NON-LOCAL BOUNDARY CONDITIONS FOR THE SPIN DIRAC OPERATOR ON SPACETIMES WITH TIMELIKE BOUNDARY

Penelope Gehring

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Hauptbetreuer:	Prof. Dr. Christian Bär
Zweitbetreuer:	Prof. Dr. Jan Metzger
Mentor:	Prof. Dr. Lars Andersson

Gutachter:	Prof. Dr. Christian Bär
	Prof. Dr. Jan Metzger
	Prof. Dr. Nadine Große

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Abstract

Non-local boundary conditions – for example the Atiyah–Patodi–Singer (APS) conditions – for Dirac operators on Riemannian manifolds are rather well-understood, while not much is known for such operators on Lorentzian manifolds. Recently, Bär and Strohmaier [15] and Drago, Große, and Murro [27] introduced APS-like conditions for the spin Dirac operator on Lorentzian manifolds with spacelike and timelike boundary, respectively. While Bär and Strohmaier [15] showed the Fredholmness of the Dirac operator with these boundary conditions, Drago, Große, and Murro [27] proved the well-posedness of the corresponding initial boundary value problem under certain geometric assumptions.

In this thesis, we will follow the footsteps of the latter authors and discuss whether the APS-like conditions for Dirac operators on Lorentzian manifolds with timelike boundary can be replaced by more general conditions such that the associated initial boundary value problems are still well-posed.

We consider boundary conditions that are local in time and non-local in the spatial directions. More precisely, we use the spacetime foliation arising from the Cauchy temporal function and split the Dirac operator along this foliation. This gives rise to a family of elliptic operators each acting on spinors of the spin bundle over the corresponding timeslice. The theory of elliptic operators then ensures that we can find families of non-local boundary conditions with respect to this family of operators. Proceeding, we use such a family of boundary conditions to define a Lorentzian boundary condition on the whole timelike boundary. By analyzing the properties of the Lorentzian boundary conditions, we then find sufficient conditions on the family of non-local boundary conditions that lead to the well-posedness of the corresponding Cauchy problems. The well-posedness itself will then be proven by using classical tools including energy estimates and approximation by solutions of the regularized problems.

Moreover, we use this theory to construct explicit boundary conditions for the Lorentzian Dirac operator. More precisely, we will discuss two examples of boundary conditions – the analogue of the Atiyah–Patodi–Singer and the chirality conditions, respectively, in our setting. For doing this, we will have a closer look at the theory of non-local boundary conditions for elliptic operators and analyze the requirements on the family of non-local boundary conditions for these specific examples.

Zusammenfassung

Über nicht-lokale Randbedingungen – zum Beispiel die Atiyah–Patodi–Singer (APS)-Bedingungen – für Dirac Operatoren auf Riemannschen Mannigfaltigkeiten ist recht viel bekannt, während für die hyperbolischen Dirac Operatoren auf Lorentz-Mannigfaltigkeiten dies noch nicht der Fall ist. Kürzlich haben Bär und Strohmaier [15] und Drago, Große und Murro [27] APS-ähnliche Bedingungen für den Spin Dirac Operator auf Lorentz-Mannigfaltigkeiten mit raumartigen bzw. zeitartigen Rand eingeführt. Während Bär und Strohmaier [15] zeigten, dass der Dirac Operator mit diesen Randbedingungen Fredholm ist, bewiesen Drago, Große und Murro [27] die Wohlgestelltheit des entsprechenden Anfangsrandwertproblems unter bestimmten geometrischen Annahmen.
In dieser Arbeit werden wir in die Fußstapfen der letztgenannten Autoren treten und diskutieren, ob die APS-ähnlichen Bedingungen für Dirac Operatoren auf Lorentz-Mannigfaltigkeiten mit zeitartigen Rand durch allgemeinere Bedingungen ersetzt werden können, sodass die zugehörigen Anfangsrandwertprobleme immer noch wohlgestellt sind.
Wir betrachten Randbedingungen, die in der Zeit lokal und in den Raumrichtungen nicht-lokal sind. Genauer gesagt verwenden wir die Raumzeitblätterung, die sich aus der Cauchy Zeitfunktion ergibt, und spalten den Dirac Operator entlang dieser Foliation auf. Daraus ergibt sich eine Familie elliptischer Operatoren, die jeweils auf Spinoren des Spinbündels über den entsprechenden Zeitschnitt wirken. Die Theorie der elliptischen Operatoren stellt dann sicher, dass wir Familien von nicht-lokalen Randbedingungen bezüglich dieser Familie von Operatoren finden können. Im weiteren Verlauf verwenden wir solche Familien von Randbedingungen, um eine Lorentzsche Randbedingung auf dem gesamten zeitartigen Rand zu definieren. Durch das Analysieren der Lorentzschen Randbedingungen finden wir dann hinreichende Bedingungen für die Familie der nicht-lokalen Randbedingungen, die zur Wohlgestelltheit der entsprechenden Cauchy-Probleme führen. Die Wohlgestelltheit selbst wird dann mit Hilfe klassischer Methoden bewiesen, einschließlich Energieabschätzungen und Annäherung durch Lösungen der regularisierten Probleme.
Außerdem verwenden wir diese Theorie, um explizite Randbedingungen für den Lorentzschen Dirac Operator zu konstruieren. Genauer gesagt werden wir zwei Beispiele für Randbedingungen diskutieren - das Analogon der Atiyah-Patodi-Singer- bzw. Chiralitäts-Bedingungen für unseren Fall. Dazu werden wir uns die Theorie der nicht-lokalen Randbedingungen für elliptische Operatoren genauer ansehen und die Anforderungen an die Familie der nicht-lokalen Randbedingungen für diese Beispiele analysieren.

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Statement of Originality

This thesis contains no material which has been accepted for the award of any other degree or diploma at any other university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. I give consent to this copy of my thesis, when deposited in the University Library, being made available for loan and photocopying.

Contents

1 INTRODUCTION

The Atiyah–Patodi–Singer index theorem [3] for elliptic first order operators on compact manifolds with boundary is one of the central mathematical discoveries of the 20th century. This index theorem requires non-local boundary conditions, which are based on the spectral decomposition of the operator induced on the boundary – the so called Atiyah–Patodi–Singer (APS) conditions. These boundary conditions are essential for making sure that the operator is indeed a Fredholm operator and, hence, that the analytical index is well-defined. Bär and Ballmann [8] and later Bär and Bandara [10] introduced general boundary conditions for elliptic first order operators and analyzed their influence on elliptic regularity and Fredholmness; in [8] the authors additionally assumed the boundary operator to be formally selfadjoint. For elliptic operators of general order, Bandara, Goffeng, and Saratchandran [7] discussed boundary conditions connected to the Calderón projectors. Their results, in the first order case, are equivalent to the results of Bär and Bandara [10].
Recently, Bär and Strohmaier [15] showed a Lorentzian index theorem for the (twisted) spin Dirac operator on spatially compact globally hyperbolic spin manifolds with boundary that consists of two disjoint smooth spacelike Cauchy hypersurfaces. Although the spin Dirac operator is hyperbolic in this case, the boundary operator is elliptic so that the APS conditions still make sense. In [15], the authors showed that under these conditions the Dirac operator becomes Fredholm and its index is given formally by the same geometric expression as in the Riemannian case. Bär and Hannes [13] investigated to what extend these boundary conditions can be replaced by more general ones and how the index changes with respect to them. Furthermore, an analogous result to [15] was obtained by Shen and Wrochna [44] for asymptotically static spacetimes with only one spacelike Cauchy hypersurface as "lower" boundary. As an application to the Lorentzian index theorem the chiral anomaly in algebraic quantum field theory on curved spacetimes was computed by Bär and Strohmaier in [14].
In physics, the Anti-de Sitter (AdS) or the asymptotically AdS spacetimes became increasingly important – especially in the context of studying the properties of Green-hyperbolic operators like

the wave, the Klein–Gordon, or the Dirac operator; see for example [5, 34, 50, 48]. The manifolds considered in these results are spacetimes with timelike boundary, i. e. spacetimes (M, g) with boundary ∂M, such that $\iota^* g$ is also a Lorentzian metric, where $\iota : \partial M \to M$ is the natural inclusion map. Further studies of Green-hyperbolic operators on spacetimes with timelike boundary were accomplished, for example, in [23, 24, 25, 29, 31]. These results are concerned with local boundary conditions. However, to establish a possible index theorem for this class of manifolds, one should also investigate the behavior of non-local boundary conditions for hyperbolic first order operators such as the Lorentzian Dirac operator.
As a first step, one should prove the well-posedness of the Cauchy problem for the Dirac operator with non-local boundary conditions. In full generality, this would require an analog of the theory of [8, 10] for hyperbolic operators on a non-compact manifold, which should be highly non-trivial since the spectrum of the boundary operator would be difficult to control. Thus, we will instead restrict ourselves to boundary conditions that are local in time and non-local in the spatial direction such that we can use for each timeslice the theory of elliptic operators.
More precisely, we consider globally hyperbolic spin manifolds with timelike boundary, i. e. $(M, g) \cong (\mathbb{R} \times \Sigma, -N^2\,\mathrm{d}t^2 + g_t)$ with $N \in C^\infty(\mathbb{R} \times \Sigma)$ strictly positive and $\{(\Sigma, g_t)\}_{t\in\mathbb{R}}$ being a smooth family of Riemannian manifolds with smooth boundary $\partial\Sigma$. On such manifolds, the Dirac operator $D : C^\infty(M, SM) \to C^\infty(M, SM)$ splits as

$$D = -\gamma(\nu)\left[\nabla^{SM}_{\nu} + iD_t - \frac{n}{2}H_t\right],$$

where ∇^{SM} is the spin connection on the complex spin bundle SM, $\gamma(\upsilon)$ the Clifford multiplication of the unit normal field ν to (Σ, g_t), H_t the mean curvature of Σ_t, and D_t a family of elliptic operators acting on spinors on $SM|_{\Sigma_t}$. This setting will be made more precise in Section 2.1.
Using the conformal change $\hat{g} = N^{-2}g$ and identifying the Cauchy hypersurfaces, we can bring D to the form

$$\tilde{D} = -\gamma(\nu)(\partial_t + i\tilde{D}_t),$$

where $\tilde{D}$ is acting on $C^\infty(\mathbb{R}, C^\infty(\Sigma, SM|_\Sigma))$ and $\tilde{D}_t$ is an elliptic operator acting on $C^\infty(\Sigma, SM|_\Sigma)$. Using the theory of [8, 10], one can find elliptic boundary conditions $B_t \subseteq H^{\frac{1}{2}}(\Sigma_t, SM|_{\Sigma_t})$ with respect to D_t. We consider certain families of boundary conditions $B := \{B_t\}_{t\in\mathbb{R}}$ called *admissible boundary conditions* (see Definition 3.3.13), which – briefly speaking – should be continuous in t in a suitable sense and induce ∞-regular selfadjoint boundary conditions $\tilde{B}_t \subseteq \check{H}(\tilde{A}_t)$, where $\tilde{A}_t$ is the boundary operator of $\tilde{D}_t$ and $\tilde{B}_t$ are the boundary conditions associated with B_t by going from D_t to $\tilde{D}_t$. A thorough investigation is presented in Chapter 3.
If we have an admissible boundary condition B, we can define the *Lorentzian boundary condition* as

$$C^\infty(\partial M, B) := \{u \in C^\infty(\partial M, SM|_{\partial M}); u|_{\partial\Sigma_t} \in B_t\}.$$

With respect to these boundary conditions, we will prove the following well-posedness result:

Main Theorem 1. *Let (M, g) be a globally hyperbolic, spatially compact spin manifold with timelike boundary ∂M. Let Σ be a spacelike Cauchy hypersurface. Let $t\colon\ M \to \mathbb{R}$ be a temporal function such that $\Sigma_0 = \Sigma$ and the gradient of t is tangential to ∂M. Let B be an admissible boundary condition with respect to t and D. Then, there exists a unique smooth solution $\psi \in C^\infty(M, SM)$ to*

$$\begin{cases} D\psi = f \in C^\infty_{cc}(M, SM) \\ \psi|_\Sigma = \psi_0 \in C^\infty_{cc}(\Sigma, SM|_\Sigma) \\ \psi|_{\partial M} \in C^\infty(\partial M, B) \end{cases} \tag{1.0.1}$$

that depends continuously on the Cauchy data (f, ψ_0).

This theorem will be proven in Chapter 4 and is a generalization of the result of Drago, Große, and Murro [27], which were the first authors to consider non-local boundary conditions in the above setting. In Chapter 5, we will discuss some examples for admissible boundary conditions. In particular, we show for boundary conditions defined over Grassmannian projections:

Main Theorem 2. *Let $(M, g) = (\mathbb{R} \times \Sigma, g = -N^2 dt^2 + g_t)$ be a globally hyperbolic, spatially compact spin manifold with timelike boundary ∂M. Let η be the unit normal field to ∂M and Σ be a spacelike Cauchy hypersurface. Let $t\colon\ M \to \mathbb{R}$ be a temporal function such that $\Sigma_0 = \Sigma$ and the gradient of t is tangential to ∂M. Let $\{P_t\}_{t\in\mathbb{R}}$ be a family of orthogonal pseudo differential operators on $L^2(\partial\Sigma_t, SM|_{\partial\Sigma_t})$ such that*

1. *$P_t = N^{-1} P_t^* N$,*
2. *$P_t = Id + \sigma_{D_t}(\eta_t^\flat) P_t \sigma_{D_t}(\eta_t^\flat)$,*
3. *$\tilde{P}_t = U N^{\frac{n}{2}} P_t N^{\frac{n}{2}} U^{-1}$ is a Grassmannian projection on $L^2(\hat{\Sigma}_0, SM|_{\partial\hat{\Sigma}_0})$, where $\hat{\Sigma}_t := (\Sigma_t, N^{-2} g_t)$ and $U\ :\ L^2(\hat{\Sigma}_t) \to L^2(\hat{\Sigma}_0)$ is the identification of the L^2 spaces with respect to the parallel transport along the t lines, and*
4. *there exists a sequence $\{k_j\}_{j\in\mathbb{N}_0}$ of non-negative integers with $k_0 = 0$ and $k_j \to \infty$ for $j \to \infty$ such that $t \mapsto \tilde{P}_t$ is H^{k_j} norm continuous for all j.*

Then $B := \{B_t\}_{t\in\mathbb{R}}$ with $B_t := P_t H^{\frac{1}{2}}(\partial\Sigma_t, SM|_{\partial\Sigma_t})$ is an admissible boundary condition and in particular there exists a unique smooth solution $\psi \in C^\infty(M, SM)$ to

$$\begin{cases} D\psi = f \in C^\infty_{cc}(M, SM) \\ \psi|_\Sigma = \psi_0 \in C^\infty_{cc}(\Sigma, SM|_\Sigma) \\ \psi|_{\partial M} \in C^\infty(\partial M, B) \end{cases} \tag{1.0.2}$$

that depends continuously on the Cauchy data (f, ψ_0).

This theorem will then be used to discuss more explicit boundary conditions like for example the Atiyah–Patodi–Singer conditions or chirality conditions.

This thesis is organized as follows:
In Chapter 2, we recall the most important facts of spin geometry and the spin Dirac operator on spacetimes with timelike boundary. In Chapter 3, we will first summarize the theory of non-local boundary conditions for elliptic operators and then apply this theory to define non-local boundary conditions in the setting of spacetimes with timelike boundary. These boundary conditions will be used in Chapter 4 to prove the well-posedness of the corresponding initial boundary value problems. In Chapter 5, we will discuss pseudo local boundary conditions and apply this discussion to some examples of boundary conditions. We will conclude this thesis by putting our results into context of recent research on boundary value problems on spacetimes in Chapter 6. In Appendix A, we briefly discuss a more general class of operators suitable for the non-local boundary conditions constructed in this work.

2 PRELIMINARIES

In this chapter, we will introduce the most important aspects about spacetimes with timelike boundary and the Lorentzian Dirac operator. In the last section, we will also give some more comments on the Riemannian Dirac operator on manifolds with or without boundary. We will require some familiarity with Lorentzian geometry and spin geometry. However, the reader may consult [39] for an introduction to Lorentzian geometry and [11, 17] for semi-Riemannian spin geometry.

2.1 Spacetimes with timelike boundary

In this section, we will introduce the basic facts about spacetimes with timelike boundary. For this let us start with their definition.

Definition 2.1.1. A *Lorentzian manifold with timelike boundary* (M, g) is a Lorentzian manifold-with-boundary such that $\iota^* g$, with $\iota\colon \partial M \to M$ being the natural inclusion, defines a Lorentzian metric on the boundary.
A *spacetime with timelike boundary* is a time-oriented Lorentzian manifold with timelike boundary.

By time-oriented one means – as in the boundaryless case – that the time-cones have been chosen continuously, i. e. locally selected by a continuous timelike field. Note that (M, g) being a spacetime with timelike boundary is equivalent to the interior and the connected components of the timelike boundary being spacetimes without boundary, which is shown in Proposition 2.4 in [1]. Let us briefly discuss some examples before we continue with our discussion of globally hyperbolic spacetimes with timelike boundary.

Example 2.1.2.

1. Consider the manifold-with-boundary $M = \overline{B_1(0)} \times \mathbb{R}$ with $\overline{B_1(0)}$ being the 2-dimensional closed unit ball. Let M be embedded into the Minkowski space $(\mathbb{R}^{1,2}, g_{Min})$, then (M, g_{Min}) is a spacetime with timelike boundary.

2. Two famous examples would be the Anti-de Sitter space (AdS) and the Anti-de Sitter Schwarzschild spacetime, which have conformal timelike boundary, i. e. after compactifying in radial direction and carefully gluing in the boundary, they become spacetimes with timelike boundary. We refer the interested reader to, for example, [45, 2].

As in the case of spacetimes without boundary, we call (M, g) *globally hyperbolic*, when it is causal and all causal diamonds $J^+(p) \cap J^-(q)$ for $p, q \in M$ are compact. Also we call a set $\Sigma \subseteq M$ a *Cauchy hypersurface* if it is intersected exactly once by every inextensible timelike curve. In the boundaryless case, Bernal and Sánchez [18] characterized globally hyperbolic spacetimes, but also in the setting of spacetimes with timelike boundary there is an analogous result:

Theorem 2.1.3 (Theorem 1.1 in Aké–Flores–Sánchez [1]). *Let (M, g) be a spacetime with timelike boundary of dimension greater or equal two. Then the following conditions are equivalent:*

1. *(M, g) is globally hyperbolic,*

2. *(M, g) possesses a Cauchy hypersurface, and*

3. *(M, g) is isometric to $\mathbb{R} \times \Sigma$ endowed with*

$$g = -N^2\, \mathrm{d}t^2 + g_t,$$

where $t : M \to \mathbb{R}$ is a Cauchy temporal function, whose gradient is tangential to ∂M, $N \in C^\infty(\mathbb{R}\times\Sigma)$ being strictly positive while $\mathbb{R} \ni t \mapsto (\Sigma_t := \{t\}\times\Sigma, g_t)$ identifies a one-parameter family of Riemannian manifolds with boundary. Each Σ_t is a Cauchy hypersurface for (M, g).

If one compares Theorem 2.1.3 with its boundaryless analogue in [18], one notices two differences. Firstly, the family of spacelike Cauchy hypersurfaces $\{\Sigma_t\}_{t\in\mathbb{R}}$ consists of Riemannian manifolds with boundary, while in the boundaryless case they have no boundary. Secondly, the Cauchy temporal function is chosen such that its gradient is tangential to the boundary. This assumption is related to the method of identifying the Cauchy hypersurfaces by the integral curves of ∂_t. If ∂_t is not tangential to the boundary, the integral curves do not exist for all time and hence also not the parallel transport along these curves, see Figure 2.1 for an intuition.

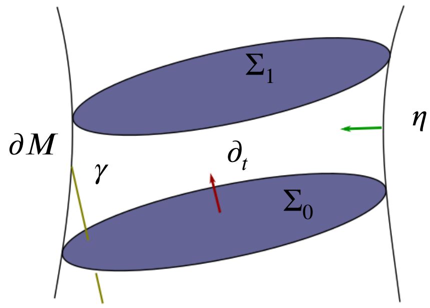

Figure 2.1: The integral curve γ of ∂_t hitting the boundary ∂M.

Example 2.1.4. Let us put Example 2.1.2 into the context of globally hyperbolic manifolds. While the first example is clearly globally hyperbolic, the discussion for the second one is more involved. The boundaryless AdS space has compact time, hence it is obviously non-globally hyperbolic. By carefully compactifying the AdS space and gluing in the conformal boundary, one ends up with a spacetime with infinite time and timelike boundary, which indeed is globally hyperbolic as well. Note, that without gluing in the boundary the compactified AdS is not globally hyperbolic in the sense of boundaryless spacetimes.

2.2 The spin Dirac operator on spacetimes with timelike boundary

Let us now introduce the Lorentzian spin Dirac operator. For this let (M, g) be a globally hyperbolic spin manifold with timelike boundary and let us denote the dimension of M by $n + 1$. Note that for the physical relevant case $n = 3$, globally hyperbolic spacetimes are always spin.
From now onwards, we will always assume M to be spatially compact, i. e. all Cauchy hypersurfaces are compact.
Let $SM \to M$ be the complex spin bundle with its invariantly defined non-degenerate inner product $\langle\cdot,\cdot\rangle_{SM}$ and ∇^{SM} its metric connection. Let us denote with $C^\infty(M, SM)$ the space of smooth section of SM. Furthermore, let us denote for tangent vectors $X \in T_xM$ the *Clifford multiplication* by $\gamma(X) : S_xM \to S_xM$.
The Clifford multiplication is symmetric with respect to $\langle\cdot,\cdot\rangle_{SM}$ and also satisfies the Clifford relation

$$\gamma(X)\gamma(Y) + \gamma(Y)\gamma(X) = -2g(X, Y) \tag{2.2.1}$$

for all $x \in M$ and $X, Y \in T_xM$. Furthermore, note that the Clifford multiplication is parallel with respect to ∇^{SM} and the Levi-Civita connection ∇ on TM, i. e. for $X, Y \in C^\infty(M, TM)$ and

$\psi \in C^\infty(M, SM)$ one has

$$\nabla_X^{SM}(\gamma(Y)\psi) = \gamma(Y)\nabla_X^{SM}\psi + \gamma(\nabla_X Y)\psi. \tag{2.2.2}$$

Since M is globally hyperbolic, we can choose a temporal function, which induces a spacetime foliation $\{\Sigma_t\}_{t\in\mathbb{R}}$ consisting of Riemannian manifolds with boundary. Let ν be the past pointing timelike unit vector field perpendicular to $\{\Sigma_t\}_{t\in\mathbb{R}}$.
Let us briefly discuss how the spin bundle $S\Sigma_t$, its Clifford multiplication γ_t and its metric $\langle\cdot,\cdot\rangle_{S\Sigma_t}$ are related to the corresponding objects on $SM|_{\Sigma_t}$. If n is even then

$$SM|_{\Sigma_t} \cong S\Sigma_t$$

with $\gamma(X) = -i\gamma(\nu)\gamma_t(X)$ and $\langle\cdot,\cdot\rangle_{SM} = \langle\gamma(\nu),\cdot,\cdot\rangle_{S\Sigma_t}$.
On the other hand, if n is odd,

$$SM|_{\Sigma_t} \cong S\Sigma_t \oplus S\Sigma_t,$$

with

$$\gamma(X) = \begin{pmatrix} 0 & i\gamma(\nu)\gamma_t(X) \\ -i\gamma(\nu)\gamma_t(X) & 0 \end{pmatrix},$$

and

$$\langle\cdot,\cdot\rangle_{SM} = \left\langle \begin{pmatrix} 0 & \gamma(\nu) \\ \gamma(\nu) & 0 \end{pmatrix}\cdot,\cdot \right\rangle_{S\Sigma_t\oplus S\Sigma_t}.$$

Remark 2.2.1.

1. Let us briefly discuss the intuition behind the case of n being odd. Here, we have $SM = S^-M \oplus S^+M$, where $S^\pm M$ are the bundles of negative and positive chirality. Then $S^+M|_{\Sigma_t} \cong S\Sigma_t$ with $\gamma(X) = i\gamma(\nu)\gamma_t(X)$, while for $S^-M|_{\Sigma_t} \cong S\Sigma_t$ we have $\gamma(X) = -i\gamma(\nu)\gamma_t(X)$. Hence, one sees that with respect to the splitting $SM|_{\Sigma_t} \cong S\Sigma_t \oplus S\Sigma_t$ Clifford multiplication splits as
$$\gamma(X) = \begin{pmatrix} 0 & \gamma(X) \\ \gamma(X) & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\gamma(\nu)\gamma_t(X) \\ -i\gamma(\nu)\gamma_t(X) & 0 \end{pmatrix}.$$
Similarly, one can see how the inner products relate.

2. Note that for both cases, n being odd or even, since $\langle\cdot,\cdot\rangle_{S\Sigma_t}$ is positive definite, the inner product
$$\langle\cdot,\cdot\rangle_0 := \langle\gamma(\nu)\cdot,\cdot\rangle_{SM}. \tag{2.2.3}$$
is positive definite on $SM|_{\Sigma_t}$. We denote the corresponding norm by $|\cdot|_0 := \sqrt{\langle\cdot,\cdot\rangle_0}$. For each $t \in \mathbb{R}$, let $L^2(\Sigma_t, SM|_{\Sigma_t})$ be the canonical L^2-space on the spinor bundle $SM|_{\Sigma_t}$ defined using the volume form $\mathrm{d}\mu_{\Sigma_t}(g_t)$ and the metric $\langle\cdot,\cdot\rangle_0$.

On the complex spin bundle SM, we consider the *Lorentzian spin Dirac operator*

$$D\colon C^\infty(M,SM)\to C^\infty(M,SM).$$

Locally, if $e_0,e_1,\dots,e_n$ is a Lorentzian orthogonal tangent frame, the Lorentzian spin Dirac operator is given by

$$D=\sum_{j=0}^n g^{jj}\gamma(e_j)\nabla^{SM}_{e_j},$$

where $g^{jj}=g(e_j,e_j)^{-1}$. It is easy to see that this does not depend on the choice of orthogonal tangent frame. If $e_0,\dots,e_n$ is even orthonormal, then D simplifies to

$$D=\sum_{j=0}^n \varepsilon_j\gamma(e_j)\nabla^{SM}_{e_j},$$

where $\varepsilon_j=g(e_j,e_j)=\pm1$. Let η be the inward pointing spacelike unit normal field to ∂M, then the divergence theorem implies for all $\psi,\phi\in C_c^\infty(M,SM)$[1]

$$\int_M \langle D\psi,\phi\rangle_{SM}+\langle\psi,D\phi\rangle_{SM}\,\mathrm{d}\mu_M=-\int_{\partial M}\langle\gamma(\eta)\psi,\phi\rangle_{SM}\,\mathrm{d}\mu_{\partial M}, \tag{2.2.4}$$

where $\mathrm{d}\mu_M$ is the volume element on M with respect to g and $\mathrm{d}\mu_{\partial M}$ is the induced one on ∂M.

Remark 2.2.2. Note that Equation 2.2.4, does not imply that D is formally anti-selfadjoint or symmetric. Since the Green formula above is using an inner product that is not positive definite, the integral over it does not define a L^2-scalar product on M. Hence, talking about formally anti-selfadjointness in this case does not make sense.

Using the Gauß formula for ∇^{SM}, i. e.

$$\nabla^{SM}_X\psi=\nabla^{S\Sigma_t}_X\psi-\frac{1}{2}\gamma(\nu)\gamma(W(\cdot))\psi,$$

where W is the Weingarten map of the Levi-Civita connection on the tangent bundle, we get

$$D=-\gamma(\nu)\left[\nabla^{SM}_\nu+iD_t-\frac{n}{2}H_t\right], \tag{2.2.5}$$

where H_t is the mean curvature of Σ_t with respect to ν and $D_t=\not{D}_t$ or $D_t=\begin{pmatrix}\not{D}_t & 0\\ 0 & -\not{D}_t\end{pmatrix}$ for n even and odd, respectively, where $\not{D}_t$ is the Riemannian spin Dirac operator on $S\Sigma_t$. Hence, D_t is a Dirac type operator (see 3.1.4) and is in particular elliptic with principal symbol of D_t is given by

$$\sigma_{D_t}(\zeta)=-i\sigma_D(\nu)\sigma_D(\zeta),$$

for $\zeta\in T_x\Sigma_t$.

[1] $C_c^\infty(M,SM)$ is the space of smooth spinors with compact support

Remark 2.2.3. Note that the principal symbol σ_{D_t} is symmetric with respect to $\langle\cdot,\cdot\rangle_{SM}$ but is skew-symmetric with respect to $\langle\cdot,\cdot\rangle_0$. Furthermore, D_t is formally selfadjoint with respect to the L^2-scalar product arising from $\langle\cdot,\cdot\rangle_0$.

2.3 The spin Dirac operator on closed Riemannian manifolds

Since D_t is a Dirac operator in the sense of Gromov and Lawson, see [9], or simply is the Riemannian Dirac operator, one can discuss the splitting along the hypersurface $\partial\Sigma_t \subseteq \Sigma_t$, which is given as follows

$$D_t = \sigma_{D_t}(\eta_t^\flat)^{-1}\left(\nabla^{S\Sigma_t}_{\eta_t} + A_t - \frac{n-1}{2}H_t^{\partial\Sigma_t}\right), \tag{2.3.1}$$

where $\eta_t := \eta(t,\cdot)$ is the unit normal field η to the timelike boundary ∂M restricted to Σ_t[2], $H_t^{\partial\Sigma_t}$ is the mean curvature of $\partial\Sigma_t$ inside Σ_t with respect to η_t and A_t is the double of the Riemannian Dirac operator on $\partial\Sigma_t$. Since $\partial\Sigma_t$ is a closed manifold, we see that A_t is essentially selfadjoint.

For later use, let us recall one important fact on Dirac operators on closed Riemannian manifolds.

Theorem 2.3.1. *Let X be a closed Riemannian spin manifold with Dirac operator A. Then the spectrum of A is real and discrete. Furthermore, all eigenspaces $E(A,\lambda)$ of A are finite dimensional, consist of smooth sections, and one has*

$$L^2(X,SX) = \overline{\bigoplus_\lambda E_\lambda(A)}^{L^2}$$

as Hilbert space sum decomposition.

The proof of this theorem is classical, but can, for example, be found in [21]. Furthermore, note that Theorem 2.3.1 is still true for selfadjoint elliptic first order operators on closed manifolds.

[2]Since ν is tangential to ∂M, i. e. $\eta \perp \nu$, we see that $\eta(t,\cdot)$ is tangent to Σ_t for all t.

2.4 Standard setup on spacetimes with timelike boundary

Except we mention otherwise, we are working in the following setting:

- (M, g) is a globally hyperbolic spatially compact spin manifold with timelike boundary ∂M; η is the interior unit normal field to ∂M,
- Σ is a compact spacelike Cauchy hypersurface of M,
- $t\colon M \to \mathbb{R}$ is a temporal function with $\Sigma_0 = \Sigma$ and its gradient being tangential to ∂M; ν is the past pointing unit normal field perpendicular to the Σ_t,
- with respect to t, the metric is given by $g = -N^2\,\mathrm{d}t^2 + g_t$ and $\{\Sigma_t\}_{t\in\mathbb{R}}$ is the smooth foliation of M by compact Riemannian manifolds with boundary as in Theorem 2.1.3,
- $D\colon C^\infty(M, SM) \to C^\infty(M, SM)$ is the Lorentzian spin Dirac operator, and
- $D_t\colon C^\infty(\Sigma_t, SM|_{\Sigma_t}) \to C^\infty(\Sigma_t, SM|_{\Sigma_t})$ is the induced Dirac type operator on Σ_t.

3

Boundary Conditions

In this chapter, we will discuss non-local boundary conditions, first for Riemannian manifolds and then for spacetimes with timelike boundary. This will be done in three parts. First, we will summarize known results for the Riemannian setting (see [8, 9, 10]). Next, we will have some more detailed technical discussion regarding elliptic regularity (see Subsection 3.1.3) and continuity of of functional calculus depending on families of operators and boundary conditions (see Section 3.2), respectively. In the last part of this chapter we will use the first two parts to define Lorentzian boundary conditions.

3.1 Boundary conditions for elliptic operators

The goal of this section is to give a brief summary of the theory of non-local boundary conditions. We will restrict ourselves only to a special case, but the theory can be done in more generality.

3.1.1 Motivation

Before discussing the theory of non-local boundary conditions in the following subsections, let us spend a little bit of time on the question why we are interested in these kind of boundary conditions for first order operators. For this we follow [38], that includes a nice overview on known results regarding index theory for manifolds with boundary and corners. Since we did not introduce any theory yet, we will not go into too many details here.
Let us start with a brief discussion of Fredholm operators on *closed* Riemannian manifolds. First, recall the abstract definition of a Fredholm operator on Hilbert spaces.

Definition 3.1.1 (Fredholm)**.** Let $\mathcal{H}_i$ be Hilbert spaces for $i = 1, 2$ and let $D\colon \operatorname{dom}(D) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$ be an unbounded operator with domain $\operatorname{dom}(D)$ being dense in $\mathcal{H}_1$. Then D is called a *Fredholm operator* if and only if

1. its range $\operatorname{ran}(D)$ is closed in $\mathcal{H}_2$,
2. its kernel is finite dimensional, and
3. its cokernel $\operatorname{coker} D := \mathcal{H}_2/\operatorname{ran} D$ is also finite dimensional.

If D is a Fredholm operator then we call the integer

$$\operatorname{ind}(D) := \dim(\ker D) - \dim(\operatorname{coker} D)$$

the *index* of D.

Example 3.1.2. Let M be a 2-dimensional closed Riemannian manifold. Then the *Gauß–Bonnet operator*[1]

$$D_{GB} := d + d^*\colon H^1(M, \Lambda^{ev} M) \to L^2(M, \Lambda^{odd} M)$$

is Fredholm and has index

$$\operatorname{ind}(D_{GB}) = \chi(M) = \frac{1}{2\pi} \int_M K,$$

where $\chi(M)$ is the Euler characteristic and K is the Gauß curvature of M.

The generalization of the example above is the Atiyah–Singer index theorem [4], which is one of the main mathematical achievements of the 20th century due to its many applications and because of the conceptual insights it provides. Atiyah and Singer [4] computed the index of an elliptic first order operator on a compact manifold without boundary and in particular show that every Dirac type operator (see Definition 3.1.4) is Fredholm from H^1 to L^2.

If M is a compact Riemannian manifold with boundary, then the situation is totally different. One can see that a Dirac type operator $D\colon H^1 \to L^2$ can never be Fredholm. In fact, it is surjective and has infinite dimensional kernel, which is shown for example in [20]. This is the point where boundary conditions come into play. The choice of boundary conditions for first order operators is more involved than for operators of order two. For an intuition, consider the following example.

Example 3.1.3. Let $M = \overline{B_1(0)} \subseteq \mathbb{R}^2$ be the unit disk with Riemannian metric $g = \mathrm{d}r^2 + \mathrm{d}\theta^2$ in polar coordinates (r, θ). The elliptic first order operator $D\colon C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C})$ with

$$D = \partial_r + i\partial_\theta,$$

[1] or Euler operator

has kernel consisting of all holomorphic functions on M. Hence, the kernel is clearly infinite dimensional and in particular the operator cannot be Fredholm. But on the other hand the real and imaginary parts of an element in the kernel are harmonic and they determine each other up to a constant. Thus for most smooth functions $h : \partial M \to \mathbb{C}$ on the boundary the Dirichlet problem

$$\begin{cases} Df = 0 \\ f|_{\partial M} = h \end{cases}$$

is not solvable.

The example above illustrates that local boundary conditions are often "too strong" for first order operators. In the following subsections we will discuss which boundary conditions are "good" in the sense of restricting the domain in such a way that the operator becomes Fredholm.

3.1.2 The range of the restriction map and boundary conditions

In this subsection we will introduce non-local boundary conditions for elliptic first order operators. The general theory in [8, 10] is for general elliptic operators on Riemannian manifolds with compact boundary. For our purpose the case of formally selfadjoint Dirac type operators on compact Riemannian manifolds will be sufficient, so we will reduce the theory to this class of operators. In the following subsections, we will mostly follow [8, 9].

Let (Σ, g) be a compact Riemannian manifold with smooth boundary $\partial\Sigma$, whose unit conormal field will be denoted by τ. Furthermore, let $(E, h_E) \to \Sigma$ be a Hermitian bundle.

Definition 3.1.4 (formally selfadjoint Dirac type operator)**.**
Let $D : C^\infty(\Sigma, E) \to C^\infty(\Sigma, E)$ be a formally selfadjoint differential operator of order one. Then we call D a *Dirac type operator* if its principal symbol σ_D satisfies the *Clifford relations*

$$\sigma_D(\zeta)\sigma_D(\eta) + \sigma_D(\eta)\sigma_D(\zeta) = -2g(\zeta, \eta) \cdot id_{E_x}, \tag{3.1.1}$$

for all $x \in \Sigma$ and $\zeta, \eta \in T^*_x\Sigma$.

Remark 3.1.5. The Riemannian spin Dirac operator is an important example for a Dirac type operator. Another important subclass of operators are the Dirac operators in the sense of Gromov and Lawson as in [30], [36] and [9].

Note, that these operators are *elliptic*, i. e. $\sigma_D(\zeta)$ is invertible for all $x \in \Sigma$ and all $0 \neq \zeta \in T^*_x\Sigma$, since by the Clifford relations the inverse can be explicitly written down as

$$\sigma_D(\zeta)^{-1} = -|\zeta|_g^{-2}\sigma_D(\zeta), \quad 0 \neq \zeta \in T^*_x\Sigma. \tag{3.1.2}$$

Before discussing boundary conditions for Dirac type operators, let us spend some time on introducing the maximal and minimal domain of an operator and the range of the restriction map to the boundary mapping from these domains.

The *graph norm* $\|\cdot\|_D$ is defined by

$$\|\cdot\|_D^2 := \|D\cdot\|_{L^2}^2 + \|\cdot\|_{L^2}^2$$

and the *maximal domain* by

$$\mathrm{dom}(D_{\max}) := \{\psi \in L^2(\Sigma, E);\ D\psi \in L^2(\Sigma, E)\}.$$

The *minimal domain* is the closure of $C^\infty_{cc}(\Sigma, E)$ with respect to the graph norm of D, where $C^\infty_{cc}(\Sigma, E)$ is the space of smooth sections compactly supported in the interior of Σ, i. e. $\operatorname{supp}\psi \cap \partial\Sigma = \emptyset$ for $\psi \in C^\infty(\Sigma, E)$. The maximal operator $D_{\max}$ and the minimal operator $D_{\min}$ are the maximal and minimal closed extensions of $D\colon\ C^\infty_{cc}(\Sigma, E) \to C^\infty_{cc}(\Sigma, E)$. In particular, $(\mathrm{dom}(D_{\max}), \|\cdot\|_D)$ and $(\mathrm{dom}(D_{\min}), \|\cdot\|_D)$ are Banach spaces.

Now we will introduce the candidate for the range of the restriction map to the boundary mapping from the maximal domain. For this, we start by discussing the boundary operator.

Definition 3.1.6. An operator $A : C^\infty(\partial\Sigma, E|_{\partial\Sigma}) \to C^\infty(\partial\Sigma, E|_{\partial\Sigma})$ is called a *boundary operator* for D if its principal symbol is given by

$$\sigma_A(x, \zeta) = \sigma_D(x, \tau(x))^{-1} \circ \sigma_D(x, \zeta)$$

for all $x \in \partial\Sigma$ and $\zeta \in T^*_x\partial\Sigma$.

Remark 3.1.7.

1. In this setting, we can choose A to anti-commute with $\sigma_D(\tau)$ and selfadjoint, see Lemma 2.2. in [9], but in general that is not necessarily the case. For Dirac operators in the sense of Gromov and Lawson there is a natural choice of a selfadjoint boundary operator A, which anti-commutes with $\sigma_D(\tau)$. This operator has a lower order term depending on the mean curvature of the boundary, see for example [9]. This can also be seen directly for the spin Dirac operator as we discussed in Section 2.3.

2. The compactness of the boundary implies that the elliptic selfadjoint operator A has discrete and real spectrum, see Theorem 2.3.1. If $\partial\Sigma$ is not compact the behavior of the spectrum of A can be hard to control. In the case of non-compact boundary, [32] discussed boundary values of the Dirac operator associated with a spin^c-structure, when Σ and $\partial\Sigma$ are complete and geometrically bounded in a suitable way.

Let $r \notin \operatorname{spec}(A)$ and define the operator $A_{(r)} := A - r$, which is then invertible.
Let $\chi^{\pm}(A_{(r)}) \colon L^2(\partial\Sigma, E|_{\partial\Sigma}) \to L^2(\partial\Sigma, E|_{\partial\Sigma})$ be the spectral projections on the spectral subspaces corresponding to the eigenvalues with positive and negative real parts, respectively. These are pseudo-differential operators of order zero and therefore

$$\chi^{\pm}(A_{(r)})H^s(\partial\Sigma, E|_{\partial\Sigma})$$

are closed subspaces of the Sobolev spaces $H^s(\partial\Sigma, E|_{\partial\Sigma})$ for all $s \in \mathbb{R}$.

Definition 3.1.8. We define the *check space* corresponding to the boundary operator A as

$$\check{H}(A_{(r)}) := \chi^-(A_{(r)})H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma}) \oplus \chi^+(A_{(r)})H^{-\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma}),$$

with norm

$$\|\psi\|^2_{\check{H}(A_{(r)})} := \left\|\chi^-(A_{(r)})\psi\right\|^2_{H^{\frac{1}{2}}} + \left\|\chi^+(A_{(r)})\psi\right\|^2_{H^{-\frac{1}{2}}}.$$

Since the check space does not depend on the chosen $r \notin \operatorname{spec}(A)$, see [8], in the following we will drop the r in the notation of the space. Using the check space, Bär and Ballmann [8] showed, that one can uniquely extend the trace map[2] $R \colon C^\infty(\Sigma, E) \to C^\infty(\partial\Sigma, E|_{\partial\Sigma})$, $\psi \mapsto \psi|_{\partial\Sigma}$ to the maximal domain of the operator, such that the image of the extension is the check space. More precisely:

Theorem 3.1.9 (Theorem 6.7 in [8]). *In the setting introduced above, the following claims hold:*

1. $C^\infty(\Sigma, E)$ *is dense in* $\operatorname{dom}(D_{\max})$ *with respect to* $\|\cdot\|_D$,
2. *the trace map extends uniquely to a continuous surjection* $R \colon \operatorname{dom}(D_{\max}) \to \check{H}(A)$ *with kernel* $\ker R = \operatorname{dom}(D_{\min})$*, and in particular* R *induces an isomorphism*

$$\check{H}(A) \cong {}^{\operatorname{dom}(D_{\max})}\!\big/\!{}_{\operatorname{dom}(D_{\min})},$$

3. *for all* $\phi, \psi \in \operatorname{dom}(D_{\max})$

$$\int_\Sigma h_E(D_{\max}\phi, \psi) - h_E(\phi, D_{\max}\psi)\,\mathrm{d}\mu_\Sigma = -\int_{\partial\Sigma} h_E(\sigma_D(\tau)R\phi, R\psi)\,\mathrm{d}\mu_{\partial\Sigma}, \text{ and} \tag{3.1.3}$$

4. $H^1(\Sigma, E) \cap \operatorname{dom}(D_{\max}) = \{\psi \in \operatorname{dom}(D_{\max});\ R\psi \in H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma})\}$.

[2] Or restriction map.

Note that the pairing in Equation 3.1.3 is well defined because $\sigma_D(\tau)$ maps $\check{H}(A)$ to $\check{H}(-A)$, since it anti-commutes with A.

Theorem 3.1.9 gives us the necessary tools to define boundary conditions:

Definition 3.1.10. A *boundary condition* is a closed linear subspace $B \subseteq \check{H}(A)$. The domains of the associated operators are

$$\begin{aligned} \mathrm{dom}(D_{\max,B}) &:= \{\psi \in \mathrm{dom}(D_{\max});\ R\psi \in B\}, \quad \text{and} \\ \mathrm{dom}(D_B) &:= \{\psi \in \mathrm{dom}(D_{\max}) \cap H^1(\Sigma, E);\ R\psi \in B\}. \end{aligned}$$

Remark 3.1.11. Since $R : {}^{\mathrm{dom}(D_{\max})}\!/_{\mathrm{dom}(D_{\min})} \to \check{H}(A)$ is an isomorphism, the check space $\check{H}(A)$ as a topological space does not depend on the choice of boundary operator A. Furthermore, we have a one to one relation between boundary conditions and closed extensions of D between the minimal and maximal domain. Hence, $(\mathrm{dom}(D_{\max,B}), \|\cdot\|_D)$ is a Banach space for all boundary conditions B.
Moreover, a boundary condition B satisfies $B \subseteq H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma})$ if and only if $D_B = D_{\max,B}$.

Motivated by the third part of Theorem 3.1.9, the associated *adjoint boundary condition* denoted by B^* is given by

$$B^* := \{\phi \in \check{H}(A);\ \int_{\partial\Sigma} h_E(\sigma_D(\tau)\psi, \phi)\,\mathrm{d}\mu_{\partial\Sigma} = 0\ \forall\, \psi \in B\}.$$

We call a boundary condition B *selfadjoint*, if $B^* = B$. By Subsection 7.2 in [8], the domain of the adjoint of $D_{\max,B}$ is given by

$$\mathrm{dom}((D_{\max,B})^*) = \{\psi \in \mathrm{dom}(D_{\max});\ \psi|_{\partial\Sigma} \in B^*\} = \mathrm{dom}(D_{\max,B^*}).$$

In particular, if B is a selfadjoint boundary condition then $D_{\max,B}$ is a selfajoint operator.

3.1.3 Elliptic and ∞-regular boundary conditions

The goal of this subsection is to discuss the connection between boundary conditions and the regularity of sections in the corresponding domains. For this we will consider elliptic and ∞-regular boundary condition, which we will introduce in the following.
Let us denote $V^s := V \cap H^s(\partial\Sigma, E|_{\partial\Sigma})$ for $V \subseteq L^2(\partial\Sigma, E|_{\partial\Sigma})$ and $s \in \mathbb{R}$.

Definition 3.1.12. Let $B \subseteq H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma})$ be a linear subspace and $r \notin \mathrm{spec}(A)$. Suppose

1. $W_\pm, V_\pm$ are mutually complementary subspaces of $L^2(\partial\Sigma, E|_{\partial\Sigma})$ such that

$$V_\pm \oplus W_\pm = \chi^\pm(A_{(r)})L^2(\partial\Sigma, E|_{\partial\Sigma}),$$

2. $W_\pm$ are finite dimensional with $W_\pm, W_\pm^* \subseteq H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma})$, and

3. there exists a bounded linear map $g : V_- \to V_+$ with $g(V_-^{\frac{1}{2}}) \subseteq V_+^{\frac{1}{2}}$ and $g^*((V_+^*)^{\frac{1}{2}}) \subseteq (V_-^*)^{\frac{1}{2}}$ such that
$$B = W_+ \oplus \{v + gv; v \in V_-^{\frac{1}{2}}\}.$$

Then we say that B can be *elliptically decomposed with respect to* r.

Remark 3.1.13. A priori Definition 3.1.12 depends on $r \notin \operatorname{spec}(A)$, but Theorem 2.9 in [10] shows that if B can be elliptically decomposed with respect to some r then it can be with respect to all.

Definition 3.1.14. Let $B \subseteq H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma})$ be a linear subspace such that $B \subseteq \check{H}(A)$ is closed and $B^* \subseteq H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma})$, then B is called *elliptic*.

The following result proven in [8], shows that B being elliptic is equivalent to B being elliptically decomposed:

Theorem 3.1.15 (Theorem 7.11 [8])**.** *Let $B \subseteq H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma})$ be a subspace. Then the following are equivalent:*

1. $\operatorname{dom}(D_{\max,B}) \subseteq H^1(\Sigma, E)$ *and* $\operatorname{dom}(D_{\max,B^*}) \subseteq H^1(\Sigma, E)$,
2. *B is elliptic,*
3. *B can be elliptically decomposed.*

Moreover, for an elliptic boundary condition B, we have that B^* is an elliptic boundary condition as well.

The following is a direct consequence of Theorem 3.1.15:

Corollary 3.1.16. *Let B be an elliptic boundary condition. Then* $\operatorname{dom}(D_{\max,B})$ *is a closed subspace of $H^1(\Sigma, E)$. Moreover, $\|\cdot\|_D$ and $\|\cdot\|_{H^1}$ are equivalent on* $\operatorname{dom}(D_{\max,B})$.

Proof. We first show that $\operatorname{dom}(D_{\max,B})$ is closed in $H^1(\Sigma, E)$. Let
$$\{\psi_n\}_{n\in\mathbb{N}} \subseteq \operatorname{dom}(D_{\max,B}) \subseteq H^1(\Sigma, E) \subseteq \operatorname{dom}(D_{\max})$$
such that $\psi_n \to \psi$ in $H^1(\Sigma, E)$. Since $\|\cdot\|_D \leq C\|\cdot\|_{H^1}$ and $(\operatorname{dom}(D_{\max,B}), \|\cdot\|_D)$ is a Banach space, we know that $\psi_n \to \psi \in \operatorname{dom}(D_{\max,B})$ and, thus, $\operatorname{dom}(D_{\max,B})$ is closed in $H^1(\Sigma, E)$.
In particular, $(\operatorname{dom}(D_{\max,B}), \|\cdot\|_D)$ and $(\operatorname{dom}(D_{\max,B}), \|\cdot\|_{H^1})$ are both Banach spaces with $\|\cdot\|_D \leq C\|\cdot\|_{H^1}$. Hence, the identity map $id : (\operatorname{dom}(D_{\max,B}), \|\cdot\|_{H^1}) \to (\operatorname{dom}(D_{\max,B}), \|\cdot\|_D)$ is continuous and bijective. By the open mapping theorem this is an isomorphism, which implies the second claim of the corollary. □

The closedness of the domain of D_B for B being an elliptic boundary condition implies in particular the following: Recall that the initial motivation for defining non-local boundary conditions was to analyze the Fredholmness for elliptic operators on manifolds with boundary. It turns out that elliptic boundary conditions are the right ones to look at:

Theorem 3.1.17 (Theorem 8.5 [8])**.** *Let $B \subseteq H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma})$ be an elliptic boundary condition for D. Then*

$$D_B : \operatorname{dom}(D_B) \to L^2(\Sigma, E)$$

is a Fredholm operator and its index is given by

$$\operatorname{ind}(D_B) = \dim(\ker(D_B)) - \dim(\ker D_{B^*}) \in \mathbb{Z}.$$

In particular, for B being a selfadjoint boundary condition the index of D_B vanishes.

In Subsection 3.3.3 we will define admissible boundary conditions, where we consider for each timeslice a selfadjoint boundary condition. Hence, let us briefly discuss the existence of selfadjoint boundary conditions for formally selfadjoint Dirac type operators.

Theorem 3.1.18 (Theorem 3.12 [9])**.** *Let $D : C^\infty(\Sigma, E) \to C^\infty(\Sigma, E)$ be a formally selfadjoint Dirac type operator and let A be a selfadjoint boundary operator that anti-commutes with $\sigma_D(\tau)$. Then an elliptic boundary condition B is selfadjoint if and only if there is*

1. *an orthogonal decomposition $\chi^-(A)L^2_{(-\infty,0)}(\partial\Sigma, E|_{\partial\Sigma}) = V \oplus W$, where W is a finite dimensional subspace of $C^\infty(\partial\Sigma, E|_{\partial\Sigma})$,*
2. *an orthogonal decomposition* $\ker A = L \oplus \sigma_D(\tau)L$*,*
3. *and a selfadjoint operator $g : V \oplus L \to V \oplus L$ of order zero such that*

$$B = \sigma_D(\tau)W \oplus \{v + \sigma_D(\tau)gv;\ v \in V^{\frac{1}{2}} \oplus L\}.$$

Remark 3.1.19. Note that in Theorem 3.1.18, the case of trivial kernel of A is not excluded. In this case, the representation of B is unique since $V = \chi^-(A)B$ and W is the orthogonal complement in $\chi^-(A)L^2(\partial\Sigma, E|_{\partial\Sigma})$. Furthermore, the orthogonal decomposition of the kernel requires that the kernel of A is even dimensional.

The following definition gives us a useful notion to allow for higher regularity of solutions up to the boundary:

Definition 3.1.20. We say an elliptic boundary condition B is ∞-*semi-regular* if $W_+ \subseteq H^s(\partial\Sigma, E|_{\partial\Sigma})$ and $g(V_-^s) \subseteq V_+^s$ for all $s \geq \frac{1}{2}$. Here $W_\pm, V_\pm$ and g are as in Definition 3.1.12. If, in addition, B^* is also ∞-semi-regular, then we say B is ∞-*regular*.

This class of boundary conditions can now be used for higher boundary regularity:

Theorem 3.1.21 (Theorem 7.17 in [8]). *Let B be an ∞-regular boundary condition, then*

$$D_{\max}\psi \in H^k(\Sigma, E) \quad \Leftrightarrow \quad \psi \in H^{k+1}(\Sigma, E).$$

for all $k \in \mathbb{N}$ and $\psi \in \mathrm{dom}(D_B)$.

Proceeding, we will use Theorem 3.1.21 to define closed subspaces of Sobolev spaces with respect to ∞-regular *selfadjoint* boundary conditions. Let B be a fixed ∞-regular selfadjoint boundary condition. Then we define the operator

$$\Delta_B := Id + D_B^2$$

with domain

$$\begin{aligned}
\mathrm{dom}(\Delta_B) &= \mathrm{dom}(Id) \cap \mathrm{dom}(D_B^2) \\
&= \mathrm{dom}(D_B^2) \\
&= \{\psi \in \mathrm{dom}(D_{\max,B});\, D\psi \in \mathrm{dom}(D_{\max,B})\} \\
&= \{\psi \in \mathrm{dom}(D_{\max,B});\, D\psi \in \mathrm{dom}(D_{\max,B}) \cap H^1(\Sigma, E)\} \\
&= \{\psi \in \mathrm{dom}(D_{\max,B}) \cap H^2(\Sigma, E);\, R(D\psi) \in B\} \\
&= \{\psi \in H^2(\Sigma, E);\; R(\psi) \in B \text{ and } R(D\psi) \in B\} \subseteq H^2(\Sigma, E)
\end{aligned}$$

Let us summarize the most important properties of Δ_B in the following Lemma. The proof is based on methods used in [6], where similar results are shown for elliptic operators on closed manifolds.

Lemma 3.1.22.

1. *The operator Δ_B is a selfadjoint Laplace type operator,*
2. *The domain $\mathrm{dom}(\Delta_B)$ is closed in $H^2(\Sigma, E)$ with $\|\cdot\|_{B,2} := \|\Delta_B \cdot\|_{L^2} \simeq \|\cdot\|_{H^2}$ on $\mathrm{dom}(\Delta_B)$.*
3. *The spectrum of Δ_B is discrete and the eigenspaces are finite dimensional and consist of smooth sections.*

Proof.

1. Since D is a formally selfadjoint Dirac type operator and B is a selfadjoint boundary condition, Δ_B is a selfadjoint Laplace type operator by functional calculus.

2. Let $\psi_n \in \mathrm{dom}(\Delta_B) \subseteq \mathrm{dom}(D_{\max,B})$ with $\psi_n \to \psi \in H^2(\Sigma, E)$ with respect to $\|\cdot\|_{H^2}$, then for $m < n$
$$\left\|\psi_n - \psi_m\right\|_D \leq C_1 \left\|\psi_n - \psi_m\right\|_{H^1} \leq C_2 \left\|\psi_n - \psi_m\right\|_{H^2},$$
where C_i are independent of ψ and n, m. Hence, $\{\psi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_D$ and since $(\mathrm{dom}(D_{\max,B}), \|\cdot\|_D)$ is a Banach space, we directly know that $\psi_n \to \psi \in \mathrm{dom}(D_{\max,B})$. Similarly, one sees for $m < n$
$$\left\|D(\psi_n - \psi_m)\right\|_D \leq C_3 \left\|D(\psi_n - \psi_m)\right\|_{H^1} \leq C_4 \left\|\psi_n - \psi_m\right\|_{H^2},$$
hence $D\psi_n$ is again a Cauchy sequence with respect to $\|\cdot\|_D$. Since $\{D\psi_n\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}(D_{\max,B})$ we can again conclude that $D\psi \in \mathrm{dom}(D_{\max,B})$. Thus, we already know that
$$\psi \in \{\phi \in \mathrm{dom}(D_{\max,B});\, D\phi \in \mathrm{dom}(D_{\max,B})\} = \mathrm{dom}(\Delta_B),$$
and $\mathrm{dom}(\Delta_B)$ is closed in $H^2(\Sigma, E)$. Since Δ_B is a second order differential operator, we see that $\left\|\Delta_B \cdot\right\|_{L^2} \leq C \left\|\cdot\right\|_{H^2}$. Then the open mapping theorem implies (similar as in Corollary 3.1.16) that $\|\cdot\|_{B,2} \simeq \|\cdot\|_{H^2}$.

3. Since $\mathrm{res}(\Delta_B) \neq \emptyset$, we have $\zeta \in \mathrm{res}(\Delta_B)$ and we can factor the map $(\zeta - \Delta_B)^{-1} : L^2(\Sigma, E) \to L^2(\Sigma, E)$ as
$$(\zeta - \Delta_B)^{-1} : L^2(\Sigma, E) \to \mathrm{dom}(\Delta_B) \overset{\text{closed}}{\subseteq} H^2(\Sigma, E) \overset{\text{compact}}{\hookrightarrow} L^2(\Sigma, E).$$
Hence, $(\zeta - \Delta_B)^{-1}$ is a compact operator by applying Theorem 4.8 in Chapter III Section 4 in [35]. Then Theorem 6.29 in Chapter II Section 6 in [35] guarantees discrete isolated spectrum and that the eigenspaces are finite dimensional. Since B is ∞-regular, Theorem 3.1.21 implies that the eigensections are smooth up to the boundary.

$\square$

Remark 3.1.23. Let us briefly recall some functional analytic theory. For $\mathcal{H}$ being a Hilbert space and $T : \mathrm{dom}(T) \subseteq \mathcal{H} \to \mathcal{H}$ being selfadjoint and densely-defined, there is a spectral measure $d E_T$ valued in $\mathcal{H}$ such that
$$Tu = \int_{\mathbb{R}} \lambda \, d E_T(\lambda)[u]$$
for all $u \in \mathrm{dom}(T)$. The existence of this measure, often called *the spectral theorem*, is an important result of functional analysis.

For a continuous function $f : \mathbb{R} \to \mathbb{R}$, we then can define
$$\mathrm{dom}(f(T)) := \left\{ u \in \mathcal{H};\ \forall\, R > 0 : \int_{-R}^{R} |f(\lambda)|^2 \left\|d E_T(\lambda)[u]\right\|_{L^2}^2 < C \right\}, \text{ and}$$

$$f(T)u := \int_{\mathbb{R}} f(\lambda)\, dE_T(\lambda)[u].$$

Furthermore, if $\|f\|_{L^\infty} < \infty$, then $f(T)$ is a bounded operator on $\mathcal{H}$ and

$$\|f(T)\|_{\mathcal{H},\mathcal{H}} \leq \|f\|_{L^\infty},$$

where $\|\cdot\|_{\mathcal{H},\mathcal{H}}$ is the operator norm for operators that map from $\mathcal{H}$ to itself. A detailed discussion can, for example, be found in Chapter 13 in [43] by Rudin.

Applying Remark 3.1.23 to Δ_B and noting that $\operatorname{spec}(\Delta_B) = \{\lambda_i;\ i \in \mathbb{N}\} \subseteq [1, \infty)$, where we count multiplicity, we see that for $u \in \operatorname{dom}(\Delta_B)$

$$\Delta_B u = \int_{\mathbb{R}} \lambda\, dE_{\Delta_B}(\lambda)[u] = \sum_{i=1}^{\infty} \lambda_i\, P_{\lambda_i}[u]$$

where P_{λ_i} is the projection on the eigenspace of λ_i. Furthermore, we see that for any continuous $f : [1, \infty) \to \mathbb{R}$, we have

$$\operatorname{dom}(f(\Delta_B)) := \left\{ u \in L^2(\Sigma, E);\ \sum_{i=1}^{\infty} |f(\lambda_i)|^2 \left\| P_{\lambda_i}[u] \right\|_{L^2}^2 < \infty \right\}, \text{ and}$$

$$f(\Delta_B)u := \sum_{i=1}^{\infty} f(\lambda_i)\, P_{\lambda_i}[u].$$

For defining the closed subspaces of $H^k(\Sigma, E)$, let us consider the explicit continuous function f given by $f : [1, \infty) \to [1, \infty); x \mapsto x^{\frac{k}{2}}$ for $k \in \mathbb{N}$. Then we can define the following:

Definition 3.1.24. Let B be an ∞-regular boundary condition, then we can define for $k \in \mathbb{N}$

$$H_B^k(\Sigma, E) := \operatorname{dom}(\Delta_B^{\frac{k}{2}}) = \left\{ u \in L^2(\Sigma, E);\ \sum_{i=1}^{\infty} \lambda_i^k \left\| P_{\lambda_i}[u] \right\|_{L^2}^2 < \infty \right\}$$

and with scalar product

$$\langle \psi, \phi \rangle_{B,k} := \left\langle \Delta_B^k \psi, \phi \right\rangle_{L^2(\Sigma,E)}.$$

Remark 3.1.25. Similar to the discussion before Lemma 3.1.22, one can characterize the spaces $H_B^k(\Sigma, E)$ also as

$$\begin{aligned} H_B^k(\Sigma, E) &= \operatorname{dom}(\Delta_B^{\frac{k}{2}}) = \operatorname{dom}(D_B^k) \\ &= \{\psi \in H^k(\Sigma, E);\ R(D^l \psi) \in B \ \text{ for all } 0 \leq l \leq k-1\}. \end{aligned}$$

The next lemma shows that these spaces are indeed closed in the corresponding Sobolev spaces, and hence are Banach spaces themselves:

Lemma 3.1.26.

1. *$\Delta^{\frac{k}{2}}$ is a closed densely defined operator for all $k \in \mathbb{N}$, in particular $H^\infty_B(\Sigma, E) := \bigcap_{k=1}^\infty H^k_B(\Sigma, E)$ is a dense subspace of $L^2(\Sigma, E)$,*

2. *$H^l_B(\Sigma, E)$ is contained in $H^k_B(\Sigma, E)$ for all $k, l \in \mathbb{N}$ with $k < l$, and*

3. *$H^k_B(\Sigma, E)$ is closed in $H^k(\Sigma, E)$ with $\|\cdot\|_{B,k} \simeq \|\cdot\|_{H^k}$ on $H^k_B(\Sigma, E)$ for all $k \in \mathbb{N}$.*

4. *On $H^k_B(\Sigma, E)$, we have*

$$\|\cdot\|^2_{k,B} \leq C \left(\|\cdot\|^2_{B,k-1} + \|D\cdot\|^2_{B,k-1} \right), \tag{3.1.4}$$

   for a constant $C > 0$.

Proof. The first two points directly follow from Theorem 6.8 in Chapter 2 in [40]. Hence, we only have to show the last two claims. The third claims follows by Remark 3.1.25 and a similar argument as in the proof of Lemma 3.1.22. For the fourth claim, let $\psi \in H^k_B(\Sigma, E)$ and estimate

$$\begin{aligned}
\|\psi\|^2_{k,B} &= \left\langle (Id + D_B^2)^k \psi \right\rangle_{L^2} \\
&= \sum_{l=0}^{k} \left\langle \binom{k}{l} D^{2l}\psi, \psi \right\rangle_{L^2} \\
&= \left\| D^k \psi \right\|^2_{L^2} + \sum_{l=0}^{k-1} \binom{k}{l} \left\| D^l \psi \right\|^2_{L^2} \\
&\leq C_1 \left(\|D\psi\|^2_{H^{k-1}} + \|\psi\|^2_{H^{k-1}} \right) \\
&\leq C_2 \left(\|D\psi\|^2_{B,k-1} + \|\psi\|^2_{B,k-1} \right),
\end{aligned}$$

where we used that Δ_B and D (as well as their powers) are symmetric and the third claim of the same lemma. □

Remark 3.1.27. Note that the third and fourth part of Lemma 3.1.26 in particular imply that on $H^k_B(\Sigma, E) \subseteq H^{k-1}_B(\Sigma, E)$ we have the estimate

$$\|\cdot\|^2_{H^k} \leq C \left(\|\cdot\|^2_{H^{k-1}} + \|D\cdot\|^2_{H^{k-1}} \right). \tag{3.1.5}$$

Moreover, the third part of Lemma 3.1.26 in particular implies that a differential operator $D : C^\infty(\Sigma, E) \to C^\infty(\Sigma, E)$ of order l is bounded seen as

$$D : H_B^{k+l}(\Sigma, E) \to H^k(\Sigma, E).$$

It is also important to mention that $C_{cc}^\infty(\Sigma, E) \subseteq H_B^\infty(\Sigma, E)$ by construction. This will be important later, when we will look at the well-posedness of certain Cauchy problems in Section 4.3.

Later we use mollifiers to define regularized versions of our Cauchy problems, see Section 4.3. Thus, let us discuss the exponential semigroup with respect to Δ_B as a convenient example of mollifiers.

Let us consider the family of continuous functions $f_\varepsilon : [1, \infty) \to [0, \infty);\ \ x \mapsto e^{-\varepsilon x}$ for $\varepsilon > 0$ and apply to it Remark 3.1.23. Since f_ε is in L^∞ for all $\varepsilon > 0$ with norm equals to $e^{-\varepsilon}$, we see that $J_B^{(\varepsilon)} := f_\varepsilon(\Delta_B)$ is a bounded operator on L^2 with operator norm smaller or equal $e^{-\varepsilon}$. Furthermore, it can be written as

$$J_B^{(\varepsilon)}\psi = \sum_{i=1}^{\infty} e^{-2\varepsilon\lambda_i} P_{\lambda_i}[\psi].$$

Remark 3.1.28. By Theorem 3.1.30 in [40], the operators $D_B^k J_B^{(\varepsilon)}$ are bounded so that we have a bounded operator $J_B^{(\varepsilon)} : L^2(\Sigma, E) \to H_B^k(\Sigma, E)$ for every $\varepsilon > 0$ and $k \in \mathbb{N}_0$. In particular, $J_B^{(\varepsilon)}\psi \in H_B^\infty(\Sigma, E)$ for each $\psi \in L^2(\Sigma, E)$. Thus $J_B^{(\varepsilon)}$ is a smoothing operator.

Since $J_B^{(\varepsilon)}$ is a function of D_B, these two operators commute. This and $\left\| J_B^{(\varepsilon)} \right\|_{L^2,L^2} \leq e^{-\varepsilon}$ implies

$$\left\| J_B^{(\varepsilon)}\psi \right\|_{k,B}^2 \leq e^{-\varepsilon} \left\| \psi \right\|_{k,B}^2$$

for each $\psi \in H_B^k(\Sigma, E)$. Thus $J_B^{(\varepsilon)} : H_B^k(\Sigma, E) \to H_B^k(\Sigma, E)$ is a contraction.

Since the family of functions $x \mapsto e^{-\varepsilon x}$ is uniformly bounded and converges pointwise to 1, the family of operators $J_B^{(\varepsilon)}$ converges strongly to $Id_{L^2(\Sigma,E)}$ as $\varepsilon \searrow 0$. For $\psi \in H_B^k(\Sigma, E)$ we have $J_B^{(\varepsilon)}\psi \to \psi$ in $L^2(\Sigma, E)$ and $D_B^k J_B^{(\varepsilon)}\psi = J_B^{(\varepsilon)} D_B^k\psi \to D_B^k\psi$ in $L^2(\Sigma, E)$. Thus $J_B^{(\varepsilon)}\psi \to \psi$ in $H_B^k(\Sigma, E)$, i.e. the family of operators $J_B^{(\varepsilon)}$ converges strongly to $Id_{H_B^k(\Sigma,E)}$ in the space of bounded operators on $H_B^k(\Sigma, E)$.

Showing that a boundary condition is elliptic or ∞-regular can be quite hard in practice, but for some classes of boundary conditions one has convenient criteria at hand. For constructing some examples of Lorentzian boundary conditions later on, let us briefly talk about such classes of boundary conditions:

Definition 3.1.29 (pseudo local boundary condition)**.**

1. We say that a linear subspace $B \subseteq H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma})$ is a *local boundary condition* if there is a subbundle $E' \subseteq E|_{\partial\Sigma}$ such that $B = H^{\frac{1}{2}}(\partial\Sigma, E')$.

2. We say that a linear subspace $B \subseteq H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma})$ is a *pseudo local boundary condition* if there is a pseudo differential operator P of order zero, acting on sections of E over $\partial\Sigma$, which induces a projection on $L^2(\partial\Sigma, E|_{\partial\Sigma})$ such that $B = P(H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma}))$.

Note that we do not assume here that they are boundary conditions as defined in Definition 3.1.10 since they are not necessarily closed in the check space. Hence, this has to be shown additionally to ellipticity, which will both be characterized in the following handy way:

Theorem 3.1.30 (Theorem 7.20, Corollary 7.23 & 7.24 [8])**.**

1. *It is equivalent:*

   (a) $B = P(H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma}))$ *is closed in* $\check{H}(A)$ *and is elliptic,*

   (b) $P - \chi^+(A) \colon L^2(\partial\Sigma, E|_{\partial\Sigma}) \to L^2(\partial\Sigma, E|_{\partial\Sigma})$ *is an elliptic operator, and*

   (c) $P - \chi^+(A) \colon L^2(\partial\Sigma, E|_{\partial\Sigma}) \to L^2(\partial\Sigma, E|_{\partial\Sigma})$ *is a Fredholm operator.*

2. *Let* $E|_{\partial\Sigma} := E' \oplus E''$ *be a decomposition such that the boundary operator* A *interchanges* E' *and* E'' *for all* $\zeta \in T^*\partial\Sigma$*. Then* $B' := H^{\frac{1}{2}}(\partial\Sigma, E')$ *and* $B'' := H^{\frac{1}{2}}(\partial\Sigma, E'')$ *are closed in* $\check{H}(A)$ *and are elliptic.*

3. *Every pseudo local elliptic boundary condition is* ∞*-regular.*

This implies directly the following Corollary:

Corollary 3.1.31. *Let* $P : L^2(\partial\Sigma, E|_{\partial\Sigma}) \to L^2(\partial\Sigma, E|_{\partial\Sigma})$ *be a pseudo differential orthogonal projection such that*

1. $P^* = P$,

2. $P = Id + \sigma_D(\tau) P \sigma_D(\tau)$, *and*

3. $P - \chi^+(A) : L^2(\partial\Sigma, E|_{\partial\Sigma}) \to L^2(\partial\Sigma, E|_{\partial\Sigma})$ *is a Fredholm operator.*

Then $B = P(H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma}))$ *is a selfadjoint elliptic boundary condition for* D*.*

Proof. First, note that by Theorem 3.1.30 and the third assumption of Corollary 3.1.31, we see that B is an elliptic boundary condition. Thus, it is only left to show that B is also selfadjoint.
By the first two assumptions of Corollary 3.1.31, we see

$$\begin{aligned}\int_{\partial\Sigma} h_E(\sigma_D(\tau)P\psi,\phi)\,\mathrm{d}\mu_{\partial\Sigma} &= -\int_{\partial\Sigma} h_E(\sigma_D(\tau)P\sigma_D(\tau)^2\psi,\phi)\,\mathrm{d}\mu_{\partial\Sigma}\\ &= -\int_{\partial\Sigma} h_E((P-Id)\sigma_D(\tau)\psi,\phi)\,\mathrm{d}\mu_{\partial\Sigma}\\ &= \int_{\partial\Sigma} h_E(\sigma_D(\tau)\psi,(Id-P)\phi)\,\mathrm{d}\mu_{\partial\Sigma}.\end{aligned}$$

This implies

$$\begin{aligned}B^* &= \{\phi\in\check{H}(A);\ \int_{\partial\Sigma} h_E(\sigma_D(\tau)\psi,\phi)\,\mathrm{d}\mu_{\partial\Sigma}=0\ \forall\psi\in B=PH^{\frac{1}{2}}(\partial\Sigma,E|_{\partial\Sigma})\}\\ &= \{\phi\in H^{\frac{1}{2}}(\partial\Sigma,E|_{\partial\Sigma});\ \int_{\partial\Sigma} h_E(\sigma_D(\tau)P\psi,\phi)\,\mathrm{d}\mu_{\partial\Sigma}=0\ \forall\psi\in H^{\frac{1}{2}}(\partial\Sigma,E|_{\partial\Sigma})\}\\ &= \{\phi\in H^{\frac{1}{2}}(\partial\Sigma,E|_{\partial\Sigma});\ \int_{\partial\Sigma} h_E(\sigma_D(\tau)\psi,(Id-P)\phi)\,\mathrm{d}\mu_{\partial\Sigma}=0\ \forall\psi\in H^{\frac{1}{2}}(\partial\Sigma,E|_{\partial\Sigma})\}\\ &= PH^{\frac{1}{2}}(\partial\Sigma,E|_{\partial\Sigma}),\end{aligned}$$

where we used in the second line that B^* is elliptic and hence contained in $H^{\frac{1}{2}}$ and in the third line we used the first two assumptions of Corollary 3.1.31. □

We will call a pseudo differential projection as in Corollary 3.1.31 a *Grassmannian projection*.

Let us end this section by discussing some important estimates for elliptic pseudo local boundary conditions, which we will use later. First, let us discuss a direct consequence of Theorem 3.1.17:

Corollary 3.1.32. *Let $P: H^{\frac{1}{2}}(\partial\Sigma,E|_{\partial\Sigma})\to H^{\frac{1}{2}}(\partial\Sigma,E|_{\partial\Sigma})$ be a pseudo differential operator, such that $B=P(H^{\frac{1}{2}}(\partial\Sigma,E|_{\partial\Sigma}))$ is an elliptic boundary condition. Then we have for $\psi\in H^1(\Sigma,E)$ the estimate*

$$\|\psi\|^2_{H^1}\le C\left(\|\psi\|^2_D+\|(1-P)R(\psi)\|^2_{H^{\frac{1}{2}}}\right),$$

for a constant $C>0$.

Proof. This follows directly by Theorem 3.1.17 by applying Proposition A.1 and Proposition A.3 in [8]. □

Since we know by the third part of Theorem 3.1.30 that every elliptic pseudo local boundary condition is also ∞-regular, we can also show analogous estimates as in Corollary 3.1.32 for H^k-norms. For this, let us define for $k \in \mathbb{N}$ the map

$$\begin{aligned}\tilde{P} : H^k(\Sigma, E) &\to \bigoplus_{l=0}^{k-1}(1-P)H^{k-l-\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma})\\ \psi &\mapsto ((1-P)R\psi, (1-P)R(D\psi), \dots, (1-P)R(D^{k-1}\psi))\end{aligned}$$

for P being a pseudo differential operator on the boundary as above.

Corollary 3.1.33. *Let $P : H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma}) \to H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma})$ be a pseudo differential operator such that $B = P(H^{\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma}))$ is an elliptic boundary condition. Furthermore, let $k \in \mathbb{N}$. Then*

$$D \oplus \tilde{P} : H^k(\Sigma, E) \to H^{k-1}(\Sigma, E) \oplus \bigoplus_{l=0}^{k-1}(1-P)H^{k-l-\frac{1}{2}}(\partial\Sigma, E|_{\partial\Sigma}),$$

has finite dimensional kernel and closed image. In particular, we have for $\psi \in H^k(\Sigma, E)$ the estimate

$$\|\psi\|^2_{H^k} \leq C\left(\|\psi\|^2_{H^{k-1}} + \|D\psi\|^2_{H^{k-1}} + \sum_{l=0}^{k-1}\left\|(1-P)R(D^l\psi)\right\|^2_{H^{k-l-\frac{1}{2}}}\right),$$

for a constant $C > 0$.

Proof. By Proposition A.1 in [8], we see that $D\oplus\tilde{P}$ has finite dimensional kernel and closed image if and only if $D|_{\ker\tilde{P}} : \ker\tilde{P} \to H^{k-1}(\Sigma, E)$ has finite dimensional kernel and closed image. Note that by the construction of $\tilde{P}$, we have $\ker\tilde{P} = H^k_B(\Sigma, E)$.
Proposition A.3 in [8] gives us two things. Firstly, it shows that the first claim of Corollary 3.1.33 directly implies its second claim. Secondly, it gives us that $D|_{\ker\tilde{P}}$ has finite dimensional kernel and closed image if and only if every bounded sequence $\{\psi_n\}_{n\in\mathbb{N}}$ in $H^k_B(\Sigma, E)$ such that $D\psi_n$ converges in $H^{k-1}(\Sigma, E)$ has a convergent subsequence in $H^k_B(\Sigma, E)$.
So let us start with such a bounded sequence $\{\psi_n\}_{n\in\mathbb{N}} \subseteq H^k_B(\Sigma, E)$ with $D\psi_n \to \phi$ in H^{k-1}. Since $H^k_B(\Sigma, E)$ embeds compactly into $H^{k-1}(\Sigma, E)$, we know that there is a subsequence $\{\psi_{n_i}\}$ such that $\psi_{n_i} \to \psi$ in $H^{k-1}(\Sigma, E)$. Hence, we get by Estimate 3.1.5 for all $i, j \in \mathbb{N}$

$$\left\|\psi_{n_i} - \psi_{n_j}\right\|^2_{H^k} \leq C\left(\left\|\psi_{n_i} - \psi_{n_j}\right\|^2_{H^{k-1}} + \left\|D(\psi_{n_i}) - D(\psi_{n_j})\right\|^2_{H^{k-1}}\right),$$

and in particular $\{\psi_{n_i}\}$ is a Cauchy sequence with respect to the H^k-norm. But since $(H^k_B(\Sigma, E), \|\cdot\|_{H^k})$ is a Banach space, we have that $\psi_{n_i} \to \psi \in H^k_B(\Sigma, E)$. With this the first claim of the corollary follows. □

3.1.4 Examples for non-local boundary conditions

Finally, let us give some examples for elliptic and ∞-regular boundary conditions that we will use later on.

1. Let χ be a selfadjoint involution of E along $\partial\Sigma$ and $E|_{\partial\Sigma} = E^+ \oplus E^-$ be the orthogonal splitting into the eigenbundles of χ for the eigenvalues ± 1. One calls χ a *boundary chirality* (w.r.t. A) if χ anti-commutes with A. The associated boundary conditions $B_\pm = H^{\frac{1}{2}}(\partial\Sigma, E^\pm)$ are elliptic, by Theorem 3.1.30. Also we have $B^*_{\pm\chi} = \sigma_D(\tau)B_{\mp\chi}$, hence if we additionally assume that χ anti-commutes with $\sigma_D(\tau)$, we have $B^*_{\pm\chi} = B_{\pm\chi}$.

2. Let Σ be a compact Riemannian spin manifold with boundary $\partial\Sigma$. Let τ be the unit conormal along $\partial\Sigma$. Let D be the spin Dirac operator acting on spinors on Σ. The boundary operator A on $\partial\Sigma$ can be chosen in such a way that it is essentially the Dirac operator on Σ. In particular, it is selfadjoint and has real spectrum. The *Atiyah–Patodi–Singer* (APS) condition $B_{APS} = \chi^-(A)H^{\frac{1}{2}}(\partial\Sigma, S\Sigma|_{\partial\Sigma})$ is then one of the most prominent examples of non-local boundary conditions. One can see that this boundary condition is elliptic using Theorem 3.1.15 by putting $V_\pm = \chi^\pm(A)L^2(\partial\Sigma, S\Sigma|_{\partial\Sigma})$, $W_\pm = \{0\}$ and $g = 0$. This is an elliptic decomposition as in Definition 3.1.12. On the other hand, one sees that the APS boundary conditions are pseudo local, see for example Proposition 14.2 in [20]. Then by Theorem 3.1.30 the APS boundary condition is elliptic and even ∞-regular. Furthermore, note that in this setting $B^*_{APS} = B_{APS}$ if and only if $\ker A = \{0\}$.

3. Let us stay in the same setting as in the previous example. Additionally assume that there is an orthogonal decomposition $\ker A = L \oplus \sigma_D(\tau)L$ as in Theorem 3.1.18. Let us look at the following *modified Atiyah–Patodi–Singer* (mAPS) condition
$$B_{mAPS} := \chi^-(A)H^{\frac{1}{2}}(\partial\Sigma, S\Sigma|_{\partial\Sigma}) \oplus L.$$
This is an ∞-regular selfadjoint boundary condition by Theorem 3.1.18 with
$$V = \chi^-(A)L^2(\partial\Sigma, E|_{\partial\Sigma}), \quad W = \{0\} \quad \text{and} \quad g \equiv 0.$$

4. Let Σ be a closed Riemannian manifold and $E \to \Sigma$ be a Hermitian vector bundle and D be a Dirac type operator from E on itself. Let $N \subseteq \Sigma$ be a hypersurface with trivial normal bundle. Cut Σ along N to obtain a compact Riemannian manifold Σ' with boundary $\partial\Sigma' = N_1 \sqcup N_2$, where N_1 and N_2 are two copies of N with opposite relative orientations in Σ'. We get an induced vector bundle $E' \to \Sigma'$ and a Dirac type operator D' from E' to itself.
For $\psi \in H^1(\Sigma, E)$, we get $\psi' \in H^1(\Sigma', E')$ such that $\psi'|_{N_1} = \psi'|_{N_2}$. Using this one can define the *transmission conditions* for D' on Σ'. We set
$$B := \{(\psi, \psi) \in H^{\frac{1}{2}}(N_1, E|_{N_1}) \oplus H^{\frac{1}{2}}(N_2, E|_{N_2}); \psi \in H^{\frac{1}{2}}(N, E|_N)\},$$

where we identify

$$H^{\frac{1}{2}}(N_1, E|_{N_1}) = H^{\frac{1}{2}}(N_2, E|_{N_2}) = H^{\frac{1}{2}}(N, E|_N).$$

Let $A = A_0 \oplus -A_0$ be a boundary operator for D', where A_0 is a selfadjoint Dirac type operator on $C^\infty(N, E|_N)$. The transmission conditions are another example for elliptic boundary conditions. For this we put

$$\begin{aligned} V_+ &:= \chi^+(A)L^2(\partial\Sigma', E|_{\partial\Sigma'}) = \chi^+(A_0)L^2(N_1, E|_{N_1}) \oplus \chi^-(A_0)L^2(N_2, E|_{N_2}), \\ V_- &:= \chi^-(A)L^2(\partial\Sigma', E|_{\partial\Sigma'}) = \chi^-(A_0)L^2(N_1, E|_{N_1}) \oplus \chi^+(A_0)L^2(N_2, E|_{N_2}), \\ W_\pm &:= \{(\psi, \pm\psi) \in \ker(A_0) \oplus \ker(A_0)\}, \end{aligned}$$

and

$$g : V_- \to V_+, g = \begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix}.$$

With these choices, B is of the form required in Definition 3.1.12 and even ∞-regular since $W_\pm$ consist of smooth functions and g maps H^s sections to H^s sections by definition. Furthermore, note that transmission conditions are not pseudo local.

3.2 Continuity of functional calculus

Before we consider Lorentzian boundary conditions in the next section, let us briefly talk about the continuity of the functional calculus of an operator family with respect to a corresponding family of boundary condition.
Let (Σ, g) be a compact Riemannian manifold with boundary, $E \to \Sigma$ a Hermitian vector bundle and $D_t : C^\infty(\Sigma, E) \to C^\infty(\Sigma, E)$ a family of formally selfadjoint Dirac type operators with coefficients depending smoothly on $t \in \mathbb{R}$. Let τ be the interior unit conormal to $\partial\Sigma$. Furthermore, let $\{P_t\}_{t\in\mathbb{R}}$ be a family of orthogonal Grassmannian projections (see Corollary 3.1.31) and $\{B_t\}_{t\in\mathbb{R}}$ the family of corresponding boundary conditions $B_t = P_t(H^{\frac{1}{2}}(\partial\Sigma, E_{\partial\Sigma}))$.
In this setting, we will discuss conditions on the family $\{P_t\}_{t\in\mathbb{R}}$ such that the functional calculus $D^l_{t,B_t} J^{(\varepsilon)}_{B_t}$ is strongly continuous in t for all $l \in \mathbb{N}$ with respect to all H^k-norms. This will be done by first discussing the case $k = 0$ and then using an induction argument for $k > 0$.

Lemma 3.2.1. *Let $\{D_t\}_{t\in\mathbb{R}}$ be as described above, $\{P_t\}_{t\in\mathbb{R}}$ be a family of Grassmannian projections and additionally assume that $t \mapsto P_t$ is L^2 norm continuous. Then*

$$t \mapsto D^l_{t,B_t} J^{(\varepsilon)}_{B_t}$$

is L^2 norm continuous for all $l \in \mathbb{N}$ and $\varepsilon > 0$.

Proof. First note that D_{t,B_t} is a closed, selfadjoint operator on $\mathrm{dom}(D_{t,B_t})$. Using the Cayley transform $\kappa : \mathbb{R} \to \mathbb{C},\ x \mapsto \frac{x-i}{x+i}$, we can transform $\{D_{t,B_t}\}_{t\in\mathbb{R}}$ into a family of unitary bounded operators $\{U_t\}_{t\in\mathbb{R}}$ with $U_t = \kappa(D_{t,B_t})$. By Theorem 1.1 in [19] the path $t \mapsto U_t$ is L^2 norm continuous if and only if D_{t,B_t} is gap continuous (for a Definition see, for example, [35]).
Since $t \mapsto P_t$ is L^2 norm continuous, Theorem 3.9 in [19] indeed implies that D_{t,B_t} is gap continuous, hence U_t is L^2 norm continuous.
Furthermore, we have that

$$D^l_{t,B_t} J^{(\varepsilon)}_{B_t} = (x^l \cdot e^{-\varepsilon(1+x^2)})(D_l) = (x^l \cdot e^{-\varepsilon(1+x^2)})(\kappa^{-1}(U_t)) = \left(i\frac{1+z}{1-z}\right)^l \cdot e^{-\varepsilon\left(1+i\left(\frac{1+z}{1-z}\right)^2\right)}(U_t).$$

Note that the inverse of the Cayley transform is a priori not defined for $z = 1$ but since $(i\frac{1+z}{1-z})^2$ is real valued and positive for $z \in \mathrm{spec}(U_t)$, we can continuously extend $\left(i\frac{1+z}{1-z}\right)^l \cdot e^{-\varepsilon\left(1+i\left(\frac{1+z}{1-z}\right)^2\right)}(U_t)$ with zero at $z = 1$. Hence, the function

$$f : \mathbb{S}^1 \to \mathbb{R}, z \mapsto \left(i\frac{1+z}{1-z}\right)^l \cdot e^{-\varepsilon\left(1+i\left(\frac{1+z}{1-z}\right)^2\right)}$$

is continuous on the spectrum of U_t for all t. Lemma 1.2.5 in [42] then implies that $f(U_t)$ is L^2 norm continuous. □

Using this Lemma, we prove strongly continuity of $t \mapsto D^l_{t,B_t} J^{(\varepsilon)}_{B_t}$ with respect to H^k:

Lemma 3.2.2. *Let $\{D_t\}_{t\in\mathbb{R}}$ be as described above, $\{P_t\}_{t\in\mathbb{R}}$ be a family of Grassmannian projections and additionally assume that there exists a sequence of integers $\{k_j\}_{j\in\mathbb{N}_0}$ with $k_0 = 0$ and $k_j \to \infty$ for $j \to \infty$ such that $t \mapsto P_t$ is H^{k_j} norm continuous for all $j \in \mathbb{N}$. Then for all $l \in \mathbb{N}$ and $\varepsilon > 0$*

$$t \mapsto D^l_{t,B_t} J^{(\varepsilon)}_{B_t}$$

is strongly continuous with respect to H^k for all $k \in \mathbb{N}$.

Proof. We prove this lemma by induction over k. Since the case $k = 0$ is already done in Lemma 3.2.1, we now will consider the induction step $k \mapsto k+1$. Let $\psi \in H^{k+1}(\Sigma, E)$, then we can estimate for $t, t' \in \mathbb{R}$

$$\begin{aligned}
&\left\| \left(D^l_{t,B_t} J^{(\varepsilon)}_{B_t} - D^l_{t',B_{t'}} J^{(\varepsilon)}_{B_{t'}} \right) \psi \right\|^2_{H^{k+1}} \\
\leq & C_1 \Bigg(\underbrace{\left\| \left(D^l_{t,B_t} J^{(\varepsilon)}_{B_t} - D^l_{t',B_{t'}} J^{(\varepsilon)}_{B_{t'}} \right) \psi \right\|^2_{H^k}}_{=:(I)} + \underbrace{\left\| D_t \left(\left(D^l_{t,B_t} J^{(\varepsilon)}_{B_t} - D^l_{t',B_{t'}} J^{(\varepsilon)}_{B_{t'}} \right) \psi \right) \right\|^2_{H^k}}_{:=(II)}
\end{aligned}$$

$$+\underbrace{\sum_{m=0}^{k}\left\|(1-P_t)R\left(D_t^m\left(D^l_{t,B_t}J^{(\varepsilon)}_{B_t}-D^l_{t',B_{t'}}J^{(\varepsilon)}_{B_{t'}}\right)\psi\right)\right\|^2_{H^{k-m+\frac{1}{2}}}}_{=:(III)}\Bigg),$$

where we used Corollary 3.1.33. Let us now look at the terms (I) to (III) separately. The first term (I) is converging to zero for $|t-t'|\to 0$ by induction hypothesis for k. So we only need to continue estimate (II) and (III). Let us look at the second term (II):

$$\begin{aligned}(II)&\leq\left\|\left(D^{l+1}_{t,B_t}J^{(\varepsilon)}_{B_t}-D^{l+1}_{t',B_{t'}}J^{(\varepsilon)}_{B_{t'}}\right)\psi\right\|^2_{H^k}+\left\|\left((D_{t'}-D_t)D^l_{t',B_{t'}}J^{(\varepsilon)}_{B_{t'}}\right)\psi\right\|^2_{H^k}\\&\leq\underbrace{\left\|\left(D^{l+1}_{t,B_t}J^{(\varepsilon)}_{B_t}-D^{l+1}_{t',B_{t'}}J^{(\varepsilon)}_{B_{t'}}\right)\psi\right\|^2_{H^k}}_{=:(IIa)}+C_2\underbrace{\left\|D_{t'}-D_t\right\|^2_{H^{k+1},H^k}}_{=:(IIb)}\underbrace{\left\|D^l_{t',B_{t'}}J^{(\varepsilon)}_{B_{t'}}\psi\right\|^2_{H^{k+1}}}_{=:(IIc)},\end{aligned}$$

where we used the triangle inequality and that $J^{(\varepsilon)}_{B_{t'}}$ is smoothing. We see that (IIa) is converging to zero for $|t-t'|\to 0$ by induction hypothesis for k. Furthermore, since on the range of $J^{(\varepsilon)}_{B_{t'}}$ the H^{k+1}-norm is equivalent to the $H^k_{B_{t'}}$-norm, which by Lemma 3.2.1 is a continuous real function, we see that for $|t-t'|$ small enough (IIc) is bounded. This together with D_t having coefficients smoothly depending on t – which controls (IIb) – implies that (II) is converging to zero for $|t-t'|\to 0$.

It remains to control (III), which we do as follows:

$$\begin{aligned}(III)\leq&C_3\sum_{m=0}^{k}\left(\left\|(1-P_t)R\left(D^{l+m}_{t',B_{t'}}J^{(\varepsilon)}_{B_{t'}}\psi\right)\right\|^2_{H^{k-m+\frac{1}{2}}}+\left\|(1-P_t)R\left((D^m_{t'}-D^m_t)D^l_{t',B_{t'}}J^{(\varepsilon)}_{B_{t'}}\psi\right)\right\|^2_{H^{k-m+\frac{1}{2}}}\right)\\\leq&C_4\sum_{m=0}^{k}\left(\left\|(P_{t'}-P_t)R\left(D^{l+m}_{t',B_{t'}}J^{(\varepsilon)}_{B_{t'}}\psi\right)\right\|^2_{H^{k_{j_0}}}+\left\|(1-P_t)R\left((D^m_{t'}-D^m_t)D^l_{t',B_{t'}}J^{(\varepsilon)}_{B_{t'}}\psi\right)\right\|^2_{H^{k_{j_0}}}\right)\\\leq&C_5\left\|P_{t'}-P_t\right\|^2_{H^{k_{j_0}},H^{k_{j_0}}}\sum_{m=0}^{k}\left\|D^{l+m}_{t',B_{t'}}J^{(\varepsilon)}_{B_{t'}}\psi\right\|^2_{H^{k_{j_0}+\frac{1}{2}}}\\&+C_6\sum_{m=0}^{k}\left(\left\|1-P_t\right\|^2_{H^{k_{j_0}}H^{k_{j_0}}}\left\|(D^m_{t'}-D^m_t)D^l_{t',B_{t'}}J^{(\varepsilon)}_{B_{t'}}\psi\right\|^2_{H^{k_{j_0}+\frac{1}{2}}}\right)\end{aligned}$$

$$\leq C_5 \underbrace{\left\| P_{t'} - P_t \right\|^2_{H^{k_{j_0}}, H^{k_{j_0}}}}_{=:(IIIa)} \underbrace{\sum_{m=0}^{k} \left\| D^{l+m}_{t',B_{t'}} J^{(\varepsilon)}_{B_{t'}} \psi \right\|^2_{H^{k_{j_0}+\frac{1}{2}}}}_{=:(IIIb)}$$

$$+ C_7 \underbrace{\left\| 1 - P_t \right\|^2_{H^{k_{j_0}}, H^{k_{j_0}}}}_{=:(IIIc)} \sum_{m=0}^{k} \underbrace{\left\| D^m_{t'} - D^m_t \right\|^2_{H^{k_{j_0}+m+\frac{1}{2}}, H^{k_{j_0}+\frac{1}{2}}}}_{=:(IIId)} \underbrace{\left\| D^l_{t',B_{t'}} J^{(\varepsilon)}_{B_{t'}} \psi \right\|^2_{H^{k_{j_0}+\frac{1}{2}}}}_{=:(IIIe)}$$

where we used that the range of $J^{(\varepsilon)}_{B_{t'}}$ is $H^{\infty}_{B_{t'}}$, classical norm inequalities and also that there exists a $j_0 \in \mathbb{N}$ such that $k_{j_0} > k + \frac{1}{2}$. Let us now look at $(IIIa)$ to $(IIId)$. By assumption $\left\| P_{t'} - P_t \right\|_{H^{k_{j_0}}}$ converges to zero for $|t - t'| \to 0$, and since $(IIIb)$ is a continuous real valued function, we see that the first term in the last estimate is converging to zero for $|t' - t| \to 0$. Furthermore, by assumption we know that $1 - P_t$ is $H^k_{j_0}$ norm continuous and thus, $(IIIc)$ is bounded for small $|t - t'|$. As seen before also $(IIIe)$ is bounded for small $|t' - t|$ and since D^m_t has coefficients depending smoothly on t for all m we as well know that $(IIId)$ is converging to zero for $|t' - t| \to 0$. Hence, also the second term in the last estimate is converging to zero for $|t - t'| \to 0$.
In total, we showed that (I) to (III) are converging to zero for $|t - t'| \to 0$. This concludes the proof. □

We end this section by proving the following special case of Lemma 3.2.2. This will be handy later for discussing explicit examples in Chapter 5. In the following, we denote for two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ the operator norm between the by $\|\cdot\|_{\mathcal{H}_1,\mathcal{H}_2}$.

Lemma 3.2.3. *Let $\{D_t\}_{t\in\mathbb{R}}$ be as described above, let $\{A_t\}_{t\in\mathbb{R}}$ be the corresponding family of boundary operators on $\partial\Sigma$ and assume that this family can be chosen such that the coefficients of A_t depend smoothly on t. Let $\{P_t\}_{t\in\mathbb{R}}$ be a family of Grassmannian projections with $P_t A_t = A_t P_t$ and assume that $t \mapsto P_t$ is L^2 norm continuous. Then for all $l \in \mathbb{N}$ and $\varepsilon > 0$*

$$t \mapsto D^l_{t,B_t} J^{(\varepsilon)}_{B_t}$$

is strongly continuous with respect to H^k for all $k \in \mathbb{N}$.

Proof. We want to apply Lemma 3.2.2, hence, we have to show that $t \mapsto P_t$ is H^{k_j} norm continuous for a sequence $\{k_j\}_{j\in\mathbb{N}_0}$ with $k_0 = 0$ and $k_j \to \infty$ for $j \to \infty$. In this particular case, we even get that $t \mapsto P_t$ is H^k norm continuous for all $k \in \mathbb{N}$. This will be shown via induction over k.
By assumption the continuity for $k = 0$ is satisfied. Now we will do the induction step $k \mapsto k + 1$: Let $t, t' \in \mathbb{R}$, then

$$\left\| P_t - P_{t'} \right\|_{H^{k+1},H^{k+1}} \leq C_1 \left[\left\| A_t (P_t - P_{t'}) \right\|_{H^{k+1},H^k} + \left\| P_t - P_{t'} \right\|_{H^{k+1},H^k} \right]$$

$$\begin{aligned}\leq C_2 \big[& \|P_t A_t - P_{t'} A_{t'}\|_{H^{k+1},H^k} \\ & + \|A_{t'} - A_t\|_{H^{k+1},H^k} \underbrace{\|P_{t'}\|_{H^{k+1},H^{k+1}}}_{=1} + \|P_t - P_{t'}\|_{H^k,H^k} \big],\end{aligned}$$

where we used in the first line the elliptic estimate for first order operators on a closed manifold and that for $|t-t'|$ small enough the constant a priori depending on t can be bounded from above. In the second line we used triangle inequality, and that A_t commutes with P_t. Let us look at the three terms in the last estimate and see why they all are converging to zero for $|t-t'| \to 0$. The last term converges to zero by induction hypothesis and the second term converges to zero since A_t is a bounded operator from H^{k+1} to H^k and has coefficients depending smoothly on t. The first one converges to zero by both arguments combined.
Hence we see that also for $H^{k+1}k$ the projection is norm continuous and the claim follows. □

Remark 3.2.4. In Section 2.3, we discussed the natural boundary operator for the Riemannian spin Dirac operator. One can see that this choice of boundary operator depends on the spin Dirac operator itself and also on the mean curvature. Hence, in this setting D_t having coefficients depending smoothly on t implies directly that A_t has also coefficients depending smoothly on t.

Remark 3.2.5. Note that instead of assuming that P_t commutes with A_t for all $t \in \mathbb{R}$ one can also assume that P_t anti-commutes with A_t, since the change of sign does not influence the estimates in the proof of Lemma 3.2.3.

Remark 3.2.6. We can directly apply Lemma 3.2.3 to the APS boundary conditions. Let us additionally assume that $\ker A_t = \{0\}$ for all $t \in \mathbb{R}$. Since A_t has coefficients that depend smoothly on t and $\partial\Sigma$ is a closed manifold, we can use Lemma 3.3 in [37] to see that $\chi^-(A_t)$ is L^2 norm continuous. Furthermore, we know that the projection $\chi^-(A_t)$ commutes with A_t. Thus, we can use Lemma 3.2.3 to archive the continuity of the functional calculus as desired.

3.3 Lorentzian boundary conditions

In this section we will introduce Lorentzian non-local boundary conditions that will later lead to the well-posedness of the corresponding Cauchy problems. We will first motivate briefly how boundary conditions in this context should be looked at by giving an easy example in the two dimensional Minkowski space. After that we will discuss the transmission conditions and also the class of admissible boundary conditions in the following subsections.

3.3.1 MIT boundary conditions on Minkowski halfspace

Later in Chapter 4 we will use boundary conditions to show the well-posedness for the corresponding initial value problem. So let us discuss the initial boundary value problem for the Dirac operator on the Minkowski half space $(M, g) = (\mathbb{R} \times [0, \infty), g_{Min} = -\mathrm{d}t^2 + \mathrm{d}x^2)$, where we can identify the spin bundle SM with the trivial bundle $M \times \mathbb{C}^2$. Let us first have a look at the following homogeneous initial value problem

$$\begin{cases} D\psi = 0 \\ \psi|_{\{x=0\}} = \psi_0 \in C^\infty([0, \infty), \mathbb{C}^2). \end{cases} \tag{3.3.1}$$

After choosing an initial value ψ_0, we can extend it to the Minkowski space $\mathbb{R}^{1,1}$ and can solve the corresponding homogeneous initial value problem, see Appendix B.2. The restriction of such a solution to M then solves the homogeneous initial value problem on M, if we do not assume any boundary conditions on $\partial M = \{x = 0\}$. Hence, the existence of solutions is not a problem, but since there is a priori no unique way of extending the Cauchy data from M to $\mathbb{R}^{1,1}$ the uniqueness is not anymore fulfilled. The boundary conditions should now make sure that the solution is unique by putting restrictions on the possible extensions of the Cauchy data on M to $\mathbb{R}^{1,1}$. Since in this example the boundary of the Cauchy hypersurfaces each only consist of a point, we cannot talk about non-local boundary conditions, but let us have a brief look at a local one. So let us consider the chirality condition (see also Section 5.4) induced by the boundary chirality

$$\chi_t = i\sigma_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} : \{(t, 0)\} \times \mathbb{C}^2 \to \{(t, 0)\} \times \mathbb{C}^2.$$

We can solve the initial boundary value problem

$$\begin{cases} D\psi & = 0 \text{ on } M \\ \psi|_{\{t=0\}} & = \psi_0 \in C^\infty([0, \infty), \mathbb{C}^2) \\ (\chi_t - 1)\psi(t, 0) & = 0 \text{ for all } t \in \mathbb{R}, \end{cases} \tag{3.3.2}$$

where this boundary condition is also called *MIT condition*, see for example [31].

For an initial value ψ_0 let $\tilde{\psi}_0 \in C^\infty(\mathbb{R}, \mathbb{C}^2)$ be an arbitrary smooth extension of $\tilde{\psi}_0$. Then one can solve the homogeneous initial value problem on $\mathbb{R}^{1,1}$, see B.2. The boundary chirality χ_t acts on the solution $\tilde{\psi}$ restricted to $\{x = 0\}$ by

$$\chi_t(\tilde{\psi}(t, 0)) = \begin{pmatrix} -\tilde{\psi}_1(t, 0) \\ \tilde{\psi}_2(t, 0) \end{pmatrix}.$$

This gives us that $\tilde{\psi}(t,0) \in E(\chi_t, 1)$ if and only if $\tilde{\psi}_1(t,0) = 0$. This implies that if $\tilde{\psi}$ fulfill the boundary conditions, then

$$\begin{aligned}(\tilde{\psi}_0)_1(x) + (\tilde{\psi}_0)_1(-x) =& \tfrac{1}{2}\left[(\tilde{\psi}_0)_1(x) - (\tilde{\psi}_0)_2(x) + (\tilde{\psi}_0)_1(-x) + (\tilde{\psi}_0)_2(-x)\right] \\ &+ \tfrac{1}{2}\left[-(\tilde{\psi}_0)_1(x) + (\tilde{\psi}_0)_2(x) + (\tilde{\psi}_0)_1(-x) + (\tilde{\psi}_0)_2(-x)\right] \\ &= \tilde{\psi}_1(x,0) + \tilde{\psi}_1(-x,0) \\ &= 0\end{aligned}$$

and similarly

$$(\tilde{\psi}_0)_2(x) - (\tilde{\psi}_0)_2(-x) = \tilde{\psi}_1(-x,0) - \tilde{\psi}_1(x,0) = 0$$

Hence, we see that the initial data ψ_0 can only be extended as

$$\tilde{\psi}_0(x) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{\psi}_0(-x).$$

This in particular implies the uniqueness of the solutions by looking at $\psi_0 \equiv 0$ on $[0,\infty)$.
Hence, for all $\psi_0 \in C^\infty_{cc}([0,\infty), \mathbb{C}^2)$ we get a unique smooth solution to the initial value problem 3.3.2. If we want to allow the support of the initial value to touch the boundary, then we have to assume compatibility conditions. In this case, they can be written as

$$\begin{aligned}\frac{d}{dx^{2k}}(\psi_0)_1(0) = 0, \quad &\text{and} \\ \frac{d}{dx^{2k+1}}(\psi_0)_2(0) = 0 &\end{aligned}$$

for all $k \in \mathbb{N}$.
Since the solutions to the homogeneous initial value problem can easily be written in terms of the initial value one directly sees that the solutions depend continuously on the Cauchy data. Hence, we see that the homogeneous initial value problem is well-posed.

Furthermore, one could also impose the boundary condition $(\chi_t + 1)\psi(t,0) = 0$ or equivalently $\psi(t,0) \in E(\chi_t, -1)$. This also leads to the well-posedness of the corresponding initial boundary value problem. Note that by construction of the solution one could not mix the two boundary conditions, because this would lead to a non-continuous extension of the initial value and hence also to a discontinuous solution. Later we will see that one indeed has to assume some continuity assumptions on the Lorentzian boundary conditions to make sure that the solutions are continuous in time direction.

3.3.2 Transmission conditions

Let us start with some less involved boundary conditions, the transmission boundary conditions. First, let us briefly recall the definition of finite energy spaces on globally hyperbolic spin manifolds with *closed* Cauchy hypersurfaces. For this we are following [15] and [16].

Let (M, g) be a globally hyperbolic spin manifold with closed Cauchy hypersurfaces and a temporal function $t\colon M \to \mathbb{R}$ such that the metric splits as

$$g = -N^2\, \mathrm{d}t^2 + g_t,$$

where N is the lapse function and g_t is a family of Riemannian metrics on the spacelike Cauchy hypersurfaces Σ_t.
For all $k \in \mathbb{Z}$, the family $\{H^k(\Sigma_t, SM|_{\Sigma_t})\}_{t\in\mathbb{R}}$ is a bundle of Hilbert spaces over $\mathbb{R}$, which is globally trivialized by the parallel transport along the t-lines. We call continuous sections of this bundle *spinors of finite k-energy* and we denote the space of such sections by $FE^k(M, SM)$. This space can be topologized as follows: Let $I \subseteq t(M)$ be a compact subinterval, then we get the semi-norms

$$\|\psi\|_{FE^k,I} := \max_{t\in I} \|\psi(t)\|_{H^k}.$$

Letting I vary over all compact subintervals of $\mathbb{R}$, we turn $FE^k(M, SM)$ into a Fréchet space. Furthermore, let us denote the space of L^2-sections of the bundle $\{H^k(\Sigma_t, SM|_{\Sigma_t})\}_{t\in\mathbb{R}}$ by

$$L^2(\mathbb{R}, H^k(\Sigma_\bullet, SM|_{\Sigma_\bullet})),$$

where $\bullet$ is referring to the t slot being empty and to be filled. We equip this space with the corresponding semi-norm

$$\|\psi\|^2_{L^2,H^k,I} := \int_I \left\| (N^{\frac{1}{2}}(t)\psi(t))|_{\Sigma_t} \right\|^2_{H^k} \mathrm{d}t,$$

where I runs through all compact subintervals of $\mathbb{R}$. This turns $L^2(\mathbb{R}, H^k(\Sigma_\bullet, SM|_{\Sigma_\bullet}))$ into a Fréchet space. Using these spaces, we can define the following

$$FE^k(M, D) := \{\psi \in FE^k(M, SM);\ D\psi \in L^2(\mathbb{R}, H^k(\Sigma_\bullet, SM|_{\Sigma_\bullet}))\}$$

with semi-norms

$$\|u\|^2_{I,FE^k,D} := \|\psi\|^2_{FE^k,I} + \|D\psi\|^2_{L^2,H^k,I},$$

where I runs again through all compact subintervals of $\mathbb{R}$.
Now we have all the spaces to be able to write down the well-posedness of the inhomogeneous Cauchy problem for the Dirac equation on spacetimes without boundary:

Theorem 3.3.1 (Theorem 2.1 in Bär–Strohmaier [15]). *For any $t \in \mathbb{R}$ the mapping*

$$\begin{aligned} \mathrm{res}_t \oplus D : \; FE^k(M, D) &\to H^k(\Sigma_t; SM|_{\Sigma_t}) \oplus L^2(\mathbb{R}, H^k(\Sigma_\bullet, SM|_{\Sigma_\bullet})) \\ \psi &\mapsto (\psi|_{\Sigma_t}, D\psi) \end{aligned}$$

is an isomorphism of Fréchet spaces.

Using this theorem, we will show well-posedness for the Cauchy problem arising from transmission conditions. For this we again start with a global hyperbolic spin manifold (M, g), with closed Cauchy hypersurfaces. Let $N \subseteq M$ be a timelike hypersurface with trivial normal bundle. Cut M along N to obtain a globally hyperbolic spacetime M' with timelike boundary $\partial M' = N_1 \sqcup N_2$ (see Figure 3.3.2). Consider now on $SM' \to M'$ the induced bundle on M' and D' the induced Dirac operator. For any temporal function $t : M \to \mathbb{R}$ we get an induced temporal function $t' : M' \to \mathbb{R}$ such that $t'|_{N_1} = t'|_{N_2}$.

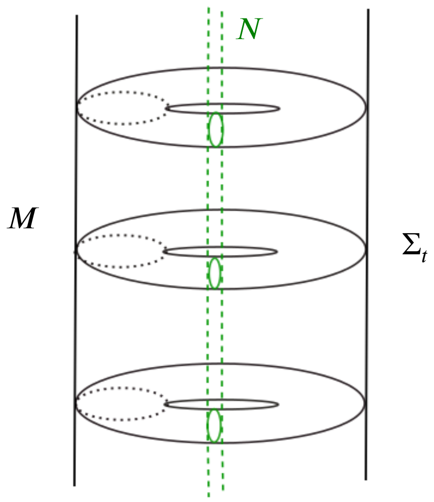

Figure 3.1: The hypersurface $N \subseteq M$.

For $\psi \in FE^1(M, SM)$, we get $\psi' \in FE^1(M', SM')$ with $\psi'|_{N_1 \cap \Sigma'_t} = \psi'|_{N_2 \cap \Sigma'_t}$ for all $t \in t'(M')$. Using this we can define the following family of boundary conditions $B = \{B_t\}_{t \in \mathbb{R}}$ with

$$B_t := \{(\psi, \psi) \in H^{\frac{1}{2}}(\Sigma'_t \cap N_1) \oplus H^{\frac{1}{2}}(\Sigma'_t \cap N_2); \psi \in H^{\frac{1}{2}}(N \cap \Sigma_t)\}$$

being a transmission condition as defined in Definition 3.1.4 (3). Hence, B_t elliptic and

$$\mathrm{dom}(D'_{t,B_t}) = \{\psi \in H^1(\Sigma'_t, SM'|_{\Sigma'_t}); \psi|_{\Sigma'_t \cap N_1} = \psi|_{\Sigma'_t \cap N_2}\},$$

where D'_t is the induced operator of D' on Σ_t, see Equation 2.2.5.
Let us now consider the following space

$$C^0(\mathbb{R}, \mathrm{dom}(D'_{t,B_t}))$$

consisting of all continuous sections that map into $\mathrm{dom}(D'_{t,B_t})$ and with semi-norms

$$\|\psi\|_{I,B} := \max_{t \in I} \|\psi\|_{D'_t},$$

which turns $C^0(\mathbb{R}, \mathrm{dom}(D'_{t,B_t}))$ into a Fréchet space. Proceeding, we define the following space

$$FE(D', B) := \{\psi \in C^0(\mathbb{R}, \mathrm{dom}(D'_{t,B_t})); D\psi \in L^2(\mathbb{R}, \mathrm{dom}(D'_{t,B_t}))\}$$

with semi-norms

$$\|\psi\|^2_{D',B,I} := \|\psi\|^2_{I,B} + \|D'\psi\|^2_{L^2,D'_t}.$$

This turns $FE(D', B)$ into a Fréchet space. Using the identification of $N_1 = N_2 = N$, we see that the following diagram commutes

$$\begin{tikzcd}
FE(D', B) \arrow[r, "\cong"] \arrow[d, "res_t \oplus D'"] & FE^1(M, D) \arrow[d, "res_t \oplus D"] \\
\mathrm{dom}(D'_{t,B_t}) \oplus L^2(\mathbb{R}, \mathrm{dom}(D'_{t,B_t})) & H^1(\Sigma_t, SM|_{\Sigma_t}) \oplus L^2(\mathbb{R}, H^1(\Sigma_\bullet, SM|_{\Sigma_\bullet})). \arrow[l, "\cong"']
\end{tikzcd}$$

Applying this diagram we get the following direct consequence of Theorem 3.3.1

Corollary 3.3.2. *Let M' and $\{B_t\}_{t\in\mathbb{R}}$ be as above. Then*

$$\begin{aligned}
res_t \oplus D' : \quad FE(D', B) &\to \mathrm{dom}(D'_{t,B_t}) \oplus L^2(\mathbb{R}, H^1(\Sigma'_\bullet, SM'|_{\Sigma'_\bullet})) \\
\psi &\mapsto (\psi|_{\Sigma'_t}, D'\psi)
\end{aligned}$$

is a Fréchet isomorphism.

Remark 3.3.3. The well-posedness result of Corollary 3.3.2 does not give much insight in the problem of defining non-local boundary conditions, since the transmission conditions are by definition the boundary conditions that relate the initial value problem on M' to the initial value problem on M. Still, it should give us a first intuition on how to define non-local boundary conditions in the setting of spacetimes with timelike boundary, since we can already see here why it is convenient to look at non-local boundary conditions inside the Cauchy hypersurfaces and then build them together to a Lorentzian boundary condition that is local in time direction and non-local in spatial direction. This idea will also be used in the next subsection, where we will define a bigger class of Lorentzian boundary conditions.

3.3.3 Admissible boundary conditions

The goal of this subsection is to define boundary conditions that will lead to the well-posedness of the associated Cauchy problem of the Lorentzian Dirac operator, which we will discuss in Section 4.1.

For constructing the desired boundary conditions, we will first simplify the form of the Lorentzian Dirac operator to get an operator of the form $\partial_t + i\tilde{D}_t$ acting on sections in $C^\infty(\mathbb{R}, C^\infty(\Sigma, SM|_\Sigma))$. This will be done in two steps. First we will do a conformal change such that ∂_t becomes geodesic and then we identify the Cauchy hypersurfaces with each other via parallel transport along ∂_t.

Conformal change

We apply the conformal transformation $\hat{g} = N^{-2}g$. Let us denote for the unit vectors X for g, $\hat{X} = N \cdot X$ the corresponding unit vectors for $\hat{g}$. Similarly, we will denote all other objects corresponding to the new metric with a hat as well. This leads to the following changes (see for example [33]):

1. the inner product $\langle \cdot, \cdot \rangle_{SM}$ is invariant,
2. $\hat{\gamma}(X) = N^{-1}\gamma(X)$, which also implies that $\hat{\gamma}(\hat{\nu}) = \gamma(\nu)$ and hence, $\langle \cdot, \cdot \rangle_0$ is also invariant,
3. $\hat{\nabla}^{SM}_X = \nabla^{SM}_X + \frac{N}{2}\left(\gamma(X)\gamma(\nabla N^{-1}) - X(N^{-1})\right)$, and
4. $\hat{D} = N^{\frac{n+2}{2}} D N^{-\frac{n}{2}}$.

Remark 3.3.4. One first could think that $\hat{D}$ being of this form cannot be formally anti-selfadjoint anymore, but note that the volume form $\mathrm{d}\mu_M(\hat{g}) = N^{-n-1}\,\mathrm{d}\mu_{\partial M}(g)$ is making sure that $\hat{D}$ is still the formally anti-selfadjoint Lorentzian Dirac operator with respect to the metric $\hat{g}$.

Remark 3.3.5. In the following we will denote $\hat{\Sigma}_t$ and $\hat{M}$ for $(\Sigma_t, \hat{g}_t)$ and $(M, \hat{g})$, respectively, to emphasize the change of metric on these manifolds.

Using Equation 2.2.5 on $\hat{D}$, we see that

$$\hat{D} = -\hat{\gamma}(\hat{\nu})\left(\nabla^{S\hat{M}}_{\hat{\nu}} + i\hat{D}_t - \frac{n}{2}\hat{H}_t\right),$$

where $\hat{D}_t = \hat{\not{D}}_t$ for n even and $\hat{D}_t = \begin{pmatrix} \hat{\not{D}} & 0 \\ 0 & -\hat{\not{D}}_t \end{pmatrix}$ for n odd. Since the Riemannian Dirac operator changes under this conformal change as $\hat{\not{D}}_t = N^{\frac{n+1}{2}} \not{D}_t N^{-\frac{n-1}{2}}$, we see that

$$\hat{D}_t = N^{\frac{n+1}{2}} D_t N^{-\frac{n-1}{2}}. \tag{3.3.3}$$

Remark 3.3.6. Note that $\hat{D}_t$ is indeed formally selfadjoint and its principal symbol is given as

$$\sigma_{\hat{D}_t}(\zeta) = N^{\frac{n+1}{2}} \sigma_{D_t}(\zeta) N^{-\frac{n-1}{2}} = N\sigma_D(\zeta), \tag{3.3.4}$$

for all $x \in \Sigma_t$ and $\zeta \in T_x^*\Sigma_t$.

Identifying the Cauchy hypersurfaces

Let $e_0, \dots, e_n$ be an orthonormal frame of $T\hat{M}$ such that $e_0 = \hat{\nu} = \partial_t$ and $e_n = \hat{\eta}$, where η is the unit normal to the timelike boundary. For $t, s \in \mathbb{R}$, let $\tau_t^s : S\hat{M}|_{\hat{\Sigma}_t} \to S\hat{M}|_{\hat{\Sigma}_s}$ be for each $y \in \hat{\Sigma}_t$ the parallel transport along the integral curves of ∂_t and we denote the same way the parallel transport $\tau_t^s : T\hat{\Sigma}_t \to T\hat{\Sigma}_s$. We denote $\tau_t := \tau_t^0$ to simplify the notation.

Remark 3.3.7.

1. Recall, that we discussed in Section 2.1 that it is essential to assume that the Cauchy temporal function has gradient tangential to the boundary, for making sure that the parallel transport is globally defined and being able to identify the Cauchy hypersurfaces with each other.

2. Another implication of ∂_t being tangential to $\partial\hat{M}$ is that the restriction of the parallel transport to the boundary $\tau_t^s|_{\partial\hat{M}}$ also identifies the spinor bundles $S\hat{M}|_{\partial\hat{\Sigma}_t}$ (as well as $T\hat{\Sigma}_t|_{\partial\hat{\Sigma}_t}$) of the Cauchy hypersurfaces of the globally hyperbolic (possibly non-connected) spacetime $\partial\hat{M}$ with each other.

3. Furthermore, since ∂_t is geodesic, we see that the Clifford multiplication on $S\hat{M}|_{\Sigma_t}$ changes under τ_t as

$$\tau_t(\hat{\gamma}(X)v) = -i\hat{\gamma}(e_0)\tau_t(\hat{\gamma}_t(X)v) = i\hat{\gamma}(e_0)\hat{\gamma}_0(\tau_t X)\tau_t v = \hat{\gamma}(\tau_t X)\tau_t v$$

for n being even and for n being odd analogously. Furthermore, since e_0 is geodesic, we have for $u, v \in S\hat{M}|_{\hat{\Sigma}_t}$

$$\tau_t^* \langle u, v\rangle_0 = \tau_t^* \langle \hat{\gamma}(e_0)u, v\rangle_{SM} = \langle \hat{\gamma}(e_0)\tau_t u, \tau_t v\rangle_{SM} = \langle \tau_t u, \tau_t v\rangle_0$$

for n being even (similar for n being odd). Hence, since τ_t is an isometry for $\langle\cdot,\cdot\rangle_{SM}$ it is also an isometry for $\langle\cdot,\cdot\rangle_0$.

Let us define the function $\rho(t) := \left|\hat{g}_0\right|^{-\frac{1}{4}} \left|\hat{g}_t\right|^{\frac{1}{4}}$, where we denote $|h| := \det(h)$ for a Riemannian metric h. Then the map

$$U(t) := \rho(t)\tau_t : L^2(\hat{\Sigma}_t, S\hat{M}|_{\hat{\Sigma}_t}) \to L^2(\hat{\Sigma}, S\hat{M}|_{\hat{\Sigma}})$$

is unitary and $U(0) = Id$.

The family $\{L^2(\hat{\Sigma}_t, S\hat{M}|_{\hat{\Sigma}_t})\}_{t\in\mathbb{R}}$ and more generally the family $\{H^s(\hat{\Sigma}_t, S\hat{M}|_{\hat{\Sigma}_t})\}_{t\in\mathbb{R}}$ of Sobolev spaces for $s \in \mathbb{R}$ can be considered as a bundle of Hilbert spaces over $\mathbb{R}$ trivialized by the parallel transport τ_t. Let $C^k(\mathbb{R}, L^2(\hat{\Sigma}_t, S\hat{M}|_{\hat{\Sigma}_t}))$ or more generally $C^k(\mathbb{R}, H^s(\hat{\Sigma}_t, S\hat{M}|_{\hat{\Sigma}_t}))$ be the space of C^k-sections of that bundle. Semi-norms of $\psi \in C^k(\mathbb{R}, H^s(\hat{\Sigma}_t, S\hat{M}|_{\hat{\Sigma}_t}))$ are by definition C^k-semi norms of $\mathbb{R} \ni t \mapsto \|\psi(t)\|_{H^s}$. Furthermore, if we set $(U\psi)(t) := U(t)\psi(t)$, then

$$U : C^k(\mathbb{R}, H^s(\hat{\Sigma}_t, S\hat{M}|_{\hat{\Sigma}_t})) \to C^k(\mathbb{R}, H^s(\hat{\Sigma}, S\hat{M}|_{\hat{\Sigma}}))$$

is an isomorphism.

Remark 3.3.8. Since e_0 is tangential to $\partial\hat{M}$, we also know that

$$U(t)|_{\partial\hat{\Sigma}_t} := \rho(t)|_{\partial\hat{\Sigma}_t}\tau_t|_{\partial\hat{\Sigma}_t} : L^2(\partial\hat{\Sigma}_t, S\hat{M}|_{\partial\hat{\Sigma}_t})) \to L^2(\partial\hat{\Sigma}, S\hat{M}|_{\partial\hat{\Sigma}})$$

is unitary, and if we set $(U|_{\partial\hat{M}}\psi)(t) := U(t)|_{\partial\hat{\Sigma}_t}\psi(t)$ then we can also have isomorphisms for the boundary sections as above.

We will use the following lemma, which is based on computations in [47], see also [50, 27], to reduce $\hat{D}$ to Hamiltonian form:

Lemma 3.3.9. *The operator $\hat{D}$ satisfies*

$$\hat{D} = -\hat{\gamma}(e_0)U^{-1}(t)(\partial_t + i\tilde{D}_t)U(t),$$

where $\tilde{D}_t := U(t)\hat{D}_t U(t)^{-1}$.

Proof. Recall that $\hat{D} = -\hat{\gamma}(e_0)(\nabla^{S\hat{M}}_{e_0} + i\hat{D}_t - \frac{n}{2}\hat{H}_t)$. For $\psi \in C^\infty(\hat{M}, S\hat{M})$ we then have

$$\begin{aligned}(\partial_t \circ U\psi)(t) &= \lim_{\varepsilon\to 0}\varepsilon^{-1}(U(t+\varepsilon)\psi|_{\hat{\Sigma}_{t+\varepsilon}} - U(t)\psi|_{\hat{\Sigma}_t})\\ &= \lim_{\varepsilon\to 0}\varepsilon^{-1}(\rho(t+\varepsilon)\tau_{t+\varepsilon}\psi|_{\hat{\Sigma}_{t+\varepsilon}} - \rho(t)\tau_t\psi|_{\hat{\Sigma}_t})\\ &= \tau_t(\lim_{\varepsilon\to 0}\varepsilon^{-1}(\tau^t_{t+\varepsilon}\psi|_{\hat{\Sigma}_{t+\varepsilon}} - \rho(t)\psi|_{\hat{\Sigma}_t}))\\ &= \tau_t((\nabla^{S\hat{M}}_{e_0}\rho(t)\psi)(t)).\end{aligned}$$

Hence, we get

$$\partial_t = \rho(t)^{-1}U(t)\nabla^{S\hat{M}}_{e_0}\circ\rho(t)U(t)^{-1}$$

which is equivalent to

$$U(t)\nabla^{S\hat{M}}_{e_0}U(t)^{-1} = \rho(t)\circ\partial_t\rho(t)^{-1}. \tag{3.3.5}$$

Furthermore, one sees that

$$-2\rho(t)^{-1}(\partial_t\rho(t)) = \operatorname{div}_{\hat{g}} e_0 = n\hat{H}_t,$$

which implies

$$U(t)\left(\frac{n}{2}\hat{H}_t\right)U(t)^{-1} = -\rho(t)^{-1}\partial_t(\rho(t)). \tag{3.3.6}$$

Noting that $U(t)^{-1}\hat{\gamma}(e_0) = \hat{\gamma}(e_0)U(t)^{-1}$, and putting Equation 3.3.5 and Equation 3.3.6 together, we get

$$\begin{aligned}\hat{D} &= \hat{\gamma}(e_0)U(t)^{-1}\left[U(t)\nabla^{S\hat{M}}_{e_0}U(t)^{-1} + i\tilde{D}_t - U(t)\frac{n}{2}\hat{H}_tU(t)^{-1}\right]U(t)\\ &= \hat{\gamma}(e_0)U(t)^{-1}(\rho(t)\circ\partial_t(\rho(t)^{-1}) + i\tilde{D}_t + \rho(t)^{-1}\partial_t(\rho(t))U(t)\\ &= \hat{\gamma}(e_0)U(t)^{-1}(\partial_t + i\tilde{D}_t)U(t),\end{aligned}$$

where we used

$$\partial_t = \rho(t)\rho(t)^{-1}\partial_t(\rho(t)\rho(t)^{-1}) = \rho(t)^{-1}\partial_t(\rho(t)) + \rho(t)\partial_t(\rho(t)^{-1}).$$

This concludes the proof. □

Remark 3.3.10. Note that $\tilde{D}_t$ has principal symbol

$$\sigma_{\tilde{D}_t}(\zeta) = \tau_t\circ\sigma_{\hat{D}_t}(\tau_t^{-1}\zeta)\circ\tau_t^{-1} \tag{3.3.7}$$

which is skew symmetric and satisfies the Clifford relations on $S\hat{M}|_{\hat{\Sigma}}$. Since U is an isometry on the L^2-spaces we also see that $\tilde{D}_t$ is again formally selfadjoint. Hence, $\tilde{D}_t$ is again a formally selfadjoint Dirac type operator.

Furthermore, by Equation 3.3.7, we see that, since D_t has coefficients depending smoothly on t[3] also $\tilde{D}_t$ has coefficients depending smoothly on t.

Remark 3.3.11. Looking at the proof of Lemma 3.3.9, one sees that ∂_t being geodesic is not needed. Even if we would not have done the conformal change first, we could have identified the Cauchy hypersurfaces and brought the operator on Hamiltonian form. But if ∂_t would not be geodesic, then U would not be an isometry and also the principal symbol of $\sigma_{\tilde{D}_t}$ would not be of the convenient form in Equation 3.3.7.

[3]Recall, that the coefficients of D_t depend on g_t, which is smooth in t.

Admissible boundary conditions

Since $\tilde{D}_t$ is a family of elliptic formally selfadjoint Dirac type operators, we can use the theory of Section 3.1 to define boundary conditions in this setting. Let $\tilde{A}_t$ be a boundary operator of $\tilde{D}_t$ such that it is formally selfadjoint and anti-commuting with $\sigma_{\tilde{D}_t}(e_n)$. Then we can consider the check space $\check{H}(\tilde{A}_t)$.

Remark 3.3.12. The principal symbol of $\tilde{A}_t$ can be computed as follows:

$$\begin{aligned}
\sigma_{\tilde{A}_t}(\zeta) &= \sigma_{\tilde{D}_t}(\hat{\eta}(0)^\flat)^{-1}\sigma_{\tilde{D}_t}(\zeta)\\
&= (\tau_t\sigma_{\hat{D}_t}(\tau_t^{-1}\hat{\eta}(0)^\flat)\tau_t^{-1})^{-1}(\tau_t\sigma_{\hat{D}_t}(\tau_t^{-1}\zeta)\tau_t^{-1})\\
&= \tau_t\sigma_{\hat{D}_t}(\tau_t^{-1}\hat{\eta}(0)^\flat)^{-1}\sigma_{\hat{D}_t}(\tau_t^{-1}\zeta)\tau_t^{-1}\\
&= \tau_t\sigma_{D_t}(\tau_t^{-1}\hat{\eta}(0)^\flat)^{-1}\sigma_{D_t}(\tau_t^{-1}\zeta)\tau_t^{-1}
\end{aligned}$$

where we used Equation 3.3.4, Equation 3.3.7 and τ_t being an isometry for g_t.

This shows that in general $\tilde{A}_t$ cannot easy be related to A_t since the principal symbols do not directly relate to each other. But assuming $N|_{\partial M} = 1$ and η being parallel transported along ∂_t, one sees that

$$\sigma_{\tilde{A}_t}(\zeta) = \tau_t\sigma_{D_t}(\eta(t)^\flat)^{-1}\sigma_{D_t}(\tau_t^{-1}\zeta)\tau_t^{-1},$$

and hence one can choose $\tilde{A}_t = UA_tU^{-1}$.

Furthermore, since $\tilde{D}_t$ is a Dirac operator in the sense of Gromov and Lawson, $\tilde{A}_t$ can be chosen naturally in terms of D_t and the mean curvature of $\partial\Sigma$. Thus, $\tilde{A}_t$ has coefficients depending smoothly on t.

Next we will define suitable families of boundary conditions to construct Lorentzian boundary conditions later on.

Definition 3.3.13. We call a family $B = \{B_t\}_{t\in\mathbb{R}}$ of boundary conditions $B_t \subseteq H^{\frac{1}{2}}(\Sigma_t, SM|_{\Sigma_t})$ an *admissible boundary condition* for D if

1. $N^{\frac{1}{2}}(t)B_t$ is a selfadjoint boundary condition for all $t \in \mathbb{R}$,
2. $\tilde{B}_t := U(t)N^{\frac{n}{2}}(t)B_t \subseteq \check{H}(\tilde{A}_t)$ is a selfadjoint ∞-regular boundary condition for all $t \in \mathbb{R}$, and
3. $t \mapsto \tilde{D}_{t,\tilde{B}_t}J^{(\varepsilon)}_{\tilde{B}_t}$ is strongly continuous with respect to $\|\cdot\|_{H^k}$ for all $k \in \mathbb{N}$.

Remark 3.3.14. Let us give an intuition for the conditions above. The first condition is needed for the boundary term in the Green formula 2.2.4 to vanish if the boundary condition is satisfied. The second condition is needed for higher regularity of solution, while the third condition is needed for finding continuous solutions in time.

We will often use the norm equivalence of $\|\cdot\|_{\tilde{B}_t,k}$ and $\|\cdot\|_{H^k}$. For example, we now define for $l, k \in \mathbb{N}$

$$C^l(\mathbb{R}, H^k_{\tilde{B}_\bullet}(\hat{\Sigma}, S\hat{M}|_{\hat{\Sigma}})) := \{\psi \in C^l(\mathbb{R}, H^k(\hat{\Sigma}, S\hat{M}|_{\hat{\Sigma}}));\ \psi(t) \in H^k_{\tilde{B}_t}(\hat{\Sigma}, S\hat{M}|_{\hat{\Sigma}})\}$$

as a subspace of $C^l(\hat{\Sigma}, S\hat{M}|_{\hat{\Sigma}})$. Semi-norms of $\psi \in C^l(\mathbb{R}, H^k_{\tilde{B}_\bullet})$ are by definition C^l-semi norms of $\mathbb{R} \ni t \mapsto \|\psi(t)\|_{H^k}$. Furthermore, $C^l(I, H^k_{\tilde{B}_\bullet}(\hat{\Sigma}, S\hat{M}|_{\hat{\Sigma}}))$ are Banach spaces for compact intervals I. In this case, we will denote the norms by

$$\|\cdot\|_{l,I,H^k}\,.$$

Let B be an admissible boundary condition. Then let us define the *Lorentzian boundary condition* as

$$C^\infty(\partial M, B) := \{\psi \in C^\infty(\partial M, SM|_{\partial M}); \psi|_{\partial\Sigma_t} \in B_t \text{ for all } t \in \mathbb{R}\}.$$

Lemma 3.3.15. *Assume the Standard Setup 2.4 and let B be an admissible boundary condition. Then*

$$\int_M \langle D\psi, \phi\rangle + \langle \psi, D\phi\rangle \,\mathrm{d}\mu_M = 0,$$

for all $\psi, \phi \in C^\infty(M, SM)$ with $\psi|_{\partial M}, \phi|_{\partial M} \in C^\infty(\partial M, B)$.

Proof. Since the gradient of the temporal function is tangential to ∂M, we have by Fubini's Theorem

$$\begin{aligned}\int_{\partial M} \langle \gamma(\eta)\psi, \phi\rangle_{SM} \,\mathrm{d}\mu_{\partial M} &= \int_{\mathbb{R}} \int_{\partial\Sigma_t} \langle \gamma(\eta)\psi, \phi\rangle\, N(t)\,\mathrm{d}\mu_{\partial\Sigma_t}\,\mathrm{d}t \\ &= -i \int_{\mathbb{R}} \left\langle \sigma_{D_t}(\eta)\psi, \phi\right\rangle_0 N(t)\,\mathrm{d}\mu_{\partial\Sigma_t}\,\mathrm{d}t \\ &= -i \int_{\mathbb{R}} \left\langle \sigma_{D_t}(\eta)N^{\frac{1}{2}}(t)\psi, N^{\frac{1}{2}}(t)\phi\right\rangle_0 \mathrm{d}\mu_{\partial\Sigma_t}\,\mathrm{d}t \\ &= 0,\end{aligned}$$

where we used additionally how the inner products and Clifford multiplications are related as well as the first part of Definition 3.3.13. Now by Equation 2.2.4, the claim follows. □

4 Initial Boundary Value Problems

In this chapter we will discuss the Cauchy problems arising from the Lorentzian boundary conditions defined in Definition 3.3.13. In the first part of this Chapter we will introduce these Cauchy problems and discuss how they relate to the Cauchy problems for the operator in Hamiltonian form. The main goal of this chapter is to show well-posedness of these Cauchy problems. For this we will show uniqueness using an L^2-estimate in Section 4.2. Proceeding, we will show the existence of smooth solutions which depend continuously on the Cauchy data in Section 4.3. For the whole chapter, we will assume the Standard Setup 2.4.

4.1 The Cauchy problems

In this section, we will use the Lorentzian boundary conditions to write down initial boundary value problems for the Lorentzian Dirac operator. For this, let $B = \{B_t\}_{t\in\mathbb{R}}$ be an admissible boundary condition as defined in Definition 3.3.13. Then consider the following Cauchy problem:

$$\begin{cases} D\psi = f \in C^\infty_{cc}(M, SM) \\ \psi|_\Sigma = \psi_0 \in C^\infty_{cc}(M, SM|_\Sigma) \\ \psi|_{\partial M} \in C^\infty(\partial M, B) \end{cases} . \tag{4.1.1}$$

Applying the reduction we did in Subsection 3.3.3, we see that $\psi \in C^\infty(M, SM)$ is a solution to the Cauchy problem 4.1.1 if and only if $\tilde{\psi} = UN^{\frac{n}{2}}\psi$ is a solution to the following Cauchy problem

$$\begin{cases} \tilde{D}\tilde{\psi} = \tilde{f} \in C^\infty_c(\mathbb{R}, C^\infty_{cc}(\hat{\Sigma}, S\hat{M}|_{\hat{\Sigma}})) \\ \tilde{\psi}(0) = N^{\frac{n}{2}}(0)\psi_0 \in C^\infty_{cc}(\hat{\Sigma}, S\hat{M}|_{\hat{\Sigma}}) \\ \tilde{\psi}|_{\partial\hat{\Sigma}} \in C^\infty(\mathbb{R}, \tilde{B}), \end{cases} \tag{4.1.2}$$

where $\tilde{f} = UN^{\frac{n+2}{2}} f$ and

$$C^\infty(\mathbb{R}, \tilde{B}) := \{\psi \in C^\infty(\mathbb{R}, C^\infty(\partial\hat{\Sigma}, S\hat{M}|_{\partial\hat{\Sigma}}));\ \psi(t) \in \tilde{B}_t\ \forall t \in \mathbb{R}\}.$$

Furthermore, note that $\tilde{\psi}$ is a solution to the Cauchy problem 4.1.2 if and only if it is a solution to

$$\begin{cases} -\hat{\gamma}(e_0)\tilde{D} = (\partial_t + i\tilde{D}_t)\tilde{\psi} = -\hat{\gamma}(e_0)\tilde{f} \\ \tilde{\psi}(0) = N^{\frac{n}{2}}(0)\psi_0 \\ \tilde{\psi}|_{\partial\Sigma} \in C^\infty(\mathbb{R}, \tilde{B}) \end{cases} . \tag{4.1.3}$$

We will abuse notation and denote $-\hat{\gamma}(e_0)\tilde{D}$ and $-\hat{\gamma}(e_0)\tilde{f}$ by $\tilde{D}$ and $\tilde{f}$ as well.

We aim for proving the well-posedness for the Cauchy problem 4.1.1, which consists of three parts; uniqueness, existence and continuous dependence on the Cauchy data. In the next subsection, we will talk about the uniqueness of the solution.

4.2 Energy estimate and uniqueness

Here we are following the approach of [12, 27], by first proving an L^2 energy estimate, which we will then use for the proof of uniqueness.
For $t_0, t_1 \in \mathbb{R}$, we will denote $M_{[t_0,t_1]} := \tau^{-1}([t_0, t_1])$ and $\partial M_{[t_0,t_1]} := M_{[t_0,t_1]} \cap \partial M$.

Proposition 4.2.1. *Assume the Standard Setup 2.4 and let B be an admissible boundary condition. Then for all $t_0, t_1 \in \mathbb{R}$ there exists a constant $C = C[t_0, t_1]$ such that*

$$\int_{\Sigma_{t_1}} |\psi|_0^2 \,\mathrm{d}\mu_{\Sigma_{t_1}} \leq e^{C(t_1-t_0)} \left[C \int_{t_0}^{t_1} \int_{\Sigma_s} |D\psi|_0^2 \,\mathrm{d}\mu_{\Sigma_s} \,\mathrm{d}s + \int_{\Sigma_{t_0}} |\psi|_0^2 \,\mathrm{d}\mu_{\Sigma_{t_0}} \right], \tag{4.2.1}$$

applies for all $\psi \in C^\infty(M, SM)$ such that $\psi|_{\partial M} \in C^\infty(\partial M, B)$.

Proof. By the divergence theorem on the set $M_{[t_0,t_1]}$ with piecewise smooth boundary

$$\partial(M_{[t_0,t_1]}) = \partial M_{[t_0,t_1]} \cup \Sigma_{t_1} \cup \Sigma_{t_0}$$

we get

$$\begin{aligned} &\int_{M_{[t_0,t_1]}} \langle D\psi, \psi\rangle_{SM} + \langle \psi, D\psi\rangle_{SM} \,\mathrm{d}\mu_M \\ &= -\int_{\partial M_{[t_0,t_1]}} \langle \gamma(\eta)\psi, \psi\rangle_{SM} \,\mathrm{d}\mu_{\partial M} + \int_{\Sigma_{t_1}} |\psi|_0^2 \,\mathrm{d}\mu_{\Sigma_{t_1}} - \int_{\Sigma_{t_0}} |\psi|_0^2 \,\mathrm{d}\mu_{\Sigma_{t_0}} \end{aligned}$$

The right hand side can be estimated as

$$\int_{M_{[t_0,t_1]}} \langle D\psi, \psi\rangle_{SM} + \langle \psi, D\psi\rangle_{SM} \, \mathrm{d}\mu_M \leq C \int_{t_0}^{t_1} |D\psi|_0^2 + |\psi|_0^2 \, \mathrm{d}\mu_{\Sigma_t} \, \mathrm{d}t,$$

where we used Cauchy–Schwarz, Fubini (where N can be bounded from above on the compact set $M_{[t_0,t_1]}$) and that $\gamma(\nu)$ is bounded. For the left hand side, note that by Lemma 3.3.15 the first term vanishes, hence we get in total

$$\int_{\Sigma_{t_1}} |\psi|_0^2 \, \mathrm{d}\mu_{\Sigma_{t_1}} \leq \int_{\Sigma_{t_0}} |\psi|_0^2 \, \mathrm{d}\mu_{\Sigma_{t_0}} + C \int_{t_0}^{t_1} |D\psi|_0^2 + |\psi|_0^2 \, \mathrm{d}\mu_{\Sigma_t} \, \mathrm{d}t.$$

Gronwall's Lemma now gives us the desired estimate.

□

Now we can use Proposition 4.2.1 to prove the following uniqueness statement:

Corollary 4.2.2. *Assume the Standard Setup 2.4 and let B be an admissible boundary condition. A section $\psi \in C^\infty(M, SM)$ with $\psi|_{\partial M} \in C^\infty(\partial M, B)$ is uniquely determined by $D\psi$ and ψ on Σ_0.*

Proof. Choose $t_1 \in \mathbb{R}$ arbitrary and $t_0 = 0$. If $\psi|_{\Sigma_0} = 0$ and $D\psi = 0$, then Proposition 4.2.1 implies that $\psi|_{\Sigma_{t_1}} = 0$. Since t_1 was chosen arbitrarily, we see that $\psi \equiv 0$. □

4.3 Well-posedness of the Cauchy problems

In this section we will prove our main result, the well-posedness of the Cauchy problem 4.1.1. The main part of the proof will be showing the existence of solutions, which will be done by an Arzelà–Ascoli argument similar as in [12, 27].

Theorem 4.3.1. *Assume the Standard Setup 2.4 and let B be an admissible boundary condition, then there exists a unique smooth solution $\psi \in C^\infty(M, SM)$ to the Cauchy problem 4.1.1, which depends continuously on the Cauchy data (f, ψ_0).*

Proof. Recall that we proved uniqueness in Corollary 4.2.2.

A. Existence of smooth solutions

For showing existence of smooth solutions to the Cauchy problem 4.1.1, we will show the existence of smooth solutions to the Cauchy problem 4.1.3. Now let us define the space

$$C^\infty(\hat{M}, \tilde{B}) := \{\psi \in C^\infty(\mathbb{R}, S\hat{M}|_{\hat{\Sigma}}); \psi(t)|_{\partial\hat{\Sigma}} \in \tilde{B}_t\}.$$

We follow the strategy of [12, 27] and show first the existence of regularized solutions. Then we will apply Arzelà–Ascoli to get a converging subsequence, whose limit will be our smooth solution. In the following, we will only work on $S\hat{M}|_{\hat{\Sigma}}$, so let us shorten the notation by setting $H^k := H^k(\hat{\Sigma}, S\hat{M}|_{\hat{\Sigma}})$ and $H^k_{\tilde{B}_t} := H^k_{\tilde{B}_t}(\hat{\Sigma}, S\hat{M}|_{\hat{\Sigma}})$.

1. *Regularized problem*
   First, let us show that for all $\varepsilon > 0$ there is a unique solution to the regularized problem
   $$(\partial_t + i\tilde{D}_{t,\tilde{B}_t} J^{(\varepsilon)}_{\tilde{B}_t})\tilde{\psi}^{(\varepsilon)}(t) = \tilde{f}(t), \tag{4.3.1}$$
   with $\tilde{\psi}^{(\varepsilon)}(0) = N^{\frac{n}{2}}(0)\psi_0$. Since $\{\tilde{D}_{t,\tilde{B}_t} J^{(\varepsilon)}_{\tilde{B}_t}\}_{t\in\mathbb{R}}$ is a H^k strongly continuous family of bounded operator from $H^k_{\tilde{B}_t}$ to $H^k_{\tilde{B}_t}$, Theorem X.69 in [41] implies the existence of a unique solution $\tilde{\psi}^{(\varepsilon),k} \in C^1(\mathbb{R}, H^k_{\tilde{B}_\bullet})$ for all $k \in \mathbb{N}$. Since $H^k_{\tilde{B}_t} \subseteq H^l_{\tilde{B}_t}$ for $l < k$, we get by the uniqueness of solutions, that there is one solution $\tilde{\psi}^{(\varepsilon)}$ independent of k. In particular $\tilde{\psi}^{(\varepsilon)}$ is smooth in spatial directions.

2. *Requirements of the Arzelà–Ascoli theorem*
   Consider now $\tilde{\psi}^{(\varepsilon)} \in C^1(\mathbb{R}, H^k_{\tilde{B}_\bullet})$. We will derive estimates for the growth of $\tilde{\psi}^{(\varepsilon)}$ in time, and the important fact is that the bounds c_j do not depend on u, u_0, f and ε. They do depend on time t, but continuously, hence they are bounded on compact time intervals. So let us compute the following t-derivative:
   $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\tilde{\psi}^{(\varepsilon)}(t)\right\|^2_{\tilde{B}_t,2k} &= \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \Delta^k_{\tilde{B}_t}\tilde{\psi}^{(\varepsilon)}(t), \Delta^k_{\tilde{B}_t}\tilde{\psi}(t)\right\rangle_{L^2} \\ &= 2\,\mathrm{Re}\left(\left\langle \partial_t\Delta^k_{\tilde{B}_t}\tilde{\psi}^{(\varepsilon)}(t), \Delta^k_{\tilde{B}_t}\tilde{\psi}^{(\varepsilon)}(t)\right\rangle_{L^2}\right) \\ &= 2\,\mathrm{Re}\left(\left\langle [\partial_t, \Delta^k_{\tilde{B}_t}]\tilde{\psi}^{(\varepsilon)}(t), \Delta^k_{\tilde{B}_t}\tilde{\psi}^{(\varepsilon)}(t)\right\rangle_{L^2} + \left\langle \Delta^k_{\tilde{B}_t}\partial_t\tilde{\psi}^{(\varepsilon)}(t), \Delta^k_{\tilde{B}_t}\tilde{\psi}^{(\varepsilon)}(t)\right\rangle_{L^2}\right).\end{aligned}$$
   Note that in contrast to [12], we do not need to differentiate the volume element, since we are working on a constant L^2-space. A priori, the commutator $[\partial_t, \nabla^k]$ is a differential operator of order $2k+1$, but we see that the principal symbol is vanishing:
   $$\sigma_{[\partial_t,\Delta^k_{\tilde{B}_t}]}(\zeta) = [\sigma_{\partial_t}(\zeta), \sigma_{\Delta^k_{\tilde{B}_t}}(\zeta)] = [\sigma_{\partial_t}(\zeta), -|\zeta|^{2k}] = 0,$$
   where we used that $\Delta_{\tilde{B}_t}$ is of Laplace type. Hence, $[\partial_t, \Delta^k_{\tilde{B}_t}]$ is of order $2k$, and we can bound the norm of the commutator by $\|\cdot\|_{2k,\tilde{B}_k}$. So we continue to estimate
   $$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\tilde{\psi}^{(\varepsilon)}(t)\right\|^2_{\tilde{B}_t,2k} \leq c_1\left\|\psi^{(\varepsilon)}(t)\right\|^2_{\tilde{B}_t,2k} + 2\,\mathrm{Re}\left(\left\langle \Delta^k_{\tilde{B}_t}\partial_t\tilde{\psi}^{(\varepsilon)}(t), \Delta^k_{\tilde{B}_t}\tilde{\psi}^{(\varepsilon)}(t)\right\rangle_{L^2}\right),$$

where we also used Cauchy–Schwarz. Now we use that $\psi^{(\varepsilon)}$ is a solution to the regularized problem 4.3.1 to estimate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\tilde{\psi}^{(\varepsilon)}(t)\right\|^2_{\tilde{B}_t,2k} \leq & c_1\left\|\tilde{\psi}^{(\varepsilon)}(t)\right\|^2_{\tilde{B}_t,2k}+2\operatorname{Re}\left(\left\langle\Delta^k_{\tilde{B}_t}\tilde{f}(t),\Delta^k_{\tilde{B}_t}\tilde{\psi}^{(\varepsilon)}(t)\right\rangle_{L^2}\right)\\ &+2\operatorname{Im}\left(\left\langle\Delta^k_{\tilde{B}_t}\tilde{D}_{t,\tilde{B}_t}J^{(\varepsilon)}_{\tilde{B}_t}\tilde{\psi}^{(\varepsilon)}(t),\Delta^k_{\tilde{B}_t}\psi^{(\varepsilon)}(t)\right\rangle_{L^2}\right).\end{aligned}$$

We estimate the second summand by

$$2\operatorname{Re}\left\langle\Delta^k_{\tilde{B}_t}\tilde{f}(t),\Delta^k_{\tilde{B}_t}\tilde{\psi}^{(\varepsilon)}\right\rangle_{L^2}\leq\|\tilde{f}(t)\|^2_{\tilde{B}_t,2k}+\left\|\tilde{\psi}^{(\varepsilon)}(t)\right\|^2_{\tilde{B}_t,2k},$$

where we again used Cauchy–Schwarz. For the third summand, we note that the C^0-semigroup $J^{(\frac{\varepsilon}{2})}_{\tilde{B}_t}$ commutes with $\Delta_{\tilde{B}_t}$ as well as with $\tilde{D}_{t,\tilde{B}_t}$ and is selfadjoint. Thus

$$\begin{aligned}2\operatorname{Im}\left\langle\Delta^k_{\tilde{B}_t}\tilde{D}_{t,\tilde{B}_t}J^{(\varepsilon)}_{\tilde{B}_t}\tilde{\psi}^{(\varepsilon)}(t),\Delta^k_{\tilde{B}_t}\psi^{(\varepsilon)}(t)\right\rangle_{L^2}&=2\operatorname{Im}\left\langle\Delta^k_{\tilde{B}_t}\tilde{D}_{t,\tilde{B}_t}J^{(\frac{\varepsilon}{2})}_{\tilde{B}_t}\tilde{\psi}^{(\varepsilon)}(t),\Delta^k_{\tilde{B}_t}J^{(\frac{\varepsilon}{2})}_{\tilde{B}_t}\tilde{\psi}^{(\varepsilon)}(t)\right\rangle_0\\ &=2\operatorname{Im}\left\langle\tilde{D}_{t,\tilde{B}_t}\Delta^k_{\tilde{B}_t}J^{(\frac{\varepsilon}{2})}_{\tilde{B}_t}\tilde{\psi}^{(\varepsilon)}(t),\Delta^k_{\tilde{B}_t}J^{(\frac{\varepsilon}{2})}_{\tilde{B}_t}\tilde{\psi}^{(\varepsilon)}(t)\right\rangle_0\\ &=0\end{aligned}$$

where we also used that $\tilde{D}_{t,\tilde{B}_t}$ commutes with its functional calculus and is formally selfadjoint on $H^k_{\tilde{B}_t}$. Putting everything together, we get the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\tilde{\psi}^{(\varepsilon)}(t)\right\|^2_{\tilde{B}_t,2k}\leq\|\tilde{f}(t)\|^2_{\tilde{B}_t,2k}+c_1\left\|\tilde{\psi}^{(\varepsilon)}(t)\right\|^2_{\tilde{B}_t,2k}.$$

Using Gronwall's Lemma we get

$$\left\|\tilde{\psi}^{(\varepsilon)}(t)\right\|^2_{\tilde{B}_t,2k}\leq\left[\left\|\psi^{(\varepsilon)}(t)\right\|^2_{\tilde{B}_t,2k}+\int_0^t\|\tilde{f}(t)\|^2_{\tilde{B}_t,2k}\,\mathrm{d}s\exp\left(\int_0^t c_1(s)\,\mathrm{d}s\right)\right]$$

and know that c_1 is independent of ε. For $t<0$ one obtains an analogue estimate by integrating over $[t,0]$. Hence, we see that for $t\in\mathbb{R}$ fixed, the set $\{\tilde{\psi}^{(\varepsilon)}(t);\varepsilon>0\}$ is bounded in $H^{2k}_{\tilde{B}_t}$, for all k and hence it is bounded in H^{2k} for all k. By Rellich–Kondracchov theorem $\{\psi^{(\varepsilon)}(t),\ \varepsilon>0\}$ is relatively compact in H^k for all k.

Now taking $\tilde{D}_{t,\tilde{B}_t} J^{(\varepsilon)}_{\tilde{B}_t}$ being bounded from $H^k_{\tilde{B}_t}$ to $H^k_{\tilde{B}_t}$ into account, we get

$$\begin{aligned}\left\|\partial_t \tilde{\psi}^{(\varepsilon)}(t)\right\|_{\tilde{B}_t,k} &= \left\|\tilde{f}(t) - i\tilde{D}_{t,\tilde{B}_t} J^{(\varepsilon)}_{\tilde{B}_t} \tilde{\psi}^{(\varepsilon)}(t)\right\|_{\tilde{B}_t,k} \\ &\leq \left\|\tilde{f}(t)\right\|_{\tilde{B}_t,k} + c_2 \left\|\tilde{\psi}^{(\varepsilon)}(t)\right\|_{\tilde{B}_t,k} \leq c_3,\end{aligned}$$

where c_3 does not depend on ε. Using that $\|\cdot\|_{\tilde{B}_t,k} \simeq \|\cdot\|_k$ on $H^k_{\tilde{B}_t}$, we see that $t \mapsto \tilde{\psi}^{(\varepsilon)}(t)$ is equicontinuous.

3. *Arzelà–Ascoli theorem*
For fixed $T > 0$ and fixed k the Arzelà–Ascoli theorem implies that $\{\tilde{\psi}^{(\varepsilon)}; \varepsilon > 0\} \subseteq C^0([-T,T], H^k)$ is relatively compact. Thus we obtain a subsequence $\tilde{\psi}^{(\varepsilon_j)}$ of $\tilde{\psi}^{(\varepsilon)}$ with $\tilde{\psi}^{(\varepsilon_j)} \to \tilde{\psi} \in C^0([-T,T], H^k)$ for $\varepsilon_j \to 0$. By a diagonal subsequence argument we can without loss of generality assume that $\tilde{\psi}^{(\varepsilon_j)} \to \tilde{\psi} \in C^0([-T,T], H^k)$ for all $k \in \mathbb{N}$, $T > 0$. Therefore the convergence $\tilde{\psi}^{(\varepsilon_j)} \to \psi$ is locally uniform in $C^0(\mathbb{R}, H^k)$ for all k. This entails in particular that $\tilde{\psi}^{(\varepsilon_j)}(t) \to \tilde{\psi}(t)$ for all $t \in \mathbb{R}$ in H^k. Since $H^k_{\tilde{B}_t} \subseteq H^k$ closed, this implies that $\tilde{\psi}(t) \in H^k_{\tilde{B}_t}$ for all t. In particular, $\tilde{\psi}$ satisfies the Lorentzian boundary condition.

4. *Solution to Cauchy problem 4.1.3*
First we see that for $t = 0$

$$\tilde{\psi}^{(\varepsilon)}(0) = N^{\frac{n}{2}}(0)\psi_0 \ \text{ for all } \ \varepsilon_j,$$

and therefore $\tilde{\psi}(0) = N^{\frac{n}{2}}(0)\psi_0$.
Showing $\tilde{D}\tilde{\psi} = \tilde{f}$ is more complicated since we also have to control the convergence of the time derivatives of $\tilde{\psi}^{(\varepsilon_j)}$ to the time derivatives of $\tilde{\psi}$. In order to get rid of the time derivatives we integrate the regularized problem 4.3.1 and we obtain

$$\tilde{\psi}^{(\varepsilon_j)}(t) - N^{\frac{n}{2}}(0)\psi_0 = \int_0^t \left[-i\tilde{D}_{s,\tilde{B}_s} J^{(\varepsilon_j)}_{\tilde{B}_s} \tilde{\psi}^{(\varepsilon_j)}(s) + \tilde{f}(s)\right] \mathrm{d}s. \tag{4.3.2}$$

Now we let $\varepsilon \to 0$. For the left hand side of Equation 4.3.2 we find $\tilde{\psi}^{(\varepsilon_j)}(t) - N^{\frac{n}{2}}(0)\psi_0 \to \tilde{\psi}(t) - N^{\frac{n}{2}}(0)\psi_0$. For the right hand side of Equation 4.3.2 we consider the first summand under the integral which is the one depending on ε_j. For all k the H^k-norm of this can be estimated as

$$\begin{aligned}\left\|\int_0^t \tilde{D}_{s,\tilde{B}_s} J^{(\varepsilon_j)}_{\tilde{B}_s} \tilde{\psi}^{(\varepsilon_j)}(s)\,\mathrm{d}s\right\|_k &\leq \int_0^t \left\|\tilde{D}_{s,\tilde{B}_s} J^{(\varepsilon_j)}_{\tilde{B}_s} \tilde{\psi}^{(\varepsilon_j)}(s)\right\|_k \mathrm{d}s \\ &\leq \int_0^t \left\|\tilde{D}_{s,\tilde{B}_s} \tilde{\psi}^{(\varepsilon_j)}(s)\right\|_k \mathrm{d}s,\end{aligned}$$

where we used that $J^{(\varepsilon_j)}_{\tilde{B}_s}$ commutes with $\tilde{D}_{s,\tilde{B}_s}$ and is a contraction. Moreover, notice that the integrand on the right hand side is a continuous function of s, which pointwise converges to $\left\|\tilde{D}_{s,\tilde{B}_s}\tilde{\psi}(s)\right\|_k$ as $j \to \infty$. A dominant convergence argument leads to

$$\tilde{\psi}(t) - UN^{\frac{n}{2}}\psi_0 = \int_0^t \tilde{f}(s)\,\mathrm{d}s - i\int_0^t \tilde{D}_{s,\tilde{B}_s}\tilde{\psi}(s)\,\mathrm{d}s,$$

which is equivalent to the fulfillment for the reduced Dirac equation 4.1.3 together with the initial value $\tilde{\psi}(0) = N^{\frac{n}{2}}(0)\psi_0$.

5. *Regularity*
So far we know continuity in time direction and smoothness in spatial direction. Next, we want to prove smoothness in time direction. We have $\tilde{\psi} \in C^0(\mathbb{R}, H^k)$ for all k and by the equation in the Cauchy problem 4.1.3 we see that $\partial_t\tilde{\psi} \in C^0(\mathbb{R}, H^{k-1})$ for all k. Hence $\tilde{\psi} \in C^1(\mathbb{R}, H^k)$ for all k. For the second derivative, we differentiate the equation in the Cauchy problem 4.1.3 in time and by comparing both sides of the equation, we see that $\tilde{\psi} \in C^2(\mathbb{R}, H^k)$. By iterating this argument, we see that $\tilde{\psi} \in C^\infty(\mathbb{R}, C^\infty(\hat{\Sigma}, S\hat{M}|_{\hat{\Sigma}}))$. Hence we showed the existence of a smooth solution to the Cauchy problem 4.1.3 and equivalently to the Cauchy problem 4.1.1.

B. Continuity of the solution map
For this part let us have a look at the Cauchy problem 4.1.1 itself, and let us denote

$$C^\infty(M, B) = \{\psi \in C^\infty(M, SM);\ \psi|_{\partial M} \in C^\infty(\partial M, B)\}.$$

The continuous dependence on the Cauchy data means in other words that the solution map

$$\begin{aligned} C^\infty_{cc}(M, SM) \oplus C^\infty_{cc}(\Sigma, SM|_\Sigma) &\to C^\infty(M, B) \\ (f, \psi_0) &\mapsto \psi \end{aligned}$$

is continuous. Let us define the following map

$$\begin{aligned} P := D \oplus \mathrm{res}_0 :\ C^\infty(M, B) &\to C^\infty(M, SM) \oplus C^\infty(\Sigma, SM|_\Sigma) \\ \psi &\mapsto (D\psi, \psi|_\Sigma), \end{aligned}$$

which is clearly continuous and linear. Fix compact sets $A_i \subseteq M$ such that $A_i \cap \partial M = \emptyset$ for $i = 1, 2$. Then define $C^\infty_{A_1}(M, SM)$ as the space of smooth sections with support in A_1, and similarly we also define $C^\infty_{A_2}(\Sigma, SM|_\Sigma)$ as the set of the smooth sections with support in $A_2 \cap \Sigma$. Then $C^\infty_{A_1}(M, SM) \oplus C^\infty_{A_2}(\Sigma, SM|_\Sigma)$ is closed in $C^\infty_{cc}(M, SM) \oplus C^\infty_{cc}(\Sigma, SM|_\Sigma)$, and hence

$$\mathcal{V}_{A_1,A_2} := P^{-1}\left(C^\infty_{A_1}(M, SM) \oplus C^\infty_{A_2}(\Sigma, SM|_\Sigma)\right)$$

is closed in $C^\infty(M, B)$. In particular all these spaces are Fréchet spaces. By step (1) and (2) of this proof, P maps $\mathcal{V}_{A_1,A_2}$ bijectively on $C^\infty_{A_1}(M, SM) \oplus C^\infty_{A_2}(\Sigma, SM|_\Sigma)$. The open mapping theorem for Fréchet spaces then gives us that

$$(P|_{\mathcal{V}_{A_1,A_2}})^{-1}; C^\infty_{A_1}(M, SM) \oplus C^\infty_{A_2}(\Sigma, SM|_\Sigma) \to \mathcal{V}_{A_1,A_2}$$

is continuous as well. The arbitrariness of the A_i shows the claim. □

Remark 4.3.2.

1. Note, that we do not prove that the solution map is surjective, which can also not be true in general, since we assume that ψ_0 and f being supported away from the boundary.

2. Note that one can extend this result to the case of not necessarily compact Cauchy hypersurfaces, but with still compact boundary. This follows by an covering argument, see for example [12, 27], where it is also shown that the well-posedness implies that the Lorentzian Dirac operator is Green-hyperbolic with respect to these boundary conditions.

5 Pseudo Local Boundary Conditions

Up to now we worked with a class of admissible boundary conditions, which fulfill properties such that the Cauchy problem is well-posed. In this chapter we will discuss the special case of pseudo local boundary conditions and how to simplify the properties of Definition 3.3.13. First we will discuss the well-posedness for pseudo local boundary conditions with respect to Grassmannian projections and then we will apply this to the discussion of the Atiyah–Patodi–Singer conditions.

5.1 Grassmannian projections

In this section, we will discuss the well-posedness of Cauchy problems with pseudo local boundary conditions with respect to Grassmannian projections. For this we will discuss the properties of Definition 3.3.13 in terms of pseudo differential operators and then show the well-posedness by using Theorem 4.3.1.

Theorem 5.1.1. *Let (M, g) be a globally hyperbolic, spatially compact spin manifold with timelike boundary ∂M. Let Σ be a spacelike Cauchy hypersurface. Let $t\colon\ M \to \mathbb{R}$ be a temporal function such that $\Sigma_0 = \Sigma$ and the gradient of t is tangential to ∂M. Let $\{P_t\}_{t\in\mathbb{R}}$ be a family of orthogonal pseudo differential operators on $L^2(\partial\Sigma_t, SM|_{\partial\Sigma_t})$ such that*

1. *$P_t = N^{-1}P_t^* N$,*
2. *$P_t = Id + \sigma_{D_t}(\eta_t^\flat)P_t\sigma_{D_t}(\eta_t^\flat)$,*
3. *$\tilde{P}_t := U N^{\frac{n}{2}} P_t N^{\frac{n}{2}} U^{-1}$ is a Grassmannian projection on $L^2(\hat{\Sigma}_0, SM|_{\partial\hat{\Sigma}_0})$, and*
4. *there exists a sequence $\{k_j\}_{j\in\mathbb{N}_0}$ of non-negative integers with $k_0 = 0$ and $k_j \to \infty$ for $j \to \infty$ such that $t \mapsto \tilde{P}_t$ is H^{k_j} norm continuous for all j.*

Then $B := \{B_t\}_{t\in\mathbb{R}}$ with $B_t := P_t H^{\frac{1}{2}}(\partial\Sigma_t, SM|_{\partial\Sigma_t})$ is an admissible boundary condition and in particular there exists a unique smooth solution $\psi \in C^\infty(M, SM)$ to

$$\begin{cases} D\psi = f \in C^\infty_{cc}(M, SM) \\ \psi|_\Sigma = \psi_0 \in C^\infty_{cc}(\Sigma, SM|_\Sigma) \\ \psi|_{\partial M} \in C^\infty(\partial M, B) \end{cases} \tag{5.1.1}$$

that depends continuously on the Cauchy data (f, ψ_0).

Proof. First note that 1. and 2. implies, with a similar argument as in Corollary 3.1.31, that $N^{\frac{1}{2}} B_t$ is a selfadjoint boundary condition. By Corollary 3.1.31, Theorem 3.1.30 and the third assumption on P_t, we see that $\tilde{B}_t$ is a selfadjoint ∞-regular boundary condition. Furthermore, by Lemma 3.2.2, we see that the fourth assumption on P_t implies that $\tilde{D}_{t,\tilde{B}_t} J^{(\varepsilon)}_{\tilde{B}_t}$ is strongly continuous with respect to all H^k-norms.
All of this together implies that $\{B_t\}_{t\in\mathbb{R}}$ is an admissible boundary condition (see Definition 3.3.13) and hence, Theorem 1 implies the well-posedness of Cauchy problem 5.1.1. □

Remark 5.1.2. Note that if we would additionally assume that P_t (anti-) commutes with the boundary operator $\tilde{A}_t$ in Theorem 5.1.1, we could us weaken the fourth assumption of the same theorem and only require the projections to be L^2 norm continuous. Then Lemma 3.2.3 would still give us that $\tilde{D}_{t,\tilde{B}_t} J^{(\varepsilon)}_{\tilde{B}_t}$ is strongly continuous with respect to all H^k-norms.

By adding additional assumptions on the geometry of the spacetime, we can simplify the assumptions on the pseudo differential operators as follows:

Corollary 5.1.3. *Let (M, g) be as in Theorem 5.1.1. Additionally assume that the normal field η to the timelike boundary is transported parallel along ∂_t and $N|_{\partial M} = 1$. Let $\{P_t\}_{t\in\mathbb{R}}$ be a family of orthogonal Grassmannian projections on $L^2(\partial\Sigma_t, SM|_{\partial\Sigma_t})$, and assume that there exists a sequence $\{k_j\}_{j\in\mathbb{N}_0}$ of non-negative integers with $k_0 = 0$ and $k_j \to \infty$ for $j \to \infty$ such that $t \mapsto \tilde{P}_t$ is H^{k_j} norm continuous for all j. Then the Cauchy problem 5.1.1 is well-posed.*

Proof. Since $N|_{\partial M} = 1$ and η is transported parallel along the integral curves of ∂_t, we see that $\tilde{P}_t$ being a Grassmannian projection in particular implies assumptions 1 and 2 of Theorem 5.1.1. Thus, the claim directly follows by Theorem 5.1.1. □

5.2 Atiyah–Patodi–Singer conditions

Recall the second part of Example 3.1.4, where we defined the APS conditions for an elliptic operator D as $B_{APS} = \chi^-(A)H^{\frac{1}{2}}(\Sigma, E)$, where A is the boundary operator of D. In the following we will construct admissible boundary conditions, which are arising from this kind of conditions.

For this recall that the Lorentzian Dirac operator has the splitting

$$D = -\gamma(\nu)\left(\nabla^{SM}_\nu + iD_t - \tfrac{n}{2}H_t\right),$$

where D_t is an elliptic differential operator acting on sections of $SM|_\Sigma$. Let A_t be a boundary operator of D_t, which can be chosen selfadjoint and such that it anti-commutes with $\sigma_{D_t}(\eta^\flat)$. Recall (see Section 2.3) that for the Riemannian Dirac operator this choice of boundary operator can be made depending on the operator itself and the mean curvature on the boundary, which also implies that the resulting operators A_t have coefficients depending smoothly on t.

Furthermore, assume that A_t has trivial kernel.
Then also the operator $\hat{A}_t = N(t)^{-\frac{1}{2}}A_tN(t)^{\frac{1}{2}}$ is a boundary operator of D_t, which also has trivial kernel and anti-commutes with $\sigma_{D_t}(\eta^\flat)$, but is not formally selfadjoint anymore.
Then $\{B_{APS,t}\}_{t\in\mathbb{R}}$ with

$$B_{APS,t} := \chi^-(\hat{A}_t)H^{\frac{1}{2}}(\partial\Sigma, SM|_{\partial\Sigma_t})$$

is a family of ∞-regular boundary conditions in $\check{H}(A_t) = \check{H}(\hat{A}_t)$.

Remark 5.2.1. Since A_t has no kernel and is selfadjoint, we see that $(\chi^-(A_t)H^{\frac{1}{2}})^* = \chi^-(A_t)H^{\frac{1}{2}}$ and hence

$$\int_{\partial\Sigma_t}\left\langle \sigma_{D_t}(\eta^\flat)N(t)^{\frac{1}{2}}\psi, N(t)^{\frac{1}{2}}\phi\right\rangle_0 \mathrm{d}\mu_{\partial\Sigma_t} = 0 \tag{5.2.1}$$

for all $\psi, \phi \in C^\infty(\Sigma_t, SM|_{\Sigma_t})$ with $\psi, \phi \in B_{APS,t}$. Then we see that B_{APS} is satisfying the first condition of Definition 3.3.13.

For making sure that also the second condition of Definition 3.3.13 is fulfilled, we make the following assumption:

Assumption 5.2.2. Let $(t, x) \in \partial M$ be the coordinates induced by the temporal function, then assume that

$$\eta^\flat(t, x) \neq -N(0, x)^2\tau_t^{-1}(\hat{\eta}(0, x)^\flat). \tag{5.2.2}$$

Remark 5.2.3. Suppose

$$\eta^\flat(t) = -N(0)^2 \tau_t^{-1} \hat{\eta}(0)^\flat,$$

then $1 = \left\|\eta(t)^\flat\right\|_{g_t} = N^2 \left\|\tau_t^{-1}\hat{\eta}(0)^\flat\right\|_{g_t} = N(0)$ implies $N(0) = 1$ and

$$\eta(t)^\flat = -\tau_t^{-1}\eta(0)^\flat.$$

Hence, if $N \neq 1$ on the boundary then Condition 5.2.2 is directly satisfied. Note that in [27], the authors assume that $N = 1$ on ∂M and ν being parallel along ∂_t, which also implies Condition 5.2.2.

Using Condition 5.2.2, we can discuss the ellipticity of $\tilde{B}_{APS,t}$.

Lemma 5.2.4. *Assume the Standard Setup 2.4 and Condition 5.2.2. Then $\{\tilde{B}_{t,APS}\}_{t\in\mathbb{R}}$ is a family of elliptic (and hence equivalently ∞-regular) boundary conditions.*

Proof. Note that one can rewrite $\tilde{B}_{APS,t}$ as

$$\tilde{B}_{APS,t} = \chi^-\left(UN(t)^{\frac{n-1}{2}} A_t N(t)^{\frac{1-n}{2}} U^{-1}\right) H^{\frac{1}{2}}(\partial\Sigma, SM|_{\partial\Sigma}), \tag{5.2.3}$$

which implies that $\tilde{B}_{APS,t} \subseteq \check{H}(\tilde{A}_t)$ is a pseudo local boundary condition. By the third part of Theorem 3.1.30, we see that it suffices to show that $\tilde{B}_{APS}$ is elliptic, which we will do by using the first part of the same theorem. For simplifying notation, let us define the pseudo differential operator

$$P_t := \chi^-\left(UN(t)^{\frac{n-1}{2}} A_t N(t)^{\frac{1-n}{2}} U^{-1}\right).$$

To apply the first part of Theorem 3.1.30, we have to show that $F_t := P_t - \chi^+(\tilde{A}_t)$ is an elliptic operator.
The principal symbol of F_t is given as follows: Let $(0, \tilde{x}) := x \in \Sigma$ and $\zeta \in T_x\Sigma \setminus \{0\}$, then

$$\begin{aligned}
\sigma_{F_t}(\zeta) &= \sigma_{P_t}(\zeta) - \sigma_{\chi^+(\tilde{A}_t)}(\zeta) \\
&= \left(1 - \frac{i\sigma_{UN(t)^{\frac{n-1}{2}} A_t N(t)^{\frac{1-n}{2}} U^{-1}}(\zeta)}{|\zeta|}\right) - \left(1 + \frac{i\sigma_{\tilde{A}_t}(\zeta)}{|\zeta|}\right) \\
&= \frac{-i}{|\zeta|}\left(\sigma_{UN(t)^{\frac{n-1}{2}} A_t N(t)^{\frac{1-n}{2}} U^{-1}}(\zeta) + \sigma_{\tilde{A}_t}(\zeta)\right).
\end{aligned}$$

Let us discuss the two principal symbols separately. For the first one, we have

$$\begin{aligned}\sigma_{UN(t)^{\frac{n-1}{2}}A_tN(t)^{\frac{1-n}{2}}U^{-1}}(\zeta) &= \tau_t\sigma_{N(t)^{\frac{n-1}{2}}A_tN(t)^{\frac{1-n}{2}}}(\tau_t^{-1}\zeta)\tau_t^{-1}\\ &= \tau_t\sigma_{A_t}(\tau_t^{-1}\zeta)\tau_t^{-1}\\ &= \tau_t\sigma_{D_t}(\eta(t)^\flat)^{-1}\sigma_{D_t}(\tau_t^{-1}\zeta)\tau_t^{-1}\\ &= -\tau_t\sigma_{D_t}(\eta(t)^\flat)\sigma_{D_t}(\tau_t^{-1}\zeta)\tau_t^{-1},\end{aligned}$$

where we first used how the the principal symbol changes under parallel transport and conformal change, and in the last step we used that σ_{D_t} satisfies the Clifford relations and is skew-symmetric. For the second one, Remark 3.3.12 and the Clifford relations give us

$$\begin{aligned}\sigma_{\tilde{A}_t}(\zeta) &= \tau_t\sigma_{D_t}(\tau_t^{-1}\hat{\eta}(0)^\flat)^{-1}\sigma_{D_t}(\tau_t^{-1}\zeta)\tau_t^{-1}\\ &= -N(0)^2\tau_t\sigma_{D_t}(\tau_t^{-1}\hat{\eta}(0)^\flat)\sigma_{D_t}(\tau_t^{-1}\zeta)\tau_t^{-1},\end{aligned}$$

Putting the two principal symbols back together to the principal symbol of F_t, we get

$$\begin{aligned}\sigma_{F_t}(\zeta) &= \frac{i}{|\zeta|}\tau_t\left(\sigma_{D_t}(\eta(t)^\flat) + N(0)^2\tau_t\sigma_{D_t}(\tau_t^{-1}\hat{\eta}(0)^\flat)\right)\sigma_{D_t}(\tau_t^{-1}\zeta)\tau_t^{-1}\\ &= \frac{i}{|\zeta|}\tau_t\sigma_{D_t}\left(\eta(t)^\flat + N(0)^2\tau_t^{-1}\hat{\eta}(0)^\flat\right)\sigma_{D_t}(\tau_t^{-1}\zeta)\tau_t^{-1}.\end{aligned}$$

Since $\zeta \neq 0$ and D_t is elliptic, we see that we only have to show that $\eta(t)^\flat + N(0)^2\tau_t^{-1}\hat{\eta}(0)^\flat \neq 0$, which is exactly Condition 5.2.2. Hence, F_t is elliptic and by using the first part of Theorem 3.1.30 this concludes our proof. □

Up to now, we only additionally assume Condition 5.2.2, which would be more general than the assumptions in [27], but unfortunately the continuity of the functional calculus needed for $\{B_{t,APS}\}$ to be a family of admissible boundary conditions, needs $\tilde{B}_{t,APS}$ to be a selfadjoint boundary condition as well. In general, this is difficult to control since we would need to show that the projection on $\tilde{B}_{t,APS}$ is still a Grassmannian projection under certain assumptions. The most handy and least technical assumptions remain the ones used in [27], which we will also use in the following corollary:

Corollary 5.2.5. *Assume the Standard Setup 2.4 and additionally assume that $N|_{\partial M} = 1$, η is parallel transported along ∂_t and the kernel of A_t is trivial for all $t \in \mathbb{R}$. Then there exists a unique smooth solution $\psi \in C^\infty(M, SM)$ to*

$$\begin{cases} D\psi = f \in C^\infty_{cc}(M, SM)\\ \psi|_\Sigma = \psi_0 \in C^\infty_{cc}(\Sigma, SM|_\Sigma)\\ \psi|_{\partial M} \in C^\infty(\partial M, B_{APS})\end{cases}$$

that depends continuously on the Cauchy data (f, ψ_0).

Proof. Using the additional assumptions on the geometry (see also Remark 3.3.12), we see that by Remark 5.2.1, Lemma 5.2.4, Lemma 3.2.3 and Remark 3.2.6, that all conditions of Corollary 5.1.3 are satisfied. Thus, the well-posedness follows. □

Remark 5.2.6. Corollary 5.2.5 coincides with the result of [27], but we filled a gap in their proof of the existence of smooth solution in Subsection 4.2.3 in [27]. In the first step – finding a regularized solution – they do not specify which kind of continuity of the operators is needed to make sure that indeed there is a solution to the regularized problem. This gap we filled in the more general setting of Grassmannian projections, see Section 3.2. A little more detailed discussion can be found in Chapter 6.

5.3 Remarks on modified Atiyah–Patodi–Singer conditions

In this section, we will discuss some possibility to generalize[1] the Atiyah–Patodi–Singer conditions for the case of non-trivial kernel. The purpose of this section is rather to give some more ideas on the class of admissible boundary conditions and will not be focused on finding new results.

As discussed in the Example 3.1.4, one possibility to generalize APS conditions to non-trivial kernel are the modified Atiyah–Patodi–Singer conditions. One then could instead of trivial kernel assume that

$$\ker A_t = L_t \oplus \sigma_{D_t}(\eta_t)L_t,$$

as in Example 3.1.4. As in the previous section, we consider $\hat{A}_t = N^{-\frac{1}{2}}(t)A_t N^{\frac{1}{2}}(t)$. Then the kernel of $\hat{A}_t$ is given by

$$\begin{aligned}
\ker \hat{A}_t &= \{\psi \in C^\infty(\partial\Sigma_t, SM|_{\partial\Sigma_t});\ N^{-\frac{1}{2}}(t)A_t N^{\frac{1}{2}}(t)\psi = 0\} \\
&= \{\psi \in C^\infty(\partial\Sigma_t, SM|_{\partial\Sigma_t});\ A_t N^{\frac{1}{2}}(t)\psi = 0\} \\
&= N^{-\frac{1}{2}}(\ker A_t) \\
&= N^{-\frac{1}{2}}(t)L_t \oplus \sigma_{D_t}(\eta_t)N^{-\frac{1}{2}}(t)L_t.
\end{aligned}$$

Define $\hat{L}_t = N^{-\frac{1}{2}}(t)L_t$. Then let us consider the $B_{mAPS} = \{B_{mAPS,t}\}_{t\in\mathbb{R}}$ with

$$B_{mAPS,t} := \chi^-(\hat{A}_t)H^{\frac{1}{2}}(\Sigma_t, SM|_{\Sigma_t}) \oplus \hat{L}_t.$$

[1]For non-trivial kernel, the APS conditions are still elliptic boundary conditions, but since they are not selfadjoint anymore they do not fit into the setting of admissible boundary conditions. So with generalization, we mean here to generalize the selfadjoint APS conditions to another selfadjoint boundary condition that does not require trivial kernel.

Remark 5.3.1. Since $\chi^-(A_t)H^{\frac{1}{2}}(\partial\Sigma_t, SM|_{\partial\Sigma_t}) \oplus L_t$ is selfadjoint, we see that

$$\int_{\partial\Sigma_t} \left\langle \sigma_{D_t}(\eta^\flat)N(t)^{\frac{1}{2}}\psi, N(t)^{\frac{1}{2}}\phi \right\rangle_0 \mathrm{d}\mu_{\partial\Sigma_t} = 0 \tag{5.3.1}$$

for all $\psi, \phi \in C^\infty(\Sigma_t, SM|_{\Sigma_t})$ with $\psi, \phi \in B_{mAPS,t}$. Thus B_{mAPS} satisfies the first condition in Definition 3.3.13.

Similar to before, the Condition 5.2.2 implies the following:

Lemma 5.3.2. *Assume the Standard Setup 2.4 and additional suppose that there is an orthogonal splitting* $\ker A_0 = L_0 \oplus \sigma_{D_t}(\eta_0)L_0$ *and that Condition 5.2.2 is satisfied.*
Then there exist an orthogonal splitting $\ker A_t = L_t \oplus \sigma_D(\eta_t)L_t$ *for all* t*. Moreover,* $\{\tilde{B}_{t,mAPS}\}_{t\in\mathbb{R}}$ *is a family of elliptic (and in particular* ∞*-regular) boundary conditions.*

Proof. Firstly, note that from $\sigma_D(\eta_t)$ anti-commuting with A_t, it follows that the spectrum of A_t is symmetric around zero. Furthermore, it also implies that for $0 \neq \lambda \in \operatorname{spec}(A_t)$ the eigenspaces of λ and $-\lambda$ have the same dimension. Since the family $\{A_t\}_{t\in\mathbb{R}}$ is smooth in t, the spectrum $\operatorname{spec}(A_t)$ is moving continuously in t. Hence, the spectral flow of the family $\{A_t\}_{t\in\mathbb{R}}$ keeps $\ker A_t$ even dimensional and since $\sigma_{D_t}(\eta_t)E(\lambda, A_t) = E(-\lambda, A_t)$, we see that the spectral flow also keeps the orthogonal splitting imposed in $t = 0$ intact. Hence, there exists an orthogonal splitting $\ker A_t = L_t \oplus \sigma_{D_t}(\eta_t)L_t$ for all $t \in \mathbb{R}$.
Next we have to show that

$$\begin{aligned}\tilde{B}_{mAPS,t} &:= U(t)N^{\frac{n}{2}}(t)(B_{mAPS})\\ &= U(t)N^{\frac{n}{2}}(t)(\chi^-(\hat{A}_t)H^{\frac{1}{2}}(\partial\Sigma_t, SM|_{\partial\Sigma_t})) \oplus U(t)N^{\frac{n}{2}}(t)\hat{L}_t,\end{aligned}$$

is elliptic for all $t \in \mathbb{R}$.
In the proof of Lemma 5.2.4 we showed that $U(t)N^{\frac{n}{2}}(t)(\chi^-(\hat{A}_t)H^{\frac{1}{2}}(\partial\Sigma_t, SM|_{\partial\Sigma_t}))$ is elliptic if Condition 5.2.2 is satisfied. Since, $U(t)N^{\frac{n}{2}}(t)\hat{L}_t$ is a finite dimensional subspace consisting of smooth section, it directly follows that $\tilde{B}_{mAPS,t}$ is elliptic and also ∞-regular as well. □

Remark 5.3.3. Similar to the APS-conditions, if we assume that $N|_{\partial M} = 0$ and that the unit normal field η to the boundary is parallel transported along ∂_t, we know that $\tilde{B}_{t,mAPS}$ is selfadjoint as well. The one thing missing is the continuity, for this we need that $\chi^-(\tilde{A}_t)$ as well as the projection onto $\tilde{L}_t$ is H^k-norm continuous. This is more involved than for the APS-conditions since in this setting there is indeed some spectral flow over zero which has to be controlled for this. This requires more detailed analysis, which we will not do at this point.

5.4 Boundary chirality

For our second example, we consider families of boundary chiralities, which we already discussed briefly in the first part of Example 3.1.4. We call $\{\chi_t\}_{t\in\mathbb{R}}$ a *selfadjoint boundary chirality* if it is a family of selfadjoint (with respect to $\langle\cdot,\cdot\rangle_0$) involutions, such that χ_t anti-commutes with $\sigma_{D_t}(\eta^\flat)$ and with A_t, respectively. Then one has the orthogonal decomposition $SM|_{\partial\Sigma_t} = S^{-1}\partial\Sigma_t \oplus S^{+1}\partial\Sigma_t$, where $S^{\pm1}\partial\Sigma_t$ is the eigenbundle of χ_t for the eigenvalues ±1.

Remark 5.4.1.

1. Recall that since χ_t anti-commutes with A_t, the boundary conditions $B_{\pm,t} := H^{\frac{1}{2}}(\partial\Sigma_t, S^{\pm1}\partial\Sigma_t)$ are ∞-regular by Theorem 3.1.30 part 2 and 3.

2. Since χ anti-commutes with $\sigma_{D_t}(\eta^\flat)$ and is selfadjoint, we see that for all $x\in\partial\Sigma_t$ and $u,v\in S^{\pm1}_x\partial\Sigma_t$
$$\left\langle \sigma_{D_t}(\eta^\flat)u, v\right\rangle_0 = 0,$$
and hence we also know that
$$\left\langle \sigma_{D_t}(\eta^\flat)N(t,x)^{\frac{1}{2}}u, N(t,x)^{\frac{1}{2}}v\right\rangle_0 = 0.$$
This implies that $B_\pm := \{B_{\pm,t}\}_{t\in\mathbb{R}}$ satisfies the first part in Definition 3.3.13.

Using Theorem 4.3.1 we get for the boundary conditions $B_\pm$ the following well-posedness result:

Corollary 5.4.2. *Assume the Standard Setup 2.4 and let $\{\chi_t\}_{t\in\mathbb{R}}$ be a selfadjoint boundary chirality which depends continuously on t. Additionally assume that $N|_{\partial M} = 1$ and η is parallel transported along the gradient of the temporal function. Then there exists a unique smooth solution $\psi\in C^\infty(M,SM)$ to the Cauchy problem*
$$\begin{cases} D\psi = f \in C^\infty_{cc}(M,SM) \\ \psi|_\Sigma = \psi_0 \in C^\infty_{cc}(\Sigma, SM|_\Sigma) \\ \psi|_{\partial M} \in C^\infty(\partial M, B_\pm) \end{cases}, \tag{5.4.1}$$
that depends continuously on the Cauchy data (f,ψ_0).

Proof. First note that
$$\tilde{B}_t = UN(t)^{\frac{n}{2}}H^{\frac{1}{2}}(\Sigma_t, S^{\pm1}\partial\Sigma_t) = UNH^{\frac{1}{2}}(\Sigma_t, S^{\pm1}\partial\Sigma_t) = H^{\frac{1}{2}}(\Sigma_0, \tilde{P}^\pm_t S\Sigma_0)$$

where

$$\tilde{P}_t^{\pm} := \frac{1}{2}\left(1 \pm \tilde{\chi}_t\right) \tag{5.4.2}$$

is a projection on the ± 1 eigenspaces of $\tilde{\chi}_t := U\chi_t U^{-1}$. Note that since U is an isometry between the L^2 spaces of the Cauchy hypersurfaces, $\tilde{\chi}_t$ is selfadjoint as well. Furthermore, by Remark 3.3.12 we see that the boundary operator $\tilde{A}_t$ of $\tilde{D}_t$ can be chosen as $\tilde{A}_t = UA_tU^{-1}$. Thus $\tilde{A}_t$ clearly anti-commutes with $\tilde{\chi}_t$. The additional geometric assumptions on N and η also imply

$$\begin{aligned}
\sigma_{\tilde{D}_t}(\hat{\eta}^\flat(0))\tilde{\chi}_t &= \tau_t\sigma_{D_t}(\eta^\flat(t))\tau_t^{-1}U(t)\chi_t U(t)^{-1} \\
&= U\sigma_{D_t}(\eta^\flat(t))\chi_t U^{-1} \\
&= -U\chi_t\sigma_{D_t}(\eta^\flat(t))U^{-1} \\
&= -\tilde{\chi}_t\sigma_{\tilde{D}_t}(\hat{\eta}^\flat(0)),
\end{aligned}$$

and hence $\{\tilde{\chi}_t\}_{t\in\mathbb{R}}$ is a selfadjoint boundary chirality. Hence, we apply the second part Theorem 3.1.30, which implies that $\tilde{B}_\pm$ is ∞-regular as well. Furthermore, since $U(t)$ depends continuously on t, we see that $\tilde{\chi}_t$ is as well continuous in t and $\tilde{P}_t^{\pm}$ is L^2 norm continuous. Remark 3.2.5 and Lemma 3.2.3 then gives us that we satisfy the requirements of Theorem 4.3.1, which implies the well-posedness of the Cauchy problem 5.4.1. □

Remark 5.4.3. The well-posedness for this kind of local boundary conditions is already known in more generality, see for example [31] for MIT conditions or [29] for Friedrich systems. We only show this result here to emphasize that also local boundary conditions are in the class of admissible boundary conditions. A more detailed discussion about the relation of Corollary 5.4.2 to these results can be found in Chapter 6.

5.5 Examples on $\mathbb{R} \times \overline{B_1(0)}$

Let us end our discussion on examples of boundary conditions with some explicit examples on a simple manifold. For this we follow [27] and consider the globally hyperbolic spin manifold $(M, g) = (\mathbb{R} \times \overline{B_1(0)}, -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2)$ with coordinates (t, r, θ). On this manifold, we can choose a spin structure such that we can identify SM with the trivial bundle $M \times \mathbb{C}^2$. In the standard coordinates on Minkowski space, the Dirac operator is given by

$$D = -\gamma(e_0)\partial_t + \gamma(e_1)\partial_x + \gamma(e_2)\partial_y,$$

where $\gamma(e_i)$ are the following Pauli matrices:

$$\gamma(e_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma(e_1) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \text{and } \gamma(e_2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Using the transformation rules for vector fields and that $\partial_r, \partial_\theta$ is not orthonormal, we can write the spin Dirac operator in spherical coordinates as follows

$$D = -\gamma(e_0)\partial_t + \big(\gamma(e_1)\cos(\theta) + \gamma(e_2)\sin(\theta)\big)\,\partial_r + \tfrac{1}{r}\,\big(\gamma(e_2)\cos(\theta) - \gamma(e_1)\sin(\theta)\big)\,\partial_\theta.$$

Using the explicit form of the Clifford multiplications, we can rewrite D as

$$D = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_t + \begin{pmatrix} 0 & ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}\partial_r + \begin{pmatrix} 0 & \frac{1}{r}e^{-i\theta} \\ -\frac{1}{r}e^{i\theta} & 0 \end{pmatrix}\partial_\theta$$

Let us now compute the induced operators D_t on $\Sigma_t = \{t\} \times \overline{B_1(0)}$ by computing the principal symbol $\sigma_{D_t}(\zeta)$ for $\zeta \in T_x^*\Sigma_t$ with $\zeta = \zeta_t\,\mathrm{d}r + \zeta_\theta\,\mathrm{d}\theta$:

$$\begin{aligned}
\sigma_{D_t}(\zeta) &= -i\sigma_D(\mathrm{d}t)^{-1}\circ\sigma_D(\zeta) \\
&= i\gamma(e_0)\circ\gamma(\zeta) \\
&= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\left[\zeta_r\begin{pmatrix} 0 & ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix} + \frac{\zeta_\theta}{r}\begin{pmatrix} 0 & e^{-i\theta} \\ -e^{i\theta} & 0 \end{pmatrix}\right] \\
&= \zeta_r\begin{pmatrix} 0 & -e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix} + \frac{\zeta_\theta}{r}\begin{pmatrix} 0 & ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}.
\end{aligned}$$

Hence, we get the operator D_t on Σ_t given by

$$D_t = \begin{pmatrix} 0 & -e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}\partial_r + \begin{pmatrix} 0 & \frac{i}{r}e^{-i\theta} \\ \frac{i}{r}e^{i\theta} & 0 \end{pmatrix}\partial_\theta.$$

The operator D_t induces the boundary operator A_t on $\partial\Sigma_t = \{t\} \times \mathbb{S}^1$, which is defined over its principal symbol σ_{A_t} given as follows: Let $\zeta \in T_x^*\partial\Sigma_t$ with $\zeta = \overline{\zeta}\,\mathrm{d}\theta$, then

$$\begin{aligned}
\sigma_{A_t}(\zeta) &= \sigma_{D_t}(\mathrm{d}r)^{-1}\circ\sigma_{D_t}(\zeta) \\
&= \overline{\zeta}\begin{pmatrix} 0 & -e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}^{-1}\begin{pmatrix} 0 & ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix} \\
&= \overline{\zeta}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\end{aligned}$$

One could choose the boundary operator

$$A_t^0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\partial_\theta,$$

but note that this operator does not anti-commute with $\sigma_{D_t}(\mathrm{d}r)$. Let us compute the anti-commutator ourselves: Let $\psi \in C^\infty(\partial\Sigma_t, SM|_{\partial\Sigma_t})$ with $\psi(\theta) = \begin{pmatrix} \psi_1(\theta) \\ \psi_2(\theta) \end{pmatrix}$, then we compute

$$
\begin{aligned}
&\left[A_t^0 \sigma_{D_t}(\mathrm{d}r) + \sigma_{D_t}(\mathrm{d}r) A_t^0\right] \psi(\theta) \\
&= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \partial_\theta \begin{pmatrix} 0 & -e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix} \begin{pmatrix} \psi_1(\theta) \\ \psi_2(\theta) \end{pmatrix} + \begin{pmatrix} 0 & -e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \partial_\theta \begin{pmatrix} \psi_1(\theta) \\ \psi_2(\theta) \end{pmatrix} \\
&= \begin{pmatrix} -i\psi_1'(\theta)e^{i\theta} - \psi_2(\theta)e^{i\theta} \\ -i\psi_2'(\theta)e^{-i\theta} + \psi_1(\theta)e^{-i\theta} \end{pmatrix} + \begin{pmatrix} ie^{i\theta}\psi_2'(\theta) \\ ie^{-i\theta}\psi_1'(\theta) \end{pmatrix} \\
&= \begin{pmatrix} -\psi_2(\theta)e^{i\theta} \\ \psi_1(\theta)e^{-i\theta} \end{pmatrix} \\
&= \sigma_{D_t}(\mathrm{d}r)\psi(\theta).
\end{aligned}
$$

Hence, we can choose the boundary operator A_t as

$$A_t = A_t^0 - \tfrac{1}{2} Id.$$

Remark 5.5.1. One could also use the theory of Dirac type operators in the sense of Gromov and Lawson (see for example [36]) and see that the lower order term to subtract is given by $\frac{1}{2}H_t$, where H_t is the mean curvature of $\partial\Sigma_t$. Since $H_t = 1$ in our case, we see that both lower order terms coincide.

The operator A_t has eigenvalues $\lambda_k = -k - \frac{1}{2}$ for $k \in \mathbb{Z}$ with eigenfunctions

$$\psi_k^1 : \theta \mapsto \begin{pmatrix} e^{ik\theta} \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_k^2 : \theta \mapsto \begin{pmatrix} 0 \\ e^{-ik\theta} \end{pmatrix}.$$

In particular A_t has trivial kernel and all eigenvalues have multiplicity two. Using these eigenfunctions, we can write down $B_{APS,t}$ simply as

$$B_{APS,t} = \operatorname{span}_{H^{\frac{1}{2}}(\partial\Sigma_t)} \left\{\psi_k^i;\ k \geq 0;\ i = 1, 2\right\}.$$

Since A_t is constant along t, this is of course also true for $B_{APS,t}$. Furthermore, since A_t has no kernel, $B_{mAPS,t} = B_{APS,t}$.

Let us briefly discuss the solutions of the homogeneous Cauchy problem

$$\begin{cases} D\psi = 0 \\ \psi|_{\{t=0\}} = \psi_0 \\ \psi|_{\{r=1\}} \in C^\infty(\partial M, B_{APS}) \end{cases} . \tag{5.5.1}$$

For this, we first will compute the eigenbasis of D_0 under the APS boundary conditions. We will look for smooth eigenfunctions of possibly generalized eigenvalues of D_0 first without boundary conditions and then we will compute how the APS conditions enforce requirements on the possible eigenvalues. Since D_0 is a Dirac type operator on a compact manifold with boundary, we know that, seen as an operator from H^1 to L^2, it has an infinite dimensional kernel, but by direct computation one can see that the periodicity condition in θ prevents the existence of non-trivial smooth elements in the kernel.
Let us do the following formal ansatz by separation of variables. For a possibly generalized eigenvalue $0 \neq \lambda \in \mathrm{spec}(D_0)$ we make the formal ansatz

$$\psi_{\lambda,k_1,k_2}(r,\theta) = \begin{pmatrix} \rho_{1,\lambda,k_1}(r)\psi_{1,k_1}(\theta) \\ \rho_{2,\lambda,k_2}(r)\psi_{2,k_2}(\theta) \end{pmatrix}, \tag{5.5.2}$$

where $k_1, k_2 \in \mathbb{Z}$ and $\rho_{i,\lambda,k}$ smooth complex valued functions. Here, we abused notation by denoting the first component of ψ_k^1 and the second component of ψ_k^2 as well by ψ_k^1 and ψ_k^2, respectively.
Since D_0 is a formally adjoint Dirac operator and the APS conditions in this case are selfadjoint boundary conditions, we can restrict to the case of $\lambda \in \mathbb{R}$ and we know that

$$-\Delta\psi_{\lambda,k_1,k_2} = \lambda^2\psi_{\lambda,k_1,k_2}. \tag{5.5.3}$$

Equation 5.5.3 is also called the *spatial Helmholtz equation for vibrating membrane* and by separation of variables and periodicity in θ, one sees that

$$\rho_{1,\lambda,k}(r) = c_{\lambda,k}J_k(\lambda r),$$

where $c_{\lambda,k}$ are constants and J_k are the k-th Bessel functions. For a more detailed discussion see, for example, [46].

Remark 5.5.2. There are different kind of Bessel functions, but here we are talking about the Bessel functions J_k of the first kind for integers $k \in \mathbb{Z}$. These satisfy the following properties:

1. $J_{-k} = (-1)^k J_k$, and
2. $\frac{d}{dr}J_k(r) - \frac{k}{r}J(r) = J_{k+1}(r)$.

For an overview on the theory of Bessel functions and their use in mathematics and physics, we refer the reader to [49].

Up to now we only know how the eigenfunctions of the Laplace equation can be written, but not all solutions to the Laplace equation are again solutions to the Dirac equation. Hence, we put ψ_{λ,k_1,k_2} back into the eigenvalue equation for D_0 and see the following:

$$\lambda \begin{pmatrix} c_{1,k_1}J_{k_1}(\lambda r)\psi_{1,k_1}(\theta) \\ c_{2,k_2}J_{k_2}(\lambda r)\psi_{2,k_2}(\theta) \end{pmatrix} = D_0 \begin{pmatrix} c_{1,k_1}J_{k_1}(\lambda r)\psi_{1,k_1}(\theta) \\ c_{2,k_2}J_{k_2}(\lambda r)\psi_{2,k_2}(\theta) \end{pmatrix}$$

$$
\begin{aligned}
&= \lambda \begin{pmatrix} -c_{2,k_2} \left[\frac{d}{dr} J_{k_2}(\lambda r) - \frac{k_2}{r\lambda} J_{k_2}(\lambda r) \right] \psi_{2,k_2+1}(\theta) \\ c_{1,k_1} \left[\frac{d}{dr} J_{k_1}(\lambda r) - \frac{k_1}{r\lambda} J_{k_1}(\lambda r) \right] \psi_{1,k_1+1}(\theta) \end{pmatrix} \\
&= \lambda \begin{pmatrix} -c_{2,k_2} J_{k_2+1}(\lambda r) \psi_{1,-k_2-1}(\theta) \\ c_{1,k_1} J_{k_1+1}(\lambda r) \psi_{2,-k_1-1}(\theta) \end{pmatrix} \\
&= \lambda \begin{pmatrix} (-1)^{k_2} c_{2,k_2} J_{-k_2-1}(\lambda r) \psi_{1,-k_2-1}(\theta) \\ (-1)^{k_1+1} c_{1,k_1} J_{-k_1-1}(\lambda r) \psi_{2,-k_1-1}(\theta) \end{pmatrix},
\end{aligned}
$$

where we used the explicit form of D_0 and the properties of the Bessel function mentioned in Remark 5.5.2. The linear independence of ψ_k^i implies

$$k_1 = -(k_2 + 1) \text{ and } c_{1,k_1} = (-1)^{k_2} c_{2,k_2}.$$

Hence, we get the following eigenfunctions for D_0:

$$\psi_{\lambda,k}(r,\theta) := \psi_{\lambda,k,-(k+1)}(r,\theta) = c_{k,1} \begin{pmatrix} J_k(\lambda r) \psi_{1,k}(\theta) \\ (-1)^k J_{k+1}(\lambda r) \psi_{2,-(k+1)}(\theta) \end{pmatrix}. \tag{5.5.4}$$

Now, we can apply the APS condition on $\psi_{\lambda,k}(1,\theta)$. We see that $\psi_{\lambda,k}(1,\theta)$ satisfies the APS condition if and only if

$$J_{k+1}(\lambda) = 0 \text{ if } k \geq 0, \text{ and } J_k(\lambda) = 0 \text{ if } k \leq -1.$$

Hence, we see that the λ's have to be the roots $\zeta_{n,k}$ of the Bessel functions J_k, which implies that

$$\operatorname{spec}(D_{0,APS}) = \{\zeta_{n,k};\ k \geq 0,\ n \in \mathbb{N}\},$$

and the domain $\operatorname{dom}(D_{0,APS})$ is spanned by the eigenbasis

$$
\begin{aligned}
\psi_{1,k,n}(r,\theta) &:= \begin{pmatrix} J_k(\zeta_{k+1,n} r) \psi_{1,k}(\theta) \\ (-1)^k J_{k+1}(\zeta_{k+1,n} r) \psi_{2,-(k+1)}(\theta) \end{pmatrix}, \text{ and} \\
\psi_{2,k,n}(r,\theta) &:= \begin{pmatrix} J_k(\zeta_{k+1,n} r) \psi_{1,-(k+1)}(\theta) \\ (-1)^k J_{k+1}(\zeta_{k+1,n} r) \psi_{2,k}(\theta) \end{pmatrix}.
\end{aligned}
$$

Hence, for any $\psi_0 \in \operatorname{dom}(D_{0,APS}) \cap C^\infty(\Sigma_0, SM|_{\Sigma_0})$, we see that

$$\psi(t,r,\theta) := e^{-it D_0} \psi_0(r,\theta) = \sum_{k,n} e^{-it\zeta_{k+1,n}} \left[a_{n,k} \psi_{1,k,n}(r,\theta) + b_{n,k} \psi_{2,k,n}(r,\theta) \right]$$

solves the Cauchy problem 5.5.1, where $a_{n,k}$ and $b_{n,k}$ are the coefficients of the expansion of ψ_0 in the eigenbasis $\psi_{i,k,n}$.

Remark 5.5.3. Let us have a brief look at an example for the chirality conditions. For this let us consider the involution $\chi_t := i\gamma(\partial_r)(x) : S_xM \to S_xM$ for $x \in \partial\Sigma_t$. By construction, we see that χ_t is anti-commuting with A_t. Furthermore, since $\gamma(\mathrm{d}r)$ is symmetric with respect to $\langle\cdot,\cdot\rangle_{SM}$ and anti-commuting with $\sigma_D(\mathrm{d}t)$, we see that χ_t is selfadjoint with respect to $\langle\cdot,\cdot\rangle_0$. Hence, χ_t is is an selfadjoint boundary chirality. Using the explicit form of the Clifford multiplication, we can write χ_t also as

$$\chi_t = \begin{pmatrix} 0 & -e^{-i\theta} \\ -e^{i\theta} & 0 \end{pmatrix}.$$

Since χ_t is an involution, its eigenvalues are ± 1. The eigenspaces are given by

$$E(\chi_t, \pm 1) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} ; y = \mp e^{-i\theta} x \right\}.$$

Hence, the boundary conditions $B_t^{\pm}$ are given by

$$B_t^{\pm} = H^{\frac{1}{2}}(\partial\Sigma_t, E(\chi_t, \pm 1)) = \{\psi \in H^{\frac{1}{2}}(\partial\Sigma_t);\ \psi_2(\theta) = \mp e^{-i\theta}\psi_1(\theta)\},$$

where $\psi(\theta) = \begin{pmatrix} \psi_1(\theta) \\ \psi_2(\theta) \end{pmatrix}$.

Remark 5.5.4. The manifold $M = \mathbb{R} \times \overline{B_1(0)}$ with the Lorentzian metric $g = -N^2\,\mathrm{d}t^2 + \mathrm{d}r^2 + f^2\,\mathrm{d}\theta^2$, where $N, f \in C^\infty(M, \mathbb{R})$ are positive functions, is a globally hyperbolic spin manifold with timelike boundary as well. We additionally assume that N and f depend only on r and t but not on θ. Then we can do the same computations as above and get the following operators:

$$D = -N^{-2}(t,r)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_t + \begin{pmatrix} 0 & ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}\partial_r + \frac{r}{f^2(r,t)}\begin{pmatrix} 0 & e^{-i\theta} \\ -e^{i\theta} & 0 \end{pmatrix}\partial_\theta,$$

$$D_t = N^2(t,r)\begin{pmatrix} 0 & -e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}\partial_r + \frac{N^2(t,r)r}{f^2(t,r)}\begin{pmatrix} 0 & ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}\partial_\theta, \text{ and}$$

$$A_t = f(t,1)^{-2}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\partial_\theta - \tfrac{N^2(t,1)}{2} Id\right].$$

Then A_t has eigenvalues $(\lambda_k)_t = -\frac{1}{f^2(t,1)}\left(k + \frac{N^2(t,1)}{2}\right)$ for $k \in \mathbb{Z}$ with eigenfunctions

$$\psi_k^1 : \theta \mapsto \begin{pmatrix} e^{ik\theta} \\ 0 \end{pmatrix}, \text{ and } \psi_k^2 : \theta \mapsto \begin{pmatrix} 0 \\ e^{-ik\theta} \end{pmatrix}.$$

Hence, we can write the APS conditions as

$$B_{APS,t} = \operatorname{span}_{H^{\frac{1}{2}}(\partial\Sigma_t)} \left\{ (\psi_k^i)_t;\ k > -\tfrac{N^2(t,1)}{2} \text{ and } i = 1,2 \right\}.$$

Furthermore note, that in this case A_t can have a non-trivial kernel. More precisely, we see

$$\ker A_t = \begin{cases} \{0\} & \text{if } \frac{1}{2}N^2(t,1) \notin \mathbb{Z}, \\ \operatorname{span}\{(\psi_k^i)_t; i = 1,2 \text{ and } k = -\frac{1}{2}N^2(t,1)\} & \text{otherwise.} \end{cases}$$

Thus, depending on the choice of N the kernel does not need to be trivial but it is always two dimensional and $\sigma_{D_t}(\partial_r)$ is mapping $\operatorname{span}\{(\psi_1)^k; k = -\frac{1}{2}N^2(t,1)\}$ to $\operatorname{span}\{(\psi_2)^k; k = -\frac{1}{2}N^2(t,1)\}$ and vice versa. This allows us to define the modified APS conditions also for this case. Note that since the lapse function N is not necessarily equal to one, we would need to look at $N^{-\frac{1}{2}}(t)(mAPS(A_t))$ to define admissible boundary conditions.

6 OUTLOOK

6.1 Related results

In this section, we will put our results into the context of current research on boundary value problems on spacetimes. We will only focus on the case of first order operators, but of course there are a lot of interesting results for operators for higher order.

6.1.1 Spacetimes with spacelike boundary

In this section, we will discuss some results on non-local boundary conditions for the case of spacelike boundary and will point out the differences to the case of timelike boundary.
Spacetimes with spacelike boundary, i. e. the induced metric on the boundary is Riemannian, are often seen as follows. Let (X, g) be a globally hyperbolic spacetime, Σ_- and Σ_+ be two disjoint smooth spacelike Cauchy hypersurfaces, and without loss of generality let Σ_- lie in the past of Σ_+. Then there exists a temporal function $t\colon\ M \to \mathbb{R}$ such that $\Sigma_{t_1} := t^{-1}(t_1) = \Sigma_-$ and $\Sigma_{t_2} := t^{-1}(t_2) = \Sigma_+$ for $t_1, t_2 \in t(M)$. One then understands $M := t^{-1}([t_1, t_2])$ as a *globally hyperbolic spacetime with spacelike boundary* $\partial M = \Sigma_+ \sqcup \Sigma_-$. If one does not want to refer to a temporal function, one could characterize M also as $M := J^+(\Sigma_-) \cap J^-(\Sigma_+)$.

For M being spatially compact, spin and even dimensional, Bär and Strohmaier [15] showed a Lorentzian index theorem for the Lorentzian Dirac operator on the complex spin bundle. Since M is even dimensional the Dirac operator D can be split with respect to the negative and positive chirality into

$$D = \begin{pmatrix} 0 & \mathcal{D} \\ \tilde{\mathcal{D}} & 0 \end{pmatrix}$$

where $\mathcal{D}\colon C^\infty(M,S^+M)\to C^\infty(M,S^-M)$ and $\tilde{\mathcal{D}}=\mathcal{D}^\dagger$. Even if in this case the operator is – as in our setting – hyperbolic, the boundary operator $D=D_{t_1}\oplus D_{t_2}$ is elliptic, hence one can make sense of non-local boundary conditions. This is a fundamental difference to our settings, since the induced operator on timelike boundary is still hyperbolic, which makes it difficult for us to define non-local boundary conditions in an analogous way.
Let us briefly discuss the boundary conditions that are considered in [18]. Recall the finite energy spaces defined in Section 3.3.2. Using these spaces one can define the following space associated to the APS conditions

$$FE_{APS}(M,\mathcal{D}):=\{\psi\in FE^0(M,\mathcal{D});P_{[0,\infty)}(t_1)(\psi|_{\Sigma_{t_1}})=0=P_{(-\infty,0]}(t_2)(\psi|_{\Sigma_{t_2}})\},$$

where for an interval $I\subseteq\mathbb{R}$ $P_I(t)$ is the spectral projection onto the eigenspaces of D_t with corresponding eigenvalues inside I, which is well-defined since D_t is chosen selfadjoint on a closed Riemannian manifold. In [15], they show that $\mathcal{D}$ is Fredholm under these boundary conditions:

Theorem 6.1.1 (Theorem 3.3 [15])**.** *The operator*

$$\mathcal{D}_{APS}:=\mathcal{D}|_{FE_{APS}}\colon FE_{APS}(M,D)\to L^2(M,S^-M)$$

is Fredholm.

Furthermore, they show that even the *anti-Atiyah–Patodi–Singer conditions* (aAPS) conditions defined as follows

$$FE_{aAPS}(M,\mathcal{D}):=\{\psi\in FE^1(M,\mathcal{D});P_{(-\infty,0]}(t_1)(\psi|_{\Sigma_{t_1}})=0=P_{[0,\infty)}(t_2)(\psi|_{\Sigma_{t_2}})\}$$

lead to Fredholmness as well:

Theorem 6.1.2 (Theorem 3.4 [18])**.** *The operator*

$$\mathcal{D}_{aAPS}:=\mathcal{D}|_{FE_{aAPS}}\colon FE_{aAPS}(M,D)\to L^2(M,S^-M)$$

is Fredholm with index

$$\mathrm{ind}[D_{aAPS}]=-\mathrm{ind}[D_{APS}].$$

This is specially interesting, since aAPS does not lead to Fredholmness in the Riemannian setting, where aAPS is not an elliptic boundary condition. Hence, the analogue of aAPS in our setting is also not a suitable boundary condition.

Furthermore, note that in [15] they do not make any assumptions on the kernel of the boundary operator. In this setting the spectral flow over zero is considered a feature that is represented in the corresponding index formula, while in our setting the spectral flow is something that we have to control to make our methods work. In Section A.1, we define Lorentzian Dirac operator in the sense of Gromov and Lawson, which include twisted operators. We require that the bundle that we are twisting with is Hermitian, which indeed is the same assumption as in the result of [15] for twisted Dirac operators. They run into the same difficulty that the induced inner product on the bundle restricted to the spacelike Cauchy hypersurface could be indefinite for more general inner products.

In the case of spacelike boundary, Bär and Hannes [13] later also investigated to what extend the (a)APS boundary conditions can be replaced by more general ones and found sufficient conditions for graph type boundary conditions, which lead to Fredholmness.

6.1.2 Spacetimes with timelike boundary

In this section, we will discuss well-posedness results of the Cauchy problem for the Dirac operator on globally hyperbolic spacetimes with timelike boundary.
In the presence of timelike boundary, one can consider the Cauchy problem with MIT boundary conditions. The *MIT boundary conditions*, first introduced in [22], are given as

$$(\gamma(\eta) - i)\psi|_{\partial M} = 0. \tag{6.1.1}$$

Note that $\gamma(\eta)^2 = -1$, hence $\gamma(\eta)$ has eigenvalues $\pm i$. Hence, the Condition 6.1.1 is equivalent to saying that $\psi|_{\partial M}(x)$ has to lie in $E(-i, \gamma(\eta(x)))$ for all $x \in \partial M$. Clearly these boundary conditions are local. Using these boundary conditions Finster and Röken [28] showed for stationary spacetimes with timelike boundary admitting a suitable timelike vector field, and Große and Murro [31] for the general case the following well-posedness:

Theorem 6.1.3. *Let (M, g) be a globally hyperbolic spin manifold with timelike boundary. Then there exists a unique spacelike compact solution $\psi \in C^\infty_{sc}(M, SM)$ to*

$$\begin{cases} D\psi = f \in C^\infty_{cc}(M, SM) \\ \psi|_{\Sigma_0} = \psi_0 \in C^\infty_{cc}(\Sigma_0, SM|_{\Sigma_0}) \\ (\gamma(\eta) - i)\psi|_{\partial M} = 0 \end{cases}$$

that depends continuously on the Cauchy data (f, ψ_0).

Note, that the MIT conditions are chirality conditions as defined in Section 5.4. While our well-posedness result Corollary 5.4.2 arises from the theory of non-local boundary conditions, Große and Murro [31] use tools for local boundary conditions, which include showing that a weak solution is a strong solution and the associated regularity theory. Both of these tools need localization not only in the interior but also at the boundary. This localization is a strong tool in the setting of local

boundary conditions, but cannot be used in the context of non-local boundary conditions. Thus, even if we also get a well-posedness result - in particular - for MIT conditions, our tools need to require some geometric assumptions, that are not needed in [31]. Later Ginoux and Murro [29] generalized this well-posedness result to a bigger class of operators - the so called Friedrich systems - and to a bigger class of suitable local boundary conditions. Furthermore, note that recently Drago, Ginoux, and Murro [26] used the well-posedness in [31] to prove the existence of Hadamard states for Dirac fields satisfying the MIT conditions.
In the context of non-local boundary conditions Drago, Große, and Murro [27] considered the Lorentzian Dirac operator and defined for a spacetime foliation $\{\Sigma_t\}_{t\in\mathbb{R}}$ the corresponding family $\{APS(A_t)\}_{t\in\mathbb{R}}$, where A_t are the induced boundary operators of the Riemannian Dirac operators D_t on Σ_t. Using these boundary conditions, they showed the following:

Theorem 6.1.4 (Theorem 1.3 [27])**.** *Let $(M, g) = (\mathbb{R}\times\Sigma, -N^2\,\mathrm{d}t^2 + g_t)$ be a globally hyperbolic spin manifold with timelike boundary. Let $\partial\Sigma$ be compact and* ker A_t *be trivial for all $t \in \mathbb{R}$. Additionally assume $N|_{\partial M} \equiv 1$ and that the unit normal η to ∂M is parallel transported along the vector field $v = N^{-1}\partial_t$, then there exists a smooth solution $\psi \in C^\infty(M, SM)$ to*

$$\begin{cases} D\psi = f \in C^\infty_{cc}(M, SM) \\ \psi|_{\Sigma_0} = \psi_0 \in C^\infty_{cc}(\Sigma_0, SM|_{\Sigma_0}) \\ \psi|_{\partial M} \in C^\infty(\partial M, APS) \end{cases}$$

which depends continuously on its Cauchy data (f, ψ_0).

Corollary 5.2.5 coincides with Theorem 6.1.4 and also the tools used in Chapter 4 are highly motivated by the tools used in the proof of Theorem 6.1.4. But on the other hand, the argument why the regularized problem 4.3.1 has a strong solution is not complete is in Subsection 4.2.3 in 6.1.4. First, they do not specify which kind of continuity the operator family $\tilde{D}_{t,\tilde{B}_t} J^{(\varepsilon)}_{\tilde{B}_t}$ is needed to make sure that there is indeed a strong solution to the regularized problem 4.3.1. Also there is no argument, why the family of APS conditions should lead to any kind of continuity of this functional calculus. We filled this gap by analyzing families of pseudo local boundary conditions and finding continuity conditions on the corresponding families of pseudo differential projections.

6.2 Open questions and possible new research directions

Let us conclude this thesis, by briefly discussing some open problems and possible future research questions. This will be done less rigorously and should be more seen as a collection of ideas that could be worked on at a later time.

1. *Analysis of class of boundary conditions:* In Section 3.3.3, we define admissible boundary conditions and in Section 5 we discuss some examples contained in this class of boundary conditions. Still, it is not yet overly clear how restricting the assumptions in Definition 3.3.13 really are. One possible ansatz to get a better understanding of these boundary conditions, could be to analyze how the check space is changing along a family of elliptic operators, or more generally establish a perturbation theory for check spaces and elliptic (∞-regular) boundary conditions. This could give us the tools to have a better understanding on how restrictive the condition $\{B_t\}_{t\in\mathbb{R}}$ being ∞-regular for all t is.

2. *Bigger class of boundary conditions:* Recall that in Definition 3.3.13, we assume that the family of boundary condition $\tilde{B}_t$ is a selfadjoint boundary condition. This is a quite restrictive assumption but also very necessary for our methods, since we need the operator $\tilde{D}_{t,\tilde{B}_t}$ to be selfadjoint on its domain. Changing the methods and going away from the classical theory, one could consider dropping the selfadjointness assumption and see if one still can obtain the well-posedness of the corresponding Cauchy problems.

3. *Lower regularity:* Our well-posedness results are (except for the transmission conditions) only making statements about smooth solutions. Another natural question to ask is if we can get lower regularity well-posedness and how the lower regularity is encoded in the boundary conditions. One ansatz could be to use an H^k energy estimate and use a density argument, but note that our Cauchy data is compactly supported away from the boundary. The space C^∞_{cc} is not dense in H^k for $k > 0$. If one wants to look at solutions to the Cauchy problems with Cauchy data in some Sobolev spaces, one has to consider compatibility conditions on the Cauchy data. This leads to the interesting question of how one could encode such compatibility conditions into the boundary conditions and if it is possible in a non-local way. Related to this, one should also mention that we are considering ∞-regular boundary conditions, but for lower regularity one would rather consider k-regular boundary conditions.

4. *More general operators:* Our methods require operators of a certain form, see Section A.1, since the reduction process to Hamiltonian form needs a good splitting of the operator into normal part and tangential part along the spacetime foliation. In the Riemannian setting, [8, 10] showed that one can write elliptic operators D in normal form, i. e.

$$D = \sigma_D(\nu)(\partial_t + A + R_t), \tag{6.2.1}$$

where R_t is an error term, whose operator norm can be controlled for small t. This motivates to look for conditions on hyperbolic operators, such that they form a similar splitting as in Equation 6.2.1 and then one could transfer our methods to such more general hyperbolic operators. The discussion of such hyperbolic operators should maybe first be done in the boundaryless case, where this could also lead to a generalization of [15].

For the reduction to Hamiltonian form, we chose explicitly the parallel transport along the t-lines for identifying the Cauchy hypersurfaces, since this fits well the lower order term consisting of the mean curvature of the Cauchy hypersurfaces. It could possible also be of help to be more flexible in this step and consider a bigger range of identifications, such that one can also look at operators whose splitting along the Cauchy hypersurfaces contain other lower order terms (as for example R_t above).

5. *Fredholmness:* Coming back to the initial motivation to look at non-local boundary conditions, let us briefly talk about Fredholmness. The well-posedness of the initial value problems is not yet suitable, since for example the homogeneous Cauchy problem

$$\begin{cases} D\psi = 0 \\ \psi|_{\Sigma_0} = \psi_0 \in C^\infty_{cc}(\Sigma) \\ \psi|_{\partial M} \in C^\infty(\partial M, B) \end{cases}$$

has solutions for all ψ_0 and hence the kernel of D with respect to the boundary condition is infinite dimensional. Hence, one also needs to put non-local boundary conditions on the initial time slice or the additional spacelike boundary.

6. *Temporal function:* The boundary conditions we defined in Definition 3.3.13 depend on the temporal function chosen at the beginning. This can also be seen in the example of APS, where the parallel transport of the unit normal to the timelike boundary along the t-lines can make the boundary condition non-elliptic. This leads to the question of how much the well-posedness itself depends on the temporal function and if there is a better description of non-local boundary conditions independent of the temporal function.

7. *Non-local in time:* Note that even if we are considering non-local boundary conditions in spatial directions, we still are local in time-direction. It would also be interesting – and also natural – to consider boundary conditions which are non-local in time. This would then require an analogue of the theory of [8, 10] for hyperbolic operators on a non-compact manifold, which should be highly non-trivial since the spectrum of the boundary operator would be difficult to control.

8. *Spatially non-compact boundary:* Recall, that for all our results, we assumed the timelike boundary to be spatially compact. This assumption is needed, since the theory of boundary

conditions for elliptic operators is (in this generality) only known for compact boundary. For special elliptic operators, [32] discussed boundary conditions for non-compact boundary with bounded geometry. Thus, it would be also interesting to see if one could for example use the theory of [32] to construct boundary conditions on not necessarily spatially compact timelike boundaries.

A

Dirac Operators in the Sense of Gromov–Lawson

In this chapter we will briefly discuss one possible generalization of our well-posedness result to a bigger class of operators. Here, we will look at the a Lorentzian equivalent of Dirac operators in the sense of Gromov and Lawson. These operators are more abstract than the spin Dirac operator but they still behave quite similar, which we will see in the following sections.

A.1 Definition and properties

The following definition is an analogue of the Dirac operators in the sense of Gromov and Lawson, which are defined like this in [9], but are first introduced by [30] and [36]. This is as in the Riemannian case a direct generalization of the spin Dirac operator.

Definition A.1.1. Let (M, g) be a Lorentzian manifold with complex vector bundle $E \to M$ with a non-degenerate sesquilinear form $\langle \cdot, \cdot \rangle_E$.
Then we call an operator $D\colon C^\infty(M, E) \to C^\infty(M, E)$ a *Lorentzian Dirac operator in the sense of Gromov–Lawson*, if there exists a metric connection ∇^E such that

1. $D = \sum_{j=0}^n \sigma_D(e_j^\flat)\nabla_{e_j}$ for any local orthonormal tangent frame $(e_1, \dots, e_n)$,

2. the principal symbol σ_D is parallel with respect to ∇^E and the Levi-Civita connection ∇ on TM,

3. the principal symbol is symmetric and satisfies for all $x \in M$ and $\zeta, \eta \in T_xM$

$$\sigma_D(\zeta)\sigma_D(\eta) + \sigma_D(\eta)\sigma_D(\zeta) = -2g(\zeta, \eta), \quad \text{and} \tag{A.1.1}$$

4. for any timelike $\tau \in T_x^* M$
$$\langle \sigma_D(\tau)\cdot, \cdot \rangle_E ,$$
is positive definite.

Remark A.1.2. This definition is quite close to the Riemannian one in [9]. The biggest difference is surely the last condition, which is in general not needed, but for our methods it is essential. For (M, g) being a globally hyperbolic spacetime, we want – like for the case of the spin Dirac operator – that the induced operators D_t on the foliating spacelike Cauchy hypersurfaces are Riemannian Dirac type operators in the sense of Gromov and Lawson acting on Hermitian vector bundles. The last condition of Definition A.1.1 ensures that the induced inner product is indeed positive definite and hence a Hermitian inner product.

Note that a Lorentzian Dirac operator in the sense of Gromov–Lawson also satisfies the analog Greens formula as in Equation 2.2.4. Furthermore note that D^2 is normally hyperbolic[1] since it has principal symbol $\sigma_{D^2}(\zeta) = -|\zeta|_g^2$.

Remark A.1.3. Following the computation in the proof of Proposition 2.5 in [30], we see that for a Lorentzian Dirac operator in the sense of Gromov–Lawson, one has
$$D^2 = (\nabla^E)^* \nabla^E + \mathcal{R}^E, \tag{A.1.2}$$
where $\mathcal{R}^E = \frac{1}{2} \sum_{i,j} \sigma_D(e_i^\flat) \circ \sigma_D(e_j^\flat) \circ R^E(e_i, e_j)$ with R^E being the curvature tensor of ∇^E.

Before we talk about some properties of these operators, let us discuss some examples.

Example A.1.4.

1. By the proprieties, discussed in Subsection 2.2, we see that the Lorentzian spin Dirac operator on the complex spin bundle is indeed a Lorentzian Dirac operator in the sense of Gromov–Lawson.

2. Let (M, g) be a Lorentzian spin manifold and $D : C^\infty(M, F) \to C^\infty(M, F)$ be a Lorentzian Dirac operator in the sense of Gromov–Lawson. Let $C \to M$ be a Hermitian vector bundle with metric connection ∇^C. Now we can define the *twisted Dirac operator*
$$D^{\nabla^C} = \sum_{j=0}^{n} (\sigma_D(e_j) \otimes id_C) \nabla^E_{e_j}.$$
Here $E = F \otimes C$ and ∇^E is the metric connection characterized by
$$\nabla^E (f \otimes c) := \nabla^F f \otimes c + f \otimes \nabla^C c.$$

[1] For a Definition see [12].

This connection is metric with respect to the natural inner product $\langle \cdot, \cdot \rangle_E$ on E, see Lemma B.1.1. The principal symbol D given by $\sigma_D(\zeta) = \sigma_D(\zeta) \otimes id_C$ is indeed parallel with respect to ∇^E and the Levi-Civita connection on TM and also satisfies the Clifford relation, see Lemma B.1.2. By Lemma B.1.3, we also see that D^{∇^C} satisfy as well an analogue Greens formula as in Equation 2.2.4. Since we twist with an Hermitian bundle, we also directly know that the last point of Definition A.1.1 is satisfied, and hence, D^{∇^C} is a Lorentzian Dirac operator in the sense of Gromov–Lawson.

This implies that for the Lorentzian spin Dirac operator on the complex spin bundle, all twisted operators with Hermitian bundles are Lorentzian Dirac operators in the sense of Gromov–Lawson.

3. If the twist bundle C is a natural bundle induced by the geometry of M itself, then the last point of Definition A.1.1 may not be necessarily satisfied. For example for $C = TM$, we have that for any timelike vector field v the inner product $\left\langle \sigma_{D^{\nabla^C}}(v)\cdot, \cdot \right\rangle_E$ is not positive definite, since for the pure tensor $f \otimes X \in F_x \otimes T_x M$ and X being timelike the inner product

$$\left\langle \sigma_{D^{\nabla^C}}(v) f \otimes X, f \otimes X \right\rangle_E = \langle \sigma_D(v) f, f \rangle_E \cdot g(X, X),$$

cannot be positive.

Now, let us assume that (M, g) is a globally hyperbolic spin manifold without boundary. Let $t\colon M \to \mathbb{R}$ a temporal function and $\{\Sigma_t\}_{t\in\mathbb{R}}$ be the corresponding foliation of M consisting of spacelike Cauchy hypersurfaces. Let v be the past directed timelike unit vector field, that is induced by the temporal function t. Following the strategy of [9], we will compute how the Lorentzian Dirac operator in the sense of Gromov–Lawson is splitting along this foliation.

Let $e_0, \dots, e_1$ be a Lorentzian orthonormal frame such that $e_0 = v$ and consider the following operator

$$D_{t,0} := -i \left(\sigma_D(v^\flat) D - \nabla^E_v \right) = -i \sigma_D(v) \sum_{j=1}^{n} \sigma_D(e_j^*) \nabla^E_{e_j}, \tag{A.1.3}$$

Then we see that the principal symbol of $D_{t,0}$ is given by

$$\sigma_{D_{t,0}}(\zeta) = -i \sigma_D(v^\flat) \sigma_D(\zeta), \tag{A.1.4}$$

which is symmetric. Furthermore, we can compute for $\phi, \psi \in C^\infty(M, E)$

$$0 = \int_M \left(\left\langle D^2 \phi, \psi \right\rangle_E - \left\langle (\nabla^E)^* \nabla^E \phi, \psi \right\rangle_E - \left\langle \mathcal{R}^E \phi, \psi \right\rangle_E \right) \mathrm{d}\mu_M$$

$$
\begin{aligned}
&= \int_M \left(\langle D\phi, -D\psi \rangle_E - \left\langle \nabla^E \phi, \nabla^E \psi \right\rangle_E - \left\langle \mathcal{R}^E \phi, \psi \right\rangle_E \right) \mathrm{d}\mu_M \\
&\quad + \int_{\Sigma_t} \left(\left\langle \sigma_D(\nu^\flat)(-D)\phi, \psi \right\rangle_E - \left\langle \sigma_{(\nabla^E)^*}(\nu^\flat)\nabla^E \phi, \psi \right\rangle_E \right) \mathrm{d}\mu_{\Sigma_t} \\
&= - \int_M \left(\langle D\phi, D\psi \rangle_E + \left\langle \nabla^E \phi, \nabla^E \psi \right\rangle_E + \left\langle \mathcal{R}^E \phi, \psi \right\rangle_E \right) \mathrm{d}\mu_M \\
&\quad + \int_{\Sigma_t} \left(- \left\langle \sigma_D(\nu^\flat) D\phi, \psi \right\rangle_E + \left\langle \nabla^E_\nu \phi, \psi \right\rangle_E \right) \mathrm{d}\mu_{\Sigma_t} \\
&= - \int_M \left(\langle D\phi, D\psi \rangle_E + \left\langle \nabla^E \phi, \nabla^E \psi \right\rangle_E + \left\langle \mathcal{R}^E \phi, \psi \right\rangle_E \right) \mathrm{d}\mu_M \\
&\quad - \int_{\Sigma_t} \left\langle i D_{t,0} \phi, \psi \right\rangle_E \mathrm{d}\mu_{\Sigma_t},
\end{aligned}
$$

where we used Σ_t as a fake boundary and applied the Weizenböck formula A.1.2 as well as the Green formula. This computation implies

$$
\int_M \left(\left\langle D^2 \phi, \psi \right\rangle_E - \left\langle (\nabla^E)^* \nabla^E \phi, \psi \right\rangle_E - \left\langle \mathcal{R}^E \phi, \psi \right\rangle_E \right) \mathrm{d}\mu_M = - \int_{\Sigma_t} \left\langle i D_{t,0} \phi, \psi \right\rangle_E \mathrm{d}\mu_{\Sigma_t},
$$

and since all objects on the right hand side are symmetric, we see that $iD_{t,0}$ is formally selfadjoint with respect to $\langle \cdot, \cdot \rangle_E$.

Now we want to find an operator D_t such that it anti-commutes with $\sigma_D(\nu^\flat)$. For this we compute the anti-commutator for $\psi \in C^\infty(M, E)$ as follows

$$
\begin{aligned}
&\left[\sigma_D(\nu^\flat) i D_{t,0} + i D_{t,0} \sigma_D(\nu^\flat) \right] \psi \\
&= \sum_{j=1}^n \left[\sigma_D(e_j^\flat) \nabla^E_{e_j} + \sigma_D(\nu^\flat) \sigma_D(e_j^\flat) \sigma_D(\nu^\flat) \nabla^E_{e_j} + \sigma_D(\nu^\flat) \sigma_D(e_j^\flat) \sigma_D(\nabla_{e_j} \nu^\flat) \right] \psi \\
&= \sigma_D(\nu^\flat) \sum_{j=1}^n \sigma_D(e_j^*) \sigma_D(\nabla_{e_j} \nu^\flat) \psi,
\end{aligned}
$$

where we see that $\nabla_{e_j} \nu$ is the Weingarten map. Now let us choose the frame $e_1, \ldots, e_n$ as the eigenbasis of the Weingarten map, then we get

$$
\begin{aligned}
&\left[\sigma_D(\nu^\flat) i D_{t,0} + i D_{t,0} \sigma_D(\nu^\flat) \right] \phi \\
&= \sigma_D(\nu^\flat) \sum_{j=1}^n (\kappa_t)_i \sigma_D(e_j^*) \sigma_D(e_j^*) \phi
\end{aligned}
$$

$$= -\sigma_D(\nu^\flat) n H_t \phi$$

where $(\kappa_t)_i$ are the principal curvatures and H_t is the mean curvature of Σ_t. Rearranging the terms above, we get

$$D = \sigma_D(\nu^\flat)\left(\nabla^E_\nu + iD_t - \frac{n}{2}H_t\right), \tag{A.1.5}$$

where now $D_t := -i\left(iD_{t,0} + \frac{n}{2}H_t\right)$ is anti-commuting with $\sigma_D(\nu^\flat)$ and it is formally selfadjoint with respect to

$$\langle\cdot,\cdot\rangle_0 = \left\langle \sigma_D(\nu^\flat)\cdot,\cdot\right\rangle_E.$$

All of the above implies that $D_t : C^\infty(\Sigma, E|_{\Sigma_t}) \to C^\infty(\Sigma, E|_{\Sigma_t})$ is a Riemannian Dirac type operator (in the sense of Definition 3.1.4), which is formally selfadjoint with respect to the positive inner product $\langle\cdot,\cdot\rangle_0$.

Remark A.1.5. The computations above show that the splitting of the spin Dirac operator also follows from the more general theory of Lorentzian Dirac operators in the sense of Gromov–Lawson.

A.2 Boundary conditions and Cauchy problems

In this section we will talk about boundary conditions for Lorentzian Dirac operators in the sense of Gromov–Lawson. Firstly, not that the transmission conditions defined in Section 3.3.2 can be directly transferred to these operators, hence let us rather talk about how the admissible boundary conditions can be looked at in this context.
Let us first discuss about the reduction of the Dirac equation to Hamiltonian form. In general, one does not know how these operators are changing under the conformal change $\hat{g} = N^{-2}g$, hence let us restrict our discussion to the ultrastatic case, i. e.

$$g = -\,\mathrm{d}t^2 + g_t.$$

So we can skip the step of the conformal change and directly look at the identification of Cauchy hypersurfaces. For $t, s \in \mathbb{R}$ let $\tau_t^s : E|_{\Sigma_t} \to E|_{\Sigma_s}$ the parallel transport along the integral curves of ν with respect to the metric connection ∇^E.

Remark A.2.1. Since σ_D is parallel and ν is parallel along itself, one sees that

$$\tau_t \sigma_{D_t}(\zeta)u = -i\tau_t\sigma_D(\nu_t)^{-1}\sigma_D(\zeta)u = -i\sigma_D(\nu_0)^{-1}\sigma_D(\tau_t\zeta)\tau_t u = \sigma_{D_0}(\tau_t\zeta)\tau_t u.$$

Furthermore, since ν is parallel along itself and ∇^E is metric the map

$$U = \rho(t)\tau_t : L^2(\Sigma_t, E|_{\Sigma_t}) \to L^2(\Sigma_0, E|_\Sigma)$$

is a unitary isomorphism.

As in Section B.1.3, one has the following Lemma:

Lemma A.2.2. *Let D be a Lorentzian Dirac operator in the sense of Gromov–Lawson, then D satisfies*

$$D = -\sigma_D(\nu)U^{-1}(t)(\partial_t + i\tilde{D}_t)U(t),$$

where $\tilde{D}_t = U(t)D_tU(t)^{-1}$.

Proof. Recall that for a Lorentzian Dirac operator in the sense of Gromov–Lawson, we have the splitting $D = \sigma_D(\nu^\flat)\left(\nabla^E_\nu + iD_t - \frac{n}{2}H_t\right)$. Using that ∇^E is metric and ν being parallel along itself and following the computation in the proof of Lemma 3.3.9, the claim follows. □

This gives us again the tools to define admissible boundary conditions:

Definition A.2.3. We call a family $B = \{B_t\}_{t\in\mathbb{R}}$ of selfadjoint boundary conditions $B_t \subseteq H^{\frac{1}{2}}(\Sigma_t, SM|_{\Sigma_t})$ an *admissible boundary condition* for D if

1. $\tilde{B}_t := U(t)B_t \subseteq \check{H}(\tilde{A}_t)$ is a selfadjoint ∞-regular boundary condition for all $t \in \mathbb{R}$, and
2. the operator $\tilde{D}_{B_t}J^{(\epsilon)}_{\tilde{B}_t}$ is strongly continuous with respect to $\|\cdot\|_{H^k}$ for all $k \in \mathbb{N}$.

As before, we can define the Lorentzian boundary condition by

$$C^\infty(\partial M, B) := \{\psi \in C^\infty(\partial M, E|_{\partial M});\ \psi|_{\partial\Sigma_t} \in B_t\ \forall t \in \mathbb{R}\}.$$

Now we can proof the following well-posedness:

Theorem A.2.4. *Let (M, g) be a globally hyperbolic, spatially compact manifold with Lapse function equals to one and timelike boundary ∂M. Let Σ be a spacelike Cauchy hypersurface and $t: M \to \mathbb{R}$ be a temporal function such that $\Sigma_0 = \Sigma$ and the gradient of t is tangential to ∂M. Let B be an admissible boundary condition with respect to t and D. Then there exists a unique smooth solution $\psi \in C^\infty(M, E)$ to*

$$\begin{cases} D\psi = f \in C^\infty(M, E) \\ \psi|_\Sigma = \psi_0 \in C^\infty(\Sigma, SM|_\Sigma) \\ \psi|_{\partial M} \in C^\infty(\partial M, B), \end{cases}$$

that depends continuously on the Cauchy data (f, ψ_0).

Proof. After reducing the Cauchy problem to Hamiltonian form, we again can use Lemma 3.1.22 to define closed subspaces of the Sobolev spaces $H^k(\Sigma_0, E|_{\Sigma_0})$ and the corresponding mollifiers. The rest of the proof works analogously to the one done for Main Theorem 1, □

Remark A.2.5. Using the same methods as in Chapter 5, one can also show the corresponding well-posedness results for APS conditions and chirality conditions.

B Some Auxiliary Discussions

B.1 Some properties of twisted operators

Let (M, g) be a Lorentzian spin manifold and $D\colon C^\infty(M, F) \to C^\infty(M, F)$ be a Lorentzian Dirac operator in the sense of Gromov and Lawson. Let $C \to M$ be a Hermitian vector bundle with metric connection ∇^C. Now we can define

$$D^{\nabla^C} = \sum_{j=0}^{n} (\sigma_D(e_j) \otimes id_C)\nabla^E_{e_j}.$$

Here $E = F \otimes C$ and ∇^E is the metric connection characterized by

$$\nabla^E(f \otimes c) := \nabla^F f \otimes c + f \otimes \nabla^C c.$$

Lemma B.1.1. *The connection ∇^E is metric with respect to the non-degenerate sesquilinear form $\langle \cdot, \cdot \rangle_E$, which is characterized by*

$$\langle f_1 \otimes c_1, f_2 \otimes c_2 \rangle_F := \langle f_1, f_2 \rangle_F \cdot \langle c_1, c_2 \rangle_C .$$

Proof. We show this for pure tensors and then by linearity the whole claim follows. Let $f_1 \otimes c_1, f_2 \otimes s_2 \in C^\infty(M, E)$ and $X \in C^\infty(M, TM)$, then we can compute pointwise

$$\begin{aligned}
\left\langle \nabla^E_X(f_1 \otimes c_1), f_2 \otimes c_2 \right\rangle_E =& \left\langle \nabla^F_X f_1 \otimes c_1 + f_1 \otimes \nabla^C_X c_1, f_2 \otimes c_2 \right\rangle_E \\
=& \left\langle \nabla^F_X f_1 \otimes c_1, f_2 \otimes c_2 \right\rangle_E + \left\langle f_1 \otimes \nabla^C_X c_1, f_2 \otimes c_2 \right\rangle_E \\
=& \left\langle \nabla^F_X f_1, f_2 \right\rangle_F \cdot \langle c_1, c_2 \rangle_C + \langle f_1, f_2 \rangle_F \cdot \left\langle \nabla^C_X c_1, c_2 \right\rangle_C \\
=& \partial_X(\langle f_1, f_2 \rangle_F) \cdot \langle c_1, c_2 \rangle_C + \langle f_1, f_2 \rangle_F \cdot \partial_X(\langle c_1, c_2 \rangle_C) \\
& - \left\langle f_1, \nabla^F_X f_2 \right\rangle_F \cdot \langle c_1, c_2 \rangle_C - \langle f_1, f_2 \rangle_F \cdot \left\langle c_1, \nabla^C_X c_2 \right\rangle_C ,
\end{aligned}$$

where we used the definition of the inner product and the connection on E and also that the connections on F and C are metric. Using now the product rule on the first two summands, we can continue to compute as follows

$$\begin{aligned}
&\left\langle \nabla^E_X(f_1 \otimes c_1), f_2 \otimes c_2 \right\rangle_E \\
&= \partial_X(\langle f_1, f_2\rangle_F \cdot \langle c_1, c_2\rangle_C) - \left\langle f_1, \nabla^F_X f_2 \right\rangle_F \cdot \langle c_1, c_2\rangle_C - \langle f_1, f_2\rangle_F \cdot \left\langle c_1, \nabla^C_X c_2 \right\rangle_C \\
&= \partial_X(\langle f_1 \otimes c_1, f_2 \otimes c_2\rangle_E) - \left\langle f_1 \otimes c_1, \nabla^F_X f_2 \otimes c_2 \right\rangle_E - \left\langle f_1 \otimes c_1, f_2 \otimes \nabla^C_X c_2 \right\rangle_E \\
&= \partial_X(\langle f_1 \otimes c_1, f_2 \otimes c_2\rangle_E) - \left\langle f_1 \otimes c_1, \nabla^E_X(f_2 \otimes c_2) \right\rangle_E,
\end{aligned}$$

where we additionally just used again the definitions of the inner product and the connection on E. □

Lemma B.1.2. *The principal symbol of D satisfies the Clifford relation and is parallel with respect to ∇^E and the Levi-Civita connection ∇.*

Proof. Let $f \otimes c \in C^\infty(M, E)$ be a pure tensor, let $X \in C^\infty(M, TM)$ and let $\zeta \in C^\infty(M, T^*M)$. Then

$$\begin{aligned}
\nabla^E_X(\sigma_{D^{\nabla^C}}(\zeta) f \otimes c) &= \nabla^F_X \sigma_D(\zeta) f \otimes c + \sigma_D(\zeta) f \otimes \nabla^C_X c \\
&= (\sigma_D(\zeta)\nabla^F_X f + \sigma_D(\nabla_X \zeta) f) \otimes c + \sigma_D(\zeta) f \otimes \nabla^C_X c \\
&= (\sigma_D(\zeta)\nabla^F_X f + \sigma_D(\zeta) f \otimes \nabla^C_X c) + \sigma_D(\nabla_X \zeta) f) \otimes c \\
&= \sigma_{D^{\nabla^C}}(\zeta)\nabla^E(f \otimes c) + \sigma_{D^{\nabla^C}}(\nabla_X \zeta)(f \otimes c),
\end{aligned}$$

where we only used that that σ_D is parallel with respect to ∇^F and ∇. This implies that $\sigma_{D^{\nabla^C}}$ is as well parallel. For showing the Clifford relations let $x \in M$ and $\zeta, \eta \in T^*_x M$. Then we can compute for a pure tensor $f \otimes c$

$$\begin{aligned}
[\sigma_{D^{\nabla^C}}(\zeta)\sigma_{D^{\nabla^C}}(\eta) + \sigma_{D^{\nabla^C}}(\eta)\sigma_{D^{\nabla^C}}(\zeta)] f \otimes c &= \sigma_{D^{\nabla^C}}(\zeta)(\sigma_D(\eta) f \otimes c) + \sigma_{D^{\nabla^C}}(\eta)(\sigma_D(\zeta) f \otimes c) \\
&= (\sigma_D(\zeta)\sigma_D(\eta) f) \otimes c + (\sigma_D(\eta)\sigma_D(\zeta) f) \otimes c \\
&= ([\sigma_D(\zeta)\sigma_D(\eta) + \sigma_D(\eta)\sigma_D(\zeta)] f) \otimes c \\
&= -2g(\zeta, \eta) f \otimes c,
\end{aligned}$$

hence also $\sigma_{D^{\nabla^C}}$ is satisfying the Clifford relations. □

Lemma B.1.3. *For all $\psi, \phi \in C^\infty_c(M, E)$ we have the Greens formula*

$$\int_M \left\langle D^{\nabla^C}\psi, \phi \right\rangle_E + \left\langle \psi, D^{\nabla^C}\phi \right\rangle_E \mathrm{d}\mu_M = -\int_{\partial M} \left\langle \sigma_{D^{\nabla^C}}(\eta)\psi, \phi \right\rangle_E \mathrm{d}\mu_{\partial M}, \tag{B.1.1}$$

where $\mathrm{d}\mu_M$ is the volume element on M with respect to g and $\mathrm{d}\mu_{\partial M}$ is the induced one on ∂M.

Proof. Note first that for pure tensors $f \otimes c \in F \otimes C$, one can rewrite D^{∇^C} as

$$D^{\nabla^C}(f \otimes c) = Df \otimes c + \sum_{j=0}^{n} \sigma_D(e_j^\flat) f \otimes \nabla^C_{e_j} c. \tag{B.1.2}$$

Using Equation B.1.2, we can compute for pure tensors the following:

$$\begin{aligned}
&\int_M \left\langle D^{\nabla^C} f_1 \otimes c_1, f_2 \otimes c_2 \right\rangle_E \mathrm{d}\mu_M \\
&= \int_M \left\langle (Df_1) \otimes c_1 + \sum_{j=0}^{n} \sigma_D(e_j^\flat) f_1 \otimes \nabla^C_{e_j} c_2, f_2 \otimes c_2 \right\rangle_E \mathrm{d}\mu_M \\
&= \int_M \langle (Df_1) \otimes c_1, f_2 \otimes c_2 \rangle_E + \left\langle \sum_{j=0}^{n} \sigma_D(e_j^\flat) f_1 \otimes \nabla^C_{e_j} c_2, f_2 \otimes c_2 \right\rangle_E \mathrm{d}\mu_M \\
&= \int_M \langle Df_1, f_2 \rangle_F \cdot \langle c_1, c_2 \rangle_C \, \mathrm{d}\mu_M + \int_M \sum_{j=0}^{n} \left\langle \sigma_D(e_j^\flat) f_1, f_2 \right\rangle_F \cdot \left\langle \nabla^C_{e_j} c_1, c_2 \right\rangle \mathrm{d}\mu_M \\
&= -\int_M \left\langle f_1, D(\langle c_1, c_2 \rangle_C f_2) \right\rangle_F \mathrm{d}\mu_M - \int_{\partial M} \left\langle \sigma_D(\eta) f_1, \langle c_1, c_2 \rangle_C f_2 \right\rangle_F \mathrm{d}\mu_{\partial M} \\
&\quad + \int_M \sum_{j=1}^{n} \left\langle f_1, \sigma_D(e_j^\flat) f_2 \right\rangle_F \left(\partial_{e_j}(\langle c_1, c_2 \rangle_C) - \left\langle c_1, \nabla^C_{e_j} c_2 \right\rangle_C \right) \mathrm{d}\mu_M \\
&= -\int_M \langle f_1, Df_2 \rangle_F \langle c_1, c_2 \rangle_C + \left\langle f_1, \sigma_D(d(\langle c_1, c_2 \rangle_C)) f_2 \right\rangle_F \mathrm{d}\mu_M - \int_M \sum_{j=1}^{n} \left\langle f_1, \sigma_D(e_j^\flat) f_2 \right\rangle_F \left\langle c_1, \nabla^C_{e_j} c_2 \right\rangle_C \\
&\quad - \left\langle f_1, \sigma_D\left(\sum_{j=0}^{n} \partial_{e_j}(\langle c_1, c_2 \rangle_C) e_j^\flat \right) f_2 \right\rangle_F \mathrm{d}\mu_M - \int_{\partial M} \left\langle \sigma_{D^{\nabla^C}}(\eta) f_1, f_2 \right\rangle_E \mathrm{d}\mu_{\partial M} \\
&= -\int_M \langle f_1, Df_2 \rangle_F \langle c_1, c_2 \rangle_C + \sum_{j=1}^{n} \left\langle f_1, \sigma_D(e_j^\flat) f_2 \right\rangle_F \left\langle c_1, \nabla^C_{e_j} c_2 \right\rangle_C \mathrm{d}\mu_M - \int_{\partial M} \left\langle \sigma_{D^{\nabla^C}}(\eta) f_1, f_2 \right\rangle_E \mathrm{d}\mu_{\partial M} \\
&\quad + \int_M \left\langle f_1, \sigma_D\left(\sum_{j=0}^{n} \partial_{e_j}(\langle c_1, c_2 \rangle_C) e_j^\flat \right) f_2 \right\rangle_F - \left\langle f_1, \sigma_D(d(\langle c_1, c_2 \rangle_C)) f_2 \right\rangle_F \mathrm{d}\mu_M \\
&= -\int_M \left\langle f_1 \otimes c_1, -D^{\nabla^C} f_2 \otimes c_2 \right\rangle_E \mathrm{d}\mu_M - \int_{\partial M} \left\langle \sigma_{D^{\nabla^C}}(\eta) f_1, f_2 \right\rangle_E \mathrm{d}\mu_{\partial M},
\end{aligned}$$

where we used the Greens formula for D, that σ_D is symmetric, that ∇^C is metric and in the last step we just used the definition of the exterior derivative in an orthonormal frame. $\square$

B.2 The spin Dirac operator on $\mathbb{R}^{1,1}$

Let us consider the Minkowski space $(M, g) = (\mathbb{R}^{1,1}, g = -\,\mathrm{d}t^2 + \mathrm{d}x^2)$, where we can identify the spin bundle $SM \simeq M \times \mathbb{C}^2$. Then we have for the orthonormal frame $\{\partial_t, \partial_x\}$ the following Dirac operator

$$D = -\sigma_0\partial_t + \sigma_2\partial_x = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \partial_t + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \partial_x,$$

where

$$\sigma_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and } \sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

represent the Clifford multiplications. Let us have a look at the homogeneous initial value problem

$$\begin{cases} D\psi = 0 \\ \psi|_{\{t=0\}} = \psi_0 \in C^\infty(\mathbb{R}, \mathbb{C}^2) \end{cases} \tag{B.2.1}$$

for $\psi \in C^\infty(M, \mathbb{C}^2)$. Firstly, note that ψ is a solution to the Cauchy problem B.2.1 if and only if it is a solution to

$$\begin{cases} \hat{D}\psi = 0 \\ \psi|_{\{t=0\}} = \psi_0 \in C^\infty(\mathbb{R}, \mathbb{C}^2) \end{cases} \tag{B.2.2}$$

where

$$\hat{D} = \partial_t + C\partial_x$$

with

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Recall, that $e^{t\partial_x}\psi_0(x) = \psi_0(x + t)$ and also note that

$$C = MWM^{-1},$$

with

$$W = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{and } M^{-1} = \frac{1}{2}M.$$

Using these two facts, the solution ψ to the reduced Cauchy problem B.2.2 can be computed as follows:

$$\psi(t, x) = e^{-tC\partial_x}\psi_0(x)$$

$$= \frac{1}{2} M \begin{pmatrix} e^{t\partial_x} & 0 \\ 0 & e^{-t\partial_x} \end{pmatrix} M \begin{pmatrix} (\psi_0)_1(x) \\ (\psi_0)_2(x) \end{pmatrix}$$

$$= \frac{1}{2} M \begin{pmatrix} e^{t\partial_x} & 0 \\ 0 & e^{-t\partial_x} \end{pmatrix} \begin{pmatrix} -(\psi_0)_1(x) + (\psi_0)_2(x) \\ (\psi_0)_2(x) + (\psi_0)_2(x) \end{pmatrix}$$

$$= \frac{1}{2} M \begin{pmatrix} -(\psi_0)_1(x+t) + (\psi_0)_2(x+t) \\ (\psi_0)_1(x-t) + (\psi_0)_2(x-t) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} (\psi_0)_1(x+t) - (\psi_0)_2(x+t) + (\psi_0)_1(x-t) + (\psi_0)_2(x-t) \\ -(\psi_0)_1(x+t) + (\psi_0)_2(x+t) + (\psi_0)_1(x-t) + (\psi_0)_2(x-t) \end{pmatrix}$$

$$= \frac{1}{2} \left[\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \psi_0(x+t) + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \psi_0(x-t) \right].$$

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