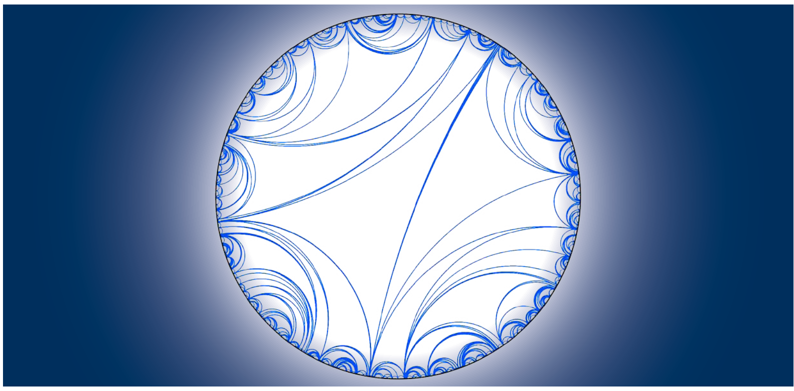
Universität Potsdam



Oleg M. Kiselev | Nikolai Tarkhanov

Scattering of Autoresonance Trajectories upon a Separatrix

Preprints des Instituts für Mathematik der Universität Potsdam
1 (2012) 2

Preprints des Instituts für Mathematik der Universität Potsdam

Preprints des Instituts für Mathematik der Universität Potsdam
1 (2012) 2

Oleg M. Kiselev | Nikolai Tarkhanov

Scattering of Autoresonance Trajectories upon a Separatrix

Universitätsverlag Potsdam

Bibliografische Information der Deutschen Nationalbibliothek
Die Deutsche Nationalbibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliografie; detaillierte bibliografische Daten sind im Internet über http://dnb.de abrufbar.

Universitätsverlag Potsdam 2012
http://info.ub.uni-potsdam.de/verlag.htm

Am Neuen Palais 10, 14469 Potsdam
Tel.: +49 (0)331 977 2533 / Fax: 2292
E-Mail: verlag@uni-potsdam.de

Die Schriftenreihe **Preprints des Instituts für Mathematik der Universität Potsdam** wird herausgegeben vom Institut für Mathematik der Universität Potsdam.

ISSN (online) 2193-6943

Kontakt:
Institut für Mathematik
Am Neuen Palais 10
14469 Potsdam
Tel.: +49 (0)331 977 1028
WWW: http://www.math.uni-potsdam.de

Titelabbildungen:
1. Karla Fritze | Institutsgebäude auf dem Campus Neues Palais
2. Nicolas Curien, Wendelin Werner | Random hyperbolic triangulation
Published at: http://arxiv.org/abs/1105.5089

Online veröffentlicht auf dem Publikationsserver der Universität Potsdam
URL http://pub.ub.uni-potsdam.de/volltexte/2012/5688/
URN urn:nbn:de:kobv:517-opus-56880
http://nbn-resolving.de/urn:nbn:de:kobv:517-opus-56880

Scattering of Autoresonance Trajectories upon a Separatrix

O. M. Kiselev [*], N. Tarkhanov [†]

November 20, 2011

1 Introduction

In this paper we study asymptotic properties of solutions to the primary resonance equation

$$\imath \frac{d\Psi}{dt} + (\varkappa(t) - |\Psi|^2)\Psi = f \tag{1.1}$$

with large amplitude $|\Psi| = O(\varepsilon^{-1})$ on a long time interval $t \in (\varepsilon^{-2}c_1, \varepsilon^{-2}c_2)$, where $0 < \varepsilon \ll 1$ and c_1, c_2 are arbitrary fixed constants satisfying $c_1 < c_2$. By f on the right-hand side is meant any positive constant.

The equations of form (1.1) are said to be primary resonance equations. They are of great importance in the study of resonance conditions in nonlinear dynamical systems. In the particular case $\varkappa(t) \equiv 0$ the primary resonance equation was derived in the classical paper of Krylov and Bogolyubov [1] who investigated resonance conditions for the solution of small amplitude to a nonlinear equation with cubic singularity. Another form of this equation with $\varkappa(t) \equiv \text{const}$ appears in the study of passing through a resonance in problems of celestial mechanics [2, 3]. In the most general form equation (1.1) is found in the study of capture into resonance of a particle in a synchrophasotron [4]. A modern view on autoresonance conditions and their numerous manifestations in physics can be found in the paper [5].

In general position one can assume that $\varkappa(t) \equiv t$. For the equation

$$\imath \frac{d\Psi}{dt} + (t - |\Psi|^2)\Psi = f \tag{1.2}$$

[*]Institute of Mathematics USC RAS (ok@ufanet.ru)
[†]Institute of Mathematics, University of Potsdam (tarkhanov@math.uni-potsdam.de)

there are known two-parameter families of solutions bounded as $t \to -\infty$, and two two-parameter families of solutions with particular behaviour as $t \to +\infty$. The first of the two consists of solutions bounded as $t \to -\infty$, and the second one consists of solutions which increase like $\sqrt{t}$ as $t \to +\infty$. Their properties are discussed e.g. in the survey [6].

In this paper we look for a connection between the parameters of the solution bounded as $t \to -\infty$ and the parameters of the two two-parameter families of solutions at $t = \infty$, one consisting of those solutions which are not captured into resonance and the other consisting of those increasing solutions which are captured into resonance. We determine the parameters of the asymptotic solution at $t = -\infty$ in which terms the families of captured and non-captured solutions are described in a manner regular with respect to ε.

In this way one has to study the passing through a separatrix for equations with a slowly varying parameter. Similar problems are intensively studied in connection with involved dynamics close to separatrices. In particular, for equation (1.1) with function $\varkappa(t)$ of special form $\varkappa'(t) \sim \varepsilon$ where $0 \ll \varepsilon \ll 1$, Neishtadt [7] evaluated the probability of capture into a resonance. For a contemporary view on probability approach to describing solutions in systems with slowly varying parameters the reader is referred to [8]. The change of variables 'action', 'phase' under crossing a separatrix in systems with degree of freedom 3/2 is studied in [9, 10, 11] up to the first terms of perturbation theory. A survey of papers devoted to the passing through a separatrix and autoresonance is given in [12].

In the articles [13, 14, 15] special solutions are analyzed which are related to the loss of stability of a slowly varying equilibrium position in equations close to (1.1).

A close in the setting problem on the connection of asymptotics and capture into resonance was treated in [15] for solutions of small amplitude to the parametric autoresonance equation. In that paper the sets of captured and non-captured solutions are determined by means of the problem on connection of asymptotics of the Painlevé-2 transzendent. In the present paper we study the structure of the set of solutions which are captured into resonance for solutions not of small, but of large amplitude, and so the approaches of [14, 15] and [16] using Painlevé-2 transzendents no longer apply.

In the plane $t = 0$ the set of initial data of those solutions which are captured into resonance looks fairly complicated, (see Fig. 1). Numerical simulations produced for instance in [6] show that the set of initial data in a neighbourhood of the origin of those solutions which increase as $t \to \infty$ has spiral structure. However, one did not succeed to understand from those results the structure of the set of initial data of solutions with large initial

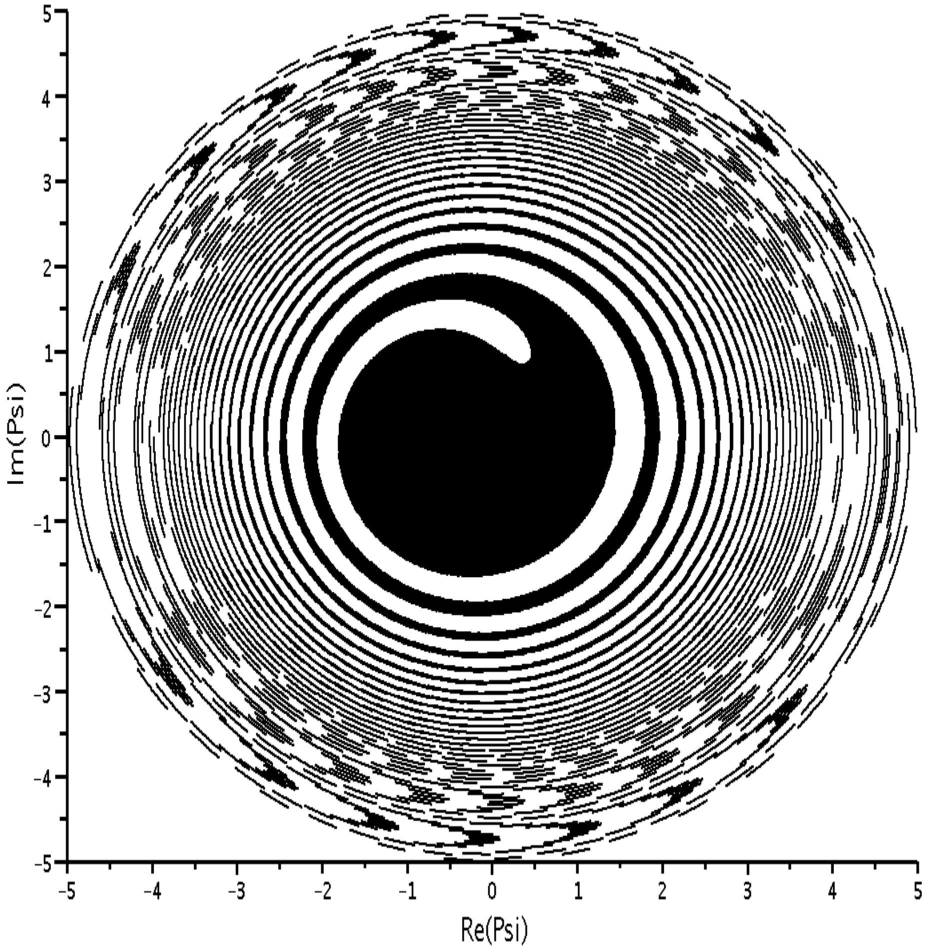


Fig. 1: The domain of initial data $\Psi\upharpoonright_{t=0}$. This is a result of numerical solution of the Cauchy problem for equation (1.2) by the Runge-Kutta method of the fourth order for $t \in [0, 30]$ with step 0.0001 and for initial data in the disk $|\Psi|\upharpoonright_{t=0} \in [0, 5]$ with step 0.01 and $\operatorname{Arg}\Psi\upharpoonright_{t=0} \in [0, 2\pi)$ with step $2\pi/2048$. The data of solutions captured into resonance are marked in black, and those of non-captured solutions are marked in white.

data.

We now dwell on the contents of the paper. The formal setting of the problem is presented in Section 2. The setting is illustrated by two results derived numerically. In Sections 3 and 4 we construct an asymptotic solution which suits for $t \ll -1$. In Section 5 we examine the domain close to $t = 1$ and construct an asymptotic solution whose principal term is a solution of the equation of mathematical pendulum with external momentum. Section 8 contains matching of the parameters of constructed asymptotic solutions away from $t = 1$ and close to $t = 1$. In Section 9 we make a comparison of asymptotic formulas obtained in this way and numerical solutions. In Section 10 we indicate once again how this paper contributes to the knowledge of primary resonance equations.

2 Setting of the problem and the main result

The purpose of the paper is to find the parameters of asymptotic solution as $t \to -\infty$, in which the sets of solutions bounded as $t \to \infty$, and solutions increasing as $t \to \infty$ are described regularly in ε for $\varepsilon \to \infty$.

Consider two numerical solution of the Cauchy problem for equation (1.2) with initial data in close proximity to each other. In Fig. 2 two trajectories of solutions of the Cauchy problem with near initial data are shown. At the initial stage these trajectories are close to each other, but at certain moment a rebuilding occurs after which the trajectories differ essentially. In the first picture the solution remains still bounded, however, it changes the direction of revolution and the amplitude of oscillations. In the second picture the solution oscillates and increases as $\sqrt{t}$ for $t \to \infty$.

Let ε be a small positive number. Then for $t < \varepsilon^{-2}$ the two-parameter family of asymptotic solutions of large amplitude to equation (1.2) has the form

$$\Psi(t,\varepsilon) = \varepsilon^{-1}(1+\varepsilon^3 r(\varepsilon^2 t,\varepsilon)) \exp \imath\Big(-\varepsilon^{-2}t + \frac{t^2}{2} - \frac{2f}{\varepsilon} + \varphi + \varepsilon^2 a(\varepsilon t,\varepsilon)\Big),$$

where

$$\begin{aligned} r(\varepsilon^2 t,\varepsilon) &\sim -\frac{f\cos\Big(-\varepsilon^{-2}t + \dfrac{t^2}{2} - \dfrac{2f}{\varepsilon} + \varphi + \varepsilon^2 a(\varepsilon t,\varepsilon)\Big)}{1-\varepsilon^2 t}, \\ a(\varepsilon^2 t,\varepsilon) &\sim \frac{f^2(2\varepsilon^2 t - 1)}{(1-\varepsilon^2 t)^2}. \end{aligned}$$

The quantities ε and φ are parameters of the asymptotic solution. In terms of parameters ε and φ the set of increasing solutions is marked in black in Fig. 3

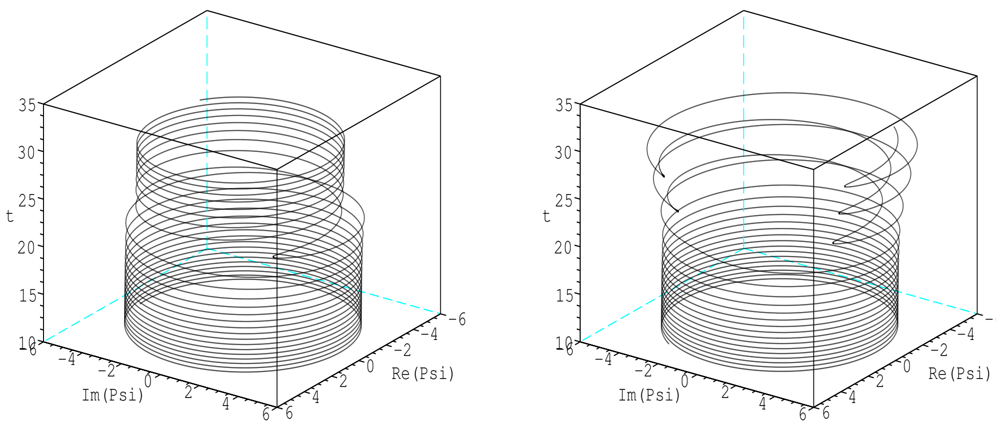


Fig. 2: Captured and nun-captured trajectories. In the first picture a trajectory segment of the solution of the Cauchy problem for equation (1.2) with initial data $\Psi\restriction_{t=0}= 5\exp(2\imath)$ is shown. At the initial stage the trajectory is close to the spiral of radius 5 which twists clockwise. At some moment close to $t = 25$ a rebuilding of the trajectory happens, namely one hundred eighty degrees turn and passing to counter-clockwise revolution over a spiral of less radius. In the second picture a trajectory segment of the solution to the Cauchy problem for equation (1.2) with initial data $\Psi\restriction_{t=0}= 5\exp(2.2\imath)$ is given. At the initial stage it is close to the spiral of radius 5. At some moment close to $t = 25$ a rebuilding of the trajectory occurs, namely a turn and passing to bananalike oscillations around a center which moves away from the origin in the complex plane of the variable Ψ. Both the Cauchy problems are solved numerically by the Runge-Kutta methods of the fourth order with step 0.0001.

The width of the domain in the plane of variables ε and φ corresponding to the solutions captured into resonance is given in terms of φ by

$$\Delta\varphi \sim \frac{8\sqrt{2f}}{3}\,\varepsilon^{5/2}.$$

The set of all bounded solutions for $t > \varepsilon^{-2}$ has the form

$$\Psi(t,\varepsilon_+) = \varepsilon_+^{-1}(1+\varepsilon_+^3 r(\varepsilon_+^2 t,\varepsilon_+))\,\exp\imath\Big(-\varepsilon_+^{-2}t + \frac{t^2}{2} + \varphi_+ + \varepsilon_+^2 a(\varepsilon_+^2 t,\varepsilon_+)\Big),$$

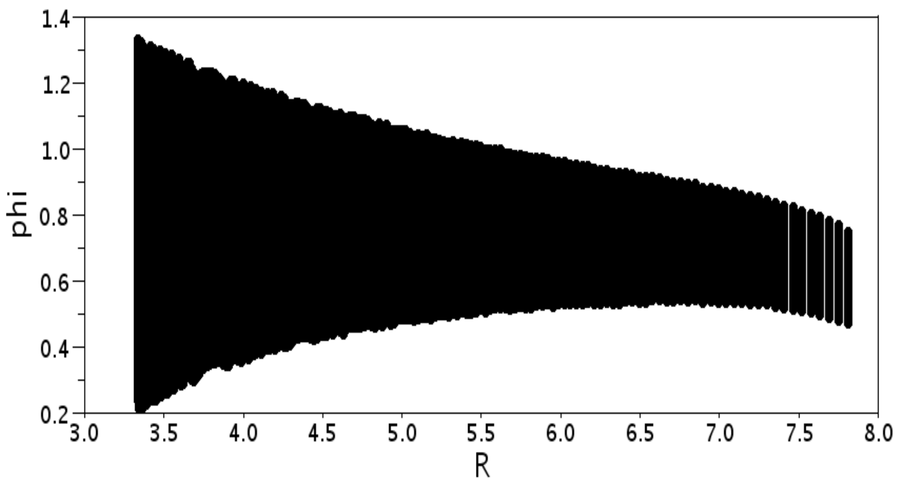


Fig. 3: The set of parameters ε, φ corresponding to those solutions which are captured into resonance.

where $\varepsilon_+ = \dfrac{\varepsilon}{1-\varepsilon^{3/2}r^+}$, with $r^+ = \dfrac{1}{4}\mathcal{I} + O(\varepsilon)$, and

$$\begin{aligned} r(\varepsilon_+^2 t, \varepsilon_+) &\sim -\frac{f\cos\Big(-\varepsilon_+^{-2}t + \dfrac{t^2}{2} + \varphi_+ + \varepsilon_+^2 a(\varepsilon_+^2 t, \varepsilon_+)\Big)}{1-\varepsilon_+^2 t}, \\ a(\varepsilon_+^2 t, \varepsilon_+) &\sim \frac{f^2\,(2\varepsilon_+^2 t - 1)}{(1-\varepsilon_+^2 t)^2} + \varphi_+, \end{aligned}$$

with

$$\varphi_+ \sim \varphi + \frac{1}{8\varepsilon}\mathcal{I}^2.$$

Here,

$$\mathcal{I} = \int_0^\infty \Big(\frac{1}{\sqrt{8f\sin(\bar\alpha + z/4\varepsilon)\sin(-z/4\varepsilon) + z}} - \frac{1}{\sqrt{z}}\Big)dz$$

and $\bar\alpha$ is the greatest root of equation $8f\sin(\bar\alpha + z/4\varepsilon)\sin(z/4\varepsilon) = z$. Note that this equation has no roots for $z > 0$.

3 Construction of outer asymptotic solution

The asymptotic solution of large amplitude is looked for in the form

$$\Psi(t) = \varepsilon^{-1}\psi(s),$$

where $0 < \varepsilon \ll 1$ and $s = \varepsilon^2 t + 1$. The primary resonance equation for the function $\psi(s)$ becomes

$$\imath\varepsilon^4 \frac{d\psi}{ds} + (s + 1 - |\psi|^2)\psi = \varepsilon^3 f. \tag{3.1}$$

For this equation we will search for solutions bounded for $s < 0$.

We wish to construct a solution to (3.1) of the form

$$\psi = (1 + \varepsilon^3 r) \exp\imath\Big(\frac{s^2}{2\varepsilon^4} + \varphi + \varepsilon^2 a)\Big),$$

where $r(s,\varepsilon)$ and $a(s,\varepsilon)$ are real-valued functions and ε, φ the solution parameters. Substitute the formula for ψ into equation (3.1). Single out the real and imaginary parts. As a result we derive immediately equations for r and a

$$\begin{aligned} \varepsilon^4 \frac{dr}{ds} &= -f \sin\Big(\frac{s^2}{2\varepsilon^4} + \varphi + \varepsilon^2 a\Big), \\ \varepsilon^3 \frac{da}{ds} &= -\frac{f \cos\Big(\frac{s^2}{2\varepsilon^4} + \varphi + \varepsilon^2 a\Big)}{1 + \varepsilon^3 r} - 2r - \varepsilon^3 r^2. \end{aligned}$$

We now construct a formal asymptotic solution of this system. For convenience we introduce the fast variable

$$S = \frac{s^2}{2\varepsilon^4} + \varphi.$$

On assuming that the solution depends on two variables, the fast variable S and the slow variable s, i.e.

$$\begin{aligned} r(s,\varepsilon) &= r(S,s,\varepsilon), \\ a(s,\varepsilon) &= a(S,s,\varepsilon), \end{aligned}$$

one can rewrite the above equations for r and a as system of partial differential equations

$$\begin{aligned} s\frac{\partial r}{\partial S} + \varepsilon^4 \frac{\partial r}{\partial s} &= -f \sin(S + \varepsilon^2 a), \\ s\frac{\partial a}{\partial S} + \varepsilon^4 \frac{\partial a}{\partial s} &= -\varepsilon\frac{f \cos(S + \varepsilon^2 a)}{1 + \varepsilon^3 r} - \varepsilon 2r - \varepsilon^4 r^2. \end{aligned} \tag{3.2}$$

One looks for an asymptotic solution to this system by the two-scales method. More precisely, write

$$
\begin{aligned}
r(S,s,\varepsilon) &= \sum_{k=0}^{\infty} \varepsilon^k \, r_k(S,s), \\
a(S,s,\varepsilon) &= \sum_{k=0}^{\infty} \varepsilon^k \, a_k(S,s)
\end{aligned}
\tag{3.3}
$$

with coefficients r_k and a_k being so far undetermined. To find the coefficients of (3.3) one substitutes the formal series for $r(S,s,\varepsilon)$ and $a(S,s,\varepsilon)$ into system (3.2). Equate the coefficients of the same powers of ε. As a result one gets differential equations for finding the dependence of the coefficients r_k and a_k upon the fast variable. In particular, equating the coefficients of $\varepsilon^0 = 1$ yields

$$
s\frac{\partial r_0}{\partial S} = -f\sin S, \quad s\frac{\partial a_0}{\partial S} = 0,
$$

and equating the coefficients of $\varepsilon^1 = \varepsilon$

$$
s\frac{\partial r_1}{\partial S} = 0, \quad s\frac{\partial a_1}{\partial S} = -f\cos S - 2r_0.
$$

In the general case, for the coefficients r_k and a_k with $k \geq 2$, one obtains the system

$$
\begin{aligned}
s\frac{\partial r_k}{\partial S} + \frac{\partial r_{k-4}}{\partial s} &= -f\sin(S)\,P_k + f\cos(S)\,Q_k, \\
s\frac{\partial a_k}{\partial S} + \frac{\partial a_{k-4}}{\partial s} + 2r_{k-1} &= -\sum_{i+j=k-4} r_i r_j + f\sin(S)\,S_k - f\cos(S)\,T_k.
\end{aligned}
\tag{3.4}
$$

Here, P_k and Q_k are polynomials of $a_0,\dots,a_{k-2}$ spanned by the monomials $a_0^{\alpha_0}\dots a_{k-2}^{\alpha_{k-2}}$ with $2\alpha_0+\dots+k\alpha_{k-2} \leq k$. And S_k, T_k are polynomials of both $a_0,\dots,a_{k-2}$ and $r_0,\dots,r_{k-2}$ spanned by $a_0^{\alpha_0}\dots a_{k-2}^{\alpha_{k-2}} r_0^{\beta_0}\dots r_{k-2}^{\beta_{k-2}}$, such that $2(\alpha_0+\beta_0)+\dots+k(\alpha_{k-2}+\beta_{k-2}) \leq k$.

For every $k = 0,1,\dots$, the equations of initial system (3.2) split up into pairs of single equations for a_k and r_k. In order to construct asymptotics which suit uniformly in S it suffices to require that the mean values of the right-hand sides in (3.4) be actually equal to their derivatives in the slow variable s. This requirement determines the dependence of the coefficients of asymptotics on s.

Lemma 3.1. *For each $k = 0, 1, \ldots,$ equations (3.4) have a two-parameter family of solutions uniformly bounded in S.*

The proof consists in consecutive building of solutions of the system beginning on $k = 0$. It is easy to see that the system is linear and, given any fixed k, the equation are decoupled. Write

$$\begin{aligned} r_k &= r'_k(S, s) + \tilde{r}_k(s), \\ a_k &= a'_k(S, s) + \tilde{a}_k(s). \end{aligned}$$

for the general solution of (3.4). The functions $\tilde{r}_k(s)$ and $\tilde{a}_k(s)$ are determined by the requirement of uniform boundedness of solutions in S. This requirement gives averaged equations

$$\frac{\partial \tilde{r}_{k-4}}{\partial s} = \tilde{F}_k, \quad \frac{\partial \tilde{a}_{k-4}}{\partial s} = -2\tilde{r}_{k-1} + \tilde{G}_k. \tag{3.5}$$

Here, $\tilde{F}_k$ and $\tilde{G}_k$ are averagings in S through the period 2π of the right-hand sides of equations (3.4), respectively.

We now give the results of computation for the first three amendments. These are

$$\begin{aligned} r_0(S, s) &= f \frac{\cos S}{s}, & a_0(S, s) &= \tilde{a}_0(s); \\ r_1(S, s) &= 0, & a_1(S, s) &= -f \frac{s+2}{s} \sin S + \tilde{a}_1(s), \\ r_2(S, s) &= -f\, \tilde{a}_0(s) \frac{\sin S}{s^3}, & a_2(S, s) &= \tilde{a}_2(s). \end{aligned}$$

One has to show that system (3.5) is actually recurrent, i.e., the equations for diverse amendments can be solved one after another in k. Rewrite system (3.5) in the form

$$\tilde{r}_{k-1} = \frac{1}{2}\Big(-\frac{\partial \tilde{a}_{k-4}}{\partial s} + \tilde{G}_k \Big), \quad \frac{\partial^2 \tilde{a}_{k-7}}{\partial s^2} - \frac{\partial \tilde{G}_{k-3}}{\partial s} + 2\tilde{F}_k = 0.$$

The right-hand side of the first equation does not contain amendments $\tilde{r}_j$ with $j > k-1$ and $\tilde{a}_j$ with $j > k-7$. Indeed, by the structure of S_k and T_k, it suffices to consider only those monomials which include amendments $\tilde{r}_j$ with $k \geq j \geq k-1$ and $\tilde{a}_j$ with $k \geq j \geq k-7$. The contribution of such monomials to the averaged right-hand sides $\tilde{F}_k$ and $\tilde{G}_{k-3}$ proves to be equal to zero.

The form of $\tilde{a}_0(s)$ is found by equating the coefficients of ε^3. At this step is determined the dependence of the amendments $r_7(S, s)$ and $r_7(S, s)$ on the fast variable S. A cumbersome but very elementary computation with the help of analytic computation program [17] yields

$$\tilde{a}_0(s) = f^2\,\frac{2s+1}{s^2} + a_0^0 + \frac{r_3^0}{2}\,s,$$

where a_0^0 and r_3^0 are arbitrary real constants, parameters of the asymptotic solution.

Lemma 3.2. *For $s \to 0$, the coefficients of expansions (3.3) have singularities $r_k(S,s) = O(s^{-k})$ and $a_k(S,s) = O(s^{-k-2})$.*

Proof. When constructing solutions to equations (3.4), one obtains the factor s^{-1} in the formulas for $r'_k(S,s)$ and $r'_k(S,s)$ at each step k. The amendments $r_0(S,s)$ and $a_0(S,s)$ described above have singularities at $s = 0$, too. Since the right-hand sides of (3.4) are nonlinear, the order of singularity at $s = 0$ becomes greater with number k. This heuristic argument is rigorously proved by induction in k. Let the assertion of the lemma be true for all $k \le k_0$. On using formulas for the right-hand sides of (3.4) one shows that the assertion holds true for $k = k_0 + 1$. □

Combining Lemmata 3.1 and 3.2 we are in a position to formulate the main result of this section.

Theorem 3.3. *Formal series (3.3) give an asymptotic solution of system (3.2) uniformly in s for $\varepsilon^{-1}|s| \gg 1$.*

In this way one derives the principal term of asymptotic solution $a(S,s,\varepsilon)$, $r(S,s,\varepsilon)$ to the system. This asymptotic solution has two parameters, namely the arbitrary constants r_3^0 and a_0^0. Without restriction of generality for the initial problem of finding an asymptotic solution to equation (3.1) one can assume that both the parameters are zero.

4 Intermediate asymptotic solution

The asymptotic solution for $s \to 0$ constructed above allows one to find variables for construction of asymptotic solution which suits for $\varepsilon^{-1}|s| \sim 1$. It has the form

$$\psi = (1 + \varepsilon^2 R(\sigma,\varepsilon))\,\exp\imath\Big(\varepsilon^{-2}\frac{\sigma^2}{2} + A(\sigma,\varepsilon)\Big),$$

where $\varepsilon\sigma = s$. For the unknown functions $R(\sigma,\varepsilon)$ and $A(\sigma,\varepsilon)$ we derive the equations

$$\begin{aligned}\varepsilon \frac{dA}{d\sigma} &= -2R - \varepsilon \frac{f \cos\Big(A + \dfrac{\sigma^2}{2\varepsilon^2}\Big)}{1+\varepsilon^2 R} - \varepsilon^2 R^2,\\ \varepsilon^2 \frac{dR}{d\sigma} &= -f \sin\Big(A + \frac{\sigma^2}{2\varepsilon^2}\Big).\end{aligned}$$

It is convenient to build an asymptotic solution by means of two-scales method. Rewrite the variable "fast time" in the form $S = \sigma^2/(2\varepsilon^2) + \varphi$. The total derivative in σ is written as sum of partial derivatives in S and σ. As a result the system takes the form

$$\begin{aligned}\sigma \frac{\partial A}{\partial S} + \varepsilon^2 \frac{\partial A}{\partial \sigma} &= -\varepsilon\, 2R - \varepsilon^2 \frac{f \cos(A+S)}{1+\varepsilon^2 R} - \varepsilon^3 R^2,\\ \sigma \frac{\partial R}{\partial S} + \varepsilon^2 \frac{\partial R}{\partial \sigma} &= -f \sin(A+S).\end{aligned} \tag{4.1}$$

The solution is searched for in the form of asymptotic series

$$\begin{aligned}A(S,\sigma,\varepsilon) &= \sum_{k=0}^{\infty} \varepsilon^k A_k(S,\sigma),\\ R(S,\sigma,\varepsilon) &= \sum_{k=0}^{\infty} \varepsilon^k R_k(S,\sigma).\end{aligned} \tag{4.2}$$

Substitute these formal decompositions into system (4.1) and equate the coefficients of the same powers of ε. As a result we derive a recurrent system for defining the coefficients A_k and R_k. In particular, we get

$$\begin{aligned}\sigma \frac{\partial A_0}{\partial S} &= 0,\\ \sigma \frac{\partial R_0}{\partial S} &= -f \sin(S + A_0).\end{aligned}$$

These equations determine the dependence of the principal term of asymptotics on the fast variable,

$$A_0(S,\sigma) = \tilde{A}_0(\sigma), \quad R_0(S,\sigma) = \frac{1}{\sigma} \cos(S + A_0) + \tilde{R}_0(\sigma).$$

Here, the functions $\tilde{A}_0(\sigma)$ and $\tilde{R}_0(\sigma)$ are still undefined. The dependence on the slow variable σ is determined by means of averaging method on taking into account that the higher amendments should be bounded in the variable S.

The equations for A_1 and R_1 are

$$\begin{aligned}\sigma \frac{\partial A_1}{\partial S} &= 2R_0, \\ \sigma \frac{\partial R_1}{\partial S} + f \cos(S + A_0)\, A_1 &= 0.\end{aligned}$$

Since we are looking for a solution to the first equation which is bounded in the fast variable S, it follows that

$$\tilde{R}_0(\sigma) \equiv 0.$$

The dependence of A_1 and R_1 on the fast variable is determined by consecutive integration of the first and the second equations. As a result we get

$$\begin{aligned}A_1(S, \sigma) &= -\frac{2}{\sigma^2} \sin(S + A_0) + \tilde{A}_1(\sigma), \\ R_1(S, \sigma) &= \frac{1}{\sigma^3} f^2 \cos(2S + 2A_0) - \frac{1}{\sigma} f \tilde{A}_1(\sigma) \sin(S + A_0) + \tilde{R}_1(\sigma),\end{aligned}$$

where $\tilde{A}_1(\sigma)$ and $\tilde{R}_1(\sigma)$ are determined from the boundedness condition of higher amendments in S.

The equations for A_k and R_k for $k \geq 2$ looks like

$$\begin{aligned}\sigma \frac{\partial A_k}{\partial S} &= -2R_{k-1} - \sum_{i+j=k-3} R_i R_j - f \sin(S+A_0)\, P_k + f \cos(S+A_0)\, Q_k - \frac{\partial A_{k-2}}{\partial \sigma}, \\ \sigma \frac{\partial R_k}{\partial S} &= -f \cos(S+A_0) A_k + f \sin(S+A_0) S_k - f \cos(S+A_0) T_k - \frac{\partial R_{k-2}}{\partial \sigma}.\end{aligned} \tag{4.3}$$

Here, P_k and Q_k are polynomials of $A_1, \ldots, A_{k-1}$ spanned by the monomials $A_1^{\alpha_1} \ldots A_{k-1}^{\alpha_{k-1}}$ with $1\alpha_1 + \ldots + (k-1)\alpha_{k-1} = k$. And S_k, T_k are polynomials of both $A_1, \ldots, A_{k-1}$ and $R_1, \ldots, R_l$ spanned by $A_1^{\alpha_1} \ldots A_{k-1}^{\alpha_{k-1}} R_1^{\beta_1} \ldots R_l^{\beta_l}$, such that

$$\sum_{j=1}^{k-1} j\alpha_j + \sum_{j=1}^{l} j\beta_j = k - 2l,$$

with l running from 0 to $[(k-1)/3]$, the integer part of $(k-1)/3$.

The dependence on the slow variable σ is determined by means of averaging the right-hand sides. The average over the period of the fast variable should vanish. The gives

$$\frac{\partial \tilde{A}_{k-2}}{\partial \sigma} = -2\tilde{R}_{k-1} + \tilde{F}_k(\sigma), \quad \frac{\partial \tilde{R}_{k-2}}{\partial \sigma} = \tilde{G}_k(\sigma), \tag{4.4}$$

or

$$\tilde{R}_{k-1} = \frac{1}{2}\Big(\frac{\partial \tilde{A}_{k-2}}{\partial \sigma} - \tilde{F}_k\Big), \quad \frac{\partial^2 \tilde{A}_{k-3}}{\partial \sigma^2} = -2\tilde{G}_k - \frac{\partial \tilde{F}_{k-1}}{\partial \sigma},$$

where $\tilde{F}_k$ and $\tilde{G}_k$ are certain functions of the independent variable σ which are obtained by averaging equations (4.3) over S in the interval $[0, 2\pi)$. In particular, for $k = 2$ the computation yields

$$\begin{aligned}
A_2(S,\sigma) &= \frac{f^2}{2\sigma^4}\sin 2(S+A_0) \\
&- \frac{2f}{\sigma^2}\cos(S+A_0)\,\tilde{A}_1 \\
&- \frac{f}{\sigma}\sin(S+A_0) \\
&+ \tilde{A}_2(\sigma), \\
R_2(S,\sigma) &= \frac{f^3}{4\sigma^5}\Big(\cos 3(S+A_0) - 5\cos(S+A_0)\Big) \\
&+ \frac{f}{\sigma^3}\Big(f\sin 2(S+A_0)\,\tilde{A}_1 + \sin(S+A_0)\Big) \\
&- \frac{f}{\sigma^2}\Big(f\cos 2(S+A_0) + \cos(S+A_0)\,\frac{\partial A_0}{\partial \sigma}\Big) \\
&- \frac{f}{\sigma}\Big(\sin(S+A_0)\,\tilde{A}_2 + \cos(S+A_0)\,\tilde{A}_1^2\Big) \\
&+ \tilde{R}_2(\sigma), \\
\tilde{R}_1(\sigma) &= -\frac{1}{2}\,\tilde{A}_0(\sigma).
\end{aligned}$$

On using the form of the right-hand sides in (4.3) we deduce the following result.

Lemma 4.1. *For each $k \geq 2$, equations (4.3) have a two-parameter family of solutions uniformly bounded in S.*

The principal term of asymptotics is of the form

$$\begin{aligned}
\tilde{A}_0(\sigma) &= \frac{1}{\sigma^2}\,f^2 + R_1^0\,\sigma + A_0^0, \\
\tilde{R}_0(\sigma) &= \frac{1}{\sigma}\,\cos(S + A_0(\sigma)),
\end{aligned}$$

where R_1^0 and A_0^0 are arbitrary constants.

System (4.3) is triangular and recurrent in k. It is integrated in S step by step, first the first equation in (4.3) and then the second one. The integration leads to the growth of singularity of higher amendments at the point $\sigma = 0$. We get

$$\begin{aligned} A_k &= O(\sigma^{-2k-2}), \\ R_k &= O(\sigma^{-2k-1}), \end{aligned}$$

as $\sigma \to 0$. These formulas allow one to describe the domain of validity of the constructed asymptotics in a neighbourhood of $\sigma = 0$. This is $\varepsilon^{-1/2}|\sigma| \gg 1$ or $|t-1| \gg \varepsilon^{-1/2}$.

5 Inner asymptotic solution

For small s we will construct an asymptotic solution of another form. To this end we introduce the new fast independent variable

$$\vartheta = \sigma\varepsilon^{-3/2}.$$

The function ψ is searched for in the form

$$\psi = \big(1 + \varepsilon^{3/2}\rho(\vartheta,\varepsilon)\big)\, e^{\imath\alpha(\vartheta,\varepsilon)}.$$

Substitute the new expression for ψ into equation (3.1). Single out equations for the real and imaginary parts. As a result we get a system of equations for ρ and α

$$\begin{aligned} \frac{d\alpha}{d\vartheta} &= -2\rho + \varepsilon\vartheta - \varepsilon^{3/2}\Big(\rho^2 + \frac{f\cos\alpha}{1+\varepsilon^{3/2}\rho}\Big), \\ \frac{d\rho}{d\vartheta} &= -f\sin\alpha. \end{aligned} \tag{5.1}$$

The asymptotic solution of this system is constructed by the method of perturbation theory in parameter ε

$$\alpha = \alpha_0(\vartheta,\varepsilon) + \varepsilon^{3/2}\alpha_1(\vartheta,\varepsilon), \quad \rho = \rho_0(\vartheta,\varepsilon) + \varepsilon^{3/2}\rho_1(\vartheta,\varepsilon).$$

The principal terms in ε in the formulas for α and ρ satisfy the system of equations

$$\begin{aligned} \frac{d\alpha_0}{d\vartheta} &= -2\rho_0 + \varepsilon\vartheta, \\ \frac{d\rho_0}{d\vartheta} &= -f\sin\alpha_0. \end{aligned} \tag{5.2}$$

System (5.2) is equivalent to the equation of mathematical pendulum with outer momentum

$$\frac{d^2\alpha_0}{d\vartheta^2} = 2f\,\sin\alpha_0 + \varepsilon. \tag{5.3}$$

It follows that (5.2) is integrable by quadratures. The first integral of system (5.2) has the form

$$E = \left(\rho_0 - \frac{\varepsilon}{2}\vartheta\right)^2 + f\cos\alpha_0 - \frac{\varepsilon}{2}\,\alpha_0. \tag{5.4}$$

The simplest solutions of system (5.2) correspond to saddle and centre points of equation (5.3). In particular, to saddle points there correspond the solutions

$$\begin{aligned}\rho_0 &= \frac{\varepsilon\vartheta}{2},\\ \alpha_{s,k} &= -\arcsin\left(\frac{\varepsilon}{2f}\right) + 2\pi k = -\frac{\varepsilon}{2f} + O(\varepsilon^3) + 2\pi k\end{aligned}$$

for $k = 0, \pm 1, \ldots$. The value of the first integral at the k-th saddle point just amounts to

$$E_{s,k} = f\cos\alpha_{s,k} - \frac{\varepsilon\alpha_{s,k}}{2} = -f + \frac{3\varepsilon^2}{8f} - \varepsilon\pi k + O(\varepsilon^4).$$

In the phase portrait of mathematical pendulum with outer momentum to centre points there correspond the solutions of system (5.2)

$$\begin{aligned}\rho_0 &= \frac{\varepsilon\vartheta}{2},\\ \alpha_{c,k} &= \arcsin\left(\frac{\varepsilon}{2f}\right) + \pi(2k-1) = \frac{\varepsilon}{2f} + O(\varepsilon^3) + \pi(2k-1),\end{aligned}$$

k being an arbitrary integer. The value of the first integral at the k-th centre point is equal to

$$E_{c,k} = f\cos\alpha_{c,k} - \frac{\varepsilon\alpha_{c,k}}{2} = f + \varepsilon\frac{\pi}{2} - \frac{3\varepsilon^2}{8f} - \varepsilon\pi k + O(\varepsilon^4).$$

Besides of saddle and centre points there is also a family of special solution to equation (5.2) called separatrices. To each saddle point there correspond three separatrices. The values of the first integral at the separatrices just amount to the values of the first integral at the saddle points to each the separatrix branches tend as $\vartheta \to \pm\infty$. Among separatrices of (5.2) there are pairs which correspond to unbounded motions, more precisely, $\alpha \to \infty$, as

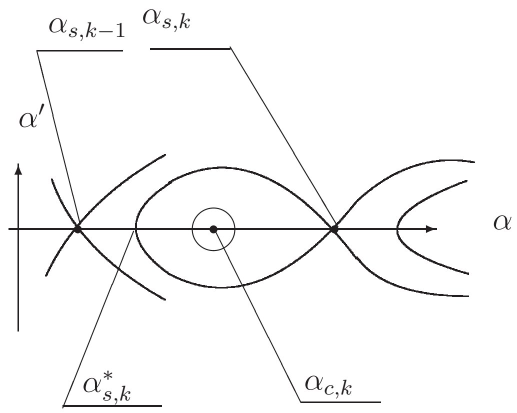


Fig. 4: A schematic sketch of the phase portrait of mathematical pendulum with outer momentum. Saddle points $\alpha_{s,k}$ and $\alpha_{s,k-1}$, a center point $\alpha_{c,k}$, and the top of separatrix loop $\alpha^*_{s,k}$.

$\vartheta \to +\infty$, and $\alpha \to \alpha_{s,k}$, as $\vartheta \to -\infty$, and the dual separatrix $\alpha \to -\infty$, as $\vartheta \to +\infty$, and $\alpha \to \alpha_{s,k}$, as $\vartheta \to +\infty$. There is moreover a loop of separatrix which goes out of its saddle point $\alpha \to \alpha_{s,k}$, as $\vartheta \to \pm\infty$, and embraces a neighbouring centre, see Fig. 4.

Denote by $\alpha^*_{s,k}$ the intersection point of the loop of separatrix going out of the k-th saddle point, and the line $\rho - \varepsilon\vartheta/2 = 0$. This is actually the top point of the separatrix loop. Within the loop in the phase plane those solutions are situated which correspond to oscillations about the k-th centre point.

Evaluate the asymptotics of $\alpha^*_{s,k}$ in ε. Let $\alpha^*_{s,k} = \alpha_{s,k} - 2\pi + \sqrt{\varepsilon}\tilde{\alpha}_{s,k}$. On substituting this formula into the the first integral at the separatrix we get readily

$$E_k = f\,\cos(\alpha_{s,k} - 2\pi + \sqrt{\varepsilon}\tilde{\alpha}_{s,k}) - \frac{\varepsilon}{2}\,(\alpha_{s,k} - 2\pi + \sqrt{\varepsilon}\tilde{\alpha}_{s,k}).$$

By the above, the values E_k and $E_{s,k}$ coincide. Equating these expressions gives

$$f\,\cos\alpha_{s,k}\left(\cos\sqrt{\varepsilon}\tilde{\alpha}_{s,k} - 1\right) - f\,\sin\alpha_{s,k}\sin\sqrt{\varepsilon}\tilde{\alpha}_{s,k} + \pi\varepsilon - \frac{\varepsilon^{3/2}}{2}\,\tilde{\alpha}_{s,k} = 0.$$

Replacing the trigonometric functions by their Taylor expansions for $\varepsilon \to 0$ we obtain as a result

$$\tilde{\alpha}^2_{s,k} = \frac{2\pi}{f} + O(\varepsilon^{3/2})$$

whence

$$\alpha^*_{s,k} = \alpha_{s,k} - 2\pi + \sqrt{\frac{\varepsilon\pi}{f}} + O(\varepsilon).$$

The gap between the top point of the loop of the separatrix related to the k-th saddle point and the preceding saddle point is

$$\Delta = \sqrt{\frac{\varepsilon\pi}{f}} + O(\varepsilon).$$

For large values of the variable ϑ the solutions of system (5.2) represent as series in the reciprocal powers of parameter ϑ. Suppose that

$$\begin{aligned} \alpha_0 &= \frac{\varepsilon\vartheta^2}{2} + \alpha_{0,1}\vartheta + \alpha_{0,0} + \sum_{k=2}^{\infty} \alpha_{0,-k}(S)\,\vartheta^{-k}, \\ \rho_0 &= \rho_{0,0} + \sum_{k=1}^{\infty} \rho_{0,-k}(S)\,\theta^{-k}, \end{aligned} \tag{5.5}$$

where $S = \dfrac{\varepsilon\vartheta^2}{2} + \alpha_{0,1}\vartheta + \alpha_{0,0}$.

Substitute the formal asymptotic expansions into system (5.2). Equate the coefficients of the same powers of the large parameter ϑ. As a result we arrive at a recurrent system for the undetermined coefficients of expansions (5.5). In particular, on equating the coefficients of ϑ^0 we conclude readily that

$$\begin{aligned} \alpha_{0,1} &= -2\,\rho_{0,0}, \\ \varepsilon\,\frac{d\rho_{0,-1}}{dS} &= -f\,\sin S. \end{aligned}$$

From these relations we get

$$\rho_{0,-1}(S) = \varepsilon^{-1} f\,\cos S + \tilde{\rho}_{0,-1},$$

where $\tilde{\rho}_{0,-1}$ is a constant to be determined below from the boundedness condition for higher amendments in the variable S. On equating the coefficients of ϑ^{-1} we get

$$\begin{aligned} \varepsilon\,\frac{d\alpha_{0,-2}}{dS} &= -2\,\rho_{0,-1}, \\ \varepsilon\,\frac{d\rho_{0,-2}}{dS} + \alpha_{0,1}\frac{d\rho_{0,-1}}{dS} &= 0. \end{aligned}$$

Substituting the explicit expression for $\rho_{0,-1}$ and integrating the equations we derive

$$\begin{aligned}\alpha_{0,-2} &= -\varepsilon^{-2}2f\,\sin S + 2\tilde{\rho}_{0,-1}S + \tilde{\alpha}_{0,-2},\\ \rho_{0,-2} &= -\varepsilon^{-2}\alpha_{0,1}f\cos S + \tilde{\rho}_{0,-2}.\end{aligned}$$

Since the amendment $\alpha_{0,-2}$ is required to be bounded, we are lead to a formula for $\tilde{\rho}_{0,-1}$, namely

$$\tilde{\rho}_{0,-1} = 0.$$

Equating the coefficients of ϑ^{-2} yields a system for determining $\alpha_{0,-3}$ and $\rho_{0,-3}$,

$$\begin{aligned}\varepsilon\,\frac{d\alpha_{0,-3}}{dS} + \alpha_{0,1}\frac{d\alpha_{0,-2}}{dS} &= -2\rho_{0,-2},\\ \varepsilon\,\frac{d\rho_{0,-3}}{dS} + \alpha_{0,1}\frac{d\rho_{0,-2}}{dS} - \rho_{0,-1} &= -f\cos S\,\alpha_{0,-2}.\end{aligned}$$

On substituting the explicit expressions for $\rho_{0,-2}$ and $\alpha_{0,-2}$ we integrate the equations. The requirement of boundedness of the coefficients in the parameter S gives readily

$$\tilde{\rho}_{0,-2} = 0,$$

and so

$$\begin{aligned}\alpha_{0,-3} &= -\varepsilon^{-3}8\rho_{0,0}f\sin S + \tilde{\alpha}_{0,-3},\\ \rho_{0,-3} &= \varepsilon^{-3}\left(-f^2\cos^2 S + 4f\rho_{0,0}^2\cos S + \varepsilon f\sin S - \varepsilon^2 f\tilde{\alpha}_{0,-2}\sin S\right) + \tilde{\rho}_{0,-3}.\end{aligned}$$

On equating the coefficients of ϑ^{-k} for arbitrary $k \geq 3$ we get in the same way

$$\begin{aligned}\varepsilon\,\frac{d\alpha_{0,-k-1}}{dS} + \alpha_{0,1}\frac{d\alpha_{0,-k}}{dS} - (k-1)\alpha_{0,-k+1} &= -2\rho_{0,-k},\\ \varepsilon\,\frac{d\rho_{0,-k-1}}{dS} + \alpha_{0,1}\frac{d\rho_{0,-k}}{dS} - (k-1)\rho_{0,-k+1} &= -f\cos S\,\alpha_{0,-k} + P_k,\end{aligned}$$

where $P_k = -f\sin S\,\alpha_{0,-2}\alpha_{0,-k+2}$ up to a polynomial of degree $k-1$ in $\alpha_{0,-2},\dots,\alpha_{0,-k+3}$ with coefficients periodic in S.

The solution of the recurrent system for $\alpha_{0,-k}$ and $\rho_{0,-k}$ has the form

$$\alpha_{0,-k}(S) = \varepsilon^{-1}F_k(S) + \tilde{\alpha}_{0,-k}, \quad \rho_{0,-k}(S) = \varepsilon^{-1}G_k(S) + \tilde{\rho}_{0,-k},$$

where $F_k(S)$ and $G_k(S)$ are certain trigonometric polynomials. The formula for $\tilde{\rho}_{0,-k}$ is established when one constructs bounded solutions to the equation for $\alpha_{0,-k-1}$. This is

$$\tilde{\rho}_{0,-k} = \frac{1}{4\pi}\int_0^{2\pi}\Big(-\alpha_{0,1}\frac{d\alpha_{0,-k}}{dS} + (k-1)\alpha_{0,-k+1}\Big)dS.$$

The formula for $\tilde{\alpha}_{0,-k}$ is established when one constructs bounded solutions to the equation for $\rho_{0,-k-3}$. When doing so, one should take into account the explicit form of $\alpha_{0,-2}$ on the right-hand side of the equation. As a result we get

$$\tilde{\alpha}_{0,-k} = \frac{1}{\varepsilon^2 f\pi}\int_0^{2\pi}\Big(-\alpha_{0,1}\frac{d\rho_{0,-k-2}}{dS} + (k+1)\rho_{0,-k-1} - f\alpha_{0,-k-2}\cos S + P_{k+2}\Big)dS.$$

We now summarize what has already been proved.

Theorem 5.1. *For $\vartheta \to \pm\infty$ there is a formal two-parameter family of solutions to system (5.2) of the form (5.5).*

The parameters of asymptotic solution (5.5) are the constants $\alpha_{0,0}$ and $\rho_{0,0}$. When having granted $\alpha_{0,0}$ and $\rho_{0,0}$, one can evaluate the first integral of system (5.2) at the asymptotic solution (5.5). For this purpose we substitute the asymptotics just constructed to the expression for the first integral and pass to the limit as $\vartheta \to -\infty$. As a result of obvious transformations we obtain

$$E = \rho_{0,0}^2 - \frac{\varepsilon}{2}\alpha_{0,0}.$$

Among constructed asymptotic solutions (5.5) there are a countable number of separatrix solutions. The parameters of these solutions are determined from the equalities

$$\rho_{0,0}^2 - \frac{\varepsilon}{2}\alpha_{0,0} = E_{s,k} \tag{5.6}$$

for $k = 0, \pm 1, \dots$.

This formula determines two branches of separatrix. On the upper branch we have $\alpha \to \infty$, as $\vartheta \to -\infty$, and $\alpha \to \alpha_{s,k}$, as $\vartheta \to +\infty$, on the lower branch we have $\alpha \to \alpha_{s,k}$, as $\vartheta \to -\infty$, and $\alpha \to \infty$, as $\vartheta \to +\infty$. The solutions, for which equality (5.6) fails to hold, are such that $\alpha'_\vartheta \to \infty$, as $\vartheta \to -\infty$, and $\alpha'_\vartheta \to -\infty$, as $\vartheta \to +\infty$.

Each constructed asymptotic has two arbitrary parameters $\alpha_{0,0}$ and $\rho_{0,0}$. For the same solution the parameters of asymptotics for $\vartheta \to -\infty$ and for $\vartheta \to +\infty$ are different in general. In order to distinguish the parameters $\alpha_{0,0}$ and $\rho_{0,0}$ of the same solution, we deduce connection formulas. To do this, we make use of the integrability in squares of equation (5.3). From the formula for the first integral we get

$$\frac{d\alpha_0}{d\vartheta} = \sqrt{4E - 4f\cos\alpha_0 + 2\varepsilon\alpha_0}.$$

Denote by $\overline{\alpha}$ the minimal value of α on the trajectory (turning point). For given E, the turning point $\overline{\alpha}$ proves to be the maximal of the solutions of equation

$$4E = 4f\,\cos\bar{\alpha} - 2\varepsilon\bar{\alpha}$$

with $-f\sin\bar{\alpha}+\varepsilon/2 \neq 0$. The equation determines the turning point while the inequality guarantees that the turning point is not a saddle point or, what is the same, the curve $E = \text{const}$ with given parameters $\alpha_{0,0}$ and $\rho_{0,0}$ fails to be a separatrix. One can show that if $E_{s,k} < E < E_{s,k+1}$ then $\alpha_{s,k} < \bar{\alpha} < \alpha^*_{s,k}$ holds.

We now turn to deriving a connection formula. The implicit formula for the general solution has the form

$$\int \frac{d\alpha}{\sqrt{4E - 4f\cos\alpha + 2\varepsilon\alpha}} = \int d\vartheta.$$

If $\vartheta \to \pm\infty$ then $\alpha \to \infty$. To get a connection formula write

$$\begin{aligned}
&\int_{\bar{\alpha}}^{\alpha_0} \frac{dx}{\sqrt{4E - 4f\cos x + 2\varepsilon x}} \\
&= \int_{\bar{\alpha}}^{\alpha_0} \Big(\frac{1}{\sqrt{4E-4f\cos x+2\varepsilon x}} - \frac{1}{\sqrt{2\varepsilon(x-\bar{\alpha})}}\Big)dx + \int_{\bar{\alpha}}^{\alpha_0} \frac{dx}{\sqrt{2\varepsilon(x-\bar{\alpha})}} \\
&= \Big(\int_{\bar{\alpha}}^{\infty} - \int_{\alpha_0}^{\infty}\Big)\Big(\frac{1}{\sqrt{4E-4f\cos x+2\varepsilon x}} - \frac{1}{\sqrt{2\varepsilon(x-\bar{\alpha})}}\Big)dx + \sqrt{\frac{2}{\varepsilon}}\sqrt{\alpha_0-\bar{\alpha}}.
\end{aligned}$$

This regularised formula contains the integrals converging in x. This form is convenient for evaluating the asymptotics of solution as $\vartheta \to \pm\infty$. The first integral in the obtained formula can be thought of as constant of integration. Substituting the regularized integral into the implicit formula for solution gives

$$-\int_{\alpha_0}^{\infty} \Big(\frac{1}{\sqrt{4E-4f\cos x+2\varepsilon x}} - \frac{1}{\sqrt{2\varepsilon(x-\bar{\alpha})}}\Big)dx + \sqrt{\frac{2}{\varepsilon}}\sqrt{\alpha_0-\bar{\alpha}} = \vartheta + \text{const}.$$

For $\alpha_0 \to +\infty$ the integral on the left-hand side tends to zero. Hence it follows that

$$\alpha_0 = \frac{\varepsilon}{2}\,(\vartheta + \text{const})^2 + \bar{\alpha},$$

as $\vartheta \to -\infty$. On the other, from (5.5) we conclude that

$$\alpha_0 = \frac{\varepsilon}{2}\vartheta^2 - 2\rho_{0,0}\vartheta + \alpha_{0,0},$$

as $\vartheta \to -\infty$. Comparing these formulas yields

$$\text{const} = -\frac{2}{\varepsilon}\,\rho_{0,0}, \qquad \alpha_{0,0} = \bar\alpha + \frac{2}{\varepsilon}\,\rho_{0,0}^2.$$

For further computations it is convenient to introduce the designation

$$I(E,\varepsilon) = \int_{\bar\alpha}^{\infty} \Big(\frac{1}{\sqrt{4E-4f\cos x+2\varepsilon x}} - \frac{1}{\sqrt{2\varepsilon(x-\bar\alpha)}}\Big)dx.$$

The parameter $\bar\alpha$ is the greatest solution of $4E-4f\cos\bar\alpha+2\varepsilon\bar\alpha=0$ for given values E and ε. From the form of the integrand function one sees that the improper integral I exists for each $\varepsilon>0$. The change of variables $y=x-\bar\alpha$ reduces it to

$$I(E,\varepsilon) = \int_{0}^{\infty} \Big(\frac{1}{\sqrt{4f\cos\bar\alpha-4f\cos(y+\bar\alpha)+2\varepsilon y}} - \frac{1}{\sqrt{2\varepsilon y}}\Big)dy.$$

From the implicit formula for the general solution we deduce as above that for $\vartheta\to+\infty$ the equality

$$\frac{2}{\varepsilon}\,(\alpha_0-\bar\alpha) = (\vartheta+\text{const}-I(E,\varepsilon))^2$$

holds whence

$$\alpha_0 = \bar\alpha + \frac{\varepsilon}{2}\left(\vartheta^2 + 2\vartheta(\text{const}-I(E,\varepsilon)) + (\text{const}-I(E,\varepsilon))^2\right).$$

Using the known values of parameters for $\vartheta\to-\infty$ we derive the parameters of asymptotics for $\vartheta\to+\infty$, namely

$$\begin{aligned}\alpha_0^+ &= \alpha_0^- + 2\rho_{0,0}I(E,\varepsilon) + \frac{\varepsilon}{2}\,I(E,\varepsilon)^2,\\ \rho_0^+ &= \rho_0^- + \frac{\varepsilon}{2}\,I(E,\varepsilon).\end{aligned} \tag{5.7}$$

Formulas (5.7) demonstrate explicit connections of asymptotics of the solution to (5.2) for $\vartheta\to\pm\infty$. Changing the variable in the integral $I(E,\varepsilon)$ by $z=2\varepsilon y$ yields $I(E,\varepsilon)=\mathcal{I}/2\varepsilon$, where

$$\mathcal{I} = \int_{0}^{\infty} \Big(\frac{1}{\sqrt{8f\sin(\bar\alpha+z/4\varepsilon)\sin(-z/4\varepsilon)+z}} - \frac{1}{\sqrt{z}}\Big)dz$$

which shows that $I(E,\varepsilon)=O(\varepsilon^{-1})$ as $\varepsilon\to 0$. It would be desirable to derive asymptotics of this integral up to $O(1)$ as $\varepsilon\to 0$ but we have not been able to do this.

6 Connection formulas for perturbed system

We now consider the perturbed system, see (5.1). Our concern will be to construct a solution to this system for large values of ϑ. Assume that the parameters $\alpha^\pm$ and $\rho^\pm$ of the solution depend slowly on ϑ. Substitute the asymptotics of the solution to the perturbed system and average the system over the fast time ϑ. As a result of integration of the averaged system we obtain

$$\begin{aligned} \alpha^\pm &= \varepsilon^{3/2}\left((\rho_0^\pm)^2\,\vartheta + O(\vartheta^{-1})\right), \\ \rho^\pm &= \rho_0^\pm + \varepsilon^{3/2}\,O(\vartheta^{-1/2}), \end{aligned}$$

where $\rho_0^- = \rho_{0,0}$ for uniformity. These formulas determine the modulation of parameters of the solution to (5.1) for large values ϑ.

Derive an equation for the evolution of parameter E for the perturbed system (5.1). To this end we differentiate the expression for E according to system (5.1). This gives

$$\frac{dE}{d\vartheta} \sim \varepsilon^{3/2}\left(f\rho^2\sin\alpha + f^2\sin\alpha\cos\alpha\right) + \frac{\varepsilon^{5/2}}{2}\left(f\cos\alpha + \rho^2\right). \qquad (6.8)$$

The derivative of E is small, hence the parameter E changes little when ϑ runs over a bounded interval. One should study the behaviour of E for large ϑ, since the changes of E may be essential on a big interval. For large values of ϑ it is convenient to use the asymptotics of α_0 and ρ_0 evaluated above. Substituting the asymptotics into (6.8) and gathering similar terms we arrive at the equation

$$\frac{dE}{d\vartheta} \sim \frac{\varepsilon^{5/2}}{2}\left(f\cos(-\varepsilon\frac{\vartheta^2}{2} + \rho_{0,0}\vartheta - \alpha_{0,0}) + \rho_{0,0}^2\right)$$

for $\vartheta \to -\infty$. Averaging over the fast variable $S = -\varepsilon\,\vartheta^2/2 + \rho_{0,0}\vartheta - \alpha_{0,0}$ leads to an equation for the slow modulation $\tilde{E}$ of the parameter E. More precisely,

$$\frac{d\tilde{E}}{d\vartheta} \sim \frac{\varepsilon^{5/2}}{2}\left(\rho_{0,0}^2 + \frac{f^2}{2\varepsilon^2}\,\frac{1}{\vartheta^2}\right). \qquad (6.9)$$

The change $s = \varepsilon^{5/2}\vartheta$ reduces this equation to $\dfrac{d\tilde{E}}{ds} \sim \dfrac{1}{2}\,\rho_{0,0}^2 + \varepsilon^3\,\dfrac{f^2}{4s^2}$. Integration gives

$$\tilde{E} \sim \text{const} + \frac{\rho_{0,0}^2}{2}\,s - \varepsilon^3\,\frac{f^2}{4s}.$$

For $\vartheta \to -\infty$, we take into account the value $\rho_{0,0} = 0$ from the outer asymptotic expansion. Adjusting the formulas with each other we get

$$\tilde{E} \sim \frac{1}{2}\varepsilon\varphi - \varepsilon^3 \frac{f^2}{4s},$$

where φ is the phase shift to be treated as parameter of the solution.

At the separatrices and saddle points we have $E = E_k$. Hence it follows that on a separatrix φ satisfies

$$\frac{1}{2}\varepsilon\varphi_k \sim -f - \varepsilon\,\pi k,$$

k being an integer number. Then,

$$\varphi_k \sim -\frac{2f}{\varepsilon} - 2\pi k$$

holds on the k-th separatrix.

The constructed outer asymptotics suits if $-\varepsilon^{-1/2}\sigma \gg 1$. In terms of the inner variable ϑ this inequality just amounts to $-\vartheta \gg \varepsilon^{-1}$. Matching of outer and inner asymptotics leads to the following assertion.

Theorem 6.1. *Given any* const > 0, *assume that* $\varepsilon^{-3/2}|E - E_k| > 0$ *for* $\sigma \to -0$. *Then the trajectories of solutions to equation (1.2) do not capture into resonance.*

The formula for E_k implies $E_{k+1} - E_k = \varepsilon\pi$. Hence, within strips of width $\varepsilon\pi$ in parameter E the asymptotic solutions with parameter values in any strip $E_{k-1} + C_1\varepsilon^{3/2} < E < E_k - C_2\varepsilon^{3/2}$, where C_1 and C_2 are arbitrary positive constants, do not capture into resonance close to centers of nonperturbed system (5.2).

The domain of validity of inner asymptotics From the view point of perturbation theory the solution of system (5.1) can be searched for in the form of solution to (5.2) with slowly varying parameters. From general considerations of dependence of the solution to (5.2) on the parameter E we derive the estimate

$$\varepsilon^{5/2}\vartheta^2 \ll 1,$$

or $|\vartheta| \ll \varepsilon^{5/4}$.

7 Capture into resonance

In this section we carry out analysis of solutions of perturbed system (5.1) in neighbourhoods of saddle points and show the width of the domain of those values of E for which the solutions are captured into oscillations about the center.

The rebuilding of solutions takes place in neighbourhoods of saddle points of nonperturbed system (5.2). Earlier we obtained an estimate for the domain of those values of E for which no capture happens. Capture into resonance for the perturbed system occurs within the domain $E - E_k = o(\varepsilon^{3/2})$. To describe the capture domain more precisely it is necessary to conduct more delicate investigations of trajectories of perturbed system (5.1) nearby saddle points.

Slowly varying equilibrium points For perturbed system (5.1), the slowly varying solutions are analogues of equilibrium points. We will look for such solutions of the form of formal series in powers of $\varepsilon^{1/2}$, the so-called Puiseax series. Substitute such series to system (5.1) rewritten in terms of $\tau = \varepsilon\vartheta$. As a result of the standard procedure of equating the coefficients of the the same powers of $\varepsilon^{1/2}$ we get a recurrent system of algebraic equations for determining the coefficients $\tilde{\alpha}_k$ and $\tilde{\rho}_k$ of these expansions. Solving the system yields

$$\begin{aligned} \tilde{\alpha}_k &\sim \pi k + \varepsilon\frac{(-1)^{k+1}}{2f} + \varepsilon^{5/2}\frac{(-1)^k\tau}{4f} + \varepsilon^3\frac{(-1)^{k+1}}{48f^3}, \\ \tilde{\rho}_k &\sim \frac{\tau}{2} - \varepsilon^{3/2}\Big(\frac{\tau^2}{8} + \frac{(-1)^k}{2} + \frac{f}{2}\Big) + \varepsilon^3\Big(\frac{\tau^3}{16} + \frac{(-1)^k f\tau}{2}\Big) \end{aligned} \tag{7.10}$$

for $k = 0, 1, \ldots$. Asymptotics (7.10) are applicable for $\tau \ll \varepsilon^{-3/2}$.

On introducing the new dependent variables $\tilde{\alpha} = \alpha - \tilde{\alpha}_k$ and $\tilde{\rho} = \rho - \tilde{\rho}_k$ and letting $\varepsilon \to 0$ in such a way that $\varepsilon\,|\vartheta| = O(1)$ we obtain

$$\begin{aligned} \frac{d}{d\vartheta}\tilde{\alpha} &\sim 2\tilde{\rho} - \varepsilon^{3/2}\left(\tilde{\rho}^2 + f\,(\cos\tilde{\alpha} - 1)\right) - \varepsilon^{5/2}\Big(\frac{\sin\tilde{\alpha}}{2} + \vartheta\rho\Big) \\ &+ \frac{\varepsilon^3}{4}\left(4f\tilde{\rho}\,(\cos\tilde{\alpha} + 1) + 2\varepsilon\vartheta f(\cos\tilde{\alpha} - 1) + (\varepsilon\vartheta)^2\,\tilde{\rho}\right), \\ \frac{d}{d\vartheta}\tilde{\rho} &\sim -f\sin\tilde{\alpha} + \varepsilon\,\frac{\cos\tilde{\alpha} - 1}{2} + \varepsilon^2\frac{\sin\tilde{\alpha}}{8} - \varepsilon^{7/2}\vartheta\,\frac{\cos\tilde{\alpha} - 1}{4}. \end{aligned} \tag{7.11}$$

One can show that for $k = 2m$ the slowly varying solutions are saddle points. A thorough analysis using the WKB method in much the same way as in [15] actually shows that the points $\tilde{\alpha} = 0$, $\tilde{\rho} = 0$ for $k = 2m + 1$ are stable focal points.

A rough conservation law Expression (5.4) changes little on solutions of system (5.1) when $\vartheta \to \infty$. This follows immediately from (6.9). However, this expression oscillates rapidly with amplitude of order $\varepsilon^{5/2}$. This is easily shown by immediate substitution of the constructed asymptotics for $\vartheta \to \infty$. In order to study solutions in small neighbourhoods of turning points it is convenient to use a modified form of equation (6.9) which oscillates in ϑ with amplitude much less than $\varepsilon^{5/2}$ and is valid for $|\vartheta| \ll \varepsilon^{-5/2}$ just as system (7.11). Set

$$\tilde{E} = \tilde{\rho}^2 + f(\cos\tilde{\alpha} - 1) - \frac{\varepsilon}{2}(\sin\tilde{\alpha} - \tilde{\alpha}) + \varepsilon^{3/2}(f\tilde{\rho}\cos\alpha + \tilde{\rho}^3 - f\tilde{\rho}) + \varepsilon^{5/2}\frac{\vartheta\tilde{\rho}^2}{2} + \varepsilon^2\frac{\tilde{\rho}^2}{8f}.$$

Differentiating in θ according to system (7.11) gives

$$\frac{d\tilde{E}}{d\vartheta} \sim \frac{\varepsilon^{5/2}}{2}\left(2\tilde{\rho}^2\cos\tilde{\alpha} + f\sin\tilde{\alpha} - \tilde{\rho}^2\right) \tag{7.12}$$

for $|\vartheta| \ll \varepsilon^{-5/2}$.

Breaking up of separatrix Consider the separatrices arriving at a point (α_{2m}, ρ_{2m}). To be specific, assume that $m = 0$. At the saddle point two separatrices arrive when $t \to \infty$. For both separatrices the limit value of $\tilde{E}$ as $t \to \infty$ is equal to $\varepsilon\pi m$. However, one of these separatrices loops the loop about the point $(\alpha_{2m+1}, \rho_{2m+1})$. From equation (7.12) it follows that the values of $\tilde{E}$ on the separatrices on the left of line $\tilde{\alpha} = 2\pi m$ differ by the value of Mel'nikov's integral [18] over the loop ℓ of the separatrix of nonperturbed system (5.2). Namely,

$$\Delta\tilde{E} \sim \frac{\varepsilon^{5/2}}{2}\int_\ell \left(2\tilde{\rho}^2\cos\tilde{\alpha} + f\sin^2\tilde{\alpha} - \tilde{\rho}^2\right) d\vartheta.$$

The integral over the loop of separatrix of the equation of mathematical pendulum with outer momentum ε tends to the sum of two integrals over the upper and lower separatrices of mathematical pendulum without outer momentum, when $\varepsilon \to 0$. The loop of separatrix for mathematical pendulum with outer momentum begins and ends at the same saddle point. For mathematical pendulum without outer momentum the upper and lower separatrices begin and end at different saddle points. Write the integral over loop of separatrix as the sum of integrals of integrand terms and consider the integrals obtained in this way separately. When obviously integrated by parts, the principal term of the first integral transforms to the form

$$2\int_\ell \tilde{\rho}^2\cos\tilde{\alpha}d\vartheta \sim -\int_\ell \tilde{\rho}\cos\tilde{\alpha}\,d\tilde{\alpha} \sim -\int_\ell \sin\tilde{\alpha}\,d\tilde{\rho} \sim f\int_\ell \sin^2\tilde{\alpha}\,d\vartheta,$$

which is due to (5.2). Hence it follows that the sum of the first and second integrals amounts to

$$\begin{aligned} f\int_\ell \sin^2\tilde\alpha\, d\vartheta &\sim \frac{1}{f}\int_\ell \Big(\frac{d\tilde\rho}{d\vartheta}\Big)^2 d\vartheta \\ &= \frac{2}{f}\int_{-\infty}^{\infty}\Big(\frac{d}{d\vartheta}\frac{2\sqrt{2f}}{\cosh(\sqrt{2f}\vartheta)}\Big)^2 d\vartheta \\ &= \frac{2}{f}\int_{-\infty}^{\infty}\Big(\frac{4f\sinh(\sqrt{2f}\vartheta)}{\cosh^2(\sqrt{2f}\theta)}\Big)^2 d\vartheta \\ &= 16\sqrt{2f}\int_{-1}^{1}\tanh^2(\sqrt{2f}\vartheta)\, d\tanh(\sqrt{2f}\vartheta) \\ &= \frac{16\sqrt{2f}}{3}. \end{aligned}$$

The principal term of the third integral is evaluated explicitly, namely

$$\frac{1}{2}\int_\ell \tilde\rho^2\, d\vartheta \sim \int_{-\infty}^{\infty}\frac{4\sqrt{2f}}{\cosh^2(\sqrt{2f}\vartheta)}\, d(\sqrt{2f}\vartheta) = 8\sqrt{2f},$$

and so the formula for Mel'nikov's integral takes the form

$$\Delta\tilde E \sim -\varepsilon^{5/2}\,\frac{8\sqrt{2f}}{3}.$$

For $\tilde E_{2m} < \tilde E < \tilde E_{2m} + \Delta\tilde E$ the trajectories prove to be captured into the neighbourhood of the focal point $\tilde\alpha = \alpha - \alpha_k$, $\tilde\rho = \rho - \rho_k$ with $k = 2m+1$, see Fig. 5.

8 Matching of asymptotics

In this section we match the parameters of asymptotics constructed for the outer and inner expansions. As a result we derive connection formulas for noncaptured asymptotic solutions and describe the domain of parameters containing those asymptotic solutions which are captured into resonance.

The parameters of outer asymptotics are ε and φ. Matching of asymptotics (3.3) and (4.2) in the domain $1 \ll \varepsilon^{-1/2}s \ll \varepsilon^{-1/2}$ yields

$$A_0^0 = \varphi, \quad R_1^0 = 0.$$

Matching of asymptotics (4.2) and (5.5) in the domain $1 \ll -\varepsilon^{-1/2}\sigma \ll \varepsilon^{-1/2}$ leads to formulas

$$\alpha_{0,0} = \varphi, \quad \rho_{0,0} = 0,$$

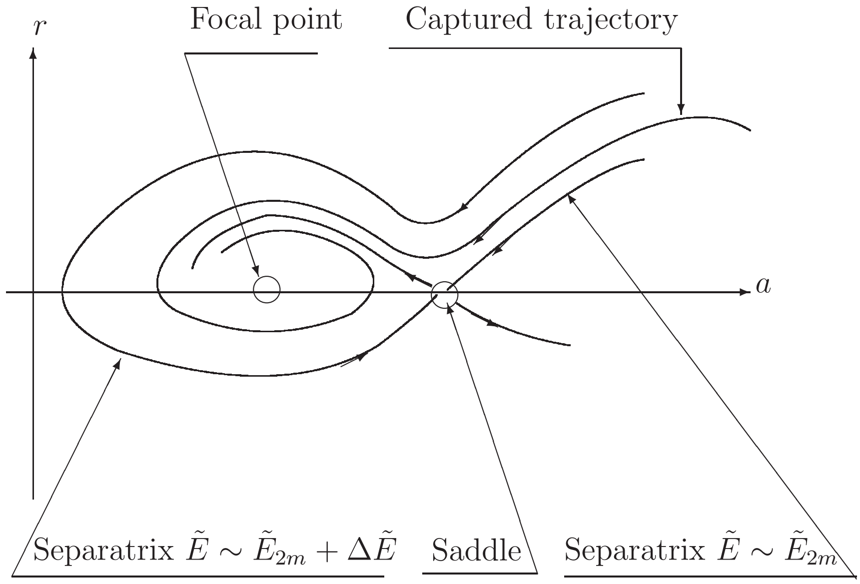


Fig. 5: Trajectory scheme of system (5.1).

The expressions for E and $\tilde{E}$ coincide in the main for $|\vartheta| \ll \varepsilon^{-5/2}$. The split separatrices differ by the quantity $\Delta\tilde{E} \sim -\varepsilon^{5/2}8\sqrt{2f}$, which does not depend on ϑ. An equivalent shift in the parameter E also causes splitting of near separatrices.

The separatrices of system (7.11) lie in the domain Ω of parameter φ given by

$$|\varphi - 2\pi k + \frac{1}{\sqrt{f}} I(E_k/(4f), \varepsilon/(4f))| = o(\varepsilon^{1/2})$$

for $k = 0, \pm 1, \dots$. For some $\varphi_1, \varphi_2 \in \Omega$, such that $\varphi_2 - \varphi_1 = 2\Delta\tilde{E}$, the trajectories with parameter φ satisfying $\varphi_1 < \varphi < \varphi_2$ are captured into resonance, i.e. $\Psi \sim \sqrt{t}$ as $t \to \infty$. The length of the interval of those φ for which the trajectories are captured into resonance is evaluated by

$$\Delta\varphi \sim 2\varepsilon^{-1}\,\Delta\tilde{E} \sim \varepsilon^{3/2}\,\frac{16\sqrt{2f}}{3}.$$

In plane of variable Ψ the area of trajectories which are captured by the time $t = O(\varepsilon^{-2})$ has the order $\Delta\varphi\,\varepsilon^{-2} = O(\varepsilon^{-1/2})$.

For noncaptured solutions the matching of parameters of asymptotics after passing the inner domain gives

$$r_3^0 \sim \frac{1}{4}\mathcal{I}, \quad \varphi^+ \sim \varphi + \frac{1}{8\varepsilon}\mathcal{I}^2.$$

Without loss of generality one can make the change $\varepsilon^+ = \varepsilon - \varepsilon^{3/2}\mathcal{I}/4$. As a result we deduce that the new value of the parameter ε in the outer expansion for $s > 0$ is actually $O(\varepsilon^{3/2})$ less than the value of ε before passing the inner domain.

9 Numerical investigations

To justify the analytical calculations we have conducted numerical simulations. We study the family of solutions to the Cauchy problem for equation (1.2) with initial data at $t = -\varepsilon^{-2}$ instead of the problem on connection of asymptotics for $t \to -\infty$ and $t \to \infty$.

In such setting there are two essential hindrances in the study of solutions with small values of parameter ε. First, the interval of numerical integration is large, $L = O(\varepsilon^{-2})$. Secondly, as asymptotic analysis shows, the solutions oscillate rapidly with frequency $\Omega = O(\varepsilon^{-4})$ far away from the origin. In this case the familiar error formula for integration by the Runge-Kutta method of order 4 with step h gives $\Delta = O(h^4\,\Omega^4\,L) = O(h^4\,\varepsilon^{-18})$.

For computations with floating point and double accuracy the value of h is chosen to be greater than 10^{-4}, since for smaller steps the discreteness of the set of double accuracy numbers adversely affect the error. This inequality and the error formula for the Runge-Kutta method Δ yield a restriction on the numerical values of parameter ε. That is, $\varepsilon > 10^{-8/9} \sim 0.1291550$ or $R < 10^{8/9} \sim 7.7426368$, where $R = \varepsilon^{-1}$.

On arguing in this way we consider the family of Cauchy problems for $\varepsilon \in [0.129, 0.3]$ with step in ε equal 0.001. For these ε we evaluated 2024 solutions on the interval $t \in [-\varepsilon^{-2}, 3/2\,\varepsilon^{-2}]$ by the Runge-Kutta method of order 4 with integration step $h = 0.0005$.

The initial conditions for the family of Cauchy problems parametrised by $N = 0, \dots, 2023$ are chosen according to the constructed asymptotic solution for $t \to -\infty$. More precisely,

$$\Psi \restriction_{t_0} = \frac{1}{\varepsilon}\, R(S,\varepsilon)\, \exp \imath\alpha(s,\varepsilon),$$

$$\alpha(s,\varepsilon) = S + \varepsilon^2\Big(\frac{f^2(2s+1)}{s^2} - \varepsilon\frac{f(s+2)\sin S}{s^2}\Big),$$

$$R(S,\varepsilon) = 1 + \varepsilon^3\Big(\frac{f\cos S}{s} - \varepsilon^2\frac{f^3(2s+1)\sin S}{s^3} + \varepsilon^3\Big(\frac{f^2}{4s^2} + \frac{f^2(4-(s+2)\cos 2S)}{4s^3}\Big)\Big)$$

where

$$\begin{aligned} s &= \varepsilon^2 t_0 - 1, \\ S &= \frac{1}{\varepsilon}\frac{s^2}{2} + \varphi_N, \end{aligned}$$

and $\varphi_N = 2\pi(\imath - N/2)/N - \dfrac{2f}{\varepsilon}$.

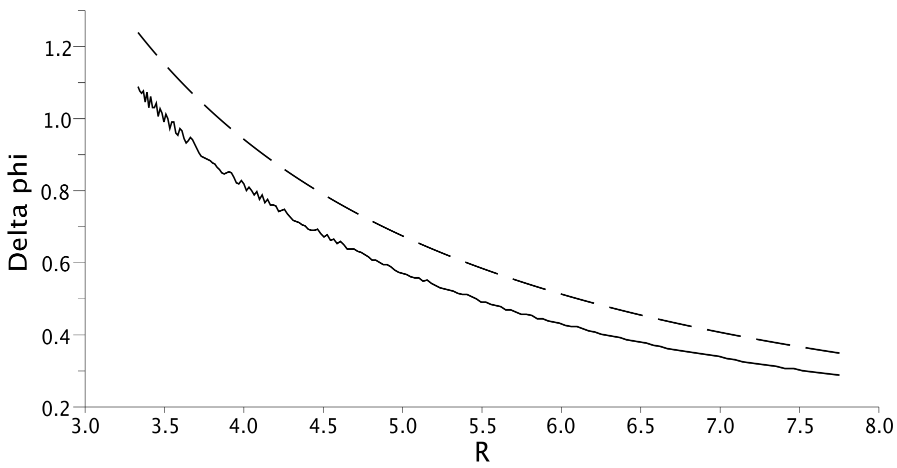


Fig. 6: The graph of $\Delta\varphi$, the width of the interval of parameters φ corresponding to captured solutions, depending on the quantity $R = \varepsilon^{-1}$. The dotted line corresponds to the constructed asymptotic formula, the continuous line is a piecewise approximation of $\Delta\varphi$ obtained numerically by data handling for the constructed family of numerical solutions to the Cauchy problems.

In Fig. 3 the domain of parameters $R = 1/\varepsilon$ and φ is painted which corresponds to captured solutions of the Cauchy problems discussed above.

In Fig. 6 we demonstrate the graph of dependence on R of the width of the interval of captured trajectories obtained by the asymptotic formula and by numerical simulation.

10 Conclusion

In the paper we derive an asymptotic formula for the connection between the parameters of those solutions which are not captured into autoresonance. Moreover, we get a formula for the measure of solutions captured into autoresonance and an estimate for the parameters before the resonance of those solutions which may be captured into autoresonance.

Acknowledgements This research was supported by the Russian Foundation for Basic Research, grant 11-01-91330, and German Research Society (DFG), grant TA 289/4-2. The authors gratefully acknowledge the many helpful suggestions of professors L. A. Kalyakin and B. I. Suleimanov.

References

[1] Krylov, N. M., and Bogolyubov, N. N., Introduction to Nonlinear Mechanics, Khar'kov, 1937, 353 pp.

[2] Allan, R. R., Evolution of Mimas-Thetys commensurability, The Astronomical Journal **72** (1969), No. 3, 497–506.

[3] Sinnclair, A. T., On the origin of the commensurabilities amongs the satellits of Saturn, Mon. Not. R. Astron. Soc. **160** (1972), 169–187.

[4] Veksler, V. I., A new method of acceleration of relativistic particles, J. Phys. USSR **9** (1945), 15–158.

[5] Fajans, J., and Friedland, L., Autoresonant (non-stationary) exitation of pendulums, Plutinos, plasmas, and other nonlinar oscillators, Amer. J. Phys. **69** (2001), No. 10, 1096–1102.

[6] Kalyakin, L. A., Asymptotic analysis of autoresonance model, Russian Math. Surweys **63** (2008), No. 5, 791–857.

[7] Neishtadt, A. I., Passing through a separatrix in a resonance problem with a slowly varying parameter, Applied Mathematics and Mechanics **39** (1975), No. 4, 621–632.

[8] Neishtadt, A., Probability phenomena in perturbed dynamical systems, In: Mechanics of the 21st Century, 2005, 241–261; DOI: 10.1007/1-4020-3559-4-16.

[9] Timofeev, A. V., On the question of constancy of the adiabatic invariant under change of the character of movement, Journal of Experimental and Theoretical Physics **75** (1978), No. 4, 1303–1308.

[10] Cary, J. R., and Scodje, R. T., Phase change between separatrix crossings, Physica D **36** (1989), 287–316.

[11] Diminnie, D. C., and Haberman, R., On the edynamic unfolding of a saddle-center bifurcation and the change in the action, Proc. Steklov Inst. Math. (2003), S91–S97.

[12] Neishtadt, A. I., The capture into resonance and scattering upon resonances in two-frequency systems, Transaction of the Moscow Mathematical Society **250** (2005), 198–218.

[13] Kiselev, O. M., Hard loss of stability in Painlevé-2 equation, Journal of Nonlinear Mathematical Physics **8** (2001), No. 1, 65–95.

[14] Kiselev, O. M., and Glebov, S. G., An asymptotic solution slowly crossing the separatrix near a saddle-center bifurcation point, Nonlinearity **16** (2003), 327–362.

[15] Kiselev, O. M., and Glebov, S. G., The capture into parametric autoresonance, Nonlinear Dynamics **48** (2007), No. 1–2, 217–230.

[16] Haberman, R., Nonlinear transition layers - second Painelevé trancendent, Studies in Applied Mathematics **57** (1977), 247–270.

[17] Maxima, a Computer Algebra System, http://maxima.sourceforge.net

[18] Mel'nikov, V. K., On stability of the centre under time periodic perturbations, Transaction of the Moscow Mathematical Society **12** (1963), 3–52.