil}} \cap \mathbb{K}(V_1 \times_\top V_2).$$

The locality relation $\top_{V_1\times_\top V_2}$ in $\mathbb{K}(V_1 \times_\top V_2)$ induces a locality relation $\top_\otimes$ on $V_1 \otimes_\top V_2$, namely the quotient locality (see Definition 4.6). Whether or not the subspace of bilinear forms $I_{\text{bil}}(V_1, V_2) \cap \mathbb{K}(V_1 \times_\top V_2)$ is locality compatible with $\top_{V_1\times_\top V_2}$, is a challenging question which we formulate as conjectural statement. In Appendix B we relate the following conjectural statements with an open problem in group theory, and provide examples and reasons to believe it is true.

Conjectural statement 5.30. [Pair of locality vector spaces] *Given a locality vector space $(E, \top)$ and two subspaces from it V_1, and V_2, the subspace $I_{\text{bil}}(V_1, V_2) \cap \mathbb{K}(V_1 \times_\top V_2) \subset \mathbb{K}(V_1 \times_\top V_2)$ is locality compatible with $\top_{V_1\times_\top V_2}$.*

Proposition 5.31. *Statement 5.30 is equivalent to the following statement:*

Let $n \geq 2$ and $V_1, \ldots, V_n$ be linear subspaces of a locality vector space $(E, \top)$. The space $I_{\rm mult,n}(V_1, \ldots, V_n) \cap \mathbb{K}(V_1 \times_\top \cdots \times_\top V_n) \subset \mathbb{K}(V_1 \times_\top \cdots \times_\top V_n)$ is locality compatible with the locality relation $\top_{V_1 \times_\top \cdots \times_\top V_n}$ (see Definition 4.9) on the space $\mathbb{K}(V_1 \times_\top \cdots \times_\top V_n)$.

Proof. We can recover Statement 5.30 from the statement of Proposition 5.31 in the case $n = 2$. Conversely, we now prove that Statement 5.30 implies that of Proposition 5.31.

For $n \geq 3$: Let x, y, z and w be elements of $\mathbb{K}(V_1 \times \cdots \times V_n)$ where $w \in I_{\rm mult,n}(V_1 \times \cdots \times V_n)$ such that $x \top_\times y$ and $(x + w) \top_\times z$. Extending linearly the usual bijection between $(V_1 \times \cdots \times V_{n-1}) \times V_n$ and $V_1 \times \cdots \times V_n$, one obtains an isomorphism of vector spaces $\mathbb{K}((V_1 \times \cdots \times V_{n-1}) \times V_n) \simeq \mathbb{K}(V_1 \times \cdots \times V_n)$. It is straightforward to show that the restriction of such an isomorphism to $\mathbb{K}((V_1 \times_\top \cdots \times_\top V_{n-1}) \times_\top V_n)$ (resp. $I_{\rm bil}(V_1 \times \cdots \times V_{n-1}, V_n)$) yields an isomorphism with $\mathbb{K}(V_1 \times_\top \cdots \times_\top V_n)$ (resp. $I_{\rm mult,n}(V_1 \times \cdots \times V_n)$). We may therefore see x, y, z and w as elements on $\mathbb{K}((V_1 \times_\top \cdots \times_\top V_{n-1}) \times_\top V_n)$ as well as view w as an element of $I_{\rm bil}(V_1 \times \cdots \times V_{n-1}, V_n)$. Assuming Statement (5.30) holds, it yields the existence of $w' \in I_{\rm bil}(V_1 \times \cdots \times V_{n-1}, V_n) \cap \mathbb{K}((V_1 \times_\top \cdots \times_\top V_{n-1}) \times_\top V_n)$ such that $(x + w') \top_\times y$ and $(x + w') \top_\times z$, which proves the statement. □

The following corollary is important to ensure the stability of locality vector spaces under tensor products and is the main reason for introducing the conjectural statement 5.30.

Corollary 5.32. Let $n \geq 1$ and $V_1, \ldots, V_n$ be linear subspaces of a locality vector space $(V, \top)$. Assuming that statement 5.30 holds true, the locality tensor product $(V_1 \otimes_\top \cdots \otimes_\top V_n, \top_\otimes)$ is a locality vector space.

Proof. This statement follows from Proposition 5.31 and Theorem 5.26. □

Even though the last statement is essential for the rest of the paper, it is not enough to make the locality tensor algebra a locality vector space, since it fails to relate the different graded components. Therefore we formulate the following conjectural statement whose consequences will be used in the sequel (see for instance Theorems 6.22 and 6.39).

Conjectural statement 5.33. [Locality tensor algebra] *Given a locality vector space $(V, \top_V)$ and any $n \in \mathbb{N}$, the subspace $\left(I^n_{\rm mult}(V) \cap \mathbb{K}(\bigcup_{k=0}^n V^{\times_\top^k})\right) \subset \mathbb{K}(\bigcup_{k=0}^n V^{\times_\top^k})$ is locality compatible with $\top_\times$ (see Definition 4.26).*

Each of the statements 5.30 and 5.33 implies the following useful property:

Proposition 5.34. *[Locality vector space] (the case $V = W$) Given a locality vector space $(V, \top_V)$ and assuming that statement 5.30 holds true, the subspace $I_{\rm bil}(V, V) \subset \mathbb{K}(V \times_\top V)$ is locality compatible with $\top_{V \times_\top V}$.*

Even though they might seem rather natural, these conjectural statements turn out to be rather challenging (see Appendix B). We devote the following paragraphs to getting a better grasp of these assumptions and their consequences.

5.5 Universal properties in the locality setup

In this section, we show how assuming that statements 5.30 and 5.33 hold true, enables us to enhance some results in the pre-locality setup to a full-fledged locality setup. In particular, Proposition 4.31 is enhanced by Proposition 5.36 where we prove that the locality tensor algebra is indeed a locality algebra. Also the universal properties in Theorems 4.14, 4.33, 4.41 and 4.48 are transposed to the locality context.

The following definition is also presented in [21, Definition 7.2], and is the locality version of Definition 1.19.

Definition 5.35. • A **graded locality algebra** is a locality algebra together with a sequence of vector spaces $\{A_n\}_{n \in \mathbb{N}}$ called the grading, such that

$$A = \bigoplus_{n \in \mathbb{N}} A_n, \quad m(A_p \otimes_\top A_q) \subset A_{p+q}, \quad u(\mathbb{K}) \subset A_0.$$

- A **filtered locality algebra** is a locality algebra together with a sequence of nested vector spaces $A^0 \subset A^1 \subset \cdots \subset A^n \subset \cdots$ called the filtration, such that

$$A = \bigcup_{n \in \mathbb{N}} A^n, \quad m(A^p \otimes_\top A^q) \subset A^{p+q}, \quad u(\mathbb{K}) \subset A^0.$$

A **graded (resp. filtered) locality Lie algebra** is a locality Lie algebra which is also a graded (resp. filtered) locality Lie algebra.

Proposition 5.36. *Assuming that conjectural statement 5.33 holds true, the locality tensor algebra over a locality vector space is a graded locality algebra.*

Proof. Let $(V, \top_V)$ be a locality vector space, it is straightforward that the locality vector space $(\mathbb{K}(V^{\times_\top^\infty}), \top_V)$, where $V^{\times_\top^\infty} := \bigcup_{k\geq 1} V^{\times_\top k}$, together with the concatenation product m_c between locally independent elements is a locality algebra. The subspace I_{mult} is moreover a locality ideal. Therefore Theorem 5.26 implies that $(\mathcal{T}_\top(V), \top_\otimes, \bigotimes)$ is a locality algebra. Since for any p and q, the concatenation product m_c preserves the grading of $\mathbb{K}(V^{\times_\top^\infty})$, namely $m_c(V^{\times_\top^p} \times_\top V^{\times_\top^q}) \subset V^{\times_\top^{p+q}}$, it follows that $\bigotimes(V^{\otimes_\top^p} \otimes_\top V^{\otimes_\top^q}) \subset V^{\otimes_\top^{p+q}}$. Finally, the convention that $V^0 = \mathbb{K}$ yields the result. □

The following theorem generalises Theorem 4.14.

Theorem 5.37 (Universal property of the locality tensor product)**.** Given $V_1 \cdots, V_n$ linear subspaces of a locality vector space $(E, \top)$, $(G, \top_G)$ a locality vector space and $f : (V_1 \times_\top \cdots \times_\top V_n, \top_\times) \to (G, \top_G)$ a locality $\top_\times$ n-linear map. Assuming that conjectural statement 5.30 holds true for the locality vector spaces $V_1, \ldots, V_n$, there is a unique locality linear map $\phi : V_1 \otimes_\top \cdots \otimes_\top V_n \to G$ such that the following diagram commutes.

$$\begin{array}{ccc} (V_1 \times_\top \cdots \times_\top V_n, \top_\times) & \xrightarrow{\ \otimes_\top\ } & (V_1 \otimes_\top \cdots \otimes_\top V_n, \top_\otimes) \\ & {\searrow}^{f_\top} \quad {\swarrow}_{\phi} & \\ & (G, \top_G) & \end{array}$$

Proof. Theorem 4.14 yields the existence and uniqueness of the linear map ϕ. Assuming the statement 5.30 holds true, Proposition 5.31 and Theorem 5.26 imply that $V_1 \otimes_\top \cdots \otimes_\top V_n$ is a locality vector space. We are only left to show that ϕ is a locality map. Recall that two equivalence classes $[a]$ and $[b]$ in $V_1 \otimes_\top \cdots \otimes_\top V_n$ verify $[a] \top_\otimes [b]$ if there are $\sum_{i=1}^N \alpha_i(x_{1,i}, \ldots, x_{n,i}) \in [a]$ and $\sum_{j=1}^M \beta_j(y_{1,j}, \ldots, y_{n,j}) \in [b]$ such that every possible pair taken from the set $\{x_{k,i} : (k,i) \in [n] \times [N]\} \cup \{y_{k,j} : (k,j) \in [n] \times [M]\}$ lies in $\top$. Since f is locality $\top_\times$ n-linear, then $f(\sum_{i=1}^N \alpha_i(x_{1,i}, \ldots, x_{n,i})) \top_A f(\sum_{j=1}^M \beta_j(y_{1,j}, \ldots, y_{n,j}))$ which amounts to $\phi([a]) \top_A \phi([b])$. Therefore ϕ is as expected. □

Assuming that conjectural statement 5.33 holds true, as a consequence of the previous theorem, we can state and prove an enhanced universal property Theorem 4.33 for the locality tensor algebra.

Theorem 5.38 (Universal property of locality tensor algebra)**.** Let $(V, \top)$ be a locality vector space, $(A, \top_A)$ a locality algebra and $f : V \to A$ a locality linear map. Assuming the conjectural statement 5.33 holds for tensor powers of V, there is a unique locality algebra morphism $\phi : \mathcal{T}_\top(V) \to A$ such that the following diagram commutes.

$$\begin{array}{ccc} (V, \top) & \xrightarrow{\ \otimes_\top\ } & (\mathcal{T}_\top(V), \top_\otimes) \\ & {\searrow}^{f} & \downarrow \phi \\ & & (A, \top_A) \end{array}$$

where $\otimes : V \to \mathcal{T}_\top(V)$ is the canonical (locality) injection map.

Proof. Since we have assumed that conjectural statement 5.33 holds true, the locality tensor algebra is a locality algebra by Proposition 5.36. By means of Theorem 4.33 and Remark 2.28 the pre-locality algebra morphism ϕ exists and is unique. Given that $\mathcal{T}_\top(V)$ and A are locality algebras, then ϕ is also a locality algebra morphism as expected. □

In order to extend Theorem 4.48 (the universal property of the locality universal enveloping algebra) to the locality setup, it is first needed that the locality universal enveloping algebra $U_\top(\mathfrak{g})$ of a locality Lie algebra $\mathfrak{g}$ is a locality algebra and not only a pre-locality algebra. In the remaining part of the chapter, assuming that the conjectural statement 5.33 holds true, we make the following further assumption which is similar in spirit to conjectural statements 5.30 and 5.33.

Conjectural statement 5.39. [Locality universal enveloping algebra] *Given a locality Lie algebra $(\mathfrak{g}, \top_\mathfrak{g}, [,])$, the ideal $J_\top(\mathfrak{g})$ of $\mathcal{T}_\top(\mathfrak{g})$ introduced in Definition 4.46 is locality compatible with $\top_\otimes$.*

Proposition 5.40. *Assuming the conjectural statement 5.39 holds true for a locality Lie algebra $(\mathfrak{g}, \top_\mathfrak{g}, [,])$, then $U_\top(\mathfrak{g})$ defines a locality algebra.*

Proof. This is a direct consequence of Theorem 5.26. □

The following theorem is the locality counterpart of Theorem 4.48.

Theorem 5.41. Let $(\mathfrak{g}, \top_\mathfrak{g}, [,])$ be a locality Lie algebra, $(A, \top_A)$ a locality algebra and $f : \mathfrak{g} \to A$ a locality Lie algebra morphism where the Lie bracket on A is the commutator defined by the product. Assuming that the conjectural statements 5.33 and 5.39 hold true for $\mathfrak{g}$, there is a unique locality algebra morphism $\phi : U_\top(\mathfrak{g}) \to A$ such that the following diagram commutes, and where $\iota_\mathfrak{g} : \mathfrak{g} \longrightarrow U_\top(\mathfrak{g})$ is the locality canonical map from $\mathfrak{g}$ to $U_\top(\mathfrak{g})$.

$$\begin{array}{ccc} (\mathfrak{g}, \top_\mathfrak{g}) & \xrightarrow{\iota_\mathfrak{g}} & (U_\top(\mathfrak{g}), \top_\mathfrak{g}) \\ & \searrow^{f} & \downarrow \phi \\ & & (A, \top_A) \end{array}$$

Proof. The proof follows from Theorem 4.48 and the fact that $U_\top(\mathfrak{g})$ and A are locality algebras. □

The following is the locality counterpart of Proposition 4.52.

Corollary 5.42. Let $(\mathfrak{g}, [,]_\mathfrak{g}, \top)$ and $(\mathfrak{g}', [,]_{\mathfrak{g}'}, \top')$ be two isomorphic locality Lie algebras, then their locality universal enveloping algebras $U_\top(\mathfrak{g})$ and $U_{\top'}(\mathfrak{g}')$ are isomorphic as locality algebras.

Proof. By means of Corollary 4.52 $U_\top(\mathfrak{g})$ and $U_{\top'}(\mathfrak{g}')$ are isomorphic as pre-locality algebras. The result follows then from Proposition 5.40. □

In order to upgrade Theorem 4.41 (universal property of the locality symmetric algebra of a pre-locality vector space) to the locality setup, it is first necessary to prove that it is indeed a locality algebra. Contrary to the previous cases, the decomposition of $\mathcal{T}_\top(V)$ into symmetric and antisymmetric tensors is enough to do so. This is a consequence of the symmetry of the locality relation which lies in the heart of the proof of Definition-Proposition 4.36.

Proposition 5.43. *Given a locality vector space $(V, \top)$. If $(\mathcal{T}_\top(V), \top_\otimes)$ is a locality algebra, then it accepts a decomposition of the type*

$$\mathcal{T}_\top(V) = \mathcal{AT}_\top(V) \oplus_\top \mathcal{ST}_\top(V),$$

i.e., $\mathcal{AT}_\top(V)$ is a strong locality complement of $\mathcal{ST}_\top(V)$ with respect to $(\mathcal{T}_\top(V), \top_\otimes)$, and thus $\mathcal{AT}_\top(V)$ is locality compatible with $\top_\otimes$.

Proof. This is a direct consequence of item 2. in Definition-Proposition 4.36 and the fact that strong locality complement implies locality compatibility (Proposition 5.23). □

In grounds of the previous proposition we can upgrade the locality symmetric algebra in the prelocality context to the locality context.

Proposition 5.44. *Let* $(V, \top)$ *be a locality vector space. If* $(\mathcal{T}_\top(V), \top_\otimes)$ *is a locality algebra, then* $(S_\top(V), \top_S)$ *is a locality graded, connected (unital, associative) algebra.*

Proof. The fact that $S_\top(V)$ is a locality algebra follows from Proposition 5.43 and Theorem 5.26. The gradedness and connectedness follow from the fact that $S_\top(V)$ is a subspace of $S(V)$ (Proposition 4.39) which is graded and connected (see Paragraph 1.4). □

Theorem 5.45 (Universal property of the locality symmetric algebra)**.** Let $(V, \top)$ be a locality vector space, $(A, \top_A)$ a commutative locality algebra and $f : V \to A$ a locality linear map. If $\mathcal{T}_\top(V)$ is a locality algebra, then there exists a unique morphism of commutative locality algebras $\phi_f : S_\top(V) \to A$ such that the following diagram commutes:

$$\begin{array}{ccc} (V,\top) & \xrightarrow{\iota_S} & (S_\top(V), \top_S) \\ & \searrow^{f} & \downarrow \phi_f \\ & & (A, \top_A) \end{array}$$

Proof. The statement follows from Theorem 4.41 and the fact that $S_\top(V)$ and A are locality algebras (Proposition 5.44). □

6 Locality versions of the Milnor-Moore and Poincaré-Birkhoff-Witt theorems

The main objective of this section is to state and prove the locality versions of the Milnor-Moore theorem (Theorem 6.22), also known as Cartier-Quillen-Milnor-Moore, and the Poincaré-Birkhoff-Witt theorem (Theorem 6.39). The Milnor-Moore theorem first appear in some seminar notes from Cartier [14] in 1957 and was later popularised by Milnor and Moore in 1965 [73] using a rather categorical language. The idea of the proof we follow stems from Cartier [15] and Patras [80]. We found a nice explanation of this proof in some lecture notes by Loïc Foissy [33], on which Paragraphs 6.2 and 6.4 are based. However, that proof uses Corollary 1.55 of the Poincaré-Birkhoff-Witt theorem in its version regarding a basis for $U(\mathfrak{g})$ (Theorem 1.53) and, as we have already mentioned, such version is in general not possible to adapt to the locality setup since basis and locality are not always compatible, in particular, a basis of a locality vector space does not always induces a basis of its locality tensor algebra. Therefore, in Proposition 6.20 we present an alternative proof in the locality context of the result of Corollary 1.55 which do not rely on the Poincaré-Birkhoff-Witt theorem but rather uses Zorn's lemma (Lemma 1.1). The consequence is that we present a proof of the locality Milnor-Moore theorem, which also applies to the non-locality case when considering the trivial locality relation, and that does not use the Poincaré-Birkhoff-Witt theorem as most of the proofs do.

On the other hand, the Poincaré-Birkhoff-Witt theorem first appeared with this names in Bourbaki's book "Lie groups and Lie algebras" in 1960 [11]. The formulation we generalise, which provides a coalgebra isomorphism between the symmetric algebra and the universal enveloping algebra of a Lie algebra, was proven by Quillen [82, Appendix B]. We base our proof in that presented in [17, Section 4.2]. However, that proof relies on the fact that idempotent tensors generate the subalgebra of symmetric tensors of a tensor algebra which represents a problem for the locality case since, in general, locality relations are not symmetric and thus idempotent tensors might not lie in the locality tensor algebra. Therefore, we deviate from their proof in Lemmas 6.33, 6.34, and 6.36 to make up for such situation. We also complete what, to our opinion, is a gap in the proof by [17, Section 4.2] since the universal enveloping algebra is a filtered algebra and thus the reduced coproduct, iterated to the degree of the filtration is not well defined. For that purpose, and inspired by [29, Section 2.1.4], we introduce Lemmas 6.37 and 6.38 to correctly define the iterated reduced coproduct as a map from the locality universal enveloping algebra to the locality tensor algebra.

In Paragraph 6.1 we introduce the necessary concepts to state the theorem, namely those of locality coagebra, locality bialgebra and locality Hopf algebras. In Paragraph 6.2 we present some Lemmas necessary for the proofs of the main theorems. In Paragraph 6.3 we endow the locality symmetric algebra and the locality universal enveloping algebra with the structure of locality Hopf algebras. Paragraph 6.4 is devoted to the proof of the locality Milnor-Moore theorem and Paragraph 6.5 to some of its consequences. Finally, Paragraph 6.6 is devoted to the proof of the locality Poincaré-Birkhoff-Witt theorem. Some paragraphs of this section follow closely [21, Part 3]. In the sequel, we assume that the conjectural statements 5.33 and 5.39 hold true, and recall that we assume all locality vector spaces (resp. locality algebras, resp. locality coalgebras, resp. locality bialgebras, resp. locality Hopf algebras) have the same underlying filed $\mathbb{K}$ of characteristic zero.

6.1 Graded connected locality Hopf algebras

In this paragraph we recall the definition of locality coalgebra, locality bialgebra and locality Hopf algebra introduced in [22]. We discuss after Definition 6.1 why such concepts require the locality and not the pre-locality context. It is worth mentioning that the concepts of locality coideals, locality sub-coalgebras, together with most results presented in this paragraph are not in [22] but are original of this research. Particularly interesting is Lemma 6.7 which, contrarily to the usual (non-locality) setup (Proposition 1.29), it requires stronger conditions for the kernel of a locality coalgebra morphism to be a locality coideal.

Locality coalgebras

Let us recall some definitions from [22]. Just as a locality algebra was defined as a locality vector space equipped with a partial product and a unit compatible with the locality relation (see Definition 2.26), a locality coalgebra is a locality vector space equipped with partial coproduct and a counit compatible with the locality relation.

Definition 6.1. [22, Definition 4.3]

- A locality $\mathbb{K}$-coalgebra is a quadruple $(C, \Delta, \epsilon, \top)$ where $(C, \top)$ is a locality $\mathbb{K}$-vector space, $\Delta : (C, \top) \to (C \otimes_\top C, \top_\otimes)$ and $\epsilon : C \to \mathbb{K}$ are linear maps such that
  - the coproduct Δ is coassociative, namely $(Id_C \otimes \Delta) \circ \Delta = (\Delta \otimes Id_C) \circ \Delta$ on C;
  - and compatible with the locality structure i.e., for any $U \subset C$, $\Delta(U^\top) \subset U^\top \otimes_\top U^\top$;
  - the counit $\epsilon : C \to \mathbb{K}$ satisfies $(Id_C \otimes \epsilon)\Delta = (\epsilon \otimes Id_C)\Delta = Id_C$.
- A **graded locality $\mathbb{K}$-coalgebra** is a locality $\mathbb{K}$-coalgebra together with a sequence of vector spaces $\{C_n\}_{n\in\mathbb{N}}$ called a **grading**, such that
$$C = \bigoplus_{n\in\mathbb{N}} C_n, \quad \Delta(C_n) \subset \bigoplus_{p+q=n} C_p \otimes_\top C_q, \quad \bigoplus_{n\geq 1} C_n \subset \ker(\epsilon).$$
  For $x \in C$, we denote by $|x|$ the degree of x.

  Moreover we call a graded locality $\mathbb{K}$-coalgebra **connected** if C_0 has dimension 1 and therefore $\bigoplus_{n\geq 1} C_n = \ker(\epsilon)$.
- A **filtered locality $\mathbb{K}$-coalgebra** is a locality $\mathbb{K}$-coalgebra together with a nested sequence of vector spaces $C^0 \subset C^1 \subset \cdots \subset C^n \ldots$, called a **filtration**, such that
$$C = \bigcup_{n\in\mathbb{N}} C^n, \quad \Delta(C^n) \subset \sum_{p+q=n} C^p \otimes_\top C^q, \quad \bigoplus_{n\geq 1} C^n \subset \ker(\epsilon).$$

Remark 6.2. The second condition implies that $\Delta : (C, \top) \to (C \otimes_\top C, \top_\otimes)$ is a locality map. Indeed for $a\top b$, Condition 2 implies that there are $a_{1,i} \otimes a_{2,i} \in \{b\}^\top \otimes_\top \{b\}^\top$ for $1 \leq i \leq N$ such that $\Delta(a) = \sum_{i=1}^N a_{1,i} \otimes a_{2,i}$. Applying Condition 2 once more, there are $b_{1,j} \otimes_\top b_{2,j} \in \{a_{1,i}, a_{2,i} : 1 \leq i \leq N\}^\top \otimes_\top \{a_{1,i}, a_{2,i} : 1 \leq i \leq N\}^\top$ for $1 \leq j \leq M$ such that $\Delta(b) = \sum_{j=1}^M b_{1,j} \otimes_\top b_{2,j}$. It follows that $\Delta(a)\top_\otimes\Delta(b)$ as expected.

Notice that the above definition only makes sense in the locality framework, where the polar set $U^\top$ of a set is required to be a vector space. In the pre-locality setup, the fact that this condition is relaxed prevents us from building locality tensor products such as $U^\top \otimes_\top U^\top$. This suggests that the Milnor-Moore theorem we are about to prove does not hold in the more general pre-locality setup.

Similar to the non-locality case (Proposition 1.28), it is possible to take quotients of locality coalgebras which inherits naturally a locality coalgebraic structure. For that purpose, the following definition provides the coalgebraic counterpart of a locality ideal (see (1.25)) and a locality morphism, (see (1.26)).

Definition 6.3. 1. A locality linear subspace J of a locality coalgebra $(C, \top, \Delta)$ is called a **left**, (resp. **right**) **locality coideal** of C, if

$$\Delta(J) \subset J \otimes_\top C; \,(\text{resp.}\, \Delta(J) \subset C \otimes_\top J) \quad \text{and} \quad \epsilon(J) = \{0\}. \tag{2.32}$$

We call it a **locality coideal** if

$$\Delta(J) \subset J \otimes_\top C + C \otimes_\top J \quad \text{and} \quad \epsilon(J) = \{0\}. \tag{2.33}$$

Note that the condition $\epsilon(J) = \{0\}$ does not involve the locality relation.

2. Given two locality coalgebras $(C_i, \top_i, \Delta_i, \epsilon_i), i = 1, 2$, a locality linear map $f : C_1 \to C_2$ is called a **locality coalgebra morphism** if

$$(f \otimes f) \circ \Delta_1 = \Delta_2 \circ f, \quad \text{and} \quad \epsilon_1 = \epsilon_2 \circ f. \tag{2.34}$$

In other words, it is a coalgebra morphism which is a locality map. An **isomorphism of locality coalgebras** is a bijective morphism of locality coalgebras, the inverse of which is also an isomorphism of locality coalgebras.

3. Let $(C, \top, \Delta)$ be a locality coalgebra. A **locality coalgebra** $(C_1, \top_1, \Delta_1)$ with $C_1 \subset C$ is a locality sub-coalgebra of $(C, \top, \Delta)$ if the inclusion map $\iota : (C_1, \top_1, \Delta_1) \hookrightarrow (C, \top, \Delta)$ is a coalgebra morphism.

It is convenient to notice that this definition of locality sub-coalgebra is more general than the one given in [22]. This level of generalisation will be needed later. The following lemma will be of use in the sequel.

Lemma 6.4. Let $(C, \top, \Delta, \epsilon)$ and $(C', \top', \Delta', \epsilon')$ be two locality coalgebras and $f : C \to C'$ a bijective morphism of locality coalgebras. Then $f^{-1} : C' \to C$ is a morphism of coalgebras, but it is not necessarily a locality map and thus not necessarily an isomorphism of locality coalgebras.

Proof. Since $f : C \to C'$ is a linear bijective map, then f^{-1} is a linear map too. Precomposing $\epsilon = \epsilon' \circ f$ with f^{-1} follows that $\epsilon \circ f^{-1} = \epsilon'$. Finally, for any $c' \in C'$, set $c := f^{-1}(c)$, then

$$\begin{aligned}
\Delta(f^{-1}(c')) &= (f^{-1} \otimes f^{-1}) \circ (f \otimes f) \circ \Delta(f^{-1} \circ f(c)) \\
&= (f^{-1} \otimes f^{-1}) \circ \Delta' \circ f(c) && (f \text{ is a coalgebra morphism}) \\
&= (f^{-1} \otimes f^{-1})(\Delta'(c')),
\end{aligned}$$

and thus $\Delta \circ f^{-1} = (f^{-1} \otimes f^{-1}) \circ \Delta'$ which finishes the proof. □

The following example illustrates that the inverse of a bijective coalgebra morphism is not necessarily a locality map.

Counter-example 6.5. Let $V \neq \{0\}$ be a vector space and consider $C := V \oplus \mathbb{K}$. Consider moreover as the counit ϵ the projection of C onto $\mathbb{K}$, and the coproduct $\Delta : C \to \mathbb{K}$ given by $\Delta(1_\mathbb{K}) = 1_\mathbb{K} \otimes 1_\mathbb{K}$ and $\Delta(v) = 0 \otimes 0$ for every $v \in V$. Consider finally the locality relations on C, $T_1 := C \times C$ and $T_2 := C \times \{0\} \cup \{0\} \times C$. It is then straightforward to check that $(C, \top_1, \Delta, \epsilon)$ and $(C, \top_2, \Delta, \epsilon)$ are locality algebras, and that the identity map $\mathrm{Id}_C : (C, \top_2) \to (C, \top_1)$ is a locality coalgebra morphism making $(C, \top_2, \Delta, \epsilon)$ a locality sub-coalgebra of $(C, \top_1, \Delta, \epsilon)$. However the inverse map $\mathrm{Id}_C : (C, \top_1) \to (C, \top_2)$ is not a locality map.

The following is the locality version of Lemma 1.18. As in Proposition 4.21, we require a compatibility between the locality relation and direct sums.

Proposition 6.6. *Let $(E,\top)$ and $(F,\top_F)$ be two locality vector spaces. For $i\in\{1,2\}$, let $f_i:V_i\to W_i$ be locality linear maps from locality subspaces $V_i\subset E$ to vector subspaces W_i of F. We moreover assume that f_1 and f_2 are mutually locally independent and the existence of surjective projections $\pi_i:V_i\to\ker(f_i)$ such that π_i and Id_{V_j} are locally independent for $i\neq j$ (see Proposition 4.21). Then*

$$\ker(f_1\otimes f_2)\cap(V_1\otimes_\top V_2)=\ker f_1\otimes_\top V_2+V_1\otimes_\top \ker f_2. \tag{2.35}$$

Proof. We know from Lemma 1.18 that $\ker(f_1\otimes f_2)=\ker f_1\otimes V_2+V_1\otimes\ker f_2$. Taking the intersection with $V_1\otimes_\top V_2$ yields

$$\ker(f_1\otimes f_2)\cap(V_1\otimes_\top V_2)=(\ker f_1\otimes V_2+V_1\otimes\ker f_2)\cap(V_1\otimes_\top V_2).$$

From elementary linear algebra, we obtain $V_1=\ker(f_1)\oplus\ker(\pi_1)$. Since by hypothesis π_1 and Id_{V_2} are locally independent and since $(E,\top)$ is a locality vector space, by the second point of Proposition 4.21, the projection $\tilde\pi_1:V_1\mapsto\ker(\pi_1)$ onto $\ker(\pi_1)$ along $\ker(f_1)$ is also independent of Id_{V_2}. Thus we can use Corollary 4.22 with $\ker(f_1)$, V_1 and V_2 respectively playing the roles of V_1, V and W in Corollary 4.22. This yields $(\ker f_1\otimes V_2)\cap(V_1\otimes_\top V_2)=\ker(f_1)\otimes_\top V_2$. Similarly, $(V_1\otimes\ker f_2)\cap(V_1\otimes_\top V_2)=V_1\otimes_\top\ker(f_2)$ and hence

$$\ker(f_1\otimes f_2)\cap(V_1\otimes_\top V_2)=(\ker f_1\otimes V_2+V_1\otimes\ker f_2)\cap(V_1\otimes_\top V_2)=\ker f_1\otimes_\top V_2+V_1\otimes_\top\ker f_2.\quad\square$$

The following lemma is the coalgebraic counterpart of Lemma 2.29.

Lemma 6.7. Let $(C_i,\Delta_i,\top_i), i\in\{1,2\}$ be locality coalgebras. The range of a locality coalgebra morphism $f:C_1\longrightarrow C_2$ is a locality subcoalgebra of C_2. Moreover, if there is a projection $\pi:C_1\to\ker(f_1)$, which is locally independent of the identity map Id_{C_1} on C_1, then $\ker(f_1)$ is a locality coideal of C_1.

Proof.

- We prove that the kernel $\ker(f)$ is a locality coideal. Let $c\in\ker(f)\subset C_1$. Since f is a locality coalgebra morphism, by (2.34), we have $(f\otimes f)(\Delta_1 c)=\Delta_2(f(c))=0$. Since Δ_1 is a locality coproduct, $\Delta_1(\ker(f))\subset(C_1\otimes_{\top_1}C_1)\cap\ker(f\otimes f)$. We now apply Proposition 6.6 to $f_i=f$ and $V_i=C_1$, with $f\otimes f$ acting on $C_1\otimes_{\top_1}C_1$. Since f is a locality morphism, it follows that $\ker(f\otimes f)|_{C_1\otimes_{\top_1}C_1}=\ker(f)\otimes_{\top_1}C_1+C_1\otimes_{\top_1}\ker(f)$. Consequently, $\Delta_1(\ker(f))\subset\ker(f)\otimes_{\top_1}C_1+C_1\otimes_{\top_1}\ker(f)$. We are left to show that $\epsilon_1(\ker(f))=\{0\}$. This follows from the fact that $\epsilon_1=\epsilon_2\circ f$, since $\epsilon_2\circ f(\ker(f))=\{0\}$. Therefore $\ker(f_1)$ is a locality coideal of C_1.
- To prove that the range $\mathrm{Im}(f)$ is a locality coalgebra, for any $c\in C_1$ such that $\Delta_1(c)=\sum_{(c)}c_1\otimes c_2$ and $c_1\top_1 c_2$, using (2.34) we write $\Delta_2 f(c)=(f\otimes f)\circ\Delta_1 c=\sum_{(c)}f(c_1)\otimes f(c_2)$. Since f is a locality map, $f(c_1)\top_2 f(c_2)$, which proves that $\Delta_2(\mathrm{Im}(f))\subset\mathrm{Im}(f)\otimes_{\top_2}\mathrm{Im}(f)$, showing that $\mathrm{Im}(f)$ is a subcoalgebra of C_2.

$\square$

Locality bialgebras and locality Hopf algebras

We proceed to define the concepts of locality bialgebras and locality Hopf algebras.

Definition 6.8.

- [22, Section 5.1] A **locality bialgebra** is a sextuple $(B,\top,m,u,\Delta,\epsilon)$ consisting of a locality algebra $(B,m,u,\top)$ and a locality coalgebra $(B,\Delta,\epsilon,\top)$ that are locality compatible in the sense that Δ and ϵ are locality algebra morphisms (1.26) and m and u are locality coalgebra morphisms[1] i.e.,

$$\Delta\circ m|_{B\otimes_\top 2}=\underbrace{(m\otimes m)}_{\text{domain } B^{\otimes_\top^4}}\circ(\mathrm{Id}_B\otimes\tau_{23}\otimes\mathrm{Id}_B)\circ\underbrace{(\Delta\otimes\Delta)|_{B^{\otimes_\top^2}}}_{\text{range } B^{\otimes_\top 4}};\quad \epsilon\circ m=\epsilon\otimes\epsilon;\quad \Delta\circ u=u\otimes u;\quad \epsilon\circ u=\mathrm{Id}_{\mathbb K},$$

where $B^{\otimes_\top^n}$ was defined in (1.21), and $\tau_{23}:B^{\otimes_\top^4}\to B^{\otimes_\top^4}$ is the map that switches the terms on the second and third position of the tensor.

[1] This condition was missing in [22].

- Let $(B_i, \top_i, m_i, u_i, \Delta_i, \epsilon_i)$ $(i \in \{1,2\})$ be two locality $\mathbb{K}$-bialgebras. A **locality bialgebra morphism** from B_1 to B_2 is a locality map $f : B_1 \longrightarrow B_2$ that is a morphism of locality algebras and of locality coalgebras.

- [22, Proposition 4.9] Let $(B, \top, m, u, \Delta, \epsilon)$ be a locality bialgebra, and $\phi, \psi : B \to B$ two mutually independent locality linear maps. The locality convolution product of ϕ and ψ is a locality linear map $B \to B$ defined by
$$(\phi \star \psi) = m(\phi \otimes \psi)\Delta.$$
Remark. The locality of $\phi\star\psi$ follows from the locality of ϕ, ψ, m, and Δ, together with Definition-Proposition 4.18 and Proposition 2.3.

- [22, Definition 5.3 and Remark 5.4] A **locality Hopf algebra** is a locality bialgebra $(H, \top, m, u, \Delta, \epsilon)$ together with a locality linear map $S : H \to H$ such that S and Id_H are mutually independent and
$$S \star Id_H = Id_H \star S = u{\circ}\epsilon.$$

- Let $(H_i, \top_i, m_i, u_i, \Delta_i, \epsilon_i, S_i)$ for $i \in \{1,2\}$ be two locality Hopf algebras. A **locality Hopf algebra morphism** between H_1 and H_2 is a morphism of locality bialgebras $f : H_1 \longrightarrow H_2$ such that $f \circ S_1 = S_2 \circ f$.

- A **locality Hopf sub-algebra** of H of a locality Hopf algebra $(H, \top, m, u, \Delta, \epsilon, S)$ is a locality Hopf algebra $(H', \top', m', u', \Delta', \epsilon', S')$ contained in H such that the injection map $f : H \hookrightarrow H'$ is a locality Hopf algebra morphism.

- A **graded (resp. filtered) locality Hopf algebra** is a locality Hopf algebra together with a grading (resp. filtration) which makes it a graded (resp. filtered) locality algebra and graded (resp. filtered) locality coalgebra and such that
$$S(H_n) \subset H_n \text{ (resp. } S(H^n) \subset H^n).$$
A **connected locality Hopf algebra** is a graded (resp. filtered) locality algebra such that H_0 (resp. H^0) has dimension 1.

Notice that, as in the algebra and coalgebra cases, our definition of locality Hopf sub-algebra is more general than the one used in [22], in that the locality relation on the locality Hopf sub-algebra can be coarser than the one in the bigger locality Hopf algebra. This fact is proper to the locality setup and has interesting consequences which will be studied at the end of this section. The following is an illustrative example of a locality Hopf algebra.

Example 6.9 (The locality tensor algebra as a locality Hopf algebra)**.** *For $(V, \top)$ a locality vector space, $\mathcal{T}_\top(V)$ is a locality Hopf algebra in the sense of [22, Definition 5.3] when equipped with the tensor product restricted to pairs in $\top_\otimes$ and the deschuffle coproduct defined on $x \in V$ by $\Delta_{\sqcup\sqcup}(x) = 1 \otimes x + x \otimes 1$ and inductively on the degree by*
$$\Delta_{\sqcup\sqcup}(x_{i_1} \otimes \cdots \otimes x_{i_n}) = \sum_{J \subset (i_1,\dots,i_n),\, w_J \top_\otimes w_{\bar{J}}} w_J \otimes w_{\bar{J}}, \tag{2.36}$$
where we have set $w_J := x_{j_1} \otimes \cdots \otimes x_{j_k}$ for $J = (j_1, \dots, j_k)$ and where $\bar{J}$ stands for the complement of J in $\{i_1, \dots, i_n\}$. The counit is defined by $\epsilon(x) = 0$ for $x \in V$ and the antipode is given by $S(x_{j_1} \otimes \cdots \otimes x_{j_k}) = (-1)^k\, x_{j_k} \otimes \cdots \otimes x_{j_1}$.

It is a connected graded cocommutative locality Hopf algebra of finite type.

The usual result for the existence of an antipode in a graded connected bialgebra also holds in the locality setup.

Proposition 6.10. *[22, Proposition 5.5] Let $(B, \top, m, u, \Delta, \epsilon)$ be a graded, connected, locality bialgebra. There exists an antipode $S : B \to B$ such that $(B, \top, m, u, \Delta, \epsilon, S)$ is a locality Hopf algebra.*

We end this paragraph with a transposition to the locality setup of the known Lie algebra structure of the space $\mathrm{Prim}(B)$ of primitive elements of a bialgebra B (see Example 1.50). Recall from Definition 1.30 item 5, that an element $x \in B$ is called **primitive** if, and only if $\Delta x = x \otimes 1 + 1 \otimes x$ and that a **graded locality Lie algebra** is a locality Lie algebra which is also a graded algebra for the Lie bracket (see Definitions 4.44 and 5.35).

Proposition 6.11. *The space* $(\mathrm{Prim}(B), \top_{\mathrm{Prim}(B)}, [\cdot;\cdot])$ *of a (resp. graded) locality bialgebra* $(B, \top)$ *equipped with* $\top_{\mathrm{Prim}(B)} = \top\big|_{\mathrm{Prim}(B)}$ *the restriction of the locality relation* $\top$ *to primitive elements and the usual commutator* $[x,y] = xy - yx$ *(resp.* $[x,y] = xy - (-1)^{|x|\cdot|y|}yx$*), is a (resp. graded) locality Lie algebra.*

Proof. We carry out the proof for the graded case, since the ungraded case can be obtained by setting all degrees to zero.

Let m be the locality product of the locality bialgebra $(B, \top)$. Since for any $U \subseteq B$, $m(U^\top \otimes_\top U^\top) \subset U^\top$, we have that $[[x,y],z]$ (and its permutations) is well-defined for any triplet $(x,y,z) \in B^{\times_\top 3}$. The rest of the proof goes exactly as for the non locality case. In particular, for any $(x,y) \in \mathrm{Prim}(B) \times_\top \mathrm{Prim}(B)$, since Δ is a locality algebra morphism, we have:

$$\begin{aligned}\Delta(xy) &= (m \otimes m) \circ \tau_{23} \circ (\Delta \otimes \Delta)(x,y) \\ &= (m \otimes m) \circ \tau_{23}((x \otimes 1 + 1 \otimes x) \otimes (y \otimes 1 + 1 \otimes y)) \quad \text{since } x \text{ and } y \text{ are primitive elements} \\ &= xy \otimes 1 + (-1)^{|x||y|} y \otimes x + x \otimes y + 1 \otimes xy.\end{aligned}$$

$\Delta(yx)$ is obtained by exchanging x and y in the previous computation. Putting everything together we obtain by linearity of Δ

$$\Delta([x,y]) = [x,y] \otimes 1 + 1 \otimes [x,y] \in B \otimes_\top B,$$

and thus $[x,y] \in \mathrm{Prim}(B)$. This, together with the fact that Δ is linear, implies that $(\mathrm{Prim}(B), \top, [;])$ is a locality Lie algebra. □

6.2 Reduced coproduct and primitive elements

This paragraph reviews preliminary well-known technical results, which we transpose to the locality setup. As before, the underlying field $\mathbb{K}$ of every vector space (or locality algebra, locality coalgebra, locality bialgebra or locality Hopf algebra) is a commutative field of characteristic zero.

Lemma 6.12. Let H be a graded, connected locality bialgebra. The projection onto $\ker(\epsilon)$ along $\mathbb{K}(1_H)$

$$\begin{array}{rccc}\rho: & H & \longrightarrow & H \\ & x & \longmapsto & x - \epsilon(x)1_H\end{array} \tag{2.37}$$

is a locality linear map, which is independent of Δ in the following sense:

$$x \top y \Longrightarrow \rho(x) \top_\otimes \Delta(y) \tag{2.38}$$

with $\top_\otimes$ the locality relation on $\mathcal{T}_\otimes(H) \supseteq H$ of Definition 4.26.

Proof. Let $x \top y$. Since $\mathbb{K} \subset H^\top$ we have $\epsilon(x)1_H \top y$ implying by linear locality that $(x - \epsilon(x)1_H)\top y$. Since $\epsilon(y)1_H \top (x - \epsilon(x)1_H)$, again by linear locality, we deduce that $x - \epsilon(x)1_H \top y - \epsilon(y)1_H$ and conclude that $\rho(x)\top\rho(y)$.

To check the mutual independence of ρ and Δ, we consider again $x \in H$ and $y \in H$ such that $x \top y$. We write $\Delta y = \sum y_i \otimes y_i'$, so that Equation (2.38) amounts to show that $\rho(x)\top y_i$ and $\rho(x)\top y_i'$. Since $\rho(x) = x - \epsilon(x)\,1_H$ and $\mathbb{K}\,1_H \top H$, by linearity, it suffices to show that $x \top y_i$ and $x \top y_i'$. But this follows from the fact that Δ maps $\{x\}^\top$ to $\{x\}^\top \otimes_\top \{x\}^\top$. □

Similar to the non-locality case (see Lemma 1.33), in a graded, connected, locality bialgebra H, for every x in H

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{(x)} x' \otimes x'',$$

where $x' \otimes x'' \in \ker(\epsilon) \otimes_\top \ker(\epsilon)$. We then consider the coassociative locality linear map $\tilde{\Delta} : H \to \ker(\epsilon) \otimes_\top \ker(\epsilon)$ defined as $\tilde{\Delta}(1) = 0$, and for $x \in \ker(\epsilon)$ as $\tilde{\Delta}(x) = \Delta(x) - 1 \otimes x - x \otimes 1$. Moreover, for $n \geq 0$, we define inductively $\tilde{\Delta}^{(n)} : H \to H^{\otimes_\top (n+1)}$ by:

- $\tilde{\Delta}^{(0)} = \rho$.
- $\tilde{\Delta}^{(1)} = \tilde{\Delta}$.
- $\tilde{\Delta}^{(n+1)} = (\tilde{\Delta} \otimes Id^{\otimes n}) \circ \tilde{\Delta}^{(n)}$.

Since Δ and ρ are locality linear maps, $\tilde{\Delta}$ and $\tilde{\Delta}^{(n)}$ are also locality maps. Moreover, ρ and Δ are mutually independent, and therefore ρ and $\tilde{\Delta}$ are also mutually independent in the sense of (2.38).

Proposition 6.13. *Let $(H, \top)$ with $H = \oplus_{k \in \mathbb{Z}_{\geq 0}} H_k$, be a graded, connected, locality bialgebra. For every $n \geq 1$*

$$\tilde{\Delta}^{(n)} = (\rho \otimes \cdots \otimes \rho) \circ \Delta^{(n)}.$$

Proof. Recall that $H = u(\mathbb{K}) \oplus \ker(\epsilon) \simeq \mathbb{K} \oplus \ker(\epsilon)$. We proceed by induction over n. For $n = 1$ we have $(\rho \otimes \rho) \circ \Delta(1) = \rho(1) \otimes \rho(1) = 0 = \tilde{\Delta}(1)$. By linearity, this extends replacing 1 for any $k \in \mathbb{K}$. Let $x \in \ker(\epsilon)$. Since $\tilde{\Delta}(x) \in \ker(\epsilon) \otimes_\top \ker(\epsilon)$ and $(\rho \otimes \rho) \circ \tilde{\Delta} = \tilde{\Delta}$, it follows that $(\rho \otimes \rho) \circ \Delta(x) = \rho(1) \otimes \rho(x) + \rho(x) \otimes \rho(1) + (\rho \otimes \rho) \circ \tilde{\Delta}(x) = \tilde{\Delta}(x)$, so that the proposition holds for $n = 1$.

Suppose now that the proposition holds true for $n-1$. We note that $\tilde{\Delta}(1) = 0$ implies that $\tilde{\Delta} \circ \rho = \tilde{\Delta}$. Hence,

$$\begin{aligned}
(\rho \otimes \cdots \otimes \rho) \circ \Delta^{(n)} &= ((\rho \otimes \rho) \circ \Delta \otimes \rho \otimes \cdots \otimes \rho) \circ \Delta^{(n-1)} \\
&= (\tilde{\Delta} \otimes \rho \otimes \cdots \otimes \rho) \circ \Delta^{(n-1)} \\
&= (\tilde{\Delta} \otimes Id \otimes \cdots \otimes Id) \circ (\rho \otimes \cdots \otimes \rho) \circ \Delta^{(n-1)} \quad \text{by the initial induction step} \\
&= (\tilde{\Delta} \otimes Id \otimes \cdots \otimes Id) \circ \tilde{\Delta}^{(n-1)} \quad \text{by the induction step} \\
&= \tilde{\Delta}^{(n)}.
\end{aligned}$$

□

The subsequent proposition combines elementary known results, which we recall for the sake of completeness. If $H = \bigoplus_{n \in \mathbb{Z}} H_n$ is a graded algebra, we say $|x| = n$ if $x \in H$ is a homogeneous component of degree n, i.e. $x \in H_n$.

Proposition 6.14. *Let $(H, \top)$ with $H = \oplus_{n \in \mathbb{Z}_{\geq 0}} H_n$, be a graded, connected, locality bialgebra, and let $k \geq 1$.*

1. *For any $x \in H$ such that $\tilde{\Delta}^{(k-1)}(x) \neq 0$, we have $\tilde{\Delta}^{(k)}(x) = 0 \Longrightarrow \tilde{\Delta}^{(k-1)}(x) \in \mathrm{Prim}(H)^{\otimes_\top k}$.*
2. *For every $x \in H$, $\tilde{\Delta}^{(k)}(x) = 0$ if $k \geq |x|$ and $\tilde{\Delta}^{(k-1)}(x) \in \mathrm{Prim}(H)^{\otimes_\top k}$ if $|x| = k$.*
3. *Let $n \geq 2$. For any $(v_1, \ldots, v_n) \in \mathrm{Prim}(H)^{\times_\top n}$, we have $\tilde{\Delta}^{(k)}(v_1 \cdots v_n) = 0$ for any $k \geq n$ and the following refinement of the first item holds:*

$$\tilde{\Delta}^{(n-1)}(v_1 \cdots v_n) = \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \ \in \mathrm{Prim}(H)^{\otimes_\top^n}, \tag{2.39}$$

where $\mathfrak{S}_n$ is the n-th symmetric group.

Proof. 1. The coassociativity of $\tilde{\Delta}$ implies that

$$(Id^{\otimes_\top (i-1)} \otimes \tilde{\Delta} \otimes Id^{\otimes_\top (k-i)}) \circ \tilde{\Delta}^{(k-1)}(x) = \tilde{\Delta}^{(k)}(x) = 0 \quad \forall 1 \leq i \leq k.$$

We infer that

$$\tilde{\Delta}^{(k-1)}(x) \in \ker(Id^{\otimes_\top (i-1)} \otimes \tilde{\Delta} \otimes Id^{\otimes_\top (k-i)}) = H^{\otimes_\top (i-1)} \otimes_\top (\mathbb{K} \oplus \mathrm{Prim}(H)) \otimes_\top H^{\otimes_\top (k-i)}.$$

Assuming that $\tilde{\Delta}^{(k-1)}(x) \neq 0$, since $\tilde{\Delta}^{(k-1)}(x) \in \ker(\epsilon)^{\otimes_\top k}$ (which is a consequence of $\mathrm{Im}(\rho) \subseteq \ker(\epsilon)$ and Proposition 6.13), it follows that

$$\tilde{\Delta}(x) \in H^{\otimes_\top (i-1)} \otimes_\top \mathrm{Prim}(H) \otimes_\top H^{\otimes_\top (k-i)} \quad \forall 1 \leq i \leq k.$$

Thus, $\tilde{\Delta}^{(k-1)}(x)$ lies in the intersection of all such spaces, which yields the statement.

2. Using the coassociativity of $\widetilde{\Delta}$, for any $x \in H$ with $|x| = n$, we have (by induction on k) $\widetilde{\Delta}^{(k)}(x) = \sum_x x^{(1)} \otimes \cdots \otimes x^{(k+1)}$, with $(x^{(1)}, \dots, x^{(k+1)}) \in H^{\times_\top k+1}$, $\sum_{j=1}^{k+1} |x^{(j)}| = |x| = n$. If $k \geq n$, this imposes that $\widetilde{\Delta}^{(k)}(x) = 0$. In particular, $\widetilde{\Delta}^{(n)}(x) = 0$. It then follows from the previous item that $\tilde{\Delta}^{(n-1)}(x) \in \mathrm{Prim}(H)^{\otimes_\top n}$.

3. Let $(v_1, \dots, v_n) \in \mathrm{Prim}(H)^{\times_\top^n}$ and $x := v_1 \cdots v_n$. To compute $\tilde{\Delta}^{(k)}(x)$ for any $k \leq n-1$, we proceed by induction on k. We prove first that $\Delta(v_1 \cdots v_n) = \sum_{I \subset [n]} v_I \otimes v_{I^C}$ by induction over n, setting $v_\emptyset = 1$. For $n = 1$,

$$\Delta(v_1) = v_1 \otimes 1 + 1 \otimes v_1 = \sum_{I \subset [1]} v_I \otimes v_{I^C} \quad \in \mathrm{Prim}(H)^{\otimes_\top 2}.$$

Now assume it is true for $n-1$. The compatibility of the product and the coproduct yields

$$\begin{aligned}\Delta(v_1 \cdots v_{n-1} v_n) &= \Delta(v_1 \cdots v_{n-1}) \Delta(v_n) \\ &= \left(\sum_{I \subset [n-1]} v_I \otimes v_{I^C} \right) (v_n \otimes 1 + 1 \otimes v_n) \\ &= \sum_{I \subset [n-1]} v_I \cdot v_n \otimes v_{I^C} + \sum_{I \subset [n-1]} v_I \otimes v_{I^C} \cdot v_n \\ &= \sum_{I \subset [n]} v_I \otimes v_{I^C} \quad \in \mathrm{Prim}(H)^{\otimes_\top^2}.\end{aligned}$$

It then follows by induction on k that

$$\Delta^{(k)}(v_1 \cdots v_n) = \sum_{I_1 \sqcup \cdots \sqcup I_{k+1} = [n]} v_{I_1} \otimes \cdots \otimes v_{I_{k+1}} \quad \in \mathrm{Prim}(H)^{\otimes_\top^{k+1}}.$$

Using Proposition 6.13, we then easily derive the expression of $\tilde{\Delta}^k(v_1 \cdots v_n)$ composing with $\rho^{\otimes(k+1)}$. Note that for $I = \emptyset$, $\rho(v_I) = 0$, otherwise $\rho(v_I) = v_I$. We have

$$\tilde{\Delta}^{(k)}(v_1 \cdots v_n) = \sum_{\substack{I_1 \sqcup \cdots \sqcup I_{k+1} = [n] \\ I_1, \dots, I_{k+1} \neq \emptyset}} v_1 \otimes \cdots \otimes v_{I_{k+1}} \quad \in \mathrm{Prim}(H)^{\otimes_\top^{k+1}}.$$

For $k = n-1$ each of the sets I_j only contains one element so that we get the expected formula. For $k \geq n$ this expression vanishes since some of the sets are empty. This ends the proof of the statement.

□

The following result will be useful in the sequel.

Proposition 6.15. *Let H be a filtered, connected, locality bialgebra and $J \neq \{0\}$ a locality left, right or two-sided coideal of H. Then J contains non zero primitive elements of H.*

Proof. Consider the filtration $J^n = J \cap H^n, n \in \mathbb{Z}_{\geq 0}$ on J induced by the one on H. Since $J \neq \{0\}$, there is some element $0 \neq x \in J$ of minimum degree k among the elements of J. Explicitly, $x \in J^k$ and $J^n = \{0\}$ for every $n < k$. The existence of x is guaranteed. Indeed, let us write

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$$

with $\sum x' \otimes x'' \in \ker(\epsilon) \otimes_\top \ker(\epsilon)$. Since Δ respects the filtration, if x' and x'' are non zero, there are integers $0 < m, n < k$ such that $x' \in H^n$ and $x'' \in H^m$ with $n + m \leq k$. If J is a left, resp. right coideal, then x', resp. x'' lies in J, which contradicts the minimality of k. Therefore at least one of the two elements x' or x'' vanishes, which implies that $\sum x' \otimes x'' = 0$, and thus x is primitive. □

6.3 Hopf algebraic structure of the locality symmetric algebra and of the locality universal enveloping algebra

On the one hand, the Milnor-Moore theorem relates a Hopf algebra H with a Hopf algebra built from the universal enveloping algebra of the Lie algebra of primitive elements of H. On the other hand, the Poincaré-Birkhoff-Witt theorem relates the coalgebraic structure of the symmetric algebra with that on the universal enveloping algebra of a Lie algebra. In order to extend both theorems to the locality setup, we need to show that the locality symmetric algebra of a locality vector space, and the locality universal enveloping algebra of a locality Lie algebra indeed admit a locality Hopf algebra structure. Both constructions are parallel in structure, we make use of the universal property of the locality symmetric algebra and of the locality universal enveloping algebra to build the coproduct, counit and antipode.

Proposition 6.16. *Given a locality vector space $(V, \top)$ and assuming conjectural statement 5.33 holds true for $(V, \top)$, the locality symmetric algebra $(S_\top(V), \top_S, m_S, u_s)$ can be endowed with a graded, connected, commutative, cocommutative Hopf algebra structure where $V \sim (S_\top(V))_1 \subset \mathrm{Prim}(S_\top(V))$.*

Proof. By means of Proposition 5.44 and if conjectural statement 5.33 holds for $(V, \top)$, then $(S_\top(V), \top_S, m_S, u_S)$ is a graded, connected, commutative locality algebra. Consider the map $\delta : V \to S_\top(V) \otimes S_\top(V)$ defined by $\delta(x) = x \otimes 1 + 1 \otimes x$. Since it is linear and $S_\top(V) \otimes S_\top(V)$ is commutative, by means of Theorem 5.45, there is a unique morphism of commutative locality algebras $\Delta_S : S_\top(V) \to S_\top(V) \otimes S_\top(V)$ which extends δ. This is the so called unshuffle coproduct. Notice that by construction Δ_S is cocommutative. For the counit, consider the function which maps V to $0 \in \mathbb{K}$, then Theorem 5.45 yields the existence of the map $\epsilon : S_\top(V) \to \mathbb{K}$ which is the only morphism of commutative locality algebras vanishing identically in V. We have described how $S_\top(V)$ is naturally endowed with the structure of a graded, connected, locality bialgebra. By means of Proposition 6.10, it is a graded, connected, commutative locality Hopf algebra which moreover happens to be cocommutative. □

Proposition 6.17. *Given a locality Lie algebra $(\mathfrak{g}, \top)$ and assuming the statement 5.39 holds true for $(\mathfrak{g}, \top)$, the universal enveloping algebra $U_\top(\mathfrak{g})$ together with the locality relation $\top_U$ (from Definition 4.46) can be equipped with a filtered, cocommutative locality Hopf algebra structure where $\iota_\mathfrak{g}(\mathfrak{g}) \subset \mathrm{Prim}(U_\top(\mathfrak{g}))$, where $\iota_\mathfrak{g} : \mathfrak{g} \to U_\top(\mathfrak{g})$ is the canonical map.*

Proof. So far, assuming that the conjectural statement 5.39 holds true for $(\mathfrak{g}, \top)$, we know from Proposition 5.40, that

$$(U_\top(\mathfrak{g}), \top_U, m_U, u)$$

is an associative unital filtered locality algebra. In order to equip it with a coproduct, we consider the map $\delta : \mathfrak{g} \to U_\top(\mathfrak{g}) \otimes_\top U_\top(\mathfrak{g})$ defined by $\delta(x) := \iota_\mathfrak{g}(x) \otimes 1 + 1 \otimes \iota_\mathfrak{g}(x)$. One can check that it is locality linear and $\delta([x, y]) = \delta(x)\delta(y) - \delta(y)\delta(x)$. Hence, Theorem 5.41 (which applies since we have assumed that the conjectural statement 5.39 holds true for $(\mathfrak{g}, \top)$) gives the existence and uniqueness of a locality algebra morphism $\Delta : U_\top(\mathfrak{g}) \to U_\top(\mathfrak{g}) \otimes_\top U_\top(\mathfrak{g})$ which extends δ. Note that by construction the elements in $\iota_\mathfrak{g}(\mathfrak{g})$ are primitive and Δ is cocommutative. For the counit we consider the zero map from $\mathfrak{g}$ to $\mathbb{K}$. This is indeed a locality Lie algebra morphism and once again by Theorem 5.41, there is a unique locality algebra morphism $\epsilon : U_\top(\mathfrak{g}) \to \mathbb{K}$ which vanishes identically on $\iota_\mathfrak{g}(\mathfrak{g})$. Therefore $U_\top(\mathfrak{g})$ with this coproduct and counit is a filtered connected bialgebra over $\mathbb{K}$. Consider the locality Lie algebra morphism $\sigma : \mathfrak{g} \to U_\top(\mathfrak{g})$ defined by $\sigma(x) = -\iota_\mathfrak{g}(x)$. Once more, Theorem 5.41 gives the existence and uniqueness of a locality algebra morphism $S : U_\top(\mathfrak{g}) \to U_\top(\mathfrak{g})$ which extends σ. To prove that it is an antipode, for $\alpha \in \mathbb{K} \subset U_\top(\mathfrak{g})$, we see that $S \star I(\alpha) = \alpha\,(S(1)I(1)) = \alpha$ and for $x_i \in \iota_\mathfrak{g}(\mathfrak{g})$, we have $S \star I(x_1 \cdots x_k) = \sum_{J \subset [k]} (-1)^{|J|} x_1 \cdots x_k = 0$ which shows that $S \star I = u \circ \epsilon$. Similarly, one shows that $I \star S = u \circ \epsilon$, so that S is an antipode and $U_\top(\mathfrak{g})$ is a locality Hopf algebra. □

Proposition 6.18. *Let $(\mathfrak{g}, \top, [,]_\top)$ be a locality Lie algebra and $(\mathfrak{g}', \top', [,]\big|_{\top'})$ a locality Lie sub-algebra of $\mathfrak{g}$. Assuming that the conjectural statement 5.39 holds true for $\mathfrak{g}'$ and $\mathfrak{g}$, then $U_{\top'}(\mathfrak{g}')$ is a locality Hopf sub-algebra of $U_\top(\mathfrak{g})$.*

Proof. By means of Proposition 4.47, $U_{\top'}(\mathfrak{g}')$ is a locality sub-algebra of $U_\top(\mathfrak{g})$. The inclusion map $\mathfrak{g}' \hookrightarrow \mathfrak{g}$ induces an injective map $\iota : U_{\top'}(\mathfrak{g}') \hookrightarrow U_\top(\mathfrak{g})$. It is easy to see that the counits relate by $\epsilon' = \epsilon \circ \iota$.

Moreover, since $\iota_{\mathfrak{g}}(\mathfrak{g})$ (resp $\iota_{\mathfrak{g}'}(\mathfrak{g}')$) generate $U_\top(\mathfrak{g})$ (resp. $U_{\top'}(\mathfrak{g}')$) as locality algebras and the coproducts are locality algebra morphisms, it is easy to see that $U_\top(\mathfrak{g}')$ is a locality sub-bialgebra of $U_\top(\mathfrak{g})$. Let $S_{U'}$ be the antipode on $U_\top(\mathfrak{g}')$, then $S_{U'} = S_U \circ \iota$ follows from the fact that the antipodes are completely determined by their action on $\iota_g(\mathfrak{g})$ (resp. $\iota_{\mathfrak{g}'}(\mathfrak{g}')$), and for $x \in \iota_{\mathfrak{g}'}(\mathfrak{g}')$,

$$S_{U'}(x) = -x = -\iota(x) = S_U(\iota(x)).$$

This proves that $U_\top(\mathfrak{g}')$ is indeed a locality Hopf sub-algebra of $U_\top(\mathfrak{g})$. □

Corollary 6.19. Let $(\mathfrak{g}, \top, [,]_\top)$ and $(\mathfrak{g}', \top', [,]'_\top)$ be two isomorphic locality Lie algebras, then $U_\top(\mathfrak{g})$ and $U_{\top'}(\mathfrak{g}')$ are isomorphic as locality Hopf algebras.

Proof. By means of Corollary 5.42 there is an isomorphism of locality algebras $\phi : U_\top(\mathfrak{g}) \to U_{\top'}(\mathfrak{g}')$. Since the coproduct, counit, and antipode are locality algebra morphisms, $\phi(\iota_{\mathfrak{g}}(\mathfrak{g})) = \iota_{\mathfrak{g}'}(\mathfrak{g}')$, and $\iota_{\mathfrak{g}}(\mathfrak{g})$ (resp. $\iota_{\mathfrak{g}'}(\mathfrak{g}')$) generate $U_\top(\mathfrak{g})$ (resp. $U_{\top'}(\mathfrak{g}')$) as a locality algebra, then we only need to check that the counit, coproduct and antipode commute with ϕ on $\iota_{\mathfrak{g}}(\mathfrak{g})$. Indeed, it is true by construction, since all the elements in $\iota_{\mathfrak{g}}(\mathfrak{g})$ (resp. $\iota_{\mathfrak{g}'}(\mathfrak{g}')$) are primitive elements of $U_\top(\mathfrak{g})$ (resp. $U_{\top'}(\mathfrak{g}')$); they lie in the kernel of ϵ (resp. ϵ'); and $\phi(S(\iota_{\mathfrak{g}}(g))) = -\phi(\iota_{\mathfrak{g}}(g)) = S'(\phi(\iota_{\mathfrak{g}}(g)))$. Thus $U_\top(\mathfrak{g})$ and $U_{\top'}(\mathfrak{g}')$ are isomorphic as locality Hopf algebras as expected. □

The final ingredient we need before stating and proving the locality version for the Milnor-Moore theorem is a description of the primitive elements of the universal enveloping algebra, which is often presented as a corollary of the Poincaré-Birkhof-Witt theorem. However, there is no locality version of this theorem available yet. We therefore provide an alternative proof using Zorn's lemma. We must notice that the following proof works also for the trivial locality relation in $\mathfrak{g}$, namely $\top = \mathfrak{g} \times \mathfrak{g}$, thus, it is valid also in the classical setting without the use of the Poincaré-Birkhof-Witt theorem.

Proposition 6.20. *Given a locality Lie algebra $(\mathfrak{g}, \top)$ over $\mathbb{K}$, if the conjectural statement 5.39 holds true, the set of primitive elements of the locality universal enveloping algebra $U_\top(\mathfrak{g})$ coincides with $\iota_{\mathfrak{g}}(\mathfrak{g})$:*

$$\mathrm{Prim}(U_\top(\mathfrak{g})) = \iota_{\mathfrak{g}}(\mathfrak{g}),$$

where $\iota_{\mathfrak{g}}$ is the canonical map from $\mathfrak{g}$ to $U_\top(\mathfrak{g})$.

Proof. By the very construction of the coproduct on $U_\top(\mathfrak{g})$, we have $\iota_{\mathfrak{g}}(\mathfrak{g}) \subset \mathrm{Prim}(U_\top(\mathfrak{g}))$ (See Proposition 6.17). For the other inclusion, for any $n \in \mathbb{N}$ we consider the set

$$G_n := \{x_1^{k_1} \cdots x_m^{k_m} \in U_\top(\mathfrak{g}) |\ (x_1, \ldots, x_m) \in \iota_{\mathfrak{g}}(\mathfrak{g})^{\times_{\top_U}^m} \ \wedge\ k_i \in \mathbb{N} \ \wedge \sum_{i=[m]} k_i \leq n\}. \tag{2.40}$$

which by construction, generates the space $(U_\top(\mathfrak{g}))^n$ of filtration degree n in the natural filtration of $U_\top(\mathfrak{g})$. Lemma 1.5 applied to $G_1 = \mathbb{K}\,1$ yields the existence of a vector space basis B_1 of $(U_\top(\mathfrak{g}))^1$ such that $\{1\} \subset B_1 \subset G_1$, so for filtration degree 1. Since $B_1 \subset G_2$, Lemma 1.5 yields a basis B_2 of $(U_\top(\mathfrak{g}))^2$ such that $B_1 \subset B_2 \subset G_2$. We proceed inductively to build $B := \bigcup_{n\in\mathbb{N}} B_n$ which is a Hamel (vector space) basis of $U_\top(\mathfrak{g})$. We use the simplified notation $\vec{x}^{\vec{k}} := x_1^{k_1} \cdots x_n^{k_n}$ and $|\vec{k}| := k_1 + \cdots + k_n$. Note that for $\vec{x}^{\vec{k}} \in B$ with $|\vec{k}| = n$, it is linearly independent of B_{n-1}.

A primitive element y of $U_\top(\mathfrak{g})$ can be expressed in terms of the basis B as

$$y = \sum_{\vec{x}^{\vec{k}} \in B} \alpha_{\vec{x}^{\vec{k}}}\, \vec{x}^{\vec{k}}$$

where only finitely many $\alpha_{\vec{x}^{\vec{k}}}$ are non zero. Let $N = \max\{|\vec{k}| : \alpha_{\vec{x}^{\vec{k}}} \neq 0\}$. If $N = 1$ we have $y \in \iota_{\mathfrak{g}}(\mathfrak{g})$ as required. Let us now assume that $N > 1$. Then $\tilde{\Delta}^{(N-1)}(y) = 0$ since y is primitive. By (2.39), one can write

$$0 = \tilde{\Delta}^{(N-1)}(y) = \sum_{|\vec{k}|=N} \alpha_{\vec{x}^{\vec{k}}} \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(N)}.$$

with $x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(N)} \in \mathrm{Prim}(U(\mathfrak{g}))^{\otimes_\top^N}$.

Applying the product m yields

$$0 = m^{(N-1)}(\tilde{\Delta}^{(N-1)}(y)) = \sum_{|\vec{k}|=N} \alpha_{\vec{x}^{\vec{k}}} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(N)} \in U_\top(\mathfrak{g}).$$

Since $\iota_{\mathfrak{g}}(\mathfrak{g}) = (U_\top(\mathfrak{g}))^1 \ni [x_i, x_j] = x_i x_j - x_j x_i$ for every i and j, we may reorder the x_i's to get the original elements of B at the cost of adding some lower order terms (l.o.t.) (with respect to the natural filtration of $U_\top(\mathfrak{g})$ given by the sets (2.40)). The resulting products arising in the new linear combination are linearly independent of the leading term due to the very manner the basis B was constructed. Hence, we have

$$0 = \sum_{|\vec{k}|=N} \frac{\alpha_{\vec{x}^{\vec{k}}}}{N!} \vec{x}^{\vec{k}} + \text{l.o.t.}$$

Since the elements of the basis B are linearly independent, we may conclude that all $\alpha_{\vec{x}^{\vec{k}}} = 0$ except if $N = 1$. Therefore $\mathrm{Prim}(U_\top(\mathfrak{g})) \subset \iota_{\mathfrak{g}}(\mathfrak{g})$. Thus $\mathrm{Prim}(U_\top(\mathfrak{g})) = \iota_{\mathfrak{g}}(\mathfrak{g})$. ☐

Remark 6.21. In the non-locality case, the injectivity of the canonical map $\iota_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})$, follows from the Poincaré-Birkhof-Witt theorem. However, this is not necessarily true in the locality setup. In general $\iota_{\mathfrak{g}}(\mathfrak{g})$ is a quotient of $\mathfrak{g}$, with the property that $U_\top(\mathfrak{g}) \simeq U_\top(\iota_{\mathfrak{g}}(\mathfrak{g}))$.

A case which will be of particular interest in the following section, is when $\mathfrak{g}$ is the Lie algebra of primitive elements of a locality bialgebra $(B, \top)$. In that case, the universal property of the locality universal enveloping algebra (Theorem 4.48) yields the existence of a pre-locality algebra morphism $\phi : U_\top(\mathfrak{g}) \to B$ which extends the canonical injection $\iota : \mathfrak{g} \to B$. Therefore, the canonical map $\iota_{\mathfrak{g}} : \mathfrak{g} \to U_\top(B)$ is indeed injective in such cases.

6.4 A locality Cartier-Quillen-Milnor-Moore theorem

Assuming that the conjectural statement 5.33 holds true, we can now prove a locality version of the Cartier-Quillen-Milnor-Moore theorem.

Theorem 6.22 (Locality Cartier-Quillen-Milnor-Moore theorem)**.** Let $(H, \top)$ be a graded, connected, cocommutative locality Hopf algebra such that its primitive elements $\mathrm{Prim}(H)$ satisfy the conjectural statements 5.39 and 5.33. In that case, we have the following isomorphism of locality Hopf algebras:

$$(H, \top) \sim (U_\top(\mathrm{Prim}(H)), \top_U) \tag{2.41}$$

with $\top_U$ the locality relation of Definition 4.46 in the case $\mathfrak{g} = \mathrm{Prim}(H)$.

Proof.

- We first prove that H is generated by its primitive elements $\mathrm{Prim}(H)$ as a locality algebra. Let H' be the locality subalgebra of H generated by $\mathrm{Prim}(H) \cup \{1\}$ and let $0 \neq x \in H$. By Proposition 6.14, 2, for k big enough, $\tilde{\Delta}^{(k)}(x) = 0$. Set $\deg_p(x)$ to be the minimum of all such integers k. Using induction over $\deg_p(x)$, we show that $H \subset H'$. If $\deg_p(x) = 0$, then by definition of $\tilde{\Delta}^{(0)}$, $\rho(x) = 0$. Hence $x \in \mathbb{K} \subset H'$. If $\deg_p(x) = 1$, then x is a primitive element so that $x \in H'$. If $n = \deg_p(x) > 0$, by means of Proposition 6.14, 1. We have

$$\tilde{\Delta}^{(n-1)}(x) = \sum_i x_1^{(i)} \otimes \cdots \otimes x_n^{(i)}$$

where all the $x_j^{(i)}$ are primitive elements. The cocommutativity of H implies invariance under the natural action of the symmetric group so we have:

$$\tilde{\Delta}^{(n-1)}(x) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sum_i x_{\sigma(1)}^{(i)} \otimes \cdots \otimes x_{\sigma(n)}^{(i)}$$

Exchanging the two sums (both over finite sets), we then recognize in the innermost sum the expression of $\tilde{\Delta}^{(n-1)}(x_1^{(i)} \cdots x_n^{(i)})$ given by Proposition 6.14 3. The linearity of $\tilde{\Delta}^{(n-1)}$ then yields

$$\tilde{\Delta}^{(n-1)}(x) = \tilde{\Delta}^{(n-1)}\left(\frac{1}{n!}\sum_i x_1^{(i)} \cdots x_n^{(i)}\right) \iff \tilde{\Delta}^{(n-1)}\left(x - \frac{1}{n!}\sum_i x_1^{(i)} \cdots x_n^{(i)}\right) = 0.$$

Hence, by definition of the degree,

$$\deg_p\left(x - \frac{1}{n!}\sum_i x_1^{(i)} \cdots x_n^{(i)}\right) < n.$$

By the induction hypothesis, this element lies in H', and so does $x \in H'$. We infer that $H' = H$, this means that H is generated by its primitive elements $\mathrm{Prim}(H)$ as a locality algebra.

- To build the isomorphism (2.41) we assume that the conjectural statement 5.33 holds true. By Proposition 6.11, the set $\mathrm{Prim}(H)$ of primitive elements of H has a graded locality Lie algebra structure. We can therefore build its enveloping algebra $U_\top(\mathrm{Prim}(H))$. Assuming that the conjectural statement 5.33 holds true, we can apply the universal property (Theorem 5.41) to extend the locality Lie algebra morphism given by the injection $i : \mathrm{Prim}(H) \to H$ to a locality algebra morphism
$$\phi : U_\top(\mathrm{Prim}(H)) \to H,$$
which stabilizes the elements in $\mathrm{Prim}(H)$.

  Moreover, since ϕ is a locality algebra morphism, again by Lemma 2.29 the range of ϕ is a locality subalgebra of H which contains H'. Hence, by the first part of this proof, ϕ is surjective.

- Let us show that ϕ is a coalgebra morphism. The coproducts Δ_U on $U_\top(\mathrm{Prim}(H))$ and Δ on H are by definition algebra morphisms. Since they coincide on the set $\mathrm{Prim}(H)$, $(\phi \otimes \phi) \circ \Delta_U = \Delta \circ \phi$ on $\mathrm{Prim}(H)$. This identity extends everywhere since $\mathrm{Prim}(H)$ generates both $U_\top(\mathrm{Prim})$ and H as locality algebras. We still need to show $\epsilon_U = \epsilon \circ \phi$, with ϵ_U the counit of $U_\top(\mathrm{Prim}(H))$ and ϵ the counit of H. Again, since $\mathrm{Prim}(H)$ generates both $U_\top(\mathrm{Prim})$ and H as locality algebras, it is enough to show that these maps coincide on $\mathrm{Prim}(H)$. On the one hand, by definition of ϵ_U (see the proof of Proposition 6.17), ϵ_U vanishes on $\mathrm{Prim}(H)$. On the other hand, for any $h \in \mathrm{Prim}(H)$, using the property of ϵ and the canonical identification $\mathbb{K} \otimes H \simeq H$, we have:
$$h = (\epsilon \otimes \mathrm{Id}_H) \circ \Delta(h) = \epsilon(h) \otimes 1_H + \epsilon(1) \otimes h = \epsilon(h) 1_H + h.$$
Therefore ϵ vanishes on $\mathrm{Prim}(H)$ as required and ϕ is a coalgebra morphism.

- We prove the injectivity of ϕ ad absurdum. The fact that ϕ is a locality coalgebra morphism, implies by Lemma 6.7, that its kernel $\ker(\phi)$ is a locality coideal of $U_\top(\mathrm{Prim}(H))$. Since ϕ is an algebra morphism, $\phi(1) = 1$ so $\ker(\phi) \cap \mathbb{K}\,1 = \{0\}$. By Proposition 6.15 applied to $J = \ker(\phi)$, assuming the latter is non trivial, it must contain a primitive element of $U_\top(\mathrm{Prim}(H))$. However this leads to a contradiction since, by Proposition 6.20, $\mathrm{Prim}(U_\top(\mathrm{Prim}(H))) = \mathrm{Prim}(H)$ which is fixed by ϕ, therefore none of them lies in the kernel. Hence ϕ is injective.

- To show that ϕ is a locality isomorphism of Hopf algebras, we still need to prove that $\phi^{-1} : H \longrightarrow U_\top(\mathrm{Prim}(H))$ is a locality algebra morphism. The previous items give the existence of the inverse map ϕ^{-1}. By definition of ϕ, on any element h of H, ϕ^{-1} acts as:
$$\phi^{-1}(h) = \sum_i x_1^{(i)} \otimes \cdots \otimes x_n^{(i)} \tag{2.42}$$
for some $n \in \mathbb{N}^*$ and primitive elements $x_j^{(i)}$ such that $h = \sum_i x_1^{(i)} \cdots x_n^{(i)}$. The right-hand-side of (2.42) actually stands for an equivalence class of tensor products, which we write as a tensor to simplify notations. We analyise each of the sumands separately, and distinguish two cases.

If $n = 1$, then h is a primitive element of H and ϕ^{-1} restricted to primitive elements is simply the projection from H to $U_\top(\text{Prim}(H))$. This map is a locality map by definition of the locality on the quotient space $U_\top(\text{Prim}(H))$.

If $n \geq 2$, using Equation (2.39) of Proposition 6.14, we have

$$\tilde{\Delta}^{(n-1)}(h) = \tilde{\Delta}^{(n-1)}(x_1 \cdots x_n) = \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} = \frac{1}{n!} x_1 \otimes \cdots \otimes x_n,$$

where, same as before, the last equality is a consequence of $U_\top(\mathfrak{g})$ and H being cocommutative. The fact that ϕ^{-1} is a locality map then follows from combining together the fact that $\tilde{\Delta}^{(n-1)}$ is a locality map, and the definitions of the locality relations on $\mathcal{T}_\top(\text{Prim}(H))$ and $U_\top(\text{Prim}(H))$.

- Then by Proposition 6.17, $U_\top(\text{Prim}(H))$ is a Hopf algebra, and since ϕ is a morphism of graded locality bialgebras and an isomorphism, it is an isomorphism of locality Hopf algebras. □

6.5 Consequences of the Milnor-Moore theorem

We now present some useful consequences of the locality Milnor-Moore theorem (Theorem 6.22). The first rather direct consequence is that the locality relation on a graded, connected, cocommutative Hopf algebra is entirely determined by the locality relation on its primitive elements as the following corollary states.

Corollary 6.23. Let $(H_1, m_1, \Delta_1, \top_1)$ and $(H_2, m_2, \Delta_2, \top_2)$ be two graded, connected, cocommutative locality Hopf algebras. Then if

$$(\text{Prim}(H_1), [,]_1, \tilde{\top}_1) \simeq (\text{Prim}(H_2), [,]_2, \tilde{\top}_2)$$

as locality Lie algebras (here $[,]_i$ is the antisymmetrisation of the product m_i, and where we have set $\tilde{\top}_i := \top_i\big|_{\text{Prim}(H_i)\times\text{Prim}(H_i)}$), then

$$(H_1, m_1, \Delta_1, \top_1) \simeq (H_2, m_2, \Delta_2, \top_2)$$

as locality Hopf algebras.

Proof. By means of Corollary 6.19 the isomorphism $(\text{Prim}(H_1), [,]_1, \tilde{\top}_1) \simeq (\text{Prim}(H_2), [,]_2, \tilde{\top}_2)$ implies that the universal enveloping algebras of these Lie algebras are isomorphic as locality Hopf algebras. The result then follows from Theorem 6.22. □

This further leads to the observation that locality Hopf algebras are not generally speaking ordinary locality Hopf algebras with an "added" locality relation, and a restricted product. In other words, whenever the locality Milnor-Moore theorem applies, one cannot simply "turn on" locality.

Corollary 6.24. Let (H, m, Δ) be a graded, connected, cocommutative Hopf algebra. The trivial locality relation $\top = H \times H$ is the only locality relation $\top$ on H such that $(H, \top, m|_\top, u, \Delta, \epsilon, S)$ is a locality Hopf algebra.

Proof. We proceed by contradiction, assuming such a non-trivial locality relation $\top$ exists. It follows that there are primitive elements a and b such that $a \not\top b$. Indeed, if this were not true, then $\top|_{\text{Prim}(H)\times\text{Prim}(H)} = \text{Prim}(H) \times \text{Prim}(H)$ would be the trivial locality on $\text{Prim}(H)$ and hence on H thanks to Corollary 6.23. Let a and b be primitive elements such that $a \not\top b$, then

$$\Delta(m(a, b)) = \Delta(a)\Delta(b) = (a \otimes 1 + 1 \otimes a)(b \otimes 1 + 1 \otimes b) = m(a, b) \otimes 1 + a \otimes b + b \otimes a + 1 \otimes m(a, b) \notin H \otimes_\top H.$$

However, this contradicts the inclusion $\Delta(H) \subset H \otimes_\top H$, which follows from the fact that the coproduct Δ of the Hopf algebra coincides with the locality coproduct of the locality Hopf algebra. □

Theorem 6.22 also allows to describe examples of locality Hopf algebras.

Example 6.25. *With the notations of Example 6.9, let P_V be the subspace of primitive elements of $\mathcal{T}_\top(V)$. Endowed with the locality Lie bracket induced by the commutator, namely $[a,b] = a\otimes b - b\otimes a$ whenever $a\top_\otimes b$, P_V is a locality Lie algebra. It then follows from Theorem 6.22 that*

$$(\mathcal{T}_\top(V), \top_\otimes) \simeq (\mathcal{U}_\top(P_V), \top_U),$$

this last isomorphism is an isomorphism of locality Hopf algebras.

The locality Milnor-Moore theorem (Theorem 6.22) allows to build locality Hopf sub-algebras from locality Lie subalgebras.

Corollary 6.26. Let $(H,\top)$ be a graded, connected, cocommutative locality Hopf algebra whose set $\mathfrak{g} := \mathrm{Prim}(H)$ of primitive elements obeys the conjectural statements 5.33 and 5.39. There is a one-to-one correspondence

$$((H,\top)\supset(H',\top')) \quad \longleftrightarrow \quad ((\mathfrak{g}',\top')\subset(\mathfrak{g},\top)) \tag{2.43}$$

between graded, connected, cocommutative locality Hopf sub-algebras of H and locality Lie sub-algebras of $(\mathfrak{g} = \mathrm{Prim}(H),\top)$ where the localities $\top$ and $\top'$ on the right hand side are the restrictions of the ones of the left hand side.

Proof. Let $\mathfrak{g}'$ be a subset of $\mathfrak{g}$ =Prim(H) with locality $\top'\subset\top$ so that $(\mathfrak{g}',\top',[,]|_{\top'})$ is locality Lie sub-algebra of $(\mathfrak{g},\top,[,]_\top)$. Then by Proposition 6.18, $U_{\top'}(\mathfrak{g}')$ is a locality sub-algebra of $U_\top(\mathfrak{g})$. It is therefore isomorphic to a locality Hopf sub-algebra $(H',\top')\simeq U_{\top'}(\mathfrak{g}')$ of $(H,\top)\simeq U_\top(\mathfrak{g})$.

Conversely, let H' be a locality Hopf sub-algebra of H with locality $\top'\subseteq\top$. Then $\mathfrak{g} := \mathrm{Prim}(H')\subseteq \mathfrak{g} = \mathrm{Prim}(H)$. By Proposition 6.11, setting $\top'' := \top'\cap(\mathfrak{g}'\times\mathfrak{g}')$, then $(\mathfrak{g}',\top'',[,]|_{\top''})$ is a locality Lie sub-algebra of $\mathfrak{g}$. Thus we have a map from locality Hopf sub-algebras of H and locality Lie sub-algebras of $\mathfrak{g}$.

The two maps built above are inverse of each other by construction, which proves the corollary.

□

Remark 6.27. It follows from Corollary 6.24 that $H = H'$ and $\mathfrak{g} = \mathfrak{g}'$ implies $\top = \top'$. Note that this statement is trivially satisfied in the usual (non-locality) setup where $\top = \top'$ is the trivial locality. The subsequent Proposition 6.29 illustrates the case when $\mathfrak{g} = \mathfrak{g}'$ but $H \neq H'$, which is specific to the locality setup, since it cannot occur in the non-locality setup due to the Milnor-Moore theorem.

We can apply Corollary 6.26 to the Hopf algebra of rooted forests in Example 1.41. In order to introduce locality, we first decorate the rooted forests. Recall that, for a set Ω, an **Ω-decorated** forest is a pair (F,d_F) with F a forest and $d_F : V(F)\longrightarrow\Omega$ a map. We often omit the d_F to lighten the notation. We also write $\mathcal{F}_\Omega$ for the set of Ω-decorated forests. The notions of Definition 1.40 easily generalise to decorated forests.

Definition 6.28. [23, Definition 3.1] Let $(\Omega,\top)$ be a locality set. A **properly** Ω-decorated forest is a Ω-decorated forest (F,d_F) such that any disjoint pair of vertices of F are decorated by independent elements of Ω:

$$\forall(v_1,v_2)\in V(F)\times V(F),\ v_1\neq v_2 \implies d_F(v_1)\top d_F(v_2).$$

We denote by $\mathcal{F}_\Omega^{\mathrm{prop}}$ the set of finite linear combinations of properly Ω-decorated forests. We endow $\mathcal{F}_\Omega^{\mathrm{prop}}$ with a locality relation $\top_{\mathcal{F}_\Omega}$ induced from the relation $\top$ on Ω:

$$(F_1,d_1)\top_{\mathcal{F}_\Omega}(F_2,d_2) \ :\Longleftrightarrow\ \forall(v_1,v_2)\in V(F_1)\times V(F_2),\ d_1(F_1)\top d_2(F_2),$$

and extend it linearly to finite linear combinations of forests.

Since the linear combination of trees are the primitive elements of the Grossman-Larson Hopf algebra, the locality relation $\top_{\mathcal{F}_\Omega}$ restricted to properly Ω-decorated trees induces a locality Hopf algebra structure on $\mathcal{F}_\Omega^{\mathrm{prop}}$:

Proposition 6.29. *Given a locality set $(\Omega,\top)$, the quadruple $(\mathcal{F}_\Omega^{\mathrm{prop}},\top_{\mathcal{F}_\Omega},*_{\top_{\mathcal{F}_\Omega}},\Delta_*)$ is a graded, connected, cocommutative locality Hopf algebra equipped with the product $*_{\top_{\mathcal{F}_\Omega}}$ given by the restriction of the $*$ product of Definition 1.40 to the graph $\top_{\top_{\mathcal{F}_\Omega}}$ of the locality relation.*

Proof. Applying Corollary 6.26 to the locality Lie algebra $(\mathfrak{g} := \mathrm{Prim}(\mathcal{F}_\Omega^{\mathrm{prop}}), \top_{\mathrm{triv}})$ equipped with the trivial locality relation $\top_{\mathrm{triv}} := \mathfrak{g} \times \mathfrak{g}$ and $(\mathfrak{g}' := \mathrm{Prim}(\mathcal{F}_\Omega^{\mathrm{prop}}), \top_{\mathcal{F}_\Omega})$ shows that $\mathcal{F}_\Omega^{\mathrm{prop}} := \mathcal{U}_{\top_{\mathcal{F}_\Omega}}(\mathfrak{g}')$ is a locality sub-Hopf algebra of $(\mathcal{F}_\Omega = \mathcal{U}_{\top_{\mathrm{triv}}}(\mathfrak{g}), \top_{\mathrm{triv}})$. $\square$

The locality Milnor-Moore theorem 6.22 also provides refinements of properties in the ordinary setup, here a decomposition involving mutually independent arguments.

Corollary 6.30. Given a locality set $(\Omega, \top)$, any properly Ω-decorated rooted forest $F = T_1 \cdots T_n$ can be expressed as a linear combination of $*$-products of finitely many pairwise independent properly Ω-decorated rooted trees $t_j^{(i)}$:

$$F = T_1 \cdots T_n = \sum_{n=1}^{N} \alpha_n t_1^{(n)} * \cdots * t_{p_n}^{(n)}$$

for $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$.

Proof. For the sake of simplicity, throughout the proof, we set $\top := \top_{\mathcal{F}_\Omega}$. Let $\phi : U_\top(\mathrm{Prim}(\mathcal{F}_\Omega^{\mathrm{prop}})) \longrightarrow \mathcal{F}_\Omega^{\mathrm{prop}}$ be the isomorphism of locality Hopf algebra given by Theorem 6.22. Since $\mathrm{Prim}(U_\top(\mathrm{Prim}(\mathcal{F}_\Omega^{\mathrm{prop}})))$ is the set of Ω-decorated rooted trees, for any such tree T we have $\phi([T]) = T$.

Since the map ϕ is an isomorphism, for any properly Ω-decorated rooted forest $F = T_1 \cdots T_n$ there is an element $\sum_{n=1}^{N} \alpha_n t_1^{(n)} \star \cdots \star t_{p_n}^{(n)} \in U_\top(\mathrm{Prim}(\mathcal{F}_\Omega^{\mathrm{prop}})$ (where we write $\star$ for the product of $U_\top(\mathrm{Prim}(\mathcal{F}_\Omega^{\mathrm{prop}}))$ such that

$$F = T_1 \cdots T_n = \phi\left(\sum_{n=1}^{N} \alpha_n t_1^{(n)} \star \cdots \star t_{p_n}^{(n)}\right) = \sum_{n=1}^{N} \alpha_n \phi([t_1^{(n)}]) * \cdots * \phi([t_{p_n}^{(n)}]) = \sum_{n=1}^{N} \alpha_n t_1^{(n)} * \cdots * t_{p_n}^{(n)},$$

where we have used that ϕ is a morphism of algebras. $\square$

6.6 A locality Poincaré-Birkhoff-Witt theorem

As it was mentioned in the introduction, we prove a locality version of the Poincaré-Birkhoff-Witt theorem in Quillen's version, this is, as a morphism of locality coalgebras instead of providing a basis for the universal enveloping algebra starting from a basis of the locality Lie algebra. This is mainly because as it was mentioned before, bases of locality vector spaces do not always behave well with the locality relations. The proof we provide is based mostly in [17, Section 4.2]. However, Cartier and Patras make use of the fact that idempotent vectors generate the symmetric tensors. Since the locality relation is not required to be reflexive, then idempotent tensors do not necessarily lie in the locality tensor algebra, thus the adaption to the locality setup requires some steps to be shown differently. More precisely, Lemma 6.36 for which we provide a combinatorial proof developed in Lemma 6.34. Also for the proof of the Poincaré-Birkhoff-Witt theorem itself, we complete what we consider could be a gap in the proof provided in [17] since the universal enveloping algebra is a priori not graded, but only filtered, and thus the iterated reduced coproduct based on the filtration is not well defined. For this purpose, we prove Lemmas 6.37 and 6.38 which are inspired from [29, Section 2.4].

Assumption: From the rest of the paragraph we will consider only locality Lie algebras $(\mathfrak{g}, \top_\mathfrak{g})$ for which the map $\iota_\mathfrak{g} : \mathfrak{g} \to U_\top(\mathfrak{g})$ is injective. The existence of such locality Lie algebras is granted by the universal property of the locality universal enveloping algebra (Theorem 5.41) as shown in the following proposition.

Proposition 6.31. *Let $(\mathfrak{g}, \top_\mathfrak{g})$ be a locality Lie algebra and $f : \mathfrak{g} \to \mathrm{Prim}(H)$ an injective morphism of locality Lie algebras where H is a connected Hopf algebra. Then, the canonical map $\iota_\mathfrak{g} : \mathfrak{g} \to U_\top(\mathfrak{g})$ is injective.*

Proof. Setting ι as the injection of $\mathrm{Prim}(H)$ into H, the map $\iota \circ f : \mathfrak{g} \to H$ is an injective morphism of locality Lie algebras, where the Lie bracket in H is the commutator. By means of the universal property of the locality universal enveloping algebra (Theorem 5.38), there exists a unique morphism of locality algebras $\phi_f : U_\top(\mathfrak{g}) \to H$ such that $f = \phi_f \circ \iota_g$. It follows from the injectivity of f that both ι_g and ϕ_f are injective. $\square$

In particular, the locality Lie algebras of primitive elements of a connected Hopf algebra is always injected into its universal enveloping algebra.

For the purpose of distinguishing the elements of $S_\top(\mathfrak{g})$, $\mathcal{T}_\top(\mathfrak{g})$, and $U_\top(\mathfrak{g})$, we stick to the conventions $g_1\otimes\cdots\otimes g_n$ for an element of $\mathcal{T}_\top(\mathfrak{g})$, $g_1\odot\cdots\odot g_n := \pi_S(g_1\otimes\cdots\otimes g_n)\in S_\top(\mathfrak{g})$, and $g_1\cdots g_n := \pi_U(g_1\otimes\cdots\otimes g_n)\in U_\top(\mathfrak{g})$, where the map π_S is the canonical map from V to $S_\top(V)$ (see Definition 4.38).

Consider for $n>0$ the morphism of $\mathfrak{S}_n$-modules $\Theta_n : (S_\top(V))_n \to V^{\otimes_\top^n}$, the existence of which is granted by Proposition 1.44, and which is described by

$$x_1\odot\cdots\odot x_n \mapsto \Theta(x_1\odot\cdots\odot x_n) := \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}. \tag{2.44}$$

The direct sum $\Theta = \bigoplus_{n\in\mathbb{Z}_{\geq 0}}\Theta_n : S_\top(V)\to\mathcal{T}_\top(V)$, where $\Theta_0(1_\mathbb{K}) = 1_\mathbb{K}$, is a graded linear map, and is a right inverse for π_S, namely

$$\pi_S\circ\Theta(x_1\odot\cdots\odot x_n) = x_1\odot\cdots\odot x_n.$$

Therefore Θ is injective.

Lemma 6.32. Let $(V,\top)$ be a locality vector space. If $\mathcal{T}_\top(V)$ is a locality algebra, the map $\Theta : (S_\top(V),\top_S)\longrightarrow(\mathcal{T}_\top(V),\top_\otimes)$ is a locality linear map.

Proof. The linearity of Θ is given by construction. For the locality consider $x = x_1\odot\cdots\odot x_n$ and $y = y_1\odot\cdots\odot y_m$ in $S_\top(V)$ such that $x\top_S y$. By definition of $\top_S$, there are $\sigma\in\mathfrak{S}_n$ and $\tau\in\mathfrak{S}_m$ such that $x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}\top_\otimes y_{\tau(1)}\otimes\cdots\otimes y_{\tau(m)}$. By means of Lemma 4.35 $x_{\sigma'(1)}\otimes\cdots\otimes x_{\sigma'(n)}\top_\otimes y_{\tau'(1)}\otimes\cdots\otimes y_{\tau'(m)}$ for every $\sigma'\in\mathfrak{S}_n$ and every $\tau'\in\mathfrak{S}_m$. Then, since $\mathcal{T}_\top(V)$ is a locality algebra, then

$$\Big(\Theta(x) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}\Big)\top_\otimes\Big(\frac{1}{m!}\sum_{\tau\in\mathfrak{S}_m} y_{\tau(1)}\otimes\cdots\otimes y_{\tau(m)} = \Theta(y)\Big)$$

which yields the result. □

We proceed to introduce the map which will serve as morphism of locality coalgebras between $S_\top(\mathfrak{g})$ and $U_\top(\mathfrak{g})$.

Lemma 6.33. The locality linear map $\Phi := \pi_U\circ\Theta = S_\top(\mathfrak{g})\to U_\top(\mathfrak{g})$ is surjective and preserves the filtration, where we are considering the filtration of $S_\top(V)$ induced by the grading.

Proof. Φ preserves the filtration as a consequence of Θ being a graded map and π_U being a filtered map. For the surjectivity, notice first that $U_\top^n(\mathfrak{g})$ is generated as vector space by the elements of the form $g_1\cdots g_k$ where $k\leq n$ and $g_i\in\mathfrak{g}$. We then only have to prove that every element of the form $g_1\cdots g_k$ lies in $\Phi(\bigoplus_{j=0}^n(S_\top(\mathfrak{g}))_j)$. The statement is true for $n=1$ since $\Phi(g)=g$ for every $g\in\mathfrak{g}$. Assume now that the statement is true for $k<n$ and consider $g_1\cdots g_n\in U_\top^n(\mathfrak{g})$. Notice that

$$\begin{aligned}\pi_U\circ\Theta(g_1\odot\cdots\odot g_n) &= \pi_U\Big(\frac{1}{n!}\sum_{\sigma\in S_n} g_{\sigma(1)}\otimes\cdots\otimes g_{\sigma(n)}\Big)\\ &= \frac{1}{n!}\sum_{\sigma\in S_n} g_{\sigma(1)}\cdots g_{\sigma(n)}\\ &= g_1\cdots g_n + \text{lower order terms.}\end{aligned}$$

By induction, all the lower order terms lie in $\Phi(\bigoplus_{j=0}^{n-1}(S_\top(\mathfrak{g}))_j)\subset\Phi(\bigoplus_{j=0}^{n}(S_\top(\mathfrak{g}))_j)$, thus $g_1\cdots g_n$ lies in $\Phi(\bigoplus_{j=0}^{n}(S_\top(\mathfrak{g}))_j)$ which proves the result. □

We recall that given two positive integers n and m, the set of (n,m)-Shuffles is defined as

$$(n,m)Sh := \{\sigma\in\mathfrak{S}_{n+m} : \sigma(1)<\sigma(2)<\cdots<\sigma(n),\ \text{and}\ \sigma(n+1)<\sigma(n+2)<\cdots<\sigma(n+m)\}.$$

It is well known [36] that the cardinality of the set $(n,m)Sh$ is

$$|(n,m)Sh| = \binom{n+m}{n} = \frac{(n+m)!}{n!m!}.$$

Lemma 6.34. Let n be a positive integer, and $g_1, \dots, g_n$ elements in a locality Lie algebra $(\mathfrak{g}, \top)$, then for a fixed integer $0 \leq k \leq n$ it follows that

$$\sum_{\substack{I' \sqcup J' = [n] \\ |I'| = k}} \sum_{\sigma \in \mathfrak{S}_n} g_{\sigma(I')} \otimes g_{\sigma(J')} = \sum_{\substack{I \sqcup J = [n] \\ |I| = k}} \frac{n!}{k!(n-k)!} \sum_{\substack{\mu \in \mathfrak{S}_k \\ \tau \in \mathfrak{S}_{n-k}}} g_{\mu_{\mathrm{Id}}(I)} \otimes g_{\tau_{\mathrm{Id}}(J)}.$$

Here we have set $g_{\sigma(I')} = g_{\sigma(i'_1)} \cdots g_{\sigma(i'_k)} \in U_\top(\mathfrak{g})$ where $I' := \{i'_1, \dots, i'_k\}$, and $i'_1 < \cdots < i'_k$, similarly for $g_{\sigma(J')}$. We also use the shortened notations $\mu_{\mathrm{Id}}(I) := \mu_{12\cdots k}(i_1 \cdots i_k)$, where $I := \{i_1, \dots, i_k\}$, and $i_1 < \cdots < i_k$ (resp. $\tau_{\mathrm{Id}}(J) := \tau_{12\cdots(n-k)}(j_1 \cdots j_{n-k})$ where $J := \{j_1, \dots, j_{n-k}\}$, and $j_1 < \cdots < j_{n-k}$.) Finally, we take the convention $g_{\sigma(\emptyset)} = g_{\tau_{\mathrm{Id}}(\emptyset)} = g_{\mu_{\mathrm{Id}}(\emptyset)} = 1$.

We illustrate the previous technical lemma with an example.

Example 6.35. *Let $n = 3$ and $k = 1$. Then the values that I (resp. I') can take are $\{1\}, \{2\}$, and $\{3\}$. Also $\mathfrak{S}_3 = \{123, 132, 213, 321, 312, 231\}$, $\mathfrak{S}_1 = \{1\}$, and $\mathfrak{S}_2 = \{12, 21\}$. Then for $I = I' = \{2\}$, $\sigma = 231$, $\mu = 1$, and $\tau = 21$, we have*

$$\begin{aligned} g_{\sigma(I')} \otimes g_{\sigma(J')} &= g_3 \otimes g_2 g_1, \qquad \text{and} \\ g_{\mu(I)} \otimes g_{\tau(J)} &= g_2 \otimes g_3 g_1. \end{aligned}$$

Then the whole sum of Lemma 6.34 is

$$\begin{aligned} \sum_{\substack{I' \sqcup J' = [3] \\ |I'| = 1}} \sum_{\sigma \in \mathfrak{S}_3} g_{\sigma(I')} \otimes g_{\sigma(J')} = & g_1 \otimes g_2 g_3 + g_1 \otimes g_3 g_2 + g_2 \otimes g_1 g_3 + g_3 \otimes g_2 g_1 + g_3 \otimes g_1 g_2 + g_2 \otimes g_3 g_1 + \\ & + g_2 \otimes g_1 g_3 + g_3 \otimes g_1 g_2 + g_1 \otimes g_2 g_3 + g_2 \otimes g_3 g_1 + g_1 \otimes g_3 g_2 + g_3 \otimes g_2 g_1 + \\ & + g_3 \otimes g_1 g_2 + g_2 \otimes g_1 g_3 + g_3 \otimes g_2 g_1 + g_1 \otimes g_3 g_2 + g_2 \otimes g_3 g_1 + g_1 \otimes g_2 g_3. \end{aligned}$$

Each line of the previous equation corresponds to each possible choice of I, namely the sets $\{1\}, \{2\}$, and $\{3\}$, together with the six possible permutations $123, 132, 213, 321, 312$, and 231. Notice that the three lines have the same 6 terms in different order, and thus it is easy to see that the whole sum is equal to

$$3(g_1 \otimes g_2 g_3 + g_1 \otimes g_3 g_2 + g_2 \otimes g_1 g_3 + g_2 \otimes g_3 g_1 + g_3 \otimes g_1 g_2 + g_3 \otimes g_2 g_1).$$

On the other hand

$$\sum_{\substack{I \sqcup J = [3] \\ |I| = 1}} \frac{3!}{1!(2)!} \sum_{\substack{\mu \in \mathfrak{S}_1 \\ \tau \in \mathfrak{S}_2}} g_{\mu(I)} \otimes g_{\tau(J)} = 3(g_1 \otimes g_2 g_3 + g_1 \otimes g_3 g_2 + g_2 \otimes g_1 g_3 + g_2 \otimes g_3 g_1 + g_3 \otimes g_1 g_2 + g_3 \otimes g_2 g_1).$$

Proof. Let us first consider (I', J') be such that $I' \sqcup J' = [n]$ and $|I'| = k$. We put $I' = \{i'_1 < \ldots < i'_k\}$ and $J' = \{j'_1 < \ldots < j'_{n-k}\}$. For any (I, J) such that $I \sqcup J = [n]$ and $|I| = k$, putting $I = \{i_1 < \ldots < i_k\}$ and $J = \{j_1 < \ldots < j_{n-k}\}$, let $\sigma^{I,J}_{I',J'} \in \mathfrak{S}_n$ be defined by $\sigma^{I,J}_{I',J'}(i'_p) = i_p$ and $\sigma^{I,J}_{I',J'}(j'_q) = j_q$ for any suitable p and q. For any permutation $\sigma \in \mathfrak{S}_n$, there exists a unique pair (I, J), $\alpha \in \mathfrak{S}(I)$ and $\beta \in \mathfrak{S}(J)$ such that $\sigma = (\alpha \sqcup \beta) \circ \sigma^{I,J}_{I',J'}$, where

$$\alpha \sqcup \beta(x) = \begin{cases} \alpha(x) \text{ if } x \in I, \\ \beta(x) \text{ if } x \in J. \end{cases}$$

In particular, $I = \sigma(I')$ and $J = \sigma(J')$. In other terms, we have a bijection

$$\left\{ \begin{array}{rcl} \{(I, J, \alpha, \beta) \mid I \sqcup J = [n],\ |I| = k,\ \alpha \in \mathfrak{S}(I),\ \beta \in \mathfrak{S}(J)\} & \longrightarrow & \mathfrak{S}_n \\ (I, J, \alpha, \beta) & \longmapsto & (\alpha \sqcup \beta) \circ \sigma^{I,J}_{I',J'}. \end{array} \right.$$

Therefore:

$$\sum_{\sigma\in\mathfrak{S}_n} g_{\sigma(I')}\otimes g_{\tau(J')} = \sum_{\substack{I\sqcup J=[n]\\ |I|=k}} \sum_{(\alpha,\beta)\in\mathfrak{S}(I)\times\mathfrak{S}(J)} g_{\alpha(I)}\otimes g_{\beta(J)}$$
$$= \sum_{\substack{I\sqcup J=[n]\\ |I|=k}} \sum_{(\mu,\tau)\in\mathfrak{S}_k\times\mathfrak{S}_{n-k}} g_{\mu_I d(I)}\otimes g_{\tau_I d(J)}.$$

Summing over all possible (I', J'), we obtain the result, as there are $\binom{n}{k}$ such pairs.

□

Lemma 6.36. The map $\Phi : S_\top(\mathfrak{g}) \to U_\top(\mathfrak{g})$ from Lemma 6.33 is a surjective morphism of locality coalgebras.

Proof. We denote by Δ_S and ϵ_S (resp. Δ_U and ϵ_U) the coproduct and the counit on $S_\top(V)$ (resp. $U_\top(V)$) to avoid ambiguity. By means of Lemma 6.33, Φ is a filtered surjective linear map. We are only left to prove that

$$\Delta_U\circ\Phi = (\Phi\otimes\Phi)\circ\Delta_S, \tag{2.45}$$

and $\epsilon_s = \epsilon_U\circ\Phi$. Since Φ is filtered, it follows that $\Phi((S_\top(\mathfrak{g}))_0 = U^0_\top(\mathfrak{g})$, this together with the fact that both $S_\top(\mathfrak{g})$ and $U_\top(\mathfrak{g})$ are connected (Propositions 6.16 and 6.17) impliey that $\epsilon_s = \epsilon_U\circ\Phi$ as expected. Notice that it is enough to prove (2.45) for elements of the form $g_1\odot\cdots\odot g_n$ with $(g_1,\ldots,g_n)\in\mathfrak{g}^{\times^n_\top}$ since they span all $S_\top(\mathfrak{g})$. On the one hand, using the notations of Lemma 6.34

$$\Delta_U\circ\Phi(g_1\odot\cdots\odot g_n) = \Delta_U\Big(\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} g_{\sigma(1)}\cdots g_{\sigma(n)}\Big)$$
$$= \frac{1}{n!}\sum_{I'\sqcup J'=[n]} g_{\sigma(I')}\otimes g_{\sigma(J')}. \tag{2.46}$$

On the other hand

$$(\Phi\otimes\Phi)\circ\Delta_S(g_1\odot\cdots\odot g_n) = (\Phi\otimes\Phi)\Big(\sum_{I\sqcup J=[n]} g_I\otimes g_J\Big)$$
$$= \sum_{I\sqcup J=[n]}\left(\Big(\frac{1}{|I|!}\sum_{\mu\in\mathfrak{S}_{|I|}} g_{\mu_{\mathrm{Id}}(I)}\Big)\otimes\Big(\frac{1}{|J|!}\sum_{\tau\in\mathfrak{S}_{|J|}} g_{\tau_{\mathrm{Id}}(J)}\Big)\right)$$
$$= \sum_{k=0}^n \frac{1}{k!(n-k)!}\sum_{\substack{I\sqcup J=[n]\\ |I|=k}}\sum_{\substack{\mu\in\mathfrak{S}_k\\ \tau\in\mathfrak{S}_{n-k}}} g_{\mu_{\mathrm{Id}}(I)}\otimes g_{\tau_{\mathrm{Id}}(J)}. \tag{2.47}$$

By means of Lemma 6.34, both (2.46) and (2.47) are equal, which proves that Φ is a surjective morphism of filtered coalgebras as expected. □

Lemma 6.37. Let $(\mathfrak{g},\top)$ be a locality Lie algebra, n be a positive integer, and $\{g_{1,i}\otimes\cdots\otimes g_{n,i}\}_{i=1}^M$ a family of elements in $\mathfrak{g}^{\otimes^n_\top}$. The sum

$$\sum_{i=1}^M \pi_U(g_{1,i}\otimes\cdots\otimes g_{n,i}) = \sum_{i=1}^M g_{1,i}\cdots g_{n,i}\in U^{n-1}_\top(\mathfrak{g})$$

if, and only if there exist permutations $\sigma_1,\ldots,\sigma_M$ in $\mathfrak{S}_n$ such that

$$\sum_{i=1}^M \Xi^{\sigma_i}_\top(g_{1,i}\otimes\cdots\otimes g_{n,i}) = 0. \tag{2.48}$$

Here the map $\Xi^\sigma_\top$ is the same as in (2.11).

Proof. The result follows directly from the definition of the locality ideal $J_\top(\mathfrak{g}) \subset \mathcal{T}_\top(\mathfrak{g})$. Indeed, when commuting two consecutive elements $g_{j-1,i}$ and $g_{j,i}$ in $g_{1,i}\cdots g_{n,i}$ a term of lower degree in the filtration arises. Thus, the leading term only vanishes if there is a set of permutations such that the sum on the left hand side of (2.48) is equal to zero. $\square$

Lemma 6.38. Let $n \geq 1$ and set $(\mathcal{ST}_\top(\mathfrak{g}))_n := V^{\otimes_\top^n} \cap \mathcal{ST}_\top(\mathfrak{g})$ the symmetric n-tensors. Then

$$U_\top^n(\mathfrak{g}) = \pi_U\big((\mathcal{ST}_\top(\mathfrak{g}))_n\big) \oplus_\top U_\top^{n-1}(\mathfrak{g}),$$

i.e., the decomposition is in strong locality complements (see Definition 5.17).

Proof. By means of Definition-Proposition 4.36

$$\mathfrak{g}^{\otimes_\top^n} = (\mathcal{ST}_\top(\mathfrak{g}))_n \oplus (\mathcal{AT}_\top(\mathfrak{g}))_n,$$

where we have set $(\mathcal{AT}_\top(\mathfrak{g}))_n := \mathcal{AT}_\top(\mathfrak{g}) \cap \mathfrak{g}^{\otimes_\top^n}$ as before. Then

$$\begin{aligned} U_\top^n(\mathfrak{g}) =& \pi_U\big(\bigoplus_{k=0}^{n-1} g^{\otimes_\top^k} \oplus g^{\otimes_\top^n}\big) \\ =& \pi_U\big(\bigoplus_{k=0}^{n-1} g^{\otimes_\top^k} \oplus (\mathcal{ST}_\top(\mathfrak{g}))_n \oplus (\mathcal{AT}_\top(\mathfrak{g}))_n\big) \\ =& \pi_U\big(\bigoplus_{k=0}^{n-1} g^{\otimes_\top^k}\big) + \pi_U\big((\mathcal{ST}_\top(\mathfrak{g}))_n\big) + \pi_U\big((\mathcal{AT}_\top(\mathfrak{g}))_n\big). \end{aligned}$$

It follows from π_U being filtered that $\pi_U\big(\bigoplus_{k=0}^{n-1} g^{\otimes_\top^k}\big) \subset U_\top^{n-1}(\mathfrak{g})$, and by means of Lemma 6.37 $\pi_U\big((\mathcal{AT}_\top(\mathfrak{g}))_n\big)$ is a subset of $U_\top^{n-1}(\mathfrak{g})$, and thus $U_\top^n(\mathfrak{g}) = \pi_U\big((\mathcal{ST}_\top(\mathfrak{g}))_n\big) + U_\top^{n-1}(\mathfrak{g})$. We proceed to prove that the sum is indeed a direct sum, namely that the intersection of both spaces is equal to $\{0\}$. This follows also from Definition-Proposition 4.36 and Lemma 6.37. Indeed, Lemma 6.37 states that the only elements of $\mathfrak{g}^{\otimes_\top^n}$ which are mapped by π_U to $U_\top^{n-1}(\mathfrak{g})$ are those in $(\mathcal{AT}(\mathfrak{g}))_n$, which, by means of Definition-Proposition 4.36, is in direct sum with $\mathcal{ST}_\top(\mathfrak{g}))_n$ as expected. We are left to prove that the decomposition is in strong locality complements, i.e., that the projection $\pi_n : U_\top^n(\mathfrak{g}) \to \pi_U\big((\mathcal{ST}_\top(g))_n\big)$ parallel to $U_\top^{n-1}(\mathfrak{g})$ is locality independent of the identity map $\mathrm{Id}_{U_\top^n(\mathfrak{g})}$. For that purpose consider two elements x and y in $U_\top^n(\mathfrak{g})$ such that $x \top_U y$. By the definition of $\top_U$ as a quotient locality (see Definition 4.6) it is possible to write $x = \sum_{i\in I} x_{i1}\cdots x_{in} + x'$ where $x' \in U_\top^{n-1}(\mathfrak{g})$ and $y = \sum_{j\in J} y_{j1}\cdots y_{jn_j}$ where $n_j \leq n$, the x_{ik} and y_{jl} lie in $\mathfrak{g}$ such that $x' \top_U y$ and $x_{ik} \top_{\mathfrak{g}} y_{jl}$ for every $i \in I$, $j \in J$, $1 \leq k \leq n$, and every $1 \leq l \leq n_j$. By means of Lemma 4.35, for every $\sigma \in \mathfrak{S}_n$,

$$x_{i\sigma(1)} \otimes \cdots \otimes x_{i\sigma(n)} \top_\otimes \sum_{j\in J} y_{j1} \otimes \cdots \otimes y_{jn_j}$$

and thus

$$x_{i\sigma(1)} \cdots x_{i\sigma(n)} \top_U \sum_{j\in J} y_{j1} \cdots y_{jn_j}.$$

Since $x = \frac{1}{n!}\sum_{i\in I}\sum_{\sigma\in\mathfrak{S}_n} x_{i\sigma(1)}\cdots x_{i\sigma(n)} + x' + x_{\text{l.o.t.}}$ where $x_{\text{l.o.t}}$ are the terms in $U_\top^{n-1}(\mathfrak{g})$ which arise when commuting two successive x_{ij} and x_{il}, then $\pi_n(x) = \frac{1}{n!}\sum_{i\in I}\sum_{\sigma\in\mathfrak{S}_n} x_{i\sigma(1)}\cdots x_{i\sigma(n)}$ which remains locality independent of y. Thus, π_U is locality independent of $\mathrm{Id}_{U_\top^n(\mathfrak{g})}$ and the result follows. $\square$

Theorem 6.39. [Locality Poincaré-Birkhoff-Witt Theorem.] Let $(\mathfrak{g}, \top)$ be a locality Lie algebra which such that the canonical map $\iota_{\mathfrak{g}} : \mathfrak{g} \to U_\top(\mathfrak{g})$ is injective. If $\mathcal{T}_\top(\mathfrak{g})$ and $U_\top(\mathfrak{g})$ are locality algebras, then the map $\Phi : S_\top(\mathfrak{g}) \to U_\top(\mathfrak{g})$ is a filtered isomorphism of locality coalgebras.

Proof. By means of Lemmas 6.36 and 6.4, we are only left to prove that Φ is injective, and Φ^{-1} is a locality map. For the injectivity we build a left inverse of Φ. Consider for every $n \geq 1$ the map

$$\Psi_n := \pi_S \circ \tilde{\Delta}^n : \pi_U(\mathcal{ST}_\top(\mathfrak{g}))_n \to (S_\top(\mathfrak{g}))_n$$

where $\tilde{\Delta}^n$ is the iterated reduced coproduct introduced in Paragraph 1.5. For $g_1 \odot \cdots \odot g_n \in (S_\top(\mathfrak{g}))$

$$\begin{aligned}\Psi_n \circ \Phi(g_1 \odot \cdots \odot g_n) =&\Psi_n\Big(\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} g_{\sigma(1)}\cdots g_{\sigma(n)}\Big)\\ =&\pi_S \circ \tilde{\Delta}^n\Big(\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} g_{\sigma(1)}\cdots g_{\sigma(n)}\Big)\\ =&\pi_S\Big(\sum_{\sigma\in\mathfrak{S}_n} g_{\sigma(1)}\otimes\cdots\otimes g_{\sigma(n)}\Big)\\ =&g_1\odot\cdots\odot g_n.\end{aligned}$$

Thus Ψ_n is a left inverse to $\Phi\big|_{(S_\top(\mathfrak{g}))_n}$. It follows from Lemma 6.38 that the map $\Psi := \bigoplus_{n\geq 0}\Psi_n$ is well defined where we set $\Psi_0 = u_S \circ \epsilon_U\big|_{U^0_\top(\mathfrak{g})}$, i.e., it maps the unit of $U_\top(\mathfrak{g})$ to the unit of $S_\top(\mathfrak{g})$. Since Ψ is a left inverse for Φ, then Φ is injective as expected.

For the locality of Φ^{-1}, we prove first that $\Psi = \Phi^{-1}$. Indeed given an element $x \in U_\top(\mathfrak{g})$, $x = \sum_{n=1}^N x_n$ where $N < \infty$ and $x_n \in \pi_U\big((\mathcal{ST}_\top(\mathfrak{g}))_n\big)$, i.e., the x_n are of the form

$$x_n = \sum_{i\in I}\sum_{\sigma\in\mathfrak{S}_n} x_{i\sigma(1)}\cdots x_{i\sigma(n)}$$

for some finite set I. Then

$$\begin{aligned}\Phi\circ\Psi(x_n) =&\Phi\circ\pi_S\circ\tilde{\Delta}^{(n)}\Big(\sum_{i\in I}\sum_{\sigma\in\mathfrak{S}_n} x_{i\sigma(1)}\cdots x_{i\sigma(n)}\Big)\\ =&\Phi\circ\pi_S\Big(n!\sum_{i\in I}\sum_{\sigma\in\mathfrak{S}_n} x_{i\sigma(1)}\otimes\cdots\otimes x_{i\sigma(n)}\Big)\\ =&\Phi\Big(n!\sum_{i\in I}\sum_{\sigma\in\mathfrak{S}_n} x_{i\sigma(1)}\odot\cdots\odot x_{i\sigma(n)}\Big)\\ =&\pi_U\circ\Theta\Big(n!\sum_{i\in I}\sum_{\sigma\in\mathfrak{S}_n} x_{i\sigma(1)}\odot\cdots\odot x_{i\sigma(n)}\Big)\\ =&\pi_U\Big(\sum_{i\in I}\sum_{\sigma\in\mathfrak{S}_n} x_{i\sigma(1)}\otimes\cdots\otimes x_{i\sigma(n)}\Big)\\ =&\sum_{i\in I}\sum_{\sigma\in\mathfrak{S}_n} x_{i\sigma(1)}\cdots x_{i\sigma(n)}\\ =&x_n,\end{aligned}$$

and thus $\Phi^{-1} = \Psi$. Consider now $x = \sum_{n=1}^N x_n$ and $y = \sum_{m=1}^M y_m$ in $U_\top(\mathfrak{g})$ where $N, M < \infty$, $x_n \in \pi_U\big((\mathcal{ST}_\top(\mathfrak{g}))_n\big)$, and $y_m \in \pi_U\big((\mathcal{ST}_\top(\mathfrak{g}))_m\big)$, and such that $x\top_U y$. By means of Lemma 6.38, the projections of $U_\top(\mathfrak{g})$ onto each of the subspaces $(\mathcal{ST}_\top(\mathfrak{g}))_n$ is locality independent to the identity $\mathrm{Id}_{U_\top(\mathfrak{g})}$, and thus $x_n\top_U y_m$ for every n and m. For fixed $m' \in [M]$ and $n' \in [N]$, $x'_n\top_U y'_m$ implies that it is possible to write $x'_n = \sum_{i\in I}\sum_{\sigma\in\mathfrak{S}'_n} x_{i\sigma(1)}\cdots x_{i\sigma(n')}$ and $y'_m = \sum_{j\in J}\sum_{\tau\in\mathfrak{S}'_m} y_{j\tau(1)}\cdots y_{j\tau(m')}$ such that $x_{i\sigma(k)}\top_{\mathfrak{g}} y_{j\tau(l)}$ for every i, j, k, and l. Therefore

$$\Big(\Phi^{-1}(x'_n) = n'!\sum_{i\in I}\sum_{\sigma\in\mathfrak{S}_{n'}} x_{i\sigma(1)}\odot\cdots\odot x_{i\sigma(n')}\Big)\top_S\Big(m'!\sum_{j\in J}\sum_{\tau\in\mathfrak{S}_{m'}} x_{j\tau(1)}\odot\cdots\odot x_{j\tau(m')} = \Phi^{-1}(y'_m)\Big).$$

Since n' and m' are arbitrary and $S_\top(V)$ is a locality vector space, then $\Phi^{-1}(x)\top_S\Phi^{-1}(y)$ which finishes the proof. □

Similarly to the non-locality case, we can summarize the locality versions of the Milnor-Moore and Poincaré-Birkhoff-Witt theorems in the following result.

Theorem 6.40. For any locality cocommutative Hopf algebra H over a filed $\mathbb{K}$ of characteristic zero, the following are equivalent.

1. H is connected.
2. There is an isomorphism of locality Hopf algebras $H \sim U(\text{Prim}(H))$.
3. There is an isomorphism of connected locality coalgebras $H \sim S(\text{Prim}(H))$.

Proof. The implication 1. $\Rightarrow$ 2. is the Milnor-Moore theorem (Theorem 6.22). Implication 2. $\Rightarrow$ 3. is the Quillen version of Poincaré-Birkhoff-Witt theorem (Theorem 6.39), and 3. $\Rightarrow$ 1. is straightforward. □

Chapter 3

Shintani zeta functions

In this chapter we present the results regarding the meromorphic continuation of the Shintani zeta functions, namely Theorems 7.10 and 7.11 in Section 7, which provide a domain of absolute convergence and meromorphic continuation of some class of functions, Theorems 8.7 and 8.18 in Section 8, which provide the description of the polar structure of the Shintani zeta functions, and Theorem 9.1 in Section 9, the proof of which provides an algorithm to distribute a multidimensional weight over the vertices of a graph such that the weight on each vertex is always bigger than a given bound. Even though at a first glance the topic of Section 9 is far away from the rest of the chapter, the results provided there are essential to prove Theorem 8.18, according to which the possible hyperplanes carrying the Shintani zeta functions have normal vectors with coefficients 0 or 1 when written in terms of the canonical basis with integer and mutually coprime coefficients (Theorem 8.18). This implies that the poles at zero are similar to the ones of generic Feynman amplitudes studied in [92, 28].

7 Mellin transform of rational functions damped by a Schwartz function

In this first section of the chapter, we introduce the necessary tools in order to build later a meromorphic continuation of the Shintani zeta functions. In Paragraph 7.1 we build the space of rational functions damped by a Schwartz function together with the function $\mathfrak{S}$ which lies in it. This function will be essential for the merommorphic continuation of the Shintani zeta functions. In Paragraph 7.2 we study the domain of convergence and meromorphic continuations of some class of functions, in particular of the rational functions damped by a Schwartz function. The main results of that Paragraph are Theorems 7.10 and 7.11 which are extensions of Theorems 3.20 and 3.21, and that yield the existence of a domain of absolute convergence and a meromorphic continuation for the function $\mathfrak{S}$ of Definition-Proposition 7.5.

7.1 The space of rational functions damped by a Schwartz function

In this paragraph we recall the definition of Schwartz functions, and define the space of rational functions damped by Schwartz functions which will contain the functions, the Mellin transform of which will provide a meromorphic continuation of the Shintani zeta functions.

We recall what a Schwartz function is: For $\mathcal{O}$ an unbounded connected region of $\mathbb{R}^n$, a function $\phi : \mathcal{O} \to \mathbb{R}$ is said to be Schwartz, denoted by $\phi \in \mathcal{S}(\mathcal{O})$, if it is smooth on $\mathcal{O}$ and for every pair $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{Z}_{\geq 0}^{2n}$ the following is satisfied:

$$\lim_{\substack{\boldsymbol{\epsilon} \to \infty \\ \boldsymbol{\epsilon} \in \mathcal{O}}} \boldsymbol{\epsilon}^{\boldsymbol{\alpha}} \partial^{(\boldsymbol{\beta})} \phi(\boldsymbol{\epsilon}) = 0.$$

Notice that in particular $\phi \in \mathcal{S}(\mathbb{R}_+^n)$ is not necessarily bounded while $\psi \in \mathcal{S}(\mathbb{R}_{\geq 0}^n)$ is. In this paper we use the space of rational functions damped by a Schwartz function. We make this precise in the following definition.

Definition 7.1.

- We say that a function $\phi : \mathbb{R}_+^n \to \mathbb{R}$ lies in the space $\mathcal{C}_b^\infty(\mathbb{R}_+^n)$ if, and only if there exists a polynomial p in n real variables such that the product $p\phi$ extends to a bounded smooth function on $\mathbb{R}_{\geq 0}^n$ with all its derivatives bounded.

- We say that a function $\phi : \mathbb{R}_+^n \to \mathbb{R}$ lies in the space $\mathcal{MS}(\mathbb{R}_+^n)$ if, and only if there exists a polynomial p in n real variables such that $p\phi$ extends to a function in $\mathcal{S}(\mathbb{R}_{\geq 0}^n)$. In other words $\mathcal{MS}(\mathbb{R}_+^n)$ is the set of rational functions damped by a Schwartz function on $\mathbb{R}_+^n$.

In [43] the space of germs of meromorphic functions with linear poles is used. However, the authors implicitly use a subspace of $\mathcal{MS}(\mathbb{R}_+^n)$, namely functions of the type $\frac{\phi}{\prod L_i}$ where $\phi \in \mathcal{S}(\mathbb{R}_{\geq 0}^n)$ is analytic at zero and the L_i's lie in $(\mathbb{R}^n)^*$. In other words, they use the space of Schwartz functions on $\mathbb{R}_+^n$ which can be extended, in a neighborhood of zero, to a meromorphic function with linear poles.

Proposition 7.2. *$\mathcal{C}_b^\infty(\mathbb{R}_+^n)$ (resp. $\mathcal{MS}(\mathbb{R}_+^n)$) is an $\mathbb{R}$-algebra (resp. non-unital $\mathbb{R}$-algebra) for the point-wise product of functions.*

Proof. We prove first that $\mathcal{C}_b^\infty(\mathbb{R}_+^n)$ is an algebra. Let ϕ and ψ in $\mathcal{C}_b^\infty(\mathbb{R}_+^n)$, i.e, there are polynomials p and q such that $p\,\phi$ and $q\,\psi$ are bounded smooth functions on $\mathbb{R}_{\geq 0}^n$ with all their derivatives bounded. The fact that the space of bounded smooth functions with all its derivatives bounded is an algebra implies that for $(\lambda, \mu) \in \mathbb{R}^2$, $\lambda\, p\,\phi + \mu\, q\,\psi$ remains in this space. Also $p\,q\,\phi\,\psi$ remains in this space proving that $\mathcal{C}_b^\infty(\mathbb{R}_+^n)$ is an algebra. The unit is the function identical to 1.

We prove now that $\mathcal{MS}(\mathbb{R}_+^n)$ is a non unital algebra. Let ϕ and ψ in $\mathcal{MS}(\mathbb{R}_+^n)$, i.e, there are polynomials p and q such that $p\,\phi \in \mathcal{S}(\mathbb{R}_{\geq 0}^n)$ and $q\,\psi \in \mathcal{S}(\mathbb{R}_{\geq 0}^n)$. The fact that $\mathcal{S}(\mathbb{R}_{\geq 0}^n)$ is a non-unital algebra implies that

$$\lambda\, p\,\phi + \mu\, q\,\psi \in \mathcal{S}(\mathbb{R}_{\geq 0}^n) \quad \forall (\lambda, \mu) \in \mathbb{R}^2,$$

and

$$p\,q\,\phi\,\psi \in \mathcal{S}(\mathbb{R}_{\geq 0}^n),$$

and thus $\mathcal{MS}(\mathbb{R}_+^n)$ is a non-unital algebra, since the function identical to 1 is not in $\mathcal{MS}(\mathbb{R}_+^n)$. ☐

Example 7.3. *Let $L(\boldsymbol{\epsilon}) = \sum_{i=1}^n a_i \epsilon_i \in (\mathbb{R}^n)^*$ where every $a_i > 0$. Let moreover $h : \mathbb{R}_{\geq 0}^n \to \mathbb{R}$ be a smooth function which is bounded together with all its derivatives. Then $\left(\boldsymbol{\epsilon} \mapsto \frac{h(\boldsymbol{\epsilon})\, e^{-L(\boldsymbol{\epsilon})}}{L(\boldsymbol{\epsilon})}\right) \in \mathcal{MS}(\mathbb{R}_+^n)$.*

We prove a lemma which will be useful in the sequel.

Lemma 7.4. The Todd function defined by $z \mapsto \mathrm{Td}(z) = \frac{z}{e^z - 1} = \frac{ze^{-z}}{1 - e^{-z}}$ is a Schwartz function when restricted to $\mathbb{R}_{\geq 0}$.

Proof. In this proof z represent a complex variable and x a real one. Notice first that $\frac{1}{e^z - 1}$ only has poles when $z = 2\pi i k$ where $k \in \mathbb{Z}$, and moreover these poles are simple. Therefore Td is a meromorphic function with simple poles at $z = 2\pi i k$ where $k \in \mathbb{Z} \setminus \{0\}$. In particular it is smooth for every $x \in \mathbb{R}$ and all its derivatives are bounded on the closed interval $[0, R]$ for any $R > 0$. We are only left to prove that $\lim_{x\to\infty} x^\alpha \mathrm{Td}^{(\beta)}(x) = 0$ where α and β are non negative integers. We see first that, for $x \in \mathbb{R}$

$$\lim_{x\to\infty} \frac{x^p e^{-qx}}{(1 - e^{-x})^r} = 0 \quad \forall (p, r) \in \mathbb{Z}_{\geq 0}^2 \text{ and } q \geq 1, \tag{3.1}$$

which follows from $\lim_{x\to\infty} \frac{1}{1 - e^{-x}} = 1$, and $x \mapsto e^{-qx}$ being a Schwartz function on $\mathbb{R}_{\geq 0}$. We show now that for every $\beta \in \mathbb{Z}_{\geq 0}$, $|\mathrm{Td}^{(\beta)}|$ can be expressed as a linear combination with real coefficients of fractions like the one on the left hand side of (3.1). Indeed, it is true for $\beta = 0$ since $\mathrm{Td}(x) = \frac{xe^{-x}}{1 - e^{-x}}$. Moreover

$$\frac{d}{dx}\left(\frac{x^p e^{-qx}}{(1 - e^{-x})^r}\right) = \frac{px^{p-1} e^{-qx}}{(1 - e^{-x})^r} - \frac{qx^p e^{-qx}}{(1 - e^{-x})^r} - \frac{rx^p e^{-(q+1)x}}{(1 - e^{-x})^{r+1}}$$

which yields the result by induction. ☐

Using the column representation of the matrices described in Definition 3.12, consider the set $\mathcal{C}_n$ of column vectors with n non negative arguments, and with at least one of them positive. Recall that, with some abuse of notation, for $C \in \mathcal{C}_n$ we denote also by C the linear form defined by $C(\boldsymbol{\epsilon}) = \langle C, \boldsymbol{\epsilon} \rangle$.

Definition-Proposition 7.5. The following map defined on $\mathcal{C}_n$ takes values in $\mathcal{C}_b^\infty(\mathbb{R}_+^n)$

$$\mathfrak{S} : C \mapsto \left(\mathbb{R}_+^n \ni \boldsymbol{\epsilon} \mapsto \mathfrak{S}(C)(\boldsymbol{\epsilon}) := \sum_{m=1}^{\infty} e^{-mC(\boldsymbol{\epsilon})} = \frac{e^{-C(\boldsymbol{\epsilon})}}{1 - e^{-C(\boldsymbol{\epsilon})}} \right). \tag{3.2}$$

We extend it linearly to $\mathbb{R}\mathcal{C}_n$ which is the real vector space freely generated by $\mathcal{C}_n$. (Notice that the sum in $\mathbb{R}\mathcal{C}_n$ is NOT the usual sum of vectors. For the latter $\mathfrak{S}$ is not a linear map.)

Proof. Notice that $C\,\mathfrak{S}(C) = \frac{C}{e^C - 1} = \mathrm{Td} \circ C$ where Td is the Todd function. Lemma 7.4 yields the result. □

We identify matrices with the tensor product of its columns. Namely, we identify $A = \{a_{i,j}\}_{1 \le i \le n, 1 \le j \le r} \in \Sigma_{n \times r}(\mathbb{R}_{\ge 0})$ with $C_1 \otimes \cdots \otimes C_r$, where $C_j = \{a_{i,j}\}_{1 \le i \le n}$ is the j-th column of A. Notice that this identification does not preserve the structure of vector space, particularly if a matrix is multiplied by a scalar k, the tensor product of the columns should be multiplied by k^r. However, matrices here are only used as parameters and the vector space structure is not considered, therefore this identification is not problematic.

In order to extend the definition of $\mathfrak{S}$ for matrices, we recall the definition and universal property of the tensor algebra. Recall from (1.7) that the tensor algebra of an $\mathbb{R}$-vector space V is defined as the direct sum of the non-negative integer tensor powers of V, namely

$$\mathcal{T}(V) := \bigoplus_{k \ge 0} V^{\otimes k}, \tag{3.3}$$

where $V^{\otimes 0} = \mathbb{R}$. The product on $\mathcal{T}(V)$ is the concatenation of vectors, which we denote by $\otimes$. Notice that the concatenation product respects the grading in (3.3), making it a graded algebra. The unit is the inclusion $u : \mathbb{R} \to V^{\otimes 0} \subset \mathcal{T}(V)$.

By means of the universal property of the tensor algebra (Theorem 1.25), $\mathfrak{S}$ defined in Equation (3.2) extends uniquely to an algebra morphism from the tensor algebra $\mathcal{T}(\mathbb{R}\mathcal{C}_n)$ to $\mathcal{C}_b^\infty(\mathbb{R}_+^n)$ (see Proposition 7.2). Namely, for $C_1 \otimes \cdots \otimes C_r$

$$\mathfrak{S}(C_1 \otimes \cdots \otimes C_r)(\boldsymbol{\epsilon}) = \prod_{j=1}^{r} \mathfrak{S}(C_j)(\boldsymbol{\epsilon}) = \sum_{\boldsymbol{m} \in \mathbb{Z}_+^r} e^{-\langle \boldsymbol{m}, \boldsymbol{C}(\boldsymbol{\epsilon}) \rangle},$$

where $\boldsymbol{C}(\boldsymbol{\epsilon}) = (C_1(\boldsymbol{\epsilon}), \ldots, C_r(\boldsymbol{\epsilon}))$.

Proposition 7.6. *Let $A \in \Sigma_{n \times r}(\mathbb{R}_{\ge 0})$, and using the identification a matrix with the tensor product of its columns, we have that*

$$\boldsymbol{\epsilon} \mapsto \mathfrak{S}(A)(\boldsymbol{\epsilon}) := \mathfrak{S}(C_1 \otimes \cdots \otimes C_r)(\boldsymbol{\epsilon}) = \sum_{\boldsymbol{m} \in \mathbb{Z}_{>0}^r} e^{-\langle A\boldsymbol{m}, \boldsymbol{\epsilon} \rangle} \tag{3.4}$$

lies in $\mathcal{MS}(\mathbb{R}_+^n)$.

Proof. Notice that for every column C_j of A, $C_j(\boldsymbol{\epsilon})\mathfrak{S}(C_j)(\boldsymbol{\epsilon}) = \mathrm{Td}(C_j(\boldsymbol{\epsilon}))$. Since $A \in \Sigma_{n \times r}(\mathbb{R}_{\ge 0})$ there is at least one non-zero element in each row, and therefore if $\boldsymbol{\epsilon} \to \infty$, there is at least one $j' \in [r]$ such that $C_{j'} \to \infty$. By means of Lemma 7.4 $\boldsymbol{\epsilon} \mapsto C_j(\boldsymbol{\epsilon})\mathfrak{S}(C_j)(\boldsymbol{\epsilon}) = \mathrm{Td}(C_{j'}(\boldsymbol{\epsilon}))$ lies in $\mathcal{S}(\mathbb{R}_{\ge 0}^n)$. Definition-Proposition 7.5 implies that $\boldsymbol{\epsilon} \mapsto C_j(\boldsymbol{\epsilon})\mathfrak{S}(C_j(\boldsymbol{\epsilon}))$ is smooth and bounded together with all its derivatives. Thus

$$\Big(\prod_{j \in [r]} C_j \Big) \mathfrak{S}(A) = \prod_{j \in [r]} C_j \mathfrak{S}(C_j) \in \mathcal{S}(\mathbb{R}_{\ge 0}^n)$$

which yields the result. □

Recall that a cone $\{\mathbb{R}_{\geq 0}a_1 + \cdots + \mathbb{R}_{\geq 0}a_m : \forall i \in [c]\, a_i \in \mathbb{R}^n\} \subset \mathbb{R}^n$ is called smooth if $m = n$ and $\{a_i\}_{i\in[n]}$ is a basis of $\mathbb{Z}^n$. Consider a matrix $A \in \Sigma_{n\times n}(\mathbb{R}_{\geq 0})$ which also belongs to $\mathrm{SL}_n(\mathbb{Z})$, then the rows of A are the edges of a smooth cone $\mathcal{C}$ on the lattice $\mathbb{Z}^n_{\geq 0}$, namely the columns of A correspond to the vectors a_i previously mentioned. In that case $\mathfrak{S}(A)$ amounts to the discrete Laplace transform of the characteristic function of $\mathcal{C} \cap \mathbb{Z}^n$. We refer the reader to [42] for a more complete treatment of smooth cones and conical zeta values.

7.2 The Mellin transform of classes of rapidly decreasing functions

The Mellin transform (Definition 3.9) will play a central role in the sequel, since it is the main tool we use to determine meromorphic continuations. In this paragraph we study the domain of convergence and meromorphic continuation of the Mellin transform of some specific functions. We also adapt Theorems 3.20 and 3.21 to the Mellin transforms of rational functions damped by a Schwartz function.

Domain of convergence of the Mellin transform of rapidly decreasing functions

We introduce a class of functions, whose behavior at infinity ensures a non empty domain of convergence of its Mellin transform.

Definition 7.7. We call a function $\phi : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ rapidly decreasing if for every $\alpha \in \mathbb{Z}^n$,

$$\lim_{\boldsymbol{\epsilon}\to\infty} \boldsymbol{\epsilon}^{\alpha}\phi(\boldsymbol{\epsilon}) = 0.$$

Notice that this definition differs from that of Schwartz functions, in that here it is not required the function to be smooth and its derivatives to be rapidly decreasing. For the rest of this section we adopt the notation $\mathbb{C}^n_+ := \{\boldsymbol{s} \in \mathbb{C}^n : \Re(\boldsymbol{s}) > 0\}$, where $\Re(\boldsymbol{s}) > 0$ means that $\Re(s_i) > 0$ for every i.

Proposition 7.8. *Let $g : \mathbb{R}^n_+ \to \mathbb{C}$ be a function, the Mellin transform of which*

$$\mathcal{M}_g(\boldsymbol{s}) = \int_{\mathbb{R}^n_+} \boldsymbol{\epsilon}^{\boldsymbol{s}-1} g(\boldsymbol{\epsilon}) d\boldsymbol{\epsilon}$$

is absolutely convergent for values of $\boldsymbol{s}$ in some non-empty set $D_g \subset \mathbb{C}^n$. For any given bounded, measurable function $\phi : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ which is rapidly decreasing, the Mellin transform of $g\phi$ given by

$$\mathcal{M}_{g\phi}(\boldsymbol{s}) = \int_{\mathbb{R}^n_+} \boldsymbol{\epsilon}^{\boldsymbol{s}-1} g(\boldsymbol{\epsilon})\phi(\boldsymbol{\epsilon}) d\boldsymbol{\epsilon}$$

is absolutely convergent on the set $D_g + \mathbb{C}^n_+$ where the sum is understood as the usual sum of subsets of a vector space (Minkowski sum). It is moreover analytic on the interior of $D_g + \mathbb{C}^n_+$.

Proof. Let $\boldsymbol{s}$ in $D_g + \mathbb{C}^n_+$, then there is $\boldsymbol{s}_0 \in D_g$ and $\boldsymbol{\alpha} \in \mathbb{C}^n_+$ such that $\boldsymbol{s} = \boldsymbol{s}_0 + \boldsymbol{\alpha}$. Let $\bar{B}_R(0)$ be a closed ball of $\mathbb{R}^n_{\geq 0}$ centered at 0 with radius $R > 0$ and set $\boldsymbol{\sigma} := \Re(\boldsymbol{s})$, $\boldsymbol{\sigma}_0 := \Re(\boldsymbol{s}_0)$, and $\boldsymbol{a} = \Re(\boldsymbol{\alpha})$, then formally

$$|\mathcal{M}_{g\phi}(\boldsymbol{s})| \leq \int_{\bar{B}_R(0)} \boldsymbol{\epsilon}^{\boldsymbol{\sigma}_0+\boldsymbol{a}-1} |g(\boldsymbol{\epsilon})\phi(\boldsymbol{\epsilon})| d\boldsymbol{\epsilon} + \int_{\mathbb{R}^n_+\setminus\bar{B}_R(0)} \boldsymbol{\epsilon}^{\boldsymbol{\sigma}_0+\boldsymbol{a}-1} |g(\boldsymbol{\epsilon})\phi(\boldsymbol{\epsilon})| d\boldsymbol{\epsilon}. \tag{3.5}$$

It is therefore enough to prove the convergence of the two integrals on the right hand side of (3.5) to prove that $\mathcal{M}_{g\phi}(\boldsymbol{s})$ is convergent. The facts that ϕ is bounded, $\boldsymbol{a}$ lies in $\mathbb{R}^n_+$, and $\bar{B}_R(0)$ is compact, imply that there is $M_a > 0$ such that for every $\boldsymbol{\epsilon} \in \bar{B}_R(0)$ the inequality $\boldsymbol{\epsilon}^{\boldsymbol{a}}|\phi(\boldsymbol{\epsilon})| < M_a$ holds, and therefore

$$\int_{\bar{B}_R(0)} \boldsymbol{\epsilon}^{\boldsymbol{\sigma}_0+\boldsymbol{a}-1} |g(\boldsymbol{\epsilon})\phi(\boldsymbol{\epsilon})| d\boldsymbol{\epsilon} \leq M_a \int_{\bar{B}_R(0)} \boldsymbol{\epsilon}^{\boldsymbol{\sigma}_0-1} |g(\boldsymbol{\epsilon})| d\boldsymbol{\epsilon} \leq M_a \int_{\mathbb{R}^n_+} \boldsymbol{\epsilon}^{\boldsymbol{\sigma}_0-1} |g(\boldsymbol{\epsilon})| d\boldsymbol{\epsilon}$$

which is convergent by assumption.

For the second integral on the right hand side of (3.5), since ϕ is rapidly decreasing, for R big enough $\boldsymbol{\epsilon}^{\boldsymbol{a}}|\phi(\boldsymbol{\epsilon})| < 1$ for every $\boldsymbol{\epsilon} \in \mathbb{R}^n_+ \setminus \bar{B}_R(0)$ and thus

$$\int_{\mathbb{R}^n_+\setminus\bar{B}_R(0)} \boldsymbol{\epsilon}^{\boldsymbol{\sigma}_0+\boldsymbol{a}-1} |g(\boldsymbol{\epsilon})\phi(\boldsymbol{\epsilon})| d\boldsymbol{\epsilon} \leq \int_{\mathbb{R}^n_+\setminus\bar{B}_R(0)} \boldsymbol{\epsilon}^{\boldsymbol{\sigma}_0-1} |g(\boldsymbol{\epsilon})| d\boldsymbol{\epsilon} \leq \int_{\mathbb{R}^n_+} \boldsymbol{\epsilon}^{\boldsymbol{\sigma}_0-1} |g(\boldsymbol{\epsilon})| d\boldsymbol{\epsilon},$$

which is again convergent by assumption. Therefore both integrals on the right hand side of (3.5) are convergent implying the expected result. The analyticity follows from Lemma 3.10. □

Remark 7.9. Notice that the case $g(\boldsymbol{\epsilon}) = 1$ is not covered by the former proposition since $\mathcal{M}_1(\boldsymbol{s})$ does not converge for any $\boldsymbol{s} \in \mathbb{C}^n$. However it is easy to see that for ϕ bounded, measurable and rapidly decreasing, $\mathcal{M}_\phi(\boldsymbol{s})$ converges and is analytic for $\Re(\boldsymbol{s}) > 1$, by this we mean $\Re(s_i) > 1$ for every $i \in [n]$. Indeed

$$\mathcal{M}_\phi(\boldsymbol{s}) = \int_{B_1(0)\cap\mathbb{R}^n_+} \boldsymbol{\epsilon}^{\boldsymbol{s}-1}\phi(\boldsymbol{\epsilon})d\boldsymbol{\epsilon} + \int_{\mathbb{R}^n_+\setminus B_1(0)} \boldsymbol{\epsilon}^{\boldsymbol{s}-1}\phi(\boldsymbol{\epsilon})d\boldsymbol{\epsilon}, \tag{3.6}$$

for $\Re(\boldsymbol{s}) > 1$ the first integral on (3.6) converges since the integrand is bounded and $B_1(0)$ is compact. For the second integral it is enough to realize that $|\boldsymbol{\epsilon}^{\boldsymbol{s}-1}\phi(\boldsymbol{\epsilon})| \leq C\,\frac{1}{|\boldsymbol{\epsilon}|^{n+1}}$ for some $C > 0$.

We proceed to adapt Theorem 3.20 [77, Theorem 1] to the Mellin transform of rational functions damped by rapidly decreasing functions. The proof we provide follows very closely the one in [77].

Theorem 7.10. Let $\phi : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ be a measurable, bounded, rapidly decreasing function, and p a polynomial completely non-vanishing on $\mathbb{R}^n_+$, then the Mellin transform of ϕ/p

$$\mathcal{M}_{\phi/p}(\boldsymbol{s}) = \int_{\mathbb{R}^n_+} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}-1}\phi(\boldsymbol{\epsilon})}{p(\boldsymbol{\epsilon})}\, d\boldsymbol{\epsilon},$$

converges absolutely and defines an analytic function $\boldsymbol{s} \mapsto \mathcal{M}_{\phi/p}(\boldsymbol{s})$ on the tube domain $\Re(\boldsymbol{s}) = \boldsymbol{\sigma} \in \Delta_p + \mathbb{R}^n_+$, where Δ_p is the Newton polytope of p (see Definition 3.13).

Proof. Whenever Δ_p has no empty interior, the statement is a direct consequence of Theorem 3.20 and Proposition 7.8. For the case $\mathrm{int}(\Delta_p) = \emptyset$, we adapt the proof of [77, Theorem 1]. Consider the change of variable $\epsilon_i \mapsto e^{x_i}$, then

$$\mathcal{M}_{\phi/p}(s) = \int_{\mathbb{R}^n} \frac{e^{\langle \boldsymbol{s},\boldsymbol{x}\rangle}\phi(e^{\boldsymbol{x}})}{p(e^{\boldsymbol{x}})} d\boldsymbol{x}.$$

Similar to Remark 7.9, it is then enough to prove that there is a bounded, measurable rapidly decreasing function ψ such that

$$\frac{e^{\langle \boldsymbol{\sigma},\boldsymbol{x}\rangle}|\phi(e^{\boldsymbol{x}})|}{|p(e^{\boldsymbol{x}})|} \leq \psi(\boldsymbol{x}) \tag{3.7}$$

for every $\boldsymbol{x} \in \mathbb{R}^n \setminus K$ where K is a compact set. We prove this by induction over the dimension n. For $n = 1$, since $\dim(\Delta_p) = 0$ then $p(e^x) = a_\alpha e^{\alpha x}$ where $\Delta_p = \alpha \in \mathbb{Z}$. Let $M > 0$ be such that $|\phi(e^x)| < M$ for every x. For negative values of x

$$\frac{e^{\sigma x}|\phi(e^x)|}{|a_\alpha e^{\alpha x}|} \leq \frac{M}{|a_\alpha|} e^{-(\sigma-\alpha)|x|}.$$

Since $e^{-(\sigma-\alpha)|x|}$ is bounded, measurable and rapidly decreasing for $x \leq 0$, this yields the result. On the other hand, $|e^{(\sigma-\alpha)x}\phi(e^x)|$ is rapidly decreasing for $x > 0$, thus our claim is true for $n = 1$.

For the inductive step assume that inequality (3.7) holds for dimensions smaller than n, and consider a polynomial in n variables $p(\boldsymbol{\epsilon}) = \sum_{\alpha\in\mathcal{A}} a_\alpha \boldsymbol{\epsilon}^\alpha$, where $\mathcal{A} \subset \mathbb{Z}^n$, and with $\dim(\Delta_p) < n$. Let $\boldsymbol{\sigma} \in \Delta_p + \mathbb{R}^n_+$, then in particular $\boldsymbol{\sigma} \notin \Delta_p$ (because $0 \notin \mathbb{R}_+$). We build a family of cones, the union of which covers $\mathbb{R}^n \setminus K$ where K is a compact set. Recall that for a set $X \subset \mathbb{R}^n$, $\mathrm{conv}(X)$ refers to the convex hull generated by the elements in X.

- For every face Γ of $\Delta_p + \mathbb{R}^n_+$, choose $\boldsymbol{\sigma}_\Gamma \in \mathrm{int}(\Gamma)$. Define $\Delta_\Gamma = \mathrm{conv}((\mathcal{A} \setminus \Gamma) \cup \{\boldsymbol{\sigma}_\Gamma, \boldsymbol{\sigma}\})$. Consider then the cone
$$\widetilde{C}_\Gamma := \{\boldsymbol{x} \in \mathbb{R}^n : (\forall \boldsymbol{\xi} \in \Delta_\Gamma)\, \langle \boldsymbol{\xi} - \boldsymbol{\sigma}_\Gamma, \boldsymbol{x} - \boldsymbol{\sigma}_\Gamma\rangle \leq 0\}.$$
- Consider the polytope $\Delta_{\boldsymbol{\sigma}} := \mathrm{conv}(\mathcal{A} \cup \{\boldsymbol{\sigma}\})$. Then define the cone
$$\widetilde{C}_{\boldsymbol{\sigma}} := \{\boldsymbol{x} \in \mathbb{R}^n : (\forall \boldsymbol{\xi} \in \Delta_{\boldsymbol{\sigma}})\, \langle \boldsymbol{\xi} - \boldsymbol{\sigma}, \boldsymbol{x} - \boldsymbol{\sigma}\rangle \leq 0\}.$$

Notice that $\mathbb{R}^n \setminus \left(\bigcup_\Gamma \widetilde{C}_\Gamma \cup \widetilde{C}_{\boldsymbol{\sigma}}\right)$ is a bounded set, where Γ takes values on the set of faces of $\Delta_p + \mathbb{R}^n_+$. Even more, consider for every Γ (resp. for $\boldsymbol{\sigma}$) a slightly smaller closed, convex cone C_Γ (resp. $C_{\boldsymbol{\sigma}}$) contained in the interior of $\widetilde{C}_\Gamma$ (resp. $\widetilde{C}_{\boldsymbol{\sigma}}$) with vertex in $\boldsymbol{\sigma}_\Gamma$ (resp. $\boldsymbol{\sigma}$) such that $\mathbb{R}^n \setminus (\bigcup_\Gamma C_\Gamma \cup C_{\boldsymbol{\sigma}})$ is still a bounded set. It is then enough to prove the estimate (3.7) for every $\boldsymbol{x}$ in C_Γ and for every $\boldsymbol{x}$ in $C_{\boldsymbol{\sigma}}$ with norm $|\boldsymbol{x}|$ chosen sufficiently large.

Fix a face Γ of $\Delta_p + \mathbb{R}^n$ and consider $\boldsymbol{x} \in \mathbb{C}_\Gamma$. With a slight abuse of notation we call p_Γ the sum of the monomials of p, the exponents of which lie in Γ. Notice that this differs from the truncated polynomials in Definition 3.19 in that here Γ is not necessarily a face of Δ_p but rather one of $\Delta_p + \mathbb{R}^n_+$. Let $q_\Gamma = p - p_\Gamma$. Then

$$\frac{e^{\langle \boldsymbol{\sigma}, \boldsymbol{x}\rangle}\phi(e^{\boldsymbol{x}})}{p(e^{\boldsymbol{x}})} = \frac{e^{\langle \boldsymbol{\sigma} - \boldsymbol{\sigma}_\Gamma, \boldsymbol{x}\rangle}}{(\phi(e^{\boldsymbol{x}}))^{-1} e^{-\langle \boldsymbol{\sigma}_\Gamma, \boldsymbol{x}\rangle} p_\Gamma(e^{\boldsymbol{x}}) + (\phi(e^{\boldsymbol{x}}))^{-1} e^{-\langle \boldsymbol{\sigma}_\Gamma, \boldsymbol{x}\rangle} q_\Gamma(e^{\boldsymbol{x}})}. \tag{3.8}$$

Let us now determine adequate bounds for the numerator and for each of the sumands in the denominator of the fraction on the right hand side of (3.8).

- Bound for $e^{\langle \boldsymbol{\sigma} - \boldsymbol{\sigma}_\Gamma, \boldsymbol{x}\rangle}$: Set $\boldsymbol{y} := \boldsymbol{x} - \boldsymbol{\sigma}_\Gamma$ and $k := \min\{\langle \boldsymbol{\sigma}_\Gamma - \boldsymbol{\sigma}, \boldsymbol{y}\rangle \,:\, |\boldsymbol{y}| = 1,\ \boldsymbol{\sigma}_\Gamma + \boldsymbol{y} = \boldsymbol{x} \in C_\Gamma\}$. By construction of C_Γ we have $k > 0$. Then

$$\begin{aligned} e^{\langle \boldsymbol{\sigma}_\Gamma - \boldsymbol{\sigma}, \boldsymbol{x}\rangle} &= e^{\langle \boldsymbol{\sigma}_\Gamma - \boldsymbol{\sigma}, \boldsymbol{\sigma}_\Gamma\rangle} e^{\langle \boldsymbol{\sigma}_\Gamma - \boldsymbol{\sigma}, \frac{\boldsymbol{y}}{|\boldsymbol{y}|}\rangle |\boldsymbol{y}|} \\ &\geq e^{\langle \boldsymbol{\sigma}_\Gamma - \boldsymbol{\sigma}, \boldsymbol{\sigma}_\Gamma\rangle} e^{k|\boldsymbol{y}|} \\ &\geq e^{\langle \boldsymbol{\sigma}_\Gamma - \boldsymbol{\sigma}, \boldsymbol{\sigma}_\Gamma\rangle} e^{k(|\boldsymbol{x}| - |\boldsymbol{\sigma}_\Gamma|)} \\ &\geq c_1 e^{k|\boldsymbol{x}|} \end{aligned}$$

  where we have set $c_1 := e^{\langle \boldsymbol{\sigma}_\Gamma - \boldsymbol{\sigma}, \boldsymbol{\sigma}_\Gamma\rangle} e^{-k|\boldsymbol{\sigma}_\Gamma|} > 0$. Setting $\psi(\boldsymbol{x}) := (c_1 e^{k|\boldsymbol{x}|})^{-1}$ implies that $|e^{\langle \boldsymbol{\sigma} - \boldsymbol{\sigma}_\Gamma, \boldsymbol{x}\rangle}| < \psi(x)$ where ψ is a bounded, measurable, rapidly decreasing function.

We now find a constant $c > 0$ such that the denominator of (3.8)

$$(\phi(e^{\boldsymbol{x}}))^{-1} e^{-\langle \boldsymbol{\sigma}_\Gamma, \boldsymbol{x}\rangle} p_\Gamma(e^{\boldsymbol{x}}) \ + \ (\phi(e^{\boldsymbol{x}}))^{-1} e^{-\langle \boldsymbol{\sigma}_\Gamma, \boldsymbol{x}\rangle} q_\Gamma(e^{\boldsymbol{x}})) > c$$

for $|\boldsymbol{x}| \in C_\Gamma$ large enough.

- Bound for $(\phi(e^{\boldsymbol{x}}))^{-1} e^{-\langle \boldsymbol{\sigma}_\Gamma, \boldsymbol{x}\rangle} p_\Gamma(e^{\boldsymbol{x}})$: Notice that $\dim(\Delta_{p_\Gamma}) = m < n$. Therefore, a change of coordinates, the transformation matrix of which has determinant 1 (in dimension 2 and 3 it is equivalent to a rotation of coordinates) can be made, such that in the new coordinates $\boldsymbol{x}'$ the polytope Δ_{p_Γ} is in a (possibly affine) subspace parallel to the span of the first m coordinates $x'_1, \ldots, x'_m$, and the remaining $n - m$ coordinates are orthogonal to Δ_{p_Γ}. We write $\boldsymbol{x}' = (\boldsymbol{x}'_1, \boldsymbol{x}'_2)$ where $\boldsymbol{x}'_1$ are the first m coordinates and $\boldsymbol{x}'_2$ are the last $n - m$ coordinates, we write similarly $\boldsymbol{\sigma}_\Gamma = (\boldsymbol{\sigma}_{\Gamma 1}, \boldsymbol{\sigma}_{\Gamma 2})$. Then $p_\Gamma(e^{\boldsymbol{x}'}) = e^{\langle \boldsymbol{\sigma}_{\Gamma 2}, \boldsymbol{x}'_2\rangle} p_\Gamma(e^{\boldsymbol{x}'_1})$. Indeed, since all the exponents of the polynomial p_Γ (seen as vectors on $\mathbb{Z}^n$) lie in Δ_{p_Γ}, then their last $n - m$ components in the coordinates $\boldsymbol{x}'$ are equal to $\boldsymbol{\sigma}_{\Gamma_2}$, because Δ_{p_Γ} is constant in those components. It follows that

$$|(\phi(e^{\boldsymbol{x}}))^{-1} e^{-\langle \boldsymbol{\sigma}_\Gamma, \boldsymbol{x}\rangle} p_\Gamma(e^{\boldsymbol{x}})| = |(\phi(e^{\boldsymbol{x}}))^{-1}||e^{-\langle \boldsymbol{\sigma}_{\Gamma 1}, \boldsymbol{x}'_1\rangle} p_\Gamma(e^{\boldsymbol{x}'_1})|.$$

  Since the right hand side depends on $m < n$ variables, by means of the induction hypothesis, there is a constant $c_2 > 0$ such that $|(\phi(e^{\boldsymbol{x}}))^{-1} e^{-\langle \boldsymbol{\sigma}_\Gamma, \boldsymbol{x}\rangle} p_\Gamma(e^{\boldsymbol{x}})| > c_2$ for $|\boldsymbol{x}|$ large enough.

- Bound for $(\phi(e^{x}))^{-1} e^{-\langle \boldsymbol{\sigma}_\Gamma, \boldsymbol{x}\rangle} q_\Gamma(e^{\boldsymbol{x}})$: We now proceed to prove that for $|\boldsymbol{x}|$ large enough

$$|\phi^{-1}(e^{\boldsymbol{x}}) e^{-\langle \boldsymbol{\sigma}_\Gamma, \boldsymbol{x}\rangle} q_\Gamma(e^{\boldsymbol{x}})| < c_2/2.$$

  Recall that

$$q_\Gamma(e^{\boldsymbol{x}}) = \sum_{\alpha \in \mathcal{A} \setminus \Gamma} a_\alpha e^{\langle \alpha, \boldsymbol{x}\rangle}.$$

  Since $\alpha \in \Delta_\Gamma$ and C_Γ is closed, the following constant exists and is positive

$$k_\alpha := \min\{\langle \boldsymbol{\sigma}_\Gamma - \alpha, \boldsymbol{y}\rangle :\ |\boldsymbol{y}| = 1,\ \boldsymbol{x} = \boldsymbol{\sigma}_\Gamma + \boldsymbol{y} \in C_\Gamma\}.$$

Hence

$$|a_\alpha e^{\langle\alpha,\boldsymbol{x}\rangle} e^{-\langle\boldsymbol{\sigma}_\Gamma,\boldsymbol{x}\rangle}| = |a_\alpha e^{\langle\alpha,\boldsymbol{\sigma}_\Gamma\rangle - |\boldsymbol{\sigma}_\Gamma|^2} e^{-\langle\boldsymbol{\sigma}_\Gamma - \alpha, \frac{\boldsymbol{y}}{|\boldsymbol{y}|}\rangle|\boldsymbol{y}|}|$$
$$\leq |a_\alpha e^{\langle\alpha,\boldsymbol{\sigma}_\Gamma\rangle - |\boldsymbol{\sigma}_\Gamma|^2} e^{-k_\alpha|\boldsymbol{y}|}|.$$

The last term tends to zero as $|\boldsymbol{y}|$ tends to infinity. Therefore, for $|\boldsymbol{x}|$ large enough $|(\phi(e^{\boldsymbol{x}}))^{-1} e^{-\langle\boldsymbol{\sigma}_\Gamma,\boldsymbol{x}\rangle} q_\Gamma(e^{\boldsymbol{x}})| < \frac{c_2}{2}$ as expected.

It follows that for $\boldsymbol{x} \in C_\Gamma$ with $|\boldsymbol{x}|$ large enough, the estimate (3.7) is satisfied. We are only left to prove the same estimate for $\boldsymbol{x} \in C_{\boldsymbol{\sigma}}$. For that purpose notice that for $\boldsymbol{x}$ with large enough norm, $\boldsymbol{x} \in C_{\boldsymbol{\sigma}}$ implies that $\boldsymbol{x} \notin \mathbb{R}^n_{\leq 0}$. It follows then that $e^{\boldsymbol{x}} \mapsto \phi(e^{\boldsymbol{x}})$ is rapidly decreasing for $\boldsymbol{x} \in C_{\boldsymbol{\sigma}}$. On the other hand, since p is a polynomial completely non-vanishing in $\mathbb{R}^n_+$, then $C_{\boldsymbol{\sigma}} \ni \boldsymbol{x} \mapsto |p(e^{\boldsymbol{x}})|$ is bounded from below by a positive constant. Therefore

$$C_{\boldsymbol{\sigma}} \ni \boldsymbol{s} \mapsto \frac{e^{\langle\boldsymbol{\sigma},\boldsymbol{x}\rangle}|\phi(e^{\boldsymbol{x}})|}{|p(e^{\boldsymbol{x}})|}$$

is a bounded, measurable rapidly decreasing function, which completes the proof. □

Meromorphic continuation of the Mellin transform

In this paragraph we build a meromorphic continuation for the Mellin transform of rational functions damped by a Schwartz function (see Definition 7.1). For that purpose we need a slightly adapted form of Theorem 3.21, the proof of which follows closely the one in [77] with a small change to take into account a Schwartz function.

Similar to (1.36), there are integers ν_k, and vectors $\boldsymbol{\mu}_k$ which lie in $\mathbb{Z}^n_{\geq 0}$ on the inward normal direction of the facets of $\Delta_p + \mathbb{R}^n_+$ with mutually coprime coordinates, such that

$$\Delta_p + \mathbb{R}^n_+ := \bigcap_{k=1}^N \{\boldsymbol{\sigma} \in \mathbb{R}^n; \langle\boldsymbol{\mu}_k, \boldsymbol{\sigma}\rangle \geq \nu_k\}. \tag{3.9}$$

Theorem 7.11. Let p be a completely non-vanishing polynomial on the positive orthant $\mathbb{R}^n_+$ and $\phi \in \mathcal{S}(\mathbb{R}^n_{\geq 0})$. Then the Mellin transform

$$\mathcal{M}_{\phi/p}(\boldsymbol{s}) = \int_{\mathbb{R}^n_+} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}}\ \phi(\boldsymbol{\epsilon})}{p(\boldsymbol{\epsilon})} \frac{d\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}} \tag{3.10}$$

admits a meromorphic continuation of the form

$$\mathcal{M}_{\phi/p}(\boldsymbol{s}) = \Phi(\boldsymbol{s}) \prod_{k=i}^N \Gamma(\langle\boldsymbol{\mu}_k, \boldsymbol{s}\rangle - \nu_k),$$

where Φ is an entire function, and N, $\boldsymbol{\mu}_k$ and ν_k are as in (3.9).

Proof. This proof closely follows that of Nilsson and Passare. We use here the following notation introduced by them: For a given $\boldsymbol{\gamma} \in \mathbb{Z}^N$

$$\Delta(\boldsymbol{\gamma}) := \bigcap_{k=1}^N \{\boldsymbol{\sigma} \in \mathbb{R}^n : \langle\boldsymbol{\mu}_k, \boldsymbol{\sigma}\rangle \geq \gamma_k\},$$

in particular $\Delta(\boldsymbol{\nu}) = \Delta_p + \mathbb{R}^n_+$, where $\boldsymbol{\nu}$ is a vector, the components of which are the ν_k as in Equation (3.9).

We claim that for every $\boldsymbol{m} \in \mathbb{Z}^N_{\geq 0}$, there are functions $\phi_{\boldsymbol{m},i} \in \mathcal{S}(\mathbb{R}^n_{\geq 0})$ and polynomials $q_{\boldsymbol{m},i}$ whose Newton polytopes satisfy $\Delta_{q_{\boldsymbol{m},i}} \subset \Delta(|\boldsymbol{m}|\boldsymbol{\nu} + \boldsymbol{m})$, such that

$$\mathcal{M}_{\phi/p}(\boldsymbol{s}) = \frac{1}{\prod_{j=1}^N u_{\boldsymbol{m},j}(\boldsymbol{s})} \left(\sum_{i=1}^{N_{\boldsymbol{m}}} \int_{\mathbb{R}^n_+} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}} q_{\boldsymbol{m},i}(\boldsymbol{\epsilon})\ \phi_{\boldsymbol{m},i}(\boldsymbol{\epsilon})}{p(\boldsymbol{\epsilon})^{1+|\boldsymbol{m}|}} \frac{d\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}} \right) \tag{3.11}$$

Here $N_{\boldsymbol{m}} \in \mathbb{N}$, $|\boldsymbol{m}| = m_1 + \cdots + m_N$, and $u_{\boldsymbol{m},j}(\boldsymbol{s}) = \prod_{l=0}^{m_j-1}(\langle \boldsymbol{\mu}_j, \boldsymbol{s}\rangle - \nu_j + l)$. We prove this by induction over $|\boldsymbol{m}|$. Let $\boldsymbol{m} = e_k$, where $\{e_i\}_{i=1}^N$ is the canonical basis of $\mathbb{R}^N$: for $\lambda > 1$, introducing the change of coordinates $(\epsilon_1, \ldots, \epsilon_n) \mapsto (\lambda^{\mu_{k1}}\epsilon_1, \ldots, \lambda^{\mu_{kn}}\epsilon_n)$ in (3.10) we obtain

$$\mathcal{M}_{\phi/p}(\boldsymbol{s}) = \lambda^{\langle \boldsymbol{\mu}_k, \boldsymbol{s}\rangle - \nu_k} \int_{\mathbb{R}_+^n} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}}}{\lambda^{-\nu_k} p(\lambda^{\boldsymbol{\mu}_k}\boldsymbol{\epsilon})} \phi(\lambda^{\boldsymbol{\mu}_k}\boldsymbol{\epsilon}) \frac{d\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}}. \tag{3.12}$$

After differentiating (3.12) with respect to λ and evaluating at $\lambda = 1$, it follows that

$$0 = (\langle \boldsymbol{\mu}_k, \boldsymbol{s}\rangle - \nu_k)\mathcal{M}_{\phi/p}(\boldsymbol{s}) - \int_{\mathbb{R}_+^n} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}} q_{e_k}(\boldsymbol{\epsilon})}{p^2(\boldsymbol{\epsilon})} \phi(\boldsymbol{\epsilon}) \frac{d\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}} + \int_{\mathbb{R}_+^n} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}}}{p(\boldsymbol{\epsilon})} \left(\sum_{l=1}^n \epsilon_l \mu_{k_l} \frac{\partial \phi}{\partial \epsilon_l}\Big|_{\boldsymbol{\epsilon}} \right) \frac{d\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}},$$

where we have set

$$q_{e_k}(\boldsymbol{\epsilon}) = \frac{d}{d\lambda} \left(\lambda^{-\nu_k} p(\lambda^{\boldsymbol{\mu}_k}\boldsymbol{\epsilon}) \right)\Big|_{\lambda=1}.$$

That implies

$$\mathcal{M}_{\phi/p}(\boldsymbol{s}) = \frac{1}{\langle \boldsymbol{\mu}_k, \boldsymbol{s}\rangle - \nu_k} \left(\int_{\mathbb{R}_+^n} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}} q_{e_k}(\boldsymbol{\epsilon})}{p^2(\boldsymbol{\epsilon})} \phi(\boldsymbol{\epsilon}) \frac{d\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}} - \sum_{l=1}^n \mu_{k_l} \int_{\mathbb{R}_+^n} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}} p(\boldsymbol{\epsilon})\epsilon_l}{p^2(\boldsymbol{\epsilon})} \frac{\partial \phi}{\partial \epsilon_l}\Big|_{\boldsymbol{\epsilon}} \frac{d\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}} \right).$$

Since $\frac{\partial \phi}{\partial \boldsymbol{\epsilon}_l} \in \mathcal{S}(\mathbb{R}^n_{\geq 0})$, we are only left to prove $\Delta_{q_{e_k}} \subset \Delta(\boldsymbol{\nu} + e_k)$ and $\Delta_{p\,\epsilon_l} \subset \Delta(\boldsymbol{\nu} + e_k)$. For the first inclusion, let Γ_{e_k} be the facet of $\Delta_p + \mathbb{R}^n_+$ contained in the hyperplane $\langle \boldsymbol{\mu}_k, \boldsymbol{\sigma}\rangle = \nu_k$. Notice that q_{e_k} contains only the monomials from p whose exponents do not lie on the facet Γ_{e_k}, therefore $\Delta_{q_{e_k}} \subset \Delta(\boldsymbol{\nu} + e_k)$. We prove the second inclusion, namely $\Delta_{p\,\epsilon_l} \subset \Delta(\boldsymbol{\nu} + e_k)$: recall that the Newton polytope of the product of two polynomials is equal to the sum of the Newton polytopes of each polynomial, thus $\Delta_{p\,\epsilon_l}$ is Δ_p translated by the vector e_l. Then if $\mu_{k_l} \neq 0$, $\boldsymbol{\sigma} \in \Delta_{p\,\epsilon_l}$ implies that for every $j \in [N]$, $\langle \boldsymbol{\sigma} - e_l, \boldsymbol{\mu}_j\rangle \geq \nu_j$ or equivalently $\langle \boldsymbol{\sigma}, \boldsymbol{\mu}_j\rangle \geq \nu_j + \mu_{jl} \geq \nu_j + \delta_{j,k}$ where the last inequality is a consequence of $\boldsymbol{\mu}_k \in \mathbb{Z}^n_{\geq 0}$ and $\mu_{k_l} \neq 0$. This proves that $\Delta_{p(\epsilon)\epsilon_i} \subset \Delta(\boldsymbol{\nu} + e_k)$ and our claim for the case $|\boldsymbol{m}| = 1$.

For the inductive step assume that (3.11) holds for a vector $\boldsymbol{m}$. We prove that it also holds for $\boldsymbol{m}' := \boldsymbol{m} + e_k$. Consider on each of the integrals on the right hand side of (3.11) the change of variables $(\epsilon_1, \ldots, \epsilon_n) \mapsto (\lambda^{\mu_{k1}}\epsilon_1, \ldots, \lambda^{\mu_{kn}}\epsilon_n)$. From differentiating with respect to λ and making $\lambda = 1$, it follows that

$$0 = (\langle \boldsymbol{\mu}_k, \boldsymbol{s}\rangle - \nu_k + m_k) \int_{\mathbb{R}_+^n} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}} q_{\boldsymbol{m},i}(\boldsymbol{\epsilon})\ \phi_{\boldsymbol{m},i}(\boldsymbol{\epsilon})}{p(\boldsymbol{\epsilon})^{1+|\boldsymbol{m}|}} \frac{d\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}} - \int_{\mathbb{R}_+^n} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}} q_{\boldsymbol{m}',i}(\boldsymbol{\epsilon})\phi_{\boldsymbol{m},i}(\boldsymbol{\epsilon})}{p(\boldsymbol{\epsilon})^{2+|\boldsymbol{m}|}} \frac{d\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}} + \int_{\mathbb{R}_+^n} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}} q_{\boldsymbol{m},i}(\boldsymbol{\epsilon})}{p(\boldsymbol{\epsilon})^{1+|\boldsymbol{m}|}} \left(\sum_{l=1}^n \epsilon_l \mu_{k_l} \frac{\partial \phi_{\boldsymbol{m},i}}{\partial \epsilon_p}\Big|_{\boldsymbol{\epsilon}} \right) \frac{d\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}},$$

implying

$$\int_{\mathbb{R}_+^n} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}} q_{\boldsymbol{m},i}(\boldsymbol{\epsilon})\ \phi_{\boldsymbol{m},i}(\boldsymbol{\epsilon})}{p(\boldsymbol{\epsilon})^{1+|\boldsymbol{m}|}} \frac{d\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}} = \frac{1}{\langle \boldsymbol{\mu}_k, \boldsymbol{s}\rangle - \nu_k + m_k} \left(\int_{\mathbb{R}_+^n} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}} q_{\boldsymbol{m}',i}(\boldsymbol{\epsilon})\phi_{\boldsymbol{m},i}(\boldsymbol{\epsilon})}{p(\boldsymbol{\epsilon})^{2+|\boldsymbol{m}|}} \frac{d\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}} - \sum_{l=1}^n \mu_{k_l} \int_{\mathbb{R}_+^n} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}} q_{\boldsymbol{m},i}(\boldsymbol{\epsilon})p(\boldsymbol{\epsilon})\epsilon_l}{p(\boldsymbol{\epsilon})^{2+|\boldsymbol{m}|}} \frac{\partial \phi_{\boldsymbol{m},i}}{\partial \epsilon_l}\Big|_{\boldsymbol{\epsilon}} \frac{d\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}} \right)$$

where $q_{\boldsymbol{m}',i}(\boldsymbol{\epsilon}) = (1 + |\boldsymbol{m}|)q_{e_k}(\boldsymbol{\epsilon})q_{\boldsymbol{m},i}(\boldsymbol{\epsilon}) - p(\boldsymbol{\epsilon})\tilde{q}_{\boldsymbol{m},i}(\boldsymbol{\epsilon})$, and

$$\tilde{q}_{\boldsymbol{m},i}(\boldsymbol{\epsilon}) = \frac{d}{d\lambda} \left(\lambda^{-|\boldsymbol{m}|\nu_k - m_k} q_{\boldsymbol{m},i}(\lambda^{\boldsymbol{\mu}_k}\boldsymbol{\epsilon}) \right)\Big|_{\lambda=1}.$$

To prove our claim, we are only left to show that $\Delta_{q_{\boldsymbol{m}',i}} \subset \Delta(|\boldsymbol{m}'|\boldsymbol{\nu} + \boldsymbol{m}')$ and $\Delta_{q_{\boldsymbol{m},i}p\,\epsilon_l} \subset \Delta(|\boldsymbol{m}'|\boldsymbol{\nu} + \boldsymbol{m}')$. For the first inclusion, notice that $\Delta_{q_{e_k}q_{\boldsymbol{m},i}} \subset \Delta(\boldsymbol{\nu} + e_k) + \Delta(|\boldsymbol{m}|\boldsymbol{\nu} + \boldsymbol{m}) \subset \Delta(|\boldsymbol{m}'|\boldsymbol{\nu} + \boldsymbol{m}')$ where we have use the general inclusion $\Delta(\boldsymbol{\alpha}) + \Delta(\boldsymbol{\beta}) \subset \Delta(\boldsymbol{\alpha} + \boldsymbol{\beta})$. Moreover, since any of the monomials in the polynomial $\tilde{q}_{\boldsymbol{m},i}$ has exponents on the hyperplane $\langle \boldsymbol{\mu}_k, \boldsymbol{\sigma}\rangle = |\boldsymbol{m}|\nu_k + m_k$, it follows that $\Delta_{p\,\tilde{q}_{\boldsymbol{m},i}} \subset \Delta(\boldsymbol{\nu}) + \Delta(|\boldsymbol{m}|\boldsymbol{\nu} + \boldsymbol{m} + e_k) \subset \Delta(|\boldsymbol{m}'|\boldsymbol{\nu} + \boldsymbol{m}')$. This yields the first inclusion. For the second inclusion, recall that if $\mu_{k_l} \neq 0$, $\Delta_{p\,\epsilon_l} \subset \Delta(\boldsymbol{\nu} + e_k)$, then

$$\Delta_{q_{\boldsymbol{m},i}p\,\epsilon_l} = \Delta_{q_{\boldsymbol{m},i}} + \Delta_{p\,\epsilon_l} \subset \Delta(|\boldsymbol{m}|\boldsymbol{\nu} + \boldsymbol{m}) + \Delta(\boldsymbol{\nu} + e_k)\Delta(|\boldsymbol{m}'|\boldsymbol{\nu} + \boldsymbol{m}')$$

proving the second inclusion. Doing the same procedure for each of the integrals on the right hand side of (3.11) we obtain

$$\mathcal{M}_{\phi/p}(\boldsymbol{s}) = \frac{1}{\prod_{j=1}^N u_{\boldsymbol{m}',j}(\boldsymbol{s})} \left(\sum_{i=1}^{N_{\boldsymbol{m}'}} \int_{\mathbb{R}_+^n} \frac{\boldsymbol{\epsilon}^{\boldsymbol{s}} q_{\boldsymbol{m}',i}(\boldsymbol{\epsilon})\ \phi_{\boldsymbol{m}',i}(\boldsymbol{\epsilon})}{p(\epsilon)^{1+|\boldsymbol{m}'|}} \frac{d\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}} \right),$$

as expected.

We prove now that each of the domains of convergence of each of the integrals on the right hand side of (3.11) contains $\Delta(\boldsymbol{\nu} - \boldsymbol{m})$. Fix i in $[N_{\boldsymbol{m}}]$, by means of Theorem 7.10, the i-th integral in (3.11) converges on

$$\bigcap_{\boldsymbol{\tau} \in \Delta_{q_{\boldsymbol{m},i}}} \left((1 + |\boldsymbol{m}|)\Delta(\boldsymbol{\nu}) - \boldsymbol{\tau}\right). \tag{3.13}$$

We show then that $\Delta(\boldsymbol{\nu} - \boldsymbol{m})$ is a subset of (3.13). Indeed, if $\boldsymbol{\sigma} \in \Delta(\boldsymbol{\nu} - \boldsymbol{m})$, then for every j in $[N]$

$$\langle \boldsymbol{\sigma}, \boldsymbol{\mu}_j \rangle \geq \nu_j - m_j,$$

moreover, the inclusion $\Delta_{q_{\boldsymbol{m},i}} \subset \Delta(|\boldsymbol{m}|\boldsymbol{\nu} + \boldsymbol{m})$ yields

$$\langle \boldsymbol{\tau}, \boldsymbol{\mu}_j \rangle \geq |\boldsymbol{m}|\nu_j + m_j, \quad \forall \tau \in \Delta_{q_{\boldsymbol{m},i}}.$$

Both inequalities imply that

$$\langle \boldsymbol{\sigma} + \boldsymbol{\tau}, \boldsymbol{\mu}_j \rangle \geq (1 + |\boldsymbol{m}|)\nu_j,$$

and thus $\boldsymbol{\sigma}$ lies on the intersection (3.13).

Inside the domain $\{\boldsymbol{s} \in \mathbb{C}^n : \boldsymbol{\sigma} \in \Delta(\boldsymbol{\nu} - \boldsymbol{m}) + \mathbb{R}_+^n\}$ the only poles of $\mathcal{M}_{\phi/p}$ are given by $u_{\boldsymbol{m},j}(\boldsymbol{s}) = 0$, and are simple, which coincide with the poles of the product $\prod_k \Gamma(\langle \boldsymbol{s}, \boldsymbol{\mu}_k \rangle - \nu_k)$. Therefore, by the theorem of removable singularities 3.5, $\Phi(\boldsymbol{s}) = \mathcal{M}_{\phi/p}(\boldsymbol{s})/(\prod_k \Gamma(\langle \boldsymbol{s}, \boldsymbol{\mu}_k \rangle - \nu_k))$ is an entire function which yields the expected result. □

8 Polar structure of Shintani zeta functions

The main objective of this section is to prove Theorems 8.7 and 8.18 which provide a better description of the polar locus of the Shintani zeta functions based on the Newton polytopes of the polynomials induced by the columns of the parametrising matrix $A \in \Sigma_{n\times r}(\mathbb{R}_{\geq 0})$. To that end, we show that the Shintani zeta functions can be expressed as the quotient of the Mellin transform of some function $\mathfrak{G}$ introduced in Paragraph 7.1 over some Gamma functions. We then use Theorems 7.10 and 7.11 to find a domain of convergence and a meromorphic continuation for the Mellin transform in question and thus for the Shintani zeta functions. Finally, in Paragraph 8.2 we prove that the possible hyperplanes carrying the Shintani zeta functions have normal vectors with coefficients 0 or 1 when written in terms of the canonical basis with integer and mutually coprime coefficients (Theorem 8.18). This proves that the poles at zero are similar to the ones of generic Feynman amplitudes studied in [92, 28].

8.1 Polar locus and Newton polytopes

In this paragraph we prove the main result of this chapter which gives a precise description of the polar loci of the Shintani zeta functions ζ_A associated to a matrix A, as in Definition 3.12 (Theorem 8.7). For this purpose, we bring together the results of Paragraphs 7.1 and 7.2. We recall that for a matrix $A \in \Sigma_{n\times r}(\mathbb{R}_{\geq 0})$ with columns are $C_1, \ldots, C_r$, $\mathfrak{G}(A) = \mathfrak{G}(C_1 \otimes \cdots \otimes C_r)$ (see (3.4)). We also recall that for a given column vector C_j we also denote, with some abuse of notation, by C_j the linear form defined by $C_j(\boldsymbol{\epsilon}) := \langle \boldsymbol{\epsilon}, C_j \rangle$.

Notice that given a matrix $A \in \Sigma_{n\times r}(\mathbb{R}_{\geq 0})$ which, under permutations of the rows and columns, can be expressed as a matrix by blocks of the form

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_M \end{pmatrix}$$

the Shintani zeta function ζ_A is the product of the Shintani zeta functions ζ_{A_k} for every $k \in [M]$ as the following example illustrates.

Example 8.1. *Consider the following matrix:*

$$A = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}$$

The sum in ζ_A can be expressed as the product

$$\begin{aligned}
\zeta_A(\boldsymbol{s}) &= \sum_{\boldsymbol{m}\in\mathbb{Z}^3_{\geq 1}} (a_{11}m_1 + a_{13}m_3)^{-s_1}(a_{22}m_2)^{-s_2}(a_{31}m_1 + a_{33}m_3)^{-s_3} \\
&= \sum_{m_1\geq 1}\sum_{m_3\geq 1} (a_{11}m_1 + a_{13}m_3)^{-s_1}(a_{31}m_1 + a_{33}m_3)^{-s_3} \sum_{m_2\geq 1} (a_{22}m_2)^{-s_2} \\
&= \zeta_{A_1}(s_1, s_3)\zeta_{A_2}(s_2)
\end{aligned}$$

where A_1 and A_2 are the two blocks of the matrix obtained after exchanging the second row with the third row, and the second column with the third column. The block matrix obtained after such permutations is

$$\begin{pmatrix} a_{11} & a_{13} & 0 \\ a_{31} & a_{33} & 0 \\ 0 & 0 & a_{22} \end{pmatrix}$$

For the rest of this document, we only consider matrices $A \in \Sigma_{n\times r}(\mathbb{R}_{\geq 0})$ which cannot be expressed as block matrices under permutations of the rows and the columns, since the analysis of such matrices might lead to several fake poles.

Proposition 8.2. *Let $A \in \Sigma_{n\times r}(\mathbb{R}_{\geq 0})$, then for $\boldsymbol{s}$ in the domain of convergence of ζ_A*

$$\zeta_A(\boldsymbol{s})\Gamma(\boldsymbol{s}) = \mathcal{M}_{\mathfrak{S}(A)}(\boldsymbol{s}).$$

Moreover, for every $A \in \Sigma_{n\times r}(\mathbb{R}_{\geq 0})$, ζ_A is absolutely convergent whenever $\mathcal{M}_{\mathfrak{S}(A)}(\boldsymbol{s})$ is absolutely convergent.

Proof. The statement follows from:

$$\begin{aligned}
\mathcal{M}_{\mathfrak{S}(A)}(\boldsymbol{s}) = \mathcal{M}_{\mathfrak{S}(C_1\otimes\cdots\otimes C_r)}(\boldsymbol{s}) &= \int_{\mathbb{R}^n_+} \boldsymbol{\epsilon}^{\boldsymbol{s}-1} \left(\prod_{j=1}^{r} \sum_{\boldsymbol{m}\in\mathbb{Z}^r_{\geq 1}} e^{-m_j C_j(\boldsymbol{\epsilon})} \right) d\boldsymbol{\epsilon} \\
&= \int_{\mathbb{R}^n_+} \boldsymbol{\epsilon}^{\boldsymbol{s}-1} \left(\sum_{\boldsymbol{m}\in\mathbb{Z}^r_{\geq 1}} e^{-\langle A^t\boldsymbol{\epsilon}, \boldsymbol{m}\rangle} \right) d\boldsymbol{\epsilon} \\
&= \sum_{\boldsymbol{m}\in\mathbb{Z}^r_{\geq 1}} \int_{\mathbb{R}^n_+} \boldsymbol{\epsilon}^{\boldsymbol{s}-1} e^{-\langle \boldsymbol{\epsilon}, A\boldsymbol{m}\rangle} d\boldsymbol{\epsilon} \\
&= \sum_{\boldsymbol{m}\in\mathbb{Z}^r_{\geq 1}} \prod_{i=1}^{n} \int_{\mathbb{R}_+} \epsilon_i^{s_i-1} e^{-L_i(\boldsymbol{m})\epsilon_i} d\epsilon_i \\
&= \sum_{\boldsymbol{m}\in\mathbb{Z}^r_{\geq 1}} \prod_{i=1}^{n} L_i(\boldsymbol{m})^{-s_i} \int_{\mathbb{R}_+} t_i^{s_i-1} e^{-t_i} dt_i = \zeta_A(\boldsymbol{s})\Gamma(\boldsymbol{s}).
\end{aligned}$$

Here $\Gamma(\boldsymbol{s}) = \Gamma(s_1)\cdots\Gamma(s_n)$, and the L_i's are the lines of the matrix A. As for the columns, we denote with some abuse of notation $L_i(\boldsymbol{m}) = \langle \boldsymbol{m}, L_i\rangle$. In the third line Fubini's theorem is used since $e^{-\langle \boldsymbol{\epsilon}, A\boldsymbol{m}\rangle}$ is positive. The statement of absolute convergence follows also from Fubini's theorem.

□

We focus on the study of $\mathcal{M}_{\mathfrak{S}(A)}$ in order to build a meromorphic continuation for ζ_A. For that purpose we prove two lemmas.

Lemma 8.3. The map $x \mapsto e^{-x}h(x)$ lies in $\mathcal{S}(\mathbb{R}_{\geq 0})$, where $h : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is defined as $h(x) = \frac{\mathrm{Td}(x)e^x - 1}{x}$ or equivalently

$$\frac{\mathrm{Td}(x)}{x} = e^{-x}\left(\frac{1}{x}]h(x)\right).$$

Proof. Indeed, $x \mapsto e^{-x}h(x) = \frac{\mathrm{Td}(x)}{x} - \frac{e^{-x}}{x}$, thus it lies in $\mathcal{S}(\mathbb{R}_+)$ as a consequence of Lemma 7.4. Moreover, in a neighborhood of zero

$$\begin{aligned} e^{-x}h(x) &= \frac{\mathrm{Td}(x)}{x} - \frac{e^{-x}}{x} \\ &= \sum_{n=-1}^{\infty} B_{n+1}\frac{x^n}{(n+1)!} - \sum_{n=-1}^{\infty}\frac{x^n}{(n+1)!} \\ &= \sum_{n=0}^{\infty}(B_{n+1}-1)\frac{x^n}{(n+1)!}. \end{aligned}$$

Here B_n are the Bernoulli numbers, and we used the fact that $B_0 = 1$. Therefore $x \mapsto e^{-x}h(x)$ is analytic at zero with $h(0) = -\frac{3}{2}$, and in particular $x \mapsto e^{-x}h(x) \in \mathcal{S}(\mathbb{R}_{\geq 0})$. □

For the rest of this paragraph we set $\phi(x) := e^{-x} \in \mathcal{S}(R_{\geq 0})$ in order to simplify the notation.

Lemma 8.4. For $A \in \Sigma_{n\times r}(\mathbb{R}_{\geq 0})$, and $J \subset [r]$, we have that $\phi(C_{[r]}(\boldsymbol{\epsilon}))h(C_J(\boldsymbol{\epsilon})) \in \mathcal{S}(\mathbb{R}^n_{\geq 0})$, where we used the compact notation $h(C_J(\boldsymbol{\epsilon})) := \prod_{j\in J} h(C_j(\boldsymbol{\epsilon}))$ and $\phi(C_{[r]}(\boldsymbol{\epsilon})) := \prod_{j=1}^{r}\phi(C_j(\boldsymbol{\epsilon}))$. Furthermore, we have

$$\mathcal{M}_{\mathfrak{S}(A)}(\boldsymbol{s}) = \mathcal{M}_{\mathfrak{S}(C_1\otimes\cdots\otimes C_r)}(\boldsymbol{s}) = \sum_{I\sqcup J=[r]}\int_{\mathbb{R}^n_+}\boldsymbol{\epsilon}^{\boldsymbol{s}-1}\frac{\phi(C_{[r]}(\boldsymbol{\epsilon}))h(C_J(\boldsymbol{\epsilon}))}{C_I(\boldsymbol{\epsilon})}d\boldsymbol{\epsilon}.$$

Proof. Fix $J \subset [r]$, we prove that $\phi(C_{[r]}(\boldsymbol{\epsilon}))h(C_J(\boldsymbol{\epsilon})) \in \mathcal{S}(\mathbb{R}^n_{\geq 0})$. Since $A \in \Sigma_{n\times r}(\mathbb{R}_{\geq 0})$, there is at least a non-zero element in each row. The latter implies that for every path in which $\boldsymbol{\epsilon}$ tends to ∞, there is at least a $j \in [r]$ such that $C_j(\boldsymbol{\epsilon}) \to \infty$. Assume $j \in J$, then Lemma 8.3 implies that $\phi(C_J(\boldsymbol{\epsilon}))h(C_J(\boldsymbol{\epsilon}))\longrightarrow 0$ as $\boldsymbol{\epsilon} \to \infty$. Since $\phi(C_{[r]\setminus J})$ is bounded, this yields the result. Assume otherwise $j \in [r] \setminus J$, then $\phi(C_{[r]\setminus J}(\boldsymbol{\epsilon})) \longrightarrow 0$ as $\boldsymbol{\epsilon} \to \infty$. By Lemma 8.3 $\phi(C_J)h(C_J)$ is bounded and therefore $\phi(C_{[r]}(\boldsymbol{\epsilon}))h(C_J(\boldsymbol{\epsilon})) \in \mathcal{S}(\mathbb{R}^n_{\geq 0})$.

On the other hand, since $\mathfrak{S}(C_j) = \frac{\mathrm{Td}(C_j)}{C_j}$ (see Definition-Proposition 7.5), with the notations of Lemma 8.3, it follows that:

$$\begin{aligned} \mathcal{M}_{\mathfrak{S}(C_1\otimes\cdots\otimes C_r)}(\boldsymbol{s}) &= \int_{\mathbb{R}^n_+}\boldsymbol{\epsilon}^{\boldsymbol{s}-1}\mathfrak{S}(C_1\otimes\cdots\otimes C_r)(\boldsymbol{\epsilon})d\boldsymbol{\epsilon} \\ &= \int_{\mathbb{R}^n_+}\boldsymbol{\epsilon}^{\boldsymbol{s}-1}\prod_{j=1}^{r}\mathfrak{S}(C_j(\boldsymbol{\epsilon}))d\boldsymbol{\epsilon} \\ &= \int_{\mathbb{R}^n_+}\boldsymbol{\epsilon}^{\boldsymbol{s}-1}\prod_{j=1}^{r}\left(\frac{e^{-C_j(\boldsymbol{\epsilon})}}{C_j(\boldsymbol{\epsilon})} + e^{-C_j(\boldsymbol{\epsilon})}h(C_j(\boldsymbol{\epsilon}))\right)d\boldsymbol{\epsilon} \\ &= \sum_{I\sqcup J=[r]}\int_{\mathbb{R}^n_+}\boldsymbol{\epsilon}^{\boldsymbol{s}-1}\frac{\phi(C_{[r]}(\boldsymbol{\epsilon}))h(C_J(\boldsymbol{\epsilon}))}{C_I(\boldsymbol{\epsilon})}d\boldsymbol{\epsilon}, \end{aligned}$$

which proves the statement. □

We show what is the domain of convergence for a sum of the type shown in (1.33).

Proposition 8.5. *Let $A \in \Sigma_{n\times r}(\mathbb{R}_{\geq 0})$. The sum on the right hand side of the Shintani zeta function ζ_A as defined in* (1.33) *is absolutely convergent for*

$$\boldsymbol{\sigma} \in \bigcap_{I\subset[r]} \left(\Delta_{C_I} + \mathbb{R}^n_+\right), \tag{3.14}$$

where $\boldsymbol{\sigma} = \Re(\boldsymbol{s})$.

Proof. From Proposition 8.2, it is enough to prove that $\mathcal{M}_{\mathfrak{S}(A)}$ is absolutely convergent for $\boldsymbol{\sigma}$ lying in the intersection on the right hand side of (3.14). The result follows from Lemma 8.4 and Theorem 7.10. □

As a corollary, we recall a well known result which we prove for the sake of completeness.

Corollary 8.6. Let $A \in \Sigma_{n\times r}(\mathbb{R}_{\geq 0})$, the sum on the right hand side of the Shintani zeta function ζ_A as defined in (1.33) is absolutely convergent for $\Re s_i > r$ for every $1 \leq i \leq r$.

Proof. By means of Proposition 8.5, we have to show that any $\boldsymbol{s}$ such that $\Re\boldsymbol{s} = \boldsymbol{\sigma}$ lies in the interior of the intersection

$$\bigcap_{I\subset[r]} \left(\Delta_{C_I} + \mathbb{R}^n_+\right). \tag{3.15}$$

Since $\Delta_{C_i} \subset \{(x_1,\ldots,x_n) \in \mathbb{R}^n_{\geq 0} : x_1+\cdots+x_n = 1\}$, then for any $I \subset [r]$, $\Delta_{C_I} \subset \{(x_1,\ldots,x_n) \in \mathbb{R}^n_{\geq 0} : x_1+\cdots+x_n = |I|\}$. We prove that

$$\Delta_{C_I} + \mathbb{R}^n_+ \supset \{(x_1,\ldots,x_n) \in \mathbb{R}^n_+ : \forall i \in [n], x_i > |I|\}.$$

Indeed, for any $\boldsymbol{\sigma} \in \{(x_1,\ldots,x_n) \in \mathbb{R}^n_{\geq 0} : \forall i \in [n], x_i > |I|\}$ and every $\boldsymbol{\sigma}_0 \in \{(x_1,\ldots,x_n) \in \mathbb{R}^n_{\geq 0} : x_1+\cdots+x_n = |I|\}$, it is straightforward to see that $\boldsymbol{\sigma} - \boldsymbol{\sigma}_0 \in \mathbb{R}^n_+$. In particular, the set $\{\boldsymbol{\sigma} \in \mathbb{R}^n : \sigma_i > r\}$ is a subset of the intersection in (3.15) which proves the result. □

As mentioned in the introduction, our aim is to build a meromorphic continuation of (1.33) to the whole space $\mathbb{C}^n$ and to give a precise description of its poles. In doing so, we refine a result by Matsumoto [69] that we have recalled in the Paragraph 3.2. Our result follows from Theorem 7.11.

Theorem 8.7. Let $A \in \Sigma_{n\times r}(\mathbb{R}_{\geq 0})$ and write $C_J(\boldsymbol{\epsilon}) = \prod_{j\in J} C_j(\boldsymbol{\epsilon})$ where $J \subset [r]$ and C_j the j-th column of A. The Shintani zeta function

$$\zeta_A(\boldsymbol{s}) = \sum_{m_1\geq 1}\cdots\sum_{m_r\geq 1} (a_{11}m_1+\cdots+a_{1r}m_r)^{-s_1} \times\cdots\times (a_{n1}m_1+\cdots+a_{nr}m_r)^{-s_n}$$

admits a meromorphic continuation to $\mathbb{C}^n$ with possible simple poles located on the hyperplanes

$$\langle \boldsymbol{\mu}_k^J, \boldsymbol{s}\rangle = \nu_k^J - l.$$

Here $\boldsymbol{\mu}_k^J$ are the vectors in $\mathbb{Z}^n$ on the inward normal direction of the facets of $\Delta_{C_J} + \mathbb{R}^n_+$ with mutually coprime coordinates, ν_k^J are integers as described in (3.9), and $l \in \mathbb{Z}_{\geq 0}$.

Moreover if $\boldsymbol{\mu}_k^J = e_i$ for some vector of the canonical basis e_i, then only the hyperplanes where $0 \leq l \leq \nu_k^J - 1$ carry poles.

Proof. By means of Proposition 8.2 and Lemma 8.4, it is enough to build a meromorphic continuation and describe the polar locus for the integrals

$$\int_{\mathbb{R}^n_+} \boldsymbol{\epsilon}^{\boldsymbol{s}-1} \frac{\phi(C_{[r]}(\boldsymbol{\epsilon}))h(C_{[r]\setminus J}(\boldsymbol{\epsilon}))}{C_J(\boldsymbol{\epsilon})} d\boldsymbol{\epsilon}$$

for every $J \subset [r]$. It follows from Theorem 7.11 that for J fixed the aforementioned integral admits a meromorphic continuation to the whole space $\mathbb{C}^n$ where the poles are the same as those of the functions $\Gamma(\langle \boldsymbol{\mu}_k^J, \boldsymbol{s}\rangle - \nu_k^J)$. This proves the first part of the theorem.

Moreover, if there is $\boldsymbol{\mu}_k^J = e_i$ for any e_i by means of the removable singularities theorem, the function $\zeta_A(\boldsymbol{s}) = \frac{1}{\Gamma(\boldsymbol{s})}\mathcal{M}_{\mathfrak{S}(C_1\otimes\cdots\otimes C_r)}(\boldsymbol{s})$ has no poles on the hyperplanes $\langle \boldsymbol{\mu}_k^J, \boldsymbol{s}\rangle = -l$ where $l \geq \nu_k^J$ which yields the result.

□

Using the polar description of the Shintani zetas provided in the previous theorem, we can describe the change of the polar structure under some linear transformations of $\boldsymbol{s}$. Notice that this amounts to a transformation of the polyhedra which induce the polar structure.

Corollary 8.8. Let A be a matrix in $\Sigma_{n\times r}(\mathbb{R}_{\geq 0})$ and B an $n\times n$ real matrix, then the function $\boldsymbol{s}\mapsto \zeta_A(B\boldsymbol{s})$ admits a meromorphic continuation to the space $\mathbb{C}^n$ with possible simple poles located on the hyperplanes $\langle B^t\boldsymbol{\mu}_k^J, \boldsymbol{s}\rangle = \nu_k^J - l$. Here B^t is the transpose of B, and $\boldsymbol{\mu}_k^J$, ν_k^J and l are as in Theorem 8.7.

Proof. It is immediate from Theorem 8.7. □

The following corollary shows how the pole structure of a Shintani zeta function ζ_A associated to a matrix A only depends on the positions of the zeros in the matrix. It does not depend on the values of the other arguments as long as they are positive. In particular, it is enough to consider the matrices $S\in\Sigma_{n\times r}(\mathbb{R}_{\geq 0})$ with arguments 0 or 1 to study all possible configurations of the poles for Shintani zeta functions. This is in sharp contrast with the convergent case, since the values of ζ_A outside the poles depend strongly on the values of the coefficients of A.

Corollary 8.9. Let $A\in\Sigma_{n\times r}(\mathbb{R}_{\geq 0})$ and consider a new matrix $\bar{A}$ defined by $\bar{a}_{ij}=0$ if $a_{ij}=0$, and $\bar{a}_{ij}=1$ if $a_{ij}\neq 0$. Then ζ_A and $\zeta_{\bar{A}}$ have the same pole structure.

Proof. Let C_j (resp. $\bar{C}_j$) be the columns of the matrix A (resp. $\bar{A}$). Once again with some abuse of notation, we denote by the same symbol the linear forms $C_j(\boldsymbol{\epsilon}) = \langle\boldsymbol{\epsilon}, C_j\rangle$ and $\bar{C}_j(\boldsymbol{\epsilon}) = \langle\boldsymbol{\epsilon}, \bar{C}_j\rangle$. Since $C_j(\epsilon) = \sum_{i=1}^n a_{ij}\epsilon_i$ and $\bar{C}_j(\epsilon) = \sum_{i=1}^n \bar{a}_{ij}\epsilon_i$ where $a_{ij}=0$ if, and only if $\bar{a}_{ij}=0$, it follows from the definition of Newton polytope (Definition 3.13) that $\Delta_{C_j} = \Delta_{\bar{C}_j}$. Since the Newton polytope of the product of polynomials is equal to the Minkowski sum of their Newton polytopes, it follows that $\Delta_{C_J} = \Delta_{\bar{C}_J}$ for any $J\subset [r]$. Theorem 8.7 then yields the result.

□

Remark 8.10. As mentioned in Section 3.2, for $\boldsymbol{b}\in\mathbb{R}_+^n$, the map

$$\boldsymbol{s}\mapsto \zeta_{A,b}(\boldsymbol{s}) := \sum_{m_1\geq 0}\cdots\sum_{m_r\geq 0}(a_{11}m_1+\cdots+a_{1r}m_r+b_1)^{-s_1}\times\cdots\times(a_{n1}m_1+\cdots+a_{nr}m_r+b_n)^{-s_n}$$

admits a meromorphic continuation with the same pole structure as ζ_A. Indeed, for the map

$$\mathfrak{S}_{\boldsymbol{b}}(A) := e^{-\langle\boldsymbol{\epsilon},\boldsymbol{b}\rangle}\left(\prod_{j=1}^r\sum_{\boldsymbol{m}\in\mathbb{Z}_{\geq 0}^r} e^{-m_jC_j(\boldsymbol{\epsilon})}\right)$$

a similar computation to that in the proof of Proposition 8.2 shows that $\zeta_{A,b}(\boldsymbol{s})\Gamma(\boldsymbol{s}) = \mathcal{M}_{\mathfrak{S}_{\boldsymbol{b}}(A)}(\boldsymbol{s})$, and thus $\zeta_{A,b}$ is absolutely convergent whenever $\mathcal{M}_{\mathfrak{S}_{\boldsymbol{b}}(A)}(\boldsymbol{s})$ is absolutely convergent. Moreover, analogous to Lemma 8.4

$$\mathcal{M}_{\mathfrak{S}_{\boldsymbol{b}}(A)}(\boldsymbol{s}) = \sum_{I\sqcup J=[r]}\int_{\mathbb{R}_+^n}\boldsymbol{\epsilon}^{\boldsymbol{s}-1}\frac{e^{-\langle\boldsymbol{\epsilon},\boldsymbol{b}\rangle}\phi(C_{[r]}(\boldsymbol{\epsilon}))h(C_J(\boldsymbol{\epsilon}))}{C_I(\boldsymbol{\epsilon})}d\boldsymbol{\epsilon}.$$

Our claim follows from replacing $\phi(C_{[r]}(\boldsymbol{\epsilon}))$ for $\phi_{\boldsymbol{b}}(C_{[r]}(\boldsymbol{\epsilon})) := e^{-\langle\boldsymbol{\epsilon},\boldsymbol{b}\rangle}\phi(C_{[r]}(\boldsymbol{\epsilon}))$ in the proof of Theorem 8.7.

8.2 Family of meromorphic germs spanned by the Shintani zeta functions

In this paragraph, we study the space of meromorphic germs at zero spanned by the Shintani zeta functions. For that purpose, we give a better description of the vectors μ_K^J obtained in Theorem 8.7. The main result of this section is Theorem 8.18 which states that the arguments of the vectors μ_K^J are either 0 or 1 if represented in the canonical basis of $\mathbb{R}^n$. This proves that the poles at zero are similar to the ones of generic Feynman amplitudes studied in [92, 28]. However, the family of meromorphic germs spanned by those containing Shintani zeta functions is bigger that the one spanned by those containing generic Feynman amplitudes, since the latter are described by singular families (see [92]).

Proposition 8.11. *For every set of vectors $S = \{\boldsymbol{\mu}_1 \cdots \boldsymbol{\mu}_m\}$ in $\{0,1\}^n$ such that each vector $\boldsymbol{\mu}_j$ has at least two of its arguments different from zero, there is a matrix $A_S \in \Sigma_{n\times r}(\mathbb{R}_{\geq 0})$ such that the meromorphic continuation of ζ_{A_S} has all its poles located at the hyperplanes $\langle \boldsymbol{\mu}_j, s\rangle = 0$ for every $1 \leq j \leq m$.*

Proof. For a set S as in the statement, consider the $n\times(m+1)$ matrix A_S where $\boldsymbol{\mu}_j$ is the j-th column for $1 \leq j \leq m$ and the last column is full of ones. We claim that ζ_{A_S} has poles at $\langle \boldsymbol{\mu}_j, s\rangle = 0$ for every j. Indeed, $\Delta_{\langle \boldsymbol{\mu}_j, \boldsymbol{\epsilon}\rangle} + \mathbb{R}^n_+$ is the intersection of the half spaces $\langle e_i, \boldsymbol{\sigma}\rangle > 0$ and $\langle \boldsymbol{\mu}_j, \boldsymbol{\sigma}\rangle > 1$, where $\{e_i\}_{i=1}^n$ is the canonical basis of $\mathbb{R}^n$. The result then follows from Theorem 8.7. □

Remark 8.12. Notice that in the previous proposition, the case where $\boldsymbol{\mu}$ is an element of the canonical basis $\{e_i\}_{i=1}^n$ is not considered. This is a consequence of Theorem 8.7 where the hyperplanes of the type $\langle e_i, \boldsymbol{s}\rangle = \nu_k^J - l$ can carry poles only when $0 \leq l \leq \nu_k^J - 1$. The reason for this is that $\zeta_A = \frac{\mathcal{M}_{\mathfrak{S}(A)}}{\Gamma}$ (see Proposition 8.2), and therefore the poles of the Gamma functions cancel those of $\mathcal{M}_{\mathfrak{S}(A)}$ lying on the hyperplanes $\langle e_i, \boldsymbol{s}\rangle = l$ with $l \in \mathbb{Z}_{\leq 0}$.

The following results aim to prove a statement converse to that on Proposition 8.11. For this purpose, let us introduce some notation first. For any subset I of $[n]$, we denote by $\pi_I : \mathbb{R}^n \to \mathbb{R}^{n-|I|}$ the projection orthogonal to the subspace $\bigoplus_{i\in I}\mathbb{R}(e_i)$. We also denote by $\iota_I : \mathbb{R}^{n-|I|} \to \mathbb{R}^n$ the injection map such that $Im(\iota_I) = \bigoplus_{i\in[n]\setminus I}\mathbb{R}(e_i)$ and

$$\pi_I \circ \iota_I = \mathrm{Id}_{\mathbb{R}^{n-|I|}}.$$

Notice also that, when restricted to $\bigoplus_{i\in[n]\setminus I}\mathbb{R}(e_i) \subset \mathbb{R}^n$,

$$\iota_I \circ \pi_I|_{\bigoplus_{i\in[n]\setminus I}\mathbb{R}(e_i)} = \mathrm{Id}_{\bigoplus_{i\in[n]\setminus I}\mathbb{R}(e_i)}. \tag{3.16}$$

For example, if $n = 4$ and $I = \{2,4\}$, then $\pi_I((v_1,v_2,v_3,v_4)) = (v_1,v_3)$, and $\iota_I((w_1,w_2)) = (w_1,0,w_2,0)$.

Lemma 8.13. Let D be a subset of $\mathbb{R}^n_+$ such that $D + \mathbb{R}^n_+ = \iota_I(\pi_I(D+\mathbb{R}^n_+)) + \sum_{i\in I}\mathbb{R}_+(e_i)$. If there is a $\hat{\boldsymbol{\mu}} \in \mathbb{R}^{n-|I|}$ such that

$$\pi_I(D+\mathbb{R}^n_+) = \mathbb{R}^{n-|I|}_+ \cap \{\tilde{\boldsymbol{\sigma}} \in \mathbb{R}^{n-|I|} : \langle \tilde{\boldsymbol{\sigma}}, \hat{\boldsymbol{\mu}}\rangle > 1\},$$

then

$$D + \mathbb{R}^n_+ = \mathbb{R}^n_+ \cap \{\boldsymbol{\sigma} \in \mathbb{R}^n : \langle \boldsymbol{\sigma}, \boldsymbol{\mu}\rangle > 1\}, \tag{3.17}$$

where $\boldsymbol{\mu} = \iota_I(\hat{\boldsymbol{\mu}})$, i.e, $\pi(\boldsymbol{\mu}) = \hat{\boldsymbol{\mu}}$ and $\mu_i = 0$ for every $i \in I$.

Proof. We prove the inclusion from left to right in (3.17). For $\boldsymbol{\sigma} \in D + \mathbb{R}^n_+$, it clearly implies that $\boldsymbol{\sigma} \in \mathbb{R}^n_+$. Notice that, for $\tilde{\boldsymbol{\sigma}} := \pi_I(\boldsymbol{\sigma})$, we have $\langle \tilde{\boldsymbol{\sigma}}, \hat{\boldsymbol{\mu}}\rangle = \langle \boldsymbol{\sigma}, \boldsymbol{\mu}\rangle$ since $\pi_I(\boldsymbol{\mu}) = \hat{\boldsymbol{\mu}}$ and $\mu_i = 0$ for every $i \in I$. It follows that $\langle \boldsymbol{\sigma}, \boldsymbol{\mu}\rangle > 1$ which yields the inclusion.

We proceed to prove the inclusion from right to left in (3.17). Let $\boldsymbol{\sigma} \in \mathbb{R}^n_+$ be such that $\langle \boldsymbol{\sigma}, \boldsymbol{\mu}\rangle > 1$, we claim $\boldsymbol{\sigma} \in \iota_I(\pi_I(D+\mathbb{R}^n_+)) + \sum_{i\in I}\mathbb{R}_+(e_i)$. Indeed, it follows from $\langle \tilde{\boldsymbol{\sigma}}, \hat{\boldsymbol{\mu}}\rangle = \langle \boldsymbol{\sigma}, \boldsymbol{\mu}\rangle$, and since $\boldsymbol{\sigma} \in \mathbb{R}^n_+$ its i-th coordinate is positive for every $i \in I$. The result follows from the equality $D + \mathbb{R}^n_+ = \iota_I(\pi_I(D + \mathbb{R}^n_+)) + \sum_{i\in I}\mathbb{R}_+(e_i)$. □

Recall that for a column vector C_j we denote, with some abuse of notation, by the same symbol the polynomial $C_j(\boldsymbol{\epsilon}) = \langle \boldsymbol{\epsilon}, C_j\rangle$, and then its Newton polytope Δ_{C_j} is well defined.

Lemma 8.14. Let A be a matrix in $\Sigma_{n\times r}(\mathbb{R}_{\geq 0})$, with C_j the columns of A. For each $j \in [r]$, consider the vector $\boldsymbol{\mu}_j \in \{0,1\}^n$ with its i-th coordinate equal to zero if, and only if, the i-th element of the column C_j is a zero. Then

$$\Delta_{C_j} + \mathbb{R}^n_+ = \bigcap_{i\in[n]}\{\boldsymbol{\sigma} \in \mathbb{R}^n : \langle \boldsymbol{\sigma}, e_i\rangle > 0\} \cap \{\boldsymbol{\sigma} \in \mathbb{R}^n : \langle \boldsymbol{\sigma}, \boldsymbol{\mu}\rangle > 1\} = \mathbb{R}^n_+ \cap \{\boldsymbol{\sigma} \in \mathbb{R}^n : \langle \boldsymbol{\sigma}, \boldsymbol{\mu}_j\rangle > 1\}. \tag{3.18}$$

Remark 8.15. Notice that the first set of intersections in the middle term in Equation (3.18) amount to $\mathbb{R}^n_+$, and therefore the middle and last terms are trivially equal. This formulation will be useful for Lemma 8.16.

Proof. We analyze two different cases:

1. Assume that $C_j(\boldsymbol{\epsilon}) = \sum_{i=1}^n a_{ij}\epsilon_j$ is such that $a_{ij} \neq 0$ for every i. Then

$$\Delta_{C_j} = \Big\{\boldsymbol{\sigma} = \sum_{i\in[n]} \lambda_i e_i \in \mathbb{R}_+^n : \sum_{i\in n} \lambda_i = 1\Big\}$$

is an $n-1$ dimensional simplex. Setting $\boldsymbol{\mu} = (\underbrace{1,\dots,1}_{n-\text{times}})$, it follows that

$$\Delta_{C_j} + \mathbb{R}_+^n = \mathbb{R}_+^n \cap \{\boldsymbol{\sigma} \in \mathbb{R}^n : \langle \boldsymbol{\sigma}, \boldsymbol{\mu}_j\rangle > 1\},$$

and the result follows.

2. Consider now the case where $a_{ij} = 0$ for $i \in I_j$ where $I_j \subset [n]$ and $I_j \neq [n]$. It implies that $\Delta_{C_j} = \iota_{I_j}(\pi_{I_j}(\Delta_{C_j}))$. In words it means that Δ_{C_j} is contained in the subspace $\bigoplus_{i\in[n]\setminus I_j} \mathbb{R}_+(e_i) \subset \mathbb{R}^n$, and thus

$$\Delta_{C_j} + \mathbb{R}_+^n = \Delta_{C_j} + \sum_{i\in[n]\setminus I_j} \mathbb{R}_+(e_i) + \sum_{i\in I_j} \mathbb{R}_+(e_i) = \iota_{I_j}(\pi_{I_j}(\Delta_{C_j} + \mathbb{R}_+^n)) + \sum_{i\in I_j} \mathbb{R}_+(e_i), \tag{3.19}$$

where we used (3.16) and the fact that $\sum_{i\in I_j} \mathbb{R}_+(e_i) \subset \ker(\pi_{I_j})$. Notice also that $\pi_{I_j}(\Delta_{C_j})$ is the Newton polytope of $\mathbb{R}^{n-|I_j|} \ni \boldsymbol{\epsilon} \mapsto p(\boldsymbol{\epsilon}) = \sum_{i=1}^{n-|I_j|} \epsilon_i$, then the previous case implies that

$$\pi_{I_j}(\Delta_{C_j}) + \mathbb{R}_+^{n-|I_j|} = \mathbb{R}_+^{n-|I_j|} \cap \{\boldsymbol{\sigma} \in \mathbb{R}^{n-|I_j|} : \langle \boldsymbol{\sigma}, \hat{\boldsymbol{\mu}}\rangle > 1\} \tag{3.20}$$

with $\hat{\boldsymbol{\mu}} = (\underbrace{1,\dots,1}_{n-|I_j|\text{ times}})$. The result follows from (3.19), (3.20) and Lemma 8.13 with $D = \Delta_{C_j}$.

The case where all $a_{ij} = 0$ is not considered since $A \in \Sigma_{n\times r}(\mathbb{R}_{\geq 0})$ implies that each line and each row has at least an argument different from zero (Definition 3.12). □

We proceed to generalize Lemma 8.14 for $\Delta_{C_J} + \mathbb{R}_+^n$ where $|J| > 1$. To that end we introduce some notations: Let $\boldsymbol{\mu} \in \{0,1\}^n$, we call the **support** of $\boldsymbol{\mu}$, denoted by $\mathrm{Supp}(\boldsymbol{\mu})$, the subset of $[n]$ given by

$$\mathrm{Supp}(\boldsymbol{\mu}) := \{i \in [n] : \mu_i \neq 0\}.$$

Notice that for every subset S of $[n]$ there is a unique vector $\boldsymbol{\mu} \in \{0,1\}^n$ such that $\mathrm{Supp}(\boldsymbol{\mu}) = S$. For a matrix A in $\Sigma_{n\times r}(\mathbb{R}_{\geq 0})$, and $J \subset [r]$, we denote by $\boldsymbol{\mu}_J$ the only vector in $\{0,1\}^n$ which satisfies $\mathrm{Supp}(\boldsymbol{\mu}_J) = \bigcup_{j\in J} \mathrm{Supp}(\mu_j)$. In particular if $J = \{j\}$, then $\boldsymbol{\mu}_j = \boldsymbol{\mu}_{\{j\}}$. We also write S_j short for $\mathrm{Supp}(\boldsymbol{\mu}_j)$ and S_J short for $\mathrm{Supp}(\boldsymbol{\mu}_J)$.

With the notations previously introduced, Lemma 8.14 implies that

$$\boldsymbol{\sigma} \in \Delta_{C_j} + \mathbb{R}_+^n \Leftrightarrow \boldsymbol{\sigma} \in \mathbb{R}_+^n \wedge \sum_{i\in S_j} (\boldsymbol{\sigma})_i > 1, \tag{3.21}$$

where $(\boldsymbol{\sigma})_i$ denotes the i-th coordinate of the vector $\boldsymbol{\sigma}$.

Lemma 8.16. Let A be a matrix in $\Sigma_{n\times r}(\mathbb{R}_{\geq 0})$ and $J \subset [r]$. Then

$$\Delta_{C_J} + \mathbb{R}_+^n \subset \Big(\bigcap_{K\subset J \,|\, K\neq\emptyset} \{\boldsymbol{\sigma} \in \mathbb{R}^n : \langle \boldsymbol{\sigma}, \boldsymbol{\mu}_K\rangle > |K|\}\Big) \cap \mathbb{R}_+^n,$$

where $C_J(\boldsymbol{\epsilon}) = \prod_{j\in J} \langle \boldsymbol{\epsilon}, C_j\rangle$ as before.

Proof. Using the fact that the Newton polytope of a product of polynomials is equal to the Minkowski sum of the Newton polytopes of each of the polynomials, one sees that $\Delta_{C_J} = \sum_{j \in J} \Delta_{C_j}$. Furthermore, since $\mathbb{R}^n_+ = \mathbb{R}^n_+ + \mathbb{R}^n_+$, we have $\Delta_{C_J} + \mathbb{R}^n_+ = \sum_{j\in J}\left(\Delta_{C_j} + \mathbb{R}^n_+\right)$. We prove then the following inclusion which is equivalent to the one in the statement of the lemma.

$$\sum_{j\in J}\left(\Delta_{C_j} + \mathbb{R}^n_+\right) \subset \Big(\bigcap_{K\subset J \,|\, K\neq\emptyset} \{\boldsymbol{\sigma}\in\mathbb{R}^n : \langle \boldsymbol{\sigma}, \boldsymbol{\mu}_K\rangle > |K|\}\Big) \cap \mathbb{R}^n_+.$$

For each $j \in J$, consider $\boldsymbol{\sigma}_j \in \Delta_{C_j} + \mathbb{R}^n_+$. It is clear that $\sum_{j\in J} \boldsymbol{\sigma}_j \in \mathbb{R}^n_+$ since every $\Delta_{C_j} \subset \mathbb{R}^n_+$. On the other hand, for $K \subset J$ with $K \neq \emptyset$, (3.21) implies that

$$\langle \boldsymbol{\sigma}_j, \boldsymbol{\mu}_K\rangle = \sum_{i\in S_K} (\boldsymbol{\sigma}_j)_i > 1$$

if $j \in K$. Otherwise

$$\langle \boldsymbol{\sigma}_j, \boldsymbol{\mu}_K\rangle > 0$$

since $\boldsymbol{\sigma}_j \in \mathbb{R}^n_+$. The last two inequalities imply that $\langle \sum_{j\in J} \boldsymbol{\sigma}_j, \boldsymbol{\mu}_K\rangle > |K|$, and thus $\sum_{j\in J} \boldsymbol{\sigma}_j \in \left(\bigcap_{K\subset J,\, K\neq\emptyset} \{\boldsymbol{\sigma}\in\mathbb{R}^n : \langle \boldsymbol{\sigma}, \boldsymbol{\mu}_K\rangle > |K|\}\right) \cap \mathbb{R}^n_+$, which completes the inclusion.

□

The next proposition states that the inclusion in Lemma 8.16 is actually an equality. Its proof borrows tools from graph theory which will be discussed in Section 9.

Proposition 8.17. *Let A be a matrix in $\Sigma_{n\times r}(\mathbb{R}_{\geq 0})$ and $J \subset [r]$. Then*

$$\Delta_{C_J} + \mathbb{R}^n_+ = \Big(\bigcap_{K\subset J \,|\, K\neq\emptyset} \{\boldsymbol{\sigma}\in\mathbb{R}^n : \langle \boldsymbol{\sigma}, \boldsymbol{\mu}_K\rangle > |K|\}\Big) \cap \mathbb{R}^n_+.$$

Proof. The inclusion from left to right was proven in Lemma 8.16. For the other direction we prove the equivalent inclusion

$$\sum_{j\in J}\left(\Delta_{C_j} + \mathbb{R}^n_+\right) \supset \Big(\bigcap_{K\subset J \,|\, K\neq\emptyset} \{\boldsymbol{\sigma}\in\mathbb{R}^n : \langle \boldsymbol{\sigma}, \boldsymbol{\mu}_K\rangle > |K|\}\Big) \cap \mathbb{R}^n_+, \tag{3.22}$$

where the $\boldsymbol{\mu}_j$s are as in Lemma 8.14. Notice also that for an element $\boldsymbol{\sigma} \in \mathbb{R}^n_+$ which lies in the set on the right hand side of (3.22), the equivalence (3.21) implies that for every $K \subset J$ with $K \neq \emptyset$

$$\sum_{i\in S_K} (\boldsymbol{\sigma})_i > |K|.$$

It follows from Corollary 9.6 below that for every $j \in [n]$ there is $\boldsymbol{\sigma}_j \in \mathbb{R}^n_+$, such that

$$\sum_{i\in S_j} (\boldsymbol{\sigma}_j)_i > 1,$$

and $\boldsymbol{\sigma} = \sum_{j\in[n]} \boldsymbol{\sigma}_j$. The latter together with (3.21) imply that $\boldsymbol{\sigma}$ lies in the left hand side of (3.22), which yields the result. □

We now state the main theorem of this section which provides a better description of the type of vectors $\boldsymbol{\mu}_J^K$ that parametrise the poles of a Shintani zeta function according to Theorem 8.7.

Theorem 8.18. Let A be a matrix in $\Sigma_{n\times r}(\mathbb{R}_{\geq 0})$, with $\{C_j\}_{j\in[r]}$ the set of columns of A, and write $C_J(\boldsymbol{\epsilon}) := \prod_{j\in J} \langle \boldsymbol{\epsilon}, C_j\rangle$ for every $J \subset [r]$. Consider also for every $J \subset [r] \setminus \emptyset$ the vector $\boldsymbol{\mu}_J \in \{0,1\}^n$ which is the only one that satisfies $\mathrm{Supp}(\boldsymbol{\mu}_J) = \bigcup_{j\in J} \mathrm{Supp}(\boldsymbol{\mu}_j)$. Then the possible singularities of the meromorphic extension of ζ_A are located on the hyperplanes

$$\langle \boldsymbol{\mu}_J, \boldsymbol{s}\rangle = |J| - l,$$

where $l \in \mathbb{Z}_{\geq 0}$. If moreover $\boldsymbol{\mu}_J$ is a vector of the canonical basis of $\mathbb{R}^n$, then l only takes values on the set $\{1, 2, \ldots, |J| - 1\}$.

Proof. The vectors $\boldsymbol{\mu}_k^J$ from Theorem 8.7 are the normal vectors to the facets of the polyhedra $\Delta_{C_J} + \mathbb{R}_+^n$ with integer, mutually coprime coefficients. These vectors correspond to the $\boldsymbol{\mu}_K$s from Proposition 8.17. The result follows from Proposition 8.17. □

We proceed to describe the family of germs of meromorphic functions at zero, spanned by the germs containing the Shintani zeta functions. Recall that a germ of meromorphic functions at zero is an equivalence class of meromorphic functions determined by the following relation

$$f \sim g \Leftrightarrow (\exists \mathcal{O} \text{ open}) : 0 \in \mathcal{O} \wedge f|_{\mathcal{O}} = g|_{\mathcal{O}}.$$

Let $M_{\zeta,n}$ be the family of germs of meromorphic functions at zero spanned by those containing Shintani zeta functions of matrices with n rows. In particular the germs in $M_{\zeta,n}$ depend on n complex variables. Using the natural embeddings $\iota_{n,m} : M_{\zeta,n} \to M_{\zeta,m}$ whenever $n < m$, we define the direct limit $M_\zeta := \varinjlim M_{\zeta,n}$.

Using the formalism developed in [45], and given that the germs in M_ζ have linear poles as it was proven by Matsumoto [69] and in Theorem 8.7, we can describe a Laurent expansion of $[\zeta_A] \in M_\zeta$ for any $A \in \Sigma_{n\times r}(\mathbb{R}_{\geq 0})$. Indeed, the germ $[\zeta_A]$ can be written as

$$[\zeta_A] = \sum_{i\in I} S_i + h,$$

where h is a holomorphic germ at zero and the S_i are fractions of the form

$$\boldsymbol{s} \mapsto S_i(\boldsymbol{s}) = \frac{h_i(\boldsymbol{s})}{\prod \langle \boldsymbol{\mu}_k^J, \boldsymbol{s}\rangle},$$

where h_i are holomorphic germs depending on variables orthogonal to their respective denominator, and the $\boldsymbol{\mu}_k^J$ are as in Theorem 8.7.

As a direct consequence of the previous discussion and of Theorem 8.18, we have the following result.

Proposition 8.19. *The germs on M_ζ are of the type $\sum_{i\in I} S_i + h$, where h is a holomorphic germ and the S_i are fractions of the form*

$$\boldsymbol{s} \mapsto S_i(\boldsymbol{s}) = \frac{h_i(\boldsymbol{s})}{\prod_{J\in\mathcal{J}_i}(\sum_{j\in J} s_j)},$$

where $\mathcal{J}_i$ is a finite collection of subsets of $[n]$ with more than one element.

We finally give some examples of how to calculate the domain of absolute convergence and the possible singularities of a Shintani zeta function ζ_A from the matrix A. By means of Corollary 8.9 we will only consider matrices with arguments equal to 0 or 1.

Example 8.20. *Consider the matrix*

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We calculate the vectors $\boldsymbol{\mu}_1 = (1,1) = \boldsymbol{\mu}_2$ from Lemma 8.14 related to the first and second columns of the matrix A. It follows that $\boldsymbol{\mu}_{\{1,2\}} = (1,1)$. By means of Propositions 8.5 and 8.17 the sum ζ_A is absolutely convergent whenever $\boldsymbol{\sigma} = \Re\boldsymbol{s}$ satisfies

$$\sigma_1 + \sigma_2 > 2.$$

By means of Theorem 8.18 the possible singularities of ζ_A are located on the hyperplanes

$$s_1 + s_2 = 2 - l,$$

where l takes values in $\mathbb{Z}_{\geq 0}$.

Example 8.21. *Consider the matrix*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The vectors μ_J are as follows

$$\boldsymbol{\mu}_1 = (1,1,0), \quad \boldsymbol{\mu}_2 = (0,1,1), \quad \boldsymbol{\mu}_3 = (0,1,0), \quad \boldsymbol{\mu}_{\{1,2\}} = (1,1,1),$$
$$\boldsymbol{\mu}_{\{1,3\}} = (1,1,0), \quad \boldsymbol{\mu}_{\{2,3\}} = (0,1,1), \quad \text{and} \quad \boldsymbol{\mu}_{\{1,2,3\}} = (1,1,1).$$

It follows from Propositions 8.5 and 8.17 that the sum ζ_A is absolutely convergent whenever the following inequalities are satisfied

$$\sigma_2 > 1, \quad \sigma_1 + \sigma_2 > 2, \quad \sigma_2 + \sigma_3 \geq 2, \quad \text{and} \quad \sigma_1 + \sigma_2 + \sigma_3 \geq 3.$$

By means of Theorem 8.18 the possible singularities of ζ_A are located on the hyperplanes

$$s_2 = 1, \quad s_1 + s_2 = 2 - l, \quad s_2 + s_3 = 2 - l, \quad \text{and} \quad s_1 + s_2 + s_3 = 3 - l$$

where l takes values in $\mathbb{Z}_{\geq 0}$.

9 Distributing weight over a graph

Interestingly the proof of Theorem 8.18, more precisely the proof of Proposition 8.17, borrows tools from graph theory, which is the reason why in this section we do an excursion to this theory. Our main objective is to prove Corollary 9.6 which is essential for the proof of Proposition 8.17. For an introduction to graph theory we refer the reader to one of the many good references in this subject, for instance [73]. The main result of this section is Theorem 9.1, the proof of which provides an algorithm to "distribute" weight over an intersection graph of a family of sets, such that the weight at each vertex is never lower than an imposed bound. This algorithm is, to the author's knowledge, new.

Notation: Throughout this section we use the round brackets to refer to the coordinate of a vector in the canonical basis of $\mathbb{R}^n$. More precisely, for $\boldsymbol{\sigma} \in \mathbb{R}^n$, $(\boldsymbol{\sigma})_k$ is the k-th component of $\boldsymbol{\sigma}$ in the canonical basis of $\mathbb{R}^n$.

It is easy to check that any real number $\sigma \geq m$, with $m \in \mathbb{Z}_{\geq 0}$, can be expressed as a sum $\sigma = \sum_{j=1}^m \sigma_j$ where each σ_j is greater than or equal to 1. A way of generalizing the previous statement is to consider $\boldsymbol{\sigma}$ in $\mathbb{R}^2_{\geq 0}$, such that $(\boldsymbol{\sigma})_1 \geq 1$, $(\boldsymbol{\sigma})_2 \geq 1$ and $(\boldsymbol{\sigma})_1 + (\boldsymbol{\sigma})_2 \geq 3$. One can check that $\boldsymbol{\sigma}$ admits a splitting of the form $\boldsymbol{\sigma} = \sum_{j=1}^3 \boldsymbol{\sigma}_j$, with the vectors $\boldsymbol{\sigma}_j$ lying in $\mathbb{R}^2_{\geq 0}$, and such that $(\boldsymbol{\sigma}_1)_1 \geq 1$, $(\boldsymbol{\sigma}_2)_2 \geq 1$ and $(\boldsymbol{\sigma}_3)_1 + (\boldsymbol{\sigma}_3)_2 \geq 1$. Indeed, one possible solution is to set $\boldsymbol{\sigma}_1 = (1,0)$, $\boldsymbol{\sigma}_2 = (0,1)$ and $\boldsymbol{\sigma}_3 = \boldsymbol{\sigma} - \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2$. The following theorem is a generalization of this fact to any finite dimension n and any number of vectors m.

For m and n positive integers, consider an application which associates to every j in $[m]$ a set $S_j \subset [n]$.

Theorem 9.1. Any $\boldsymbol{\sigma}$ in $\mathbb{R}^n_{\geq 0}$, such that for every $K \subset [m]$

$$\sum_{i \in \bigcup_{k \in K} S_k} (\boldsymbol{\sigma})_i \geq |K|,$$

can be written as a sum $\boldsymbol{\sigma} = \sum_{j \in [m]} \boldsymbol{\sigma}_j$, where the vectors $\boldsymbol{\sigma}_j$ lie in $\mathbb{R}^n_{\geq 0}$, and such that

$$\sum_{i \in S_j} (\boldsymbol{\sigma}_j)_i \geq 1. \tag{3.23}$$

Remark 9.2. Notice that the family of sets S_j used in Theorem 9.1 can be represented by a matrix $\mathfrak{A}$ with n rows and m columns, built using the characteristic functions of the sets S_j. More precisely, if χ_j is the characteristic function of the set S_j, then the argument on the i-th row and j-th column of $\mathfrak{A}$ is $\chi_j(i)$.

In particular, if in Proposition 8.17 $J = [r]$, and the sets S_j are the supports of the vectors $\boldsymbol{\mu}_j$, then $\mathfrak{A}$ would be an $n \times r$ matrix with zeros in the same positions than the matrix A, and ones in the rest of the arguments. It follows from Corollary 8.9 that ζ_A and $\zeta_{\mathfrak{A}}$ have the same polar structure.

We do the proof of the previous theorem by induction over m, using an algorithm which uses the intersection graph of the sets S_j. To illustrate how it works, we present an example before giving the actual proof.

Example 9.3. *Set $n = 6$, $m = 5$ and consider the sets $S_1 = \{2, 3\}$, $S_2 = \{1, 4, 6\}$, $S_3 = \{1, 5\}$, $S_4 = \{6\}$ and $S_5 = \{3, 4, 5\}$. The matrix built with the characteristic functions of the sets S_j as in Remark 9.2 is*

$$\mathfrak{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Consider the vector $\boldsymbol{\sigma} = (1.9, 0.6, 0.6, 0.8, 0.2, 1.4)$. One can check that it satisfies

$$\sum_{i \in \bigcup_{k \in K} S_k} (\boldsymbol{\sigma})_i \geq |K|$$

for every $K \subset [5]$ with $K \neq \emptyset$.

Since our proof is by induction, consider the vectors $\boldsymbol{\sigma}_1 = (0, 0.6, 0.6, 0, 0, 0)$, $\boldsymbol{\sigma}_2 = (0.1, 0, 0, 0.8, 0, 0.3)$, $\boldsymbol{\sigma}_3 = (1.8, 0, 0, 0, 0.2, 0)$ and $\boldsymbol{\sigma}_4 = (0, 0, 0, 0, 0, 1.1)$, and therefore

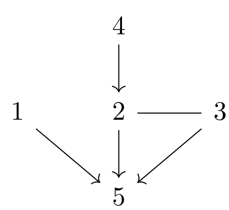
$$\boldsymbol{\sigma} = \sum_{j=1}^{4} \boldsymbol{\sigma}_j.$$

Notice that for every $j \in [4]$, the vector $\boldsymbol{\sigma}_j$ satisfies $\sum_{i \in S_j} (\boldsymbol{\sigma}_j)_i \geq 1$. We proceed to describe an algorithm to build another vector $\boldsymbol{\sigma}_5$ out of the current vectors $\boldsymbol{\sigma}_j$ with $1 \leq j \leq 4$ such that it satisfies the conditions on the statement of Theorem 9.1.

Build the intersection graph $\mathcal{G}$ of the family of sets S_j with $j \in [5]$ (see for instance [73]), i.e. a non oriented graph whose set of vertices is $[5]$ and whose set of edges is

$$E(\mathcal{G}) := \{(i, j) \in [5]^2 : S_i \cap S_j \neq \emptyset\}.$$

Namely,



Since at this point, the only coordinates of the vectors $\boldsymbol{\sigma}_j$ which are greater than zero are the coordinates in S_j, this graph represents the way the vectors can "share" some amount to the vector $\boldsymbol{\sigma}_5$. This is represented by the arrows, but keep in mind that the graph is not oriented. Take for instance the vertex 1. It has an edge connecting to the vertex 5. This means $S_1 \cap S_5 \neq \emptyset$. Indeed $\{3\} \subset S_1 \cap S_5$, and therefore we can subtract an amount λ from the third coordinate of $\boldsymbol{\sigma}_1$ and add it to $\boldsymbol{\sigma}_5$. This value λ cannot be bigger than 0.6 since $(\boldsymbol{\sigma}_1)_3 = 0.6$ and then $\boldsymbol{\sigma}_1$ would leave $\mathbb{R}^6_+$. It also cannot be greater than 0.2 since $\sum_{i \in S_1} (\boldsymbol{\sigma}_1)_i = 0.6 + 0.6$ must remain greater than 1. We therefore set $\lambda = 0.2$, and set the new vectors

$$\boldsymbol{\sigma}_1 = (0, 0.6, 0.4, 0, 0, 0)$$

and

$$\boldsymbol{\sigma}_5 = (0, 0, 0.2, 0, 0, 0).$$

Proceed to the vertex 2, which is still connected by an edge to the vertex 5 since $4 \in S_2 \cap S_5$. By a similar analysis as before, we realize that we can subtract $\lambda = 0.2$ again tho the fourth coordinate of $\boldsymbol{\sigma}_2$ and add it to $\boldsymbol{\sigma}_5$, having then

$$\boldsymbol{\sigma}_2 = (0.1, 0, 0, 0.6, 0, 0.3),$$

and

$$\boldsymbol{\sigma}_5 = (0, 0, 0.2, 0.2, 0, 0).$$

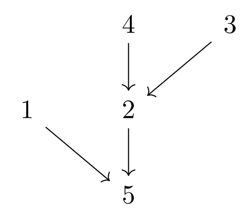
We proceed to the vertex 3 and by the same analysis we subtract $\lambda = 0.2$ to the fifth coordinate of $\boldsymbol{\sigma}_3$ and add it to the fifth coordinate of $\boldsymbol{\sigma}_5$ setting

$$\boldsymbol{\sigma}_3 = (1.8, 0, 0, 0, 0, 0),$$

and

$$\boldsymbol{\sigma}_5 = (0, 0, 0.2, 0.2, 0.2, 0).$$

However this time, even with $\sum_{i \in S_3} (\boldsymbol{\sigma}_3)_i = 1.8$ the vector $\boldsymbol{\sigma}_3$ cannot share anything else with $\boldsymbol{\sigma}_5$ since $\sum_{i \in S_3 \cap S_5} (\boldsymbol{\sigma}_3)_i = 0$. For this reason we cut the vertex between vertices 3 and 5. The new graph $\mathcal{G}$ is therefore



and vertex 3 is now at least at 2 steps from vertex 5.

We proceed now to analyze vertices which are not direct neighbors of 5. Consider the vertex 4: by a similar argumentation as before the sixth coordinate of $\boldsymbol{\sigma}_4$ can share a maximum of 0.1 to $\boldsymbol{\sigma}_2$, which then can share 0.1 of its fourth coordinate with $\boldsymbol{\sigma}_5$. Then the new vectors are

$$\boldsymbol{\sigma}_4 = (0, 0, 0, 0, 0, 1),$$

$$\boldsymbol{\sigma}_2 = (0.1, 0, 0, 0.5, 0, 0.4)$$

and

$$\boldsymbol{\sigma}_5 = (0, 0, 0.2, 0.3, 0.2, 0).$$

We finally revisit vertex 3 but now as an indirect neighbor of the vertex 5. We see now that we may subtract 0.8 from the first component of $\boldsymbol{\sigma}_3$ and add it to $\boldsymbol{\sigma}_2$. Then $\boldsymbol{\sigma}_2$ can share a maximum of 0.5 from its fourth component with $\boldsymbol{\sigma}_5$, and then

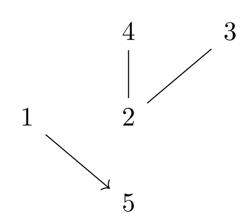
$$\boldsymbol{\sigma}_3 = (1, 0, 0, 0, 0, 0),$$

$$\boldsymbol{\sigma}_2 = (0.9, 0, 0, 0, 0, 0.4),$$

and

$$\boldsymbol{\sigma}_5 = (0, 0, 0.2, 0.8, 0.2, 0).$$

Since $\sum_{i \in S_2 \cap S_5} (\boldsymbol{\sigma}_2)_i = 0$ we cut the edge between the the vertices 2 and 5 and the graph now looks like this:



The vectors we end up with are:

$$\begin{aligned}\boldsymbol{\sigma}_1 &= (0, 0.6, 0.4, 0, 0, 0)\\ \boldsymbol{\sigma}_2 &= (0.9, 0, 0, 0.5, 0, 0.4)\\ \boldsymbol{\sigma}_3 &= (1, 0, 0, 0, 0, 0)\\ \boldsymbol{\sigma}_4 &= (0, 0, 0, 0, 0, 1)\\ \boldsymbol{\sigma}_5 &= (0, 0, 0.2, 0.8, 0.2, 0),\end{aligned}$$

which clearly satisfy the requirements. Indeed $\boldsymbol{\sigma} \in \mathbb{R}^6_+$, $\sum_{i\in S_j}(\boldsymbol{\sigma}_j)_i \geq 1$, *and* $\boldsymbol{\sigma} = \sum_{j=1}^5 \boldsymbol{\sigma}_j$. *Notice also that the final graph has two separated components. This means, as it can be checked, that the vectors corresponding to vertices in the components not containing 5 (namely the vertices 2, 3, and 4) have no intersection in their supports with* S_5, *or the coordinates in the intersection are equal to* 0. *The latter implies that they cannot share any more with the vector* $\boldsymbol{\sigma}_5$.

After having illustrated in the example how the algorithm works we shall provide the actual proof of Theorem 9.1

Proof of Theorem 9.1. The proof is by induction over m. The case $m = 1$ is trivial setting $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}$. For the inductive step assume the statement is true for $m-1$ and consider a vector $\boldsymbol{\sigma} \in \mathbb{R}^n_{\geq 0}$ which, for every $K \subset [m]$, satisfies

$$\sum_{i\in\bigcup_{k\in K} S_k} (\boldsymbol{\sigma})_i \geq |K|.$$

It follows from the induction hypothesis that there exist vectors $\boldsymbol{\sigma}_j \in \mathbb{R}^n_{\geq 0}$ for $1 \leq j \leq m-1$ satisfying (3.23), and such that $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{j_1} + \cdots + \boldsymbol{\sigma}_{j_{m-1}}$.

Define the intersection graph $\mathcal{G}$ for the sets S_j, namely the graph whose set of vertices is $[m]$ and whose set of edges is

$$E(\mathcal{G}) := \{(i,j) \in [m]^2 : \mathrm{Supp}(\boldsymbol{\mu}_i) \cap \mathrm{Supp}(\boldsymbol{\mu}_j) \neq \emptyset\}.$$

Consider then the distance map $d_m : [m] \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$, where $d_m(j)$ is the minimum number of steps one has to give in the graph $\mathcal{G}$ to go from the vertex j to the vertex m. If there is no path from j to m we set $d_m(j) = \infty$. Notice that this distance map does not define an order on the connected component of the graph containing the vertex m, since the antisymmetry might fail. For every $j \in [m-1]$ such that $d_m(j) \neq \infty$ define $S_{j,<} := \{i \in S_j : (\exists l \in [m])\, i \in S_l \ \wedge\ d_m(l) = d_m(j) - 1\}$. In words, $S_{j,<}$ is the subset of elements in S_j which are also in some S_l, where l is a vertex closer to m in the the graph $\mathcal{G}$.

Finally set a counter $\alpha = 1$ which will help us analyze the vertices of $\mathcal{G}$ depending on their distance to the vertex m.

We use the notation $:\models$ to redefine a parameter on the left hand side using its old value on the right hand side. For instance $\alpha :\models \alpha + 1$ means we add 1 to the value of the counter α.

1. Take a vertex j such that $d_m(j) = \alpha$, and let $\rho(j)$ be the set of paths of length α going from j to m.

2. Choose a path $\varrho = (h_1, \ldots, h_\alpha) \in \rho(j)$, this means that $h_1 = j$ and $h_\alpha = m$, and set a counter $\beta = 1$.

3. For the vertex h_β, consider

$$r = \min\left\{\sum_{i\in S_{h_\beta}} (\boldsymbol{\sigma}_{h_\beta})_i - 1, \sum_{i\in S_{h_\beta}\cap S_{h_{\beta+1}}} (\boldsymbol{\sigma}_{h_\beta})_i\right\}$$

   and subtract from the coordinates $(\boldsymbol{\sigma}_{h_\beta})_i$ with $i \in S_{h_\beta} \cap S_{h_{\beta+1}}$ a total amount of r, which will be added to the same coordinates of the vector $\boldsymbol{\sigma}_{h_{\beta+1}}$. This is, for $i \in S_{h_\beta} \cap S_{h_{\beta+1}}$ redefine

$$\begin{aligned}(\boldsymbol{\sigma}_{h_\beta})_i &:\models (\boldsymbol{\sigma}_{h_\beta})_i - \lambda_i, \quad \text{and}\\ (\boldsymbol{\sigma}_{h_{\beta+1}})_i &:\models (\boldsymbol{\sigma}_{h_{\beta+1}})_i + \lambda_i\end{aligned}$$

where $0 \leq \lambda_i < (\boldsymbol{\sigma}_{h_\beta})_i$, and such that $\sum_{i \in S_{h_\beta} \cap S_{h_{\beta+1}}} \lambda_i = r$.

Notice at this point that the existence of the λ_is is granted by the way r is defined. Notice also that the new vectors $\boldsymbol{\sigma}_{h_\beta}$ and $\boldsymbol{\sigma}_{h_{\beta+1}}$ still satisfy (3.23), except if $h_{\beta+1} = m$.

If $\beta < \alpha - 1$, add one to the counter β, namely $\beta :\models \beta + 1$ and repeat step 3. If $\beta = \alpha - 1$ continue with the next step.

4. We have three options at this point:

   - If $\sum_{i \in S_{h_\beta,<}} (\boldsymbol{\sigma}_{h_\beta})_i > 0$ and $\beta > 1$, subtract 1 to the value of β, namely
   $$\beta :\models \beta - 1,$$
   and repeat step 4.
   - If $\sum_{i \in S_{h_\beta,<}} (\boldsymbol{\sigma}_{h_\beta})_i = 0$, it means that the vector $\boldsymbol{\sigma}_{h_\beta}$ has shared everything from the coordinates in $S_{h_\beta,<}$. It implies that it cannot share any more to a vertex closer to m. In this case remove the edges $(h_\beta, h_{\beta+1})$ and $(h_{\beta+1}, h_\beta)$ from the set of edges of the graph $\mathcal{G}$, recalculate the values of the distance map d_m for every vertex and go to step 1, without resetting the value of α.
   - If $\sum_{i \in S_{h_\beta,<}} (\boldsymbol{\sigma}_{h_\beta})_i > 0$ and $\beta = 1$, proceed to the next step.

5. We have again two options:

   - If there is a path $\varrho \in \rho(j)$ that we haven't followed in step 2, choose that path and go back to step 2.
   - If the process from step 2 has been followed for every $\varrho \in \rho$, then proceed to the next step.

6. We have again three options:

   - If there is any other vertex j with $d_m(j) = \alpha$ that hasn't been chosen in step 1, choose that vertex j and go back to step 1.
   - If the process in step 1 has been followed for every vertex j such that $d_m(j) = \alpha$, and α is not maximal yet, namely $\alpha < \max\{d_m(j) : j \in J \wedge d_m(j) < \infty\}$, then add 1 to the value of α
   $$\alpha :\models \alpha + 1$$
   and go back to step 1.
   - If the process from step 1 has been followed for every vertex j such that $d_m(j) = \alpha$, and α is maximal, proceed to the last step.

7. For every j in $[m-1]$, consider the coordinates $i \notin S_j$ and redefine
   $$(\boldsymbol{\sigma}_j)_i :\models 0, \quad \text{and}$$
   $$(\boldsymbol{\sigma}_m)_i :\models (\boldsymbol{\sigma}_m)_i + (\boldsymbol{\sigma}_j)_i,$$
   and finish the algorithm.

The algorithm is clearly finite since the graph $\mathcal{G}$ is finite. We proceed to prove that the vectors $\boldsymbol{\sigma}_j$ obtained after following the algorithm satisfy the requirements

$$\boldsymbol{\sigma} = \sum_{j=1}^{m} \boldsymbol{\sigma}_j, \quad \boldsymbol{\sigma}_j \in \mathbb{R}^n_{\geq 0}, \quad \text{and} \quad \sum_{i \in S_j} (\boldsymbol{\sigma}_j)_i \geq 1, \tag{3.24}$$

for every j in $[m]$. The first condition is satisfied by assumption and the fact that every amount that was subtracted from a coordinate of a vector, was added to the same coordinate of the other vector. The second condition is a consequence of having added positive values to the coordinates of the vectors. Only in step 3 a positive real number λ_i is subtracted from the coordinates in the support of the vector $\boldsymbol{\sigma}_{h_\beta}$, but $\lambda_i \leq (\boldsymbol{\sigma}_{h_\beta})_i$ and therefore all coordinates remain non negative. The fact that (3.24) is satisfied for all $l \in [m-1]$ is also a consequence of the way r is chosen in step 3, since $r < \sum_{i \in S_{h_\beta}} (\boldsymbol{\sigma}_{h_\beta})_i - 1$, then the vector $\boldsymbol{\sigma}_{h_\beta}$ still satisfies (3.24).

Finally we have to check that

$$\sum_{i \in S_m} (\boldsymbol{\sigma}_m)_i \geq 1.$$

Notice first that for all vertices j not in the connected component of the graph $\mathcal{G}$ containing the vertex m, the value of their coordinates $i \in S_m$ is equal to 0. Let K be the subset of $[m]$ whose vertices are on the connected component containing m. As a consequence of the algorithm, for every $k \in K$, $\sum_{i \in S_k} (\boldsymbol{\sigma}_k)_i = 1$. Indeed, if it is still connected to the vertex m, step 4 indicates that $\sum_{i \in S_{k,<}} (\boldsymbol{\sigma}_k)_i > 0$. This is only possible if at some point the third step of the algorithm gives $r = \sum_{i \in S_k} (\boldsymbol{\sigma}_k)_i - 1$, which means that at the end of the algorithm $\boldsymbol{\sigma}_k$ is such that

$$\sum_{i \in S_k} (\boldsymbol{\sigma}_k)_i = 1.$$

On the other hand, by assumption

$$\begin{aligned} |K| &\leq \sum_{i \in \bigcup_{k \in K} S_k} (\boldsymbol{\sigma})_i \\ &= \sum_{k \in K \setminus \{m\}} \Big(\sum_{i \in S_k} (\boldsymbol{\sigma}_k)_i \Big) + \sum_{i \in S_m} (\boldsymbol{\sigma}_m)_i. \end{aligned}$$

The first term in the last line is equal to $(|K|-1)$, which implies that $\sum_{i \in S_m} (\boldsymbol{\sigma}_{j_m})_i \geq 1$ as expected. This finishes the proof.

□

Remark 9.4. Notice that the process described in the proof is not an algorithm in a strict sense since it is not completely deterministic. For instance, there is some freedom when choosing the order in which one analyzes the vertices with the same d_m in step 1, or the different paths in step 2. Also there might be a freedom in the way the λ_is are chosen in step 3. Different choices might lead to different constructions of the vectors $\boldsymbol{\sigma}_m$, but it doesn't matter because they all satisfy the requirements of the theorem.

We must also warn that this process is not optimal in computation time since many redundancies may occur. It is written with the only purpose of demonstrating the existence of a solution to the equation $\boldsymbol{\sigma} = \sum_{j=1}^m \boldsymbol{\sigma}_j$ satisfying the conditions $\sum_{i \in S_j} (\boldsymbol{\sigma}_j)_i \geq 1$ and $\boldsymbol{\sigma}_j \in \mathbb{R}^n_{\geq 0}$, provided that for every $K \subset [m]$

$$\sum_{i \in \bigcup_{k \in K} S_k} (\boldsymbol{\sigma})_i \geq |K|.$$

Remark 9.5. Theorem 9.1 is related to Hall's marriage theorem [47] and the theory of optimal transport. In this direction Thierry Champion could give an alternative proof of the result using von Neumann's minmax theorem.

We now prove that Theorem 9.1 is also true if the inequalities in the statement are strict. Recall our assumption that for m and n positive integers, we fix an application which associates to every j in $[m]$ a set $S_j \subset [n]$.

Corollary 9.6. Any $\boldsymbol{\sigma}$ in $\mathbb{R}^n_+$, such that for every $K \subset [m]$

$$\sum_{i \in \bigcup_{k \in K} S_k} (\boldsymbol{\sigma})_i > |K|, \tag{3.25}$$

can be written as a sum $\boldsymbol{\sigma} = \sum_{j \in [m]} \boldsymbol{\sigma}_j$, where the vectors $\boldsymbol{\sigma}_j$ lie in $\mathbb{R}^n_+$, and such that

$$\sum_{i \in S_j} (\boldsymbol{\sigma}_j)_i > 1. \tag{3.26}$$

Proof. The result follows from Theorem 9.1 and the fact that the set of $\boldsymbol{\sigma} \in \mathbb{R}^n_+$ satisfying (3.25) is an open set. More precisely, for each $i \in [n]$ consider $\lambda_i > 0$ small enough such that $\boldsymbol{\sigma}' := \boldsymbol{\sigma} - \boldsymbol{\lambda}$ still satisfies (3.25), and $\boldsymbol{\sigma}' \in \mathbb{R}^n_+$. By means of Theorem 9.1 there are $\boldsymbol{\sigma}'_j \in \mathbb{R}^n_{\geq 0}$ for $1 \leq j \leq m$ satisfying (3.23), and such that $\boldsymbol{\sigma}' = \sum_{j=1}^m \boldsymbol{\sigma}'_j$. It follows that the vectors $\boldsymbol{\sigma}_j := \boldsymbol{\sigma}_j + \frac{1}{m}\boldsymbol{\lambda} \in \mathbb{R}^n_+$ satisfy (3.26) and the result follows. □

Appendix A

Alternative locality tensor product

In this appendix we explore an alternative definition for the locality tensor product. Such definition was already suggested in [22] and made more precise in [21, Appendix A], however, both in [22] and in [21] the locality tensor product from Definition 2.14 is used since the alternative tensor product is in general not a subspace of the usual (non-locality) tensor product. Notice that this alternative locality tensor product induces an alternative definition of $\top_\times$-bilinearity as announced in Paragraph 2.2. This appendix follows closely [21, Appendix A].

A.I An alternative view of locality bilinearity

Let $(E, \top)$ be a pre-locality vector space and V and W two linear subspaces of it. Recall Definition 4.11 where we defined a $\top_\times$-bilinear map $f : V \times_\top W \to G$ as a map, the unique linear extension $\bar{f} : \mathbb{K}(V \times_\to W) \to G$ of which, vanishes at $\mathbb{K}(V \times_\top W) \cap I_{\rm bil}(V, W)$, where $I_{\rm bil}(V, W)$ is defined in Equations (1.2) to (1.5). Alternatively, we can consider the linear subspace $I_{{\rm bil},\top}(V, W) \subseteq \mathbb{K}(V \times_\top W)$ generated by all elements in $V \times_\top W$ of the forms (1.2) to (1.5) such that each argument in the linear combinations (1.2)–(1.5), lies in $\top$. In some cases $I_{{\rm bil},\top}(V, W)$ will coincide with $I_{\rm bil}(V, W) \cap \mathbb{K}(V \times_\top W)$ but as we see in the following example, this is not always the case.

Example A.I.1. *[21][Example A.1] Consider the locality vector space $V = \mathbb{R}^2$, with locality relation*

$$\top = \mathbb{R}^2 \times \{0\} \cup \{0\} \times \mathbb{R}^2 \cup \mathbb{K}(e_1+e_2) \times \mathbb{K}(e_1) \cup \mathbb{K}(e_1+2e_2) \times \mathbb{K}(e_2) \cup \mathbb{K}(e_1) \times \mathbb{K}(e_1+e_2) \cup \mathbb{K}(e_2) \times \mathbb{K}(e_1+2e_2)$$

and the element of $\mathbb{K}(V \times_\top V)$

$$y = (-e_1 - e_2, e_1) + (-e_1 - 2e_2, e_2) + (e_1, e_1 + e_2) + (e_2, e_1 + 2e_2).$$

Consider the following elements of $I_{\rm bil}(V)$:

$$\begin{aligned} y_1 :=& -(e_1 + e_2, e_1) - (-e_1 - e_2, e_1),\\ y_2 :=& (e_1 + e_2, e_1) - (e_1, e_1) - (e_2, e_1),\\ y_3 :=& -(-e_1 - 2e_2, e_2) - (e_1 + 2e_2, e_2),\\ y_4 :=& (e_1 + 2e_2, e_2) - (e_1, e_2) - (2e_2, e_2)\\ y_5 :=& (e_1, e_1 + e_2) - (e_1, e_1) - (e_1, e_2)\\ y_6 :=& (e_2, e_1 + 2e_2) - (e_2, e_1) - (e_2, 2e_2)\\ y_7 :=& (e_2, 2e_2) - (2e_2, e_2). \end{aligned}$$

We can then write

$$\begin{aligned} y =& (-e_1 - e_2, e_1) + (-e_1 - 2e_2, e_2) + (e_1, e_1 + e_2) + (e_2, e_1 + 2e_2)\\ =& y_1 + y_2 - (e_1, e_1) - (e_2, e_1) + y_3 + y_4 - (e_1, e_2) - (2e_2, e_2) - y_5 + (e_1, e_1) + (e_1, e_2) - y_6 + (e_2, e_1) - y_7 + (2e_2, e_2)\\ =& y_1 + y_2 + y_3 + y_4 - y_5 - y_6 - y_7. \end{aligned}$$

Thus $y \in I_{\text{bil}} \cap \mathbb{K}(V \times_\top V)$. However, y_2, y_4, y_5, y_6, and y_7 do not lie in $\mathbb{K}(V \times_\top V)$. Therefore $y \notin I_{\text{bil},\top}(V)$.

The last example illustrates how the bilinear forms from Equations (1.2) to (1.5) may "leave" the locality relation (namely $\mathbb{K}(V \times_\top W)$) and come back afterwards. Thus, in general only the inclusion $I_{\text{bil},\top}(V,W) \subset I_{\text{bil}}(V,W) \cap \mathbb{K}(V \times_\top V)$ holds, which suggests another definition of the bilinearity in the locality setup, and therefore an alternative definition of a locality tensor product.

Definition A.I.2. Let $(E, \top)$ be a pre-locality vector space, and V and W two subspaces of E, and G another vector space.

1. A map $f : V \times_\top W \to G$ is called $\top^\times$**-bilinear** if $\bar{f}(I_{bil,\top}) = \{0\}$.
2. The **alternative locality tensor product** of V and W is defined as

$$V \otimes^\top W := {}^{\mathbb{K}(V \times_\top W)}/_{I_{\text{bil},\top}(V,W)} \tag{A.1}$$

The definition of $\top_\times$-bilinear map conicides with locality bilinear maps from [22], i.e., a map $f : V \times_\top W \to G$ is $\top^times$-bilinear if, and only if

$$f(v_1 + v_2, w_1) = f(v_1, w_1) + f(v_2, w_2), \quad f(v_1, w_1 + w_2) = f(v_1, w_1) + f(v_1, w_2),$$

$$f(kv_1, w_1) = kf(v_1, w_1), \quad f(v_1, kw_1) = kf(v_1, w_1),$$

for all $v_1, v_2 \in V$, $w_1, w_2 \in W$, and $k \in \mathbb{K}$ such that all above terms are defined.

Notice also that in general the alternative locality tensor product is not a subspace of the usual tensor product since $I_{\text{bil},\top}(V,W)$ is smaller than $I_{\text{bil}}(V,W) \cap \mathbb{K}(V \times_\top W)$, as the following example illustrates.

Example A.I.3. *Going back to Example (A.I.1), the alternative locality tensor product is*

$$V \otimes^\top V = \mathbb{K}\{(e_1 + e_2) \otimes e_1, (e_1 + 2e_2) \otimes e_2, e_1 \otimes (e_1 + e_2), e_2 \otimes (e_1 + 2e_2)\} \not\subseteq V \otimes V$$

As a consequence of the computations in Example A.I.1, the 4 terms spanning $V \otimes^\top V$ are linearly independent. Thus $V \otimes^\top V$ has dimension 4 like the usual tensor product, but is not the same as the usual one, because in the usual tensor product those 4 elements mentioned above are not linearly independent. However, the locality tensor product in this case is

$$V \otimes_\top V = \mathbb{K}\{(e_1 + e_2) \otimes e_1, (e_1 + 2e_2) \otimes e_2, e_1 \otimes (e_1 + e_2)\} \subset V \otimes V.$$

It has dimension 3 as a consequence of Example A.I.1. Indeed, the element $e_2 \otimes (e_1 + 2e_2)$ can be written as a linear combination in $V \otimes_\top V$ of the three elements spanning $V \otimes_\top V$.

A.II Associativity of the alternative locality tensor product

Using the same methods as in Chapter 2, one can show that this alternative tensor product also enjoys universal properties analogous to Theorems 4.14, 4.33, and 4.48 but replacing $\top_\times$- bilinearity for $\top^\times$-bilinearity. However the equivalence between the usual universal properties and those with the locality tensor product, namely Corollaries 4.16, and 4.34, do not have an analogous with the alternative locality tensor product since their proof relies in the inclusion $V \otimes_\top V \subset V \otimes V$.

Another important result, the proof of which depends on the aforementioned inclusion, is the associativity of the locality tensor product, namely Proposition 4.30. We provide here an alternative proof which is new, but nevertheless closely follows [21, Lemma 3.7 & Theorem 3.9].

Definition A.II.1.
- We define the relation $\top_{\times m,n} \subset (V_1 \times_\top \cdots \times_\top V_m) \times (V_{m+1} \times_\top \cdots \times_\top V_{m+n})$ as follows

$$(x_1, \ldots, x_m) \top_{\times m,n} (y_1, \ldots, y_n) :\Longleftrightarrow \forall (i,j) \in [m] \times [n], x_i \top y_j, \tag{A.2}$$

extend it linearly to $\mathbb{K}\left(V_1 \times_\top \cdots \times_\top V_m\right) \times \mathbb{K}\left(V_{m+1} \times_\top \cdots \times_\top V_{m+n}\right)$ as follows

$$\left(\sum_{k\in K} \alpha_k\, x_k, \sum_{l\in L} \beta_l\, y_l\right) \in \mathbb{K}\left(V_1 \times_\top \cdots \times_\top V_m\right) \times_{\top_{\times m,n}} \mathbb{K}\left(V_{m+1} \times_\top \cdots \times_\top V_{m+n}\right)$$
$$:\Longleftrightarrow \forall (k,l) \in K \times L, x_k \top_{\times m,n} y_l,$$

for any K and L finite sets. Here $(\alpha_k, \beta_l) \in (\mathbb{K} \setminus \{0_{\mathbb{K}}\})^2$ for every $k \in K$ and every $l \in L$.

- Finally, we define the relation $\top_{\otimes m,n} \subset \left(V_1 \otimes^\top \cdots \otimes^\top V_m\right) \times \left(V_{m+1} \otimes^\top \cdots \otimes^\top V_{m+n}\right)$ induced from $\top_{\times m,n}$ as follows:

$$(x,y) \in \top_{\otimes m,n} \Leftrightarrow (\exists x' \in x) \wedge (\exists y' \in y) :\, (x',y') \in \top_{\times m,n}, \tag{A.3}$$

  where $x \in V_1 \otimes^\top \cdots \otimes^\top V_m$ and $y \in V_{m+1} \otimes^\top \cdots \otimes^\top V_{m+n}$ (recall that the elements on the locality tensor product are equivalence classes so that the notation $x' \in x$ makes sense).

Remark A.II.2. Notice that $\top_{\times m,n}$ and $\top_{\otimes m,n}$ are not locality relations since they are not in general subsets of a set of the form $S \times S$. Instead, $\top_{\times m,n}$ can be seen as a relation between $\mathbb{K}(V_1 \times_\top \cdots \times_\top V_m)$ and $\mathbb{K}(V_{m+1} \times_\top \cdots \times_\top V_{m+n})$; and $\top_{\otimes m,n}$ as a relation between $V_1 \otimes^\top \cdots \otimes^\top V_m$ and $V_{m+1} \otimes^\top \cdots \otimes^\top V_{m+n}$. In contrast, $\top_\otimes(V_1 \otimes^\top \cdots \otimes^\top V_n)$, defined analogously to $\top_\otimes(V_1 \otimes_\top \cdots \otimes_\top V_n)$ (Definition 4.9), is a locality relation on $V_1 \otimes^\top \cdots \otimes^\top V_n$.

Lemma A.II.3. The map

$$\Psi_{m,n} : \begin{cases} \mathbb{K}\left(V_1 \times_\top \cdots \times_\top V_m\right) \times_\top \mathbb{K}\left(V_{m+1} \times_\top \cdots \times_\top V_{m+n}\right) & \longrightarrow & \mathbb{K}\left(V_1 \times_\top \cdots \times_\top V_{m+n}\right) \\ ((x_1,\ldots,x_m),(y_1,\ldots,y_n)) & \longmapsto & (x_1,\ldots,x_m,y_1,\ldots,y_n) \end{cases} \tag{A.4}$$

linearly extends to a surjective morphism of pre-locality vector spaces:

$$\Psi_{m,n} : \mathbb{K}\Big(\mathbb{K}\left(V_1 \times_\top \cdots \times_\top V_m\right) \times_\top \mathbb{K}\left(V_{m+1} \times_\top \cdots \times_\top V_{m+n}\right)\Big) \to \mathbb{K}\left(V_1 \times_\top \cdots \times_\top V_{m+n}\right). \tag{A.5}$$

Remark A.II.4. Note that $\Psi_{m,n}$ is not expected to be an isomorphism. Let us take $m = n = 1$ to simplify. A basis of $\mathbb{K}\Big(\mathbb{K}(V_1) \times_\top \mathbb{K}(V_2)\Big)$ is given by elements $(k_1 v_1 + \ldots + k_p v_p, l_1 w_1 + \ldots + l_q w_q)$, with p, q in $\mathbb{N}$, (v_i, w_j) in $V_1 \times_\top V_2$ and k_i, l_j in $\mathbb{K}$ non zero for all indices i, j. A basis of $\mathbb{K}(V_1 \times_\top V_2)$ is given by pairs $(v_1, v_2) \in V_1 \times_\top V_2$. Since

$$\Psi_{1,1}((k_1 v_1 + \cdots + k_p v_p, l_1 w_1 + \cdots + l_q w_q)) = \sum_{i,j} k_i l_j (v_i, w_j) = \Psi_{1,1}\left(\sum_{i,j} k_i l_j (v_i, w_j)\right),$$

$\Psi_{1,1}$ is not injective.

Proof. The fact that $\Psi_{m,n}$ extends to a surjective morphism of vector spaces is a classical result of linear algebra. To show that $\Psi_{m,n}$ is a locality morphism we just need to check that it is a locality map, which is an easy consequence of the following equivalence: for any $x := (x_1, \ldots, x_m) \in V_1 \times_\top \cdots \times_\top V_m$ and $y := (y_1, \ldots, y_n) \in V_{m+1} \times_\top \cdots \times_\top V_{m+n}$, we have

$$\begin{aligned}(x,y) \in \top_{\times m,n} &\Longleftrightarrow (x_i, y_j) \in \top \quad \forall (i,j) \in [m] \times [n] \\ &\Longleftrightarrow \Psi(x,y) = (x_1,\ldots,x_m,y_1,\ldots,y_n) \in \mathbb{K}\left(V_1 \times_\top \cdots \times_\top V_{m+n}\right)\end{aligned}$$

One simply needs to take two independent pairs (x,y) and (x',y') and work with these equivalences. We omit here the detailed proof which is straightforward but rather cumbersome to write. ☐

The locality morphism (A.5) induces a locality morphism between locality tensor products.

Theorem A.II.5. For any subspaces $V_1, \ldots, V_{m+n}$ of the pre-locality space $(E, \top)$, we set

$$\begin{aligned}&\left(V_1 \otimes^\top \cdots \otimes^\top V_m\right) \otimes^\top \left(V_{m+1} \otimes^\top \cdots \otimes^\top V_{m+n}\right)\\ &:= \mathbb{K}\left(\left(V_1 \otimes^\top \cdots \otimes^\top V_m\right) \times_{\top_{\otimes m,n}} \left(V_{m+1} \otimes^\top \cdots \otimes^\top V_{m+n}\right)\right) \big/ \left(I_{\mathrm{bil},\top}\left(\left(V_1 \otimes_\top \cdots \otimes_\top V_m\right), \left(V_{m+1} \otimes_\top \cdots \otimes_\top V_{m+n}\right)\right)\right).\end{aligned}$$

and Definition 4.6 yields a pre-locality relation on this quotient which we denote by $\top_\otimes^{m,n}$.

There is an isomorphism of pre-locality vector spaces

$$\begin{aligned}\Phi_{m,n} : &\left(\left(V_1 \otimes^\top \cdots \otimes^\top V_m\right) \otimes^\top \left(V_{m+1} \otimes^\top \cdots \otimes^\top V_{m+n}\right), \top_\otimes^{m,n}\right)\\ &\xrightarrow{\sim} \left(V_1 \otimes^\top \cdots \otimes^\top V_m \otimes^\top V_{m+1} \otimes^\top \cdots \otimes^\top V_{m+n}, \top_{\otimes(m+n)}\right).\end{aligned} \tag{A.6}$$

Here $\top_{\otimes(m+n)}$ denotes an analogous to $\top_\otimes(V_1, \ldots, V_{m+n})$ from Definition 4.9 but for the alternative locality tensor product.

Remark A.II.6. Notice that the locality relation $\top_{\otimes(m+n)}$ is different from the relation $\top_{\otimes m,n}$ of Definition A.II.1, and also different from the newly introduced locality relation $\top_\otimes^{m,n}$.

Proof. We build the isomorphism $\Phi_{m,n}$ from the morphism $\Psi_{m,n}$ defined in (A.4).

- An element $[y]$ of $\left(V_1 \otimes^\top \cdots \otimes^\top V_m\right) \otimes^\top \left(V_{m+1} \otimes^\top \cdots \otimes^\top V_{m+n}\right)$ reads

$$[y] = \left[\sum_{i \in I} \alpha_i \left(\sum_{j \in J} \beta_j^i v_{j,1}^i \otimes \cdots \otimes v_{j,m}^i, \sum_{k \in K} \gamma_k^i v_{k,1}^i \otimes \cdots \otimes v_{k,n}^i\right)\right],$$

  for some vectors $v_{j,r}^i \in V_r$ and $v_{k,s}^i \in V_s$ and scalars $\alpha_i, \beta_j^i, \gamma_k^i$ in $\mathbb{K}$ with I, J, K three finite sets, such that $\sum_{j \in J} \sum_{k \in K} \beta_j^i \gamma_k^i \left(v_{j,1}^i, \ldots, v_{j,m}^i, v_{k,1}^i, \ldots, v_{k,n}^i\right) \in (V_1 \times_\top \cdots \times_\top V_m) \times_{\top n,m} (V_{m+1} \times_\top \cdots \times_\top V_{m+n}) \simeq (V_1 \times_\top \cdots \times_\top V_{m+n})$.

- The linear map $\Phi_{m,n} : \left(V_1 \otimes^\top \cdots \otimes^\top V_m\right) \otimes^\top \left(V_{m+1} \otimes^\top \cdots \otimes^\top V_{m+n}\right) \longrightarrow \left(V_1 \otimes^\top \cdots \otimes^\top V_{m+n}\right)$ is defined by the following action on an element $[y]$ of $\left(V_1 \otimes^\top \cdots \otimes^\top V_m\right) \otimes_{\top n,m} \left(V_1 \otimes^\top \cdots \otimes^\top V_n\right)$:

$$\Phi_{m,n}([y]) = \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \alpha_i \beta_j^i \gamma_k^i \left(v_{j,1}^i \otimes \cdots \otimes v_{j,n}^i \otimes v_{k,1}^i \otimes \cdots \otimes v_{k,m}^i\right).$$

  Notice that $\left(v_{j,1}^i, \ldots, v_{j,n}^i, v_{k,1}^i, \ldots, v_{k,m}^i\right)$ is an element of $V_1 \times_\top \cdots \times_\top V_{m+n}$ by definition of $\left(V_1 \otimes^\top \cdots \otimes^\top V_m\right) \otimes^\top \left(V_{m+1} \otimes^\top \cdots \otimes^\top V_{m+n}\right)$, and that the difference of two representatives of $[y]$ lies in I_{bil} whose image by Φ lies in $I_{\mathrm{mult}}(V_1, \ldots, V_{m+n})$. Thus $\Phi_{m,n}$ is well-defined.

- The injectivity of $\Phi_{m,n}$ follows from the commutativity of the following diagram in which the vertical arrows are quotient maps.

$$\begin{array}{ccc}
\mathbb{K}\Big(\mathbb{K}\left(V_1 \times_\top \cdots \times_\top V_m\right) \times_{\top_{\times m,n}} \mathbb{K}\left(V_{m+1} \times_\top \cdots \times_\top V_{m+n}\right)\Big) & \xrightarrow{\Psi_{m,n}} & \mathbb{K}(V_1 \times_\top \cdots \times_\top V_{m+n}) \\
\big\downarrow{\scriptstyle \pi_m \times \pi_n} & & \\
\mathbb{K}\Big(\left(V_1 \otimes^\top \cdots \otimes^\top V_m\right) \times_{\top_{\otimes m,n}} \left(V_{m+1} \otimes^\top \cdots \otimes^\top V_{m+n}\right)\Big) & & \big\downarrow{\scriptstyle \pi_{m+n}} \\
\big\downarrow & & \\
\left(V_1 \otimes^\top \cdots \otimes^\top V_m\right) \otimes^\top \left(V_{m+1} \otimes^\top \cdots \otimes^\top V_{m+n}\right) & \xrightarrow{\Phi_{m,n}} & V_1 \otimes^\top \cdots \otimes^\top V_{m+n}
\end{array}$$

  Assume $\Phi_{m,n}([y]) = 0$ for some $[y] \in \left(V_1 \otimes^\top \cdots \otimes^\top V_m\right) \otimes^\top \left(V_{m+1} \otimes^\top \cdots \otimes^\top V_{m+n}\right)$. Then the preimage of $\Phi_{m,n}([y])$ under the quotient map π_{m+n} in $V_1 \times_\top \cdots \times_\top V_{m+n}$ lies in $I_{\mathrm{mult},\top}(V_1, \ldots, V_{m+n})$. This implies that its preimage under $\pi_{m+n} \circ \Psi_{m,n}$ lies in

$$\mathbb{K}(I_{\mathrm{mult},\top}(V_1, \ldots, V_m) \times_{\top m,n} I_{\mathrm{mult},\top}(V_{m+1}, \ldots, V_{m+n})) \tag{A.7}$$

  by construction of $\Psi_{m,n}$ in Lemma A.II.3. This element of (A.7) is the preimage of $[y]$ under the two projections on the left column of the previous diagram, and therefore $[y] = 0$.

- The surjectivity of $\Phi_{m,n}$ follows from that of $\Psi_{m,n}$ combined with the commutativity of the diagram above. Indeed, any element $y \in V_1 \otimes^\top \cdots \otimes^\top V_{m+n}$ has a preimage $\tilde{y} \in \mathbb{K}(\mathbb{K}(V_1 \times_\top \cdots \times_\top V_m) \times_{\top n,m} \mathbb{K}(V_{m+1} \times_\top \cdots \times_\top V_{m+n}))$ under $\pi_{m+n} \circ \Psi_{m,n}$ since $\Psi_{m,n}$ and $\pi_{m,n}$ are surjections. Taking the image of $\tilde{y}$ under the two projections of the left column of the diagram above, we obtain a preimage of y in $\big(V_1 \otimes^\top \cdots \otimes^\top V_m\big) \otimes^\top \big(V_{m+1} \otimes^\top \cdots \otimes^\top V_{m+n}\big)$. $\square$

Regarding Conjectural statements 5.30 and 5.34, their analogous versions for the space $I_{\text{bil},\top}$ are stronger since $I_{\text{bil},\top}(V,W) \subset I_{\text{bil}}(V,W) \cap \mathbb{K}(V \times_\top V)$. This is another reason why we prefer the locality tensor product used in Chapter 2. Under those analogous conjectural assumptions, analogous results to Theorems 5.37, 5.38, 5.41, and 5.45 can be proved.

Appendix B

Conjectural statements

B.III A group theoretic interpretation

Question (2.15), namely whether or not a quotient of locality vector spaces is a locality vector space, can be rephrased as whether or not a union of subgroups of a group is again a subgroup, as we will show in this Appendix. More precisely, when is the union of subgroups of a larger group closed under the group operation. This question in full generality, is to our knowledge, still an open question. A good description of the state of the art on the question of when a group is the union of n proper subgroups is given in [9]. An answer was provided in [27] for $n \leq 6$ and in [97] for $n = 7$. To our knowledge, the question for any n or for infinite unions of groups remains open.

Notation: In this paragraph, given a locality set $(S, \top)$ we will use the polar map

$$P^\top : \mathcal{P}(S) \longrightarrow \mathcal{P}(S) \tag{B.1}$$

$$U \longmapsto U^\top. \tag{B.2}$$

By convention, we take $P^\top(\emptyset) := S$. Furthermore, in a small abuse of notation we write $P^\top(u)$ instead of $P^\top(\{u\})$ if $U = \{u\}$ has only one element. The following elementary result characterizes locality vector spaces using the notation recently introduced.

Lemma B.III.1. Given a pre-locality vector space $(V, \top)$, it is a locality vector space if, and only if, $P^\top(u)$ is a vector subspace of V for any $u \in V$.

Proof. By Definition 2.21, $(V, \top)$ is a locality vector space if, and only if, $P^\top(U)$ is a vector subspace of V for any subset $U \subseteq V$. If this holds, then it is obvious that $P^\top(u)$ is a vector subspace of V for any $u \in V$.

Conversely, assume $P^\top(u)$ is a linear subspace of V for any $u \in V$. Then for any subset $U \subseteq V$, we have

$$P^\top(U) := \bigcap_{u \in U} P^\top(u)$$

which is an intersection of vector spaces and thus a vector space. □

In order to make a precise statement, let us first recall some basic facts. Given a locality vector space $(V, \top)$, $W \subseteq V$ a linear subspace of V and $\pi : V \longrightarrow V/W$ the linear quotient map, we write $[u] := \pi^{-1}(\pi(u)) \subseteq V$. We have $[u] = \{\bar{u} + w, w \in W\} = \bar{u} + W$ where $\bar{u}$ is any representative in V of the class $\pi(u)$. Note that $0_V \in [u]$ implies $[u] = W$. We denote the elements of V/W, as $[u]$ or $\pi(u)$ indifferently, as it is more convenient.

We need one other intermediate result:

Lemma B.III.2. Let $(V, \top)$ be a locality vector space, $W \subseteq V$ a linear subspace of V and $\pi : V \longrightarrow V/W$ the quotient map. Then

$$P^{\overline{\top}}([u]) = \bigcup_{u' \in [u]} \pi(P^\top(u'))$$

for any u in V (see (B.1)).

Proof. Let $(V, \top)$, W and π be as in the statement and let u be any element of V.

- The inclusion from left to right: Let $[\alpha]$ be in V/W such that $[\alpha] \in P^{\overline{\top}}([u])$ i.e., such that $[\alpha]\overline{\top}[u]$. Then by definition of $\overline{\top}$ there is some $u' \in [u]$ and some $\alpha' \in [\alpha]$ such that $\alpha' \top u' \Leftrightarrow \alpha' \in P^{\top}(u')$. This implies that
$$[\alpha] = \pi(\alpha) = \pi(\alpha') \in \pi(P^{\top}(u')) \subset \bigcup_{u' \in [u]} \pi(P^{\top}(u')).$$
Thus $P^{\overline{\top}}([u]) \subseteq \bigcup_{u' \in [u]} \pi(P^{\top}(u'))$.

- The inclusion from right to left: Let $[\alpha]$ be in V/W such that $[\alpha] \in \bigcup_{u' \in [u]} \pi(P^{\top}(u'))$. Then
$$\exists u' \in [u] : [\alpha] \in \pi(P^{\top}(u')) \implies \exists (u', w) \in [u] \times W : (\alpha + w) \in P^{\top}(u').$$
In V, we have $\alpha' := (\alpha + w) \in [\alpha]$ and this in turn implies that
$$\exists (u', \alpha') \in [u] \times [\alpha] : \alpha' \top u' \implies [\alpha]\overline{\top}[u].$$
Thus $[\alpha] \in P^{\overline{\top}}([u])$ and $\bigcup_{u' \in [u]} \pi(P^{\top}(u')) \subseteq P^{\overline{\top}}([u])$ which proves the statement. □

The following theorem relates the situation of a quotient of a locality vector space being again a locality vector space, with the question of when the union of subgroups of a bigger group is again a subgroup.

Theorem B.III.3. Let $(V, \top)$ be a locality vector space, $W \subseteq V$ a linear subspace of V and $\pi : V \longrightarrow V/W$ the quotient map. Then the following statements are equivalent:

1. $(V/W, \overline{\top})$ is a locality vector space,
2. For any u in V, the set
$$H_u := \bigcup_{u' \in [u]} \pi(P^{\top}(u'))$$
is a commutative group for the internal operation $+$ induced on the quotient space by the internal operation $+$ on V.
3. The set H_u is a commutative semigroup (for the same product) for any u in V.

Remark B.III.4. Notice that $P^{\top}(u')$ is a subset of V, thus $\pi(P^{\top}(u'))$ is a subset of V/W, thus the $\bigcup$ notation in the Theorem.

Proof.
- 2. $\Leftrightarrow$ 3. If H_u is a group, it is in particular a semigroup. Let us show that the converse is also true. Assume that H_u is a semigroup (i.e. that it is closed under summation) for any u in V and observe that by Lemma B.III.2 $H_u = P^{\overline{\top}}([u])$. Since $(V, \top)$ is a locality vector space we have $[0] \ni 0 \top u \in [u]$ thus $[0] = 0_{V/W} \in P^{\overline{\top}}([u]) = H_u$ by definition of $\overline{\top}$ (Definition 4.6). Hence H_u is a monoid for any u in V. We are left to show that if H_u is stable under addition, it is also stable under taking the inverse, that is multiplication by the scalar -1.

  For any $[\alpha] \in V/W$ we have:
$$[\alpha] \in P^{\overline{\top}}([u]) \implies \exists (\alpha', u') \in [\alpha] \times [u] : \alpha' \top u' \implies \exists (\alpha', u') \in [\alpha] \times [u] : \lambda\alpha' \top u' \ \forall \lambda \in \mathbb{K}$$
since $P^{\top}(u)$ is a vector subspace of V. Then since $\alpha' \in [\alpha] \Rightarrow \lambda\alpha' \in \lambda[\alpha]$ we deduce that
$$[\alpha] \in P^{\overline{\top}}([u]) \implies \lambda[\alpha] \in P^{\overline{\top}}([u]) \ \forall \lambda \in \mathbb{K}$$
and in particular $H_u = P^{\overline{\top}}([u])$ is a group.

- 1. $\Leftrightarrow$ 2. Let V, W and π be as in the statement of the theorem. By Lemma B.III.1, we know that $(V/W, \overline{\top})$ is a locality vector space if, and only if, $P^{\overline{\top}}([u]) = H_u$ is a vector subspace of V/W for any $[u] \in V/W$.

  We have already shown that H_u is stable by scalar multiplication, thus it is a vector space if, and only if, it is a group for the addition. □

B.IV Attempted algorithmic proof

We devote this last paragraph to present an algorithmic attempt to prove Conjectural statement 5.30, which we recall now.

Conjectural statement. *Given a locality vector space* $(E, \top)$ *and two subspaces from it* V_1*, and* V_2*, the subspace* $I_{\text{bil}}(V_1, V_2) \cap \mathbb{K}(V_1 \times_\top V_2) \subset \mathbb{K}(V_1 \times_\top V_2)$ *is locality compatible with* $\top_{V_1 \times_\top V_2}$.

Recall from Definition 2.14 that the locality tensor product $V_1 \otimes_\top V_2$ is a quotient space

$$\mathbb{K}(V_1 \times_\top V_2) \big/ (I_{\text{bil}}(V_1, V_2) \cap \mathbb{K}(V_1 \times_\top V_2)).$$

For any $(x, y) \in V_1 \times_\top V_2$, in the equivalence class $[(x, y)]$ we consider the following relation

$$(x', y') \preceq (x'', y'') \Leftrightarrow \big((\forall z \in \mathbb{K}(V_1 \times_\top V_2)) : (x'', y'') \top_{V_1 \times_\top V_2} z \Rightarrow (x', y') \top_{V_1 \times_\top V_2} z\big). \tag{B.3}$$

It is straightforward to see that it defines a preorder (it is reflexive and transitive).

Our claim is that for every pair of elements in the equivalence class $[(x, y)]$ endowed with the preorder mentioned above, there is at least one element which is smaller than the original two. Notice that this claim is enough to prove the Conjectural statement 5.30. Indeed, if we denote $\mathsf{x} := (x', y')$ and $\mathsf{x}+\mathsf{w} := (x'', y'')$ for some$\mathsf{x} \in \mathbb{K}(V_1 \times_\top V_2)$ and $\mathsf{w} \in I_{\text{bil}}(V_1, V_2) \cap \mathbb{K}(V_1 \times_\top V_2)$, such that $\mathsf{x} \top \mathsf{y}$ and $\mathsf{x}+\mathsf{w} \top \mathsf{z}$, then the existence of an element $(p, q) \in [(x', y')]$ which is smaller than (x', y') and that (x'', y'') implies that $\mathsf{x}+\hat{\mathsf{w}} := (p, q)$ is locality independent to y and to z as expected (see definition of locality compatibility in 5.21).

We use the following natural notations:

- $(x, y) \succeq (x', y') :\Leftrightarrow (x', y') \preceq (x, y)$.
- $(x, y) \sim (x', y') :\Leftrightarrow (x', y') \preceq (x, y) \wedge (x, y) \preceq (x', y')$.
- $(x, y) \prec (x', y') :\Leftrightarrow (x, y) \preceq (x', y') \wedge (x, y) \not\sim (x', y')$.
- $(x, y) \succ (x', y') :\Leftrightarrow (x, y) \succeq (x', y') \wedge (x, y) \not\sim (x', y')$.

For an element $x \in \mathbb{K}(V_1 \times_\top V_2)$, we call the **support** of x, and denote it by $\text{supp}(x)$, at the unique subset of $V_1 \times_\top V_2$ such that

$$x = \sum_{(v,w) \in \text{supp}(x)} \alpha_{(v,w)}(v, w)$$

where all $\alpha_{(v,w)} \neq 0$. We call the **length** of x, denoted by $\text{len}(x)$, at $|\text{supp}(x)|$.

Let $(x, y, z) \in \mathbb{K}(V_1 \times_\top V_2)$ and $w \in I_{\text{bil}}(V_1, V_2) \cap \mathbb{K}(V_1 \times_\top V_2)$ such that $x \top_{V_1 \times_\top V_2} y$ and $x+w \top_{V_1 \times_\top V_2} z$.

By definition $I_{\text{bil}}(V_1, V_2) \subset \mathbb{K}(V_1 \times V_2)$ is generated by elements of the form

$$(a_1 + a_2, b) - (a_1, b) - (a_2, b) \quad , \quad (a, b_1 + b_2) - (a, b_1) - (a, b_2), \tag{B.4}$$

and

$$(ka, b) - (a, kb) \quad , \quad k(a, b) - (ka, b) \quad , \quad k(a, b) - (a, kb). \tag{B.5}$$

We will call a generator of the type described in Equation (B.4) a **Q-type** generator, and the type described in Equation (B.5) a **P-type** generator.

Remark B.IV.1. Notice that if w is a Q-type generator, and $\{(a_1, b), (a_2, b)\} \subset \text{supp}(w)$, then the third element in the support of w is either $(a_1 + a_2, b), (a_1 - a_2, b)$, or $(-a_1 + a_2, b)$. Moreover since only one of the three elements on $\text{supp}(w)$ has coefficient $+1$ in w, the three options correspond to the three ways the coefficients can be arranged. Namely

$$(a_1 + a_2, b) - (a_1, b) - (a_2, b), \quad -(a_1 - a_2, b) + (a_1, b) - (a_2, b), \quad \text{and} \quad -(-a_1 + a_2, b) - (a_1, b) + (a_2, b).$$

Analogously if $\{(a, b_1), (a, b_2)\} \subset \text{supp}(w)$, then the third element in the support of w is either $(a, b_1 + b_2), (a, b_1 - b_2)$, or $(a, -b_1 + b_2)$ depending on the position of the $+1$ coefficient.

Proposition B.IV.2. *Let w_1 and w_2 two Q-type generators such that $|\mathrm{supp}(w_1) \cap \mathrm{supp}(w_2)| = 2$ which we write as $\mathrm{supp}(w_1) = \{\gamma_1, \gamma_2, \alpha\}$ and $\mathrm{supp}(w_2) = \{\gamma_1, \gamma_2, \beta\}$. Then, for any $(\lambda_1, \lambda_2) \in \mathbb{K}^2 \setminus (0, 0)$, it follows that either*

$$\gamma_i \in \mathrm{supp}(\lambda_1 w_1 + \lambda_2 w_2) \quad \textit{for some} \quad i \in \{1, 2\},$$

or

$$\alpha^{\top_{V_1 \times_\top V_2}} = \beta^{\top_{V_1 \times_\top V_2}}.$$

Proof. As a consequence of Remark B.IV.1 and the fact that $|\mathrm{supp}(w_1) \cap \mathrm{supp}(w_2)| = 2$ it follows that either γ_1 or γ_2 does not have the same coefficients in w_1 and w_2. We analise four possible case:

1. In the case that the coefficient of γ_1 is different in w_1 and in w_2 while the coefficient of γ_2 is -1 in both w_1 and w_2. Then in the sum $\lambda_1 w_1 + \lambda_2 w_2$ at least one of them is not canceled.

2. In the case that the coefficient of γ_2 is different in w_1 and in w_2 while the coefficient of γ_1 is -1 in both w_1 and w_2. The argument is analogous to the item before.

3. In the case that the coefficient of γ_1 is 1 in w_1 and -1 in w_2, while the coefficient of γ_2 is -1 in γ_1 and 1 in γ_2. By means of Remark B.IV.1, the polar sets of α and β are the same as expected.

4. In the case that the coefficient of γ_2 is 1 in w_1 and -1 in w_2, while the coefficient of γ_1 is -1 in γ_1 and 1 in γ_2. The argument is analogous to the item before.

Since those are all the possible cases where $|\mathrm{supp}(w_1) \cap \mathrm{supp}(w_2)| = 2$, the proof is complete. □

We state some simple results which will useful in the sequel.

Lemma B.IV.3. Let $x \top_{V_1 \times_\top V_2} y$, and w a generator of $I_{\mathrm{bil}}(V_1, V_2)$ (i.e., it is an element of the form (B.4) or (B.5)).

1. If w is a Q-type generator and $|\mathrm{supp}(x) \cap \mathrm{supp}(w)| \geq 2$, then $(x + \lambda w) \top_{V_1 \times_\top V_2} y$ for any $\lambda \in \mathbb{K}$.

2. If w is a P-type generator and $|\mathrm{supp}(x) \cap \mathrm{supp}(w)| \geq 1$, then $(x + \lambda w) \top_{V_1 \times_\top V_2} y$ for any $\lambda \in \mathbb{K}$.

In particular, each of the above cases imply that

$$(x + w) \preceq x,$$

with the preorder defined in (B.3).

Proof. Let $x = \sum_{i=1}^{N} \alpha_i (v_i, w_i)$ where $\mathrm{supp}(x) = (v_i, w_i)_{i=1}^{N}$. Assume without lost of generality that $y = (p, q)$. Assume for item (1) that w is of the form $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$. Since $|\mathrm{supp}(x) \cap \mathrm{supp}(w)| \geq 2$, there are i and j such that (v_i, w_i) and (v_j, w_j) are in the support of w, for instance $(v_i, w_i) = (a_1, b)$ and $(v_j, w_j) = (a_2, b)$. This implies that

$$a_1 \top_E b \quad , \quad a_2 \top_E b \quad , \quad a_1 \top_{V_1} p \quad , \quad \text{and} \quad b \top_{V_2} q.$$

Since $(E, \top)$ is a locality vector space, it follows that $(a_1 + a_2) \top_E b$, and $(a_1 + a_2) \top_{V_1} p$, and hence $x + w \in \mathbb{K}(V_1 \times_\top V_2)$ and $(x + y) \top y$. A similar argument applies for any Q-type generator w.

Assume for item (2) that w has the form $(ka, b) - (a, kb)$. Since $|\mathrm{supp}(x) \cap \mathrm{supp}(w)| \geq 1$, there is at least one $(v_i, w_i) = (ka, b)$ or $(v_i, w_i) = (a, kb)$. Either case, the fact that $(E, \top)$ is a locality vector space implies that $a \top_E b$, $p \top_{V_1} a$ and $q \top_{V_2} b$, and hence $(ka, b) - (a, kb) \in \mathbb{K}(V_1 \times_\top V_2)$ and $(x + w) \top_{V_1 \times_\top V_2} y$. A similar argument works for every P-type generator w. □

Corollary B.IV.4. Let $(x + w) \top_{V_1 \times_\top V_2} y$, with w a generator of $I_{\mathrm{bil}}(V_1, V_2)$ (i.e., it is an element of the form (B.4) or (B.5)).

1. If w is a Q-type generator and $|\mathrm{supp}(x) \cap \mathrm{supp}(w)| \leq 1$, then $x \top_{V_1 \times_\top V_2} y$.

2. If w is a P-type generator and $|\mathrm{supp}(x) \cap \mathrm{supp}(w)| \leq 1$, then $x \top_{V_1 \times_\top V_2} y$.

In particular, each of the above cases implies that

$$x \preceq (x + w),$$

with the preorder defined in (B.3).

Proof. The proof follows from Lemma B.IV.3 writing $x' = x + w$ and $w' = -w$. □

Let $x \in \mathbb{K}(V_1 \times_\top V_2)$ and $\omega = \sum_{i=1}^{N} \lambda_i w_i \in I_{\mathrm{bil}}(V_1, V_2) \cap \mathbb{K}(V_1 \times V_2)$, where every w_i is a generator of $I_{\mathrm{bil}}(V_1, V_2)$, we say that $(\lambda_1 w_1, \ldots, \lambda_N w_N)$ is a path between x and $x + \omega$. Lemma B.IV.3 and Corollary B.IV.4 imply that given an element $\omega \in I_{\mathrm{bil}}(V_1, V_2) \cap \mathbb{K}(V_1 \times V_2)$, and a path $(\lambda_1 w_1, \ldots, \lambda_N w_N)$ from x to $x + \omega$, it is possible to follow each step i of the path in the graph of the preorder $\preceq$ defined in (B.3) by considering the intersection $\mathrm{supp}(w_i) \cap \mathrm{supp}(x + \sum_{j=1}^{i-1} \lambda_j w_j)$. We say that the step i is descending (resp. strictly descending, resp. ascending, resp. strictly ascending, resp. flat) if $x + \sum_{j=1}^{i-1} \lambda_i w_i \preceq x + \sum_{j=1}^{i} \lambda_i w_i$ (resp. $x + \sum_{j=1}^{i-1} \lambda_i w_i \prec x + \sum_{j=1}^{i} \lambda_i w_i$, resp. $x + \sum_{j=1}^{i-1} \lambda_i w_i \succeq x + \sum_{j=1}^{i} \lambda_i w_i$, resp. $x + \sum_{j=1}^{i-1} \lambda_i w_i \succ x + \sum_{j=1}^{i} \lambda_i w_i$, resp. $x + \sum_{j=1}^{i-1} \lambda_i w_i \sim x + \sum_{j=1}^{i} \lambda_i w_i$).

The idea is then to reorganize or change the w_is into some w_i's such that $\omega = \sum_{i=1}^{N'} \lambda_i' w_i'$ and for which the path $(\lambda_1' w_1', \ldots, \lambda_{N'}' w_{N'}')$ has a minimum element in the preorder $\preceq$ (not necessarily unique since it is only a preorder). The first step is to pull to the beginning of the path, whenever it is possible, all generators w_i which satisfy conditions of Lemma B.IV.3, i.e. all generators which make $\preceq$ descend or at least don't ascend. More precisely, consider a counter $\hat{N} = 0$ and $\alpha = 1$. For the generator w_α consider the possible scenarios:

1. If w_α doesn't satisfy the conditions of Lemma B.IV.3 for $x = x + \sum_{i=1}^{\alpha-1} \lambda_i w_i$ and $w = \lambda_\alpha w_\alpha$, add 1 to the value of α and start again analyzing the next element in the path.
2. If w_α satisfies the conditions of Lemma B.IV.3 for $x = x + \sum_{i=1}^{\alpha-1} \lambda_i w_i$ and $w = \lambda_\alpha w_\alpha$, and $\hat{N} = \alpha - 1$, then add 1 to the value of $\hat{N}$ and also to the value of α. Then start again analyzing the next element in the path.
3. If w_α satisfies the conditions of Lemma B.IV.3 for $x = x + \sum_{i=1}^{\alpha-1} \lambda_i w_i$ and $w = \lambda_\alpha w_\alpha$, and $\hat{N} < \alpha - 1$, consider two possibilities:
   (a) If $\mathrm{supp}(w_\alpha) \cap \mathrm{supp}(w_{\alpha-1}) = \emptyset$, then $x + \sum_{i=1}^{\alpha-2} \lambda_i w_i$ and $w = \lambda_\alpha w_\alpha$ will also satisfy the conditions of Lemma B.IV.3. Then exchange the indices of $\lambda_{i-1} w_{i-1}$ and $\lambda_i w_i$, subtract 1 to the value of α, and continue analyzing the new element w_α.

   In other words, we pull w_α 1 step closer to the beginning of the path if its support does not intersect that of $w_{\alpha-1}$.

   (b) If $\mathrm{supp}(w_\alpha) \cap \mathrm{supp}(w_{\alpha-1}) \neq \emptyset$, then add 1 to the value of α and continue analyzing the next element of the path.

The process stops when $\alpha = N + 1$. Notice that at the end, the first N' of the path will be descending, i.e. $x + \sum_{i=1}^{j} \lambda_i w_i \succeq x + \sum_{i=1}^{j+1} \lambda_i w_i$ for $1 \leq j \leq \hat{N}$, and also that the step $\hat{N} + 1$ is strictly increasing, i.e. $x + \sum_{i=1}^{\hat{N}} \lambda_i w_i \prec x + \sum_{i=1}^{\hat{N}+1} \lambda_i w_i$. However, we may not conclude yet that the last $N - \hat{N}$ steps of the path are ascending because of the case described in item (3b). Every step of the path which satisfies the conditions of item 3b represents a "bump" in the ascending path from $x + \sum_{i=1}^{\hat{N}} \lambda_i w_i$ to $x + \omega$. We now proceed to analyse some type of "bumps" that can occur and to show how they can be "flattened".

Bumps of length 2

We proceed to consider all "bumps" where the step $i - 1$ is strictly ascending and the step i is strictly descending for $i - 1 > \hat{N}$. Namely $x + \sum_{j=1}^{i-2} \lambda_j w_j \prec x + \sum_{j=1}^{i-1} \lambda_j w_j$, and $x + \sum_{j=1}^{i-1} \lambda_j w_j \succ x + \sum_{j=1}^{i} \lambda_j w_j$. Our goal is to prove that $x + \sum_{j=1}^{i-2} \lambda_j w_j \preceq x + \sum_{j=1}^{i} \lambda_j w_j$, and thus the bump can be "flattened". Let us first prove some technical result regarding the behavior of the support of the sum of two generators of $I_{\mathrm{bil}}(V_1, V_2)$.

Lemma B.IV.5. Let w_1 and w_2 be two generators of $I_{\text{bil}}(V_1, V_2)$, $(\lambda_1, \lambda_2) \in (\mathbb{K} \setminus \{0\})^2$, and $z \in K(V_1 \times_\top V_2)$.

1. If both w_1 and w_2 are Q-type generators with $\text{supp}(w_1) = \text{supp}(w_2)$, then $w_1 = w_2$.
2. If both w_1 and w_2 are P-type generators with $\text{supp}(w_1) = \text{supp}(w_2)$, then $w_1 = w_2$.
3. If both w_1 and w_2 are Q-type generators with $|\text{supp}(w_1) \cap \text{supp}(w_2)| = 2$, then at least one of the following is satisfies:
   - there is at least one element $\gamma \in \text{supp}(w_1) \cap \text{supp}(w_2)$ such that $\gamma \in \text{supp}(\lambda_1 w_1 + \lambda_2 w_2)$, or
   - $\big(\text{supp}(w_1) \setminus (\text{supp}(w_1) \cap \text{supp}(w_2))\big)^{\top_{V_1 \times_\top V_2}} = \big(\text{supp}(w_2) \setminus (\text{supp}(w_1) \cap \text{supp}(w_2))\big)^{\top_{V_1 \times_\top V_2}}$.
4. If both w_1 and w_2 are P-type generators with $|\text{supp}(w_1) \cap \text{supp}(w_2)| = 1$, and $z \top_{V_1 \times_\top V_2} \gamma$ for some $\gamma \in \text{supp}(w_1) \cup \text{supp}(w_2)$ then $z \in (\text{supp}(w_1) \cup \text{supp}(w_2))^{\top_{V_1 \times_\top V_2}}$. In particular,
$$z \top_{V_1 \times_\top V_2} \lambda_1 w_1 + \lambda_2 w_2.$$
5. If w_1 is a P-type generator and w_2 is a Q-type generator with $|\text{supp}(w_1) \cap \text{supp}(w_2)| = 2$, then, there is at least one $\gamma \in \text{supp}(w_1)$ such that $\gamma \in \text{supp}(\lambda_1 w_1 + \lambda_2 w_2)$. Even more if for some $\beta \in \text{supp}(w_1) \cup \text{supp}(w_2)$, $z \top_{V_1 \times_\top V_2} \beta$, then $z \in (\text{supp}(w_1) \cup \text{supp}(w_2))^{\top_{V_1 \times_\top V_2}}$. In particular
$$z \top_{V_1 \times_\top V_2} \lambda_1 w_1 + \lambda_2 w_2.$$

Proof. The first two items follow from the definition of Q and P-type elements respectively.

3. Call γ_1 and γ_2 the elements of $\text{supp}(w_1) \cap \text{supp}(w_2)$. From Remark B.IV.1 we consider two options:
   - The sign of one of the γs (say γ_1) is negative in both w_1 and w_2 while γ_2 is positive in one (say w_1) and negative in w_2. In this case for every pair $(\lambda_1, \lambda_2) \in (\mathbb{K} \setminus \{0\})^2$ at least one γ_i will remain in the sum $\lambda_1 w_1 + \lambda_2 w_2$.
   - The sign of γ_1 is negative in w_1 and positive in w_2 while γ_2 is positive in w_1 and negative in w_2. Using the notations of Remark B.IV.1 we can say that $\gamma_1 = (a_1, b)$, and $\gamma_2 = (a_2, b)$. It follows that the other element in $\text{supp}(w_1)$ is $(-a_1 + a_2, b)$, while the other element in $\text{supp}(w_2)$ is $(a_1 - a_2, b)$. But from linear locality we conclude that $\{(a_1 - a_2, b)\}^{\top_{V_1 \times_\top V_2}} = \{(-a_1 + a_2, b)\}^{\top_{V_1 \times_\top V_2}}$ which proves the result. The same analysis applies if the difference between γ_1 and γ_2 is in the second coordinate.
4. If both w_1 and w_2 are P-type generators with $|\text{supp}(w_1) \cap \text{supp}(w_2)| = 1$, by construction of the P-type generators it follows that the three elements in $\text{supp}(w_1) \cup \text{supp}(w_2)$ are of the form $\gamma_i = (\kappa_i a, \vartheta_i b)$ for some κ_i and ϑ_i in $\mathbb{K}$ and some $(a, b) \in V_1 \times_\top V_2$. Therefore, if $z \top_{V_1 \times_\top V_2} \gamma_i$ for some $i \in \{1, 2, 3\}$, then $z \top_{V_1 \times_\top V_2} \gamma_i$ for all $i \in \{1, 2, 3\}$.
5. Let γ_1 and γ_2 be the elements in $\text{supp}(w_1)$ and γ_3 the only element in $\text{supp}(w_2) \setminus \text{supp}(w_1)$. By construction of the Q-type generators, only one coordinate varies in all γ_i, we will assume it is the first coordinate. The same analysis is valid for the second coordinate. Then w_1 is of the form $k(a, b) - (ka, b)$ for some $k \in \mathbb{K} \setminus \{0, 1\}$, and thus, by means of Remark B.IV.1, γ_3 is either $((k+1)a, b)$, $((k-1)a, b)$, or $((-k+1)a, b)$. In either case, $z \top_{V_1 \times_\top V_2} \gamma_i$ for some $i \in \{1, 2, 3\}$ implies that $z \top_{V_1 \times_\top V_2} \gamma_i$ for all $i \in \{1, 2, 3\}$, and thus
$$z \top_{V_1 \times_\top V_2} \lambda_1 w_1 + \lambda_2 w_2.$$
   On the other hand $\lambda_1 w_1 + \lambda_2 w_2 = (\lambda_1 k + \lambda_2)(a, b) + (\lambda_1 + \lambda_2)(ka, b) + \lambda_2 \gamma_3$. Since $k \notin \{0, 1\}$, then $\lambda_1 \neq 0$, $(\lambda_1 k + \lambda_2)$ and $(\lambda_1 + \lambda_2)$ are not zero at the same time which implies that at least one element in $\text{supp}(w_1)$ is in $\text{supp}(\lambda_1 w_1 + \lambda_2 w_2)$.

$\square$

In the following proposition we flatten all bumps of length 2 where w_{i-1} and w_i are both P-type generators. Since we consider the step $i-1$ to be strictly ascending, it follows from Lemma B.IV.3 and Corollary B.IV.4 that $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})=\emptyset$. Similarly, since the step i is strictly descending, we also assume that $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|=2$.

Proposition B.IV.6 (P-P)**.** *Let w_{i-1} and w_i be two P-type generators in a path $(\lambda_1 w_1,\ldots,\lambda_N w_N)$ from x to $x+\sum_{j=1}^{N}\lambda_j w_j$, such that* $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})=\emptyset$, *and* $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|=2$.

1. *If* $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=2$, *then* $\sum_{j=1}^{i-2}\lambda_j w_j\preceq\sum_{j=1}^{i}\lambda_j w_j$.
2. *If* $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=1$, *then* $\sum_{j=1}^{i-2}\lambda_j w_j\sim\sum_{j=1}^{i}\lambda_j w_j$. *In particular, exchanging the steps $i-1$ and i both steps turn flat.*

Proof. 1. By means of Lemma B.IV.5 item (2) $w_{i-1}=w_i$, and $\lambda_{i-1}w_{i-1}+\lambda_i w_i=(\lambda_{i-1}+\lambda_i)w_{i-1}$. Since $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})=\emptyset$, Corollary B.IV.4 yields the result.

2. Since $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})=\emptyset$, and $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|=2$ it follows that there is at least a γ (resp. β) in $\text{supp}(w_{i-1})\cup\text{supp}(w_i)$ such that $\gamma\in\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)$ (resp. $\beta\in\text{supp}(x+\sum_{j=1}^{i}\lambda_j w_j)$). Then $z\top_{V_1\times_\top V_2}x+\sum_{j=1}^{i-2}\lambda_j w_j$ (resp. $y\top_{V_1\times_\top V_2}x+\sum_{j=1}^{i}\lambda_j w_j$) implies that $z\top_{V_1\times_\top V_2}\gamma$ (resp. $y\top_{V_1\times_\top V_2}\beta$). By means of Lemma B.IV.5 item (4) $z\top_{V_1\times_\top V_2}x+\sum_{j=1}^{i}\lambda_j w_j$ (resp. $y\top_{V_1\times_\top V_2}x+\sum_{j=1}^{i-2}\lambda_j w_j$) which yields the result.

□

We proceed with the case where w_{i-1} is a P-type generator, and w_i is a Q-type generator. Once again, since the step $i-1$ is strictly ascending and the step i is strictly descending, Lemma B.IV.3 and Corollary B.IV.4 imply that $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})=\emptyset$, and $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|\geq 2$.

Proposition B.IV.7 (P-Q)**.** *Let w_{i-1} be a P-type generator and w_i a Q-type generator in a path $(\lambda_1 w_1,\ldots,\lambda_N w_N)$ from x to $x+\sum_{j=1}^{N}\lambda_j w_j$, such that* $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})=\emptyset$, *and* $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|\geq 2$.

1. *If* $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=2$ *and* $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|=3$, *then* $\sum_{j=1}^{i-2}\lambda_j w_j\sim\sum_{j=1}^{i}\lambda_j w_j$.
2. *If* $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=2$ *and* $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|=2$, *then* $\sum_{j=1}^{i-2}\lambda_j w_j\preceq\sum_{j=1}^{i}\lambda_j w_j$.
3. *If* $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=1$ *and* $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|=3$, *then exchanging steps $i-1$ and i, it follows that the new step $i-1$ is descending while the step i is flat.*
4. *If* $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=1$ *and* $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|=2$, *then exchanging steps $i-1$ and i, it follows that the new step $i-1$ is ascending while the step i is flat.*

Proof. 1. Let γ_1 and γ_2 be the two elements in $\text{supp}(w_{i-1})$ and γ_3 the only element in $\text{supp}(w_i)\setminus\text{supp}(w_{i-1})$. By means of Lemma B.IV.5 item (5) γ_1 or γ_2 is in $\text{supp}(x+\sum_{j=1}^{i}\lambda_j w_j)$, and thus, also by means of Lemma B.IV.5 item (5) $z\top_{V_1\times_\top V_2}(x+\sum_{j=1}^{i}\lambda_j w_j)$ implies that $z\top_{V_1\times_\top V_2}\gamma_3$. Since γ_3 is the only element of $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)$ which might not be in $\text{supp}(x+\sum_{j=1}^{i}\lambda_j w_j)$, then $z\top_{V_1\times_\top V_2}x+\sum_{j=1}^{i-2}\lambda_j w_j)$ and $\sum_{j=1}^{i-2}\lambda_j w_j\preceq\sum_{j=1}^{i}\lambda_j w_j$.

Analogously if $y\top_{V_1\times_\top V_2}(x+\sum_{j=1}^{i-2}\lambda_j w_j)$, in particular $z\top_{V_1\times_\top V_2}\gamma_3$, and by means of Lemma B.IV.5 item (5) $y\top_{V_1\times_\top V_2}\gamma_i$ for $i\in\{1,2\}$. Since γ_1 and γ_2 are the only elements possibly in $\text{supp}(x+\sum_{j=1}^{i}\lambda_j w_j)$ which are not in $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)$, then $z\top_{V_1\times_\top V_2}(x+\sum_{j=1}^{i}\lambda_j w_j)$ and $\sum_{j=1}^{i-2}\lambda_j w_j\succeq\sum_{j=1}^{i}\lambda_j w_j$ as expected.

2. Since the support of w_{i-1} has two elements, none of which is in $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)$, then $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=2$ and $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|=2$ imply that $\text{supp}(w_{i-1})\cap\text{supp}(w_i)=\text{supp}(w_{i-1})$, and thus $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_i)=\emptyset$. Then, $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)$ is strictly contained in $x+\sum_{j=1}^{i}\lambda_j w_j$ and the result follows.

3. Let $\{\gamma_1,\gamma_2\}=\text{supp}(w_i)\setminus\text{supp}(w_{i-1})$ and γ_3 be the only element in $\text{supp}(w_i)\cap\text{supp}(w_{i-1})$. Setting $\lambda_i' w_i' := \lambda_{i-1}w_{i-1}$ and $\lambda_{i-1}' w_{i-1}' := \lambda_i w_i$, i.e, exchanging the steps i and $i-1$, it follows that $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1}')=\{\gamma_1,\gamma_2\}$, and $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j+\lambda_{i-1}' w_{i-1'})\cap\text{supp}(w_i')=\{\gamma_3\}$. Lemma B.IV.3 and Corollary B.IV.4 yield the result.

4. Let γ_1 be the only element in $\text{supp}(w_i)\cap\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)$ and γ_2 the only element in $\text{supp}(w_i)\cap\text{supp}(w_{i-1})$. Setting $\lambda_i' w_i' := \lambda_{i-1}w_{i-1}$ and $\lambda_{i-1}' w_{i-1}' := \lambda_i w_i$, i.e, exchanging the steps i and $i-1$, it follows that $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1}')=\{\gamma_1\}$, and $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j+\lambda_{i-1}' w_{i-1'})\cap\text{supp}(w_i')=\{\gamma_2\}$. Lemma B.IV.3 and Corollary B.IV.4 yield the result.

□

We proceed with the case where w_{i-1} is a Q-type generator, and w_i is a P-type generator. Once again, since the step $i-1$ is strictly ascending and the step i is strictly descending, Lemma B.IV.3 and Corollary B.IV.4 imply that $|\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})|\le 1$, and $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|=2$.

Proposition B.IV.8 (Q-P)**.** *Let w_{i-1} be a Q-type generator and w_i a P-type generator in a path $(\lambda_1 w_1,\dots,\lambda_N w_N)$ from x to $x+\sum_{j=1}^{N}\lambda_j w_j$, such that $|\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})|\le 1$, and $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|=2$.*

1. *If $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=2$ and $|\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})|=0$, then $\sum_{j=1}^{i-2}\lambda_j w_j\preceq\sum_{j=1}^{i}\lambda_j w_j$.*

2. *If $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=2$ and $|\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})|=1$, then $\sum_{j=1}^{i-2}\lambda_j w_j\sim\sum_{j=1}^{i}\lambda_j w_j$.*

3. *If $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=1$ and $|\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})|=0$, then exchanging steps $i-1$ and i, it follows that the new step $i-1$ is flat while the step i is ascending.*

4. *If $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=1$ and $|\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})|=1$, then exchanging steps $i-1$ and i, it follows that the new step $i-1$ is flat while the step i is descending. This case might create a larger bump.*

Proof. Items 3. and 4. follow straightforward from the assumptions and from Lemma B.IV.3 and Corollary B.IV.4.

1. In this case, exchanging the steps i and $i-1$ we recover the case of Lemma B.IV.7 item (2) which yields the result.

2. Set $\{\gamma_1,\gamma_2\}=\text{supp}(w_{i-1})\cap\text{supp}(w_i)$ and $\{\gamma_3\}=\text{supp}(w_{i-1})\setminus\text{supp}(w_i)$. If the only element in $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})$ is γ_1 or γ_2 (resp. γ_3), then $\gamma_3\in\text{supp}(x+\sum_{j=1}^{i}\lambda_j w_j)$ (resp. γ_1 or γ_2 in $\text{supp}(x+\sum_{j=1}^{i}\lambda_j w_j)$ by means of Lemma B.IV.5 item 5), and thus $z\top_{V_1\times_\top V_2}\text{supp}(x+\sum_{j=1}^{i}\lambda_j w_j)$ implies that $z\top_{V_1\times_\top V_2}\gamma_3$ (resp. $z\top_{V_1\times_\top V_2}\gamma_1$ or $z\top_{V_1\times_\top V_2}\gamma_2$). By means of Lemma B.IV.5 item (5) $z\top_{V_1\times_\top V_2}\gamma_i$ for all i. Since γ_1 or γ_2 (resp. γ_3) are the only elements that might be in $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)$ which might not be in $\text{supp}(x+\sum_{j=1}^{i}\lambda_j w_j)$, then $z\top_{V_1\times_\top V_2}\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)$. On the other hand, if $y\top_{V_1\times_\top V_2}\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)$ implies that $y\top_{V_1\times_\top V_2}\gamma_1$ or $y\top_{V_1\times_\top V_2}\gamma_2$ (resp. γ_3). Once again by means of Lemma B.IV.5 item (5) $y\top_{V_1\times_\top V_2}\gamma_i$ for every i. Since the γ_is are the only elements that might be in $\text{supp}(x+\sum_{j=1}^{i}\lambda_j w_j)$ which might not be in $x+\sum_{j=1}^{i-2}\lambda_j w_j$, then $z\top_{V_1\times_\top V_2}(x+\sum_{j=1}^{i}\lambda_j w_j)$ which finishes the proof.

□

Finally we deal with the case where w_{i-1} and w_i are Q-type generators. Once again, since the step $i-1$ is strictly ascending and the step i is strictly descending, Lemma B.IV.3 and Corollary B.IV.4 imply that $|\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})|\leq 1$, and $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|\geq 2$.

Proposition B.IV.9 (Q-Q)**.** *Let w_{i-1} and w_i be Q-type generators in a path $(\lambda_1 w_1,\ldots,\lambda_N w_N)$ from x to $x+\sum_{j=1}^{N}\lambda_j w_j$, such that $|\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})|\leq 1$, and $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|\geq 2$.*

1. *If $|\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})|\leq 1$ and $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=3$, then $w_{i-1}=w_i$ and*

$$\left(x+\sum_{j=1}^{i-2}\lambda_j w_j\right)\preceq\left(x+\sum_{j=1}^{i}\lambda_j w_j\right).$$

2. *If $|\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})|=0$ and $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=2$, then*

$$\left(x+\sum_{j=1}^{i-2}\lambda_j w_j\right)\preceq\left(x+\sum_{j=1}^{i}\lambda_j w_j\right).$$

3. *If $|\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})|=0$ and $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=1$, exchanging the steps $i-1$ and i follows that the new step $i-1$ is descending and the new step i is ascending.*

4. *If $|\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})|=1$, $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|=3$ and $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=2$, then $(x+\sum_{j=1}^{i-2}\lambda_j w_j)$ and $(x+\sum_{j=1}^{i}\lambda_j w_j)$ are comparable. Namely either $(x+\sum_{j=1}^{i-2}\lambda_j w_j)\succeq(x+\sum_{j=1}^{i}\lambda_j w_j)$ or $(x+\sum_{j=1}^{i-2}\lambda_j w_j)\preceq(x+\sum_{j=1}^{i}\lambda_j w_j)$.*

5. *If $|\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})|=1$, $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|=3$ and $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=1$, exchanging the steps $i-1$ and i implies that the new steps $i-1$ and i are both descending.*

6. *If $|\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)\cap\text{supp}(w_{i-1})|=1$, $|\text{supp}(x+\sum_{j=1}^{i-1}\lambda_j w_j)\cap\text{supp}(w_i)|=2$ and $|\text{supp}(w_{i-1})\cap\text{supp}(w_i)|=2$, then $(x+\sum_{j=1}^{i-2}\lambda_j w_j)\succeq(x+\sum_{j=1}^{i}\lambda_j w_j)$.*

Proof. 1. The fact that $w_{i-1}=w_i$ follows from Lemma B.IV.5 item (1). Thus $\lambda_{i-1}w_{i-1}+\lambda_i w_i=(\lambda_{i-1}+\lambda_i)w_{i-2}$ which together with Corollary B.IV.4 yields the result.

2. Set $\{\gamma_1\}=\text{supp}(w_i)\setminus\text{supp}(w_{i-1})$, $\{\beta\}=\text{supp}(w_{i-1})\setminus\text{supp}(w_i)$ and $\{\gamma_2,\gamma_3\}=\text{supp}(w_i)\cap\text{supp}(w_{i-1})$. Let $z\top_{V_1\times_\top V_2}(x+\sum_{j=1}^{i}\lambda_j w_j)$, so in particular $z\top_{V_1\times_\top V_2}\beta$. Since γ_1 is the only element of $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)$ which might not be in $\text{supp}(x+\sum_{j=1}^{i}\lambda_j w_j)$, we must prove that $z\top_{V_1\times_\top V_2}\gamma_1$. By means of Lemma B.IV.5 item (3), either $\{\gamma_1\}^\top_{V_1\times_\top V_2}=\{\beta\}^\top_{V_1\times_\top V_2}$, in which case the result follows; or γ_2 (or γ_3) lie in $\text{supp}(\lambda_{i-1}w_{i-1}+\lambda_i w_i)$ which implies $\gamma_2\in\text{supp}(x+\sum_{j=1}^{i}\lambda_j w_j)$. The latter implies that $z\top_{V_1\times_\top V_2}\gamma_3$, then, since z is locality independent of at least two elements on the support of w_i, it is also independent of the third one, namely $z\top_{V_1\times_\top V_2}\gamma_3$. Using the same argument for w_{i-1} implies that $z\top_{V_1\times_\top V_2}\gamma_1$ as expected.

3. The result follows straightforward from the assumptions, from Lemma B.IV.3, and Corollary B.IV.4.

4. Set $\{\gamma_1,\gamma_2,\gamma_3\}=\text{supp}(w_{i-1})$ and $\{\gamma_1,\gamma_2,\beta\}=\text{supp}(w_i)$. Either both γ_3 and β lie in $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)$, or β and γ_1 lie in $\text{supp}(x+\sum_{j=1}^{i-2}\lambda_j w_j)$. In the latter case, exchanging the steps $i-1$ and i in the path, leads to two descending steps. For the other case, assume without lost of generality that $\gamma_1=(a_1,b)$ and $\gamma_2=(a_2,b)$. By means of Remark B.IV.1, γ_3 and β are either (a_1-a_2,b), $(-a_1+a_2,b)$ or (a_1+a_2,b). We consider three cases:

- In the case $\gamma_1 = (a_1 - a_2, b)$ and $\beta = (-a_1 + a_2, b)$ and $\lambda_{i-1} = \lambda_i$, then $\operatorname{supp}(\lambda_{i-1}w_{i-1} + \lambda_i w_i) = \{\gamma_3, \beta\}$, and thus $\operatorname{supp}(x + \sum_{j=1}^{i} \lambda_j w_j) \supset \operatorname{supp}(x + \sum_{j=1}^{i-2} \lambda_j w_j)$ which implies that

$$\left(x + \sum_{j=1}^{i-2} \lambda_j w_j\right) \succeq \left(x + \sum_{j=1}^{i} \lambda_j w_j\right).$$

- In the case $\gamma_1 = (a_1 - a_2, b)$ and $\beta = (-a_1 + a_2, b)$ and $\lambda_{i-1} \neq \lambda_i$, then γ_1 and γ_2 lie in $\operatorname{supp}(x + \sum_{j=1}^{i} \lambda_j w_j)$. Thus $z \top_{V_1 \times_\top V_2} (x + \sum_{j=1}^{i} \lambda_j w_j)$ implies that $z \top_{V_1 \times_\top V_2} \gamma_1$ and $z \top_{V_1 \times_\top V_2} \gamma_2$, which moreover implies that $z \top_{V_1 \times_\top V_2} \gamma_3$ and $z \top_{V_1 \times_\top V_2} \beta$. Since β and γ_3 are the only elements of $\operatorname{supp}(x + \sum_{j=1}^{i-2} \lambda_j w_j)$ that might not be in $\operatorname{supp}(x + \sum_{j=1}^{i} \lambda_j w_j)$, it follows that $z \top_{V_1 \times_\top V_2} (x + \sum_{j=1}^{i-2} \lambda_j w_j)$. Therefore

$$\left(x + \sum_{j=1}^{i-2} \lambda_j w_j\right) \preceq \left(x + \sum_{j=1}^{i} \lambda_j w_j\right).$$

- For any other possible choice of γ_3 and β, $z \in \{\gamma_3, \beta\}^{\top}_{V_1 \times_\top V_2}$, this implies that $z \top_{V_1 \times_\top V_2} \gamma_1$ and $z \top_{V_1 \times_\top V_2} \gamma_2$. Therefore, similarly as above

$$\left(x + \sum_{j=1}^{i-2} \lambda_j w_j\right) \succeq \left(x + \sum_{j=1}^{i} \lambda_j w_j\right).$$

5. The result follows straightforward from the assumptions and from Lemma B.IV.3 and Corollary B.IV.4.

6. The result follows from the assumptions and Lemma B.IV.5 item (3) in a similar way to the previous items.

□

The previous 4 propositions give us criteria to compare the elements $x + \sum_{j=1}^{i-2} \lambda_j w_j$ and $x + \sum_{j=1}^{i} \lambda_j w_j$, namely before and after the bump. Those cases in which we obtain

$$\left(x + \sum_{j=1}^{i-2} \lambda_j w_j\right) \preceq \left(x + \sum_{j=1}^{i} \lambda_j w_j\right),$$

the bump can be considered "flattened" which won't be problematic to the goal of showing $x + \sum_{j=1}^{\hat{N}} \lambda_j w_j) \preceq (x + \omega)$. In the cases where the result is $(x + \sum_{j=1}^{i-2} \lambda_j w_j) \succeq (x + \sum_{j=1}^{i} \lambda_j w_j)$, both steps $i-1$ and i can be pulled to the beginning of the path as described before.

Notice however that there could be bumps of bigger length, namely when the step i is ascending and the next k steps are descending or flat, and their supports are not disjoint. It is not difficult to see that the number of cases increases exponentially with the length of the bump, and thus the idea of proving each of them is hopeless. Nonetheless, the fact that the previous 17 cases in the last 4 Proposition "magically" work, hint at the fact that there is an algebraic reason behind it. However, this interesting question is left for a future work.

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