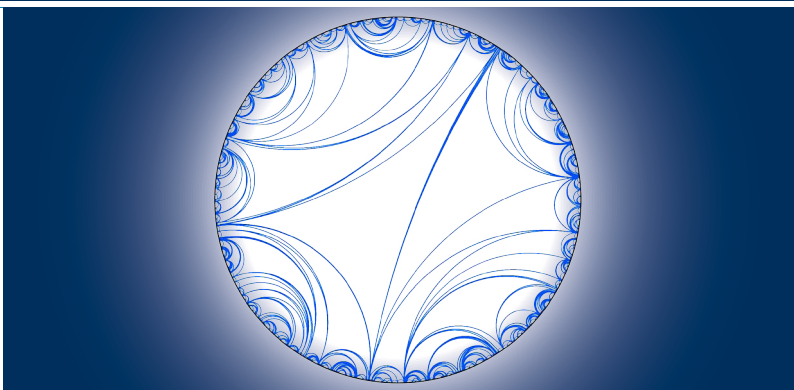




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**ON COMPLETENESS OF ROOT FUNCTIONS
OF STURM-LIOUVILLE PROBLEMS
WITH DISCONTINUOUS BOUNDARY OPERATORS**

A. SHLAPUNOV AND N. TARKHANOV

ABSTRACT. We consider a Sturm–Liouville boundary value problem in a bounded domain \mathcal{D} of \mathbb{R}^n . By this is meant that the differential equation is given by a second order elliptic operator of divergent form in \mathcal{D} and the boundary conditions are of Robin type on $\partial\mathcal{D}$. The first order term of the boundary operator is the oblique derivative whose coefficients bear discontinuities of the first kind. Applying the method of weak perturbation of compact self-adjoint operators and the method of rays of minimal growth, we prove the completeness of root functions related to the boundary value problem in Lebesgue and Sobolev spaces of various types.

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INTRODUCTION

The Hilbert space methods take considerable part in the modern theory of partial differential equations. In particular, the spectral theorem for compact self-adjoint operators attributed to Hilbert and Schmidt allows one to look for solutions of boundary value problems for formally self-adjoint operators in the form of expansions over eigenfunctions of the operator.

Non-self-adjoint compact operators fail to have eigenvectors in general. Keldysh [Keld51] (see also [Keld71] and [GK69, Ch. 5, §8] for more details) elaborated expansions over root functions for weak perturbations of compact self-adjoint operators. In particular, he applied successfully the theorem on the completeness of root

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functions to the Dirichlet problem for second order elliptic operators in divergent form.

The problem of completeness of the system of eigen- and associated functions of boundary value problems for elliptic operators in domains with smooth boundary was studied in many articles (see for instance [Bro53], [Bro59a], [Bro59b], [Agm62], [Kond99]). In a series of papers [Agr94a], [Agr94b], [Agr08], [Agr11b], [Agr11c], including two surveys [Agr02] and [Agr11a], Agranovich proved the completeness of root functions for a wide class of boundary value problems for second order elliptic equations with boundary conditions of the Dirichlet, Neumann and Zaremba type in standard Sobolev spaces over domains with Lipschitz boundary. Note that the class of Lipschitz surfaces does not include surfaces with general conical points, as they are introduced in analysis on singular spaces.

Root functions of general elliptic boundary value problems in weighted Sobolev spaces over domains with conic and edge type singularities on the boundary were studied in [EKS01] and [Tar06]. These papers used estimates of the resolvent of compact operators and the so-called rays of minimal growth. In order to realize fully to what extent the completeness criteria of [EKS01] and [Tar06] are efficient, we dwell on the concept of ellipticity on a compact manifold with smooth edges on the boundary. Such a singular space \mathcal{X} has three smooth strata, more precisely, the interior part \mathcal{X}_0 of \mathcal{X} , the smooth part \mathcal{X}_1 of the boundary and the edge \mathcal{X}_2 which is assumed to be a compact closed manifold. Pseudodifferential operators on \mathcal{X} are (3×3) -matrices \mathcal{A} whose entries $A_{i,j}$ are operators mapping functions on \mathcal{X}_j to functions on \mathcal{X}_i . To each operator \mathcal{A} one assigns a principal symbol $\sigma(\mathcal{A}) := (\sigma_0(\mathcal{A}), \sigma_1(\mathcal{A}), \sigma_2(\mathcal{A}))$ in such a way that $\sigma(\mathcal{A}) = 0$ if and only if \mathcal{A} is compact, and $\sigma(\mathcal{B}\mathcal{A}) = \sigma(\mathcal{B})\sigma(\mathcal{A})$ for all operators \mathcal{A} and \mathcal{B} whose composition is well defined. The components $\sigma_i(\mathcal{A})$ of the principal symbol are functions on the cotangent bundles of \mathcal{X}_i with values in operator spaces. They are smooth away from zero sections of the bundles and bear certain twisted homogeneity as operator families. An operator \mathcal{A} is called elliptic if its principal symbol is invertible away from the zero sections of cotangent bundles. The invertibility of $\sigma_0(\mathcal{A})$ just amounts to the ellipticity of \mathcal{A} in the interior of \mathcal{X} . The invertibility of $\sigma_1(\mathcal{A})$ is equivalent to the Shapiro-Lopatinskii condition on the smooth part of $\partial\mathcal{X}$. The invertibility of $\sigma_2(\mathcal{A})$ constitutes the most difficult problem, for this operator family is considered in weighted Sobolev spaces on an infinite cone. An operator \mathcal{A} proves to be Fredholm if and only if it is elliptic. However, from what has been said it follows that there is no efficient criteria of ellipticity on compact manifolds with edges on the boundary. In general these techniques allow one to derive at most the following result. Consider a classical boundary value problem on \mathcal{X} satisfying the Shapiro-Lopatinskii condition away from the edge \mathcal{X}_2 . It is actually given by a column of operators $A_{i,0}$ with $i = 0, 1$, where $A_{0,0}$ is an elliptic differential operator in \mathcal{X}_0 and $A_{1,0}$ a differential operator near \mathcal{X}_1 followed by restriction to \mathcal{X}_1 . We complete the column to a (2×2) -matrix A by setting $A_{0,1} = 0$ and $A_{1,1} = 0$. The Shapiro-Lopatinskii condition implies that $\sigma_2(A)(y, \eta)$ is a family of Fredholm operators on the unit sphere in $T^*\mathcal{X}_2$. Hence we can set $\sigma_2(A)(y, \eta)$ in the frame of a (3×3) -matrix $a(y, \eta)$ on the unit sphere of $T^*\mathcal{X}_2$ which is moreover invertible. A distinct quantisation procedure leads then immediately to a Fredholm operator

of the type

$$\begin{pmatrix} A_{0,0} & A_{0,2} \\ A_{1,0} & A_{1,2} \\ A_{2,0} & A_{2,2} \end{pmatrix} : \begin{matrix} C^\infty(\mathcal{X}) \\ \oplus \\ C^\infty(\mathcal{X}_2, \mathbb{C}^{l_1}) \end{matrix} \rightarrow \begin{matrix} C^\infty(\mathcal{X}) \\ \oplus \\ C^\infty(\partial\mathcal{X}, \mathbb{C}^m) \\ \oplus \\ C^\infty(\mathcal{X}_2, \mathbb{C}^{l_2}) \end{matrix}, \quad (0.1)$$

where l_1 and l_2 are non-negative integers. However, the Fredholm property of (0.1) elucidates by no means the original problem

$$\begin{cases} A_{0,0}u = f & \text{in } \mathcal{X}_0, \\ A_{1,0}u = u_0 & \text{on } \mathcal{X}_1, \end{cases}$$

unless \mathcal{X}_2 is of dimension 0. Thus, operator-valued symbols make the condition of ellipticity ineffective.

In the present paper we study the completeness of root functions for the Sturm-Liouville boundary value problems for second order elliptic operators in divergent form with Robin type boundary conditions containing oblique derivative with discontinuous coefficients. The discontinuities are of first kind along a smooth hypersurface on the boundary of \mathcal{D} . This hypersurface can be thought of as edge on the boundary, hence we are within the framework of analysis on compact manifolds with edges of codimension 1 on the boundary. The theory of [Tar06] applies in this situation provided that one is able to establish the invertibility of the edge symbol. This latter is a family of Sturm-Liouville boundary value problems in an infinite plane cone parametrized by the points of the cotangent bundle of the edge. On reducing the family to the boundary of the cone one obtains two ordinary pseudodifferential equations on the rays constituting the boundary of the cone. The invertibility of the edge symbol just amounts to the unique solvability of these equations in certain weighted Sobolev spaces on the rays. Making this the subject of our next paper, we restrict our attention here on variational solutions of the Sturm-Liouville boundary value problem. In the absence of regularity theorem we are forced to apply the weak perturbation theory, thus showing the completeness of root functions in the L^2 space.

1. WEAK PERTURBATIONS OF COMPACT SELF-ADJOINT OPERATORS

Let H be a separable (complex) Hilbert space and $A : H \rightarrow H$ a linear operator. As usual, $\lambda \in \mathbb{C}$ is said to be an eigenvalue of A if there is a non-zero element $u \in H$, such that $(A - \lambda I)u = 0$, where I is the identity operator in H . The element u is called an eigenvector of A corresponding to the eigenvalue λ . When supplemented with the zero element, all eigenvectors corresponding to an eigenvalue λ form a vector subspace $E(\lambda)$ in H . It is called an eigenspace of A corresponding to λ , and the dimension of $E(\lambda)$ is called the (geometric) multiplicity of λ . The famous spectral theorem of Hilbert and Schmidt asserts that the system of eigenvectors of a compact self-adjoint operator in H is complete.

Theorem 1.1. *Let $A : H \rightarrow H$ be compact and self-adjoint. Then all eigenvalues of A are real, each non-zero eigenvalue has finite multiplicity, and the system of all eigenvalues counted with their multiplicities is countable and has the only accumulation point $\lambda = 0$. Moreover, there is an orthonormal basis in H consisting of eigenvectors of A .*

As already mentioned, a non-self-adjoint compact operator might have no eigenvalues. However, each non-zero eigenvalue (if exists) is of finite multiplicity, see for instance [DS63]. Similarly to the Jordan normal form of a linear operator on a finite-dimensional vector space one uses the more general concept of root functions of operators.

More precisely, an element $u \in H$ is called a root vector of A corresponding to an eigenvalue $\lambda \in \mathbb{C}$ if $(A - \lambda I)^m u = 0$ for some natural number m . The set of all root vectors corresponding to an eigenvalue λ form a vector subspace in H whose dimension is called the (algebraic) multiplicity of λ .

If the linear span of the set of all root elements is dense in H one says that the root elements of A are complete in H . Aside from self-adjoint operators, the question arises under what conditions on a compact operator A the system of its root elements is complete.

If the dimension of H is finite then the completeness is equivalent to the possibility of reducing the matrix A to the Jordan normal form. Of course, this is always the case for linear operators in complex vector spaces, see, for instance, [VdW67, § 88].

In order to formulate the simplest completeness result for Hilbert spaces we need the definition of the order of a compact operator A . Since $A : H \rightarrow H$ is compact, the operator A^*A is compact, self-adjoint and non-negative. Hence it follows that A^*A possesses a unique non-negative self-adjoint compact square root $(A^*A)^{1/2}$ often denoted by $|A|$. By Theorem 1.1 the operator $|A|$ has countable system of non-negative eigenvalues $s_\nu(A)$ which are called the s -numbers of A . It is clear that if A is self-adjoint then $s_\nu = |\lambda_\nu|$, where $\{\lambda_\nu\}$ is the system of eigenvalues of A .

Definition 1.2. The operator A is said to belong to the Schatten class \mathfrak{S}_p , with $0 < p < \infty$, if

$$\sum_{\nu} |s_\nu(A)|^p < \infty.$$

Note that \mathfrak{S}_2 is the set of all Hilbert-Schmidt operators while \mathfrak{S}_1 is the ideal of all trace class operators.

The following lemma will be very useful in the sequel; it is taken from [DS63] (see also [GK69, Ch. 2, § 2]).

Lemma 1.3. *Let A be a compact operator of class \mathfrak{S}_p , with $0 < p < \infty$, in a Hilbert space H , and B be a bounded operator in H . Then the compositions BA and AB belong to \mathfrak{S}_p .*

After M.V. Keldysh a compact operator A is said to be of finite order if it belongs to a Schatten class \mathfrak{S}_p . The infimum of such numbers p is called the order of A . The following result is usually referred to as theorem on weak perturbations of compact self-adjoint operators. It was first proved in [Keld51], see also [Keld71]. Here we present its formulation from [GK69, Ch. 5, § 8].

Theorem 1.4. *Let A_0 be a compact self-adjoint operator of finite order in H . If δA is a compact operator and the operator $A_0(I + \delta A)$ is injective, then the system of root elements of $A_0(I + \delta A)$ is complete in H and, for any $\varepsilon > 0$, all eigenvalues of $A_0(I + \delta A)$ (except for a finite number) belong to the angles $|\arg \lambda| < \varepsilon$ and $|\arg \lambda - \pi| < \varepsilon$. Moreover,*

- 1) If A_0 has only a finite number of negative eigenvalues, then $A_0(I + \delta A)$ has at most a finite number of eigenvalues in the angle $|\arg \lambda - \pi| < \varepsilon$.
- 2) If A_0 has only a finite number of positive eigenvalues, then $A_0(I + \delta A)$ has at most a finite number of eigenvalues in the angle $|\arg \lambda| < \varepsilon$.

As is easy to see, both operators $A_0(I + \delta A)$ and A_0 are in fact injective under the hypothesis of Theorem 1.4.

2. THE STURM-LIOUVILLE PROBLEM

By a Sturm-Liouville problem in \mathbb{R}^n we mean any boundary value problem for solutions of second order elliptic partial differential equation with Robin-type boundary condition. The coefficients of the Robin boundary condition are allowed to have discontinuities of the first kind, and so mixed boundary conditions are included as well.

Let \mathcal{D} be a bounded domain in \mathbb{R}^n with Lipschitz boundary, i.e., the surface $\partial\mathcal{D}$ is locally the graph of a Lipschitz function. In particular, the boundary $\partial\mathcal{D}$ possesses a tangent hyperplane almost everywhere.

We consider complex-valued functions defined in the domain \mathcal{D} and its closure $\overline{\mathcal{D}}$. For $1 \leq q \leq \infty$, we write $L^q(\mathcal{D})$ for the space of all measurable functions u in \mathcal{D} , such that the integral of $|u|^q$ over \mathcal{D} is finite. For functions $u \in C^s(\overline{\mathcal{D}})$ we introduce the norm

$$\|u\|_{H^s(\mathcal{D})} = \left(\int_{\mathcal{D}} \sum_{|\alpha| \leq s} |\partial^\alpha u|^2 dx \right)^{1/2},$$

where $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, and $\partial_j = \partial/\partial x_j$. The completion of the space $C^s(\overline{\mathcal{D}})$ with respect to this norm is the Banach space $H^s(\mathcal{D})$. It is convenient to set $H^0(\mathcal{D}) := L^2(\mathcal{D})$. Then $H^s(\mathcal{D})$ is a Hilbert space with scalar product

$$(u, v)_{H^s(\mathcal{D})} = \int_{\mathcal{D}} \sum_{|\alpha| \leq s} \partial^\alpha u \overline{\partial^\alpha v} dx,$$

$u, v \in H^s(\mathcal{D})$.

Consider a second order partial differential operator A in the domain \mathcal{D} of divergence form

$$A(x, \partial)u = - \sum_{i,j=1}^n \partial_i(a_{i,j}(x)\partial_j u) + \sum_{j=1}^n a_j(x)\partial_j u + a_0(x)u,$$

the coefficients $a_{i,j}$, a_j and a_0 being of class $L^\infty(\mathcal{D})$.

Let $v(x) = (v_1(x), \dots, v_n(x))$ be a vector field on the surface $\partial\mathcal{D}$. Denote by ∂_v the oblique derivative

$$\partial_v = \sum_{j=1}^n v_j(x)\partial_j$$

and introduce a first order boundary operator $B(x, \partial) = \partial_v + b_0(x)$. The coefficients $v_1(x), \dots, v_n(x)$ and $b_0(x)$ are assumed to be bounded measurable functions on $\partial\mathcal{D}$. We allow the vector $v(x)$ to vanish on an open connected subset S of $\partial\mathcal{D}$ with piecewise smooth boundary ∂S . In this case we assume that $b_0(x)$ does not vanish on S .

Consider the following boundary value problem with Robin-type condition on the surface $\partial\mathcal{D}$. Given a distribution f in \mathcal{D} , find a distribution u in \mathcal{D} which satisfies

$$\begin{cases} Au = f & \text{in } \mathcal{D}, \\ Bu = 0 & \text{on } \partial\mathcal{D}. \end{cases} \quad (2.1)$$

In order to get asymptotic results, it is necessary to put some restrictions on the operators A and B . Suppose that the matrix

$$(a_{i,j}(x))_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$$

is Hermitian and

$$\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j \geq m|\xi|^2 \quad (2.2)$$

for all $(x, \xi) \in \mathcal{D} \times (\mathbb{R}^n \setminus \{0\})$, with m a positive constant independent on x and ξ . Estimate (2.2) is nothing but the statement that the operator A is strongly elliptic. Note that inequality (2.2) is weaker than the coercivity, i.e., the existence of a constant m such that

$$\sum_{i,j=1}^n a_{i,j}(x)\overline{\partial_i u}\partial_j u \geq m \sum_{j=1}^n |\partial_j u|^2 \quad (2.3)$$

for all $u \in C^1(\overline{\mathcal{D}})$, because u may take on complex values.

On assuming that $a_{i,j}(x)$ are continuous up to the boundary of \mathcal{D} we consider the complex vector field $c(x)$ on $\partial\mathcal{D}$ with components

$$c_j(x) = \sum_{i=1}^n a_{i,j}(x)\nu_i(x),$$

where $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ is the unit outward normal vector of $\partial\mathcal{D}$ at $x \in \partial\mathcal{D}$. From condition (2.2) it follows that there is a complex-valued function $b_1(x)$ on the boundary with the property that the difference $v(x) - b_1(x)c(x)$ is orthogonal to $\nu(x)$ for almost all $x \in \partial\mathcal{D}$. In fact, the pointwise equality $(v - b_1c, \nu) = 0$ just amounts to

$$b_1(x) = \frac{\sum_{j=1}^n v_j(x)\nu_j(x)}{\sum_{i,j=1}^n a_{i,j}(x)\nu_i(x)\nu_j(x)}$$

for $x \in \partial\mathcal{D}$. Obviously, $b_1(x)$ is a bounded measurable function on $\partial\mathcal{D}$ and the vector field $t(x) = v(x) - b_1(x)c(x)$ takes on its values in the tangent hyperplane $T_x(\partial\mathcal{D})$ of $\partial\mathcal{D}$ at x . Summarizing we conclude that if the boundary of \mathcal{D} is a Lipschitz surface then

$$B(x, \partial) = b_1(x)\partial_c + \partial_t + b_0(x), \quad (2.4)$$

where $b_1 \in L^\infty(\partial\mathcal{D})$ and $t(x)$ is a tangential vector field on $\partial\mathcal{D}$ whose components belong to $L^\infty(\partial\mathcal{D})$. By assumption, both b_1 and t vanish on S . Concerning the behavior of b_1 in $\partial\mathcal{D} \setminus S$ we require that $b_1(x) \neq 0$ for almost all $x \in \partial\mathcal{D} \setminus S$ and $1/b_1 \in L^1(\partial\mathcal{D} \setminus S)$.

Note that the Shapiro-Lopatinskii condition is violated on $\partial\mathcal{D} \setminus S$ unless the coefficients $a_{i,j}(x)$ are real-valued.

Denote by $H^1(\mathcal{D}, S)$ the subspace of $H^1(\mathcal{D})$ consisting of those functions whose restriction to the boundary vanishes on S . This space is Hilbert under the induced norm. It is easily seen that smooth functions on $\overline{\mathcal{D}}$ vanishing in a neighbourhood of \overline{S} are dense in $H^1(\mathcal{D}, S)$. Since on S the boundary operator reduces to $B = b_0$ and $b_0(x) \neq 0$ for $x \in S$, the functions of $H^1(\mathcal{D})$ satisfying $Bu = 0$ on $\partial\mathcal{D}$ belong to $H^1(\mathcal{D}, S)$.

As we want to study perturbation of self-adjoint operators we split both a_0 and b_0 into two parts

$$\begin{aligned} a_0 &= a_{0,0} + \delta a_0, \\ b_0 &= b_{0,0} + \delta b_0, \end{aligned}$$

where $a_{0,0}$ is a nonnegative bounded function in \mathcal{D} and $b_{0,0}$ a bounded function on $\partial\mathcal{D}$ satisfying $b_{0,0}/b_1 \geq 0$. If the functional

$$\|u\|_{SL} = \left(\int_{\mathcal{D}} \sum_{i,j=1}^n a_{i,j} \overline{\partial_i u} \partial_j u \, dx + \|\sqrt{a_{0,0}}u\|_{L^2(\mathcal{D})}^2 + \|\sqrt{b_{0,0}b_1^{-1}}u\|_{L^2(\partial\mathcal{D}\setminus S)}^2 \right)^{1/2}$$

defines a norm on $H^1(\mathcal{D}, S)$, we denote by $H_{SL}(\mathcal{D})$ the completion of $H^1(\mathcal{D}, S)$ with respect to this norm. Then $H_{SL}(\mathcal{D})$ is actually a Hilbert space with scalar product

$$(u, v)_{SL} = \int_{\mathcal{D}} \sum_{i,j=1}^n a_{i,j} \partial_j u \overline{\partial_i v} \, dx + (a_{0,0}u, v)_{L^2(\mathcal{D})} + (b_{0,0}b_1^{-1}u, v)_{L^2(\partial\mathcal{D}\setminus S)}.$$

From now on we assume that the space $H_{SL}(\mathcal{D})$ is continuously embedded into the Lebesgue space $L^2(\mathcal{D})$, i.e.,

$$\|u\|_{L^2(\mathcal{D})} \leq c \|u\|_{SL} \quad (2.5)$$

for all $u \in H_{SL}(\mathcal{D})$, where c is a constant independent of u . This condition is not particularly restrictive, as examples below show. If the coefficients $a_{i,j}$ are smooth up to S then the functions of $H_{SL}(\mathcal{D})$ belong actually to $H_{\text{loc}}^1(\mathcal{D} \cup S)$, for the Dirichlet problem for strongly elliptic equations satisfies the Shapiro-Lopatinski condition.

Write ι for the inclusion

$$H_{SL}(\mathcal{D}) \hookrightarrow L^2(\mathcal{D}), \quad (2.6)$$

which is continuous by (2.5). We use this inclusion to specify the dual space of $H_{SL}(\mathcal{D})$ via the pairing in $L^2(\mathcal{D})$. More precisely, let $H_{SL}^-(\mathcal{D})$ be the completion of $H^1(\mathcal{D}, S)$ with respect to the norm

$$\|u\|_{H_{SL}^-(\mathcal{D})} = \sup_{\substack{v \in H^1(\mathcal{D}, S) \\ v \neq 0}} \frac{|(v, u)_{L^2(\mathcal{D})}|}{\|v\|_{SL}}.$$

Remark 2.1. As $H^1(\mathcal{D}, S)$ is dense in $H_{SL}(\mathcal{D})$ and the norm $\|\cdot\|_{SL}$ majorises $\|\cdot\|_{L^2(\mathcal{D})}$ we see that

$$\|u\|_{H_{SL}^-(\mathcal{D})} = \sup_{\substack{v \in H_{SL}(\mathcal{D}) \\ v \neq 0}} \frac{|(v, u)_{L^2(\mathcal{D})}|}{\|v\|_{SL}}.$$

Lemma 2.2. *The space $L^2(\mathcal{D})$ is continuously embedded into $H_{SL}^-(\mathcal{D})$. If inclusion (2.6) is compact then the space $L^2(\mathcal{D})$ is compactly embedded into $H_{SL}^-(\mathcal{D})$.*

Proof. By definition and estimate (2.5) we get

$$\|u\|_{H_{SL}^-(\mathcal{D})} \leq \sup_{\substack{v \in H^1(\mathcal{D}, S) \\ v \neq 0}} \frac{\|u\|_{L^2(\mathcal{D})} \|v\|_{L^2(\mathcal{D})}}{\|v\|_{SL}} \leq c \|u\|_{L^2(\mathcal{D})}$$

for all $u \in L^2(\mathcal{D})$, i.e., the space $L^2(\mathcal{D})$ is continuously embedded into $H_{SL}^-(\mathcal{D})$ indeed.

Suppose (2.6) is compact. Then the Hilbert space adjoint $\iota^* : L^2(\mathcal{D}) \hookrightarrow H_{SL}(\mathcal{D})$ is compact, too. By Remark 2.1) we conclude that

$$\begin{aligned} \|u\|_{H_{SL}^-(\mathcal{D})} &= \sup_{\substack{v \in H^1(\mathcal{D}, S) \\ v \neq 0}} \frac{|(\iota(v), u)_{L^2(\mathcal{D})}|}{\|v\|_{SL}} \\ &= \sup_{\substack{v \in H_{SL}(\mathcal{D}) \\ v \neq 0}} \frac{|(v, \iota^*(u))_{SL}|}{\|v\|_{SL}} \\ &= \|\iota^*(u)\|_{SL} \end{aligned} \tag{2.7}$$

for all $u \in L^2(\mathcal{D})$. Therefore, any weakly convergent sequence in $L^2(\mathcal{D})$ converges in $H_{SL}^-(\mathcal{D})$, which shows the second part of the lemma. \square

Lemma 2.3. *The Banach space $H_{SL}^-(\mathcal{D})$ is topologically isomorphic to the dual space $H_{SL}(\mathcal{D})'$ and the isomorphism is defined by the sesquilinear form*

$$(v, u) = \lim_{\nu \rightarrow \infty} (v, u_\nu)_{L^2(\mathcal{D})} \tag{2.8}$$

for $u \in H_{SL}^-(\mathcal{D})$ and $v \in H_{SL}(\mathcal{D})$ where $\{u_\nu\}$ is any sequence in $H^1(\mathcal{D}, S)$ converging to u .

That is, for every fixed $u \in H_{SL}^-(\mathcal{D})$, pairing (2.8) defines a continuous linear functional f_u on $H_{SL}(\mathcal{D})$ and, for each $f \in H_{SL}(\mathcal{D})'$, there is a unique $u \in H_{SL}^-(\mathcal{D})$ with $f(v) = f_u(v)$ for all $v \in H_{SL}(\mathcal{D})$. Moreover, the conjugate linear map $u \mapsto f_u$ is an isometry.

Proof. To show that the limit on the right-hand side of (2.8) exists for each fixed function $v \in H_{SL}(\mathcal{D})$, it suffices to show that $\{(v, u_\nu)_{L^2(\mathcal{D})}\}$ is a Cauchy sequence. By definition,

$$|(v, u_\nu - u_\mu)_{L^2(\mathcal{D})}| \leq \|v\|_{SL} \|u_\nu - u_\mu\|_{H_{SL}^-(\mathcal{D})} \rightarrow 0$$

as $\nu, \mu \rightarrow \infty$, which is our claim. Clearly, this limit does not depend on the particular sequence $\{u_\nu\}$, for if $\|u_\nu\|_{H_{SL}^-(\mathcal{D})} \rightarrow 0$, then $|(v, u_\nu)_{L^2(\mathcal{D})}| \rightarrow 0$ for all $v \in H_{SL}(\mathcal{D})$.

From the definition it follows that

$$|(v, u)| \leq \|u\|_{H_{SL}^-(\mathcal{D})} \|v\|_{SL} \tag{2.9}$$

for all $u \in H_{SL}^-(\mathcal{D})$ and $v \in H_{SL}(\mathcal{D})$. Hence, for every fixed $u \in H_{SL}^-(\mathcal{D})$, the formula $f_u(v) := (v, u)$ defines a continuous linear functional f_u on $H_{SL}(\mathcal{D})$, such that

$$\|f_u\|_{H_{SL}(\mathcal{D})'} \leq \|u\|_{H_{SL}^-(\mathcal{D})}.$$

If $\{u_\nu\} \subset H^1(\mathcal{D}, S)$ approximates an element u in $H_{SL}^-(\mathcal{D})$, then equality (2.7) implies that the sequence $\{\iota^* \iota u_\nu\}$ converges to a function U in the space $H_{SL}(\mathcal{D})$ and

$$\begin{aligned} \|U\|_{SL} &= \lim_{\nu \rightarrow \infty} \|\iota^* \iota u_\nu\|_{SL} \\ &= \lim_{\nu \rightarrow \infty} \|u_\nu\|_{H_{SL}^-(\mathcal{D})} \\ &= \|u\|_{H_{SL}^-(\mathcal{D})}. \end{aligned}$$

Moreover,

$$\begin{aligned} f_u(v) &= (v, u) \\ &= \lim_{\nu \rightarrow \infty} (\iota v, \iota u_\nu)_{L^2(\mathcal{D})} \\ &= \lim_{\nu \rightarrow \infty} (v, \iota^* \iota u_\nu)_{SL} \\ &= (v, U)_{SL} \end{aligned}$$

for all $v \in H_{SL}(\mathcal{D})$. Now, the Riesz theorem yields $\|U\|_{SL} = \|f_u\|_{H_{SL}(\mathcal{D})'}$ whence $\|f_u\|_{H_{SL}(\mathcal{D})'} = \|u\|_{H_{SL}^-(\mathcal{D})}$.

It remains to show that any continuous linear functional f on $H_{SL}(\mathcal{D})$ has the form $f(v) = (v, u_f)$ for some $u_f \in H_{SL}^-(\mathcal{D})$. According to the Riesz theorem, for any $f \in H_{SL}(\mathcal{D})'$ there is a unique element $U_f \in H_{SL}(\mathcal{D})$, such that

$$f(v) = (v, U_f)_{SL}$$

for all $v \in H_{SL}(\mathcal{D})$. Besides, $\|U_f\|_{SL} = \|f\|_{H_{SL}(\mathcal{D})'}$. By definition, the operator ι is injective and its image is dense in $L^2(\mathcal{D})$. Hence, the image of the operator $\iota^* \iota$ in $H_{SL}(\mathcal{D})$ is dense, too. Pick a sequence $\{u_\nu\} \subset H_{SL}(\mathcal{D})$ with the property that $\{\iota^* \iota u_\nu\}$ converges to U_f . Then, according to (2.7), $\{u_\nu\}$ is a Cauchy sequence in $H_{SL}^-(\mathcal{D})$, and so it converges to an element u_f in this space. It is easy to see that u_f is actually independent of the particular choice of the sequence $\{u_\nu\}$. Finally, we obtain

$$\begin{aligned} (v, u_f) &= \lim_{\nu \rightarrow \infty} (\iota v, \iota u_\nu)_{L^2(\mathcal{D})} \\ &= \lim_{\nu \rightarrow \infty} (v, \iota^* \iota u_\nu)_{SL} \\ &= (v, U_f)_{SL} \\ &= f(v) \end{aligned}$$

for all $v \in H_{SL}(\mathcal{D})$, as desired. \square

Note that $H_{SL}(\mathcal{D})$ is reflexive, since it is a Hilbert space. Hence it follows that $(H_{SL}^-(\mathcal{D}))' = H_{SL}(\mathcal{D})$, i.e., the spaces $H_{SL}(\mathcal{D})$ and $H_{SL}^-(\mathcal{D})$ are dual to each other with respect to (2.8).

Given any $u \in H^2(\mathcal{D})$ and $v \in H^1(\mathcal{D})$, the Stokes formula yields

$$\int_{\partial \mathcal{D}} \bar{v} \partial_c u \, ds = \int_{\mathcal{D}} \sum_{i,j=1}^n (a_{i,j} \bar{\partial}_i v \partial_j u + \bar{v} \partial_i (a_{i,j} \partial_j u)) \, dx.$$

On integrating by parts we see that

$$(Au, v)_{L^2(\mathcal{D})} = \int_{\mathcal{D}} \sum_{i,j=1}^n a_{i,j} \overline{\partial_i v} \partial_j u \, dx + (b_1^{-1}(\partial_t + b_0)u, v)_{L^2(\partial\mathcal{D}\setminus S)} + \left(\sum_{j=1}^n a_j \partial_j u + a_0 u, v \right)_{L^2(\mathcal{D})}$$

for all $u \in H^2(\mathcal{D})$ and $v \in H^1(\mathcal{D})$ satisfying the boundary condition of (2.1). Suppose

$$\left| (b_1^{-1}(\partial_t + \delta b_0)u, v)_{L^2(\partial\mathcal{D}\setminus S)} + \left(\sum_{j=1}^n a_j \partial_j u + \delta a_0 u, v \right)_{L^2(\mathcal{D})} \right| \leq c \|u\|_{SL} \|v\|_{SL} \quad (2.10)$$

for all $u, v \in H^1(\mathcal{D}, S)$, where c is a positive constant independent of u and v . Then, for each fixed $u \in H_{SL}(\mathcal{D})$, the sesquilinear form

$$Q(u, v) = \int_{\mathcal{D}} \sum_{i,j=1}^n a_{i,j} \overline{\partial_i v} \partial_j u \, dx + (b_1^{-1}(\partial_t + b_0)u, v)_{L^2(\partial\mathcal{D}\setminus S)} + \left(\sum_{j=1}^n a_j \partial_j u + a_0 u, v \right)_{L^2(\mathcal{D})}$$

determines a continuous linear functional f on $H_{SL}(\mathcal{D})$ by $f(v) := \overline{Q(u, v)}$ for $v \in H_{SL}(\mathcal{D})$. By Lemma 2.3, there is a unique element in $H_{SL}^-(\mathcal{D})$, which we denote by Lu , such that

$$f(v) = (v, Lu)$$

for all $v \in H_{SL}(\mathcal{D})$. We have thus defined a linear operator $L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$. From (2.10) it follows that L is bounded.

The bounded linear operator $L_0 : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ defined in the same way via the sesquilinear form $(\cdot, \cdot)_{SL}$, i.e.,

$$(v, u)_{SL} = (v, L_0 u) \quad (2.11)$$

for all $u, v \in H_{SL}(\mathcal{D})$, corresponds to the case $a_j \equiv 0$ for all $j = 1, \dots, n$, $a_0 = a_{0,0}$, and $t \equiv 0$, $b_0 = b_{0,0}$.

We are thus lead to a weak formulation of problem (2.1). Given $f \in H_{SL}^-(\mathcal{D})$, find $u \in H_{SL}(\mathcal{D})$, such that

$$\overline{Q(u, v)} = (v, f) \quad (2.12)$$

for all $v \in H_{SL}(\mathcal{D})$.

It should be noted that there is no need to assume the continuity of the coefficients $a_{i,j}$ up to the boundary of D in order to define the operator L and to consider the weak formulation of problem (2.1). Now one can handle problem (2.12) by standard techniques of functional analysis, see for instance [LU73, Ch. 3, §§ 4-6]) for the coercive case.

Suppose $f \in L^2(\mathcal{D})$. If u is a solution of (2.12) then $Au = f$ holds in the sense of distributions in \mathcal{D} , for $C_{\text{comp}}^\infty(\mathcal{D}) \subset H_{SL}(\mathcal{D})$. Since A is elliptic, we readily deduce that $u \in H_{\text{loc}}^2(\mathcal{D})$ and the equality $Au = f$ is actually satisfied almost everywhere in \mathcal{D} .

If, in addition, $u \in H^2(\mathcal{D})$ then

$$((\partial_c + b_1^{-1}(\partial_t + b_0))u, v)_{L^2(\partial\mathcal{D}\setminus S)} = 0$$

for all $v \in H_{SL}(\mathcal{D})$. Since any smooth function v in $\overline{\mathcal{D}}$ whose support does not meet S belongs to $H_{SL}(\mathcal{D})$, we conclude that $(b_1\partial_c + \partial_t + b_0)u = 0$ on $\partial\mathcal{D} \setminus S$. Hence $Bu = 0$ on $\partial\mathcal{D}$, for $u = 0$ and $b_1 = 0$ on S .

Lemma 2.4. *Let estimate (2.5) hold true. Assume that $a_j \equiv 0$ for all $j = 1, \dots, n$, $\delta a_0 = 0$, and $t \equiv 0$, $\delta b_0 = 0$. Then for each $f \in H_{SL}^-(\mathcal{D})$ there is a unique solution $u \in H_{SL}(\mathcal{D})$ to problem (2.12), i.e., the operator $L_0 : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is continuously invertible. Moreover, the norms of both L_0 and its inverse L_0^{-1} equal to 1.*

Proof. Under the hypotheses of the lemma, (2.12) is just a weak formulation of problem (2.1) with A and B replaced by

$$\begin{aligned} A_0 &= - \sum_{i,j=1}^n \partial_i(a_{i,j}\partial_j) + a_{0,0}, \\ B_0 &= b_1\partial_c + b_{0,0}, \end{aligned}$$

respectively. The corresponding bounded operator in Hilbert spaces just amounts to $L_0 : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ defined by (2.11). Its norm equals 1, for, by Lemma 2.3, we get

$$\|L_0 u\|_{H_{SL}^-(\mathcal{D})} = \sup_{\substack{v \in H_{SL}(\mathcal{D}) \\ v \neq 0}} \frac{|(v, L_0 u)|}{\|v\|_{SL}} = \sup_{\substack{v \in H_{SL}(\mathcal{D}) \\ v \neq 0}} \frac{|(v, u)_{SL}|}{\|v\|_{SL}} = \|u\|_{SL} \quad (2.13)$$

whenever $u \in H_{SL}(\mathcal{D})$.

The existence and uniqueness of solutions to problem (2.12) follows immediately from the Riesz theorem on the general form of continuous linear functionals on Hilbert spaces. From (2.13) we conclude that L_0 is actually an isometry of $H_{SL}^-(\mathcal{D})$ onto $H_{SL}(\mathcal{D})$, as desired. \square

Consider the sesquilinear form on $H_{SL}^-(\mathcal{D})$ given by

$$(u, v)_{H_{SL}^-(\mathcal{D})} := (L_0^{-1}u, v)$$

for $u, v \in H_{SL}^-(\mathcal{D})$. Since

$$(L_0^{-1}u, v) = (L_0^{-1}u, L_0 L_0^{-1}v) = (L_0^{-1}u, L_0^{-1}v)_{SL} \quad (2.14)$$

for all $u, v \in H_{SL}^-(\mathcal{D})$, the last equality being due to (2.11), this form is Hermitian. Combining (2.13) and (2.14) yields

$$\sqrt{(u, u)_{H_{SL}^-(\mathcal{D})}} = \|u\|_{H_{SL}^-(\mathcal{D})}$$

for all $u \in H_{SL}^-(\mathcal{D})$. From now on we endow the space $H_{SL}^-(\mathcal{D})$ with the scalar product $(\cdot, \cdot)_{H_{SL}^-(\mathcal{D})}$.

Lemma 2.5. *Let estimates (2.5) and (2.10) be fulfilled. If moreover the constant c of (2.10) satisfies $c < 1$ then, for each $f \in H_{SL}^-(\mathcal{D})$, there exists a unique solution $u \in H_{SL}(\mathcal{D})$ to problem (2.12), i.e., the operator $L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is continuously invertible.*

Proof. If estimate (2.10) holds with $c < 1$ then the operator $L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ corresponding to problem (2.12) is easily seen to differ from L_0 by a bounded operator $\delta L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ whose norm does not exceed $c < 1$. As L_0 is

invertible and the inverse operator L_0^{-1} has norm 1, a familiar argument shows that L is invertible, too. \square

Since $C_{\text{comp}}^\infty(\mathcal{D}) \hookrightarrow H_{SL}(\mathcal{D}) \hookrightarrow L^2(\mathcal{D})$, the elements of $H_{SL}^-(\mathcal{D})$ are distributions in \mathcal{D} and any solution to problem (2.1) satisfies $Au = f$ in \mathcal{D} in the sense of distributions. Though the boundary conditions are interpreted in a weak sense, they agree with those in terms of restrictions to $\partial\mathcal{D}$ if the solution is sufficiently smooth up to the boundary, e.g. belongs to $C^1(\overline{\mathcal{D}})$. In particular, if L is invertible then the inverse operator maps $H_{SL}^-(\mathcal{D})$ to $H_{SL}(\mathcal{D})$ where elements u satisfy $Bu = 0$ on $\partial\mathcal{D}$ in a proper sense.

3. COMPLETENESS OF ROOT FUNCTIONS FOR WEAK PERTURBATIONS

We are now in a position to study the completeness of root functions related to problem (2.12). We begin with the self-adjoint operator L_0 . To this end we write ι' for the continuous embedding of $L^2(\mathcal{D})$ into $H_{SL}^-(\mathcal{D})$, as it is described by Lemma 2.2.

Lemma 3.1. *Suppose that estimate (2.5) is fulfilled and inclusion (2.6) is compact. Then the inverse L_0^{-1} of the operator given by (2.11) induces compact positive self-adjoint operators*

$$\begin{aligned} Q_1 &= \iota' \iota L_0^{-1} &: H_{SL}^-(\mathcal{D}) &\rightarrow H_{SL}^-(\mathcal{D}), \\ Q_2 &= \iota L_0^{-1} \iota' &: L^2(\mathcal{D}) &\rightarrow L^2(\mathcal{D}), \\ Q_3 &= L_0^{-1} \iota' \iota &: H_{SL}(\mathcal{D}) &\rightarrow H_{SL}(\mathcal{D}) \end{aligned}$$

which have the same systems of eigenvalues and eigenvectors. Moreover, all eigenvalues are positive and there are orthonormal bases in $H_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}^-(\mathcal{D})$ consisting of the eigenvectors.

Proof. Since ι is compact and ι' , L_0^{-1} bounded, all the operators Q_1 , Q_2 , Q_3 are compact.

Recall that we endow the space $H_{SL}^-(\mathcal{D})$ with the scalar product $(\cdot, \cdot)_{H_{SL}^-(\mathcal{D})}$. Then, by (2.14),

$$\begin{aligned} (Q_1 u, v)_{H_{SL}^-(\mathcal{D})} &= \overline{(v, \iota' \iota L_0^{-1} u)_{H_{SL}^-(\mathcal{D})}} \\ &= \overline{(L_0^{-1} v, \iota' \iota L_0^{-1} u)} \\ &= (\iota L_0^{-1} u, \iota L_0^{-1} v)_{L^2(\mathcal{D})}, \end{aligned}$$

and

$$\begin{aligned} (u, Q_1 v)_{H_{SL}^-(\mathcal{D})} &= \overline{(Q_1 v, u)_{H_{SL}^-(\mathcal{D})}} \\ &= (\iota L_0^{-1} u, \iota L_0^{-1} v)_{L^2(\mathcal{D})} \end{aligned}$$

for all $u, v \in H_{SL}^-(\mathcal{D})$, i.e., the operator Q_1 is self-adjoint.

Using (2.11) we get

$$\begin{aligned} (Q_2 u, v)_{L^2(\mathcal{D})} &= (\iota(L_0^{-1}(\iota' u)), v)_{L^2(\mathcal{D})} \\ &= (L_0^{-1}(\iota' u), \iota' v) \\ &= (L_0^{-1}(\iota' u), L_0^{-1}(\iota' v))_{SL} \end{aligned}$$

and

$$\begin{aligned} (u, Q_2 v)_{L^2(\mathcal{D})} &= \overline{(Q_2 v, u)_{L^2(\mathcal{D})}} \\ &= (L_0^{-1}(\iota' u), L_0^{-1}(\iota' v))_{SL} \end{aligned}$$

for all $u, v \in L^2(\mathcal{D})$, i.e., the operator Q_2 is self-adjoint.

On applying (2.11) once again we obtain

$$\begin{aligned} (Q_3 u, v)_{SL} &= (L_0^{-1}(\iota' \iota u), v)_{SL} \\ &= (\iota' \iota u, v) \\ &= (\iota u, \iota v)_{L^2(\mathcal{D})} \end{aligned}$$

and

$$\begin{aligned} (u, Q_3 v)_{SL} &= \overline{(Q_3 v, u)_{SL}} \\ &= (\iota u, \iota v)_{L^2(\mathcal{D})} \end{aligned}$$

for all $u, v \in H_{SL}(\mathcal{D})$, which establishes the self-adjointness of Q_3 .

Finally, as the operator L_0^{-1} is injective, so are the operators Q_1 , Q_2 and Q_3 . Hence, all their eigenvectors $\{u_\nu\}$ belong to the space $H_{SL}(\mathcal{D})$, for $L_0^{-1}u_\nu$ lies in $H_{SL}(\mathcal{D})$. From the injectivity of ι and ι' we conclude that the systems of eigenvalues and eigenvectors of Q_1 , Q_2 and Q_3 coincide. The last part of the lemma follows from Theorem 1.1. \square

Our next goal is to apply Theorem 1.4 for studying the completeness of root functions of weak perturbations of Q_j .

Theorem 3.2. *If the operator $Q_1 : H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is of finite order then, for any invertible operator of the type $L_0 + \delta L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}(\mathcal{D})$ with a compact operator $\delta L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}(\mathcal{D})$, the system of root functions of the compact operator*

$$P_1 = \iota' \iota (L_0 + \delta L)^{-1} : H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$$

is complete in the spaces $H_{SL}^-(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$.

Proof. By assumption there is a bounded inverse $(L_0 + \delta L)^{-1} : H_{SL}(\mathcal{D}) \rightarrow H_{SL}(\mathcal{D})$. Since

$$I - L_0(L_0 + \delta L)^{-1} = \delta L(L_0 + \delta L)^{-1},$$

we conclude that

$$\begin{aligned} L_0^{-1} - (L_0 + \delta L)^{-1} &= L_0^{-1}(\delta L(L_0 + \delta L)^{-1}), \\ Q_1 - P_1 &= Q_1(\delta L(L_0 + \delta L)^{-1}). \end{aligned} \tag{3.1}$$

From the compactness of δL and boundedness of $(L_0 + \delta L)^{-1}$ it follows that the operator

$$\delta L(L_0 + \delta L)^{-1} : H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$$

is compact.

Hence, P_1 is an injective weak perturbation of the compact self-adjoint operator Q_1 . If in addition the order of Q_1 is finite then Theorem 1.4 implies that the countable system $\{u_\nu\}$ of root functions related to the operator P_1 is complete in the Hilbert space $H_{SL}^-(\mathcal{D})$.

Pick a root function u_ν of the operator P_1 corresponding to an eigenvalue λ_ν . Note that $\lambda_\nu \neq 0$, for the operator $(L_0 + \delta L)^{-1}$ is injective. By definition there

is a natural number m , such that $(P_1 - \lambda_\nu I)^m u_\nu = 0$. Using the binomial formula yields

$$\sum_{j=0}^m \binom{m}{j} \lambda_\nu^{m-j} P_1^j u_\nu = 0.$$

In particular, since $\lambda_\nu \neq 0$, we get

$$u_\nu = \sum_{j=1}^m \binom{m}{j} \lambda_\nu^{-j} (\iota' \iota (L_0 + \delta L)^{-1})^j u_\nu.$$

Hence, $u_\nu \in H_{SL}(\mathcal{D})$ because the range of the operator $(L_0 + \delta L)^{-1}$ lies in the space $H_{SL}(\mathcal{D})$.

We have thus proved that $\{u_\nu\} \subset H_{SL}(\mathcal{D})$. Our next concern will be to show that the linear span $\mathcal{L}(\{u_\nu\})$ of the system $\{u_\nu\}$ is dense in $H_{SL}(\mathcal{D})$ (cf. Proposition 6.1 of [Agr11a] and [Agr11c, p. 12]). For this purpose, pick $u \in H_{SL}(\mathcal{D})$. As $L_0 + \delta L$ maps $H_{SL}(\mathcal{D})$ continuously onto $H_{SL}^-(\mathcal{D})$, we get $(L_0 + \delta L)u \in H_{SL}^-(\mathcal{D})$. Hence, there is a sequence $\{f_k\} \subset \mathcal{L}(\{u_\nu\})$ converging to $(L_0 + \delta L)u$ in $H_{SL}^-(\mathcal{D})$. On the other hand, the inverse $(L_0 + \delta L)^{-1}$ maps $H_{SL}^-(\mathcal{D})$ continuously to $H_{SL}(\mathcal{D})$, and so the sequence

$$(L_0 + \delta L)^{-1} f_k = (L_0 + \delta L)^{-1} \iota' \iota f_k$$

converges to u in $H_{SL}(\mathcal{D})$.

If now $u_{\nu_0} \in \mathcal{L}(\{u_\nu\})$ corresponds to an eigenvalue λ_0 of multiplicity m_0 then the vector $v_{\nu_0} = P_1 u_{\nu_0}$ satisfies

$$(P_1 - \lambda_0 I)^{m_0} v_{\nu_0} = (P_1 - \lambda_0 I)^{m_0+1} u_{\nu_0} + \lambda_0 (P_1 - \lambda_0 I)^{m_0} u_{\nu_0} = 0.$$

Thus, the operator P_1 maps $\mathcal{L}(\{u_\nu\})$ to $\mathcal{L}(\{u_\nu\})$ itself. Therefore, the sequence $\{\iota' \iota (L_0 + \delta L)^{-1} f_k\}$ still belongs to $\mathcal{L}(\{u_\nu\})$ and we can think of $\{(L_0 + \delta L)^{-1} f_k\}$ as sequence of linear combinations of root functions of P_1 converging to u . These arguments show that the subsystem $(L_0 + \delta L)^{-1} \mathcal{L}(\{u_\nu\}) \subset \mathcal{L}(\{u_\nu\})$ is dense in $H_{SL}(\mathcal{D})$.

Finally, since $C_{\text{comp}}^\infty(\mathcal{D}) \subset H_{SL}(\mathcal{D})$ and $C_{\text{comp}}^\infty(\mathcal{D})$ is dense in the Lebesgue space $L^2(\mathcal{D})$, the space $H_{SL}(\mathcal{D})$ is dense in $L^2(\mathcal{D})$ as well. This proves the completeness of the system of root functions in $L^2(\mathcal{D})$. \square

Similar assertions are also true for the weak perturbations of the operators Q_2 and Q_3 .

The operator $L_0 + \delta L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ with a compact operator δL fails to be injective in general, and so Theorem 3.2 does not apply. However, as L_0 is continuously invertible, we conclude that $L = L_0 + \delta L$ is Fredholm. In particular, there is a constant c , such that

$$\|u\|_{SL} \leq c \left(\|Lu\|_{H_{SL}^-(\mathcal{D})} + \|u\|_{H_{SL}^-(\mathcal{D})} \right) \quad (3.2)$$

for all $u \in H_{SL}(\mathcal{D})$.

We next extend Theorem 3.2 to Fredholm operators. To this end denote by T the unbounded linear operator $H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}(\mathcal{D})$ with domain $\mathcal{D}_T = H_{SL}(\mathcal{D})$ which maps an element $u \in \mathcal{D}_T$ to Lu . The operator T is clearly closed because of inequality (3.2). It is densely defined as $H^1(\mathcal{D}, S) \subset H_{SL}(\mathcal{D})$ is dense in $H_{SL}^-(\mathcal{D})$. It is well known that the null space of T is finite dimensional in $H_{SL}(\mathcal{D})$ and its range is closed in $H_{SL}^-(\mathcal{D})$.

When speaking on eigen- and root functions u of the operator T we always assume that $u \in \mathcal{D}_T$ and $(T - \lambda I)^j u \in \mathcal{D}_T$ for all $j = 1, \dots, m - 1$.

Let $T_0 : H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ corresponds to the self-adjoint operator L_0 . The operator T_0 is obviously continuously invertible and the inverse operator coincides with $\iota' \iota L_0^{-1} = Q_1$.

Lemma 3.3. *The spectrum of the operator T_0 consists of the points $\mu_\nu = \lambda_\nu^{-1}$ in $\mathbb{R}_{>0}$, where λ_ν are the eigenvalues of Q_1 .*

Proof. Recall that all λ_ν are positive, which is due to Lemma 3.1, and so $\mu_\nu > 0$. If $\lambda \neq 0$ then

$$(T_0 - \lambda I)u = (I - \lambda \iota' \iota L_0^{-1}) T_0 u = -\lambda \left(Q_1 - \frac{1}{\lambda} I \right) T_0 u$$

for all $u \in H_{SL}(\mathcal{D})$, showing the lemma. \square

If the spectrum of T fails to be the whole complex plane, i.e., if the resolvent $\mathcal{R}(\lambda; T) = (T - \lambda I)^{-1}$ exists for some $\lambda = \lambda_0$, then it follows from the resolvent equation (since $\mathcal{R}(\lambda_0; T)$ is compact) that $\mathcal{R}(\lambda; T)$ exists for all $\lambda \in \mathbb{C}$ except for a discrete sequence of points $\{\lambda_\nu\}$ which are the eigenvalues of T (see [Keld71, p. 17]). In the general case, however, one cannot exclude the situation where the spectrum of T is the whole complex plane.

Theorem 3.4. *Let $\delta L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ be a compact operator and the operator $Q_1 : H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ be of finite order. Then the spectrum of the closed operator $T : H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ corresponding to $L = L_0 + \delta L$, is different from \mathbb{C} and the system of root functions of T is complete in the spaces $H_{SL}^-(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$. Moreover, for any $\varepsilon > 0$, all eigenvalues of T (except for a finite number) belong to the corner $|\arg \lambda| < \varepsilon$.*

Proof. First we note that

$$T - \lambda I = L - \lambda \iota' \iota \quad (3.3)$$

on $H_{SL}(\mathcal{D})$ for all $\lambda \in \mathbb{C}$. Let us prove that there is $N \in \mathbb{N}$, such that $\lambda_0 = -N$ is a resolvent point of T . For this purpose, using (3.3) and Lemma 3.3, we get

$$T + kI = (I + \delta L(L_0 + k \iota' \iota)^{-1})(T_0 + kI) \quad (3.4)$$

for all $k \in \mathbb{N}$.

We will show that the operator $I + \delta L(L_0 + k \iota' \iota)^{-1}$ is injective for some $k \in \mathbb{N}$. Indeed, we argue by contradiction. Suppose for any $k \in \mathbb{N}$ there is $f_k \in H_{SL}^-(\mathcal{D})$, such that $\|f_k\|_{H_{SL}^-(\mathcal{D})} = 1$ and

$$(I + \delta L(L_0 + k \iota' \iota)^{-1})f_k = 0. \quad (3.5)$$

Given any $u \in H_{SL}(\mathcal{D})$ and $k \in \mathbb{N}$, an easy computation shows that

$$\begin{aligned} \|(L_0 + k \iota' \iota)u\|_{H_{SL}^-(\mathcal{D})}^2 &= \|u + k L_0^{-1}u\|_{SL}^2 \\ &= \|u\|_{SL}^2 + 2k \|u\|_{L^2(\mathcal{D})}^2 + k^2 \|L_0^{-1}u\|_{SL}^2 \\ &\geq \|u\|_{SL}^2. \end{aligned}$$

Hence, the sequence $u_k := (L_0 + k \iota' \iota)^{-1}f_k$ is bounded in $H_{SL}(\mathcal{D})$. Now the weak compactness principle for Hilbert spaces yields that there is a subsequence $\{f_{k_j}\}$ with the property that both $\{f_{k_j}\}$ and $\{u_{k_j}\}$ converge weakly in the spaces $H_{SL}^-(\mathcal{D})$ and $H_{SL}(\mathcal{D})$ to limits f and u , respectively. Since δL is compact, it follows that

the sequence $\{\delta L u_{k_j}\}$ converges to $\delta L u$ in $H_{SL}^-(\mathcal{D})$, and so $\{f_{k_j}\}$ converges to f because of (3.5). Obviously,

$$\|f\|_{H_{SL}^-(\mathcal{D})} = 1.$$

In particular, we conclude that the sequence $\{\delta L(L_0 + k_j \iota' \iota)^{-1} f_{k_j}\}$ converges to $-f$ whence

$$f = -\delta L u. \quad (3.6)$$

Further, on passing to the weak limit in the equality $f_{k_j} = (L_0 + k_j \iota' \iota) u_{k_j}$ we obtain

$$f = L_0 u - \lim_{k_j \rightarrow \infty} k_j \iota' \iota u_{k_j},$$

for the continuous operator $L_0 : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ maps weakly convergent sequences to weakly convergent sequences. As the operator $\iota' \iota$ is compact, the sequence $\{\iota' \iota u_{k_j}\}$ converges to $\iota' \iota u$ in the space $H_{SL}^-(\mathcal{D})$ and $\iota' \iota u \neq 0$ which is a consequence of (3.6) and the injectivity of $\iota' \iota$. This shows readily that the weak limit

$$\lim_{k_j \rightarrow \infty} k_j \iota' \iota u_{k_j} = L_0 u - f$$

does not exist, a contradiction.

We have proved more, namely that the operator $I + \delta L(L_0 + k \iota' \iota)^{-1}$ is injective for all but a finitely many natural numbers k . Since this is a Fredholm operator of index zero, it is continuously invertible. Hence, (3.4) and Lemma 3.3 imply that $(T - \lambda_0 I)^{-1}$ exists for some $\lambda_0 = -N$ with $N \in \mathbb{N}$.

As λ_0 is a resolvent point of T ,

$$(T - \lambda_0 I)^{-1} = (L - \lambda_0 \iota' \iota)^{-1}$$

on $H_{SL}^-(\mathcal{D})$. Since $L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is Fredholm and the inclusion ι compact, the operator $L - \lambda_0 \iota' \iota : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is Fredholm. So $(L - \lambda_0 \iota' \iota)^{-1}$ maps $H_{SL}^-(\mathcal{D})$ continuously to $H_{SL}(\mathcal{D})$. Similarly to (3.1) we get

$$L_0^{-1} - (L - \lambda_0 \iota' \iota)^{-1} = L_0^{-1} ((\delta L - \lambda_0 \iota' \iota)(L - \lambda_0 \iota' \iota)^{-1}).$$

Then, Theorem 3.2 yields that the root functions $\{u_\nu\}$ of the operator $(L - \lambda_0 \iota' \iota)^{-1}$ are complete in the spaces $H_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}^-(\mathcal{D})$.

From (3.3) it follows that the systems of root functions related to the operators $(L - \lambda_0 \iota' \iota)^{-1}$ and $T - \lambda_0 I$ coincide.

Finally, as the operators $T - \lambda_0 I$ and T have the same root functions, we conclude that $\mathcal{L}(\{u_\nu\})$ is dense in the spaces $H_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}^-(\mathcal{D})$. \square

The equality $(T - \lambda I)u = 0$ for a function $u \in H_{SL}(\mathcal{D})$ may be equivalently reformulated by saying that u is a solution in a weak sense to the boundary value problem

$$\begin{cases} Au = \lambda u & \text{in } \mathcal{D}, \\ Bu = 0 & \text{on } \partial\mathcal{D}, \end{cases} \quad (3.7)$$

where the pair (A, B) corresponds to the perturbation $L_0 + \delta L$. For $n = 1$ such problems are known as Sturm-Liouville boundary problems for second order ordinary differential equations (see for instance [Hart64, Ch. XI, § 4]). Thus, we may still refer to (3.7) as the Sturm-Liouville problem in many dimensions.

To finish the preliminary discussion we indicate a broad class of compact operators of finite orders acting in spaces of integrable functions. For this purpose, given

any positive non-integer s , we denote by $H^s(\mathcal{D})$ the Sobolev-Slobodetskii space with norm

$$\|u\|_{H^s(\mathcal{D})} = \left(\|u\|_{H^{[s]}(\mathcal{D})}^2 + \iint_{\mathcal{D} \times \mathcal{D}} \sum_{|\alpha|=[s]} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{n+2(s-[s])}} dx dy \right)^{1/2},$$

where $[s]$ is the integer part of s . The following result goes back at least as far as [Agm62].

Theorem 3.5. *Let $s > 0$ and $A : H^s(\mathcal{D}) \rightarrow H^s(\mathcal{D})$ be a compact operator. If there is $\delta s > 0$, such that A maps $H^s(\mathcal{D})$ continuously to $H^{s+\delta s}(\mathcal{D})$, then it belongs to Schatten class $\mathfrak{S}_{n/\delta s+\varepsilon}$ for each $\varepsilon > 0$.*

Proof. For the case $s > 0$ see [Agm62]. For the case $s \in \mathbb{R}$ and Sobolev spaces on a compact closed manifold see Proposition 5.4.1 in [Agr90]. \square

For $s > 0$, let $H^{-s}(\mathcal{D})$ stand for the dual space of $H^s(\mathcal{D})$ with respect to the $L^2(\mathcal{D})$ -pairing, as discussed in Section 2.

Corollary 3.6. *If for some $s > 0$ there is a continuous embedding*

$$\iota_s : H_{SL}(\mathcal{D}) \hookrightarrow H^s(\mathcal{D}) \tag{3.8}$$

then the operators Q_1, Q_2 and Q_3 belong to Schatten class $\mathfrak{S}_{n/s+\varepsilon}$ for any $\varepsilon > 0$ (and so they are of finite order).

Proof. Write $i : H^s(\mathcal{D}) \hookrightarrow L^2(\mathcal{D})$ for the natural inclusion and $i^* : L^2(\mathcal{D}) \hookrightarrow H^s(\mathcal{D})$ for the Hilbert space adjoint of i . By Rellich's theorem, the operator i is compact. The composition ii^* is a bounded linear operator in $L^2(\mathcal{D})$. Moreover, it is compact and maps $L^2(\mathcal{D})$ into $H^s(\mathcal{D})$. By Theorem 3.5, the operator ii^* is of Schatten class $\mathfrak{S}_{n/s+\varepsilon}$ for arbitrary $\varepsilon > 0$. Since the operators ii^* and i^*i have the same eigenvalues, it follows that the operator i^*i in $H^s(\mathcal{D})$ belongs to the same class $\mathfrak{S}_{n/s+\varepsilon}$, too.

The embedding $\iota : H_{SL}(\mathcal{D}) \hookrightarrow L^2(\mathcal{D})$ is assumed to factor through i , i.e., $\iota = i \circ \iota_s$ whence $\iota^* = \iota_s^* i^*$. Since i_s^* is continuous, we conclude by Lemma 1.3 that the operator $\iota^* \iota = \iota_s^* (i^* i) \iota_s$ belongs to $\mathfrak{S}_{n/s+\varepsilon}$. Arguing as above we see that the operator $\iota \iota^*$ lies in $\mathfrak{S}_{n/s+\varepsilon}$ for any $\varepsilon > 0$.

From the definition of the sesquilinear form (\cdot, \cdot) on $H_{SL}(\mathcal{D}) \times H_{SL}^-(\mathcal{D})$ and formula (2.11) it follows that

$$(v, \iota' u) = (\iota v, u)_{L^2(\mathcal{D})} = (v, \iota^* u)_{SL} = (v, L_0 \iota^* u)$$

for all $v \in H_{SL}(\mathcal{D})$ and $u \in L^2(\mathcal{D})$. Hence

$$\begin{aligned} \iota' &= L_0 \iota^*, \\ \iota' \iota &= L_0 \iota^* \iota, \end{aligned} \tag{3.9}$$

and so

$$\begin{aligned} Q_1 &= \iota' \iota L_0^{-1} = L_0 \iota^* \iota L_0^{-1}, \\ Q_2 &= \iota L_0^{-1} \iota' = \iota \iota^*, \\ Q_3 &= L_0^{-1} \iota' \iota = \iota^* \iota. \end{aligned}$$

By Lemma 1.3, the operator Q_1 lies in the class $\mathfrak{S}_{n/s+\varepsilon}$. The same is obviously true for Q_2 and Q_3 . \square

Corollary 3.7. *Suppose there is a continuous embedding (3.8) with some $s > 0$. Then any compact operator $R : H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ which maps $H_{SL}^-(\mathcal{D})$ continuously to $H_{SL}(\mathcal{D})$ is of Schatten class $\mathfrak{S}_{n/s+\varepsilon}$ for any $\varepsilon > 0$. In particular, its order is finite.*

Proof. Let R_0 be the operator R which is thought of as a bounded map of $H_{SL}^-(\mathcal{D})$ to $H_{SL}(\mathcal{D})$. Then $R = \iota' \iota R_0$. Since $\iota^* \iota \in \mathfrak{S}_{n/s+\varepsilon}$ for all $\varepsilon > 0$, it follows from (3.9) and Lemma 1.3 that

$$R = L_0 (\iota^* \iota) R_0$$

is of Schatten class $\mathfrak{S}_{n/s+\varepsilon}$, as desired. \square

Now we want to study the completeness of root functions of “small” perturbations of compact self-adjoint operators instead of the weak ones. To this end we apply the so-called method of rays of minimal growth of resolvent which leads to more general results than Theorem 1.4. This idea seems to go back at least as far as [Agm62].

4. RAYS OF MINIMAL GROWTH

We first describe briefly the method of minimal growth rays following [DS63] and Theorem 6.1 of [GK69, p. 302].

Let $L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ be the bounded linear operator constructed in Section 2. We still assume that estimates (2.6) and (2.10) hold and that the operator L is Fredholm. In the sequel we confine ourselves to those Sturm-Liouville problems for which the spectrum of the corresponding unbounded closed operator $T : H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is discrete, cf. [Agm62]. We denote by $\mathcal{R}(\lambda; T)$ the resolvent of the operator T .

Definition 4.1. A ray $\arg \lambda = \vartheta$ in the complex plane \mathbb{C} is called a ray of minimal growth of the resolvent $\mathcal{R}(\lambda; T) : H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ if the resolvent exists for all λ of sufficiently large modulus on this ray, and if, moreover, for all such λ an estimate

$$\|\mathcal{R}(\lambda; T)\|_{\mathcal{L}(H_{SL}^-(\mathcal{D}))} \leq c |\lambda|^{-1} \quad (4.1)$$

holds with a constant $C > 0$.

Theorem 4.2. *Let the space $H_{SL}(\mathcal{D})$ be continuously embedded into $H^s(\mathcal{D})$ for some $s > 0$. Suppose there are rays of minimal growth of the resolvent $\arg \lambda = \vartheta_j$, where $j = 1, \dots, J$, in the complex plane, such that the angles between any two neighbouring rays are less than $\pi s/n$. Then the spectrum of the operator T is discrete and the root functions form a complete system in the spaces $H_{SL}^-(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$.*

Proof. The proof actually follows by the same method as that in Theorem 3.2 of [Agm62].

Since the spectrum of the operator T is different from the whole complex plane it is actually discrete. It remains to show that if $g \in H_{SL}^-(\mathcal{D})$ is orthogonal to all eigen- and associated functions of the operator T then g is identically zero. By the Hahn-Banach theorem, this implies that the root functions of T are complete in $H_{SL}^-(\mathcal{D})$.

Since the operators T and $T - \lambda_0 I$ have the same root functions, we may assume without loss of generality that the origin is not in the spectrum of T . Choosing $\lambda_0 = 0$ in $R = (T - \lambda_0 I)^{-1}$, we set $R = T^{-1}$.

Consider now the function

$$f(\lambda) = \left(\mathcal{R}\left(\frac{1}{\lambda}; R\right)u, g \right)_{H_{SL}^-(\mathcal{D})}, \quad (4.2)$$

where $u \in H_{SL}^-(\mathcal{D})$ and $(\cdot, \cdot)_{H_{SL}^-(\mathcal{D})}$ stands for the scalar product in $H_{SL}^-(\mathcal{D})$. Since the resolvent of R is a meromorphic function with poles at the points of the spectrum of R , the function f is analytic for $\lambda \neq \lambda_\nu$, where $\{\lambda_\nu\}$ is the sequence of eigenvalues of $R^{-1} = T$. We shall use a familiar relation between the resolvents of the operators T and T^{-1} , namely

$$\mathcal{R}\left(\frac{1}{\lambda}; T^{-1}\right) = -\lambda I - \lambda^2 \mathcal{R}(\lambda; T). \quad (4.3)$$

Consider the expansion

$$\mathcal{R}(\lambda; T)u = \frac{f_{-N}}{(\lambda - \lambda_\nu)^N} + \frac{f_{-N+1}}{(\lambda - \lambda_\nu)^{N-1}} + \dots + \frac{f_{-1}}{\lambda - \lambda_\nu} + \sum_{k=0}^{\infty} f_k (\lambda - \lambda_\nu)^k,$$

in a neighbourhood of the point $\lambda = \lambda_\nu$, where λ_ν is a pole of $\mathcal{R}(\lambda; T)$. Here $N \geq 1$ and $f_{-N} \neq 0$, the functions $f_{-N}, \dots, f_{-1} \in H_{SL}^-(\mathcal{D})$ form a chain of associated functions of T , and $f_k \in H_{SL}^-(\mathcal{D})$ for $k \geq 0$. This expansion implies that λ_ν is a regular point of $f(\lambda)$, for g is orthogonal to all f_{-N}, \dots, f_{-1} . Therefore, $f(\lambda)$ is an entire function.

Relations (4.1), (4.2) and (4.3) imply that f is of exponential type, i.e., there is a constant $c > 0$, such that

$$|f(\lambda)| \leq c \exp |\lambda| \quad (4.4)$$

for $|\lambda| \rightarrow \infty$, provided that $\arg \lambda = \vartheta_j$ for some $j = 1, \dots, J$. We use the following lemma taken from [DS63].

Lemma 4.3. *Assume that R is a compact linear operator of Schatten class \mathfrak{S}_p , with $0 < p < \infty$, in a Hilbert space H . Then there exists a sequence ρ_j satisfying $\rho_j \rightarrow 0$, such that*

$$\|\mathcal{R}(\lambda; R)\|_{\mathcal{L}(H)} \leq \text{const} \exp(c|\lambda|^{-p})$$

for $|\lambda| = \rho_j$.

According to Corollary 3.7, the operator R belongs to $\mathfrak{S}_{n/s+\varepsilon}$ for any $\varepsilon > 0$. Then it follows from Lemma 4.3 that for any $\varepsilon > 0$ there exists a sequence $\rho_j \rightarrow 0$, such that

$$|f(\lambda)| \leq \exp\left(|\lambda|^{-\frac{n}{s}-\varepsilon}\right) \quad (4.5)$$

for all $\lambda \in \mathbb{C}$ satisfying $|\lambda| = 1/\rho_j$.

Consider $f(\lambda)$ in the closed corner between the rays $\arg \lambda = \vartheta_j$ and $\arg \lambda = \vartheta_{j+1}$. Its angle is less than $\pi s/n$. Since

$$\mathcal{R}\left(\frac{1}{\lambda}; R\right) = -\lambda I - \lambda^2 \mathcal{R}(\lambda; T)$$

and each ray $\arg \lambda = \vartheta_j$ is a ray of minimal growth, inequality (4.4) is fulfilled on the sides of the corner and inequality (4.5) on a sequence of arcs which tends to infinity.

Choosing $\varepsilon > 0$ in (4.5) sufficiently small and applying the Frøman-Lindelöf theorem we conclude that $|f(\lambda)| = O(|\lambda|)$ as $|\lambda| \rightarrow \infty$ in the whole complex plane.

Therefore, $f(\lambda)$ is an affine function, i.e., $f(\lambda) = a_0 + c_1\lambda$. On the other hand, we have

$$\mathcal{R}\left(\frac{1}{\lambda}; R\right) = -\lambda I - \lambda^2 R + \dots,$$

and so

$$f(\lambda) = -\lambda(u, g)_{H_{SL}^-(\mathcal{D})} - \lambda^2 (Ru, g)_{H_{SL}^-(\mathcal{D})} + \dots$$

Since $f(\lambda)$ is affine, we get

$$(Ru, g)_{H_{SL}^-(\mathcal{D})} = 0$$

for all $u \in H_{SL}^-(\mathcal{D})$. Hence it follows that $g = 0$, for the range of the operator R is dense in $H_{SL}^-(\mathcal{D})$. Thus, the system of root functions of the operator T is complete in $H_{SL}^-(\mathcal{D})$.

As the operators T , $(T - \lambda_0 I)$ and $(T - \lambda_0 I)^{-1}$ have the same root functions, it suffices to repeat the arguments of the proof of Theorem 3.2 to see the completeness in the spaces $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$. \square

This theorem raises the question under what conditions neighbouring rays of minimal growth are close enough. We now indicate some conditions for a ray $\arg \lambda = \vartheta$ in the complex plane to be a ray of minimal growth for the resolvent of T .

Lemma 4.4. *Each ray $\arg \lambda = \vartheta$ with $\vartheta \neq 0$ is a ray of minimal growth for $\mathcal{R}(\lambda; T_0)$ and*

$$\|(T_0 - \lambda I)^{-1}\|_{\mathcal{L}(H_{SL}^-(\mathcal{D}))} \leq \begin{cases} (|\lambda| |\sin(\arg \lambda)|)^{-1}, & \text{if } |\arg \lambda| \in (0, \pi/2), \\ |\lambda|^{-1}, & \text{if } |\arg \lambda| \in [\pi/2, \pi]. \end{cases} \quad (4.6)$$

Moreover, the operator $L_0 - \lambda \iota' \iota : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is continuously invertible and

$$\|(L_0 - \lambda \iota' \iota)^{-1}\|_{\mathcal{L}(H_{SL}^-(\mathcal{D}), H_{SL}(\mathcal{D}))} \leq \begin{cases} |\sin(\arg \lambda)|, & \text{if } |\arg \lambda| \in (0, \pi/2), \\ 1, & \text{if } |\arg \lambda| \in [\pi/2, \pi]. \end{cases} \quad (4.7)$$

Proof. According to Lemma 3.3 the resolvent

$$(T_0 - \lambda I)^{-1} : H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$$

exists for all $\lambda \in \mathbb{C}$ away from the positive real axis. As the operator $Q_3 = L_0^{-1} \iota' \iota$ is self-adjoint, the operator T_0 is symmetric, i.e.,

$$\begin{aligned} (T_0 u, g)_{H_{SL}^-(\mathcal{D})} &= (L_0 u, g)_{H_{SL}^-(\mathcal{D})} \\ &= (u, Q_3^{-1} g)_{SL} \\ &= (Q_3^{-1} u, g)_{SL} \\ &= (u, L_0 g)_{H_{SL}^-(\mathcal{D})} \\ &= (u, T_0 g)_{H_{SL}^-(\mathcal{D})} \end{aligned}$$

for all $u, v \in H_{SL}(\mathcal{D})$. If $|\arg(\lambda)| \in (0, \pi/2)$, then

$$\begin{aligned} \|(T_0 - \lambda I)u\|_{H_{SL}^-(\mathcal{D})}^2 &= \|(T_0 - \Re \lambda I)u\|_{H_{SL}^-(\mathcal{D})}^2 + |\Im \lambda|^2 \|u\|_{H_{SL}^-(\mathcal{D})}^2 \\ &\geq |\lambda|^2 |\sin(\arg \lambda)|^2 \|u\|_{H_{SL}^-(\mathcal{D})}^2 \end{aligned}$$

for all $u \in H_{SL}(\mathcal{D})$, which establishes the first estimate of (4.6). If $|\arg \lambda| \in [\pi/2, \pi]$, then $\Re \lambda \leq 0$ whence

$$\|(T_0 - \lambda I)u\|_{H_{SL}^-(\mathcal{D})}^2 \geq |\lambda|^2 \|u\|_{H_{SL}^-(\mathcal{D})}^2$$

and so the second estimate of (4.6) holds.

Now it follows from (3.3) that the operator $L_0 - \lambda \iota' \iota$ is injective for $\lambda \in \mathbb{C}$ away from the positive real axis. As this operator is Fredholm and its index is zero, it is continuously invertible. Finally, as the operator $Q_3 = L_0^{-1} \iota' \iota$ is positive, we deduce readily that

$$\begin{aligned} \|(L_0 - \lambda \iota' \iota)u\|_{H_{SL}^-(\mathcal{D})} &= \|(I - \lambda L_0^{-1} \iota' \iota)u\|_{SL} \\ &\geq |\lambda| |\Im \lambda^{-1}| \|u\|_{SL} \\ &= |\sin(\arg \lambda)| \|u\|_{SL}, \end{aligned}$$

if $|\arg \lambda| \in (0, \pi/2)$, i.e., the second estimate of (4.7) is fulfilled. Similar arguments lead to the second estimate of (4.7). \square

Theorem 4.5. *Let the space $H_{SL}(\mathcal{D})$ be continuously embedded into $H^s(\mathcal{D})$ for some $s > 0$ and estimate (2.10) be fulfilled with a constant $c < |\sin(\pi s/2n)|$. Then all eigenvalues of the closed operator $T : H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ belong to the corner $|\arg \lambda| \leq \arcsin c$, each ray $\arg \lambda = \vartheta$ with $|\vartheta| > \arcsin c$ is a ray of minimal growth for $\mathcal{R}(\lambda; T)$ and the system of root functions is complete in the spaces $H_{SL}^-(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$.*

Proof. First we note that, by Lemma 2.5, the operator $L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is invertible. Indeed, $L = L_0 + \delta L$ where $\delta L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is a bounded operator with the norm $\|\delta L\|_{\mathcal{L}(H_{SL}(\mathcal{D}), H_{SL}^-(\mathcal{D}))} < 1 = \|L_0^{-1}\|^{-1}$. In particular, by (3.3), the spectrum of the corresponding operator T does not coincide with the whole complex plane.

Fix $\vartheta \neq 0$ and set $m_\vartheta = |\sin \vartheta|$, if $|\vartheta| \in (0, \pi/2)$, and $m_\vartheta = 1$, if $|\vartheta| \in [\pi/2, \pi]$. If $m_\vartheta > c$ then

$$\|\delta L\|_{\mathcal{L}(H_{SL}(\mathcal{D}), H_{SL}^-(\mathcal{D}))} \leq c < m_\vartheta \leq \|(L_0 - \lambda \iota' \iota)^{-1}\|_{\mathcal{L}(H_{SL}(\mathcal{D}), H_{SL}^-(\mathcal{D}))}^{-1}.$$

Hence it follows that the operator $L - \lambda \iota' \iota : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is continuously invertible and

$$\|(L - \lambda \iota' \iota)^{-1}\|_{\mathcal{L}(H_{SL}^-(\mathcal{D}), H_{SL}(\mathcal{D}))} \leq (m_\vartheta - c)^{-1}. \quad (4.8)$$

In order to establish estimate (4.1) we have to show that there is a constant $C > 0$, such that

$$C |\lambda|^{-1} \|(T - \lambda I)u\|_{H_{SL}^-(\mathcal{D})} \geq \|u\|_{H_{SL}^-(\mathcal{D})}$$

for all $u \in H_{SL}(\mathcal{D})$.

If $\arg \lambda = \vartheta$ with $m_\vartheta > c$, then, by (3.3), we get

$$\begin{aligned} \|(T - \lambda I)u\|_{H_{SL}^-(\mathcal{D})} &= \|(L - \lambda \iota' \iota)u\|_{H_{SL}^-(\mathcal{D})} \\ &\geq (m_\vartheta - c) \|u\|_{SL} \\ &\geq (m_\vartheta - c) \|u\|_{H_{SL}^-(\mathcal{D})} \end{aligned}$$

for all $u \in H_{SL}(\mathcal{D})$. Therefore, given any λ on the ray $\arg \lambda = \vartheta$ with $m_\vartheta > c$, it follows that

1) The range of the operator $T - \lambda I : H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is a closed subspace of $H_{SL}^-(\mathcal{D})$.

2) The null space of the operator $T - \lambda I : H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is trivial.

By (3.3) the range of $T - \lambda I$ coincides with the range of $L - \lambda \iota' \iota$ which is the whole space $H_{SL}^-(\mathcal{D})$. Hence, the resolvent $(T - \lambda I)^{-1}$ exists for all λ away from

the corner $|\arg \lambda| \leq \arcsin c$ in the complex plane. On applying (3.3) and Lemma 4.4 we obtain

$$T - \lambda I = L_0 + \delta L - \lambda \iota' \iota = (I + \delta L(L_0 - \lambda \iota' \iota)^{-1})(T_0 - \lambda I) \quad (4.9)$$

on $H_{SL}(\mathcal{D})$ and

$$\begin{aligned} & \|(I + \delta L(L_0 - \lambda \iota' \iota)^{-1})u\|_{H_{SL}^-(\mathcal{D})} \\ & \geq \|u\|_{H_{SL}^-(\mathcal{D})} - \|\delta L\|_{\mathcal{L}(H_{SL}(\mathcal{D}), (H_{SL}^-(\mathcal{D})))} \|(L_0 - \lambda \iota' \iota)^{-1}u\|_{H_{SL}(\mathcal{D})} \\ & \geq (1 - c/m_\vartheta) \|u\|_{H_{SL}^-(\mathcal{D})}. \end{aligned}$$

Therefore the operator $I + \delta L(L_0 - \lambda \iota' \iota)^{-1}$ is continuously invertible as Fredholm operator of zero index and trivial null space. Moreover,

$$\|(I + \delta L(L_0 - \lambda \iota' \iota)^{-1})^{-1}\|_{\mathcal{L}(H_{SL}^-(\mathcal{D}))} \leq (1 - c/m_\vartheta)^{-1}.$$

Now (4.9) implies

$$\begin{aligned} & \|(T - \lambda I)^{-1}\|_{\mathcal{L}(H_{SL}^-(\mathcal{D}))} \\ & \leq \|(I + \delta L(L_0 - \lambda \iota' \iota)^{-1})^{-1}\|_{\mathcal{L}(H_{SL}^-(\mathcal{D}))} \|(T_0 - \lambda I)^{-1}\|_{\mathcal{L}(H_{SL}^-(\mathcal{D}))} \quad (4.10) \\ & \leq (1 - c/m_\vartheta)^{-1} m_\vartheta^{-1} |\lambda|^{-1} \quad (4.11) \end{aligned}$$

for all λ satisfying $\arg \lambda = \vartheta$ with $m_\vartheta > c$.

Thus, all rays outside of the corner $|\arg \lambda| \leq \arcsin c$ are rays of minimal growth. By the hypothesis of the theorem, the angles between the pairs of neighbouring rays $\arg \lambda = \vartheta$, are less than $\pi s/n$, and so the completeness of root functions follows from Theorem 4.2. \square

We are now in a position to prove the main result of this section. When compared with [Agr11c] our contribution consists in developing dual function spaces which fit the problem.

Theorem 4.6. *Let the space $H_{SL}(\mathcal{D})$ be continuously embedded into $H^s(\mathcal{D})$ for some $s > 0$, $\delta L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ be a bounded linear operator whose norm is less than $|\sin(\pi s/2n)|$, and $C : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ be compact. Then the following is true:*

- 1) *The spectrum of the operator T in $H_{SL}^-(\mathcal{D})$ corresponding to $L_0 + \delta L + C$ is discrete.*
- 2) *For any $\varepsilon > 0$, all eigenvalues of the operator T (except for a finite number) belong to the corner $|\arg \lambda| < \arcsin \|\delta L\| + \varepsilon$.*
- 3) *Each ray $\arg \lambda = \vartheta$ with*

$$|\vartheta| > \arcsin \|\delta L\| \quad (4.12)$$

is a ray of minimal growth for $\mathcal{R}(\lambda; T)$.

- 4) *The system of root functions is complete in the spaces $H_{SL}^-(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$.*

Proof. First we note that the operator $L_0 + \delta L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is continuously invertible and hence the operator $L_0 + \delta L + C : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is actually Fredholm.

Theorem 4.5 implies that all rays satisfying (4.12) are rays of minimal growth for $\mathcal{R}(\lambda; T_0 + \delta T)$ with the closed operator $T_0 + \delta T$ in $H_{SL}^-(\mathcal{D})$ corresponding to $L_0 + \delta L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$.

Fix an arbitrary $\varepsilon > 0$. Then estimates (4.8) and (4.10) imply that there are constants c_1 and c_2 depending on ε , such that

$$\|(L_0 + \delta L - \lambda \iota')^{-1}\|_{\mathcal{L}(H_{SL}^-(\mathcal{D}), H_{SL}(\mathcal{D}))} \leq c_1, \quad (4.13)$$

$$\|(T_0 + \delta T - \lambda I)^{-1}\|_{\mathcal{L}(H_{SL}^-(\mathcal{D}))} \leq c_2 |\lambda|^{-1} \quad (4.14)$$

for all λ satisfying

$$|\arg \lambda| \geq \arcsin \|\delta L\|_{\mathcal{L}(H_{SL}(\mathcal{D}), H_{SL}^-(\mathcal{D}))} + \varepsilon. \quad (4.15)$$

Then, using (3.3), (4.13) and Theorem 4.5 we obtain

$$T - \lambda I = (I + C(L_0 + \delta L - \lambda \iota')^{-1})(T_0 + \delta T - \lambda I) \quad (4.16)$$

on $H_{SL}(\mathcal{D})$ for all rays satisfying (4.15).

We now prove that there is a constant $M_\varepsilon > 0$ depending on ε , such that the operator $I + C(L_0 + \delta L - \lambda \iota')^{-1}$ is injective for all λ satisfying both (4.15) and $|\lambda| \geq M_\varepsilon$. To do this, we argue by contradiction in the same way as in the proof of Theorem 3.4. Suppose for each natural number k there are $f_k \in H_{SL}^-(\mathcal{D})$, satisfying $\|f_k\|_{H_{SL}^-(\mathcal{D})} = 1$, and λ_k , satisfying (4.15) and $|\lambda_k| \geq k$, such that

$$(I + C(L_0 + \delta L - \lambda_k \iota')^{-1}) f_k = 0. \quad (4.17)$$

It follows from (4.13) that the sequence $u_k = (L_0 + \delta L - \lambda_k \iota')^{-1} f_k$ is bounded in $H_{SL}(\mathcal{D})$. By the weak compactness principle for Hilbert spaces one can assume without restriction of generality that the sequences $\{f_k\}$ and $\{u_k\}$ converge weakly in the spaces $H_{SL}^-(\mathcal{D})$ and $H_{SL}(\mathcal{D})$ to functions f and u , respectively. Since C is compact, it follows that the sequence $\{C u_k\}$ converges to $C u$ in $H_{SL}^-(\mathcal{D})$ and so $\{f_k\}$ converges to f , which is due to (4.17). Obviously, the $H_{SL}^-(\mathcal{D})$ -norm of f just amounts to 1. In particular, we conclude that $\{C(L_0 + \delta L - \lambda_k \iota')^{-1} f_k\}$ converges to $-f$ whence

$$f = -C u. \quad (4.18)$$

Further, as $f_k = (L_0 + \delta L - \lambda_k \iota') u_k$, letting $k \rightarrow \infty$ in this formula yields readily

$$f = (L_0 + \delta L) u - \lim_{k \rightarrow \infty} \lambda_k \iota' u_k.$$

As the operator ι' is compact, the sequence $\{\iota' u_k\}$ converges to $\iota' u$ in the space $H_{SL}^-(\mathcal{D})$, and $\iota' u \neq 0$ because of (4.18) and the injectivity of ι' . Therefore, the weak limit

$$\lim_{k \rightarrow \infty} \lambda_k \iota' u_k = (L_0 + \delta L) u - f$$

fails to exist, for $\{\lambda_k\}$ is unbounded. A contradiction.

As the operator $I + C(L_0 + \delta L - \lambda \iota')^{-1}$ is Fredholm and it has index zero, this operator is continuously invertible for all $\lambda \in \mathbb{C}$ satisfying both (4.15) and $|\lambda| \geq M_\varepsilon$. Set

$$N_\varepsilon = \inf \|(I + C(L_0 + \delta L - \lambda \iota')^{-1}) f\|_{H_{SL}^-(\mathcal{D})} \geq 0,$$

the infimum being over all $f \in H_{SL}^-(\mathcal{D})$ of norm 1 and all $\lambda \in \mathbb{C}$ satisfying (4.15) and $|\lambda| \geq M_\varepsilon$. We claim that $N_\varepsilon > 0$. To show this, we argue by contradiction. If $N_\varepsilon = 0$ then there are sequences $\{f_k\}$ in $H_{SL}^-(\mathcal{D})$, each f_k being of norm 1, and $\{\lambda_k\}$ satisfying (4.15) and $|\lambda_k| \geq M_\varepsilon$, such that

$$\lim_{k \rightarrow \infty} \|(I + C(L_0 + \delta L - \lambda_k \iota')^{-1}) f_k\|_{H_{SL}^-(\mathcal{D})} = 0. \quad (4.19)$$

Again, by (4.13), the sequence $u_k = (L_0 + \delta L - \lambda_k \iota' \iota)^{-1} f_k$ is bounded in $H_{SL}(\mathcal{D})$. By the weak compactness principle for Hilbert spaces we may assume that the sequences $\{f_k\}$ and $\{u_k\}$ are weakly convergent in the spaces $H_{SL}^-(\mathcal{D})$ and $H_{SL}(\mathcal{D})$ to functions f and u , respectively. Since C is compact, the sequence $\{Cu_k\}$ converges to Cu in $H_{SL}^-(\mathcal{D})$ and so $\{f_k\}$ converges to f because of (4.19); obviously, $\|f\|_{H_{SL}^-(\mathcal{D})} = 1$. In particular, we deduce that the sequence $C(L_0 + \delta L - \lambda_k \iota' \iota)^{-1} f_k$ converges to $-f$ whence

$$f = -Cu \quad (4.20)$$

with $u \neq 0$.

If the sequence $\{\lambda_k\}$ is bounded in \mathbb{C} , then using the weak compactness principle and passing to a subsequence, if necessary, we may assume that $\{\lambda_k\}$ converges to $\lambda_0 \in \mathbb{C}$ which satisfies (4.15) and $|\lambda| \geq M_\varepsilon$. Since

$$\begin{aligned} & (L_0 + \delta L - \lambda_k \iota' \iota)^{-1} f_k - (L_0 + \delta L - \lambda_0 \iota' \iota)^{-1} f \\ &= (L_0 + \delta L - \lambda_j \iota' \iota)^{-1} (f_k - f) + ((L_0 + \delta L - \lambda_k \iota' \iota)^{-1} - (L_0 + \delta L - \lambda_0 \iota' \iota)^{-1}) f \end{aligned}$$

and

$$\begin{aligned} & \|((L_0 + \delta L - \lambda_k \iota' \iota)^{-1} - (L_0 + \delta L - \lambda_0 \iota' \iota)^{-1}) f\|_{H_{SL}^-(\mathcal{D})} \\ & \leq |\lambda_k - \lambda_0| \|(L_0 + \delta L - \lambda_k \iota' \iota)^{-1}\| \|(L_0 + \delta L - \lambda_0 \iota' \iota)^{-1}\| \|f\|_{H_{SL}^-(\mathcal{D})}, \end{aligned}$$

estimate (4.13) implies that in this case the sequence $\{(L_0 + \delta L - \lambda_k \iota' \iota)^{-1} f_k\}$ converges to $(L_0 + \delta L - \lambda_0 \iota' \iota)^{-1} f$, and so

$$(I + C(L_0 + \delta L - \lambda_0 \iota' \iota)^{-1}) f = 0$$

because of (4.19). But λ_0 satisfies (4.15) and $|\lambda| \geq M_\varepsilon$, and hence the injectivity of the operator $I + C(L_0 + \delta L - \lambda_0 \iota' \iota)^{-1}$ established above yields $f = 0$. This contradicts $\|f\| = 1$.

If $\{\lambda_k\}$ is unbounded in \mathbb{C} we can repeat the arguments above. Indeed, then $f_k = (L_0 + \delta L - \lambda_k \iota' \iota) u_k$ and on passing to the weak limit with respect to $k \rightarrow \infty$ we get

$$f = (L_0 + \delta L)u - \lim_{k \rightarrow \infty} \lambda_k \iota' \iota u_k.$$

As the operator $\iota' \iota$ is compact, the sequence $\{\iota' \iota u_k\}$ converges to $\iota' \iota u$ in the space $H_{SL}^-(\mathcal{D})$. Moreover, $\iota' \iota u \neq 0$ because of (4.20) and the injectivity of $\iota' \iota$. This shows that the weak limit

$$\lim_{n \rightarrow \infty} \lambda_k \iota' \iota u_k = (L_0 + \delta L)u - f$$

fails to exist if $\{\lambda_k\}$ is unbounded in \mathbb{C} , a contradiction. Therefore, $N_\varepsilon > 0$ and for all $\lambda \in \mathbb{C}$ satisfying (4.15) and $|\lambda| \geq M_\varepsilon$ we obtain

$$\|(I + C(L_0 + \delta L - \lambda \iota' \iota)^{-1})^{-1}\|_{\mathcal{L}(H_{SL}^-(\mathcal{D}))} \leq 1/N_\varepsilon. \quad (4.21)$$

From estimates (4.13), (4.21) and formula (4.16) it follows that, given any $\lambda \in \mathbb{C}$ satisfying (4.15) and $|\lambda| \geq M_\varepsilon$, the resolvent $\mathcal{R}(\lambda; T)$ exists and

$$\|\mathcal{R}(\lambda; T)\|_{\mathcal{L}(H_{SL}^-(\mathcal{D}))} \leq \text{const}(\varepsilon) |\lambda|^{-1}.$$

As C is compact, there are only finitely many $\lambda \in \mathbb{C}$ with $|\lambda| < N_\varepsilon$, such that the operator $(I + C(L_0 - \lambda \iota' \iota))$ is not injective. Therefore, it follows from formula (4.16) that all eigenvalues of the operator T corresponding to $L_0 + \delta L + C$ (except for a finite number) belong to the corner $|\arg \lambda| < \arcsin \|\delta L\| + \varepsilon$. Finally, since $\varepsilon > 0$ is arbitrary, all rays (4.12) are rays of minimal growth. By the hypothesis of

the theorem, the angles between the pairs of neighbouring rays $\arg \lambda = \vartheta$ satisfying (4.12) are less than $\pi s/n$, and so the statement of the theorem follows from Theorem 4.2. \square

5. A NON-COERCIVE EXAMPLE

To the best of our knowledge the completeness of root functions has been studied for elliptic boundary value problems, i.e., for those satisfying the Shapiro-Lopatinskii condition. In this section we consider an example where the Shapiro-Lopatinskii condition is violated.

Let the complex structure in $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ be given by $z_j = x_j + \sqrt{-1}x_{n+j}$ with $j = 1, \dots, n$ and $\bar{\partial}$ stand for the Cauchy-Riemann operator corresponding to this structure, i.e., the column of n complex derivatives

$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + \sqrt{-1} \frac{\partial}{\partial x_{n+j}} \right).$$

The formal adjoint $\bar{\partial}^*$ of $\bar{\partial}$ with respect to the usual Hermitian structure in the space $L^2(\mathbb{C}^n)$ is the line of n operators

$$-\frac{1}{2} \left(\frac{\partial}{\partial x_j} - \sqrt{-1} \frac{\partial}{\partial x_{n+j}} \right) =: -\frac{\partial}{\partial z_j}.$$

An easy computation shows that $\bar{\partial}^* \bar{\partial}$ just amounts to the $-1/4$ multiple of the Laplace operator

$$\Delta_{2n} = \sum_{j=1}^{2n} \left(\frac{\partial}{\partial x_j} \right)^2$$

in \mathbb{R}^{2n} .

As A we take

$$A = -\Delta_{2n} + \sum_{j=1}^n a_j(z) \frac{\partial}{\partial \bar{z}_j} + a_0(z),$$

where $a_1(z), \dots, a_n(z)$ and $a_0(z)$ are bounded functions in \mathcal{D} , \mathcal{D} being a bounded domain with Lipschitz boundary in \mathbb{C}^n . The matrix

$$(a_{i,j})_{\substack{i=1,\dots,2n \\ j=1,\dots,2n}}$$

is chosen to be

$$\begin{pmatrix} E_n & \sqrt{-1}E_n \\ -\sqrt{-1}E_n & E_n \end{pmatrix}$$

where E_n is the unity $(n \times n)$ -matrix. Obviously, $\overline{a_{i,j}} = a_{j,i}$ for all $i, j = 1, \dots, 2n$ and the corresponding conormal derivative is

$$\partial_c = 2 \sum_{j=1}^n (\nu_j - \sqrt{-1}\nu_{n+j}) \frac{\partial}{\partial \bar{z}_j} = \frac{\partial}{\partial \nu} + \sqrt{-1} \sum_{j=1}^n \left(\nu_j \frac{\partial}{\partial x_{n+j}} - \nu_{n+j} \frac{\partial}{\partial x_j} \right),$$

which is known as complex normal derivative at the boundary of \mathcal{D} .

Consider the following boundary value problem. Given a distribution f in \mathcal{D} , find a distribution u in \mathcal{D} satisfying

$$\begin{cases} \left(-\Delta_{2n} + \sum_{j=1}^n a_j(z) \frac{\partial}{\partial \bar{z}_j} + a_0(z) \right) u = f & \text{in } \mathcal{D}, \\ \partial_c u + b_0(z)u = 0 & \text{on } \partial \mathcal{D}. \end{cases} \quad (5.1)$$

In this case S is empty and $b_1 = 1$. Set $a_{0,0}(z) \equiv 1$ in \mathcal{D} and $b_{0,0}$ to be any positive constant function on the boundary. Then the corresponding Hermitian form $(\cdot, \cdot)_{SL}$ is

$$(u, v)_{SL} = (u, v)_{L^2(\mathcal{D})} + (2\bar{\partial}u, 2\bar{\partial}v)_{L^2(\mathcal{D})} + b_{0,0} (u, v)_{L^2(\partial\mathcal{D})}$$

and the space $H_{SL}(\mathcal{D})$ is defined to be the completion of $H^1(\mathcal{D})$ with respect to the norm $\|u\|_{SL} = \sqrt{(u, u)_{SL}}$.

Lemma 5.1. *The inclusion*

$$\iota : H_{SL}(\mathcal{D}) \rightarrow L^2(\mathcal{D})$$

is continuous and compact. More exactly, there are continuous embeddings

$$\begin{aligned} H^1(\mathcal{D}) &\hookrightarrow H_{SL}(\mathcal{D}) \hookrightarrow H^{1/2}(\mathcal{D}), \\ H^{-1/2}(\mathcal{D}) &\hookrightarrow H_{\overline{SL}}(\mathcal{D}) \hookrightarrow H^{-1}(\mathcal{D}). \end{aligned} \quad (5.2)$$

Proof. The lemma is a consequence of results on the Dirichlet problem for the Laplace operator in the scale of spaces $H^s(\mathcal{D})$, where $s \in \mathbb{R}$.

For strongly pseudoconvex domains $\mathcal{D} \subset \mathbb{C}^n$ the proof can be given within the framework of complex analysis. Indeed, it follows from [Str84] that

$$\|u\|_{L^2(\partial\mathcal{D})} \geq c \|u\|_{H^{1/2}(\mathcal{D})}$$

for all harmonic functions in \mathcal{D} of class $H^{1/2}(\mathcal{D})$, with c a constant independent of u . If \mathcal{D} is pseudoconvex then there is a constant $c > 0$, such that

$$\|\bar{\partial}u\|_{L^2(\mathcal{D})} \geq c \|u\|_{H^{1/2}(\mathcal{D})}$$

for all $u \in L^2(\mathcal{D})$ orthogonal to the subspace of $L^2(\mathcal{D})$ consisting of holomorphic functions in \mathcal{D} (see [Kohn79]). Since any holomorphic function is harmonic we conclude readily that there are continuous embeddings of the first line in (5.2). The second line of embeddings follows by familiar duality arguments, thus showing the lemma. \square

Remark 5.2. There is no continuous embedding $H_{SL}(\mathcal{D}) \hookrightarrow H^{1/2+\varepsilon}(\mathcal{D})$ with any $\varepsilon > 0$. This means that the Shapiro-Lopatinskii condition fails to hold for problem (5.1).

As $t = 0$ and $b_1 \equiv 1$, we deduce that estimate (2.10) is valid, provided that the functions a_j and a_0 are of class $L^\infty(\mathcal{D})$ and $b_0 \in L^\infty(\partial\mathcal{D})$. The operator L_0 corresponds to

$$\begin{cases} -\Delta_{2n}u + u = f & \text{in } \mathcal{D}, \\ \partial_c u + b_{0,0}u = 0 & \text{on } \partial\mathcal{D}. \end{cases}$$

Theorem 5.3. *The inverse L_0^{-1} of the operator L_0 induces compact positive self-adjoint operators*

$$\begin{aligned} Q_1 &= \iota' L_0^{-1} & : & H_{\overline{SL}}(\mathcal{D}) \rightarrow H_{\overline{SL}}(\mathcal{D}), \\ Q_2 &= \iota L_0^{-1} \iota' & : & L^2(\mathcal{D}) \rightarrow L^2(\mathcal{D}), \\ Q_3 &= L_0^{-1} \iota' \iota & : & H_{SL}(\mathcal{D}) \rightarrow H_{SL}(\mathcal{D}) \end{aligned}$$

which have the same systems of eigenvalues and eigenvectors. Moreover, all eigenvalues are positive and there are orthonormal bases in $H_{SL}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{\overline{SL}}(\mathcal{D})$ consisting of the eigenvectors.

Proof. For the proof it suffices to combine Lemmas 3.1 and 5.1. \square

Corollary 3.6 actually shows that the operators Q_1 , Q_2 and Q_3 are of Schatten class $\mathfrak{S}_{4n+\varepsilon}$ for any $\varepsilon > 0$, and so their orders are finite.

Theorem 5.4. *Let $t = 0$ and $\delta b_0 = 0$. Then operator $L : H_{SL}(\mathcal{D}) \rightarrow H_{\overline{SL}}(\mathcal{D})$ related to problem (5.1) is Fredholm. Moreover, the system of root functions of the corresponding closed operator T is complete in the spaces $H_{\overline{SL}}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$. For any $\varepsilon > 0$, all eigenvalues of T (except for a finite number) belong to the corner $|\arg \lambda| < \varepsilon$.*

Proof. Indeed, the operator

$$\delta L = L - L_0 : H_{SL}(\mathcal{D}) \rightarrow H_{\overline{SL}}(\mathcal{D})$$

maps $H_{SL}(\mathcal{D})$ continuously to $L^2(\mathcal{D})$, and hence it is compact. Thus, the statement follows from Theorem 3.4. \square

Corollary 5.5. *Let $t = 0$ and $\delta b_0 = 0$. Assume that the constant c of estimate (2.10) is less than 1. Then the operator $L : H_{SL}(\mathcal{D}) \rightarrow H_{\overline{SL}}(\mathcal{D})$ corresponding to problem (5.1) is continuously invertible. Moreover, the system of root functions of the compact operator $\iota' \iota L^{-1}$ in $H_{\overline{SL}}(\mathcal{D})$ is complete in the spaces $H_{\overline{SL}}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$. For any $\varepsilon > 0$, all eigenvalues of $\iota' \iota L^{-1}$ (except for a finite number) belong to the corner $|\arg \lambda| < \varepsilon$.*

Recall that estimate (2.10) in the particular case under considerations becomes explicitly

$$\left| \left(\sum_{j=1}^n a_j(z) \frac{\partial u}{\partial \bar{z}_j} + (a_0 - 1)u, v \right)_{L^2(\mathcal{D})} \right| \leq c \|u\|_{SL} \|v\|_{SL}$$

for all $u, v \in H^1(\mathcal{D})$, with c a constant independent of u and v .

Corollary 5.6. *If $t = 0$ and the constant c of estimate (2.10) is less than 1, then the closed operator T in $H_{\overline{SL}}(\mathcal{D})$ corresponding to problem (5.1) is continuously invertible. Moreover, the system of root functions of T is complete in the spaces $H_{\overline{SL}}(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$, and all eigenvalues of T (except for a finite number) lie in the corner $|\arg \lambda| < \arcsin c$.*

6. THE COERCIVE CASE

We now turn to the coercive case, i.e., we assume that estimate (2.3) is fulfilled. This is obviously the case if all the coefficients $a_{i,j}(z)$ of A are real-valued, which is due to (2.2).

Lemma 6.1. *Suppose estimate (2.3) is fulfilled. Then there are continuous embeddings*

$$\begin{aligned} H_{SL}(\mathcal{D}) &\hookrightarrow H^1(\mathcal{D}, S), \\ H^{-1}(\mathcal{D}) &\hookrightarrow H_{\overline{SL}}(\mathcal{D}) \end{aligned}$$

if at least one of the following conditions holds:

- 1) S is not empty.
- 2) $\int_{\mathcal{D}} a_{0,0}(x) dx > 0$.
- 3) $\int_{\partial \mathcal{D} \setminus S} \frac{b_{0,0}(x)}{b_1(x)} ds > 0$.

In particular, in this case inclusion (2.6) is compact.

Proof. The proof is standard, cf. for instance [Mik76, Ch. 3, § 5.6]. First we prove that there is a constant $c > 0$, such that

$$\|u\|_{H^1(\mathcal{D})} \leq c \|u\|_{SL}$$

for all $u \in H^1(\mathcal{D}, S)$. We argue by contradiction. Let for any natural number k there be a function u_k in $H^1(\mathcal{D}, S)$ satisfying $\|u_k\|_{H^1(\mathcal{D})} > k \|u_k\|_{SL}$. Then $u_k \neq 0$, and so replacing u_k by $u_k/\|u_k\|_{H^1(\mathcal{D})}$, if necessary, we can assume that $\|u_k\|_{H^1(\mathcal{D})} = 1$ and $\|u_k\|_{SL} < 1/k$ for all $k \in \mathbb{N}$. Thus, the sequence $\{u_k\}$ converges to zero in $H_{SL}(\mathcal{D})$. In particular, for each derivative $\{\partial_j u_k\}$ with $1 \leq j \leq n$, we have

$$\lim_{k \rightarrow \infty} \|\partial_j u_k\|_{L^2(\mathcal{D})} = 0 \quad (6.1)$$

because of estimate (2.3).

On the other hand, the weak compactness principle shows that on passing to a subsequence, if necessary, we may assume that the sequence $\{u_k\}$ converges weakly to an element u in $H^1(\mathcal{D})$. As the embedding $H^1(\mathcal{D}) \hookrightarrow L^2(\mathcal{D})$ is compact, the sequence $\{u_k\}$ converges to u in $L^2(\mathcal{D})$. Then it follows from (6.1) that $\{u_k\}$ converges to u in $H^1(\mathcal{D})$ whence

$$\begin{aligned} \lim_{k \rightarrow \infty} \|u_k\|_{H^1(\mathcal{D})} &= \|u\|_{H^1(\mathcal{D})} \\ &= 1 \end{aligned}$$

and the derivatives of the limit function u vanish almost everywhere in \mathcal{D} , i.e.,

$$|u| = \frac{1}{\int_{\mathcal{D}} dx} > 0. \quad (6.2)$$

If S is not empty, then the traces of u_k on S converge, by the trace theorem, to the trace of u on S . Since $u_k = 0$ on S for all k , we get $u = 0$ which contradicts (6.2).

Furthermore, as $\lim_{k \rightarrow \infty} \|u_k\|_{SL} = 0$, we see that

$$\begin{aligned} |u| \int_{\mathcal{D}} a_{0,0}(x) dx &= 0, \\ |u| \int_{\partial\mathcal{D} \setminus S} \frac{b_{0,0}(x)}{b_1(x)} ds &= 0, \end{aligned}$$

i.e., (6.2) contradicts either of the conditions 2) and 3). Therefore, the space $H_{SL}(\mathcal{D})$ is continuously embedded to the space $H^1(\mathcal{D}, S)$. The second embedding follows by duality. \square

Lemma 6.2. *Assume that estimate (2.3) is fulfilled. If one of the three conditions of Lemma 6.2 holds true and $b_{0,0}/b_1 \in L^\infty(\partial\mathcal{D} \setminus S)$, then the norms $\|\cdot\|_{SL}$ and $\|\cdot\|_{H^1(\mathcal{D})}$ are equivalent and the spaces $H_{SL}(\mathcal{D})$ and $H^1(\mathcal{D}, S)$ coincide as topological ones.*

Proof. We have already proved that, under estimate (2.3) and one of the three conditions of Lemma 6.2, the norm $\|\cdot\|_{SL}$ is not weaker than the norm $\|\cdot\|_{H^1(\mathcal{D})}$ on $H^1(\mathcal{D}, S)$. Furthermore, since $a_{i,j}$ and $a_{0,0}$ are bounded functions in \mathcal{D} and $b_{0,0}/b_1$

a bounded function on $\partial\mathcal{D} \setminus S$, we obtain

$$\begin{aligned} \int_{\mathcal{D}} \sum_{i,j=1}^n a_{i,j}(x) \overline{\partial_i u(x)} \partial_j u(x) dx &\leq c \sum_{j=1}^n \|\partial_j u\|_{L^2(\mathcal{D})}^2, \\ \int_{\mathcal{D}} a_{0,0}(x) |u(x)|^2 dx &\leq c \|u\|_{L^2(\mathcal{D})}^2, \\ \int_{\partial\mathcal{D} \setminus S} \frac{b_{0,0}(x)}{b_1(x)} |u(x)|^2 ds &\leq c \|u\|_{H^1(\mathcal{D})}^2 \end{aligned}$$

for all $u \in H^1(\mathcal{D}, S)$, the last inequality being a consequence of the trace theorem for Sobolev spaces. Here, by c is meant a constant independent of u , which can be diverse in different applications.

On combining the above estimates we deduce immediately that there is a constant c with the property that

$$\|u\|_{SL} \leq c \|u\|_{H^1(\mathcal{D})}$$

for all $u \in H^1(\mathcal{D}, S)$, as desired. \square

Now we want to understand what perturbations of a_j and $\delta b_0/b_1$, t preserve the completeness property of root functions of the operator L_0^{-1} .

Lemma 6.3. *Let estimate (2.3) and one of the three conditions of Lemma 6.1 be fulfilled. If*

$$\left| \int_{\partial\mathcal{D} \setminus S} b_1^{-1} \partial_t u \bar{v} ds \right| \leq c \|u\|_{SL} \|v\|_{SL} \quad (6.3)$$

for all $u, v \in H^1(\mathcal{D}, S)$, with a constant $c < 1$ independent on u and v , and there is $s < 1/2$ such that

$$|((b_1^{-1} \delta b_0)u, v)_{L^2(\partial\mathcal{D} \setminus S)}| \leq c \|u\|_{H^s(\partial\mathcal{D})} \|v\|_{SL} \quad (6.4)$$

for all $u, v \in H^1(\mathcal{D}, S)$, then inequality (2.10) holds and problem (2.12) is Fredholm of index zero.

Proof. Denote by $\delta L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ the operator defined by ∂_t , as is described in Section 2. By (6.3),

$$\|\delta L\|_{\mathcal{L}(H_{SL}(\mathcal{D}), H_{SL}^-(\mathcal{D}))} < 1 = \|L_0^{-1}\|_{\mathcal{L}(H_{SL}^-(\mathcal{D}), H_{SL}(\mathcal{D}))},$$

and so a familiar argument shows that the operator $L_0 + \delta L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is invertible.

The differential operator

$$\delta A = \sum_{j=1}^n a_j(x) \partial_j + \delta a_0(x)$$

maps the space $H^1(\mathcal{D})$ continuously to $L^2(\mathcal{D})$. Lemma 6.1 implies readily that δA maps also the space $H_{SL}(\mathcal{D})$ continuously to $L^2(\mathcal{D})$. In particular, we get

$$|(\delta A u, v)_{L^2(\mathcal{D})}| \leq \text{const} \|u\|_{H^1(\mathcal{D})} \|v\|_{L^2(\mathcal{D})} \leq \text{const} \|u\|_{SL} \|v\|_{SL} \quad (6.5)$$

for all $u, v \in H_{SL}(\mathcal{D})$. As the embedding $\iota' : L^2(\mathcal{D}) \hookrightarrow H_{SL}^-(\mathcal{D})$ is compact (see Lemma 2.2), the operator δA induces a compact operator

$$C_1 : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D}).$$

By (6.4),

$$|((b_1^{-1}\delta b_0)u, v)_{L^2(\partial\mathcal{D}\setminus S)}| \leq \text{const} \|u\|_{H^s(\partial\mathcal{D})} \|v\|_{SL} \leq \text{const} \|u\|_{SL} \|v\|_{SL}$$

for all $u, v \in H^1(\mathcal{D}, S)$. Combining this estimate with (6.5) and (6.3) shows that inequality (2.10) is fulfilled with c a constant independent of u and v . Moreover, the summand $((b_1^{-1}\delta b_0)u, v)_{L^2(\partial\mathcal{D}\setminus S)}$ in the weak formulation (2.12) of problem (2.1) induces a bounded linear operator

$$C_2 : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$$

satisfying

$$\|C_2 u\|_{H_{SL}^-(\mathcal{D})} \leq \text{const} \|u\|_{H^s(\partial\mathcal{D})}. \quad (6.6)$$

Now, if σ is a bounded set in $H_{SL}(\mathcal{D})$ then, by Lemma 6.1, it is also bounded in $H^1(\mathcal{D})$ and, by the trace theorem, the set $\sigma|_{\partial\mathcal{D}}$ of restrictions of elements of σ to $\partial\mathcal{D}$ is bounded in $H^{1/2}(\partial\mathcal{D})$, too. The Rellich theorem implies that $\sigma|_{\partial\mathcal{D}}$ is precompact in $H^s(\partial\mathcal{D})$, if $s < 1/2$. Hence, for any sequence $\{u_k\}$ in σ there is a subsequence $\{u_{k_j}\}$ whose restriction to $\partial\mathcal{D}$ is a Cauchy sequence in $H^s(\partial\mathcal{D})$. Using (6.6) we conclude that any sequence $\{u_k\}$ in σ has a subsequence $\{u_{k_j}\}$, such that $\{C_2 u_{k_j}\}$ is a Cauchy sequence in $H_{SL}^-(\mathcal{D})$. Thus, the operator $C_2 : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is compact.

Finally, by construction, the operator

$$L - (L_0 + \delta L) = C_1 + C_2 : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$$

is compact. Therefore, problem (2.12) is equivalent to the Fredholm-type operator equation

$$(I + (L_0 + \delta L)^{-1}(C_1 + C_2))u = (L_0 + \delta L)^{-1}f$$

in $H_{SL}(\mathcal{D})$ with compact operator $(L_0 + \delta L)^{-1}(C_1 + C_2) : H_{SL}(\mathcal{D}) \rightarrow H_{SL}(\mathcal{D})$. This establishes the lemma. \square

In fact condition (6.4) means that the multiplication operator by $b_1^{-1}\delta b_0$ maps the traces of elements of $H_{SL}(\mathcal{D})$ on the boundary continuously to $H^{-s}(\partial\mathcal{D} \setminus S)$ with some $s < 1/2$.

Lemma 6.4. *If $b_1^{-1}\delta b_0 \in L^\infty(\partial\mathcal{D} \setminus S)$ then*

$$|((b_1^{-1}\delta b_0)u, v)_{L^2(\partial\mathcal{D}\setminus S)}| \leq c \|u\|_{L^2(\partial\mathcal{D})} \|v\|_{SL}$$

for all $u, v \in H^1(\mathcal{D}, S)$, the constant c being independent of u and v .

Proof. Since $b_1^{-1}\delta b_0$ is a bounded function on $\partial\mathcal{D} \setminus S$, it follows that

$$\begin{aligned} |((b_1^{-1}\delta b_0)u, v)_{L^2(\partial\mathcal{D}\setminus S)}| &\leq c \|u\|_{L^2(\partial\mathcal{D})} \|v\|_{L^2(\partial\mathcal{D})} \\ &\leq c \|u\|_{L^2(\partial\mathcal{D})} \|v\|_{H^1(\mathcal{D})} \\ &\leq c_1 \|u\|_{L^2(\partial\mathcal{D})} \|v\|_{SL} \end{aligned}$$

for all $u, v \in H^1(\mathcal{D}, S)$, as desired. \square

Let $t_1(x), \dots, t_{n-1}(x)$ be a basis of tangential vectors of the boundary surface at a point $x \in \partial\mathcal{D}$. Then we can write

$$\partial_t u = \sum_{j=1}^{n-1} c_j(x) \partial_{t_j}$$

with c_1, \dots, c_{n-1} bounded functions on the boundary vanishing on S . As ∂_{t_j} are first order differential operators with bounded coefficients they map $H^{1/2}(\partial\mathcal{D})$ continuously to $H^{-1/2}(\partial\mathcal{D})$. Given any $u, v \in H^1(\mathcal{D}, S)$, we get

$$\begin{aligned} \left| \int_{\partial\mathcal{D} \setminus S} b_1^{-1} \partial_t u \bar{v} ds \right| &\leq c \sum_{j=1}^{n-1} \|\partial_{t_j} u\|_{H^{-1/2}(\partial\mathcal{D})} \|v\|_{H^{1/2}(\partial\mathcal{D})} \\ &\leq c \|u\|_{H^{1/2}(\partial\mathcal{D})} \|v\|_{H^{1/2}(\partial\mathcal{D})} \\ &\leq c \|u\|_{H^1(\mathcal{D})} \|v\|_{H^1(\mathcal{D})} \\ &\leq c \|u\|_{SL} \|v\|_{SL}, \end{aligned}$$

provided that $b_1^{-1} c_j \in L^\infty(\partial\mathcal{D})$ for $1 \leq j \leq n-1$, the constant c does not depend on u and v and may be diverse in different applications.

Theorem 6.5. *Let estimates (2.3), (2.10) and one of the three conditions of Lemma 6.1 be fulfilled. If the constant c of inequality (2.10) is less than 1, then problem (2.12) is uniquely solvable and the inverse operator $L^{-1} : H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}(\mathcal{D}) \subset H^1(\mathcal{D})$ is bounded.*

Proof. This follows from Lemmas 2.5 and 6.1. \square

Under the hypotheses of Theorem 6.5, if moreover $t = 0$, then the system of root functions of the compact operator $\iota' L^{-1} : H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is complete in the spaces $H_{SL}^-(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D}) \subset H^1(\mathcal{D})$. Moreover, for each $\varepsilon > 0$, all eigenvalues of $\iota' L^{-1}$ (except possibly for a finite number) belong to the corner $|\arg \lambda| < \varepsilon$.

Corollary 6.6. *If the constant c of inequality (2.10) is less than $\sin(\pi/2n)$ then the operator $T : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ corresponding to problem (2.12) is continuously invertible. Moreover, the system of root functions of the operator T is complete in the spaces $H_{SL}^-(\mathcal{D})$, $L^2(\mathcal{D})$ and $H_{SL}(\mathcal{D})$, and all eigenvalues of T belong to the corner $|\arg \lambda| \leq \arcsin c$.*

Proof. The statement follows from Theorem 4.5 and Lemmas 6.1 and 6.3. \square

Theorem 6.7. *Suppose that estimate (2.3) and one of the three conditions of Lemma 6.1 are fulfilled. If $t = 0$ and $b_1^{-1} \delta b_0 \in L^\infty(\partial\mathcal{D} \setminus S)$, then the operator $L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ related to Problem (2.12) is Fredholm. Moreover, the system of root functions of the corresponding closed operator $T : H_{SL}^-(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ is complete in the spaces $H_{SL}^-(\mathcal{D})$, $L^2(\mathcal{D})$ and $H^1(\mathcal{D}, S)$, and, for arbitrary $\varepsilon > 0$, all eigenvalues of T (except for a finite number) belong to the corner $|\arg \lambda| < \varepsilon$ in \mathbb{C} .*

Proof. For the proof it suffices to apply Theorem 3.2 combined with Lemmas 6.1 and 6.3. \square

Theorem 6.8. *Let estimate (2.3) and one of the three conditions of Lemma 6.1 be fulfilled. If $b_1^{-1} \delta b_0 \in L^\infty(\partial\mathcal{D} \setminus S)$ and the constant c of estimate (6.3) is less than $|\sin \pi/2n|$, then the operator $L : H_{SL}(\mathcal{D}) \rightarrow H_{SL}^-(\mathcal{D})$ related to problem (2.12) is Fredholm. Moreover, the system of root functions of the corresponding closed operator T in $H_{SL}^-(\mathcal{D})$ is complete in the spaces $H_{SL}^-(\mathcal{D})$, $L^2(\mathcal{D})$ and $H^1(\mathcal{D}, S)$, and, for any $\varepsilon > 0$, all eigenvalues of T (except for a finite number) lie in the corner $|\arg \lambda| < \arcsin c + \varepsilon$.*

Proof. This is a consequence of Theorem 4.6 and Lemmas 6.1 and 6.3. \square

7. ZAREMBA TYPE PROBLEMS

As but one example of boundary value problems satisfying the Shapiro-Lopatinski condition we consider the mixed problem

$$\begin{cases} -\Delta_n u + \sum_{j=1}^n a_j(x) \partial_j u + a_0(x) u = f & \text{in } \mathcal{D}, \\ u = 0 & \text{on } S, \\ \partial_\nu u = 0 & \text{on } \partial\mathcal{D} \setminus S \end{cases} \quad (7.1)$$

for a real-valued function u , where Δ_n is the Laplace operator in \mathbb{R}^n , the coefficients a_1, \dots, a_n and a_0 are assumed to be bounded functions in \mathcal{D} , and $\partial_\nu = \partial_\nu + \varepsilon \partial_t$ with a tangential vector field $t(x)$ on $\partial\mathcal{D}$ whose coefficients are bounded functions vanishing on S , and $\varepsilon \in \mathbb{C}$ (cf. [Zar10]).

In this case $a_{i,j} = \delta_{i,j}$, $b_0 = \chi_S$ is the characteristic function of the boundary set S , and $b_1 = \chi_{\partial\mathcal{D} \setminus S}$ is that of $\partial\mathcal{D} \setminus S$.

From results of the previous section it follows that the root functions related to problem (7.1) in the space $H_{SL}(\mathcal{D})$ are complete in $H_{SL}^-(\mathcal{D})$, $L^2(\mathcal{D})$ and $H^1(\mathcal{D}, S)$ for all ε of sufficiently small modulus.

We finish the paper by showing a second order elliptic differential operator in \mathbb{R}^2 for which no Zaremba-type problem is Fredholm. The idea is traced back to a familiar example of A.V. Bitsadze (1948).

Example 7.1. Let $A = \bar{\partial}^2$ be the square of the Cauchy-Riemann operator in the plane of complex variable z . We choose \mathcal{D} to be the upper half-disk of radius 1, i.e., the set of all $z \in \mathbb{C}$ satisfying $|z| < 1$ and $\Im z > 0$. As S we take the upper half-circle, i.e., the part of $\partial\mathcal{D}$ lying in the upper half-plane. Consider the function sequence

$$u_k(z) = (|z|^2 - 1) \frac{\sin(kz)}{k^s},$$

for $k = 1, 2, \dots$, where s is a fixed positive number. Each function u_k vanishes on S . Moreover, for any differential operator B of order $< s$ with bounded coefficients, the sequence $\{Bu_k\}$ converges to zero uniformly on $\partial\mathcal{D} \setminus S = [-1, 1]$. Since $|u_k(z)| \rightarrow \infty$ for all $z \in \mathcal{D}$, we deduce that no reasonable setting of Zaremba-type problem is possible.

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