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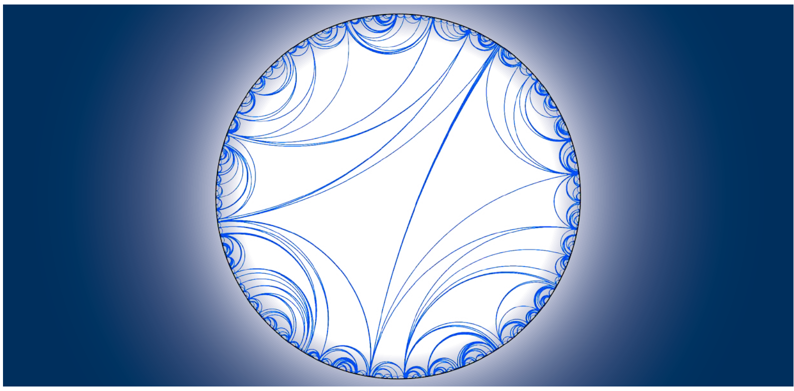

Benjamin Nehring

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Construction of Point Processes for Classical and Quantum Gases

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Abstract

We propose a new construction of point processes, which generalizes the class of infinitely divisible point processes. Examples are the quantum Boson and Fermion gases as well as the classical Gibbs point processes, where the interaction is given by a stable and regular pair potential.

Keywords: Gibbs point processes, permanental-, determinantal point processes, cluster expansion, Lévy measure, infinitely divisible point processes

1 Introduction

Main Results

For a general pair potential ϕ, which satisfies the conditions of stability and regularity as defined in section 7 and for a sufficiently small activity z we construct a point process in a general Polish space X, which will be identified as a Gibbs point process in theorem 7.5. The Gibbs property will be shown by the use of an integral equation, due to [15]. Also a tree estimate of the

Ursell function as in [16] and a generalization of an integral equation due to [11] are important elements of the proof.

In [18] Ruelle has shown that for a translation invariant pair potential which is stable and regular and for a small activity there exist infinite volume limit correlation functions, which satisfy the Kirkwood-Salzburg equations. In [19] Ruelle found that if the conditions of regularity, superstability and lower regularity are imposed on a translation invariant pair potential then there exists a Gibbs point process for *any* activity. In [6], [7] Kuna et al. obtained for a stable and regular pair potential, which additionally has the finite range property, that for a small activity the corresponding Gibbs point process exists. They establish the Gibbs property by directly verifying the equilibrium equations due to Dobrushin-Lanford-Ruelle. Here we take a more point process theoretical approach.

The idea is to verify for the limiting process the integration by parts formula already mentioned above, which is equivalent to the DLR-equation as well as to Ruelle's equilibrium equation in [19]. (A proof of this equivalence can be found in [15]). The same idea, for a different class of potentials, has been indicated already in the article of Kutoviy and Rebenko (see [8]) but without giving the detailed argument.

The Gibbs point processes as well as the Fermion and Boson point processes are examples of the existence theorem 3.2, they belong to the class of point processes with a signed Lévy functional, as introduced in the next section. For another approach to the construction of Boson and Fermion point processes we refer to Shirai, Takahashi [20] and Soshnikov [21]. In the section 4 entitled "the method of cluster expansion" we explain how the construction fits into the work [9] of Malyshev and Minlos and obtain that a weaker integrability condition is sufficient for the existence of the thermodynamic limit.

This article is based on the report in the proceedings [13] to the international mathematical conference: 50 years of IPPI. There it is shown that also the classes of Polya and immanantal point processes are covered by the existence theorem 3.2.

Notation

Let X be a Polish space, $\mathcal{B}(X)$ resp. $\mathcal{B}_0(X)$ denote the Borel resp. bounded Borel sets. $\mathcal{M}(X)$ is the space of locally finite measures on X i.e. Radon measures on X, which is Polish for the vague topology. $\mathcal{M}^{\cdot\cdot}(X)$ denotes the closed and thereby measurable subspace of Radon point measures and $\mathcal{M}^{\cdot}(X)$ denotes the measurable subspace of simple Radon point measures. Furthermore let $\mathcal{M}^{\cdot\cdot}_f(X)$ be the set of finite point measures on X. A law P on $\mathcal{M}(X)$ resp. $\mathcal{M}^{\cdot\cdot}(X)$ resp. $\mathcal{M}^{\cdot}(X)$ resp. $\mathcal{M}^{\cdot\cdot}_f(X)$ is called resp. random measure, point process, simple point process and finite point process. U is the set of all bounded non negative measurable functions on X with compact support and F_+ is the set of all non negative measurable functions, where the underlying space will be clear from the context. $\mathbf{1}_A$ denotes the indicator function of the set A and $\mathbf{1}$ the function which is identically one. An important tool for the analysis of measures K on $\mathcal{M}(X)$ is the Campbell measure $C_K(h) = \iint h(x,\mu)\,\mu(\mathrm{d}\,x)K(\mathrm{d}\,\mu)$ for $h \in F_+(X \times \mathcal{M}(X))$. The marginal $\nu^1_K(f) = C_K(f \otimes \mathbf{1})$, $f \in U$, is called the first moment measure of K. Remark that for $f \in U$, $\zeta_f : \mu \mapsto \mu(f)$ is a well defined continuous function on $\mathcal{M}(X)$ and let $\mathcal{L}_P(f) = P(e^{-\zeta_f})$, $f \in U$, be the Laplace transform of the random measure P.

2 Reminder: Infinitely Divisible Point Processes and a Generalization

Let us recall the well known (see Mecke [11]) existence result for point processes. In the sequel we will denote by $\mathcal{W}$ the class of non negative measures L on $\mathcal{M}^{\cdot\cdot}(X)$ such that $L(\mathbf{1} - e^{-\zeta_f}) < \infty$, $f \in U$, and $L(\{0\}) = 0$.

Theorem 2.1 (Mecke)**.** *Let $L \in \mathcal{W}$. Then there exists a point process P such that its Laplace transform is given by*

$$\mathcal{L}_P(f) = \exp[-L(\mathbf{1} - e^{-\zeta_f})] \text{ for } f \in U, \tag{2.1}$$

and P is infinitely divisible. Moreover if $\nu^1_L \in \mathcal{M}(X)$ then (2.1) is equivalent to

$$(\mathbf{\Sigma}_L) \qquad C_P(h) = \int_{\mathcal{M}^{\cdot\cdot}(X)} \int_{\mathcal{M}^{\cdot\cdot}(X)} \int_X h(x, \eta + \mu)\, C_L(\mathrm{d}\,x\,\mathrm{d}\,\eta)P(\mathrm{d}\,\mu)$$

for all $h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X))$*. This point process will be denoted* Ψ_L *in the sequel.*

As an aside we remark that also the converse holds true: To every infinitely divisible point process P there belongs some $L \in \mathcal{W}$ such that (2.1) holds. The measure L is called the Lévy measure of the infinitely divisible point process Ψ_L. The right hand side of $(\mathbf{\Sigma}_L)$ can be seen as a kind of convolution of C_L with P so we will denote it by $C_L \star P(h)$. Here $\star$ should not be confused with the convolution operator $*$ used below. Furthermore let $\mathcal{K}_L$ be the functional on U such that $-\log(\mathcal{K}_L(f)) = L(\mathbf{1} - e^{-\zeta_f})$, the so called *signed modified Laplace functional.* Meckes proof in [11] relies on his following result, a version of Lévy's continuity theorem for random measures

Theorem 2.2 (Mecke)**.** *Let* $(P_m)_m$ *be a sequence of laws on* $\mathcal{M}(X)$ *such that*

$$\mathcal{L}_{P_m}(f) \to \mathcal{K}(f) \text{ as } m \to \infty$$

for $f \in U$ *and the limiting functional* $\mathcal{K}$ *on* U *satisfies for any sequence* $u_n \in U$ *with* $u_n \downarrow 0$, $\mathcal{K}(u_n) \to 1$ *as* $n \to \infty$, *then there exists a law* P *on* $\mathcal{M}(X)$ *such that* $\mathcal{L}_P = \mathcal{K}$.

A natural question is whether $\mathcal{K}_L$ is still a Laplace transform of a point process if we let L be a finite signed measure on $\mathcal{M}^{\cdot\cdot}(X)$. In the monograph [10] the authors asked whether for a finite signed measure K on $\mathcal{M}^{\cdot\cdot}(X)$ with $K(\mathcal{M}^{\cdot\cdot}(X)) = 0$ does there exists a point process P such that $P = \exp(K) = \sum_{j=0}^{\infty} \frac{K^{*j}}{j!}$? Here $*$ is the usual convolution operator. It can be seen that this question is equivalent to the existence of a finite signed measure L on $\mathcal{M}^{\cdot\cdot}(X)$ with $L(\{0\}) = 0$ such that $\mathcal{L}_P = \mathcal{K}_L$. They obtained that this is the case if and only if

$$\Psi_{L^+} = \Psi_{L^-} * P \tag{2.2}$$

here L^+ resp. L^- is the positive resp. negative part in the Hahn-Jordan decomposition of L and this "is quite a complicated question" as the authors of [10] on page 79 remarked. The negative part L^- of the Lévy measure L can be interpreted as to contribute to a *deletion of points* in Ψ_{L^+}, since according to the convolution equation (2.2) to obtain a realization of Ψ_{L^+} we have to take a realization of P and superpose it independently by a realization of Ψ_{L^-}. As Matthes et al. [10] formulated, P is the convolution quotient of the infinitely divisible point processes Ψ_{L^+} and Ψ_{L^-}.

The main problem studied here: Given any two L^+, $L^- \in \mathcal{W}$, does there exists a point process P such that (2.2) holds? In the sequel we will denote $L = L^+ - L^-$, the so called *signed Lévy functional* of a point process P if either (2.2) or $\mathcal{L}_P = \mathcal{K}_L$ on U holds. But note here: L is not even a signed measure on $\mathcal{M}^{\cdot\cdot}(X)$ since undefined expressions like $\infty - \infty$ can occur whereas L is well defined on the set of functions, which are $|L|$-integrable. Here $|L|$ denotes the measure $L^+ + L^-$.

3 Existence of Point Processes with a Signed Lévy Functional

We will restrict our investigation to measures $L^+, L^- \in \mathcal{W}$ which are concentrated on $\mathcal{M}^{\cdot\cdot}_f(X)$. These can be represented as follows

$$L^\epsilon(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n} \int_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \Theta_n^\epsilon(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n), \tag{3.1}$$

for all $\varphi \in F_+(\mathcal{M}^{\cdot\cdot}_f(X))$. Here Θ_n^ϵ is a non negative measure on X^n. Certainly they can always chosen to be symmetric. Only in section 5 we will start with a family Θ_n^ϵ, which has no more symmetry properties than invariance under cyclic permutations. We also introduce $\Theta_n = \Theta_n^+ - \Theta_n^-$. We will see that under the condition $|L|(\mathbf{1} - e^{-\zeta_f}) < \infty$ for $f \in U$, $|\Theta_n| = \Theta_n^+ + \Theta_n^-$ is a Radon measure on X^n. We shall call $\{\Theta_n\}_{n=1}^\infty$ the family of *cumulant measures*[1]. Let us agree on the following convention. If we say we are considering a signed Lévy functional of the form

$$L(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n} \int_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \vartheta(x_1, \ldots, x_n)\, \lambda(\mathrm{d}\, x_1) \ldots \lambda(\mathrm{d}\, x_n), \tag{3.2}$$

where $\vartheta : \sqcup_{n=1}^\infty X^n \mapsto \mathbb{R}$ and λ is some non negative measure on X then it is always understood that we have made the canonical choice

$$\Theta_n^\epsilon = \vartheta_\epsilon(x_1, \ldots, x_n)\, \lambda(\mathrm{d}\, x_1) \ldots \lambda(\mathrm{d}\, x_n),$$

[1] Remark that Θ_n is only a well defined finite signed measure if restricted to the bounded sets of X^n. Θ_n evaluated for unbounded sets might lead to undefined expressions like $\infty - \infty$. Such objects will be called *signed Radon measures* in the sequel.

where $\vartheta_\epsilon = \max\{\epsilon\,\vartheta, 0\}$ are the positive resp. negative part of ϑ. The following combinatorial result is a direct consequence of Ruelle's algebraic approach ([18] chapter 4, eqn. (4.14)). It can also be found in the book [22] of Stanley corollary 5.1.6, where it is called the exponential formula.

Lemma 3.1. *For a sequence h_k of real numbers, such that the exponentiated series on the below left hand side converges absolutely, the series on the below right hand side converges absolutely and we have*

$$\exp\left[\sum_{k=1}^{\infty}\frac{h_k}{k!}\right] = 1 + \sum_{k=1}^{\infty}\frac{1}{k!}\sum_{\mathcal{J}\in\pi([k])}\prod_{J\in\mathcal{J}} h_{|J|}.$$

Here $\pi([k])$ denotes the set of all partitions of the set $[k] = \{1,\dots,k\}$.

Now we can formulate our existence result.

Theorem 3.2. *Let $L^+, L^- \in \mathcal{W}$ be of the form (3.1). Then $|\Theta_n|$ has to be a Radon measure. Furthermore we assume that the following signed measures*

$$\varrho_k(\otimes_{j=1}^k f_j) = \sum_{\sigma\in S_k}\prod_{\omega\in\sigma}\Theta_{\ell(\omega)}(\otimes_{i\in\omega} f_i),\ \ f_1,\dots,f_k \in U. \tag{3.3}$$

are actually non negative Radon measures. Here the above product has to be taken over all cycles ω in the permutation σ and $\ell(\omega)$ denotes the cycle length. Then there exists a point process Ψ_L such that $\Psi_{L^+} = \Psi_{L^-} * \Psi_L$ or equivalently $\mathcal{L}_{\Psi_L} = \mathcal{K}_L$ on U. We call $\{\varrho_k\}_{k=1}^\infty$ the family of Schur measures of Ψ_L.

Proof. Let $\Lambda \in \mathcal{B}_0(X)$ and $\mathcal{M}^{\cdot\cdot}(\Lambda) = \{\mu \in \mathcal{M}^{\cdot\cdot}(X) | \operatorname{supp}(\mu) \subset \Lambda\}$. The method of the proof will be to investigate the restriction $L_\Lambda(\varphi) = L(\mathbf{1}_{\mathcal{M}^{\cdot\cdot}(\Lambda)}\varphi)$, for $\varphi \in F_+(\mathcal{M}^{\cdot\cdot}(X))$ such that $|L_\Lambda|(\varphi) < \infty$, of L to point measures in Λ. By a combinatorial argument (lemma 3.1) we will then observe that there exist finite point processes Q_Λ in Λ such that $\mathcal{L}_{Q_\Lambda} = \mathcal{K}_{L_\Lambda}$ on U. As $\Lambda \uparrow X$ we will see that $\mathcal{K}_{L_\Lambda} \to \mathcal{K}_L$. By using theorem 2.2 we will obtain the assertion.
Let us first observe that L_Λ is a finite signed measure.

$$|L_\Lambda|(\mathbf{1}) = 2\,|L|(\mathbf{1}_{\mathcal{M}^{\cdot\cdot}(\Lambda)}\frac{1}{2}) \leq 2\,|L|(\mathbf{1} - e^{-\zeta_\Lambda}) < \infty.$$

For the first inequality observe that $\zeta_\Lambda \geq 1$ on $\mathcal{M}^{\cdot\cdot}(\Lambda)$, $|L|$-a.e. and $\frac{1}{2} \leq 1 - e^{-x}$ for $x \geq 1$. In particular $|L_\Lambda|(\mathbf{1}) < \infty$ implies that Θ_n^ϵ are Radon measures. Now set $\Xi(\Lambda) = \exp[L_\Lambda(\mathbf{1})]$. So we have for $f \in U$

$$\mathcal{K}_{L_\Lambda}(f) = \frac{1}{\Xi(\Lambda)} \exp[\mathcal{L}_{L_\Lambda}(f)]$$

where

$$\mathcal{L}_{L_\Lambda}(f) = \sum_{n=1}^{\infty} \frac{\Theta_n^+((\mathbf{1}_\Lambda e^{-f})^{\otimes n}) - \Theta_n^-((\mathbf{1}_\Lambda e^{-f})^{\otimes n})}{n}.$$

The above sum converges absolutely due to $|L_\Lambda|(\mathbf{1}) < \infty$. So choose in lemma 3.1 $h_n = (n-1)!\, \Theta_n((\mathbf{1}_\Lambda e^{-f})^{\otimes n})$, which yields combined with the fact that $|S_n^{cy}| = (n-1)!$ (here S_n^{cy} is the set of permutations of the set $[n]$, which consists of one cycle)

$$\exp[\mathcal{L}_{L_\Lambda}(f)] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{\omega \in \sigma} \Theta_{\ell(\omega)}((\mathbf{1}_\Lambda e^{-f})^{\otimes \ell(\omega)}).$$

From which we finally conclude

$$\mathcal{K}_{L_\Lambda}(f) = \frac{1}{\Xi(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} e^{-f(x_1)} \dots e^{-f(x_n)}\, \varrho_n(\mathrm{d}\, x_1 \dots \mathrm{d}\, x_n).$$

So we have identified $\mathcal{K}_{L_\Lambda}$ as the Laplace functional of the finite point process

$$Q_\Lambda(\varphi) = \frac{1}{\Xi(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \varphi(\delta_{x_1} + \dots + \delta_{x_n})\, \varrho_n(\mathrm{d}\, x_1 \dots \mathrm{d}\, x_n),$$

for $\varphi \in F_+(\mathcal{M}^{\cdot\cdot}_f(X))$. Now let us check that the assumptions of Mecke's theorem 2.2 are fulfilled. Since $|(L - L_\Lambda)(\mathbf{1} - e^{-\zeta_f})| \leq |L|((\mathbf{1} - \mathbf{1}_{\mathcal{M}^{\cdot\cdot}(\Lambda)})(\mathbf{1} - e^{-\zeta_f})) \downarrow 0$ as $\Lambda \uparrow X$, by dominated convergence, we obtain $\mathcal{L}_{Q_\Lambda} \to \mathcal{K}_L$ as $\Lambda \uparrow X$ on U. Similarly one can establish continuity of $\mathcal{K}_L$ at zero. Let $u_n \in U$ with $u_n \downarrow 0$ then $|L(\mathbf{1} - e^{-\zeta_{u_n}})| \leq |L|(\mathbf{1} - e^{-\zeta_{u_n}}) \downarrow 0$ as $\Lambda \uparrow X$ again justified by dominated convergence. So theorem 2.2 gives us the existence of a law Ψ_L on $\mathcal{M}(X)$ such that $\mathcal{L}_{\Psi_L} = \mathcal{K}_L$. Now since Ψ_L is the weak limit of the Q_Λ we have $\Psi_L(\mathcal{M}^{\cdot\cdot}(X)) = 1$, because the set of point processes is closed with respect to weak convergence (see [4], page 32). $\square$

Remark that Ψ_L does only depend on the difference $\Theta_n = \Theta_n^+ - \Theta_n^-$. If we have found another family $\{\tilde{\Theta}_n^\epsilon\}_{n=1}^\infty$, such that the assumptions of theorem 3.2 are satisfied and $\Theta_n = \tilde{\Theta}_n$ on bounded sets then $\Psi_L = \Psi_{\tilde{L}}$ is implied by $Q_\Lambda = \tilde{Q}_\Lambda$ for $\Lambda \in \mathcal{B}_0(X)$.

Remark 3.3. *In the proof to theorem (3.2) we have constructed finite point processes* Q_Λ *such that*

$$\Psi_{L^+_\Lambda} = \Psi_{L^-_\Lambda} * Q_\Lambda.$$

Furthermore we have shown $\Psi_{L^\epsilon_\Lambda} \Rightarrow \Psi_{L^\epsilon}$ *weakly as* $\Lambda \uparrow X$, $\epsilon = +, -$. *Now with ([10] proposition 3.2.9.) we can conclude that there exists a point process* P, *the weak limit of the* Q_Λ *such that*

$$\Psi_{L^+} = \Psi_{L^-} * P.$$

So this gives an alternative to theorem 2.2 for establishing the thermodynamic limit.

In the sequel we will need the Campbell measures of higher order. They are defined as follows. Let K be a non negative measure on $\mathcal{M}(X)$. Then its n-th order Campbell measure is given by

$$C^n_K(h) = \int_{\mathcal{M}(X)} K(\mathrm{d}\,\mu) \int_{X^n} \mu(\mathrm{d}\,x_1) \dots \mu(\mathrm{d}\,x_n)\, h(x_1, \dots, x_n; \mu)$$

for $h \in F_+(X^n \times \mathcal{M}(X))$. Let $f \in F_+(X^n)$, then the marginal $C^n_K(f \otimes \mathbf{1})$ is called the n-th order moment measure $\nu^n_K(f)$ of K. If $\nu^n_K \in \mathcal{M}(X^n)$ we say that the n-th moment of K exists. In the sequel we will also encounter expressions like $C^n_K(h)$, $h \in F_+(X^n \times \mathcal{M}(X))$ where $K = K^+ - K^-$ and K^+, K^- are non negative measures. This has to be understood as $C^n_{K^+}(h) - C^n_{K^-}(h)$ and will only be present in the well defined case $C^n_{K^+}(h), C^n_{K^-}(h) < \infty$.

Remark 3.4. *Since* $1 - e^{-x} \leq x$ *we have* $|L|(\mathbf{1} - e^{-\zeta_f}) \leq \nu^1_{|L|}(f)$, $f \in U$. *So* $|L|(\mathbf{1} - e^{-\zeta_f}) < \infty$ *is implied by the existence of the first moment of* $|L|$.

Proposition 3.5. *Let* L *be a signed Lévy functional such that* $\nu^1_{|L|}$ *exists. Then* $\nu^1_{\Psi_L}$ *exists and* Ψ_L *is a solution to*

$$(\mathbf{\Sigma}_L) \qquad C_P(h) + C_{L^-} \star P(h) = C_{L^+} \star P(h)$$

for all $h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X))$.

Proof. By using the representation of the Laplace transform of Ψ_L and applying two times $1 - e^{-x} \leq x$ for $x \in \mathbb{R}$, we obtain for $f \in U$ and $s > 0$

$$\Psi_L\left(\frac{\mathbf{1} - e^{-s\zeta_f}}{s}\right) = \frac{1 - e^{-L(\mathbf{1} - e^{-s\zeta_f})}}{s} \leq |L|(\zeta_f) = \nu^1_{|L|}(f).$$

So if we use the lemma of Fatou then we get

$$\nu^1_{\Psi_L}(f) = \Psi_L(\zeta_f) = \Psi_L\left(\liminf_{s\downarrow 0} \frac{\mathbf{1} - e^{-s\zeta_f}}{s}\right) \leq \nu^1_{|L|}(f).$$

Now it is well known (see [11]) that if the first moment of a point process P exists (this allows to interchange integration and differentiation below) then for $f, g \in U$ we have

$$C_P(f \otimes e^{-\zeta_g}) = -\frac{\mathrm{d}}{\mathrm{d}\, s}\mathcal{L}_P(sf+g)|_{s=0}.$$

So we have to compute

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}\, s}\mathcal{L}_{\Psi_L}(sf+g) &= \mathcal{L}_{\Psi_L}(sf+g)\,\frac{\mathrm{d}}{\mathrm{d}\, s}L(\mathbf{1} - e^{-\zeta_{sf+g}})\\ &= \mathcal{L}_{\Psi_L}(sf+g)L(\zeta_f e^{-\zeta_g}) = C_L(f \otimes e^{-\zeta_g})\mathcal{L}_{\Psi_L}(sf+g).\end{aligned}$$

Again the second equality holds since we are allowed to interchange differentiation and integration with respect to L since $\nu^1_{|L|}$ exists. We arrive at $C_{\Psi_L}(f \otimes e^{-\zeta_g}) = C_L \star \Psi_L(f \otimes e^{-\zeta_g})$, which certainly can be brought into the form $(\mathbf{\Sigma}_L)$. Now the same argument as in [11] chapter 4, proof of theorem 10 yields that the equality $(\mathbf{\Sigma}_L)$ can be extended from $h = f \otimes e^{-\zeta_g}$ to arbitrary $h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X))$. □

The converse to proposition 3.5 also holds: If P is a point process such that ν^1_P exists and $(\mathbf{\Sigma}_L)$ holds then $P = \Psi_L$. Remark that $(\mathbf{\Sigma}_L)$ certainly implies $C_{\Psi_L}(h) \leq C_{|L|} \star \Psi_L(h)$ for all $h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X))$. So we have in particular $\nu^1_{\Psi_L} \leq \nu^1_{|L|}$ on $F_+(X)$.

Lemma 3.6. *Assume that $\nu^1_{|L|}$ exists. Let $f, g \in F_+(X)$ such that $\nu^n_{|L|}(f^{\otimes n}) < \infty$, $n \geq 1$ then $\nu^n_{\Psi_L}(f^{\otimes n}) < \infty$, $n \geq 1$ and we have for $n \geq 1$*

$$(\mathbf{\Sigma}^n_L) \qquad C^n_{\Psi_L}(f^{\otimes n} \otimes e^{-\zeta_g}) = \mathcal{L}_{\Psi_L}(g) \sum_{\mathcal{J}\in\pi([n])} \prod_{J\in\mathcal{J}} C^{|J|}_L(f^{\otimes|J|} \otimes e^{-\zeta_g}).$$

Proof. Let us give the proof by induction. The case $n = 1$ has been dealt with in proposition 3.5. Now let us assume that $\nu^k_{\Psi_L}(f^{\otimes k}) < \infty$ and $\left(\mathbf{\Sigma}^k_L\right)$ holds for $1 \leq k \leq n-1$.

$$\begin{aligned}\nu^n_{\Psi_L}(f^{\otimes n}) &= C_{\Psi_L}(f \otimes \zeta_f^{n-1}) \leq C_{|L|} \star \Psi_L(f \otimes \zeta_f^{n-1})\\ &= \sum_{B\subset\{2,\ldots,n\}} \nu^{|B|+1}_{|L|}(f^{\otimes(|B|+1)})\, \nu^{|B^c|}_{\Psi_L}(f^{\otimes|B^c|}) < \infty.\end{aligned}$$

So that

$$C^n_{\Psi_L}(f^{\otimes n} \otimes e^{-\zeta_g}) = -\frac{\mathrm{d}}{\mathrm{d}\, s} C^{n-1}_{\Psi_L}(f^{\otimes(n-1)} \otimes e^{-\zeta_{sf+g}})|_{s=0}$$

Now $(\mathbf{\Sigma}^n_L)$ follows by just using the product rule of differentiation and the fact that $C^k_L(f^{\otimes k} \otimes e^{-\zeta_g}) = -\frac{\mathrm{d}}{\mathrm{d}\, s} C^{k-1}_L(f^{\otimes(k-1)} \otimes e^{-\zeta_{sf+g}})|_{s=0}$ for any $k \geq 1$. □

In the following we will see that besides $Q_\Lambda \Rightarrow \Psi_L$ weakly as $\Lambda \uparrow X$, we also have convergence of the Campbell measures for a sufficiently large class of test functions.

Lemma 3.7. *Let $f, g \in U$ and assume that $\nu^1_{|L|}$ exists. Then we have for $h = f \otimes e^{-\zeta_g}$*

$$C_{Q_\Lambda}(h) \to C_{\Psi_L}(h) \text{ as } \Lambda \uparrow X.$$

Proof. We have $C_{Q_\Lambda}(h) = C_{L_\Lambda}(h)\mathcal{L}_{Q_\Lambda}(g)$, since Q_Λ is a solution to $(\mathbf{\Sigma}_{L_\Lambda})$. Similarly we have $C_{\Psi_L}(h) = C_L(h)\mathcal{L}_{\Psi_L}(g)$. Since $\mathcal{L}_{Q_\Lambda}(g) \to \mathcal{L}_{\Psi_L}(g)$ is already established, $C_{L_\Lambda}(h) \to C_L(h)$ remains to be seen. But

$$|(C_L - C_{L_\Lambda})(h)| = |(L - L_\Lambda)(\zeta_f e^{-\zeta_g})| \leq |L|((\mathbf{1} - \mathbf{1}_{\mathcal{M}^{\cdot\cdot}(\Lambda)})\zeta_f e^{-\zeta_g}) \downarrow 0 \text{ as } \Lambda \uparrow X,$$

by dominated convergence. □

In the end let us give a sufficient condition for the simplicity of Ψ_L.

Proposition 3.8. *Assume that for some $\vartheta : \sqcup_{n=1}^\infty X^n \mapsto \mathbb{R}$ and $\lambda \in \mathcal{M}^\circ(X)$ a diffuse Radon measure*

$$L(\varphi) = \sum_{n=1}^\infty \frac{1}{n} \int_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \vartheta(x_1, \ldots, x_n)\, \lambda(\mathrm{d}\, x_1) \ldots \lambda(\mathrm{d}\, x_n),$$

is a signed Lévy functional such that $\nu^1_{L^+}$ exists (the choice of L^+, L^- has to be taken as explained above). Then Ψ_L is a simple point process.

Proof. The result is an immediate consequence from [10] proposition 2.2.9. which says that an infinitely divisible point process Ψ_H is simple if and only if $H(\mathcal{M}^{\cdot\cdot}(X) \setminus \mathcal{M}^{\cdot}(X)) = 0$ and $H(\{\zeta_{\{x\}} > 0\}) = 0$ for all $x \in X$. Now it is well known that for a diffuse λ the product λ^n is concentrated on $\grave{X}^n = \{(x_1, \ldots, x_n) \in X^n | \, x_i \neq x_j \text{ for } i \neq j\}$ (see [5] theorem 3), which yields $L^+(\mathcal{M}^{\cdot\cdot}(X) \setminus \mathcal{M}^{\cdot}(X)) = 0$. Moreover for $f \in U$

$$\nu^1_{L^+}(f) = \int_X f(x) \sum_{n=1}^\infty \int_{X^{n-1}} \vartheta_+(x, x_2, \ldots, x_n)\lambda(\mathrm{d}\, x_2) \ldots \lambda(\mathrm{d}\, x_n)\, \lambda(\mathrm{d}\, x).$$

So $\nu^1_{L^+}$ is a diffuse Radon measure and we obtain $L^+(\{\zeta_{\{x\}} > 0\}) \le L^+(\zeta_{\{x\}}) = 0$. Since Ψ_{L^+} is simple the equation $\Psi_{L^+} = \Psi_{L^-} * \Psi_L$ forces Ψ_{L^-} and Ψ_L to be simple. □

4 The Method of Cluster Expansion

For the historical development of this method we refer to [9] and [16]. Instead we give the concept of the method of cluster expansion in the generalized form presented here.

Cluster Representations

In theorem 3.2 we started with a family of signed Radon measures $\{\Theta_n\}_{n=1}^\infty$, the cumulant measures and obtained the family of Schur measures $\{\varrho_k\}_{k=1}^\infty$ by means of (3.3). We then say that the family of Schur measures admits a *cluster representation* in terms of the cumulant measures[2]. Certainly (3.3) gives us a duality between the Schur and cumulant measures. If we prescribe a family of non negative symmetric Radon measures $\{\varrho_k\}_{k=1}^\infty$ then we obtain the cumulant measures by a Möbius inversion as in [17]

$$\Theta_n(\otimes_{j=1}^n f_j) = \frac{1}{(n-1)!} \sum_{\mathcal{J} \in \pi([n])} (-1)^{|\mathcal{J}|-1} (|\mathcal{J}|-1)! \prod_{J \in \mathcal{J}} \varrho_{|J|}(\otimes_{j\in J} f_j), \quad (4.1)$$

where $f_1, ..., f_n \in U$. For the existence of the point process Ψ_L according to theorem 3.2 it remains to check that the signed Lévy functional L, built on the cumulant measures defined in (4.1) and for some choice of Θ_n^+, Θ_n^-, satisfies $|L|(1 - e^{-\zeta_f}) < \infty$, $f \in U$. We say that (4.1) is the *dual cluster representation* of the cumulant measures in terms of the Schur measures. In the upcoming examples it will always be the case that the Schur measures are of the form $\varrho_k = \psi(x_1, \ldots, x_n)\lambda(\mathrm{d}\, x_1)\ldots\lambda(\mathrm{d}\, x_k)$ for some symmetric $\psi : \sqcup_{n=0}^\infty X^n \mapsto [0,\infty)$. So with the help of the dual cluster representation we obtain that the corresponding signed Lévy functional has the form (3.2) with

$$\vartheta = \frac{1}{(n-1)!} \sum_{\mathcal{J} \in \pi([n])} (-1)^{|\mathcal{J}|-1} (|\mathcal{J}|-1)! \prod_{J \in \mathcal{J}} \psi((x_j)_{j\in J}). \quad (4.2)$$

[2] We propose for the $\{\varrho_k\}_{k=1}^\infty$ the name Schur measures because the cluster representation is in direct analogy to the concept of Schur functions.

To summarize: The *method of cluster expansion* consists in defining first the signed Lévy functional L by means of the given data $\{\Theta_n\}_{n=1}^{\infty}$ respectively $\{\varrho_k\}_{k=1}^{\infty}$; and next, to construct with their help the local processes $\{Q_\Lambda\}_{\Lambda\in\mathcal{B}_0(X)}$. (This construction reflects the cluster structure.) The limiting process is obtained under the condition that $|L|(1-e^{-\zeta_f})<\infty$, $f\in U$. If furthermore $\nu^1_{|L|}$ exists it is the unique solution to the integral equation $(\boldsymbol{\Sigma}_L)$, see proposition 3.5.

Comparison to the Approach of Malyshev and Minlos

Here we give some relations to the work [9] of Malyshev and Minlos. They are in the following setting: Let $\lambda\in\mathcal{M}^\circ(X)$ a diffuse Radon measure and $\vartheta:\mathcal{M}^{\cdot\cdot}_f(X)\mapsto\mathbb{R}$ a measurable function. Moreover let the cumulant measures be given by $\Theta_n=\frac{1}{(n-1)!}\vartheta(x_1,...,x_n)\,\lambda(\mathrm{d}\,x_1)...\lambda(\mathrm{d}\,x_n)$. If we introduce the following measure Π on $\mathcal{M}^{\cdot\cdot}_f(X)$

$$\Pi(\varphi)=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{X^n}\varphi(\delta_{x_1}+...+\delta_{x_n})\,\lambda(\mathrm{d}\,x_1)...\lambda(\mathrm{d}\,x_n),$$

for all $\varphi\in F_+(\mathcal{M}^{\cdot\cdot}_f(X))$. The signed Lévy functional can be written as $L(\varphi)=\Pi(\mathbf{1}_{\mathcal{M}^{\cdot\cdot}_f(X)\setminus\{0\}}\vartheta\,\varphi)$. They impose the following condition on ϑ. For any $\Lambda\in\mathcal{B}_0$ there holds

$$\int_{\mathcal{M}^{\cdot\cdot}(\Lambda)\setminus\{0\}}\Pi(\mathrm{d}\,\mu)\int_{\mathcal{M}^{\cdot\cdot}_f(X)}\Pi(\mathrm{d}\,\eta)\,|\vartheta(\mu+\eta)|<\infty. \tag{4.3}$$

Then they can show that there exists a weak limit of the Q_Λ, the local Gibbs modifications, as $\Lambda\uparrow X$ (see [9], chapter 3, theorem 2). The condition (4.3) can also be formulated in terms of the so called factorial moment measures of $|L|$. Let H be a non negative measure on $\mathcal{M}^{\cdot\cdot}(X)$ then for $f\in F_+(X^n)$

$$\check{\nu}^n_H(f)=\int H(\mathrm{d}\,\mu)\int\mu(\mathrm{d}\,x_1)(\mu-\delta_{x_1})(\mathrm{d}\,x_2)\ldots(\mu-\sum_{j=1}^{n-1}\delta_{x_j})(\mathrm{d}\,x_n)\,f(x_1,\ldots,x_n)$$

is called the n-th factorial moment measure of H. As computed in [14] we have for $\Lambda\in\mathcal{B}_0(X)$

$$\check{\nu}^n_{|L|}(\Lambda^n)=\int_{\Lambda^n}\lambda(\mathrm{d}\,y_1)\ldots\lambda(\mathrm{d}\,y_n)\int\Pi(\mathrm{d}\,\mu)\,|\vartheta(\mu+\delta_{y_1}+\ldots+\delta_{y_n})|$$

So in terms of $|L|$ (4.3) can be formulated as

$$\sum_{n=1}^{\infty} \frac{\check{\nu}^n_{|L|}(\Lambda^n)}{n!} < \infty \text{ for } \Lambda \in \mathcal{B}_0(X).$$

We required that $\nu^1_{|L|}(\Lambda) = \check{\nu}^1_{|L|}(\Lambda) < \infty$, which is a bit weaker. Let us remark that it is immediate from the proof of theorem 3.2 that the partition function $\Xi(\Lambda)$ can be either expressed as $\log(\Xi(\Lambda)) = \int \Pi(\mathrm{d}\,\mu)\, \mathbf{1}_{\mathcal{M}^{\cdot\cdot}(\Lambda)\setminus\{0\}}(\mu)\vartheta(\mu)$ or

$$\Xi(\Lambda) = \int_{\mathcal{M}^{\cdot\cdot}(\Lambda)} \Pi(\mathrm{d}\,\mu) \sum_{\mathcal{J}\in\pi([|\mu|])} \prod_{j\in\mathcal{J}} \vartheta((x_j)_{j\in J}),$$

here we have given each $\mu = \delta_{x_1} + ... + \delta_{x_{|\mu|}}$ some arbitrary numbering. Malyshev and Minlos call the above identity a cluster representation of the partition function. We remark that, due to proposition 3.8, all point processes constructed in this section are simple.

We now give two classes of point processes which can be constructed via theorem 3.2. These are the quantum gases Boson and Fermion (or permanental and determinantal) point processes as well as the point processes from classical statistical mechanics the so called Gibbs point processes.

5 Permanental and Determinantal Processes

Let $\lambda \in \mathcal{M}(X)$ and $k : X \times X \mapsto \mathbb{R}$ be a positive definite kernel, that is for every $x_1, ..., x_n \in X$ and $z_1, ..., z_n \in \mathbb{R}$ we have

$$\sum_{i,j=1}^{n} z_i k(x_i, x_j) z_j \geq 0.$$

Consider the following two families of cumulant measures, $\epsilon = +1, -1$

$$\Theta_n(\epsilon) = \epsilon^{n-1}\, k(x_1, x_2)k(x_2, x_3) \ldots k(x_{n-1}, x_n)k(x_n, x_1)\, \lambda(\mathrm{d}\, x_1) \ldots \lambda(\mathrm{d}\, x_n). \tag{5.1}$$

Let us denote by $L(\epsilon)$ the signed Lévy functional corresponding to the family $\{\Theta_n(\epsilon)\}_{n=1}^{\infty}$ of cumulant measures by formula (5.1). Furthermore let us denote by $k^{(n)}(x, y) = \int k(x, z)k^{(n-1)}(z, y)\, \lambda(\mathrm{d}\, z)$, $k^{(1)} = k$, $n \in \mathbb{N}$ the iterated kernels of k, in case the integral is well defined.

Theorem 5.1. *If k is a positive definite kernel such that $\|k\|_\infty := \sup_{x,y\in X} |k(x,y)| < \infty$ and*

$$\alpha := \sup_{x\in X} \int |k(x,y)|\,\lambda(\mathrm{d}\,y) < 1, \tag{5.2}$$

then there exist point processes $\Psi^\epsilon_{\lambda,k}$ such that $L(\epsilon)$, $\epsilon = +1, -1$ is their signed Lévy functional. Moreover $\Psi^+_{\lambda,k}$ is a permanental point process to the kernel $h_+ = \sum_{m\geq 1} k^{(m)}$ and $\Psi^-_{\lambda,k}$ is a determinantal point process to the kernel $h_- = \sum_{m\geq 1}(-1)^{m-1}k^{(m)}$.

Proof. A straightforward computation shows that

$$\nu^1_{|L(\epsilon)|}(f) = \sum_{n\geq 1} \int f(x)|k|^{(n)}(x,x)\,\lambda(\mathrm{d}\,x) \text{ for } f \in U.$$

Furthermore we have the following estimate $\||k|^{(n)}\|_\infty \leq \|k\|_\infty \alpha^{n-1}$, which yields $\nu^1_{|L(\epsilon)|}(f) < \infty$. The corresponding Schur measures have densities with respect to the products of λ and are given by

$$\tfrac{\mathrm{d}\,\varrho_n^+}{\mathrm{d}\,\lambda^n} = \mathrm{per}(K) \text{ and } \tfrac{\mathrm{d}\,\varrho_n^-}{\mathrm{d}\,\lambda^n} = \det(K),$$

where K is the matrix $\{k(x_i,x_j)\}_{1\leq i,j\leq n}$ and $\mathrm{per}(K) = \sum_{\sigma\in S_n}\prod_{j=1}^n K(j,\sigma(j))$ is the so called permanent of K. Since K is a positive definite matrix it is well known (see [2]) that $\mathrm{per}(K) \geq \det(K) \geq 0$, which implies non negativity of the Schur measures. So theorem 3.2 gives us the existence of point processes $\Psi^\epsilon_{\lambda,k}$. In [14] it is shown that if we start with a signed Lévy functional of the present form the correlation functions ρ_ϵ of $\Psi^\epsilon_{\lambda,k}$ are given by

$$\rho_+ = \mathrm{per}(H_+) \text{ and } \rho_- = \det(H_-),$$

where $H_\epsilon = \{h_\epsilon(x_i,x_j)\}_{1\leq i,j\leq n}$ and $h_\epsilon = \sum_{m\geq 1}\epsilon^{m-1}k^{(m)}$. So we have identified $\Psi^+_{\lambda,k}$ as a permanental and $\Psi^-_{\lambda,k}$ as a determinantal point process. □

For another approach to the construction of permanental and determinantal point processes we refer to [20] and [21], who use the Kolmogorov extension theorem for the existence of the thermodynamic limit. If $\lambda \in \mathcal{M}^\circ(X)$ then proposition 3.8 shows that $\Psi^\epsilon_{\lambda,k}$ is simple.
If we consider kernels $k(x,y) = \psi(x-y)$ for some positive definite function ψ then condition (5.2) takes the form $\|\psi\|_1 := \lambda(|\psi|) < 1$. Characteristic

functions ψ are positive definite and bounded. So we still need to verify $\|\psi\|_1 < 1$. In the case $X = \mathbb{R}$ Móricz [12] has given sufficient conditions for the Lebesgue integrability of ψ. Let us give two typical examples: Let $X = \mathbb{R}^d$ and

$$g_z(x) = \frac{z}{(2\pi\beta)^{d/2}} \exp\left(-\frac{\|x\|^2}{2\beta}\right), \quad x \in \mathbb{R}^d,$$

be a scaled gaussian density where $z \in (0,1)$ and $\beta > 0$ are some parameters. It is well known that g_z is positive definite and if we let λ the Lebesgue measure on X then $\|g_z\|_1 = z < 1$. The point process Ψ^+_{λ,g_z} is called the ideal Bose gas of Fichtner and was studied in [3] and Ψ^-_{λ,g_z} is the corresponding ideal Fermi gas as we call it here.
Another example is the following: Let $X = \mathbb{R}$ and λ the Lebesgue measure

$$\psi(x) = \gamma \exp(-\frac{|x|}{\alpha}), \quad x \in \mathbb{R},$$

where $\alpha, \gamma > 0$ are chosen such that $\|\psi\|_1 = 2\alpha\gamma < 1$. It is well known that ψ is positive definite.

6 Reminder: Papangelou Processes

Before proceeding to the Gibbs point processes we introduce a general form of Gibbsianess the concept of Papangelou point processes. We recall some facts on these processes from Zessin [23]. Let $\pi(\mu, \mathrm{d}\, x)$ be a kernel from $\mathcal{M}^{\cdot\cdot}(X)$ to $\mathcal{M}(X)$. We say that a point process P is a Papangelou process with Papangelou kernel π if

$$(\boldsymbol{\Sigma}'_\pi) \qquad C_P(h) = \int_{\mathcal{M}^{\cdot\cdot}(X)} \int_X h(x, \mu + \delta_x)\, \pi(\mu, \mathrm{d}\, x) P(\mathrm{d}\, \mu)$$

for all $h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X))$. For a detailed discussion of the $(\boldsymbol{\Sigma}'_\pi)$ condition the interested reader is refered to [4]. Define for $\eta \in \mathcal{M}^{\cdot\cdot}(X)$, $m \geq 1$

$$\pi^{(m)}(\eta; \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_m) = \pi(\eta, \mathrm{d}\, x_1)\pi(\eta+\delta_{x_1}, \mathrm{d}\, x_2) \ldots \pi(\eta+\delta_{x_1}+\ldots+\delta_{x_{m-1}}, \mathrm{d}\, x_m)$$

the iterated kernel $\pi^{(m)}$ from $\mathcal{M}^{\cdot\cdot}(X)$ to $\mathcal{M}(X^m)$. If for any $\eta \in \mathcal{M}^{\cdot\cdot}(X)$ $\pi^{(2)}(\eta; \mathrm{d}\, x\, \mathrm{d}\, y)$ is a symmetric measure then we say that π satisfies the cocycle condition. Let now π be a kernel from $\mathcal{M}_f^{\cdot\cdot}(X)$ to $\mathcal{M}_f(X)$ such that for some $\eta \in \mathcal{M}_f^{\cdot\cdot}(X)$

$$0 < \Xi(\eta) = \sum_{m=0}^{\infty} \frac{1}{m!} \pi^{(m)}(\eta; X^m) < \infty$$

then we say that π is η-integrable. Under the condition of η-integrability of π the following finite point process

$$P_\pi^\eta(\varphi) = \frac{1}{\Xi(\eta)} \sum_{m=0}^{\infty} \frac{1}{m!} \int\limits_{X^m} \varphi(\delta_{x_1} + \ldots + \delta_{x_m})\, \pi^{(m)}(\eta; \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_m)$$

for $\varphi \in F_+(\mathcal{M}_f^{\cdot\cdot}(X))$ is well defined. The following result from [23] will later serve as a main lemma for us.

Lemma 6.1. *Assume that π is η-integrable for some $\eta \in \mathcal{M}_f^{\cdot\cdot}(X)$ and satisfies the cocycle condition. Then P_π^η is a Papangelou process with boundary condition η. That is P_π^η is a solution to*

$$C_P(h) = \int_{\mathcal{M}_f^{\cdot\cdot}(X)} \int_X h(x, \mu + \delta_x)\, \pi(\eta + \mu, \mathrm{d}\, x) P(\mathrm{d}\, \mu)$$

for all $h \in F_+(X \times \mathcal{M}_f^{\cdot\cdot}(X))$.

So in particular we have that P_π^0 is a Papangelou process with Papangelou kernel π. Here 0 denotes the zero measure on X.

7 Gibbs Processes

Let $\phi : X \times X \mapsto \mathbb{R} \cup \{\infty\}$ be a symmetric measurable function, a so called pair potential and $\lambda \in \mathcal{M}(X)$. The energy at $x \in X$ given $\mu \in \mathcal{M}_f^{\cdot\cdot}(X)$ is defined as

$$E(\mu, x) = \int \phi(x, y)\, \mu(\mathrm{d}\, y).$$

Later it will be important to consider $E(\mu, x)$ also for infinite μ. The energy $E(\mu)$ of $\mu \in \mathcal{M}_f^{\cdot\cdot}(X)$ is recursevly given by $E(0) = 0$ and

$$E(\mu + \delta_x) = E(\mu) + E(\mu, x). \tag{7.1}$$

Remark that we have for any $x_1, \ldots, x_n \in X$,

$$E(\delta_{x_1} + \ldots + \delta_{x_n}) = \sum_{1 \le i < j \le n} \phi(x_i, x_j).$$

Furthermore let us denote by

$$\xi(\mu, \mathrm{d}\, x) = e^{-\beta E(\mu, x)}\, z\lambda(\mathrm{d}\, x)$$

for $\mu \in \mathcal{M}^{\cdot\cdot}_f(X)$ the Boltzmann kernel, where $\beta, z > 0$ are some parameters. β is called the inverse temperatur and z the activity. Remark that due to (7.1) the iterated kernels of ξ evaluated for the zero boundary configuration are given by

$$\xi^{(k)}(0; \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_k) = e^{-\beta E(\delta_{x_1} + \ldots + \delta_{x_k})}\, z\lambda(\mathrm{d}\, x_1) \ldots z\lambda(\mathrm{d}\, x_k). \tag{7.2}$$

Definition 7.1. *A pair potential ϕ is called stable, if there exists $B > 0$ such that $E(\mu) \ge -B\, \mu(X)$ for any $\mu \in \mathcal{M}^{\cdot\cdot}_f(X)$.*

Remark that stable pair potentials are bounded from below $E(\delta_x + \delta_y) = \phi(x, y) \ge -2B$.

Definition 7.2. *Furthermore we assume ϕ to be regular. That means for some $\beta > 0$ we have*

$$C(\beta) = \sup_{x \in X} \int \lambda(\mathrm{d}\, y)\, |1 - e^{-\beta\phi(x,y)}| < \infty.$$

In [18] remark to definition 4.1.2. Ruelle gave the important hint that regularity of a pair potential implies the existence of a set with finite Lebesgue measure such that the potential is absolutely integrable on its complement. Ruelle is in the setting of translation invariant potentials. So the remark below is a straightforward generalization to arbitrary pair potentials. In the sequel let us denote $\phi_x : y \mapsto \phi(x, y)$.

Remark 7.3. *Let $\epsilon > 0$. Since $|1 - e^{-t}| \ge c(\epsilon)$ for $|t| > \epsilon$ for some $c(\epsilon) > 0$ we have*

$$C(\beta) \ge \sup_{x \in X} \int_{\{\beta|\phi_x| > \epsilon\}} \lambda(\mathrm{d}\, y)\, |1 - e^{-\beta\phi_x(y)}| \ge c(\epsilon) \sup_{x \in X} \lambda(\{\beta|\phi_x| > \epsilon\}).$$

Furthermore since $|1 - e^{-t}| \ge \tilde{c}(\epsilon)|t|$ for $|t| \le \epsilon$ for some $\tilde{c}(\epsilon) > 0$ we have

$$C(\beta) \ge \sup_{x \in X} \int_{\{\beta|\phi_x| \le \epsilon\}} \lambda(\mathrm{d}\, y)\, |1 - e^{-\beta\phi_x(y)}| \ge \tilde{c}(\epsilon)\beta \sup_{x \in X} \int_{\{\beta|\phi_x| \le \epsilon\}} \lambda(\mathrm{d}\, y)\, |\phi_x(y)|.$$

Now we prescribe a family of Schur measures

$$\varrho_k = e^{-\beta E(\delta_{x_1}+\ldots+\delta_{x_k})}\, z\lambda(\mathrm{d}\,x_1)\ldots z\lambda(\mathrm{d}\,x_k),$$

which coincide with the $\xi^{(k)}(0;\cdot)$ (see eqn. (7.2)). The cumulant measures corresponding to this family of Schur measures are given by (4.1). They certainly have a density with respect to the products of λ given by (4.2), which is known as the Ursell function u (this representation of u was used for instance by Basuev in [1]). So we have

$$\Theta_n = \frac{z^n}{(n-1)!} u(x_1,\ldots,x_n)\,\lambda(\mathrm{d}\,x_1)\ldots\lambda(\mathrm{d}\,x_n).$$

Let us denote the corresponding signed Lévy functional by $\Re$. Recall from the introduction that $\Re$ is just an abbreviation for $\Re^+ - \Re^-$ and expressions like $\Re(\varphi)$ do only occur in case of $|\Re|(|\varphi|) < \infty$.

There also exists a more graph theoretical representation for the Ursell function. If we denote by $\mathcal{C}_n$ the set of all connected graphs with n vertices and $\zeta(x,y) = e^{-\beta\phi(x,y)} - 1$ then it is well known (see [18] chapter 4) that

$$u(x_1,...,x_n) = \sum_{G\in\mathcal{C}_n} \prod_{\{i,j\}\in G} \zeta(x_i,x_j),$$

where the product has to be taken over all edges in G. The following tree estimate of the Ursell function due to [16] will be fundamental in the sequel.

Theorem 7.4 (Poghosyan, Ueltshi)**.** *Let ϕ be a stable pair potential then*

$$|u(x_1,...,x_n)| \leq \left(e^{2\beta B}\right)^n \sum_{G\in\mathcal{T}_n} \prod_{\{i,j\}\in G} |\zeta(x_i,x_j)|.$$

Here $\mathcal{T}_n$ denotes the set of trees with n vertices.

As already remarked in [9] the number $|\mathcal{T}_n|$ is dominated by $c^n n!$ for some constant c, which can be taken to be $c = e$, we obtain that, uniformly in $x \in X$,

$$\int \lambda(\mathrm{d}\,x_2)...\lambda(\mathrm{d}\,x_n)\,|u(x,x_2,...,x_n)| \leq \left(e^{2\beta B}\right)^n\, e^n n!\, C(\beta)^{n-1}. \tag{7.3}$$

In order to establish existence of a point process with the above prescribed Schur measures we still have to show, according to theorem 3.2, that $\nu^1_{|\Re|}$ exists. But for $f \in U$

$$\nu^1_{|\Re|}(f) = \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!} \int\limits_{X^n} f(x)|u(x, x_2, \ldots, x_n)|\, \lambda(\mathrm{d}\, x)\lambda(\mathrm{d}\, x_2) \ldots \lambda(\mathrm{d}\, x_n)$$

is finite due to (7.3) if we let $z \in (0, \frac{e^{-2\beta B-1}}{C(\beta)})$.The next task is to identify the above constructed point process, which we call $G_{z,\phi}$, as a Gibbs point process where the interaction is given by the pair potential ϕ. Remark that $G_{z,\phi}$ is simple due to proposition 3.8 if $\lambda \in \mathcal{M}^{\circ}(X)$.

Theorem 7.5. *Let ϕ be a stable and regular pair potential. Then for $z \in (0, \frac{e^{-4\beta B-1}}{C(\beta)})$ $G_{z,\phi}$ exists, the Boltzmann kernel ξ is $G_{z,\phi}$-a.s. well defined and $G_{z,\phi}$ is a solution to $\left(\boldsymbol{\Sigma}'_{\xi}\right)$.*

Proof. Since existence of $G_{z,\phi}$ was already established it remains to be seen that it satisfies the integration by parts formula $\left(\boldsymbol{\Sigma}'_{\xi}\right)$. Since the Schur measures ϱ_k coincide with $\xi^{(k)}(0;\cdot)$ the finite point process Q_Λ coincides with the Papangelou point process $P^0_{\xi_\Lambda}$ for $\Lambda \in \mathcal{B}_0(X)$, where ξ_Λ denotes the Boltzmann kernel restricted to Λ. It is straightforward to see that ξ satisfies the cocycle condition. So with Zessin's lemma 6.1 we conclude that Q_Λ is a solution to $\left(\boldsymbol{\Sigma}'_{\xi_\Lambda}\right)$. That is

$$C_{Q_\Lambda}(h) = \int\limits_{\mathcal{M}^{\cdot\cdot}(\Lambda)} \int\limits_{\Lambda} h(x, \mu + \delta_x)\, \xi(\mu, \mathrm{d}\, x) Q_\Lambda(\mathrm{d}\, \mu), \tag{7.4}$$

for $h \in F_+(\Lambda \times \mathcal{M}^{\cdot\cdot}(\Lambda))$. In the sequel let h be of the form $f \otimes e^{-\zeta_g}$ for $f, g \in U$ and Λ such that $\mathrm{supp}(f), \mathrm{supp}(g) \subset \Lambda$. In lemma 3.7 convergence of $C_{Q_\Lambda}(h) \to C_{G_{z,\phi}}(h)$ as $\Lambda \uparrow X$ was proved. Remark that $E(\mu, x) = \zeta_{\phi_x}(\mu)$, so the right hand side of (7.4) can be written as

$$\int_X f(x) e^{-g(x)} \mathcal{L}_{Q_\Lambda}(g + \beta\phi_x)\, z\lambda(\mathrm{d}\, x). \tag{7.5}$$

Certainly in order to prove the theorem we would like to have

$$\mathcal{L}_{Q_\Lambda}(g + \beta\phi_x) \to \mathcal{L}_{G_{z,\phi}}(g + \beta\phi_x) \text{ as } \Lambda \uparrow X.$$

We already now that $\mathcal{L}_{Q_\Lambda}(f) \to \mathcal{L}_{G_{z,\phi}}(f)$ for $f \in U$. But since $g + \beta\phi_x$ can be unbounded, negative and does not need to have bounded support it is not clear whether convergence of the Laplace transforms on U imply convergence for $g + \beta\phi_x$. This difficulty will be overcome by the use of the signed modified Laplace functionals $\mathcal{K}_{\Re_\Lambda}$ resp. $\mathcal{K}_{\Re}$ of Q_Λ resp. $G_{z,\phi}$. We will first establish $\mathcal{K}_{\Re_\Lambda}(g + \beta\phi_x) \to \mathcal{K}_{\Re}(g + \beta\phi_x)$ and then show that we actually have $\mathcal{L}_{Q_\Lambda}(g+\beta\phi_x) = \mathcal{K}_{\Re_\Lambda}(g+\beta\phi_x)$ and $\mathcal{L}_{G_{z,\phi}}(g+\beta\phi_x) = \mathcal{K}_{\Re}(g+\beta\phi_x)$. The main technical result will be the following

Lemma 7.6. *Choose $\epsilon > 0$ such that $z < \frac{e^{-4\beta B-1-\epsilon}}{C(\beta)}$ and let Υ be the function on $\mathcal{M}_f^{\cdot\cdot}(X)$*

$$\Upsilon(\mu) = \begin{cases} 2\left(e^{2\beta B}\right)^{\mu(X)}, & \text{for } supp(\mu) \cap O_x \neq \emptyset \\ \beta e^{\epsilon\mu(X)}\,\mu(|\phi_x|), & \text{for } \mu \in \mathcal{M}_f^{\cdot\cdot}(O_x^c), \end{cases}$$

where $O_x = supp(g) \cup \{\beta|\phi_x| > \epsilon\}$. Then we have $|1 - e^{-\zeta_{g+\beta\phi_x}}| \leq \Upsilon$ on $\mathcal{M}_f^{\cdot\cdot}(X)$ and there exists some $\alpha \in \mathbb{R}_+$, independent of $x \in X$, such that $|\Re|(\Upsilon) \leq \alpha$.

Proof. Since $\phi_x(y) \geq -2B$ for all $y \in X$ we certainly have $|1 - e^{-\mu(g+\beta\phi_x)}| \leq 1 + \left(e^{2\beta B}\right)^{\mu(X)} \leq 2\left(e^{2\beta B}\right)^{\mu(X)}$ for all $\mu \in \mathcal{M}_f^{\cdot\cdot}(X)$. Let now $\mu \in \mathcal{M}_f^{\cdot\cdot}(O_x^c)$ since $O_x^c = \text{supp}(g)^c \cap \{\beta|\phi_x| \leq \epsilon\}$ we have $\mu(g) = 0$ and $\beta|\mu(\phi_x)| \leq \epsilon\mu(X)$. So due to $|1 - e^t| \leq |t|e^{|t|}$ we have $|1 - e^{-\mu(g+\beta\phi_x)}| \leq \Upsilon(\mu)$. Let us name $\gamma = e^{2\beta B+1}C(\beta)$ in the sequel.

$$\begin{aligned}
&|\Re|(\mathbf{1}_{\mathcal{M}_f^{\cdot\cdot}(O_x^c)}\,\Upsilon) \\
= \;& \sum_{n=1}^{\infty} \frac{z^n}{n!}\beta e^{\epsilon n} \int_{(O_x^c)^n} \sum_{i=1}^{n} |\phi_x(x_i)||u(x_1,...,x_n)|\,\lambda(\mathrm{d}\,x_1)...\lambda(\mathrm{d}\,x_n) \\
\leq \;& \beta\sum_{n=1}^{\infty} \frac{z^n e^{\epsilon n}}{(n-1)!} \int_{O_x^c} \lambda(\mathrm{d}\,y)\,|\phi_x(y)| \int_{X^{n-1}} \lambda(\mathrm{d}\,x_1)...\lambda(\mathrm{d}\,x_{n-1})\,|u(y,x_1,...,x_{n-1})| \\
\leq \;& \frac{\beta}{C(\beta)}\sum_{n=1}^{\infty} n(ze^{\epsilon}\gamma)^n \int_{O_x^c} \lambda(\mathrm{d}\,y)\,|\phi_x(y)| \\
\leq \;& \frac{\beta}{C(\beta)}\frac{1}{(1-ze^{\epsilon}\gamma)^2}\lambda\left(|\phi_x|1_{\{\beta|\phi_x|\leq\epsilon\}}\right) \leq \frac{1}{(1-ze^{\epsilon}\gamma)^2}\frac{1}{\tilde{c}(\epsilon)} =: \alpha_1.
\end{aligned}$$

For the second inequality we have used the estimate (7.3) and the last inequality is due to remark 7.3.
Since $X^n \setminus (O_x^c)^n = (O_x \times X^{n-1}) \cup (X \times O_x \times X^{n-2}) \cup ... \cup (X^{n-1} \times O_x)$ we have

$$\begin{aligned} & |\Re|(\mathbf{1}_{\{\mu \in \mathcal{M}_f^{\cdot\cdot}(X) | \mathrm{supp}(\mu) \cap O_x \neq \emptyset\}}\, \Upsilon) \\ \leq & \sum_{n=1}^{\infty} \frac{z^n}{n!} n2 \left(e^{2\beta B}\right)^n \int_{O_x} \lambda(\mathrm{d}\, y) \int_{X^{n-1}} \lambda(\mathrm{d}\, x_1)...\lambda(\mathrm{d}\, x_{n-1})\, |u(y, x_1, ..., x_{n-1})| \\ \leq & \frac{2}{C(\beta)} \sum_{n=1}^{\infty} n (z e^{2\beta B} \gamma)^n\, \lambda(O_x) \\ \leq & \frac{2}{C(\beta)} \frac{1}{(1 - z e^{2\beta B}\gamma)^2} (\lambda(\mathrm{supp}(g)) + \lambda(\{\beta|\phi_x| > \epsilon\})) \\ \leq & \frac{2}{C(\beta)} \frac{1}{(1 - z e^{2\beta B}\gamma)^2} (\lambda(\mathrm{supp}(g)) + \frac{C(\beta)}{c(\epsilon)}) =: \alpha_2. \end{aligned}$$

For the second inequality we have used the estimate (7.3). The last inequality follows by remark 7.3. And so we can choose $\alpha = \alpha_1 + \alpha_2$. □

Corollary 7.7. *We have $\mathcal{K}_{\Re_\Lambda}(g + \beta\phi_x) \to \mathcal{K}_{\Re}(g + \beta\phi_x)$ as $\Lambda \uparrow X$ and there exists a constant $\tilde{\alpha}$, independent of Λ and x, such that $\mathcal{K}_{\Re_\Lambda}(g + \beta\phi_x) \leq \tilde{\alpha}$.*

Proof. By the preceding lemma we have $|(\Re - \Re_\Lambda)(\mathbf{1} - e^{-\zeta_{g+\beta\phi_x}})| \leq |\Re|((\mathbf{1} - \mathbf{1}_{\mathcal{M}^{\cdot\cdot}(\Lambda)})\Upsilon) \downarrow 0$ as $\Lambda \uparrow X$ by dominated convergence. Furthermore we certainly have $|\Re_\Lambda(\mathbf{1} - e^{-\zeta_{g+\beta\phi_x}})| \leq |\Re|(\Upsilon) \leq \alpha$, so we can choose $\tilde{\alpha} = e^{\alpha}$. □

To shorten notation let us denote $\tilde{\phi} = g + \beta\phi_x$ in the sequel.

Lemma 7.8. *We have*

$$\mathcal{L}_{G_{z,\phi}}(\tilde{\phi}) = \mathcal{K}_{\Re}(\tilde{\phi}) \text{ and } \mathcal{L}_{Q_\Lambda}(\tilde{\phi}) = \mathcal{K}_{\Re_\Lambda}(\tilde{\phi}).$$

Proof. Let us start by establishing $\mathcal{L}_{G_{z,\phi}}(\tilde{\phi}_+) = \mathcal{K}_{\Re}(\tilde{\phi}_+)$. Let $(f_n)_{n\geq 1}$ be an increasing sequence of functions in U such that[3] $f_n \uparrow \tilde{\phi}_+$ as $n \to \infty$. Remark that due to monotone convergence there holds $\lim_{n\to\infty} \zeta_{f_n} = \zeta_{\tilde{\phi}_+}$ on $\mathcal{M}^{\cdot\cdot}(X)$. Thus with $\mathbf{1} - e^{-\zeta_{f_n}} \leq \mathbf{1} - e^{-\zeta_{\tilde{\phi}_+}} \leq \Upsilon$ we conclude by dominated convergence

[3] Let $f_n = \min\{\tilde{\phi}_+, n\}\mathbf{1}_{\Lambda_n}$ with $\Lambda_n \in \mathcal{B}_0(X)$ and $\Lambda_n \uparrow X$.

$\Re(\mathbf{1}-e^{-\zeta_{f_n}}) \to \Re(\mathbf{1}-e^{-\zeta_{\tilde{\phi}_+}})$ as $n\to\infty$, which implies the last equality in the following expression.

$$\mathcal{L}_{G_{z,\phi}}(\tilde{\phi}_+) = \lim_{n\to\infty} \mathcal{L}_{G_{z,\phi}}(f_n) = \lim_{n\to\infty} \mathcal{K}_{\Re}(f_n) = \mathcal{K}_{\Re}(\tilde{\phi}_+).$$

The first equality above can also be justified by dominated convergence and the second is due to $f_n \in U$. Now let us treat the general case.

$$\begin{aligned}\Re(\mathbf{1}-e^{-\zeta_{\tilde{\phi}}}) &= \int \left(1 - e^{-\mu(\tilde{\phi}_+)} \sum_{j=0}^{\infty} \frac{\mu(\tilde{\phi}_-)^j}{j!}\right) \Re(\mathrm{d}\,\mu) \\ &= \int \left(1-e^{-\mu(\tilde{\phi}_+)}\right) \Re(\mathrm{d}\,\mu) - \int \sum_{j=1}^{\infty} \frac{e^{-\mu(\tilde{\phi}_+)}\mu(\tilde{\phi}_-)^j}{j!} \Re(\mathrm{d}\,\mu)\end{aligned}$$

The second equality above is due to the following. Since $e^{\zeta_{\tilde{\phi}_-}} - 1 \leq \Upsilon$ we have

$$\int \sum_{j=1}^{\infty} \frac{e^{-\mu(\tilde{\phi}_+)}\mu(\tilde{\phi}_-)^j}{j!} |\Re|(\mathrm{d}\,\mu) \leq \int \left(e^{\mu(\tilde{\phi}_-)} - 1\right) |\Re|(\mathrm{d}\,\mu) < \infty. \tag{7.6}$$

According to the monotone convergence theorem we are allowed to exchange in the below equation the sum with the integrals.

$$\int \sum_{j=1}^{\infty} \frac{e^{-\mu(\tilde{\phi}_+)}\mu(\tilde{\phi}_-)^j}{j!} \Re(\mathrm{d}\,\mu) = \sum_{j=1}^{\infty} \frac{1}{j!} C_{\Re}^{j}(\tilde{\phi}_-^{\otimes j} \otimes e^{-\zeta_{\tilde{\phi}_+}}).$$

The estimate (7.6) also shows absolute convergence of the above right hand side. So we have by lemma 3.1

$$\exp\left[\sum_{j=1}^{\infty} \frac{C_{\Re}^{j}(\tilde{\phi}_-^{\otimes j} \otimes e^{-\zeta_{\tilde{\phi}_+}})}{j!}\right] = 1 + \sum_{j=1}^{\infty} \frac{1}{j!} \sum_{\mathcal{J}\in\pi([j])} \prod_{J\in\mathcal{J}} C_{\Re}^{|J|}(\tilde{\phi}_-^{\otimes|J|} \otimes e^{-\zeta_{\tilde{\phi}_+}}).$$

Collecting everything together we obtain

$$\begin{aligned}
\mathcal{K}_{\Re}(\tilde{\phi}) &= \mathcal{K}_{\Re}(\tilde{\phi}_+)\exp\left[\sum_{j=1}^{\infty}\frac{C^j_{\Re}(\tilde{\phi}_-^{\otimes j}\otimes e^{-\zeta_{\tilde{\phi}_+}})}{j!}\right] \\
&= \mathcal{L}_{G_{z,\phi}}(\tilde{\phi}_+)\left(1+\sum_{j=1}^{\infty}\frac{1}{j!}\sum_{\mathcal{J}\in\pi([j])}\prod_{J\in\mathcal{J}}C^{|J|}_{\Re}(\tilde{\phi}_-^{\otimes|J|}\otimes e^{-\zeta_{\tilde{\phi}_+}})\right) \\
&= \mathcal{L}_{G_{z,\phi}}(\tilde{\phi}_+)+\sum_{j=1}^{\infty}\frac{C^j_{G_{z,\phi}}(\tilde{\phi}_-^{\otimes j}\otimes e^{-\zeta_{\tilde{\phi}_+}})}{j!} \\
&= \int e^{-\mu(\tilde{\phi}_+)}\sum_{j=0}^{\infty}\frac{\mu(\tilde{\phi}_-)^j}{j!}\,G_{z,\phi}(\mathrm{d}\,\mu) = \mathcal{L}_{G_{z,\phi}}(\tilde{\phi}).
\end{aligned}$$

The third equation is due to lemma 3.6 since (7.6) implies $\nu^n_{|\Re|}(\tilde{\phi}_-^{\otimes n}) < \infty$ for $n \geq 1$ and therefore the assumptions of that lemma are satisfied. In particular lemma 3.6 gives $\nu^1_{G_{z,\phi}}(\tilde{\phi}_-) < \infty$ which implies $\zeta_{\tilde{\phi}_-} < \infty$, $G_{z,\phi}$ − a.s. so the conditional energy $E(\mu, x)$ is $G_{z,\phi}$ − a.s. well defined.

Certainly all arguments are valid if we replace $\Re$ by $\Re_\Lambda$ therefore we also obtain the second assertion. □

We can now finish the proof of the theorem:

With corollary 7.7 we obtain $\mathcal{L}_{Q_\Lambda}(g + \beta\phi_x) \to \mathcal{L}_{G_{z,\phi}}(g + \beta\phi_x)$ as $\Lambda \uparrow X$ and $\mathcal{L}_{Q_\Lambda}(g+\beta\phi_x) \leq \tilde{\alpha}$. So we can take the limit $\Lambda \uparrow X$ inside the integral of equation (7.5) and thus obtain that $G_{z,\phi}$ solves $(\mathbf{\Sigma}'_\xi)$ for test functions $h = f\otimes e^{-\zeta_g}$, $f, g \in U$. But again this can be extended to all $h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X))$ by the argument in [11] chapter 4, proof of theorem 10. □

In [15] Nguyen and Zessin have shown that $(\mathbf{\Sigma}'_\xi)$ is equivalent to the equilibrium equations in the sense of Dobrushin-Lanford-Ruelle. So we have identified $G_{z,\phi}$ as a Gibbs point process. Since X can be a general Polish space lattice as well as continuous systems are covered by theorem 7.5.

A small drawback is that we could construct $G_{z,\phi}$ for $z \in (0, \frac{e^{-2\beta B-1}}{C(\beta)})$ but the Gibbs property could only be shown for $z \in (0, \frac{e^{-4\beta B-1}}{C(\beta)})$. If there exists a sharper estimate than the one given by lemma 7.6 this difficulty might be overcome.

We have shown that $(\mathbf{\Sigma}_{\Re})$ implies $\left(\mathbf{\Sigma}'_{\xi}\right)$. A natural question is if also $\left(\mathbf{\Sigma}'_{\xi}\right) \Rightarrow (\mathbf{\Sigma}_{\Re})$ holds. If this is the case then no phase transition can occur since $(\mathbf{\Sigma}_{\Re})$ has a unique solution. In [19] theorem 5.7. Ruelle remarked that for small z there exists a unique solution to the equilibrium equations for his class of superstable pair potentials. So it is suggestive to ask whether this remains true in our setting.

Remark 7.9. *The proof to theorem 7.5 gives a general scheme for showing that a point process with a signed Lévy functional* L*, such that* $\nu^1_{|L|}$ *exists, is a Papangelou point process. Assume you have a candidate* π *for the Papangelou kernel of* Ψ_L *and* π *satisfies the cocycle condition as well as* $\varrho_k = \pi^{(k)}(0;\cdot)$*. Then if for* $f, g \in U$

$$\lim_{\Lambda\uparrow X} \int\limits_{\mathcal{M}^{\cdot\cdot}(\Lambda)} \pi(\mu, f) e^{-\mu(g)}\, Q_\Lambda(\mathrm{d}\,\mu) = \int\limits_{\mathcal{M}^{\cdot\cdot}(X)} \pi(\mu, f) e^{-\mu(g)}\, \Psi_L(\mathrm{d}\,\mu)$$

we can indentify π *as the Papangelou kernel of* Ψ_L*.*

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