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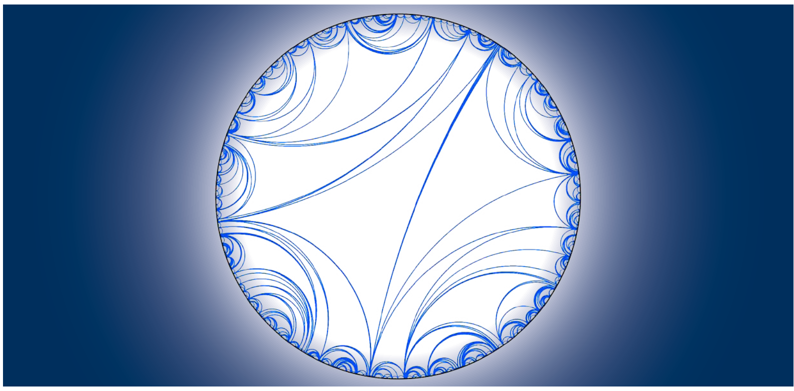

Nikolai Tarkhanov | Daniel Wallenta

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THE LEFSCHETZ NUMBER OF SEQUENCES OF TRACE CLASS CURVATURE

N. TARKHANOV AND D. WALLENTA

Abstract. For a sequence of Hilbert spaces and continuous linear operators the curvature is defined to be the composition of any two consecutive operators. This is modeled on the de Rham resolution of a connection on a module over an algebra. Of particular interest are those sequences for which the curvature is "small" at each step, e.g., belongs to a fixed operator ideal. In this context we elaborate the theory of Fredholm sequences and show how to introduce the Lefschetz number.

Contents

1. Introduction

Let F be a smooth vector bundle over a compact manifold $\mathcal{X}$ and ∂ a connection on F. This is a first order differential operator $C^\infty(\mathcal{X}, F) \to \Omega^1(\mathcal{X}, F)$ on $\mathcal{X}$ satisfying $\partial(\omega f) = d\omega\, f + \omega\, \partial f$ for all $f \in C^\infty(\mathcal{X}, F)$ and $\omega \in C^\infty(\mathcal{X})$. As usual, we denote by $\Omega^i(\mathcal{X}, F)$ the space of all smooth differential forms of degree i with coefficients in F on $\mathcal{X}$. On keeping the Leibniz rule the connection extends to a first order differential operator $\partial^i : \Omega^i(\mathcal{X}, F) \to \Omega^{i+1}(\mathcal{X}, F)$ for each $i = 1, \ldots, n$, where n is the dimension of $\mathcal{X}$. An easy computation shows that $\partial^{i+1}\partial^i f = \Omega\, f$, where Ω is a differential form of degree 2 with coefficients in $C^\infty(\mathrm{Hom}(F))$. The form Ω is said to be the curvature of the connection ∂, which generalizes to the curvature homomorphism related to a connection on a module over an algebra, see [Bou80, 2.10].

In a scale of Sobolev spaces on $\mathcal{X}$, if there is any, the operators ∂^i assemble into a sequence

$$0 \;\to\; H^s(\mathcal{X}, F^0) \;\stackrel{\partial^0}{\to}\; H^{s-1}(\mathcal{X}, F^1) \;\stackrel{\partial^1}{\to}\; \ldots \;\stackrel{\partial^{n-1}}{\to}\; H^{s-n}(\mathcal{X}, F^n) \;\to\; 0, \quad (1.1)$$

where $F^i = F \otimes \Lambda^i T^*\mathcal{X}$, $\partial^0 := \partial$, and $s \geq n$ is any fixed number. The compositions $\partial^{i+1} \circ \partial^i$ act through the embeddings $H^{s-i}(\mathcal{X}, F^{i+2}) \hookrightarrow H^{s-i-2}(\mathcal{X}, F^{i+2})$, which

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are compact by the Rellich theorem, and so $\partial^{i+1} \circ \partial^i$ are compact operators from $H^{s-i}(\mathcal{X}, F^i)$ to $H^{s-i-2}(\mathcal{X}, F^{i+2})$. On using the scale of Schatten ideals $\mathfrak{S}_p$ with $p > 0$ we can even further specify the "smallness" of the curvature of sequence (1.1), more precisely, $\partial^{i+1}\partial^i$ is of class $\mathfrak{S}_p$ with any $p > n/2$, see for instance § 15 of [Pie78].

A connection ∂ on the bundle F is said to be flat if its curvature vanishes, that is $\partial^{i+1} \circ \partial^i = 0$ for all $i = 0, \ldots, N-1$. In this case complex (1.1) possesses a well-defined cohomology, which allows one to define the Euler characteristic for (1.1). In the general case the cohomology is no longer available, and so the sequence (1.1) bears no analytical index although the topological index can be easily introduced. In [Tar07], a Fredholm complex is constructed whose differential differs from ∂^i by compact operators. Hence it follows that the Euler characteristic of this Fredholm complex does not depend on its concrete choice. In this way the analytical index is introduced for sequence (1.1), which has led to a substantial index theory, see [Wal11].

More generally, let $(L^\cdot, d)$ stand for a sequence of Hilbert spaces L^i and continuous linear operators $d^i : L^i \to L^{i+1}$. We simply write $df := d^i f$ for $f \in L^i$, if it causes no confusion. When considering bounded sequences we can certainly assume that $L^i = 0$ for i different from $0, 1, \ldots, N$, for if not, we shift the indexing. We thus arrive at

$$0 \;\to\; L^0 \;\stackrel{d^0}{\to}\; L^1 \;\stackrel{d^1}{\to}\; \ldots \;\stackrel{d^{N-1}}{\to}\; L^N \;\to\; 0. \tag{1.2}$$

To adhere to geometric language we say that the compositions $d^{i+1} \circ d^i$ characterize the curvature of the sequence $(L^\cdot, d)$. The sequences of zero curvature are not stable relative to small perturbations of the differential d. Hence we will be interested in those sequences which have "small" curvature. By "smallness" is meant that all the compositions $d^{i+1} \circ d^i$ belong to an operator ideal $\mathcal{I}(L^i, L^{i+1})$. Let $\mathcal{L}$ denote the class of all bounded linear operators acting between arbitrary Banach spaces. Loosely speaking, an operator ideal $\mathcal{I}$ is a subclass of $\mathcal{L}$, such that $\mathcal{I} + \mathcal{I} = \mathcal{I}$ and $\mathcal{L} \circ \mathcal{I} \circ \mathcal{L} = \mathcal{I}$. As but a few examples we recall the ideals of compact, trace class and absolutely summing operators, see [Pie78]. Formally the case where $\mathcal{I}$ is the zero ideal is also included.

We shall make a standing assumption on the ideals under considerations, namely that $\mathcal{I}$ is a subclass of compact operators.

2. Sequences of class $\mathcal{I}$ curvature

As mentioned, from the point of view of analysis sequences of compact curvature seem to be much more natural objects than complexes. In particular, on perturbing the differential of a complex by operators of the ideal $\mathcal{I}$ we go beyond the framework of complexes. However, the sequences of class $\mathcal{I}$ curvature survive under perturbations of the differential by operators of $\mathcal{I}$. We are thus lead to a class of sequences $(L^\cdot, d)$ bearing the property that the compositions $d^i \circ d^{i-1}$ belong to the ideal $\mathcal{I}$ for all $i = 0, 1, \ldots$.

Definition 2.1. By a (cochain) sequence $(L^\cdot, d)$ of class $\mathcal{I}$ curvature is meant any sequence of Hilbert spaces L^i, $i \in \mathbb{Z}$, and operators $d^i \in \mathcal{L}(L^i, L^{i+1})$ satisfying $d^i \circ d^{i-1} = 0$ modulo operators of $\mathcal{I}(L^i, L^{i+2})$.

For Hilbert spaces L and M, we write $\mathcal{I}(L, M)$ for the subspace of $\mathcal{L}(L, M)$ consisting of all operators $f \in \mathcal{I}$ which map L to M. This subspace fails to be closed in $\mathcal{L}(L,M)$ in general, the smallest closed nonzero "ideal" is $\mathcal{K}(L,M)$. For $f_1, f_2 \in \mathcal{L}(L,M)$ we write $f_1 \sim f_2$ if $f_1 - f_2 \in \mathcal{I}(L,M)$. Suppose that $(L^\cdot, d_L)$ and $(M^\cdot, d_M)$ are two sequences of class $\mathcal{I}$ curvature. By a cochain mapping of $(L^\cdot, d_L)$ into $(M^\cdot, d_M)$ is meant any collection of operators $f^i \in \mathcal{L}(L^i, M^i)$, $i \in \mathbb{Z}$, such that $d_M^i f^i \sim f^{i+1} d_L^i$ for all $i \in \mathbb{Z}$. In particular, $0 = (0_{L^i})_{i\in\mathbb{Z}}$ and $1 = (1_{L^i})_{i\in\mathbb{Z}}$ are cochain mappings of (L,d) into itself, and so are all their perturbations by operators of $\mathcal{I}$.

Cochain mappings $(f_0^i)_{i\in\mathbb{Z}}$ and $(f_1^i)_{i\in\mathbb{Z}}$ of $(L^\cdot, d_L)$ into $(M^\cdot, d_M)$ are said to be homotopic if there is a collection $h^i \in \mathcal{L}(L^i, M^{i-1})$, $i \in \mathbb{Z}$, with the property that $f_1^i - f_0^i \sim d_M^{i-1} h^i + h^{i+1} d_L^i$ for all $i \in \mathbb{Z}$.

The task is now to extend the concept of Fredholm complexes to the more general context of sequences of class $\mathcal{I}$ curvature. Recall that an operator $d \in \mathcal{L}(L,M)$ in Hilbert spaces is Fredholm if and only if its image in the Calkin algebra $\mathcal{L}(L,M)/\mathcal{K}(L,M)$ is invertible. Thus, the idea is to pass in a given sequence to quotients modulo spaces of operators of $\mathcal{I}$ and require exactness. To this end, we modify correspondingly the functor ϕ_Σ introduced by Putinar [Put82]. For complexes of pseudodifferential operators it specifies to what is known as complex of symbols.

For Hilbert spaces K and L, set $\phi_K(L) = \mathcal{L}(K,L)/\mathcal{I}(K,L)$. Moreover, given any $d \in \mathcal{L}(L,M)$, we define $\phi_K(d) \in \mathcal{L}(\phi_K(L), \phi_K(M))$ by the formula

$$\phi_K(d)\,(f + \mathcal{I}(K,L)) = d \circ f + \mathcal{I}(K,M)$$

for $f \in \mathcal{L}(K,L)$. Clearly, this operator is well defined. It is easily seen that $\phi_K(d^2 d^1) = \phi_K(d^2)\phi_K(d^1)$ for all $d^1 \in \mathcal{L}(L^1, L^2)$ and $d^2 \in \mathcal{L}(L^2, L^3)$. If 1_L is the identity operator on L, then $\phi_K(1_L)$ is the identity operator on $\phi_K(L)$. These remarks show that ϕ_K is actually a covariant functor in the category of Hilbert spaces.

The crucial fact is that ϕ_K vanishes on operators of ideal $\mathcal{I}$, for every Hilbert space K. Conversely, if $d \in \mathcal{L}(L,M)$ and $\phi_K(d) = 0$ for any Hilbert space K, then $d \in \mathcal{I}(L,M)$. Indeed, taking $K = L$, we deduce from

$$\begin{aligned} \phi_L(d)\,(1_L + \mathcal{I}(L,L)) &= d + \mathcal{I}(L,M) \\ &= \mathcal{I}(L,M) \end{aligned}$$

that $d \in \mathcal{I}(L,M)$.

If $(L^\cdot, d)$ is an arbitrary sequence of class $\mathcal{I}$ curvature, then $(\phi_K(L^\cdot), \phi_K(d))$ is a complex, for each Hilbert space K. Thus, the functor ϕ_K transforms sequences of class $\mathcal{I}$ curvature into ordinary complexes, i.e., ϕ_K "rectifies" curved sequences. Furthermore, cochain mappings of sequences of class $\mathcal{I}$ curvature transform under ϕ_K into cochain mappings of complexes, and ϕ_K preserves the homotopy classes of cochain mappings.

Definition 2.2. A sequence $(L^\cdot, d)$ of class $\mathcal{I}$ curvature is called Fredholm if the associated complex $(\phi_K(L^\cdot), \phi_K(d))$ is exact, for each Hilbert space K.

Let $(L^\cdot, d_1)$ and $(L^\cdot, d_2)$ be two sequences of class $\mathcal{I}$ curvature, such that $d_1^i \sim d_2^i$ for all $i \in \mathbb{Z}$. Then the complexes $(\phi_K(L^\cdot), \phi_K(d_1))$ and $(\phi_K(L^\cdot), \phi_K(d_2))$ obviously coincide, for every Hilbert space K. Therefore, $(L^\cdot, d_1)$ and $(L^\cdot, d_2)$ are simultaneously Fredholm. In other words, any class $\mathcal{I}$ perturbation of a Fredholm sequence of class $\mathcal{I}$ curvature is a Fredholm sequence of class $\mathcal{I}$ curvature.

Theorem 2.3. *A bounded above sequence* $(L^\cdot, d)$ *of class* $\mathcal{I}$ *curvature is Fredholm if and only if the identity mapping of* $(L^\cdot, d)$ *is homotopic to the zero one.*

This theorem goes back at least as far as [Put82] where the case $\mathcal{I} = \mathcal{K}$ is treated. The designation 'essential complexes' is used in [Put82] for what we call 'sequences of compact curvature' here.

Proof. Necessity. Let $(L^\cdot, d)$ be Fredholm and bounded above, i.e., $L^i = 0$ for all but $i \leq N$. Our goal is to show that there are operators $\pi^i \in \mathcal{L}(L^i, L^{i-1})$, $i \in \mathbb{Z}$, such that

$$d^{i-1}\pi^i + \pi^{i+1}d^i = 1_{L^i} - c^i \tag{2.1}$$

for all $i \in \mathbb{Z}$, where $c^i \in \mathcal{I}(L^i)$.

Set $\pi^i = 0$ for all integers $i > N$. If $i = N$, then from the exactness of the complex $(\phi_K(L^\cdot), \phi_K(d))$, $K = L^N$, at step N if follows that there is an operator $\pi^N \in \mathcal{L}(L^N, L^{N-1})$ such that $d^{N-1}\pi^N \sim 1_{L^N}$. Denoting by c^N the difference $1_{L^N} - d^{N-1}\pi^N$, we thus get $c^N \in \mathcal{I}(L^N)$.

We now proceed by induction. Suppose we have already found mappings

$$\begin{array}{lll} \pi^i, & \pi^{i+1}, & \ldots; \\ c^i, & c^{i+1}, & \ldots, \end{array}$$

such that the equality (2.1) is satisfied at steps $i, i+1, \ldots$, for some $i \leq N$. Note that

$$\begin{array}{rcl} d^{i-1}\left(1_{L^{i-1}} - \pi^i d^{i-1}\right) & = & d^{i-1} - \left(1_{L^i} - c^i - \pi^{i+1}d^i\right) d^{i-1} \\ & = & c^i d^{i-1} + \pi^{i+1} d^i d^{i-1} \\ & \sim & 0 \end{array}$$

by (2.1). From the exactness of $(\phi_K(L^\cdot), \phi_K(d))$, with $K = L^{i-1}$, at step $i-1$ it follows that there is $\pi^{i-1} \in \mathcal{L}(L^{i-1}, L^{i-2})$ such that $d^{i-2}\pi^{i-1} \sim 1_{L^{i-1}} - \pi^i d^{i-1}$. Setting $c^{i-1} = 1_{L^{i-1}} - \pi^i d^{i-1} - d^{i-2}\pi^{i-1}$, we obtain $c^{i-1} \in \mathcal{I}(L^{i-1})$ and (2.1) fulfilled at step $i-1$. This establishes the existence of solutions π^i, c^i to (2.1) for each $i \in \mathbb{Z}$, i.e., the homotopy between the identity and zero cochain mappings of (L, d).

Sufficiency. If the identity mapping $1 = (1_{L^i})_{i\in\mathbb{Z}}$ is homotopic to the zero mapping $0 = (0_{L^i})_{i\in\mathbb{Z}}$ on $(L^\cdot, d)$, then the identity mapping on the cohomology $H^i(\phi_K(L^\cdot), \phi_K(d))$ vanishes for all $i \in \mathbb{Z}$. Hence, the complex $(\phi_K(L^\cdot), \phi_K(d))$ is exact for each Hilbert space K, as required. □

Any solution $\pi^i \in \mathcal{L}(L^i, L^{i-1})$, $i \in \mathbb{Z}$, to (2.1) is called a parametrix of sequence $(L^\cdot, d)$ modulo class $\mathcal{I}$ operators. Thus, Theorem 2.3 just amounts to saying that a bounded above sequence of class $\mathcal{I}$ curvature is Fredholm if and only if it possesses a parametrix modulo class I operators. Given any Fredholm sequence $(L^\cdot, d)$ of class $\mathcal{I}$ curvature, if $f \in L^i$ satisfies $d^i f = 0$, then $f = c^i f + d^{i-1}\pi^i f$, where $(L^\cdot, \pi)$ is a parametrix for $(L^\cdot, d)$ as in (2.1). In other words the operator d^{i-1} has a right inverse π^i modulo class $\mathcal{I}$ operators on solutions to $d^i u = 0$. However, since the compositions $d^i d^{i-1}$ need not vanish for a curved sequence $(L^\cdot, d)$, the range of d^{i-1} no longer lies in solutions of the equation $d^i u = 0$. It follows that the usual cohomology does not make sense for $(L^\cdot, d)$. The question on a proper substitute of the cohomology for curved sequences seems to be considerably subtle, see [Tar07], [Wal11].

3. Reduction to a complex

Let $(L^\cdot, d)$ be a Fredholm sequence consisting of Hilbert spaces L^i which are zero for all i but $i = 0, 1, \ldots, N$, and operators $d^i \in \mathcal{L}(L^i, L^{i+1})$ with $d^{i+1} \circ d^i$ of class $\mathcal{I}$.

These spaces and operators are fit together to form a sequence of Hilbert spaces of class $\mathcal{I}$ curvature, namely,

$$0 \;\to\; L^0 \;\stackrel{d^0}{\to}\; L^1 \;\stackrel{d^1}{\to}\; \ldots \;\stackrel{d^{N-1}}{\to}\; L^N \;\to\; 0. \tag{3.1}$$

Theorem 3.1. *For every sequence (3.1) of class $\mathcal{I}$ curvature there exist bounded operators $D^i \in \mathcal{L}(L^i, L^{i+1})$ satisfying $D^i = d^i$ modulo operators of $\mathcal{I}(L^i, L^{i+1})$ and $D^{i+1} D^i = 0$ for all i.*

Proof. Set $D^{N-1} = d^{N-1}$. The Laplacian

$$\Delta^N = D^{N-1} D^{N-1*}$$

is a selfadjoint operator on L^N, and its kernel just amounts to the kernel of D^{N-1*}. By Theorem 2.3, the latter operator D^{N-1*} has a left parametrix modulo compact operators. In fact, the equality

$$\pi^{N*} D^{N-1*} = 1_{L^N} - c^{N*}$$

holds on L^N. Hence, the identity operator on $\ker D^{N-1*}$ is compact. It follows that the kernel of D^{N-1*} is finite dimensional, and so Δ^N is Fredholm.

By the abstract Hodge theory, there is a selfadjoint operator $G^N \in \mathcal{L}(L^N)$ mapping into the orthogonal complement of $\ker \Delta^N$, such that $1_{L^N} = H^N + \Delta^N G^N$ on L^N, where H^N is the orthogonal projection onto the finite-dimensional space $\ker \Delta^N = \ker D^{N-1*}$.

The space $\ker D^{N-1*}$ is thus an obstruction to the existence of a right inverse operator for D^{N-1}. The operator $\Phi^N = D^{N-1*} G^N$ is a special right parametrix for D^{N-1} in $\mathcal{L}(L^N, L^{N-1})$.

We now show that $P^{N-1} = 1_{L^{N-1}} - \Phi^N D^{N-1}$ is an orthogonal projection onto the kernel of D^{N-1}. To this end, we note that P^{N-1} is the identity operator on the kernel of D^{N-1}, and

$$\begin{aligned} D^{N-1} P^{N-1} &= D^{N-1} - \Delta^N G^N D^{N-1} \\ &= D^{N-1} - (1_{L^N} - H^N) D^{N-1} \\ &= 0, \end{aligned}$$

for $H^N D^{N-1} = (D^{N-1*} H^N)^* = 0$. From this the desired conclusion follows.

In order to construct D^{N-2} we consider the last fragment of sequence (3.1), namely

$$L^{N-2} \;\stackrel{d^{N-2}}{\to}\; L^{N-1} \;\stackrel{D^{N-1}}{\to}\; L^N.$$

Set

$$D^{N-2} = P^{N-1} d^{N-2},$$

then $D^{N-2} \in \mathcal{L}(L^{N-2}, L^{N-1})$ satisfies

$$\begin{aligned} D^{N-1} D^{N-2} &= D^{N-1} P^{N-1} d^{N-2} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} D^{N-2} &= \left(1_{L^{N-1}} - \Phi^N d^{N-1}\right) d^{N-2} \\ &= d^{N-2}, \end{aligned}$$

modulo operators in $\mathcal{I}(L^{N-2}, L^{N-1})$, as desired.

We now restrict ourselves to the suitably modified preceding fragment of the sequence (3.1), i.e.,

$$L^{N-3} \stackrel{d^{N-3}}{\to} L^{N-2} \stackrel{D^{N-2}}{\to} L^{N-1}.$$

The Laplacian $\Delta^{N-1} = D^{N-1*}D^{N-1} + D^{N-2}D^{N-2*}$ is a selfadjoint operator in $\mathcal{L}(L^{N-1})$, whose kernel is obviously $\ker D^{N-1} \cap \ker D^{N-2*}$. Our next goal is to prove that this kernel is of finite dimension. To this end, we observe that the equality

$$D^{N-2}\pi^{N-1} + \pi^N D^{N-1} = 1_{L^{N-1}} - C^{N-1}$$

holds for some compact operator $C^{N-1} \in \mathcal{I}(L^{N-1})$, since both $D^{N-1} - d^{N-1}$ and $D^{N-2} - d^{N-2}$ are of class $\mathcal{I}$. Hence, the identity operator on the cohomology $H^{N-1}(L^\cdot, D)$ is compact, and so the dimension of $H^{N-1}(L^\cdot, D)$ is finite. Since the natural embedding $\ker \Delta^{N-1} \hookrightarrow H^{N-1}(L^\cdot, D)$ is injective, we immediately deduce that the kernel of Δ^{N-1} is finite dimensional, too. This shows that the Laplacian Δ^{N-1} is Fredholm.

By the abstract Hodge theory, there is a selfadjoint operator $G^{N-1} \in \mathcal{L}(L^{N-1})$ which maps into the orthogonal complement of $\ker \Delta^{N-1}$ and fulfills

$$1_{L^{N-1}} = H^{N-1} + \Delta^{N-1}G^{N-1}$$

on L^{N-1}, where H^{N-1} is the orthogonal projection onto the finite-dimensional space $\ker \Delta^{N-1}$.

We claim that $D^{N-1}G^{N-1} = G^N D^{N-1}$. To prove this, pick an arbitrary element $u \in L^{N-1}$. Then

$$D^{N-1}u = D^{N-1}D^{N-1*}D^{N-1}G^{N-1}u$$

on the one hand, and

$$D^{N-1}u = D^{N-1}D^{N-1*}G^N D^{N-1}u$$

on the other hand. Hence it follows that $\Delta^N \left(D^{N-1}G^{N-1}u - G^N D^{N-1}u\right) = 0$, and since $D^{N-1}G^{N-1}u - G^N D^{N-1}u$ is orthogonal to $\ker \Delta^N$ we conclude that $D^{N-1}G^{N-1}u - G^N D^{N-1}u = 0$, as desired.

The composition $\Phi^{N-1} = D^{N-2*}G^{N-1}$ is thus an operator in $\mathcal{L}(L^{N-1}, L^{N-2})$ satisfying the homotopy equation

$$\Phi^N D^{N-1} + D^{N-2}\Phi^{N-1} = 1_{L^{N-1}} - H^{N-1}.$$

In other words, the pair $\{\Phi^{N-1}, \Phi^N\}$ is a special parametrix at steps $N-1$ and N for the sequence (3.1).

To construct D^{N-3} we can now argue in the same way as in the construction of D^{N-2}. Namely, let us show that $P^{N-2} = 1_{L^{N-2}} - \Phi^{N-1}D^{N-2}$ is an orthogonal projection onto the kernel of D^{N-2}. To this end, we note that P^{N-2} is the identity operator on the kernel of D^{N-2}, and

$$\begin{aligned} D^{N-2}P^{N-2} &= D^{N-2} - D^{N-2}\Phi^{N-1}D^{N-2} \\ &= D^{N-2} - (1_{L^{N-1}} - H^{N-1} - \Phi^N D^{N-1})D^{N-2} \\ &= 0, \end{aligned}$$

for $H^{N-1}D^{N-2} = (D^{N-2*}H^{N-1})^* = 0$. From this the desired conclusion readily follows.

Set
$$D^{N-3} = P^{N-2}d^{N-3},$$
then $D^{N-3} \in \mathcal{L}(L^{N-3}, L^{N-2})$ satisfies
$$\begin{aligned} D^{N-2}D^{N-3} &= D^{N-2}P^{N-2}d^{N-3} \\ &= 0 \end{aligned}$$
and
$$\begin{aligned} D^{N-3} &= \left(1_{L^{N-2}} - \Phi^{N-1}D^{N-2}\right) d^{N-3} \\ &= \left(1_{L^{N-2}} - \Phi^{N-1}d^{N-2}\right) d^{N-3} \\ &= d^{N-3} \end{aligned}$$
modulo operators in $\mathcal{I}(L^{N-3}, L^{N-2})$, as desired.

We now proceed by induction, thus completing the proof, for the sequence (3.1) terminates. □

4. Lefschetz number

Consider a Fredholm sequence (3.1) of trace class curvature, with L^i being Hilbert spaces. By Theorem 3.1, there are operators $D^i \in \mathcal{L}(L^i, L^{i+1})$, such that $D^i = d^i$ modulo trace class operators and $D^{i+1}D^i = 0$ for all i. We thus arrive at a Fredholm complex
$$0 \to L^0 \overset{D^0}{\to} L^1 \overset{D^1}{\to} \ldots \overset{D^{N-1}}{\to} L^N \to 0, \tag{4.1}$$
the latter being a consequence of the fact that Fredholm sequences of trace class curvature are stable under perturbations of trace class.

Suppose $e = (e^i)_{i\in\mathbb{Z}}$ is a cochain mapping of complex $(L^\cdot, D)$ into itself, i.e., $e^i \in \mathcal{L}(L^i, L^i)$ satisfy $D^ie^i = e^{i+1}D^i$ for all i. Such a mapping preserves the spaces of cocycles and coboundaries of complex $(L^\cdot, D)$. On passing to quotient spaces it induces the homomorphisms $(He)^i$ of cohomology $H^i(L^\cdot, D)$, for each i. Since the cohomology is finite dimensional at each step, the traces $\mathrm{tr}\,(He)^i$ of the linear mappings are well defined.

Definition 4.1. By the Lefschetz number of a cochain mapping e of $(L^\cdot, D)$ is meant
$$L(e) = \sum_{i=0}^{N} (-1)^i \ \mathrm{tr}\,(He)^i.$$

In particular, if e is the identity mapping of $(L^\cdot, D)$, then $L(e)$ is the Euler characteristic of this complex. In [Tar07] the Euler characteristic of $(L^\cdot, D)$ is proved to depend on the sequence $(L^\cdot, d)$ solely. In this way the Euler characteristic is defined not only for sequences of zero curvature but also for those of compact curvature. The question arises whether the Lefschetz number is actually independent of the complex $(L^\cdot, D)$ and is determined by $(L^\cdot, d)$. The following theorem gives a partial evidence of this fact.

Theorem 4.2. *As defined above, the Lefschetz number of the endomorphism e is given by the formula*

$$L(e) = \sum_{i=0}^{N} (-1)^i \ \mathrm{tr}\ \Big(e^i - (e^i \pi^{i+1}) d^i - d^{i-1} (e^{i-1} \pi^i)\Big), \qquad (4.2)$$

where $\{\pi^i\}$ is a parametrix of $(L^\cdot, d)$ modulo trace class operators.

Proof. Since $D^i - d^i$ is of trace class for each $i = 0, 1, \ldots, N$, it follows that $\{\pi^i\}$ is a parametrix of complex $(L^\cdot, D)$ modulo trace class operators. By the homotopy formula,

$$\pi^{i+1} D^i - D^{i-1} \pi^i = 1_{L^i} - r^i$$

for all i, where r^i is a trace class operator on L^i. On applying the cochain mapping e^i to both sides of this equality we obtain

$$(e^i \pi^{i+1}) D^i - D^{i-1} (e^{i-1} \pi^i) = e^i - e^i \circ r^i,$$

i.e., the cochain selfmappings $(e^i)_{i \in \mathbb{Z}}$ and $(e^i \circ r^i)_{i \in \mathbb{Z}}$ of complex $(L^\cdot, D)$ are homotopic. Hence, they induce the same action on the cohomology of $(L^\cdot, D)$, which gives

$$\begin{aligned} L(e) &= L(e \circ r) \\ &= \sum_{i=0}^{N} (-1)^i \ \mathrm{tr}\ e^i \circ r^i, \end{aligned}$$

the latter equality being a consequence of the Euler identity, see for instance Theorem 19.1.15 in [Hoe85]. To complete the proof, it remains to use an argument of [Fed91, p. 203], namely,

$$\begin{aligned} \sum_{i=0}^{N} (-1)^i \ \mathrm{tr}\ e^i \circ r^i &= \sum_{i=0}^{N} (-1)^i \ \mathrm{tr}\ \Big(e^i - (e^i \pi^{i+1}) D^i - D^{i-1} (e^{i-1} \pi^i)\Big) \\ &= \sum_{i=0}^{N} (-1)^i \ \mathrm{tr}\ \Big(e^i - (e^i \pi^{i+1}) d^i - d^{i-1} (e^{i-1} \pi^i)\Big), \end{aligned}$$

which is due to a familiar theorem of Lidskii, for D^i and d^i differ by trace class operators. □

Obviously, $(e^i)_{i \in \mathbb{Z}}$ is a cochain mapping of sequence $(L^\cdot, d)$, for $d^i e^i = e^{i+1} d^i$ modulo trace class operators. However, we are able to introduce the Lefschetz number only for those cochain mappings of $(L^\cdot, d)$ which are cochain mappings of some Fredholm complex $(L^\cdot, D)$ which is a perturbation of $(L^\cdot, d)$ by trace class operators. It would be desirable to show that, given any cochain mapping e of $(L^\cdot, d)$, there is a Fredholm complex $(L^\cdot, D)$, whose differential D differs from d by trace class operators and commutes with e, but we have not been able to do this. In any case we can define the Lefschetz number of arbitrary cochain mapping e of $(L^\cdot, d)$ by formula. As already mentioned, this definition will depend on the particular choice of neither parametrix π nor the differential d up to trace class operators.

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