Coupling Matter to Loop Quantum Gravity

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Zusammenfassung.

Motiviert durch neuere Vorschläge zur experimentellen Untersuchung von Quantengravitationseffekten werden in der vorliegenden Arbeit Annahmen und Methoden untersucht, die für die Vorhersagen solcher Effekte im Rahmen der Loop-Quantengravitation verwendet werden können. Dazu wird als Modellsystem ein skalares Feld, gekoppelt an das Gravitationsfeld, betrachtet.
Zunächst wird unter bestimmten Annahmen über die Dynamik des gekoppelten Systems eine Quantentheorie für das Skalarfeld vorgeschlagen. Unter der Annahme, daß sich das Gravitationsfeld in einem semiklassischen Zustand befindet, wird dann ein "QFT auf gekrümmter Raumzeit-Limes" dieser Theorie definiert. Im Gegensatz zur gewöhnlichen Quantenfeldtheorie auf gekrümmter Raumzeit beschreibt die Theorie in diesem Grenzfall jedoch ein quantisiertes Skalarfeld, das auf einem (klassisch beschriebenen) Zufallsgitter propagiert.
Sodann werden Methoden vorgeschlagen, den Niederenergieliemes einer solchen Gittertheorie, vor allem hinsichtlich der resultierenden modifizierten Dispersonsrelation, zu berechnen. Diese Methoden werden anhand von einfachen Modellsystemen untersucht.
Schliesslich werden die entwickelten Methoden unter vereinfachenden Annahmen und der Benutzung einer speziellen Klasse von semiklassischen Zustaenden angewandt, um Korrekturen zur Dispersionsrelation des skalaren und des elektromagnetischen Feldes im Rahmen der Loop-Quantengravitation zu berechnen. Diese Rechnungen haben vorläufigen Charakter, da viele Annahmen eingehen, deren Gültigkeit genauer untersucht werden muss. Zumindest zeigen sie aber Probleme und Möglichkeiten auf, im Rahmen der Loop-Quantengravitation Vorhersagen zu machen, die sich im Prinzip experimentell verifizieren lassen.

Abstract.

Motivated by recent proposals on the experimental detectability of quantum gravity effects, the present thesis investigates assumptions and methods which might be used for the prediction of such effects within the framework of loop quantum gravity. To this end, a scalar field coupled to gravity is considered as a model system.
Starting from certain assumptions about the dynamics of the coupled gravity-matter system, a quantum theory for the scalar field is proposed. Then, assuming that the gravitational field is in a semiclassical state, a "QFT on curved space-time limit" of this theory is defined. In contrast to ordinary quantum field theory on curved space-time however, in this limit the theory describes a quantum scalar field propagating on a (classical) random lattice.
Then, methods to obtain the low energy limit of such a lattice theory, especially regarding the resulting modified dispersion relations, are discussed and applied to simple model systems.
Finally, under certain simplifying assumptions, using the methods developed before as well as a specific class of semiclassical states, corrections to the dispersion relations for the scalar and the electromagnetic field are computed within the framework of loop quantum gravity. These calculations are of preliminary character, as many assumptions enter whose validity remains to be studied more thoroughly. However they exemplify the problems and possibilities of making predictions based on loop quantum gravity that are in principle testable by experiment.

Contents

1. Introduction

In modern day physics, two very different types of theories stand side by side in the description of the fundamental interactions governing our world: On the one hand, the gravitational field is described by general relativity (GR for short), a *classical* field theory in which the basic field $g_{\mu\nu}$, the metric, determines the geometry of space and time.
On the other hand, the electroweak and strong interactions are most accurately described by quantum field theories (QFT), *quantum* theories of systems with infinitely many degrees of freedom. Though many technical and conceptual parts of QFT still await deeper understanding, there is no doubt about the quantum nature of the fields, and the description of elementary particle physics via QFT as embodied in the standard model seems to be a remarkably accurate one in many respects. The apparent distinction of gravity from the other interactions has led physicists to conjecture that a quantum theory might also underly classical GR. The search for such a *quantum gravity* theory is very fascinating and an actively pursued topic in theoretical physics today. Consequently, it has been tackled from different sides. The most prominent approaches are string theory, loop quantum gravity[i] (together with its covariant offspring, the theory of spin foams) and non-commutative geometry, but many other interesting approaches exist.
Loop quantum gravity (LQG for short) is the framework for the current thesis. Unlike string theory, it is not a direct attempt to unify all fundamental interactions under a single symmetry principle in a single theory. Still it takes a big step towards unification of the fundamental forces in the following sense: The classical theories from which the standard model is derived are geometric in character in a way similar to GR. However, unlike the metric in GR, the basic field in these theories does not have the character of a field strength but is a connection – the theories governing strong and electroweak interaction are *gauge theories*. Remarkably, it is possible to cast GR in the form of a gauge theory as well, thus revealing a deep kinship to the other fundamental interactions. This reformulation discovered by Ashtekar [1] is the starting point of LQG.
The other tenet of LQG is the strict avoidance of the use of any classical background geometry in the formulation. The idea behind this is simple: In GR the split of the geometry into some immutable background and a dynamical part would spoil diffeomorphism covariance, the beautiful symmetry principle underlying the whole theory. Therefore a quantization of GR should not be based on such a split. Besides, constructing a theory with classical and quantum geometry in it, side by side, would be "stopping midway".
Background independence precludes any type of perturbative quantization. The quantum theory at the heart of LQG is therefore obtained as a Dirac type quantization. Because of its background independence, it differs considerably from standard QFT and has a remarkable geometric flavor. The first formulation was given by Rovelli and Smolin in [2]. Since then it has undergone major reformulations and is by now an established and mathematically well defined theory. Its successes include the determination of the spectrum of geometric quantities such as area and volume [3, 4, 5], the derivation of the Bekenstein formula for black hole entropy [6, 7] and quantum cosmological

[i] Loop quantum gravity is an approach with many names: "Quantized general relativity", "quantum geometry", or simply "the Ashtekar program" are in use besides "loop quantum gravity". In the present thesis, we will stick to the latter since we feel it is the most widely known one.

results [8, 9].

The topic of this thesis.

Despite the undeniable progress that has been made in the field of quantum gravity during the last 50 years, a vital ingredient is still missing: The comparison with experiments. There are several causes for this lack: On the one hand, quantum gravity effects are expected to be extremely tiny. The Planck length

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \cdot 10^{-35}\mathrm{m}$$

which is a natural candidate to set the scale for quantum gravity is smaller than the distances probed by current high energy physics experiments by fifteen orders of magnitude, for example. Therefore it is a tremendous challenge to come up with circumstances or experimental setups in which quantum gravity effects might be detectable with current day technology.
On the other hand the various theories for quantum gravity, though advanced and fascinating, are still far from being complete. Thorough understanding of many aspects on the conceptual as well as on the technical level is lacking. Therefore it is hard to come up with unambiguous predictions for experimental outcome, based on one of these theories.
During the last few years, however, tremendous progress has been made in both respects, and the detection of quantum gravity effects in experiments should not be regarded as a hopeless task anymore:
Under the heading *quantum gravity phenomenology* (see [10] for a recent review), effects are discussed which are likely to occur in any quantum theory of gravity. Prime examples of such effects are "distance fuzziness", i.e. fluctuation of physical distances due to quantum effects, and breaking of Lorenz invariance (resulting in modified dispersion relation and particle kinematics).
Also, circumstances are identified, in which the tiny quantum gravity effects become amplified to the extent that their detection is possible: A modification of particle production thresholds and decay rates might become visible in measurements of cosmic rays, due to their high energy (see for example [11, 12, 13]). γ-ray bursts are proposed as excellent candidates for the direct measurement of modified dispersion relations as well as for the detection of distance fuzziness because of their huge travel time and very short time resolution ([14, 13] and references therein). Noise due to distance fuzziness might be detectable with the next generation of laser interferometers for the detection of gravitational waves ([15] as well as references in [10]).
These impressive developments motivates the investigation of the coupling of matter fields to gravity in the framework of LQG, undertaken in the present work.

In LQG, gravity is treated as a constrained quantum system along the lines pioneered by Dirac. As in any theory with a reparametrization invariant Lagrangian, the Hamiltonian for the gravitational field coupled to matter turns out to be a constraint, itselve. Consequently, the dynamics of the theory is encoded by implementing this constraint, that is by quantizing it on a *kinematical* Hilbert space and restricting attention to its kernel, the Hilbert space of *physical* states.
In a remarkable series of papers [16, 17, 18, 19], Thiemann succeeded in achieving the quantization of the Hamiltonian of gravity coupled to the matter fields of the standard model on the kinematical Hilbert space of LQG. This is an important step towards implementing the dynamics of the theory. However, it turns out that the constraint operators are extremely complicated objects, and there is little hope that the space of solutions can be found and analyzed analytically. This does not come as a surprise – answering questions about the dynamics is already a very difficult task in the case of ordinary interacting QFT, and for gravity it becomes even more involved due to the complicated non-polynomial interaction terms and the difficulties of interpreting the solutions in the absence of

a natural time parameter (the notorious "problem of time" in quantum gravity).
The basic idea underlying the present work is to sidestep the enormous problem of finding solutions to the Hamiltonian constraint by an approximation: We will *not* treat the the matter parts in the Hamiltonian as constraints, but as Hamiltonians generating the dynamics of the matter fields in the ordinary QFT sense. With the part in the Hamiltonian describing the self interaction of the gravitational field we will deal by using *semiclassical states*, which, as we will explain, annihilate this part of the Hamiltonian constraint at least approximately. Proceeding in this way certainly only amounts to establishing an approximation to the full theory: The self interaction of gravity is only partly reflected (via the semiclassical states) and we completely neglect the back-reaction of the matter fields on gravity.
What we gain is a relatively easy to interprete *fully quantized* theory of gravity and matter fields. This way we have "a foot in the door" to the fascinating topic of interaction between quantum matter and quantum gravity and can start to discuss the conceptual issues arising, as well as take some steps towards the prediction of observable effects resulting from this interplay.
Throughout this work, we will consider two matter fields: The scalar field for its simplicity, and the electromagnetic field for its relevance in the search for possible effects. We will achieve the following:

In a first step we quantize the matter parts in the Hamiltonian to become quadratic forms in the matter fields, taking values in the operators on the kinematical Hilbert space. To achieve this we adapt the methods developed in [19] to our view of the constraints as Hamiltonians and with respect to the later use of semiclassical states.
In LQG, geometry is not continuous but "polymer like" in nature, being encoded in graphs in the space-like hypersurfaces of the spacetime. We will see that diffeomorphism covariance of the above mentioned quadratic forms requires the matter to be located on these graphs, too. Consequently the matter fields cease to be fields in the continuum, but propagate on the graphs related to the gravity degrees of freedom.

We then quantize the the matter fields. Our procedure here is inspired by methods from QFT in curved space-times. These methods can however not be applied in a straightforward way since the geometry is quantized in our approach. We are led to a theory that is based on a Fock space over the tensor product of the kinematical Hilbert space of LQG and the one particle space of the matter fields. Our definition of this theory is rather formal, however, so there is a lot of work left for the future.
We proceed to discuss how a "QFT on curved space-time limit" can be obtained from the quantum theory for matter and gravity by taking partial expectation values in the gravitational part of the Hilbert space.

In a second part of the work we address the question, if and how predictions for observable effects can be derived, from the quantization of the gravity-matter system discussed before. We focus on modification of the standard dispersion relations and give a general discussion how these modifications might arise in LQG. Then we proceed in two ways:
On the one hand we investigate a very simple model system which can be treated analytically, to see how modified dispersion relations arise for fields propagating on discrete (random) lattices.
On the other hand we motivate and detail a procedure for obtaining dispersion relations from the gravity-matter system discussed before. An essential ingredient in this procedure are so called *semi-classical states*, states of the gravitational sector in which the gravitational field behaves almost as a classical field. Such states were proposed and investigated in various works [20, 21, 22, 23, 24, 25].

To round up the present work, we finally give a model calculation of dispersion relations via the procedure mentioned before, for the scalar and the electromagnetic field, using our quantization of the matter Hamiltonians and a specific sort of semiclassical states, the *coherent states for LQG* [21].

By now it is high time to stress that the steps we will take in this work and which we have sketched above, are merely tentative. We do not claim to have a satisfactory theory, even though we decided to neglect back-reaction and to treat the dynamics of the gravitational field only very approximately. To start with details, there are some ambiguities in the quantization of the matter Hamiltonians as constraints. Also, the way we quantize the matter fields is formal and merits future investigations: Perhaps there are more straightforward quantizations of the matter possible. Another aspect which should be investigated further is the method to compute the modified dispersion relations: Although we will motivate it by physical arguments, one should be aware that it is, in its essence, heuristic, and rigorous statements about its applicability would be highly welcome. Moreover, only the future will show whether the general route taken in this thesis is right and how close the approximation will be to the actual dynamics of matter and gravity.
To summarize, rather then presenting a ready-to-use theory, we would like to show what one can do to describe the interaction of matter with gravity with the machinery of LQG available to date, and try to clearly state assumptions and issues arising in this context.

As explained above, the present work can only take a small step towards analyzing the coupled gravity-matter system. It is, however, certainly not the only one and in fact draws heavily on earlier work:
On the one hand the works [16, 17, 18, 19] in which the Hamiltonian constraint was quantized on the kinematical level present a very important step in the analysis of the full theory of LQG coupled to matter and we will make heavy use of it in our quantization of the matter Hamiltonians.
On the other hand there is ground-breaking work on the phenomenology of loop quantum gravity: In [26], Pullin and Gambini consider the propagation of electromagnetic waves in a background described by LQG. Lead by general features of the theory such as the properties of its states and the Hamilton constraint, a chiral modification to the usual dispersion relation is proposed.
In the works [27, 28, 29], a more detailed consideration along the lines of [26] is given. Though no specific states are used to compute expectation values of the gravitational operators, properties of such states that are used in the course of the computation are clearly and carefully stated. Additionally, the consideration is extended to the propagation of neutrinos, and corrections to particle production thresholds caused by the modification of the dispersion relation are computed.
The present work draws much inspiration from [26, 27]: The idea to treat the matter Hamiltonians not as constraints but as Hamiltonians generating the dynamics is implicitly contained in these works. Moreover the method used to extract the dispersion relation in [27] is very similar to the one that we will present. It should therefore be seen as one of the goals of this thesis to supplement and extend [26, 27] and putting these works on a basis as firm as possible.
Finally, the semiclassical states used in the present work are those of Thiemann [21]. These are very promising candidates for states corresponding approximately to a classical geometry, as they set out to minimize fluctuations for both, the configuration and the momentum degrees of freedom of the theory. Their semiclassical properties have been thoroughly analyzed in [22, 23, 30].

To close, we should again emphasize that despite the efforts in [16, 17, 18, 19, 26, 27, 28, 29] and those reported about in the present work, one is still far from a solid understanding of the interacting gravity-matter system and hence from unambiguous quantitative predictions. This is no surprise since the interplay of quantum gravity and quantum matter fields is a highly complicated topic and its serious investigation has only begun recently. However, it is already showing today that the research in this direction also contributes a lot to the revision and clarification of the conceptual foundations of quantum gravity as a whole, something that is very gratifying in itselve.

The structure of the text.

Let us finish the introduction with a brief overview over the content of the remaining chapters of this work:
We start out in chapter 2 with a very brief outline of the formalism of LQG. The chapter also serves to introduce the notation and conventions used throughout the rest of the text.
Chapter 3 introduces the program of the thesis in more detail and discusses its merits and issues on a conceptual level.
In chapter 4 we discuss the concept of *semiclassical states* and review the proposal for such states made by Thiemann and Winkler. These states will be used later in this work in chapter 7.
Chapter 5 is all about quantization: We propose a quantum version of matter Hamiltonians for the scalar and the electromagnetic field and a corresponding dynamical quantization of these matter fields.
In chapter 6 we discuss conceptual and technical issues encountered when trying to obtain a dispersion relation for the matter fields from the full quantum theory. On the one hand we present a model system in which we are able to partly solve the questions analytically, on the other hand we develop a scheme for extracting dispersion relations, geared to the application in LQG.
Chapter 7 contains an application of the results of the chapters 4, 5 and 6: Under some simplifying assumptions, we compute dispersion relations for the scalar and electromagnetic field based on coherent states for LQG and the quantization of the Hamiltonians.
We close with a discussion of the results of this work in chapter 8, along with a list of open problems arising in the different topics touched in this thesis.
In an appendix, a laborious calculation of expectation values in semiclassical states used in chapter 7 is presented.

2. Briefing on loop quantum gravity

The present chapter serves as as brief introduction to the formalism of LQG, as well as to notation and conventions used in this work. In the first two sections we will be concerned with notation associated with geometry and Lie groups, respectively. That done, we can turn to LQG proper in sections 2.3 and 2.4.

2.1. Manifolds, metrics, graphs

In the following, we will often have to deal with a classical spacetime M. Technically speaking, M will be a four dimensional, analytic, pseudo-Riemannian manifold. Its metric, which we denote by $g_{\mu\nu}$, shall carry a signature $\operatorname{Tr} g = -2$. Spacetime indices will be denoted by lowercase Greek letters, and, as usual, raised and lowered by g.

M is assumed to have a decomposition $M = \mathbb{R} \times \Sigma$ such that Σ is a space-like, embedded analytic submanifold. We let q_{ab} denote the space-like three metric $q = -g|_\Sigma$, where for convenience we have changed signs such that $\operatorname{Tr} q = 3$. Lowercase Latin letters are generally employed for indices corresponding to the (co)vector spaces on Σ. They are raised and lowered by q.

We will have frequent opportunity to use the notion of graphs embedded in Σ:

Definition 2.1.1. By an *edge* e in Σ we shall mean an equivalence class of analytic maps $[0, 1] \longrightarrow \Sigma$, where two such maps are equivalent if they differ by an orientation preserving reparametrization.
A *graph* in Σ is defined to be a set of edges such that two distinct ones intersect at most in their endpoints.

There is some notation in connection to graphs that we will use frequently:
The endpoints of an edge e will be called *vertices* and denoted by $e(0)$ (the point e is emanating from), $e(1)$ (the point e is "running into").
The set of edges of a graph γ will be denoted by $E(\gamma)$, the set of vertices of its edges (the *vertices of the graph* for short) by $V(\gamma)$.
Given a graph γ, we will denote the edges of γ having v as vertex by $E(\gamma, v)$ or $E(v)$ if it is clear which graph we are referring to.
Given a graph γ, a vertex $v \in V(\gamma)$ and an edge $e \in E(v)$ we define

$$\sigma(v, e) = \begin{cases} +1 & \text{if } e \text{ is } \textit{out}\text{going with respect to } v \\ -1 & \text{if } e \text{ is } \textit{in}\text{going with respect to } v \end{cases}.$$

Thus $e^{\sigma(v,e)}$ is always outgoing with respect to v. We will also use the shorthand $e_{\triangleright} \doteq e^{\sigma(v,e)}$ if it is clear from the context relative to which vertex v we work.

We will call a graph γ *adapted* to a surface S, if all non-transversal points of intersections of γ with S are vertices of γ.

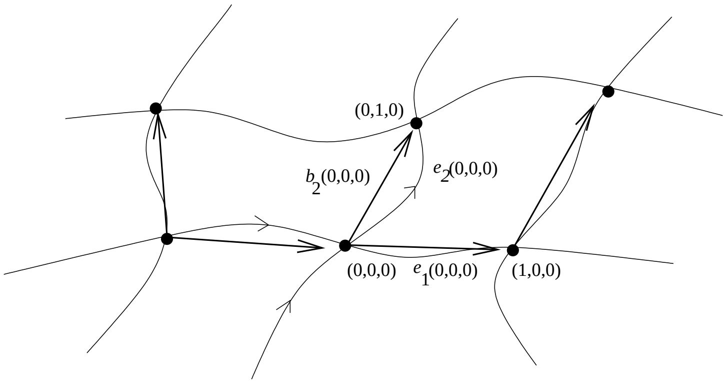

Figure 2.1.: “Slice” of a graph of cubic topology

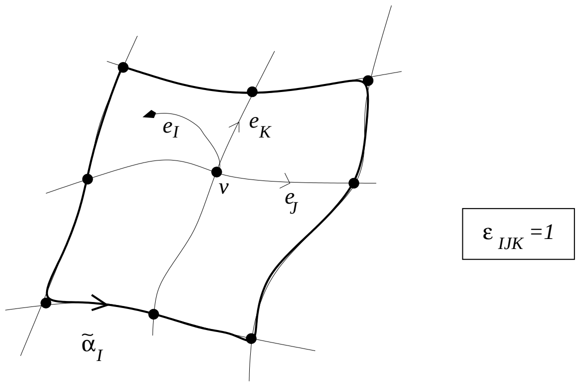

Figure 2.2.: The loop $\widetilde{\alpha}_I(v)$

The set of graphs in Σ is a partially ordered directed set. We write $\gamma' \geq \gamma$ if $E(\gamma) \subset E(\gamma')$. For details we refer to [31].

The family of edges e embedded in Σ forms a groupoid: If the endpoint $e(1)$ of an edge coincides with the starting point $e'(0)$ of another edge e', their product can be defined as their concatenation. The inverse of an edge e consists of the same submanifold of Σ but with its orientation reversed.

Finally we introduce some notation related to graphs of *cubic topology*. By a graph of cubic topology we mean a graph in which each vertex is six-valent with three edges ingoing and three outgoing. A “slice of such a graph is depicted in figure 2.1. We denote the outgoing edges by e_I, $I = 1, 2, 3$ and choose an ordering, such that the tangents of e_1, e_2, e_3 form a right handed triple wrt. the given orientation of Σ. The vertices can be labeled by elements $\underline{n}$ of $\mathbb{Z}^3$. We denote by a_I the three basis vectors in the $\mathbb{Z}^3$ lattice and write $e_I^+(n) := e_I(n),\ e_I^-(n) := e_I(n - a_I)$.
Furthermore we define $\widetilde{\alpha}_I(v)$ to be the loop “in the I-plane, around v” – see figure 2.2. If the graph is embedded in Euclidean space, we can also define the vectors $\vec{b}_I(\underline{z}) = e_I(\underline{z})(1) - e_I(\underline{z})$. These are also depicted in 2.1.

2.2. Lie groups

In the framework of LQG, Lie groups, especially the group SU(2) of unitary 2 by 2 matrices with determinant equal to one, play a prominent role. Although we will deal almost exclusively with the groups U(1) and SU(2) in this thesis, we will consider an arbitrary compact connected Lie group G in the present section.

By $\mathfrak{g}$ we will denote the Lie algebra of G. $\mathfrak{g}$ comes with an Ad_G-invariant quadratic form $\langle\cdot,\cdot\rangle_{\mathfrak{g}} = \mathrm{Tr}(\cdot,\cdot)$, *Cartan-Killing form* which in the case of a compact G is positive definite. We choose a basis τ^I for $\mathfrak{g}$ such that $\langle\tau^I,\tau^J\rangle = N\delta^{IJ}$ where N is the rank of G and denote the structure constants with respect to this basis by c_{IJ}^K. As already done in the last few lines, we will always use uppercase Latin letters to denote Lie algebra indices.
$\mathfrak{g}$ can also be viewed as the algebra of right invariant vector-fields on G. We denote the basis for these vector-fields corresponding to the basis τ^I by X^I. These vector-fields certainly come with a natural action on the differentiable functions $C^1(G)$ on G.
Every Lie group carries a unique measure invariant under right and left translations, the *Haar measure*, which we will denote by μ_{H} in the following. Using this measure one can define the Hilbert space

$$\mathcal{H}_G \doteq L^2(G, d\mu_{\mathrm{H}})$$

over G.
For compact G, all its irreducible unitary representations are finite dimensional. We will denote them by π_ν where ν is in some index set. The theorem of Peter and Weyl states that an orthonormal basis for $\mathcal{H}_G$ is provided by the normalized matrix elements in these representations,

$$e_{\nu mn}(g) \doteq \sqrt{\dim\pi_\nu}\,(\pi_\nu(g))_{mn}. \tag{2.1}$$

The characters

$$\chi_\nu(g) \doteq \mathrm{Tr}[\pi_\nu(g)]$$

form an orthonormal basis for the space of Ad_G-invariant functions in $\mathcal{H}_G$.
The right invariant vector fields X^I multiplied with the imaginary unit i turn out to be symmetric operators on $C^1(G) \subset \mathcal{H}_G$.

2.3. The classical theory

In this and the following section we want to review the formalism of LQG, mainly to fix our notation and conventions. In the present section we concentrate on the classical part, in the next one on the quantum theory.
Since we will be very brief, the reader not familiar with LQG is apologized to and referred to [32] as an very instructive nontechnical review, the excellent, very detailed recent review [33], or to [34] as a short and pedagogical exposition.

To begin with, recall that in the *new variables*-formulation [1] of general relativity, the canonical consists of a $SU(2)$ connection one-form A and a frame field E_I with a nontrivial density weight. Both of these take values on a spacial slice Σ of the four manifold M. The connection to the ADM variables q^{ab}, the (inverse) spacial metric on Σ, and K_{ab}, Σ's exterior curvature, is

$$\det(q)q^{ab} = \iota E^a_I E^{bI}, \qquad A^I_a = \Gamma^I_a - \frac{\iota}{\sqrt{\det(q)}} K_{ab}E^{bI}.$$

Here, Γ is the spin connection corresponding to the triad E, i.e. $\nabla_a^{(\Gamma)} E^a = 0$ where $\nabla^{(\Gamma)}$ is the covariant derivative defined by Γ. ι is the *Barbero-Immirzi* parameter which was originally chosen to be $\iota = i$ but which can in principle take any nonzero value in $\mathbb{C}$ [35, 36]. A convenient choice for ι is also 1 which was advertised for the first time in [37], as it renders the connection A real valued. The space of smooth connections A is usually denoted by $\mathcal{A}$.
As for units, we choose $[A] = \text{meter}^{-1}$. As a consequence, E will be dimensionless.

In any Hamiltonian formulation of GR known to date, the basic fields are constrained. In the variables given above, the constraints read

$$\begin{aligned} G_I(E) &= \nabla_a E^a_I, \\ D_a(A,E) &= E^b_I F^I_{ab}, \\ H(A,E) &= \epsilon^{IJK} E^a_I E^b_J F_{abK} - 2\frac{\iota^2+1}{\iota^2} E^a_{[I} E^b_{J]} (A^I_a - \Gamma^I_a)(A^J_b - \Gamma^J_b), \end{aligned}$$

where ∇, F are covariant derivative and curvature of the connection A. G_I is the so called *Gauß constraint* corresponding to the SU(2) gauge freedom introduced with the triads E, the *diffeomorphism constraint* D_a generates diffeomorphisms of Σ, and H is called the *Hamilton constraint* and generates diffeomorphisms orthogonal to Σ. The Hamiltonian of GR vanishes on the constraint surface, it reads

$$H_{\text{grav}}[\Lambda, \Lambda^a, A, E] = \frac{1}{\kappa} \int_\Sigma \Lambda H(A,E) + \Lambda^a D_a,$$

where $\Lambda(p)$, $\Lambda^A(p)$ are the lapse function and the shift vector-field. κ is the coupling constant

$$\kappa = \frac{8\pi G}{c^3}.$$

A decisive advantage of the new variables is that both connection and triad alow for a metric-independent way of integrating them to form more regular functionals on the classical phase space and hence make a quantization feasible:
Being a one-form, A can be integrated naturally (that is, without recurse to background structure) along differentiable curves e in Σ, to form *holonomies*

$$h_e[A] = \mathcal{P} \exp\left[i \int_e A \right] \qquad \in \mathrm{SU}(2)\,.$$

The density weight of E on the other hand is such that, using an additional real (co-)vector field f^I it can be naturally integrated over surfaces S to form a quantity

$$E_{S,f} = \int_S f_I (*E)^I$$

analogous to the electric flux through S. In the following, we will designate the set of classical quantities $E_{S,f}$ by $\mathcal{E}$.

It is convenient to consider a class of functionals of the connection A a bit more general:

Definition 2.3.1. A functional $f[A]$ of the connection is called *cylindrical with respect to a graph* γ if there is a function

$$f : \mathrm{SU}(2)^{|E(\gamma)|} \longrightarrow \mathbb{C}$$

such that

$$f[A] = f(h_{e_1}[A], h_{e_2}[A], \ldots), \qquad e_1, e_2, \ldots \in E(\gamma). \tag{2.2}$$

Strictly speaking, one should define cylindrical functions as equivalence classes of the functions defined above under pullback from smaller to larger graphs. But as we will not be concerned with such technical intricacies until we refer the reader to the original work [31] for details.

From the canonical Poisson brackets of A and E one can compute the Poisson brackets for the $c \in \mathrm{Cyl}$, $E_{S,f} \in \mathcal{E}$:

$$\{E_{S,f}, c\} = X_{S,f}[c], \qquad \text{where} \qquad X_{S,f}[c] = \sum_{v \in S \cap \gamma} \sum_{e \in E(v)} \sigma(v,e) f_I(v) X_e^I[c]. \tag{2.3}$$

X_e is the right resp. left invariant vector-field on SU(2) when e lies above resp. below S (wrt. its orientation), acting on the entry corresponding to e of c written as a function (2.2) on $\mathrm{SU}(2)^{|E(\gamma)|}$. Also, in the above formula, we have assumed without loss of generality that γ is adapted to S.

2.4. Quantum theory

One of the defining aspects of the program of loop quantum gravity is to base the quantization on the set on holonomies (or, slightly more general, on cylindrical functions) and the triads smeared over two dimensional surfaces.
The first successful quantization of this kind of variables was accomplished by Rovelli and Smolin in [2]. Though not talking directly about two dimensional smeared triads, the *loop representation* they found is in essence a representation of the Poisson structure (2.3).
Since then, much work has gone into extracting the essence of this quantization and putting it onto firm mathematical ground. Key ideas in this context were the usage of C^*-algebraic methods [38] and projective limit techniques [31] resulting in what is now called the *connection representation*. In the present thesis we will work with this modern incarnation of LQG.

In the connection representation, quantization of the classical functions Cyl and $\mathcal{E}$ is achieved on a Hilbert space

$$\mathcal{H}_{\mathrm{AL}} = L^2(\overline{\mathcal{A}}, d\mu_{\mathrm{AL}}).$$

It is based on the compact Hausdorff space $\overline{\mathcal{A}}$ of generalized connections which is a suitable enlargement of the space of smooth connections $\mathcal{A}$ in the following sense:
As a set, $\overline{\mathcal{A}}$ can be characterized as the set of *all* groupoid morphisms from the groupoid of edges embedded in Σ to SU(2), "all" meaning that there is no continuity requirement on the group element associated to an edge e under variation of e.
It is useful to briefly review one way of defining μ_{AL}. The reader should consult the original work [31] for proofs and details.
Because of the identification of cylindrical functions with functions on a finite product of copies of SU(2), it is natural to introduce the following net:

$$\gamma \longmapsto \mathcal{A}_\gamma \cong \mathrm{SU}(2)^{|E(\gamma)|}. \tag{2.4}$$

This net is functorial in the following sense: Let $\gamma' \leq \gamma$ and denote the edges of γ by $e_1, e_2, \ldots, e_M$ and those of γ' by $e'_1, e'_2, \ldots, e'_{M'}$. Since γ is bigger then γ', for every e'_i there is a representation $e'_i = e_{j_1^i}^{n_1^i} \circ e_{j_2^i}^{n_2^i} \circ \ldots$ and one can define

$$p_{\gamma'\gamma} : \mathcal{A}_\gamma \longrightarrow \mathcal{A}_{\gamma'}, \qquad \left(g_{e_j}\right)_{j=1,2,\ldots M} \longmapsto \left(g_{j_1^i}^{n_1^i} g_{j_2^i}^{n_2^i} \ldots\right)_{i=1,2,\ldots M'}. \tag{2.5}$$

This turns out to be a well defined projection. Since the set of graphs is directed, one can define the projective limit of the net (2.4) with projections (2.5), which is nothing else then $\overline{\mathcal{A}}$.
The above characterization of $\overline{\mathcal{A}}$ can now be used to define measures on this space:

For every graph γ let a measure μ_γ on $\mathcal{A}_\gamma$ be given. Call a family $\{\mu_\gamma\}$ of measures consistent if for any $\gamma \leq \gamma'$

$$\int_{\mathcal{A}_\gamma} f_\gamma \, d\mu_\gamma = \int_{\mathcal{A}_\gamma} (p_{\gamma'\gamma})^* f_\gamma d\mu_{\gamma'} \qquad \text{for all} \qquad f_\gamma \in \mathrm{Cyl}_\gamma \, .$$

It turns out that the correspondence between measures on $\overline{\mathcal{A}}$ and consistent families of measures is one to one [39]:

Proposition 2.4.1. *Every consistent family of measures on the $\mathcal{A}_\gamma$ defines a unique measure on $\overline{\mathcal{A}}$ and every measure on $\overline{\mathcal{A}}$ can be obtained from a consistent family of measures on the $\mathcal{A}_\gamma$.*

Choosing the measure μ_γ to be the Haar measure on the product group, one obtains a consistent family of measures which in turn define a measure on $\overline{\mathcal{A}}$. This measure is nothing else then the Ashtekar-Lewandowski measure.

After this digression on measures on $\overline{\mathcal{A}}$, we come back to the promised representation of the Poisson structure (2.3): The cylindrical function simply act by multiplication:

$$\mathrm{Cyl} \ni f_\gamma : \psi[\overline{A}] \longmapsto (f_\gamma \psi)[\overline{A}] \doteq f_\gamma[\overline{A}]\psi[\overline{A}] \tag{2.6}$$

where the extension of the cylindrical functions from $\mathcal{A}$ to $\overline{\mathcal{A}}$ is afforded by the characterization of elements $\overline{\mathcal{A}}$ as groupoid morphisms.
It is not surprising that the fluxes $E_{S,f}$ are represented by the right and left invariant vector fields. Let γ be adapted to S (without loss of generality) and f_γ any function cylindrical on γ which is differentiable if viewed as a function on $\mathcal{A}_\gamma$. Then[i]

$$\widehat{E}_{S,f}[f_\gamma] = i\hbar\kappa \sum_{v \in S\cap\gamma} \sum_{e \in E(v)} \sigma(v,e) f_I(v) X_e^I[f_\gamma]. \tag{2.7}$$

The operators defined by (2.6) and (2.7) are symmetric and implement the Poisson relation (2.3) in a precise sense [40].

2.5. Constraints

To finish the quantization of the gravitational field, the constraints (2.3) have to be implemented, i.e. they have to be turned into operators, the biggest common subset of their kernels then serves as physical Hilbert space. Some technical complications arise if zero is in the continuous spectrum of the constraint operators: Then the physical Hilbert space is not a subspace of the kinematical Hilbert space anymore, but some subspace of its topological dual.

The gauge invariant states are easy to identify. They form a closed subset of $\mathcal{H}_{\mathrm{AL}}$ [41].
More work has to be done for the diffeomorphism constraint: The group D of diffeomorphisms φ of Σ naturally acts on functions in Cyl via $U(\varphi)f_\gamma \doteq f_{\varphi(\gamma)}$, where $f_{\varphi(\gamma)} \in \mathrm{Cyl}_{\varphi(\gamma)}$ is defined by

$$f_{\varphi(\gamma)}[A] = f\left(h_{\varphi(e_1)[A]}, h_{\varphi(e_2)[A]}, \ldots\right), \qquad\qquad e_1, e_2, \ldots \in E(\gamma).$$

[i] In this formula we use the brackets "[·]" to emphasize that we mean the action of an operator on a vector, not the product of two operators. We will use this notation wherever confusion is possible.

Since μ_{AL} is defined in a diffeomorphism invariant fashion, this action is unitary. It turns out that the diffeomorphism invariant states do not lie in $\mathcal{H}_{\mathrm{AL}}$ anymore, but in a subset of its topological dual [42]. The difficulty that comes with the implementation of the diffeomorphism constraint is the following: For pure gravity, the known diffeomorphism invariant quantities are rather complicated and therfore hard to quantize. This problem gets alleviated when matter is coupled to the gravitational field. Roughly speaking, the matter can serve to define submanifolds of Σ (for example the surfaces S for the $E_{S,f}$) in a diffeomorphism independent way, thus alowing for the construction of diffeomorphism invariant observables [43, 44]. Indeed we will see that this also applies to the Hamiltonian for gravity coupled matter: The corresponding operator constructed in the next section will be diffeomorphism invariant. This is important for the following reason: Since the diffeomorphisms of Σ are implemented unitarily on $\mathcal{H}_{\mathrm{AL}}$, the expectation value of a diffeomorphism invariant operator does not differ from its expectation value in the state resulting from projecting the original one to the diffeomorphism invariant Hilbert space (via group averaging) [42]. Therefore as long as we work with diffeomorphism invariant operators on $\mathcal{H}_{\mathrm{AL}}$ we do not have to bother about implementing the diffeomorphism constraint.

Implementation of the Hamilton constraint is very difficult because of its complicated non-polynomial structure. Remarkably, there is a clever proposal for its quantization due to Thiemann [16], and some of its solutions have been given in [17]. The interpretation of the resulting theory is notoriously hard.
As we will explain in the next chapter, in this work we try to circumvent the implementation of the Hamilton constraint by the use of semiclassical states on the one hand, and by regarding the matter Hamiltonians as generating the dynamics in the ordinary QFT sense. We will however use the ideas of [16] in the quantization of the matter Hamiltonians in chapter 5.

3. The general scheme

In this section we want to discuss the issues related to the coupling of matter to the gravitational field in the setting of LQG and explain how we will deal with them in the present work. Let us start by describing the "ideal" procedure for obtaining the fully quantized theory and the prediction of observable effects:

Certainly, a computation of quantum gravity corrections from QGR from first principles should be carried out in a setting in which both the gravitational field and the matter is described by a quantum theory. To obtain such a composite quantum theory, one should start from a classical formulation of the matter theories as similar to that used for gravity as possible, for reasons of consistency. Then, to stay in keeping with the spirit of loop quantum gravity, a Dirac type quantization should be performed. Thus, the steps that have to be taken are roughly as follows:

Step 1: A (kinematical) quantization of the matter field theory in the spirit of LQG has to be given on a Hilbert space $\mathcal{H}^{\mathrm{kin}}_{\mathrm{matter}}$. To prepare for the implementation of the constraints, a quantization of the classical Hamiltonian constraint of the coupled gravity-matter system must be obtained as an operator on the Hilbert space $\mathcal{H}^{\mathrm{kin}}_{\mathrm{grav}} \otimes \mathcal{H}^{\mathrm{kin}}_{\mathrm{matter}}$.

Step 2: The constraints of the theory have to be implemented, i.e. (generalized) vectors in the kernels of the constraint operators have to be found. Among the solutions, those corresponding roughly to the matter fields propagating in a fixed background geometry (flat space, say) must be identified.

Step 3: The theory obtained in the steps so far has to be investigated: Dispersion relations or other equations characterizing the phenomenology of the system in the limit where the energy of the non-gravitational fields are small have to be derived.

Certainly these steps are interrelated or even overlapping, but let us for the sake of the presentation stick to this sub-division and discuss the individual steps in more detail in what follows.

Step 1.

The kinematical quantization in the first step is fairly straightforward and unambiguous. The reason for that are the fundamental principles of LQG which have to be obeyed: The quantum theory should be formulated in a background free and diffeomorphism covariant fashion.

If the matter field is a gauge field with compact gauge group, we can quantize it with exactly the same methods that are used in LQG for the gravitational field. This way, we obtain a neat unified description of gravity and the other gauge fields. Also for fermions or scalar fields, a representation should be used that is background independent. This rules out the usual Fock representation. New representations for fermionic and scalar fields in keeping with the principles of LQG were proposed in [45].

The quantization of the Hamiltonian of the coupled system is a rather nontrivial task, due to its complicated non-polynomial dependence on the basic variables of the theory. Remarkably, a scheme for the quantization for densities of weight one has been worked out by Thiemann in [16, 19], which affords the task. The resulting operators are quite complicated but perfectly well defined and lead to reasonable results in a symmetry reduced context [8, 46]. Another very encouraging aspect of the scheme is that it works *precisely due* to the density character of the classical quantities and not only *despite* of it.

Quantization of the matter Hamiltonian in a fashion similar to that of Thiemann was assumed in [26], and [27] uses Thiemanns methods directly.
We will also use the methods of [16, 19], although slightly modified for our purposes: Our operators will not change the graphs when casting on cylindrical functions and we will not quantize the matter fields along with the gravity degrees of freedom, but in a second step. The reason for this is our changed viewpoint on the matter parts of the Hamilton constraint as being Hamiltonians for the matter in their own right, and will be elaborated on when discussing step 2.
Also, we will choose different c-number coefficients in the quantized matter Hamiltonians as compared to Thiemann. This is to insure good semiclassical behavior in connection with the semiclassical states from [21, 22, 23] that we are going to use later. We refer to [30] and the discussion in chapter 4 for details.

Step 2.

This step in the calculation is by far the most difficult one since it corresponds to solving the dynamics of a fantastically complicated system of coupled quantum fields. We will therefore not be able to solve this problem exactly but only in some approximation. The constraints that have to be considered are: A Gauß constraint for Gravity and for every matter gauge field, the diffeomorphism constraint of gravity, and, finally and most importantly, the Hamilton constraint of the coupled gravity-matter system.

The implementation of the Gauß constraints is easy: The gauge invariant vectors form a closed subspace in the Hilbert space of functionals of the corresponding connection and are explicitely known [47, 48].

The implementation of the diffeomorphism constraint is technically more complicated than that of the Gauß constraint since its solutions do not lie in the original Hilbert space, but can nevertheless be accomplished [42]. As we have explained in section 2.5, since diffeomorphisms are implemented unitarily on $\mathcal{H}_{\rm AL}$, we do not have to care about the diffeomorphism constraint as long as we just consider diffeomorphism invariant operators. This applies to the Hamiltonians constructed in the next chapter, and so we will not bother about the diffeomorphism constraint anymore.

We now turn to the implementation of the Hamilton constraint. Even for pure gravity, this is a very difficult topic. Though solutions have been found [17, 18], they are notoriously hard to interpret due to the lack of gauge invariant observables and a thorough understanding of the "problem of time". The problem of finding solutions to the Hamilton constraint for gravity coupled to matter has not been treated before.

Since our ultimate goal is to explore ways for computing quantum gravity corrections to field propagation on Minkowski space (or any other classical background spacetime for that matter), the task presented to us is even harder: Not only do we have to find *some* solutions to the Hamilton constraint, but we are interested in *specific* solutions in which the gravitational field is in a state "close to Minkowski". In the light of these difficulties, we propose to proceed along slightly different lines. To give an idea what we are aiming at, imagine we ought to compute corrections to the interaction of some quantum system (an atom, say) with an electromagnetic field, which are due to the quantum nature of the electromagnetic field. Ultimately this is a problem in quantum electrodynamics and therefore certainly not solvable in full generality. What can be done?

For the free Maxwell field, there is a family of states describing configurations of the quantum field close to classical ones, the *coherent states*: Expectation values for field operators yield the classical values and the quantum mechanical uncertainties are minimal in a specific sense. Such states could be used to model the classical electromagnetic field. Certainly these coherent states are no viable states for the full quantum electrodynamics treatment in any sense. They do not know anything about the dynamics of the full theory. The key point now is that though being in some sense "kinematical", the coherent states for the Maxwell field are nevertheless a very good starting point to compute approximate quantum corrections as testified by the computations in the framework of quantum optics.

In the present work we will proceed in the same spirit: We will not seek states which are solutions to the constraint *and* represent some sort of quantum Minkowski space, but rather start by considering *kinematical* states in the gravity sector which are close to Minkowski space. Such states are usually called *semiclassical states*.

Consequently, and thereby slightly changing the content of step 1, we will give a treatment of the Hamiltonians of the matter fields as Hamiltonians in the sense of ordinary QFT, *not* as constraints, and write down a dynamical quantization of the matter fields accordingly. More precisely, we will quantize the gravitational degrees of freedom in the matter Hamiltonians along the lines of [16, 19] as explained above, yielding operator valued quadratic forms in the classical matter fields. Then we canonically quantize the matter fields in a fashion similar to ordinary QFT. However, since gravity is also quantized we will find that the resulting matter quantum fields act on the Fock space over the tensor product of the gravity- and the one particle Hilbert space, not only on the Fock space over the latter.

It is hard to judge the validity of this approach as compared to the desirable full fledged solution of the Hamilton constraint. In simple quantum mechanical model systems with nonlinear dynamics such as two coupled Harmonic oscillators, predictions obtained with coherent states on the kinematical level numerically differ from the results of a treatment using dynamical coherent states. However, the qualitative picture obtained with the kinematical semiclassical states persists on the dynamical level. We hope that the same holds true in the present situation. Although precise numerical predictions might not be possible with this simplified treatment, it nevertheless encodes many essential qualitative features of LQG and it is therefore not implausible that qualitative features of the quantum corrections such as their order of magnitude, the rough relative magnitude of different correction terms, and maybe the fact that that certain types corrections turn out to be absent may be correctly predicted.
We have to note, however, that dealing with the dynamics in the way described above means to treat time fundamentally different then space: The time coordinate remains classical and continuous, whereas the geometry of space is quantum and discrete. It is implausible that this picture will remain valid in case the full dynamics of the theory is implemented in some way, and therefore important aspects of quantum gravity might not be visible within our approach.

Another issue raised by the treatment outlined above is that much depends on the choice of the state that is employed to play the role of the semiclassical state. We will discuss this issue in chapter 4 and only make some brief remarks here:
All candidate semiclassical states proposed so far are graph based states, i.e. cylindrical functions in $\mathcal{H}_{\mathrm{AL}}$. Consequently, this is assumed to be the case in the present work. The picture might

however change substantially if ideas such as the averaging over infinitely many graph based states advocated in [24] could be employed.
The works [26, 27] also assume that semiclassical states are based on graphs, but they do not work with a specific sort of such states. In contrast to that, we will use the gauge theory coherent states constructed in [21, 22, 23] in our computation of dispersion relations in chapter 7, and thus obtain more specific results.

Step 3.

The last step in the program is to obtain testable predictions from the theory constructed in step 2. Despite its importance it has not yet been thoroughly analyzed in the literature so far. In 6 we will for definiteness concentrate on modifications of the dispersion relations of matter fields. We will see that there are at least two mechanisms by which such modifications can arise in LQG. One has to do with the back-reaction of the matter on the geometry, and we are not able to analyze it within the present setting. The other one is the fundamental discreteness predicted by LQG, due to the one dimensional nature of the excitations of the quantum gravitational field. As soon as a semiclassical state for the gravity sector is chosen, translation and rotation symmetry is heavily broken on short scales due to the discreteness of the underlying graph. The theory for the matter fields then describes fields propagating on random lattices, bearing a remarkable similarity to models considered in lattice gauge theory [49, 50, 51]. Due to the lack of symmetry on short scales, notions such as plane waves and hence dispersion relations can at best be defined in some large scale or low energy limit. We will show that the problem of treating these limits is by no means trivial and requires careful physical considerations. It is closely related to the condensed matter physics problem of computing macroscopic parameters of an amorphous (i.e. locally anisotropic and inhomogeneous) solid from the parameter of its microscopic structure.
To get a feeling for the problem, we will start by studying a one dimensional model system for which we are able to find exact solutions. We will then turn to general fields on random lattices and describe a procedure to obtain dispersion relations valid in the long wavelength regime. In chapter 7 we will apply this procedure, together with the coherent states for LQG and the quantization of 5 (and some rather drastic simplifying assumptions) to arrive at dispersion relations for matter fields coupled to LQG.

4. Semiclassical States

4.1. General features of semiclassical states

In the present chapter we consider the problem of finding states in the gravity sector which are close to some given classical geometry in a specific sense, so called *semiclassical states*.
Semiclassical states are an important and fascinating topic. Unlike in other QFT, for gravity the global state the world is in does not seem to be close to the vacuum (i.e. the state with the lowest energy) but a highly excited state. Therefore, in any attempt to make contact to large scale physics, some sort of semiclassical state is likely to play a prominent role. Also, semiclassical states provide a way to define a sort of $\hbar$ goes to zero limit (in which LQG should reduce to GR) and therefore a test of the quantum theory [21].
As explained in the introduction, the task of finding *dynamical* states (in the sense that they are annihilated by $\widehat{H}_{\text{grav}}$) which correspond to classical geometries is a very hard one, and at present no meaningful way of tackling it is known. The states we consider in this chapter will therefore be states in the kinematical Hilbert space $\mathcal{H}_{\text{AL}}$ of the theory. We will see, however, that semiclassical states can be constructed that are at least "approximately" dynamical, in the sense that the action of H_{grav} on them results in states that have tiny norm. One can therefore entertain the hope that results obtained upon using these kinematical semiclassical states, although quantitatively wrong, will nevertheless have realistic qualitative features.[i]

Let us state more precisely what is usually meant by the term *semiclassical state*: Consider some set of classical observables $\mathcal{O}$. Minimal requirements for a state $\Psi_{A,E}$ to be close to some classical configuration (A, E) with respect to observables contained in $\mathcal{O}$ can be stated as follows:

1. Operators $\widehat{O}$ corresponding to observables $O \in \mathcal{O}$ should have expectation values close to the classical values in the given phase space point, i.e. $\langle\widehat{O}\rangle_{\Psi_{(A,E)}} \approx O(A, E)$.

2. Operators $\widehat{O}$ corresponding to observables $O \in \mathcal{O}$ should have small quantum mechanical fluctuations, i.e. $\langle\widehat{O}^2\rangle_{\Psi_{(A,E)}} - \langle\widehat{O}\rangle^2_{\Psi_{(A,E)}}$ should be tiny.

Let us make a few remarks on these requirements:
First of all it is evident that only a careful choice of the class of observables $\mathcal{O}$ will lead to states which are semiclassical in a physical sense: It is to be expected that a realistic semiclassical state would behave classical when probed at low energies or large scales, but significant deviations from the behavior expressed in 1. and 2. would become evident at very high energies and small scales.
In LQG the choice of observables is usually done via specification of a macroscopic length scale L which is thought to be very large compared to Planck length l_P. $\mathcal{O}$ is then chosen as a set of geometric observables (i.e. areas, volumes) whose classical values at the phase space point (A, E)

[i] This holds true at least for simple (albeit nonlinear) quantum mechanical models such as two coupled Harmonic oscillators.

of interest range on a scale equal to or larger then L.
Also note that the phrases "close to" and "tiny" have to be specified more precisely. One way to do this is to talk about the *relative* quantities

$$\frac{\langle\widehat{O}\rangle_{\Psi_{(A,E)}} - O(A,E)}{O(A,E)}, \qquad \frac{\langle\widehat{O}^2\rangle_{\Psi_{(A,E)}} - \langle\widehat{O}\rangle^2_{\Psi_{(A,E)}}}{\langle\widehat{O}\rangle^2_{\Psi_{(A,E)}}}.$$

This works as long as the classical value $O(A,E)$ and the expectation value are nonzero. If that is not the case, an additional scale has to be introduced, relative to which things should be tiny.
Let us finally say that ultimately a more concise and univocal definition of a semiclassical state is desirable: Many states will meet the above requirements but differ significantly in other respects, ultimately also in predictions obtained from them.

An important aspect which has to be considered in connection with semiclassical states for LQG is the fact that "almost all" states in $\mathcal{H}_{\mathrm{AL}}$ are cylindrical with respect to some graph γ. Integral part of any construction of a semiclassical state will therefore be the specification of the underlying graph.
To insure a reasonably continuous behavior of expectation values of macroscopic observables under the isometries of the underlying geometry, the graph has to be large (i.e. zillions of vertices) and the length of its edges, as measured with the metric to be approximated, as tiny. The length scale on which the graph has its structures, the typical edge length say, is usually called the *microscopic scale* and we will denote it by ϵ.
As the scales L and ϵ are so different, what matters of a given graph is not so much its precise shape as its average properties: Quantities relevant to the expectation values of macroscopic observables, such as the number of intersections of surfaces of macroscopic size, or valence of vertices in macroscopic regions will be well described by their average values (average intersection number per area, average valence, ...) of the *given* graph. In the following, we will call these kind of averages *graph averages*[ii].
Hence, to describe a semiclassical state it is not absolutely necessary to specify a single graph. Any graph of an ensemble Γ of graphs with the same graph averages will lead to similar results. We will usually refer to such a graph as a *random graph*, since it is specified by its graph averages and its construction of can be regarded as a random process. We will also speak of a "random graph prescription" meaning a random process to generate a graph, or, equivalently an ensemble Γ of graphs with equal graph averages.
The graphs will certainly have to respect the symmetries of the geometry to be approximated by the semiclassical state. Let us make this more precise for the case of Minkowski space:

Assumption 4.1.1.
1. Averages of geometric properties over parts of the graph contained in regions small compared to L but large as compared to ϵ should be equal to the corresponding graph averages (*homogeneity*).
2. Graph averages of *tensors* derived from the geometric properties of the random graph should be invariant under rotations (*isotropy*).

We will have occasion to use these properties in the calculations of chapter 6.

The one dimensional nature of the states in $\mathcal{H}_{\mathrm{AL}}$ also poses some questions concerning the construction of semiclassical states which are not yet satisfactorily answered:
The first concerns the holonomies: Given a graph γ, a generic edge e in Σ will not be contained in

[ii] These graph averages should not be confused with *ensemble averages* appearing in random geometry. There one considers probability measures on an ensemble Γ of graphs and computes averages with respect to this measure.

γ. Therefore, the expectation value of the holonomy along a generic edge e in a semiclassical state based on γ will not be close to the classical but just zero. Thus even if a state is a good semiclassical one with respect to holonomies contained in the underlying graph, it will do poorly for almost all the configuration variables of the theory![iii]
The triad operators $\widehat{E}_{S,f}$ are better behaved in this respect: A generic surface with dimensions larger then the typical edge-length ϵ of a given graph γ *will* intersect γ and $\widehat{E}_{S,f}$ will therefore have nontrivial expectation value in some semiclassical state based on γ.

The problems with graph based semiclassical states mentioned above and even more so the general problem of the freedom in choice of such states suggest that it might also be useful to consider more general possibilities of implementing semi-classicality. For some ideas in this direction we refer to [24].
For the rest of the present chapter, we will however stick to graph based semiclassical states.

Several proposals for the construction of semiclassical states have been put forward up to now. We mention the *weave states* [20] as the earliest attempt, and the *shadow states* [25]. The latter represent a very recent go at constructing semiclassical states. Although graph based states, they take a step towards independence of the underlying graph in that they derive from a graph independent object, a measure on the space $\overline{\mathcal{A}}$ of generalized connections. As the shadow states are still being investigated with respect to their semiclassical properties, we will turn in the present work to a class of semiclassical states, the *coherent states for LQG* [21, 22, 23] and we will review their construction and properties in detail in the following sections.

4.2. Coherent states

It is probably fair to say that *coherent states for LQG* [21, 22, 23, 30] are the only fully worked out proposal for semiclassical states for LQG, to date. As weave states, coherent states for loop quantum gravity (LQC for short) are cylindrical states, obtained as a product of functions cylindrical over the edges of the graph. In contrast to weave states, however, the functions on the edges are *not* eigenstates of the geometry, but are carefully chosen such that good classical behavior is obtained for *both* configuration *and* momentum degrees of freedom. In a precise technical sense, functions cylindrical on the edges are chosen to be *coherent states on* SU(2).

As in the case of weaves, the choice of the underlying graph is decisive for achieving a good semiclassical behavior. Details will be discussed in section 4.2.4.

In the following sections we will go through the construction of the LQC in detail.

4.2.1. Coherent states in quantum mechanics

Since the group coherent states used in the construction of LQC are closely analogous to the coherent states used in quantum mechanics, it is worthwhile to briefly review their construction and properties, before turning to the LQC. This is what we are going to do in the present section.
Consider quantum mechanics of a particle on the real line, without specifying the potential it is moving in. The basic observables are configuration and momentum, $\widehat{X}, \widehat{P}$, with

$$\left[\widehat{X}, \widehat{P}\right] = i\hbar 1. \tag{4.1}$$

[iii]In addition to that there seem to be problems with *fluctuations* of holonomies along large edges. We refer to [52] for details.

From these, we can build an annihilation operator

$$\widehat{a} = \sqrt{\frac{\omega}{2\hbar}}\widehat{X} + i\frac{1}{\sqrt{2\hbar\omega}}\widehat{P}$$

whose classical counterpart we denote by z:

$$z \doteq \sqrt{\frac{\omega}{2\hbar}}X_0 + i\frac{1}{\sqrt{2\hbar\omega}}P_0.$$

Here (X_0, P_0) is a point in the classical phase space. Also, the reader might have noticed that we introduced an additional parameter ω to balance units in the above formulae. Indeed

$$[\omega] \stackrel{!}{=} \left[\frac{P}{X}\right],$$

so ω translates between the unit of P and that of X. We will comment on its meaning in more detail below. We can now define the coherent state associated with (X_0, P_0):

$$\Psi_z \doteq e^{-\frac{1}{2}|z|^2} \sum_n \frac{z^n}{n!} \left(\widehat{a}^\dagger\right)^n |0\rangle,$$

where the state $|0\rangle$ is defined by $\widehat{a}|0\rangle = 0$. To discuss the properties of these states, we write them down in configuration space and momentum space representation:

$$\psi_z(x) = \sqrt{\frac{\omega}{\pi\hbar}} e^{-\left[\frac{\omega}{2\hbar}(x-X_0)^2 - \frac{i}{\hbar}xP_0\right]}, \qquad \psi_z(p) = \sqrt{\frac{\hbar}{\pi\omega}} e^{-\left[\frac{\hbar}{2\omega}(p-P_0)^2 + \frac{i}{\hbar}pX_0\right]}.$$

Upon introducing $t = \hbar/\omega$ this can also be written as

$$\psi_z(x) = \frac{1}{\sqrt{\pi t}} e^{-\left[\frac{1}{2t}(x-X_0)^2 - \frac{i}{\hbar}xP_0\right]}, \qquad \psi_z(p) = \frac{t}{\sqrt{\pi}} e^{-\left[\frac{t}{2}(p-p_0)^2 + \frac{i}{\hbar}xP_0\right]}. \tag{4.2}$$

From these formulae we can read off the most important properties of coherent states: In both configuration and momentum representation the wavefunctions are *Gaussian distributions*, centered at

$$\langle \widehat{X} \rangle_{\Psi_z} = X_0 \quad \text{resp.} \quad \langle \widehat{P} \rangle_{\Psi_z} = P_0.$$

Furthermore we can see that the width of the distribution in the configuration representation is inversely proportional to that in momentum representation. More precisely: Denoting the *fluctuation* of an observable $\widehat{O}$ by

$$\Delta_\Psi(\widehat{O}) \equiv \left(\langle \widehat{O}^2 \rangle_\Psi - \langle \widehat{O} \rangle_\Psi^2\right)^{\frac{1}{2}},$$

we find that

$$\Delta_{\Psi_z}(\widehat{X})\Delta_{\Psi_z}(\widehat{P}) = \frac{\hbar}{2}, \qquad \frac{\Delta_{\Psi_z}(\widehat{P})}{\Delta_{\Psi_z}(\widehat{X})} = \omega. \tag{4.3}$$

The first equation shows that the uncertainty product takes the smallest value allowed by (4.1). One can show quite generally [53] that coherent states of the type (4.2) are the only states with this property for quantum mechanics on $\mathbb{R}$. Therefore, Ψ_z is commonly interpreted as the quantum state of a particle *as close as possible* to that of a *classical* particle at X_0 moving with momentum P_0. It is this interpretation that suggests the use of coherent states in the construction of semiclassical states for LQG.

The second equation sheds light on the meaning of the parameter ω: It balances the amount of uncertainty between $\widehat{X}$ and $\widehat{P}$. How can ω be fixed? There are different possibilities:

(a) There is a parameter present in the description of the physical system under consideration that can play the role of the ω. This is for example the case if we consider the harmonic oscillator.

(b) In the case that both X_0 and P_0 are nonzero, a possible choice would be

$$\omega = \frac{P_0}{X_0}.$$

(c) The value of ω can be chosen arbitrarily. This amounts to saying that there is not one coherent state but a one parameter family.

We remark that in the light of ω distributing the uncertainty between $\widehat{X}$ and $\widehat{P}$, (b) amounts to saying that the *relative* uncertainties be equal, i.e.

$$\frac{\Delta_{\Psi_z}(\widehat{X})}{\langle\widehat{X}\rangle_{\Psi_z}} \stackrel{!}{=} \frac{\Delta_{\Psi_z}(\widehat{P})}{\langle\widehat{P}\rangle_{\Psi_z}}.$$

Coherent states have many more interesting properties and a wide range of applications which we refrain from discussing here. We refer the interested reader to [54]. To lead over to the contents of the next chapter there is, however, a final important observation to make: As can be guessed from (4.2), there is a connection between the coherent states and the *kernel* of the *heat operator*

$$e^{-t\Delta}\delta_y(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{2t}(x-y)^2}. \tag{4.4}$$

In fact, the coherent states can be obtained as *analytic continuation* of the heat kernel:

$$\psi_z(x) \sim \left[e^{-t\Delta}\delta_y\right]_{y\to z}(x). \tag{4.5}$$

4.2.2. Group coherent states

In the present section, G will denote a compact, connected Lie group. For the notation and conventions used in this context, we remind the reader of section 2.2.

As mentioned at the end of the last section, coherent states can be obtained as analytic continuation of the heat kernel on $\mathbb{R}$. In [55], Hall observed that this fact can be used to give a *definition* of coherent states on compact, connected Lie groups. Indeed, such a group G possesses all the necessary ingredients to write down an equation like (4.5) from the last section:

1. From the left invariant vector-fields and the positive definite inner product we can construct a second order differential operator, which in our basis reads

$$\Delta_G \doteq \sum_{IJ} X^I X^J.$$

Δ_G is independent of the basis used in its definition up to a constant, and it is nothing else then the Laplace-Beltrami operator coming from the metric induced on G by the one we chose on $\mathfrak{g}$. It follows that $-\Delta_G$ is symmetric and positive definite as operator in $L^2(G, d\mu_H)$.

2. The heat operator $\exp(-t\Delta_G)$ has an analytic kernel $H(t, g, g')$ with respect to the Haar measure on G, i.e.
$$e^{-t\Delta_G} f(g) = \int_g H(t, g, g') f(g')\, d\mu_H(g').$$
t is an arbitrary dimensionless parameter which plays a role comparable to the t introduced in (4.2). An expansion of this heat kernel can be given using the Peter-Weyl decomposition:
$$H(t, g, g') = \sum_\pi \dim(\pi) e^{-t\lambda_\pi} \chi_\pi \left[g(g')^{-1}\right]$$
where the sum is over all irreducible representations of G (one representation picked from every isomorphism class), $\dim(\pi)$ is the dimension and $\chi_\pi(\cdot) = \mathrm{Tr}(\pi(\cdot))$ the character of the representation.

3. There is a well defined Lie group $G^{\mathbb{C}}$, the *complexification* of G. $G^{\mathbb{C}}$ is characterized by being the smallest Lie group with Lie algebra $\mathfrak{g} + i\mathfrak{g}$.

Using these ingredients, Hall [55] shows that there is indeed a unique analytic continuation of the heat kernel on G to its complexification $G^{\mathbb{C}}$. Based on this, group coherent states (GCS for short) can be defined as
$$\Psi_u^t(g) = H(t, g, g')|_{g' \longrightarrow u} \qquad \in L^2(G, d\mu_H) \tag{4.6}$$
where (unique) analytic continuation is understood in the element g' of $G^{\mathbb{C}}$.
Many interesting mathematical results about these states as well as a generalization of the *Segal-Bargmann transform* induced by them can be found in [55, 56]. We just mention that for example the analogy to the usual coherent states can even be carried further: As $\mathbb{C}$ can be regarded as a phase space, $G^{\mathbb{C}}$ is isomorphic to the cotangent bundle T^*G, and the whole theory can be developed from this perspective.
For the purpose of using the GCS in the construction of semiclassical states for LQG, however, it is of utmost importance to understand whether they have peakedness properties with respect to a classical phase space point, analogous to the ones for the usual coherent states as expressed by (4.2). What precisely do we mean by this? In the case of the GCS, the role of the configuration variable operator $\widehat{Q}$ is played by the multiplication operators $\widehat{g}_{AB}$ on G (i.e. $\widehat{g}_{AB} f(g) = (\pi(g))_{AB} f(g)$, where π is the defining representation of G). Hence the GCS should be peaked in what could be dubbed their *configuration representation*, i.e. their representation as functions on the group (4.6). Pushing the analogy to the usual coherent states further, one would expect the width of the peak to be roughly given by t.
At the same time, the GCS should be peaked in a suitably defined *momentum representation* in which the right invariant vector-fields act by multiplication. The width of the peak in this representation should be proportional to t^{-1}.
The question whether the above expectations are actually met by the GCS were answered affirmatively by Thiemann and Winkler in [22]. In their work they consider the case $G = \mathrm{SU}(2)$ which is the relevant one for applications in LQG. We would like to give a rough sketch of their findings. Let us start to consider GCS as they are defined, i.e. as functions on the group, i.e. in the configuration representation. We make use of the following parametrizations:
$$\begin{aligned} g &= e^{\tau_I x^I} \qquad \in \mathrm{SU}(2) \\ u &= e^{i\tau_I p_0^I} e^{\tau_J x_0^J} \qquad \in \mathrm{SL}(2, \mathbb{C}) \quad (= \mathrm{SU}(2)^{\mathbb{C}}). \end{aligned}$$

In [22], an estimate is derived, which, among other things, implies

$$\frac{\left|\Psi_u^t(g)\right|^2}{\left\|\Psi_u^t\right\|^2} \leq c\,[1+O(\underline{x}-\underline{x}_0)]\exp-\frac{1}{t}\left[\left|\underline{x}-\underline{x}_0\right|^2+O(\underline{x}-\underline{x}_0)\right] \tag{4.7}$$

where c as well as the terms abbreviated by the O symbols also depend on $\underline{p}_0$. Since these estimates are sharp (at least in the leading order behavior), the GCS fulfill the expectations in that they are exponentially peaked with respect to the group element parametrized by $\underline{x}_0$ in the configuration representation. The width of the peak is given by t. We should however also note an important difference to the coherent states on $\mathbb{R}$: In the case of SU(2) it is the *parametrization* of the group elements that shows up in the exponent, whereas in the case $G=\mathbb{R}$, it is the group elements *themselves*.
Next, we want to consider the GCS in their *momentum representation*. What do we mean by that? In quantum mechanics on the real line, going over to momentum space amounts to Fourier transforming the states. The analog of the Fourier transform we will use here is that of decomposition into the Peter-Weyl basis (2.1) of $L^2(G,d\mu_H)$ in which the right invariant vector-fields act as multiplication operators. That is, we are interested in peakedness of $\Psi_u^t(j,m,n)$, where

$$\Psi_u^t(j,m,n)=\langle e_{j,m,n},\Psi_u^t\rangle, \qquad \text{i.e.} \qquad \Psi_u^t(g)=\sum_{j,m,n}\Psi_u^t(j,m,n)e_{j,m,n}(g).$$

We will again use the parametrization

$$u=e^{i\tau_I p_0^I}e^{\tau_J x_0^J}\equiv e^{i\tau_I p_0^I}g_0 \qquad \in \mathrm{SL}(2,\mathbb{C}) \quad (=\mathrm{SU}(2)^{\mathbb{C}})$$

and define in addition

$$p'^I_0=\mathrm{Tr}(g_0\tau_I g_0^{-1}\tau_J)p_0^J.$$

With this notation, it is again a result of [22] that

$$\begin{aligned}\frac{\left|\Psi_u^t(j,m,n)\right|^2}{\left\|\Psi_u^t\right\|^2} &\leq c[1+O(j)+O(m)+O(n)]\times\\ &\times\exp-\left[\frac{j}{2}\left(\frac{(m/j-p_0^3/\left|\underline{p}_0\right|)^2}{1-(p_0^3/\left|\underline{p}_0\right|)^2}+\frac{(n/j-p'^3_0/\left|\underline{p}'_0\right|)^2}{1-p'^3_0/\left|\underline{p}'_0\right|)^2}\right)+t\left[(j+1/2)-\frac{1}{t}\left|\underline{p}_0\right|\right]^2\right].\end{aligned} \tag{4.8}$$

This estimate shows that indeed $\Psi_u^t(j,m,n)$ is peaked at $j\approx 1/t\left|\underline{p}_0\right|$, $tm\approx p_0^3$, $tn\approx p'^3_0$. The details of the shape of the peak depend on the point u, but its width is still roughly given by t^{-1}. The estimates of [22] are more general then the ones displayed here in that care is taken to keep track of the behavior for small t, for it is the limit $t\longrightarrow 0$ that Thiemann and Winkler use to define their notion of *classical limit*.
Summarizing the present section, GCS seem well suited for semiclassical considerations due to their peakedness properties, which are displayed in (4.7), (4.8). With respect to the operators $\widehat{g}$ and X^i they essentially behave like the ordinary coherent states do with respect to the operators $\widehat{X}$ and $\widehat{P}$. To actually use GCS to construct semiclassical states for LQG however, another problem has to be addressed.

4.2.3. Application to Gravity

It was already mentioned above that the general idea to obtain LQC is as follows: As in the case of the weave, LQC should be defined as cylindrical functions over graphs γ, each of them being a product of functions cylindrical over a single edge. In the case of LQC, these functions are chosen to be GCS:

$$\Psi^t_\gamma\left(g_{e_1},\ldots,g_{e_{|E(\gamma)|}}\right)=\prod_i \Psi^t_{u(e_i)}(g_{e_i}).$$

We have seen in the last chapter that GCS inherit certain peakedness properties which should be exploited. More precisely, given a classical geometry (A_0, E_0), the

$$u(e)=e^{i\tau_I p^I(e)}g(e),\qquad g(e)\in \mathrm{SU}(2) \tag{4.9}$$

should be chosen such that the above state is peaked with respect to certain operators corresponding to the canonical variables A resp. E of LQG. Let us consider A first: In the last section we have seen that the GCS are peaked with respect to the multiplication operator $\widehat{g}$. In LQG, the operator $\widehat{h}_e$ quantizing the holonomy along edge e is represented precisely by this multiplication operator on the cylindrical subspace associated to e. It is thus very natural to choose $u(e)$ such that the corresponding GCS is peaked at $g = h_e(A_0)$. This fixes $g(e)$ in (4.9).
Can a similar thing be done with respect to the peakedness of the GCS in the momentum representation? More precisely we would like to find to each edge e a classical quantity depending on E such that its quantization on the cylindrical subspace corresponding to e would be given by X^I. On the one hand, this seems to be feasible, because the smeared triads $E_{S,f}$ are indeed quantized using the invariant vector-fields. On the other hand, it is one of the basic features that E makes sense as an operator when smeared over a two surface, and there is no natural way to associate a specific two-surface to a given edge e.
Remarkably, Thiemann [57] has worked out a detailed proposal how to deal with this difficulty. It can roughly be summarized as follows (for the many details we refer the reader to the original work [57]): To each graph γ fix once and for all a dual 2-complex $\Delta(\gamma)$, i.e. roughly speaking a set of surfaces $(S_e)_{e\in E(\gamma)}$ which intersect each other in common boundaries at most and such that the edge e of γ intersects only S_e and that this intersection is transversal. The surfaces S_e shall be given an orientation according to the orientations of the edges e, i.e. the pairing between the orientation two form on S_e with the tangent vector field on e at the intersection point should be positive.
Also to each point p lying in a surface S_e fix an analytic path $\rho(p)$ connecting the intersection point $S_e\cap e$ with p and denote the part of e from $e(0)$ to $S_e \cap e$ by e^{in}.
With the help of these structures, Thiemann can now define the quantity

$$P^I_e(A,E)=-\operatorname{Tr}\left[\tau^I h_{e^{\mathrm{in}}}\left(\int_{S_e} h_{\rho(p)}E^a(p)h^{-1}_{\rho(p)}\epsilon_{abc}\,dS^{bc}(p)\right)h^{-1}_{e^{\mathrm{in}}}\right]. \tag{4.10}$$

It is a close relative of the $E_{S,f}$ in that it is also the integral of E over a two-surface. The only difference is the following: Since E is a section in a bundle, values at different points can not be added in a well defined way. To achieve a well defined expression for $E_{S,f}$, the E at different points get contracted with the vector-field f before being added, whereas for the P^I_e, the values of E at different points get parallel transported to the same point $S_e\cap e$ before being added.
The key feature of this new variable P^I_e is that

$$\left\{P^I_e, h_{e'}\right\}=\delta_{e,e'}\frac{\tau^I}{2}h_{e'},\qquad \left\{P^I_e,P^J_{e'}\right\}=\delta_{e,e'}c^{IJ}_K P^K_e \tag{4.11}$$

where c_{IJ} are the structure constants of SU(2). Therefore, if h_e is represented by the multiplication operator $\widehat{g}$ on the cylindrical subspace corresponding to e, P_e^I can be represented by the right invariant vector-field $i\kappa\hbar X^I$ acting on the cylindrical subspace corresponding to e.
In this way, Thiemann obtains the desired momentum observables with respect to which the GCS should be peaked: He essentially proposes to choose the $p^I(e)$ of equation (4.9) as

$$p^I(E) = \frac{t}{\hbar\kappa} P_e^I(A_0, E_0). \tag{4.12}$$

Equation (4.7) implies that t will be a measure of peakedness for the $\widehat{h}_e$, (4.8) together with (4.12) that $\hbar\kappa/t$ measures peakedness for the $\widehat{P}_e^I$. Comparing with the corresponding formulae (4.3) for the ordinary coherent states, we see that the quantity $\hbar\kappa/t^2 \equiv l_P^2/t^2$ corresponds to the parameter ω, translating between the units of the configuration and the momentum functions. From this point of view, it would be natural to set $t = 1$: l_P is the natural parameter to do the translation. On the other hand, picking other values for t will prove to be useful later, so we will leave t unspecified for now.
In [22], the freedom in the choice of t is phrased a bit different. There, a length parameter a is introduced such that

$$t = \frac{\hbar\kappa}{a^2}.$$

We now briefly want to discuss the quantization of more complicated observables like the volume of regions or the area of surfaces in space. In view of the family $\{P_e^I\}$ of momentum observables introduced by Thiemann, two attitudes towards the quantization of further observables could be assumed:

Standpoint 1: The quantization of more complicated observables should still be based on holonomies and the family $\{\widehat{E}_{S,f}\}$ as elementary ones. Thus one would obtain the usual area and volume quantization [5, 4] etc.

Standpoint 2: We regard the P_e^I as the basic momentum observables of the theory. Therefore, the quantization of more complicated observables should be based on holonomies and them.

The first point of view has the advantage of simplicity: No new operators have to be introduced, quantization of geometric operators [5, 4] as well as Hamiltonians [16] can be taken over from the literature. The disadvantage is that expectation values of these observables in LQC might differ considerably from their classical values. We refer to the discussion in [30] for details.
The second point of view, though resulting in a more complicated theory, can not be dismissed on general grounds. Any choice of basic observables should be permitted, as long as it leads to a reasonable theory. The advantage of the second standpoint is that the P_e^I, and hence the more complicated operators constructed from it are adapted to the coherent states in that they have the right expectation values.

4.2.4. Random graphs for coherent states

As one LQC is defined for every given graph γ, the question arises as to which graphs are appropriate, i.e. render the LQC into reasonable semiclassical states. To give an extreme example, an LQC based on a graph containing just a single edge will obviously not be a good semiclassical state in any sense.

As discussed above, there are at least two requirements for a state to be called semiclassical, namely that expectation values of macroscopic observables should be close to the classical values and their uncertainties should be small. In the following we want to give an order of magnitude estimate of deviations from classical values and uncertainties for LQC for the case that flat space should be approximated. These will relate the macroscopic scale L, the typical edge-length ϵ, the parameter t and the Planck length. The outcome of the estimate is very important as it will reflect the "mechanics" of the different scales resulting in the order of magnitude of quantum gravity corrections. As we will be very general, these considerations will apply to any semiclassical state similar to the LQC, i.e. any semiclassical state that

1. is based on a graph and
2. is a tensor product of states on the edges that approximate the local degrees of freedom in a "coherent fashion", i.e. with the fluctuations of the momentum degrees of freedom inversely proportional to that of the configuration degrees of freedom.

For simplicity we do the calculation for U(1) instead of SU(2) as gauge group. That makes the reasoning much more transparent while not affecting the outcome, since only very general estimates are used. In the same spirit will be cavalier with factors of order of unity, abbreviating all of them with $\not{c}$.

So, let a graph γ with typical edge-length ϵ and a classical phase space point (A_0, E_0) be given. The fields A_0 and E_0 should vary on a length scale L which is thought to be macroscopic. Finally let t be a dimensionless constant which we leave unspecified for now. Then the LQC $\Psi^t_{\gamma,(A_0,E_0)}$ can be constructed as described in the previous section. We will denote this state by Ψ_γ in the following.

As observables we will consider electric and magnetic fluxes, since observables of this type are typical for LQG:

$$E_S \doteq \int \epsilon_{abc} E^a \, dS^{bc} \qquad \text{and} \qquad B_S \doteq \int_S \mathrm{d}A,$$

where S is some surface in Σ which has an area $\not{c} L^2$ of macroscopic size.

Let us consider the electric flux first: There are different ways to quantize E_S. The standard one was given in 2.4. One could, however also quantize E_S by first approximating it by P_{S_e} (4.10), the S_e being surfaces in the dual polyhedral decomposition associated to γ, and then quantize the P_{S_e}. We refer the reader to the discussion at the end of section 4.2.3. In both cases, the quantization of E_S will act via the right invariant vector-fields (times l_P^2) on edges of the graph which are intersected by, or whose dual surfaces approximate, the surface S. Since the LQC are tensor products of states cylindrical over single edges, we find

$$\Delta^2_{\Psi_\gamma}(\widehat{E}_S) = \sum_{e\in\gamma\,:\,e\cap S\neq\emptyset} \not{c}\, l_P^4 \Delta^2_{\Psi_e}(X^2) = \sum_{e\in\gamma\,:\,e\cap S\neq\emptyset} \not{c}\, \frac{l_P^4}{t} = \not{c}\, \frac{L^2}{\epsilon^2}\frac{l_P^4}{t}.$$

Note that since each $\Delta^2_{\Psi_e}(X^2)$ contributes to the same order (and essentially independent of E_S) to the fluctuation, it is proportional to the number of edges intersected by or used to approximate S, and hence inversely proportional to ϵ.

Essentially the same happens for B_S. First we have to discuss its quantization. Since the basic configuration variable is not the connection itself but its holonomies, one would first approximate B_S through holonomies and then quantize them. For example $1/2i(h_\alpha - h_\alpha^{-1})$ would be an approximation of the magnetic flux through a surface which has the loop α as its border. One would, however, have to quantize in a graph dependent way, because generic edges will never be contained in γ. This problem was discussed in 4.1.

For *any* quantization along these lines we find

$$\Delta^2_{\Psi_\gamma}(\widehat{B}_S) = \sum_{\substack{e\in\gamma:\\ e \text{ needed for approx.}}} \not{c}\,\Delta^2_{\Psi_e}(h_e - h_e^{-1}) = \sum_{\substack{e\in\gamma:\\ e \text{ needed for approx.}}} \not{c}\,t = \not{c}\,t\frac{L^2}{\epsilon^2}.$$

We define e_0 and b_0 via

$$E_S = e_0 \mathrm{Area}(S), \qquad B_S = b_0 \mathrm{Area}(S)$$

and compute for the relative uncertainties

$$\frac{\Delta^2_{\Psi_\gamma}(\widehat{E}_S)}{E_S} = \not{c}\,\frac{l_P^4}{t e_0^2 \epsilon^2 L^2}, \qquad \frac{\Delta^2_{\Psi_\gamma}(\widehat{B}_S)}{B_S} = \not{c}\,\frac{t}{b_0^2 \epsilon^2 L^2}.$$

We assume here that e_0 and b_0 are nonzero. If that is not the case one would have to use some dimension-full constant to compare the absolute uncertainties. This could be l_P (which would, however change the outcome of the estimates quite drastically) or some constant derived from the physical situation considered. We discussed this issue for the coherent states used in quantum mechanics in section 4.2.1.
We want the relative fluctuations to be equal. That fixes t to be

$$t = \not{c}\,\frac{b_0 l_P^2}{e_0}.$$

b_0 and e_0 are macroscopic quantities, their ratio will thus generically be $e_0/b_0 = \not{c}\,L^2$. Thus a natural choice for t is

$$t = \frac{l_P^2}{L^2},$$

a very tiny number. For the relative uncertainties, we find

$$\frac{\Delta^2_{\Psi_\gamma}(\widehat{E}_S)}{E_S} = \frac{\Delta^2_{\Psi_\gamma}(\widehat{B}_S)}{B_S} = \not{c}\,\frac{l_p^2}{b_0 e_0 L^2 \epsilon^2},$$

which decreases with increasing ϵ as expected.

Now, the second issue concerning semiclassicality has to be addressed: The expectation values of $\widehat{B}_S$ and $\widehat{E}_S$ will deviate from the classical values B_S, E_S. These deviations have two sources. The first is that the LQC are geared to yield the right expectation values for the microscopic degrees of freedom associated to the edges. When the macroscopic E_S and B_S get quantized, they effectively get expressed in terms of the microscopic ones. But in this process, an error of approximation occurs. The situation is analogous to that occuring when approximating an integral by a discrete sum. Clearly, the resulting deviations generically get smaller, the smaller ϵ and the bigger L is.
The other source is that if an observable like E_S is quantized without reference to the S_e of the dual polyhedral decomposition of γ, the expectation values in LQC may be systematically off the classical values by a factor of order unity (in [30], this was dubbed *staircase problem*).
We do not want to go into a detailed discussion of the size of these two effects but assume

$$\frac{(\langle \widehat{E}_S\rangle_{\Psi_\gamma} - E_S)^2}{E_S^2} = \not{c} + \not{c}\left(\frac{\epsilon}{L}\right)^{2\beta}$$

for some positive constant α and similarly for B_S. We will say something about the value of β below.

The important observation now is that fluctuations in LQC get smaller with bigger ϵ whereas expectation values get worse. The scale ϵ can therefore be fixed by somehow weighting the classical error against the quantum uncertainties and finding the optimum value.
But how should this weighting be done? A simple and natural way is to just add relative fluctuations squared relative deviations and minimize the sum. This leads to

$$\epsilon = \not{c}(\alpha e_0)^{-\frac{1}{\alpha+2}} L^{\frac{\alpha-2}{\alpha+2}} b_0^{-\frac{1}{\alpha+2}} l_P^{\frac{2}{\alpha+2}}$$

or, upon setting $\alpha = \not{c}$ and using $e_0/b_0 = \not{c} L^2$

$$\epsilon = \not{c} l_P^{\frac{1}{\beta+1}} L^{\frac{\beta}{\beta+1}} =: l_P^{\alpha} L^1 - \alpha.$$

We see that the value of α is very important to determine the nature of the underlying graph: Small α will require a graph whose typical edge-length is l_P, bigger values of α lead to graphs with typical edge-length closer to the macroscopic scale L. In the work [30], a value of $\alpha = 1/6$ was obtained from some rough estimate.

5. The Hamilton constraint of Matter coupled to gravity

In the classical theory, the coupling between matter fields and geometry is contained in Einstein's famous field equations

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = \frac{8\pi G}{c^2} T^{\mu\nu}. \tag{5.1}$$

Here, $g^{\mu\nu}$ is the four dimensional metric, $R, R^{\mu\nu}$ its scalar and Ricci curvature respectively, and $T^{\mu\nu}$ the stress-energy tensor of the matter. As is obvious, the dependence is mutual: Matter curves space-time, geometry of space-time tells matter where to go.
In the present chapter, we address the question how the system of gravity coupled to matter can be quantized, starting from the quantization of the gravitational field achieved in LQG.
Since LQG is in essence a canonical quantization, it is natural to address this problem in a Hamiltonian setting. It is well known that the field equations (5.1) can be obtained from a Hamiltonian

$$H = H_{\text{grav}}(A, E) + H_{\text{matter}}(A, E, \text{matter configurations and momenta}).$$

In what follows, we will concentrate on the Klein-Gordon and the Maxwell field as matter: The former is ideally suited as a model system due to its simplicity, the latter is important for the prediction of actual effects, for example in the case of the γ-ray bursts.
The relevant classical Hamiltonians are therefore

$$\begin{aligned} H_{KG} &= \frac{1}{2Q_{KG}} \int_\Sigma N(p) \left[\frac{\pi^2}{\sqrt{\det(q)}} + \sqrt{\det(q)} [q^{ab} \phi_{,a} \phi_{,b} + K^2 \phi^2] \right] d^3p, \\ H_{EM} &= \frac{1}{2Q_{EM}} \int_\Sigma N(p) \frac{q_{ab}}{\sqrt{\det(q)}} \left[E^a E^b + \epsilon^{acd} A_{d,c} \epsilon^{bef} A_{f,e} \right] d^3p. \end{aligned} \tag{5.2}$$

Here, Q_{KG} and Q_{EM} are the coupling constants in the respective case (electron charge squared in the latter), and K is the rest Compton wave number. π/Q_{KG} is conjugate to ϕ. We will take ϕ to be dimension-free, thus $\pi \propto \dot{\phi}$ has dimension meter^{-1}. It follows that $\hbar Q_{KG}$ has dimension meter^2. E^a/Q_{EM} is canonically conjugate to the (local section of the) $U(1)$ connection A_a. We will take A_a to have dimension meter^{-1}, thus $E^a \propto \dot{A}_a$ has dimension meter^{-2}. It follows that the Feinstruktur constant $\hbar Q_{EM}$ is dimension-free.

As already explained in chapter 3, we will view (5.2) not as constraints but as ordinary Hamiltonians, consequently delete the lapse $N(p)$ from the expressions. In a first step, we regulate and quantize the gravitational variables in the matter Hamiltonians along the lines of [16, 19] to obtain operator valued quadratic forms in the matter fields. In section 5.2 we will then turn to the quantization of the matter fields.

5.1. Regularization of the matter Hamiltonians

The present section is devoted to the regularization of the matter Hamiltonians of the Einstein-Klein-Gordon and Einstein-Maxwell theory (5.2) and quantization of the gravitational degrees of freedom appearing in them. For convenience of presentation, let us pick and label the relevant terms from (5.2) as follows:

$$\begin{aligned} F_1(\phi) \doteq \int \sqrt{\det q}\,\phi^2, \qquad F_2(\phi) \doteq \int \sqrt{\det q}\,q^{ab}\phi_{,a}\phi_{,b}, \qquad F_3(\pi) \doteq \int \frac{\pi^2}{\sqrt{\det q}}, \\ F_4(E) \doteq \int \frac{q_{ab}}{\sqrt{\det q}} E^a E^b, \qquad F_5(A) \doteq \int \frac{q_{ab}}{\sqrt{\det q}} \epsilon^{acd} A_{d,c} \epsilon^{bef} A_{f,e}. \end{aligned} \tag{5.3}$$

As already announced, to achieve quantization of these quadratic forms, we will follow the regularization prescription described in detail in [16, 19]. Our explanations will therefore be very brief.

Two ingredients play a fundamental role in the work of Thiemann. The first is an elaborate point splitting procedure which regulates the classical expression. Care is taken that this procedure respects the density character of the constituents of the expression to be quantized. It is in fact one of the great strengths of Thiemanns procedure that it works precisely because the terms to be quantized have an overall density weight equal to one.

The second ingredient is the application of certain identities on the classical level: The first is that the density weight zero triad

$$e_a^I \doteq \sqrt{|\det E|}\left(E^{-1}\right)_a^I \tag{5.4}$$

can be written as a Poisson bracket

$$e_a^I = \frac{2}{\gamma\kappa}\left\{A_a^I, V\right\}. \tag{5.5}$$

This opens up the possibility of quantizing it as the commutator of the quantized connection with the volume operator.
Since all the classical expressions could in principle be written in terms of the usual triad E_I^a for which a quantization exists, direct quantization of the e_a^I is not mandatory but still offers enormous simplification. However, it becomes essential, when one wants to employ the second observation used in [19]:
In the course of the quantization of expressions like $F_1, \ldots, F_5$, negative powers of the volume V_R of little cells R frequently arise. These can not be quantized as negative powers of the respective volume operator, since it possesses a huge kernel and its inverse is therefore not existent. In this situation, the simple identity

$$\{A, B^\alpha\} = \alpha B^{\alpha-1}\{A, B\} \tag{5.6}$$

comes to rescue: If in an expression containing negative powers of the volume, (5.5) has already been used in the quantization process thus producing a Poisson bracket involving the volume, one can absorb them into the bracket via (5.6) which in turn can be quantized by a commutator.
One can even go one step further and insert an arbitrary positive power of

$$1 = \frac{(\det e)^2}{|\det E|} \tag{5.7}$$

into expressions not containing the triad e_i^I, to treat negative powers of the volume with the same trick.

The present section is devoted to the quantization of the quadratic forms $F_1, \ldots, F_5$ using the methods described above. Before we start, we have to discuss one final issue: As discussed towards the end of section 4.2.3, in the light of the later application of LQC two different points of view regarding quantization could be taken: To take standpoint 1 would mean that we should quantize $F_1, \ldots, F_5$ without reference to the $P_e^I(v)$, that is to follow [16, 19] in detail.
In the following we will rather focus on standpoint 2, i.e. we will describe quantizations based on holonomies and the variables P_e^I. This has some impact on the coefficients showing up in the regularization process so that the formulae obtained this way slightly differ from those of [16, 19].

We will start by briefly considering volume quantization in terms of the $\widehat{P}_e^I(v)$. Then we turn to the regularization of $F_1, \ldots, F_5$ as promised.

5.1.1. Volume quantization

As explained in the preceeding section, Thiemanns quantization prescription for densities of weight one is based on the assumption that the volume V_R of a region R has already been quantized. Indeed, there are quantizations of the volume by Rovelli and Smolin [3] and by Ashtekar and Lewandowski [4], which in spite of some differences are very similar in spirit. To remind the reader, we recall the quantization from [4]: The volume operator $\widehat{V}(R)$ of a region R acts on a function f_γ cylindrical on γ via

$$\widehat{V}(R) f_\gamma = \sum_{v \in V(\gamma) \cap R} \widehat{V}_v f_\gamma \tag{5.8}$$

where

$$\widehat{V}_v f_\gamma = c_0 l_P^2 \sqrt{\frac{1}{48} \left| \sum_{\{e,e',e''\}} \epsilon^{ee'e''} \epsilon_{IJK} X_e^I X_{e'}^J X_{e''}^K \right|} f_\gamma . \tag{5.9}$$

Here c_0 is an overall constant, which is usually set to 1. This volume operator, or the version of [3], can certainly be used.
However, in the light of future applications of the LQC, it can also be useful to take the point of view expressed in “Standpoint 2” at the end of section 4.2.3, namely that the quantization of all operators should be based on the elementary h_e and $\widehat{P}_e^I$. In the following, we will give a quantization of the volume $\widehat{V}_{R_v}$ of a cell R_v belonging to the dual polyhedral decomposition of some graph in this spirit. We will see that the resulting operator differs from the one given above, that the constant c_0 in (5.9) will be chosen to depend on the topology of the vertex in a specific way. On six valent graphs however, both quantizations will be seen to coincide.

Let γ be some graph, v a vertex of γ and χ a coordinate chart containing R_v. In the following we work in coordinates defined by χ. At the moment, χ is completely arbitrary. We will however discuss the choice of these charts in section 5.1.4 and dispose of them in a specific way.

What we want to do is to quantize the volume of R_v in terms of the variables $P_I^e(v)$. Classically, if E does not change too much within R_v,

$$V(R_v) = \int_{\chi^{-1}(R_v)} \sqrt{\left| [\det E]_\chi \right|}\, d^3x \approx \nu(v) \det E(v)$$

is a good approximation, where we have introduced the coordinate volume

$$\nu(v) \doteq \int_{\chi^{-1}(R_v)} d^3x .$$

If furthermore the S_e do not excessively bend in the coordinates given by χ,

$$P^e_I \approx \mu_a(e) E^a_I(v) \text{ with } \mu_a(e) = \int_{\chi^{-1}(S_e)} dA_a(x).$$

Using this approximation, we compute

$$\frac{1}{3!} \sum_{e,e',e'' \in \{e_1,e_2,e_3\}} \epsilon_{ee'e''} \epsilon^{IJK} P^e_I P^{e'}_J P^{e''}_K \approx \det \mu \, \det E(v)$$

for three edges e_1, e_2, e_3 incident at v with linear independent tangents. To treat all edges incident at v on the same footing, we sum over all possible triples:

$$\frac{1}{3!} \sum_{\{e_1,e_2,e_3\} \in E(v)^3} \sum_{e,e',e'' \in \{e_1,e_2,e_3\}} \epsilon_{ee'e''} \epsilon^{IJK} P^e_I P^{e'}_J P^{e''}_K \approx \mu(v) \det E$$

with

$$\mu(v) \doteq \sum_{\{e_1,e_2,e_3\} \in E(v)^3} \sum_{e,e',e'' \in \{e_1,e_2,e_3\}} \epsilon_{ee'e''} \epsilon^{abc} \mu_a(e) \mu_b(e') \mu_c(e'').$$

So we find as a classical approximation of the quantity to be quantized

$$V(R_v) \approx \nu(v) \sqrt{\frac{1}{3!\mu(v)} \sum_{\{e_1,e_2,e_3\}} \sum_{e,e',e'' \in \{e_1,e_2,e_3\}} \epsilon_{ee'e''} \epsilon^{IJK} P^e_I P^{e'}_J P^{e''}_K}.$$

This suggests the following quantization: Let f_γ be a function cylindrical on γ. Then

$$\widehat{V}_v f_\gamma \doteq \nu(v) \sqrt{\frac{1}{3!\mu(v)} \sum_{\{e_1,e_2,e_3\}} \sum_{e,e',e'' \in \{e_1,e_2,e_3\}} \epsilon_{ee'e''} \epsilon^{IJK} X^e_I X^{e'}_J X^{e''}_K} \, f_\gamma. \tag{5.10}$$

$$\widehat{V}(R) f_\gamma \doteq \sum_{v \in V(\gamma) \cap R} \widehat{V}_v f_\gamma. \tag{5.11}$$

There remain two things to say about this quantization: Firstly, we have not yet specified the charts used in the regions around the vertices on which the quantization depends. We will do this in section 5.1.4 in such a way that the operator becomes covariant under diffeomorphisms.
Secondly, equations (5.10),(5.11) do not yet define an operator on $\mathcal{H}_{\mathrm{AL}}$ but a family of operators on the cylindrical subspaces, which is not yet consistent due to the appearance of coefficients depending on the valence etc. of the vertices. We will deal with this by simply requiring (5.10),(5.11) to define the action of $\widehat{V}$ on the spin network basis of $\mathcal{H}_{\mathrm{AL}}$. By defining the action of the operator on a basis, it is automatically defined (consistently) on every element of the Hilbert space. We will do the same for the operators to be defined below, however without explicitely mentioning it.

5.1.2. KG-Hamiltonian

Now we come to the quantization of F_1, F_2, F_3.

The remarks made at the end of the last section concerning cylindrical consistency and diffeomorphism covariance will also apply to the operators in this and the following section. Diffeomorphism

covariance will be dealt with in sections 5.1.4 and consistency is insured by defining the operators on spin networks.
Let γ be a graph and assume that charts have been chosen in neighborhoods of the vertices.

Quantization of F_1:

We start with the easiest term and display all steps in detail: Let $R_\epsilon(u)$ be the sphere around u with radius ϵ (in some chart and with respect to some fiducial metric) and χ_ϵ be the corresponding characteristic function. Then in this chart

$$\begin{aligned} F_1(\phi) &= \int \sqrt{\det q}(x)\phi^2(x)\, d^3x = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\iint \sqrt{\det q}(x)\phi^2(u)\chi_\varepsilon(x,u)\, d^3x\, d^3u \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int V(R_\varepsilon(u))\phi^2(u)\, d^3u. \end{aligned}$$

This suggests the following quantization:

$$\begin{aligned} \widehat{F}_1(\phi)f_\gamma &\doteq \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int \widehat{V}(R_\varepsilon(u))\phi^2(u)\, d^3u\, f_\gamma = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int \sum_{v\in V(\gamma)\cap R_\varepsilon(u)} \widehat{V}_v f_\gamma \phi^2(u)\, d^3u \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\sum_{v\in V(\gamma)} \widehat{V}_v f_\gamma \int_{R_\varepsilon(v)} \phi^2(u)\, d^3u = \sum_{v\in V(\gamma)} \phi^2(v)\widehat{V}_v f_\gamma \end{aligned}$$

where $\widehat{V}_v$ is either defined by (5.9) (standard volume quantization) or by (5.10). We can also write this quantization in terms of a quantized density:

$$\widehat{F}_1(\phi) = \iint \phi(p)\phi(q)\widehat{F}_1(p,q)\, dp\, dq \qquad \text{with} \qquad \widehat{F}_1(p,q)|_\gamma = \sum_{v\in V(\gamma)} \delta(p,v)\delta(q,v)\widehat{V}_v|_\gamma.$$

Quantization of F_2:

Now we come to the form F_2. We note first that

$$\sqrt{\det q}q^{ab} = \frac{E^a_I E^{bI}}{\sqrt{\det q}} = \frac{1}{4}\epsilon^{acd}\epsilon^{bef}\epsilon_{IJK}\epsilon^I_{\ LM}\frac{e^J_c e^K_d\, e^L_e e^M_f}{\sqrt{\det q}}. \tag{5.12}$$

To replace the triads e^I_a by Poisson brackets, we will now derive an analog of equation (5.5) in terms of the volume V_{R_v}. We begin by observing

$$\begin{aligned} \left\{h_{e_0}, V_v^2\right\} &= \frac{\nu^2(v)}{3!\mu(v)}\sum_{\{e_1,e_2,e_3\}\in E(v)^3}\ \sum_{e,e',e''\in\{e_1,e_2,e_3\}} \epsilon_{ee'e''}\epsilon^{IJK}\left\{h_{e_0}, P^e_I P^{e'}_J P^{e''}_K\right\} \\ &= \frac{l_P^2\nu^2(v)}{2\mu(v)}\sum_{\{e_2,e_3\}\in E(v)^2}\ \sum_{e',e''\in\{e_2,e_3\}} \epsilon_{e_0e'e''}\epsilon^{IJK}\tau_I h_{e_0} P^{e'}_J P^{e''}_K \end{aligned}$$

where we have made use of the Poisson structure (4.11),

$$\approx \frac{l_P^2\nu^2(v)}{2\mu(v)}\sum_{\{e_2,e_3\}\in E(v)^2}\ \sum_{e',e''\in\{e_2,e_3\}} \epsilon_{e_0e'e''}\epsilon^{IJK}\tau_I h_{e_0}\mu_b(e')\mu_c(e'')E^b_J E^c_K$$

by (5.11). We note that

$$(M^{-1})^i_l = \frac{1}{\det M}\epsilon^{ijk}\epsilon_{lmn}M^j_m M^k_n \tag{5.13}$$

for any invertible 3 by 3 matrix M and continue

$$\begin{aligned}
\operatorname{Tr}\left[\tau_I h^{-1}_{e_0}\left\{h_{e_0},V_v\right\}\right] &= \frac{l_P^2\nu^2(v)}{4\mu(v)}V_v^{-1}\sum_{\{e_2,e_3\}\in E(v)^2}\ \sum_{e',e''\in\{e_2,e_3\}}\epsilon_{e_0e'e''}\epsilon^{IJK}\mu_b(e')\mu_c(e'')E^b_J E^c_K\\
&= \frac{l_P^2\nu^2(v)}{8\mu(v)}V_v^{-1}\sum_{\{e_2,e_3\}\in E(v)^2}\ \sum_{e',e''\in\{e_2,e_3\}}\epsilon_{e_0e'e''}\epsilon^{abc}\epsilon_{ade}\epsilon^{IJK}\mu_b(e')\mu_c(e'')E^d_J E^e_K\\
&= \frac{l_P^2\nu^2(v)}{2\mu(v)}V_v^{-1}\sum_{e_2,e_3}\epsilon_{e_0e_2e_3}\epsilon^{abc}\mu_b(e_2)\mu_c(e_3)\sqrt{|\det E|}e^I_a\\
&\doteq \frac{l_P^2\nu^2(v)l_p^2}{2\mu(v)}V_v^{-1}2\omega^a(e_0)\sqrt{|\det E|}e^I_a.
\end{aligned}$$

Equation (5.13) was used in the second line and, together with the definition (5.4) of the triad e^I_i, in the last but one. The last line served to define $\omega^a(e)$.
In order to solve for e^I_a we observe that

$$\sum_e \omega^a(e)\mu_b(e) = \delta^a_b\mu(v)$$

and finish with

$$\begin{aligned}
e^I_a &\approx \frac{1}{l_P^2\nu^2(v)}\frac{V_v}{\sqrt{\det E}}\sum_{e_0}\mu_a(e_0)\operatorname{Tr}\left[\tau^I h^{-1}_{e_0}\left\{h_{e_0},V_v^2\right\}\right] &(5.14)\\
&\approx \frac{1}{l_P^2\nu(v)}\sum_{e_0}\mu_a(e_0)\operatorname{Tr}\left[\tau^I h^{-1}_{e_0}\left\{h_{e_0},V_v^2\right\}\right]. &(5.15)
\end{aligned}$$

This is the analog of (5.5). Since it will be used frequently in the following, we provide an abbreviation for the trace on the right hand side, as well as for its quantization: For e in $E(v)$ and α positive real,

$$Q_I(v,e,\alpha) \doteq \operatorname{Tr}\left[\tau_I h_e^{-1}\left\{h_e,V_v^\alpha\right\}\right], \qquad \widehat{Q}_I(v,e,\alpha) \doteq \frac{1}{i\hbar}\operatorname{Tr}\left[\tau_I h_e^{-1}\left[h_e,\widehat{V}_v^\alpha\right]\right]. \tag{5.16}$$

For future use, we note the following

Lemma 5.1.1. *For any positive real α the operator $\widehat{Q}_I(v,e,\alpha)$ on $\mathcal{H}_{AL}$ defined by (5.16) is essentially self-adjoint with core given by the core of $\widehat{V}_v$.*

Proof. Since $(h_e)_{AB}$ is a bounded operator it suffices to show that $\widehat{Q}_I(v,e,\alpha)$ is symmetric with dense domain the core of $\widehat{V}_v$.
Using that $[(h_e)_{AB}]^\dagger = (h_e^{-1})_{BA}$ and $\overline{(\tau_J)_{AB}} = -(\tau_J)_{BA}$ we find

$$\begin{aligned}
[\widehat{Q}_J(v,e,\alpha)]^\dagger &= i\hbar\overline{(\tau_J)_{AB}}\left[((h_{e\rhd})_{CA})^\dagger, V_v^\alpha\right]\left((h^{-1}_{e\rhd})_{BC}\right)^\dagger = -i\hbar(\tau_j)_{BA}\left[\left((h^{-1}_{e\rhd})_{AC}\right), V_v^\alpha\right]((h_{e\rhd})_{CB})\\
&= -i\hbar\operatorname{Tr}\left(\tau_J\left[\left(h^{-1}_{e\rhd}\right),\widehat{V}_v^\alpha\right]h_{e\rhd}\right) = -i\hbar\operatorname{Tr}\left(\tau_J h^{-1}_{e\rhd}\widehat{V}_v^\alpha h_{e\rhd}\right)\\
&= \operatorname{Tr}\left(\tau_J h^{-1}_{e\rhd}\left[h_{e\rhd},V_v^\alpha\right]\right) = \widehat{Q}_J(v,e,\alpha)
\end{aligned}$$

because $\operatorname{Tr}(\tau_J)\widehat{V}_v = 0$. □

Now we pause and think about the derivatives $\phi_a\phi$ of ϕ. At a first glance it seems natural to leave them untouched and simply quantize $\sqrt{\det q}q^{ab}$. But that is counter to the spirit of the quantization procedure proposed here, because we would have to fix an additional coordinate system which would define what the quantization of $\sqrt{\det q}q^{ab}$ really is. We thus procede differently: Let

$$\partial_e^+\phi(v) \doteq \phi(e_\triangleright(1)) - \phi(v)$$

denote the discrete (forward) derivative of ϕ along e. In the coordinates chosen around v we find that

$$\partial_e^+\phi(v) \approx t^i(v,e)\partial_i\phi(v), \qquad \text{where} \qquad t^i(v,e) = (e_\triangleright(1) - v)^i\,.$$

We would like to solve this for $\partial_i\phi(v)$. To that end we proceed analogously to the case of the e_a^I above: Let

$$\Omega_a(v,e) \doteq \frac{1}{2}\sum_{e,e',e''}\epsilon_{e'e''}\epsilon_{abc}t^b(e')t^c(e'')$$
$$\tau(v) \doteq \frac{1}{3!}\sum_{\{e_1,e_2,e_3\}\in E(v)^3}\ \sum_{e,e',e''\in\{e_1,e_2,e_3\}}\epsilon_{ee'e''}\epsilon_{abc}t^a(e)t^b(e')t^c(e'')$$

and observe that

$$\sum_{e\in E(v)} t^a(v,e)\Omega_b(v,e) = \tau(v)\delta_b^a.$$

Using this we find

$$\partial_i\phi(v) \approx \frac{1}{\tau(v)}\sum_{e\in E(v)}\Omega_i(v,e)\partial_e^+\phi(v).$$

What we have achieved is to express the partial derivatives through discrete ones and some coefficients depending on the coordinates chosen around v. This done, we can proceed to the quantization of F_2.
For a start note that for any density μ with weight one we have

$$\int_{R_v}\mu \approx \nu(v)\mu(v). \tag{5.17}$$

Specifically,

$$V_{R_v} \approx \nu(v)\sqrt{\det q}(v). \tag{5.18}$$

We can therefore approximate F_2 as follows:

$$\begin{aligned}
F_2(\phi) &= \int\sqrt{\det q}q^{ab}\partial_a\phi\partial_b\phi \overset{(5.12)}{=} \frac{1}{4}\sum_{v\in V(\gamma)}\int_{R_v}\partial_a\phi\partial_b\phi\epsilon^{acd}\epsilon^{bef}\epsilon_{IJK}\epsilon^I_{LM}\frac{e_c^Je_d^K\,e_e^Le_f^M}{\sqrt{\det q}} \\
&\overset{(5.17)(5.18)(5.15)}{\approx} \frac{1}{4}\sum_{v\in V(\gamma)}\frac{\nu(v)^2}{(\kappa\nu(v))^4\tau(v)^2}\Big(\sum_{e\in E(v)}\Omega_a(v,e)\partial_e^+\phi(v)\Big)\Big(\ldots{}_b\ldots\Big) \\
&\qquad\cdot\ \epsilon^{acd}\epsilon^{bef}\epsilon_{IJK}\epsilon^I_{LM}\frac{1}{V_v}\Big(\sum_{e\in E(v)}\mu_c(e)Q^J(v,e,1)\Big)\Big(\ldots{}^K_d\ldots\Big)\Big(\ldots{}^L_e\ldots\Big)\Big(\ldots{}^M_f\ldots\Big) \\
&\overset{(5.6)}{=} 3\sum_{v\in V(\gamma)}\frac{1}{\kappa^4\nu(v)^2\tau(v)^2}\Big(\sum_{e\in E(v)}\Omega_a(v,e)\partial_e^+\phi(v))\Big)\Big(\ldots{}_b\ldots\Big) \\
&\qquad\cdot\ \epsilon^{acd}\epsilon^{bef}\epsilon_{IJK}\epsilon^I_{LM}\Big(\sum_{e\in E(v)}\mu_c(e)Q^J(v,e,3/4)\Big)\Big(\ldots{}^K_d\ldots\Big)\Big(\ldots{}^L_e\ldots\Big)\Big(\ldots{}^M_f\ldots\Big).
\end{aligned}$$

Quantization is now achieved by replacing $q_I(v,e,3/4)$ with $\widehat{Q}_I(v,e,3/4)$. However we have to make a choice about how to order the factors. We dispose of this in such a way that the operator acting at a vertex becomes symmetric. Again we can write this in terms of an operator valued density:

$$\widehat{F}_2(\phi) = \iint \phi(p)\phi(q)\widehat{F}_2(p,q)$$

with

$$\widehat{F}_2(p,q)|_\gamma = 3 \sum_{v\in V(\gamma)} \frac{1}{\kappa^4 \nu(v)^2 \tau(v)^2} \delta(p,q) \Big(\sum_{e\in E(v)} \Omega_a(v,e)\,[\delta(p,e_\triangleright(1)) - \delta(p,v)] \Big) \Big(\dots\, {}_b\, \dots \Big)$$
$$\cdot\ \epsilon^{acd}\epsilon^{bef}\epsilon_{IJK}\epsilon^I_{LM} \left[\Big(\sum_{e\in E(v)} \mu_c(e)\widehat{Q}^J(v,e,3/4)|_\gamma \Big) \Big(\dots\, {}^K_d\, \dots \Big) \right]^\dagger \Big(\dots\, {}^L_e\, \dots \Big) \Big(\dots\, {}^M_f\, \dots \Big). \quad (5.19)$$

Quantization of F_3:

The quantization of F_3 is relatively simple again. First, we approximate

$$\begin{aligned} F_3(\pi) = \int_\sigma \frac{\pi^2(p)}{\sqrt{\det q}(p)} &\overset{(5.7)}{=} \sum_{v\in V(\gamma)} \int_{R_v} \pi^2(p) \frac{\det e}{\det q^{\frac{3}{4}}}(p) \frac{\det e}{\det q^{\frac{3}{4}}}(p) \\ &\overset{(5.17)(5.18)}{\approx} \sum_{v\in V(\gamma)} \pi_v^2 \frac{\nu(v)^{\frac{5}{2}} \det e(v)}{V_v^{\frac{3}{2}}} \frac{\nu(v)^{\frac{5}{2}} \det e(v)}{V_v^{\frac{3}{2}}} \\ &\overset{(5.15)(5.6)}{\approx} \left(\frac{3}{\kappa}\right)^6 \sum_{v\in V(\gamma)} \pi_v^2 \frac{1}{\nu(v)} \det\left[\sum_{e\in E(v)} \mu_a(e) Q_I(v,e_\triangleright,1/3) \right]^2 \end{aligned}$$

where we have introduced

$$\pi_v \doteq \int_{R_v} \pi.$$

Again we replace q by $\widehat{Q}$ and order the factors in such a way that the expression at a vertex becomes symmetric:

$$\widehat{F}_3(\pi) f_\gamma \doteq \left(\frac{3}{\kappa}\right)^6 \sum_{v\in V(\gamma)} \pi_v^2 \frac{1}{\nu(v)} \det\left[\sum_{e\in E(v)} \mu_a(e)\widehat{Q}_I(v,e_\triangleright,1/3) \right]^\dagger \det\Big[\dots\Big] f_\gamma$$

which can also be written in terms of an operator valued scalar function:

$$\widehat{F}_3(\pi) = \int \widehat{F}_3(p,q)\pi(p)\pi(q)\,dp\,dq$$

with

$$\widehat{F}_3(p,q)|_\gamma = \left(\frac{3}{\kappa}\right)^6 \sum_{v\in V(\gamma)} \chi_{R_v}(p)\chi_{R_v}(q) \frac{1}{\nu(v)} \det\left[\sum_{e\in E(v)} \mu_a(e)\widehat{Q}_I(v,e_\triangleright,1/3)|_\gamma \right]^\dagger \det\Big[\dots\Big]. \quad (5.20)$$

This completes the quantization of the gravity parts in the KG-Hamiltonian.

5.1.3. Maxwell Hamiltonian

Now we turn to the quantization of the quadratic forms F_4 and F_5 showing up in the Maxwell Hamiltonian. We will procede in the by now familiar fashion

Quantization of F_4:

Classically we have

$$E_e = \int_{S_e} *E \approx E^a(v)\mu_a(e), \qquad \text{hence} \qquad \sum_{e\in E(v)} \omega^a(e)E_e = \mu(v)E^a(v).$$

Thus we can regularize

$$\begin{aligned} F_4(E) &= \int \frac{q_{ab}(p)}{\sqrt{\det q}(p)} E^a(p)E^b(p)\, d^3p = \sum_v \int_{R_v} \frac{e^I_a e_{bI}(p)}{\sqrt{\det q}(p)} E^a(p)E^b(p)\, d^3p \\ &\approx \sum_v \nu(v)^2 \frac{e^I_a}{\sqrt{V_v}} \frac{e_{bI}(v)}{\sqrt{V_v}} E^a(v)E^b(v) \\ &\approx \frac{4}{\kappa^2} \sum_v \frac{1}{\mu(v)^2} \Big(\sum_{e\in E(v)} \omega^a(e)E_e \Big)\Big(\sum_{e'\in E(v)} \omega^b(e')E_{e'} \Big) \\ &\qquad \cdot \Big(\sum_{e''\in E(v)} \mu_a(e'')Q^I(e,v,1/2) \Big)\Big(\sum_{e'''\in E(v)} \mu_b(e'')Q_I(e,v,1/2) \Big) \end{aligned}$$

where E_e is the electric field integrated over the two surface dual to the edge e. Quantization is done in the by now familiar way:

$$\begin{aligned} \widehat{F}_4(E) \doteq \frac{4}{\kappa^2} \sum_v \frac{1}{\mu(v)^2} \Big(\sum_{e\in E(v)} \omega^a(e)E_e \Big)\Big(\sum_{e'\in E(v)} \omega^b(e')E_{e'} \Big) \\ \cdot \Big(\sum_{e''\in E(v)} \mu_a(e'')\widehat{Q}^I(e,v,1/2) \Big)^\dagger \Big(\sum_{e'''\in E(v)} \mu_b(e'')\widehat{Q}_I(e,v,1/2) \Big). \end{aligned}$$

Quantization of F_5:

For the regularization of this term we need some preparations: Let $\sigma(e,\alpha) = \pm 1$ whenever $\alpha = \ldots \circ e^{\pm 1} \circ \ldots$ Then the magnetic field integrated along a loop α is

$$B_\alpha \doteq \int_{S_\alpha} \mathrm{d}A = \int_\alpha A = \sum_{e\subset\alpha} \sigma(e,\alpha)A_e.$$

Now we need some definition.

Definition 5.1.2. By a *minimal loop* based at v we mean a loop α in γ which

- starts along an edge e of γ incident at v and ends along a different edge e' of γ incident at v,
- does not self-overlap,

- the number of edges used by α except e, e' cannot be reduced without breaking the loop into pieces,
- the tangents of the starting and ending edges e, e' are linearly independent at v.

Now we can approximate

$$B_\alpha = \int_{S_\alpha} B^a(p) n_a(p)\, d^2p \approx B^a(v)\rho_a(\alpha) \qquad \text{with} \qquad \rho_a(\alpha) \doteq \int_{S_\alpha} n_a(p)\, d^2p. \tag{5.21}$$

where here and in the following α denotes a minimal loop. Note that $\rho_a(\alpha)$ plays the role for S_α that $\mu_a(e)$ plays for S_e. Extending the analogy further, we define

$$\begin{aligned}
\rho(\alpha, \alpha', \alpha'') &\doteq \sum_{\beta,\beta',\beta'' \in \{\alpha,\alpha',\alpha''\}} \frac{1}{3!} \epsilon_{\beta\beta'\beta''} \epsilon_{IJK} \rho_I(\alpha)\rho_J(\alpha')\rho_K(\alpha''), \\
\rho(v) &\doteq \sum_{\alpha,\alpha',\alpha''} \rho(\alpha, \alpha', \alpha''), \\
\xi^I(\alpha) &\doteq \sum_{\alpha',\alpha''} \frac{1}{2} \epsilon_{\alpha\alpha'\alpha''} \epsilon^{IJK} \rho_J(\alpha')\rho_K(\alpha'').
\end{aligned}$$

Whence we get

$$\sum_{v=\partial\alpha} \rho_I(\alpha)\xi^J(\alpha) = \rho(v)\delta_I^J.$$

So we can solve (5.21) for $B^a(v)$ to get

$$B^a(v) \approx \frac{1}{\rho(v)} \sum_\alpha B_\alpha \xi^a(\alpha)$$

and proceed to the regularization of F_5:

$$\begin{aligned}
F_5(B) &= \int \frac{q_{ab}(p)}{\sqrt{\det q(p)}} B^a(p) B^b(p)\, d^3p \\
&\approx \frac{4}{\kappa^2} \sum_v \frac{1}{\rho(v)^2} \Big(\sum_\alpha \xi^a(\alpha) B_\alpha\Big)\Big(\sum_{\alpha'} \xi^b(\alpha') B_{\alpha'}\Big) \\
&\qquad \cdot \Big(\sum_{e\in E(v)} \mu_a(e) Q^I(e, v, 1/2)\Big)\Big(\sum_{e'\in E(v)} \mu_b(e') Q_I(e, v, 1/2)\Big).
\end{aligned}$$

Quantization is again done in the familiar fashion:

$$\begin{aligned}
\widehat{F}_5(B) \doteq \frac{4}{\kappa^2} \sum_v \frac{1}{\rho(v)^2} &\Big(\sum_\alpha \xi^a(\alpha) B_\alpha\Big)\Big(\sum_{\alpha'} \xi^b(\alpha') B_{\alpha'}\Big) \\
&\cdot \Big(\sum_{e\in E(v)} \mu_a(e) \widehat{Q}^I(e, v, 1/2)\Big)^\dagger \Big(\sum_{e'\in E(v)} \mu_b(e') \widehat{Q}_I(e, v, 1/2)\Big).
\end{aligned} \tag{5.22}$$

5.1.4. On the choice of the coordinate systems

In the quantization of the gravitational parts of the matter Hamiltonians presented in the last section, coordinate charts in regions around each vertex of a graph chosen at will entered the definitions in the form of coefficients $\mu(v), \nu(v), \ldots$. This introduces a lot of arbitrariness in the definition of the operators. Moreover, the operators thus defined will in general not transform covariantly under diffeomorphisms of Σ. In the case of the volume operator, one would for example require

$$U(\sigma)\widehat{V}_R f_\gamma \stackrel{!}{=} \widehat{V}_{\sigma(R)} U(\sigma)[f_\gamma] \equiv \widehat{V}_{\sigma(R)} f(h_{\sigma(e_1)}, h_{\sigma(e_2)}, \ldots)$$

where σ is an arbitrary diffeomorphism of Σ and $f_\gamma = f(h_{e_1}, h_{e_2}, \ldots)$. The equation above is certainly satisfied for the standard volume quantization (5.8),(5.9), but it is in general not valid for the quantization (5.10),(5.11) of the volume adapted to the dual tetrahedral decomposition, since the chart chosen around a vertex v would in general have nothing to do with the chart chosen around $\sigma(v)$ and therefore the coefficients μ and ν in (5.10) would change.
Similar definitions for the gauge covariance of $\widehat{F}_1, \ldots, \widehat{F}_5$ can be given and the same problem occurs.

In the following we will present a way to fix some of the arbitrariness in the choice of the coordinate charts in such a way that diffeomorphism covariance is restored:
Let us start by introducing the following notation: Let G be the set of all graphs γ (+ their dual polyhedral decomposition, system of paths,...) such that

- there is at most one vertex v of γ which has a valence higher than one,
- if such a vertex exists, all edges of γ start or end in v,
- if there are just one valent vertices, γ just contains a single edge.

Furthermore, we denote by $[\gamma]$ the equivalence class of an element γ (+ dual polyhedral decomposition, system of paths,...) of G.
Now we pick

- a chart χ onto some region R of Σ,
- from each equivalence class $[\gamma]$ in G a representative γ_R lying entirely within R,
- for each element γ' of $[\gamma]$ a diffeomorphism $\sigma_{\gamma,\gamma'}$ such that $\sigma_{\gamma,\gamma'}(\gamma_R) = \gamma'$ (and similar equations for the dual polyhedral decomposition etc.).

Let v be a vertex of some graph γ_0 and denote by γ_0' the graph obtained from γ_0 by deleting all edges which do not intersect v. The chart χ_v around v showing up in the constructions of the previous sections should now be chosen as

$$\chi_v = \sigma_{\gamma_0',\gamma_0'} \circ \chi.$$

5.2. Representation of the matter fields

This section deals with the quantization of the matter ingredients in the Hamiltonians (5.2). We will aim at a theory in which the Hamiltonians unitarily generate the dynamics in much the same way as in ordinary QFT. In the following we will concentrate on the scalar field, because of its simplicity, and make some comments on the problems occuring for the electromagnetic field at the end of this section.

Let us start by noting that the quantities $\phi_v \doteq \phi(v)$ and π_v appearing in the regularization of the Klein Gordon Hamiltonian have canonical Poisson brackets

$$\{\phi_v, \phi_{v'}\} = \{\pi_v, \pi_{v'}\} = 0, \qquad \{\pi_v, \phi_{v'}\} = Q_{\mathrm{KG}}\delta_{vv'}. \tag{5.23}$$

It is therefore natural to attempt a canonical quantization of these quantities.

Let us recall how this is done in the usual canonical quantization procedure for the KG field. We need some notation for this purpose: Let us denote by $\mathcal{F}_S[\mathcal{H}]$ the symmetric Fock space over some Hilbert space $\mathcal{H}$. The annihilation resp. creation operators on $\mathcal{F}_S[\mathcal{H}]$ will be written as $\widehat{a}(f)$,$\widehat{a}^\dagger(f)$, respectively, and we denote the second quantization map, turning operators on $\mathcal{H}$ into operators on $\mathcal{F}_S[\mathcal{H}]$ by Γ. Finally we will distinguish the operators on $\mathcal{H}$ from that on $\mathcal{F}_S[\mathcal{H}]$ by typesetting the former in boldface and letting the latter carry a "$\widehat{\ }$".
Now we can start: We assume that ϕ and π are canonically conjugate fields on Σ,

$$\{\phi(p), \phi(q)\} = \{\pi(p), \pi(q)\} = 0, \qquad \{\phi(p), \pi(q)\} = \delta(p, q), \tag{5.24}$$

with respect to the Hamiltonian:

$$H = \frac{1}{2}\left(\langle \pi, \mathbf{O}_1\pi\rangle_1 + \langle \phi, \mathbf{O}_2\phi\rangle_1\right).$$

Here, $\langle\,\cdot\,,\,\cdot\,\rangle_1$ is a scalar product on functions on Σ which defines a Hilbert space $\mathcal{H}_1$, usually called *one particle Hilbert space*. $\mathbf{O}_1, \mathbf{O}_2$ are operators on this Hilbert space which are assumed to be selfadjoint on a certain domain.
The key to the quantization of this system is to find a decomposition of the Hamiltonian H into complex conjugate functions $z(\phi,\pi), \overline{z}(\phi,\pi)$ linear in ϕ, π

$$H = \langle \overline{z}, \mathbf{h}z\rangle_1, \tag{5.25}$$

where $\mathbf{h}$ is a suitable operator, such that $z, \overline{z}$ fulfill the Poisson algebra

$$\{z(p), z(p')\} = \{z(p), \overline{z}(p')\} = 0, \qquad \{z(p), \overline{z}(p')\} = i\delta(p, p'). \tag{5.26}$$

The operator $\mathbf{h}$ is also called the *one particle Hamiltonian.*
Once such functions $z(p), \overline{z}(p)$ are found, one can quantize them as the creation and annihilation operators $\widehat{a}(p)$, $\widehat{a}^\dagger(p)$ on the symmetric Fock space $\mathcal{H} = \mathcal{F}_S[\mathcal{H}_1]$ over the one particle space. This in turn yields a quantization of ϕ and π by solving $z(p), \overline{z}(p)$ for ϕ,π. The quantization $\widehat{H}$ of H is obtained as the second quantization of $\mathbf{h}$.

We disregard the functional analytic niceties (domains, selfadjointness, ...) concerning functions of the operators $\mathbf{O}_1, \mathbf{O}_1$ in stating the following proposition, well known from QFT in curved spacetimes:

Proposition 5.2.1. *The choice*

$$\mathbf{h} \doteq \sqrt{\sqrt{\mathbf{O}_1}\mathbf{O}_2\sqrt{\mathbf{O}_1}}, \qquad z \doteq \frac{1}{\sqrt{2}}\left(\sqrt{\mathbf{h}}\frac{1}{\sqrt{\mathbf{O}_1}}\pi - i\frac{1}{\sqrt{\mathbf{h}}}\sqrt{\mathbf{O}_1}\phi\right),$$

satisfies the equations (5.25), (5.26). *As a consequence, the quantization*

$$\widehat{\phi}=\frac{i}{\sqrt{2}}\frac{1}{\sqrt{\mathbf{O}_1}}\sqrt{\mathbf{h}}(\widehat{a}-\widehat{a}^\dagger),\qquad \widehat{\pi}=\frac{1}{\sqrt{2}}\sqrt{\mathbf{O}_1}\frac{1}{\sqrt{\mathbf{h}}}(\widehat{a}+\widehat{a}^\dagger) \tag{5.27}$$

implement the Poisson relations (5.24), *the second quantization* $\widehat{H}\doteq\Gamma(\mathbf{h})$ *of* $\mathbf{h}$ *is a positive operator, the time dependent field*

$$\widehat{\phi}(t,f)\doteq e^{it\widehat{H}}\widehat{\phi}(f)e^{-it\widehat{H}}$$

fulfills the field equations

$$\ddot{\widehat{\phi}}(t,f)=\widehat{\phi}(t,\mathbf{h}^2 f).$$

As long as domain questions etc. are ignored, the proof of the proposition consists of a straightforward and well known computation and is therefore not reproduced here.

There are two obstacles which hinder the implementation of the quantization sketched above for the scalar field coupled to quantized gravity in a straightforward way.

1. As we can not presuppose any background metric, it is unclear how we should obtain a useful measure to turn some set of functions on Σ into the one particle Hilbert space $\mathcal{H}_1$. Furthermore, even if one would define a measure using some fiducial metric, the quadratic forms $\widehat{F}_1$,$\widehat{F}_2$,$\widehat{F}_3$ are not likely to define operators since they are not very regular.
2. The quadratic forms $\widehat{F}_1, \widehat{F}_2, \widehat{F}_3$ of which the Hamiltonian consist in our case are *operator valued*. Thus even if the aforementioned problem could be solved it is not clear on which space the operators $\mathbf{F}_1$,$\mathbf{F}_2$,$\mathbf{F}_3$ should be expected to act.

Both problems do not come unexpected in a background independent setting: The first highlights the fact that the $\phi(f)$ usually used as basic variable, are not a good choice in the present setting because ϕ can not be integrated against a test function without recurse to a background metric. The second has to do with the fact that the definition of the ground state of a quantum field makes heavy use of the background metric. Upon quantizing gravity, it will therefore become something more complicated then just a state in the matter Hilbert space.

We would like, however, to give some idea of what one is able to do within the given state of affairs: When one restricts consideration to a single given graph, a quantization of the matter fields can be achieved at least formally, as follows.
Given a graph γ we can define $\mathcal{H}_1$ to be the space of functions $f:V(\gamma)\longrightarrow\mathbb{C}$ with the scalar product

$$\langle f,f'\rangle\doteq\sum_{v\in V(\gamma)}\overline{f(v)}f'(v).$$

This is a way to overcome the first difficulty: As measure we use the counting measure, which is defined independently of any metric information.

It is not hard to see from the expressions obtained in the last section that the quadratic forms $\widehat{F}_i$ can also be expressed via operator valued functions on $V(\gamma)\times V(\gamma)$:

$$\widehat{F}_i(f)=\sum_{v,v'\in V(\gamma)}\widehat{F}_i(v,v')f(v)f(v').$$

But even more can be said:

Lemma 5.2.2. *The quadratic forms $\widehat{F}_1$,$\widehat{F}_2$,$\widehat{F}_3$ define symmetric operators $\mathbf{F}_1$,$\mathbf{F}_2$,$\mathbf{F}_3$ on $\mathcal{H} \doteq \mathcal{H}_1 \otimes \mathcal{H}_{AL}$ via*

$$\mathbf{F}_i[f \otimes c_\gamma] \doteq \sum_{v,v' \in V(\gamma)} \delta(\cdot, v) f(v') \otimes \widehat{F}_i(v, v') c_\gamma.$$

Proof. It is straightforward to check that the $\mathbf{F}_i$ are well defined linear maps with this definition. To prove symmetry, we calculate

$$\begin{aligned}\langle f_1 \otimes c_1, \mathbf{F}_i\left[f_2 \otimes c_2\right]\rangle_{\mathcal{H}} &= \sum_{v,v'} \overline{f_1(v)} f_2(v') \langle c_1, \widehat{F}_i(v,v') c_2\rangle_{\mathcal{H}_{\mathrm{AL}}},\\ \langle \mathbf{F}_i\left[f_1 \otimes c_1\right], f_2 \otimes c_2\rangle_{\mathcal{H}} &= \sum_{v,v'} \overline{f_1(v')} f_2(v) \langle \widehat{F}_i(v,v') c_1, c_2\rangle_{\mathcal{H}_{\mathrm{AL}}}.\end{aligned}$$

So what is left to show is that $F_i(v,v') = [F_i(v',v)]^\dagger$. $\widehat{F}_1(v,v')$ and $\widehat{F}_3(v,v')$ are nonzero only on the diagonal, anyway, and certainly symmetric operators there. Checking $F_2(v,v') = [F_2(v',v)]^\dagger$ consists in a short, straightforward computation. □

Using this lemma, a quantization analogous to that given in proposition 5.2.1 is possible:

Proposition 5.2.3. *Let*

$$\mathbf{O}_1 \doteq \mathbf{F}_3, \qquad \mathbf{O}_2 \doteq K^2 \mathbf{F}_1 + \mathbf{F}_2, \qquad \mathbf{h} \doteq \sqrt{\sqrt{\mathbf{O}_1}\mathbf{O}_2\sqrt{\mathbf{O}_1}}$$

and denote by $\widehat{a}(\,\cdot\,)$,$\widehat{a}^\dagger(\,\cdot\,)$ the annihilation and creation operators on the Fock space $\mathcal{F}_S\left[\mathcal{H}_{AL} \otimes \mathcal{H}_1\right]$. Then the field operators

$$\widehat{\phi} \doteq \frac{i}{\sqrt{2}} \frac{1}{\sqrt{\mathbf{O}_1}} \sqrt{\mathbf{h}}(\widehat{a} - \widehat{a}^\dagger), \qquad \widehat{\pi} \doteq \frac{1}{\sqrt{2}} \sqrt{\mathbf{O}_1} \frac{1}{\sqrt{\mathbf{h}}}(\widehat{a} + \widehat{a}^\dagger) \tag{5.28}$$

fulfill the commutation relations

$$\begin{aligned}\left[\widehat{\phi}(c_\gamma \otimes f), \widehat{\phi}(c'_\gamma \otimes f')\right] &= \left[\widehat{\pi}(c_\gamma \otimes f), \widehat{\pi}(c'_\gamma \otimes f')\right] = 0,\\ \left[\widehat{\pi}(c_\gamma \otimes f), \widehat{\phi}(c'_\gamma \otimes f')\right] &= i\hbar \langle c_\gamma, c'_\gamma\rangle_{\mathcal{H}_{AL}} \langle f, f'\rangle_{\mathcal{H}_1}\end{aligned}$$

for c_γ, c'_γ cylindrical functions on γ and $f, f' \in \mathcal{H}_1$. The second quantization $\widehat{H} \doteq \Gamma(\mathbf{h})$ of $\mathbf{h}$ is a positive operator, and the time dependent field

$$\widehat{\phi}(t,f) \doteq e^{it\widehat{H}} \widehat{\phi}(f) e^{-it\widehat{H}}$$

fulfills the field equations

$$\ddot{\widehat{\phi}}(t,f) = \widehat{\phi}(t, \mathbf{h}^2 f).$$

Again, the proof is straightforward as long as domain problems etc. are not taken into account.

It is at first surprising that the KG field is represented on the Hilbert space $\mathcal{F}_S\left[\mathcal{H}_{\mathrm{AL}} \otimes \mathcal{H}_1\right]$ and not on $\mathcal{F}_S\left[\mathcal{H}_1\right]$. It is however unavoidable that the quantization of the gravitational field "mixes" in one way or another with the quantum theory of the matter fields, as already on classical backgrounds the geometry enters in the definition of commutation relations as well as the ground state in a decisive way. Also it might serve to reassure the reader that a representation of the KG-field on $\mathcal{F}_S[\mathcal{H}_1]$ can be derived from the one given above as we will show in the next section.

Let us finish with some remarks concerning the quantization of the electromagnetic field: For the "dynamical quantization" of the Maxwell field one would proceed analogously to what we have sketched for the KG-field, above: One would replace the electromagnetic holonomies appearing in (5.22) via

$$h_e = \exp i \int_e A \approx 1 + iA_a(e(0))(e(1)-e(0))^a$$

and the smeared electric fields via

$$\sum_{e\in E(v)} \omega_a(e) E^a(v) \approx \int_{R_v} E^a \doteq E^a_v$$

whence one has again introduced matter variables E_e, A_e which (upon choosing Coulomb gauge) have canonical Poisson brackets:

$$\{A_a(v), A_b(v)\} = \left\{E^a_v, E^b_{v'}\right\} = 0, \qquad \{E^a_v, A_b(v')\} = Q_{\rm EM}\delta^a_b\delta_{vv'}. \tag{5.29}$$

However, it is well known that transforming the analogue of (5.29) for the fields $A(x), E(x)$ at a point into a relation for operators by replacing Poisson brackets with commutators does not lead to a well defined theory as it is not consistent with the gauge condition. Instead the delta function has to be replaced by what is called the *transverse delta function*. However, this transverse delta function makes use of the flat background metric, so it would have to be replaced with a quantized version of it. Even worse, since we are not dealing with fields in the continuum anymore, we would have to come up with a "quantum lattice transverse delta function". Trying to do this would lead to a theory on such a level of formality we do not dare to write it down.
So, to achieve a "dynamical quantization" for the electromagnetic field, one would either have to develop a background independent gauge fixing and implement it already on the classical level, or to somehow manage to write down a well defined background independent quantum theory in which the gauge fixing can then be implemented. This is however beyond the scope of the present thesis.

5.3. The "QFT on curved space-time limit"

In this section we demonstrate how a representation of the scalar on the space $\mathcal{F}_S[\mathcal{H}_1]$ can be recovered from the one given above as soon as one fixes a state in the gravitational Hilbert space. We call this representation the *QFT on curved space-time limit* because this is precisely what it represents from a physical viewpoint if the state chosen for the gravitational field is a semiclassical state representing some background geometry. We do however remind the reader that the one particle Hilbert space $\mathcal{H}_1$ is based on a *discrete* set and therefore the QFT on curved space-time limit will behave as an ordinary QFT on curved space-time only in the low energy regime.

Let $\Psi \in \mathrm{Cyl}_\gamma \subset \mathcal{H}_{\rm AL}$ with $\|\Psi\|_{\rm AL} = 1$ be given. Then one can define maps

$$F^{(n)}_\Psi : \bigotimes\nolimits^n_S \mathcal{H}_1 \longrightarrow \bigotimes\nolimits^n_S \mathcal{H}_\gamma \otimes \mathcal{H}_1, \qquad P^{(n)}_\Psi : \bigotimes\nolimits^n_S \mathcal{H}_\gamma \otimes \mathcal{H}_1 \longrightarrow \bigotimes\nolimits^n_S \mathcal{H}_1$$

via

$$F_\Psi^{(n)}\left[\sum_i f_{1i}\otimes f_{2i}\otimes\ldots\otimes f_{ni}\right]=\sum_i \Psi\otimes f_{1i}\otimes\Psi\otimes f_{2i}\otimes\ldots\otimes\Psi\otimes f_{ni},$$
$$P_\Psi^{(n)}\left[\sum_i c_{1i}\otimes f_{1i}\otimes c_{2i}\otimes f_{2i}\otimes\ldots\otimes c_{ni}\otimes f_{ni}\right]=\sum_i\langle\Psi,c_{1i}\rangle_\gamma\ldots\langle\Psi,c_{ni}\rangle_\gamma f_{1i}\otimes\ldots\otimes f_{ni}$$

and in turn

$$F_\Psi:\mathcal{F}_S\left[\mathcal{H}_1\right]\longrightarrow\mathcal{F}_S\left[\mathcal{H}_\gamma\otimes\mathcal{H}_1\right],\qquad P_\Psi:\mathcal{F}_S\left[\mathcal{H}_\gamma\otimes\mathcal{H}_1\right]\longrightarrow\mathcal{F}_S\left[\mathcal{H}_1\right]$$

by

$$F_\Psi\left[c\oplus f^{(1)}\oplus f^{(2)}\oplus\ldots\right]=c\oplus F_\Psi^{(1)}(f^{(1)})\oplus F_\Psi^{(2)}(f^{(2)})\oplus\ldots,$$
$$P_\Psi\left[c\oplus v^{(1)}\oplus v^{(2)}\oplus\ldots\right]=c\oplus P_\Psi^{(1)}(v^{(1)})\oplus P_\Psi^{(2)}(v^{(2)})\oplus\ldots.$$

Note that since Ψ has unit norm, $P_\Psi F_\Psi=1$. A more standard quantization of the KG field can now be obtained:

$$\widehat{\phi}_\Psi,\widehat{\pi}_\Psi:\mathcal{H}_1\longrightarrow\mathcal{L}\left(\mathcal{F}_S\left[\mathcal{H}_1\right]\right),\qquad\qquad \widehat{H}_\Psi\in\mathcal{L}\left(\mathcal{F}_S\left[\mathcal{H}_1\right]\right),$$

$$\widehat{\phi}_\Psi[f]\doteq P_\Psi\widehat{\phi}[\Psi\otimes f]F_\Psi,\qquad \widehat{\pi}_\Psi[f]\doteq P_\Psi\widehat{\pi}[\Psi\otimes f]F_\Psi,\qquad \widehat{H}_\Psi\doteq P_\Psi\widehat{H}F_\Psi.$$

A tedious but straightforward computation reveals that the representation $\widehat{\phi}_\Psi,\widehat{\pi}_\Psi,\widehat{H}_\Psi$ can also be obtained by using the operators $\langle\mathbf{F}_3\rangle_\Psi$,$\langle K^2\mathbf{F}_1+\mathbf{F}_2\rangle_\Psi$ on $\mathcal{H}_1$ obtained as *partial expectation values* wrt. Ψ as the operators $\mathbf{O}_1,\mathbf{O}_2$ of lemma 6.1.1 and doing the associated second quantization.

Let us finally remark that a less fundamental but more easily feasible way to arrive at a QFT on curved spacetime limit is to *first* compute the gravitational expectation values and *then* form the operators $\mathbf{O}_1,\mathbf{O}_2$. More precisely, one could start from a completely classical Hamiltonian

$$H_{\mathrm{KG}}^{(\Psi)}\doteq\frac{1}{2}\sum_{v,v'}\left(\langle\widehat{F}_3(v,v')\rangle_\Psi\pi_v\pi_{v'}+\langle\widehat{F}_2(v,v')\rangle_\Psi\phi_v\phi_{v'}+K^2\langle\widehat{F}_1(v,v')\rangle_\Psi\phi_v\phi_{v'}.\right)\tag{5.30}$$

and then construct a QFT for the fields ϕ,π via proposition 5.2.1.

6. Towards dispersion relations

Dispersion relations are the relations between the frequency ω and the wave vector $\vec{k}$ of waves of a field of some sort, traveling in vacuum or through some medium. In quantum mechanical systems, the dispersion relation is the the relation between the momentum and the energy of particles. The form of the dispersion relations appearing in fundamental physics is dictated by Lorenz invariance. Since this invariance is likely to be broken in quantum gravity, modification of dispersion relations is conjectured to be an observable effect of quantum gravity. In this chapter we would like to explain why loop quantum gravity would indeed lead to modified dispersion relations, and how one can proceed in a calculation of these modifications.

There are at least two mechanisms by which modified dispersion relations arise in the context of loop quantum gravity, and it is important to keep them apart. Let us start to discuss the first one by considering an analogous effect in another branch of physics:
A prime example coming to mind when thinking about modified dispersion relations is the propagation of light in materials. The mechanism which causes these modification is roughly as follows: The electromagnetic field of the in-falling wave acts on the charges in the material, they are accelerated and in turn create electromagnetic fields. These fields interfere with the in-falling ones, the net effect of this is a wave with modified phase and therefore, a phase velocity differing from the one in vacuum. The precise relation between the force acting on the charges and the fields induced by them depends on the properties of the material and also on the frequency of the wave, and thus gives rise to a frequency dependent phase velocity and, hence, a nontrivial dispersion relation. Under some simplifying assumptions, this relation looks as follows:

$$\omega(|k|) = |k| \left(1 + \frac{1}{\omega_0^2 - \omega^2(|k|) + i\rho\omega(|k|)}\right)$$

where ω_0 and ρ are properties of the material. As is to be expected, if the energy of the in-falling wave is very low compared with the binding energies ($\sim \omega_0$) of the charges, the frequency dependence of the phase velocity will also be very small.
In loop quantum gravity, modified dispersion relations can be expected from the interplay between matter and quantum gravity by an analogous mechanism: The propagating matter wave causes changes in the local geometry, which in turn affect the propagation of the wave. Again, if the energy of the wave is very small, so will be the modification of the dispersion relation as compared to the standard one.
The calculation of the actual form and magnitude of this effect within loop quantum gravity requires a detailed understanding of the dynamics of the coupled matter gravity system, an understanding that we are still lacking. As we have pointed out in the introduction, the successful quantization of the relevant Hamiltonian constraints [16, 17] is only a first, albeit very important step. Identification and analysis of solutions would have to follow. Therefore this effect can up to now not be analyzed at a sufficient level of confidence, and the present work will be no exception in that respect.

There is, however, a second source of modifications to the dispersion relations: The inherent discreteness of geometry found in loop quantum gravity. This effect has nothing to do with back-reaction

of the geometry on the matter and it is the contribution of this effect to the dispersion relations that we will consider in the rest of this chapter.
Let us again start by briefly reviewing an analogous phenomenon from a different branch of physics, the propagation of lattice vibrations (sound) in crystals. We consider an extremely simple model, a one dimensional chain of atoms. We assume that all atoms have the same mass m and that each of them acts on its two neighbors with an attractive force proportional to the mutual distance. If we denote by ϵ the interatomic distance in the equilibrium situation, by $q(z)$ the displacement of atom z from its equilibrium position ϵz and set $p(z) = \dot{q}(z)$, the Hamiltonian for the system reads

$$H = \frac{1}{2}\sum_{z\in\mathbb{Z}} \frac{1}{m}p^2(z) + K\left(q(z+1) - q(z)\right)^2 . \tag{6.1}$$

The corresponding equations of motion are simple, a complete set of solutions is given by

$$q(t,z) = \exp i\left(\epsilon z k - \omega(k)t\right), \qquad \text{with} \qquad \omega^2(k) = \frac{2K}{m}\left(1 - \cos k\epsilon\right). \tag{6.2}$$

As the solutions are straightforward analogues of plane waves in the continuum, $\omega^2(k)$ is readily interpreted as the dispersion relation for the system. We see that it contains the "linear" term proportional to k^2 expected for sound waves in the continuum, as well as higher order corrections due to the discreteness of the lattice.

Let us reconsider (6.1): The fact that the $q(z)$ are displacements of atoms is not explicitly visible. H could as well be the Hamiltonian of a field q with a certain form of potential, propagating on a regular lattice! Having made that observation, we are already very close to the models developed in the preceeding chapter. Consider for example the scalar field: Upon choosing a semiclassical state Ψ_{flat} for the gravitational field, describing flat Euclidean space, the Hamiltonian (5.30) becomes

$$H_{\text{KG}}^{(\Psi_{\text{flat}})} \doteq \frac{1}{2}\sum_{v,v'}\left(\langle\widehat{F}_3(v,v')\rangle_{\Psi_{\text{flat}}}\pi_v\pi_{v'} + \langle\widehat{F}_2(v,v')\rangle_{\Psi_{\text{flat}}}\phi_v\phi_{v'} + K^2\langle\widehat{F}_1(v,v')\rangle_{\Psi_{\text{flat}}}\phi_v\phi_{v'}.\right). \tag{6.3}$$

There are however two things to say about the analogy between (6.3) and (6.1):

Firstly, on a fundamental level we should regard the scalar field as a quantum field. Therefore the proper way towards a dispersion relation would not be to derive equations of motion from (6.3) and analyze their solutions, but instead to consider the spectrum of the one particle Hamiltonian constructed from $\langle F_1\rangle, \langle F_2\rangle, \langle F_3\rangle$. If even more ambitious, one would consider the spectrum of the operator $\langle\widehat{h}^2\rangle_{\Psi_{\text{flat}}}$, the partial expectation value of the square of the *fully quantized* one particle Hamiltonian $\widehat{h}$ of section 5.2.
We maintain, however that at least as long as one considers only processes of very low energy, it should not matter much whether one treats the matter fields as being quantum or as classical fields and therfore, in a first approximation, it is reasonable to analyze the equations of motion deriving from (6.3).

The second thing we want to discuss is that an important difference between (6.1) and (6.3) lies in the following: In (6.1), the coefficients of the fields do not depend on the vertex. This is the reason why one can explicitly calculate solutions to the equations of motion. In contrast to that, $\langle F_1\rangle, \langle F_2\rangle, \langle F_3\rangle$ will in general depend on v, v', even if the state Ψ_{flat} employed to compute the gravity expectation values is a good semiclassical state. As a result, the field equations will be complicated and, most important for us, not have "plane wave" solutions

$$q_{\vec{k}}(t,v) = \exp i(\vec{k}\vec{x}(v) - \omega t) \tag{6.4}$$

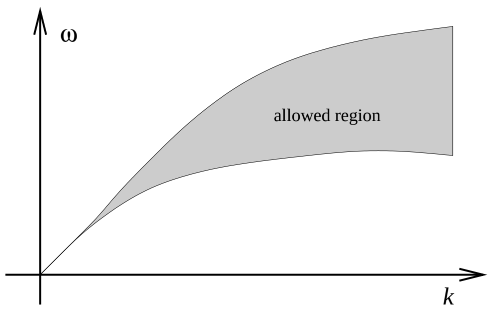

Figure 6.1.: Fourier transform of EOM. The support of solutions has to lie in the shaded region.

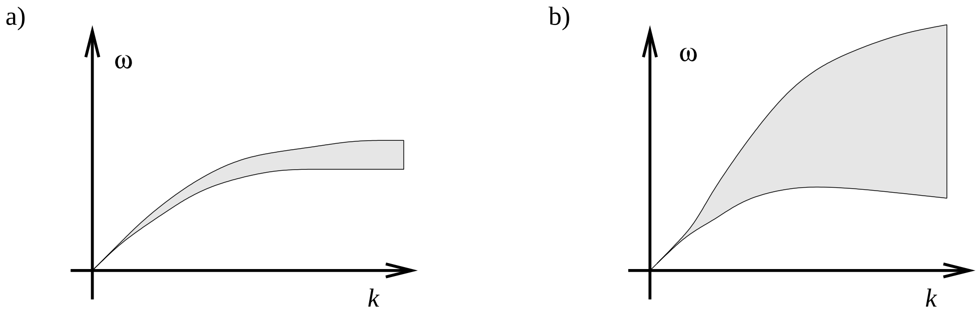

Figure 6.2.: Can higher order corrections to the dispersion relation be given? a) Yes, approximately. b) No, there is no meaningful notion of dispersion relation beyond linear order.

any more. Hence if we would Fourier decompose solutions of the field equations with respect to (6.4), the support of the resulting functions will not be confined by a dispersion relation to some line in the ω-$|k|$ plane, anymore.

However, for a good semiclassical state, symmetry, which is absent due to the vertex dependence of the coefficients, will be approximately restored on a large length scale. For example, if the vertex dependent coefficients would be averaged over large enough regions of Σ the average would be independent of the specific choice of the region. Therefore, for long wavelength, plane waves (6.4) should at least be approximate solutions to the field equations. The following scenario is conceivable: Although there is no exact dispersion relation, the support of the Fourier transform of a solution might be confined to some region in the ω-$|k|$ plane, or the Fourier transform has at least to be peaked there. This region should get more and more narrow for longer wavelength, leading to an ordinary dispersion relation in the limit (see figure 6.1). We have to note, however that even if this is true, there is no guarantee that a dispersion relation with corrections to the linear term makes sense as an approximate description for long wavelength. We tried to visualize this in figure 6.2. So, to conclude, it is very plausible that a nonlinear dispersion relation will turn out to be a good approximate description of the physical contents of (6.3) for long wavelength in this sense. But issues such as the one depicted in figure 6.2 definitely merit further studies.

In view of both, difficulties and prospects of the determination of dispersion relations from the LQG based models of chapter 5, we will proceed in two different directions for the rest of the chapter: In the next section, we consider a simple generalization of the system of springs and masses characterized above by (6.1).

In section 6.2 we will come back to the LQG models and turn to the practical question of how a nonlinear dispersion relation can actually be computed from (6.3).

6.1. A toy model

To get a feeling for the problems involved in treating the propagation of waves on random lattices, in this section we will consider a simple model.
We start with a one dimensional version of the Hamiltonian (6.3), assume $\langle F_3\rangle(v,v')=\delta_{v,v'}$, $K^2=0$, and $\mathbf{F}_2$ to contain only nearest neighbor interactions. Such a Hamiltonian can be written in the form

$$H(\phi,\pi)=\frac{1}{2}\sum_{z\in\mathbb{Z}}\left[\pi_z^2+g_z\left(\partial^+\phi)_z^2\right)\right]. \tag{6.5}$$

We mention two interpretations of (6.5):

1. ϕ is the KG field on a random lattice with standard lattice Hamiltonian. Then g can be interpreted as the metric, and for flat space we would have

$$g_z=[x(z+1)-x(z)]^{-2}\doteq l_z^{-2} \tag{6.6}$$

where $x(z)$ is the position of the lattice point z.

2. Consider a chain of unit masses at positions $\ldots,x_{z-1},x_z,x_{z+1},\ldots$ where neighboring masses are coupled to each other via springs with (varying) spring constants g_z. Then (6.5) is the Hamiltonian of that system.

In the following we will adopt the first interpretation, all of the formulae can however be rewritten easily in terms of the the second one.

If all the l_i are equal (to l, say), it is easy to solve the equations of motion of (6.5). The solutions are "plane waves"

$$\phi_z(t,k)=e^{i(klz-\omega(k)t)},\qquad \omega^2(k)=\frac{2}{l^2}\left(1-\cos(kl)\right)=k^2-\frac{l^2}{12}k^4+O(k^6). \tag{6.7}$$

For the general case on the other hand, with generically all l_i different, it is not possible to explicitely write down any solution to the EOM.
The analysis we are aiming at in the present section lies somewhere in-between these two extreme cases: We will make an assumption on the l_i under which we are able to treat the system analytically and try to remove it at the end of the analysis: Let us assume that the system is periodic with $N\in\mathbb{N}$ the length of period. More precisely we assume that $g_{z+N}=g_z$ for all $z\in\mathbb{Z}$. We introduce the notation $\phi_n^{(z)}\doteq\phi_{n+zN}$ with $n\in\{0,1,\ldots,N-1\}$ and make the ansatz

$$\phi_n^{(z)}(t)=c_n\exp i(zk-\omega t). \tag{6.8}$$

This ansatz turns the equations of motion induced by (6.5) into an eigenvalue problem for $\underline{c}$ and ω: (6.8) is a solution iff

$$\underline{\underline{M}}\,\underline{c}=\omega^2\underline{c}\text{ where }\underline{\underline{M}}=\begin{pmatrix} g_{N-1}+g_0 & -g_0 & 0 & \ldots & 0 & -g_{N-1}e^{ik}\\ -g_0 & g_0+g_1 & -g_1 & 0 & \ldots & 0\\ 0 & -g_1 & g_1+g_2 & -g_2 & 0 & \ldots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ -g_{N-1}e^{-ik} & 0 & \ldots & 0 & -g_{N-2} & g_{N-2}+g_{N-1}\end{pmatrix}. \tag{6.9}$$

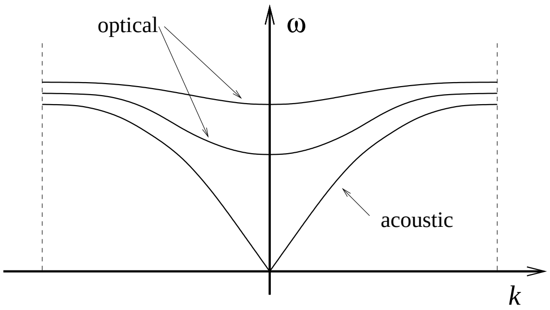

Figure 6.3.: Optical and acoustic branches in the dispersion relation

The eigenvalues $\omega_0 \ldots \omega_{N-1}$ represent the different branches of the dispersion relation. One can show on very general grounds that there is at least one branch, denoted ω_{ac} in the following, with $\omega_{\mathrm{ac}}(k) \to 0$ for $k \to 0$. Following the custom of condensed matter physics, we call this branch *acoustic* in contrast to the *optical* branches nonzero at $k = 0$. The situation is sketched in figure 6.3. As we are interested in the low energy (i.e. small ω) behavior of the field, the acoustic branch is the relevant one for our purpose and we will compute its small k behavior in the following.
Let us start by making the ansatz

$$\omega_{\mathrm{ac}}^2(k) = w[1]\,|k| + w[2]k^2 + w[3]\,|k|^3 + \ldots, \tag{6.10}$$

explicitly forcing $\omega_{\mathrm{ac}}(0)$ to be zero. Accordingly, we expand $\det(\underline{\underline{M}} - \omega^2 1)$:

$$\det(\underline{\underline{M}} - \omega^2 1) = \sum_{i=0}^{N-1} \omega^{2i} \sum_{j=0}^{\infty} w[i,j]\,|k|^j\,. \tag{6.11}$$

By plugging (6.10) and (6.11) into the equation $\det(\underline{\underline{M}} - \omega^2 1) = 0$ one easily obtains the following

Lemma 6.1.1. *For* (6.10) *to yield a solution to* $\det(\underline{\underline{M}} - \omega^2 1) = 0$*,* $w[0,0]$ *has to be zero. Furthermore the lowest coefficients in* (6.10) *have to be*

$$\begin{aligned}
w[1] &= -\frac{w[0,1]}{w[1,0]},\\
w[2] &= -\frac{1}{w[1,0]}(w[1]w[1,1] + w[1]^2 w[2,0] + w[0,2]),\\
w[3] &= -\frac{1}{w[1,0]}(w[2]w[1,1] + w[1]w[1,2] + w[1]^2 w[2,1] + w[3,0]),\\
w[4] &= -\frac{1}{w[1,0]}(w[3]w[1,1] + w[2]w[1,2] + w[1]w[1,3] + w[2]^2 w[2,0] + w[1]^2 w[2,2] + w[0,4]).
\end{aligned}$$

To see how these coefficients depend on the metric g, we have to actually compute the series expansion (6.11). Remarkably, at least the lowest order coefficients can be explicitely written down. We present the results in the following

Proposition 6.1.2. *For $\underline{\underline{M}}$ of the form* (6.9),

$$\begin{aligned}\det(\underline{\underline{M}} - \omega^2 1) = &- 2g_0 \dots g_{N-1}(1-\cos k)\\ &+ \omega^2 N g_0 \dots g_{N-1} \sum_{i=0}^{N-1} g_i^{-1}\\ &+ \omega^4 g_0 \dots g_{N-1} \sum_{0\le i<j\le N-1} (j-i)[N-(j-i)] g_i^{-1} g_j^{-1}\\ &+ O(\omega^6).\end{aligned}$$

Proof. It is an elementary combinatorial fact that

$$\det(\underline{\underline{M}} - \omega^2 1) = \sum_{i=0}^{N-1} (-1)^i \omega^{2i} \begin{pmatrix} \text{sum over all } (N-i)\times(N-i) \\ \text{principal sub determinants of } \underline{\underline{M}} \end{pmatrix}. \tag{6.12}$$

Therefore the proof of the theorem reduces to the calculation of numerous sub-determinants of $\underline{\underline{M}}$. These calculations are tedious but elementary. As a preparation, we observe that for n in $\{0, 1, \dots, N-1\}$

$$\begin{vmatrix} g_0+g_1 & -g_1 & & & 0 \\ -g_1 & g_1+g_2 & -g_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -g_n \\ 0 & & & -g_n & g_n \end{vmatrix} = g_0 \dots g_n \tag{6.13}$$

by repeatedly adding all other columns to the first one and pulling out factors. Repeated use of the linearity of the determinant in the last column of (6.13) yields

$$\begin{vmatrix} g_0+g_1 & -g_1 & & & 0 \\ -g_1 & g_1+g_2 & -g_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -g_{n-1} \\ 0 & & & -g_{n-1} & g_{n-1}+g_n \end{vmatrix} = g_0 \dots g_n \sum_{i=0}^{n} g_i^{-1}, \tag{6.14}$$

another identity which will be used frequently. Finally we introduce the abbreviation $q \doteq \exp(-ik)$. We turn now to the calculation of the lowest order coefficients in (6.12).

Calculation of $\det \underline{\underline{M}}$**:**

$$\det \underline{\underline{M}} = \begin{vmatrix} g_{N-1}(2-q-q^{-1}) & & & & & g_{N-1}(1-q^{-1}) \\ & g_0+g_1 & -g_1 & & & \\ & -g_1 & g_1+g_2 & -g_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & -g_{N-2} \\ g_{N-1}(1-q) & & & & -g_{N-2} & g_{N-2}+g_{N-1} \end{vmatrix}$$

by adding all columns to the first column and subsequently all rows to the first one,

$$= g_{N-1}(2-q-q^{-1}) \begin{vmatrix} g_0+g_1 & -g_1 & & & 0 \\ -g_1 & g_1+g_2 & -g_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -g_{N-2} \\ 0 & & & -g_{N-2} & g_{N-2} \end{vmatrix}$$

by pulling out a factor and eliminating the entry in the upper right and lower left corners,

$$= g_{N-1}(2-q-q^{-1})g_0 \begin{vmatrix} g_1+g_2 & -g_2 & & & 0 \\ -g_2 & g_2+g_3 & -g_3 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -g_{N-2} \\ 0 & & & -g_{N-2} & g_{N-2} \end{vmatrix}$$

by adding all columns to the first column and subsequently all rows to the first one and pulling out a factor,

$$= g_0 \dots g_{N-1}(2-q-q^{-1})$$

by applying (6.13).

Calculation of the $(N-1)\times(N-1)$ sub-determinants:

Let $0 < n < N-1$. We consider computing the sub-determinant of $\underline{\underline{M}}$ where row and column n are deleted.

$$
\begin{aligned}
\det \underline{\underline{M}}'_n &= \begin{vmatrix}
g_{N-1}+g_0 & -g_0 & & & -g_{N-1}q^{-1} \\
\ddots & \ddots & \ddots & & \\
& g_{n-1}+g_n & 0 & & \\
& 0 & 1 & 0 & \\
& & 0 & g_{n+1}+g_{n+2} & \\
& & \ddots & \ddots & \ddots \\
-g_{N-1}q & & & -g_{N-2} & g_{N-2}+g_{N-1}
\end{vmatrix} \\
&= \begin{vmatrix}
g_{N-1}+g_0 & -g_0 & \\
-g_0 & & \\
& \ddots & \\
& & -g_{n-1} \\
& -g_{n-1} & g_{n-1}+g_n
\end{vmatrix}
\begin{vmatrix}
g_{n+1}+g_{n+2} & -g_{n+2} & \\
-g_{n+2} & & \\
& \ddots & \\
& & -g_{N-2} \\
& -g_{N-2} & g_{N-2}+g_{N-1}
\end{vmatrix} \\
&\quad - q^{-1}g_{N-1} \begin{vmatrix}
g_{N-1}+g_0 & -g_0 & & & 1 \\
-g_0 & \ddots & & & \\
& & 1 & & \\
& & & \ddots & 0 \\
0 & & & -g_{N-2} & 0
\end{vmatrix}
- qg_{N-1} \begin{vmatrix}
0 & -g_0 & & & 0 \\
0 & \ddots & & & \\
& & 1 & & \\
& & & \ddots & -g_{N-2} \\
1 & & & 0 & 0
\end{vmatrix} \\
&\quad - g_{N-1}^2 \begin{vmatrix}
g_0+g_1 & -g_1 & & & 0 \\
-g_1 & \ddots & & & \\
& & 1 & & \\
& & & \ddots & -g_{N-3} \\
0 & & & -g_{N-3} & g_{N-3}+g_{N-2}
\end{vmatrix}
\end{aligned}
$$

by expanding in the first and last column of the matrix. It is not hard to see that the determinants in the terms proportional to q and q^{-1} vanish: The corresponding matrices can be brought to a form where they contain a zero column by simple column operations. The remaining determinants can be treated using (6.14):

$$
\begin{aligned}
&= g_{N-1}g_0\dots g_n \left(g_{N-1}^{-1} + \sum_{i=0}^{n} g_i^{-1} \right) g_{n+1}\dots g_{N-1} \left(\sum_{i=n+1}^{N-1} g_i^{-1} \right) \\
&\quad - g_{N-1}^2 g_0\dots g_n \left(\sum_{i=0}^{n} g_i^{-1} \right) g_{n+1}\dots g_{N-2} \left(\sum_{i=n+1}^{N-2} g_i^{-1} \right) \\
&= N g_0 \dots g_{N-1} \sum_{i=0}^{N-1} g_i^{-1}
\end{aligned}
$$

The cases where $n=0$ and $n=N-1$ have to be treated separately, either by an explicit calculation or by appealing to the symmetry of the problem under cyclic permutations of $g_0\dots g_{N-1}$. They yield the same result.

Calculation of the $(N-2)\times(N-2)$ sub-determinants:

The calculation of the $(N-2)\times(N-2)$ sub-determinants proceeds analogously to that in the last paragraph. Let $0<n<m<N-1$ and consider the sub-determinant of $\underline{\underline{M}}$ where row and column m and n are deleted. Again we start by expanding linearly in the first and the last column:

$$
\det \underline{\underline{M}}'_{nm} =
$$
$$
\begin{vmatrix}
g_{N-1}+g_0 & -g_0 & & \\
-g_0 & \ddots & & \\
& & & -g_{n-1} \\
& & -g_{n-1} & g_{n-1}+g_n
\end{vmatrix}
\begin{vmatrix}
g_{n+1}+g_{n+2} & -g_{n+2} & & \\
-g_{n+2} & \ddots & & \\
& & & -g_{N-2} \\
& & -g_{m-1} & g_{m-1}+g_m
\end{vmatrix}
$$
$$
\cdot \begin{vmatrix}
g_{m+1}+g_{m+2} & -g_{m+2} & & \\
-g_{n+2} & \ddots & & \\
& & & -g_{N-2} \\
& & -g_{N-2} & g_{N-2}+g_{N-1}
\end{vmatrix}
$$
$$
-q^{-1}g_{N-1}
\begin{vmatrix}
g_{N-1}+g_0 & -g_0 & & & & & 1 \\
-g_0 & \ddots & & & & & \\
& & 1 & & & & \\
& & & \ddots & & & \\
& & & & 1 & & \\
& & & & & \ddots & 0 \\
0 & & & & & -g_{N-2} & 0
\end{vmatrix}
- q g_{N-1}
\begin{vmatrix}
0 & -g_0 & & & & & 0 \\
0 & \ddots & & & & & \\
& & 1 & & & & \\
& & & \ddots & & & \\
& & & & 1 & & \\
& & & & & \ddots & -g_{N-2} \\
1 & & & & & 0 & 0
\end{vmatrix}
$$
$$
-g_{N-1}^2
\begin{vmatrix}
g_0+g_1 & -g_1 & & & & & 0 \\
-g_1 & \ddots & & & & & \\
& & 1 & & & & \\
& & & \ddots & & & \\
& & & & 1 & & \\
& & & & & \ddots & -g_{N-3} \\
0 & & & & & -g_{N-3} & g_{N-3}+g_{N-2}
\end{vmatrix}
$$

Again it is not hard to see that the determinants in the terms proportional to q and q^{-1} vanish. By means of (6.14) we get

$$
\begin{aligned}
&= g_{N-1}g_0\cdots g_n\left(g_{N-1}^{-1}+\sum_{i=0}^{n}g_i^{-1}\right)g_{n+1}\cdots g_m\left(\sum_{i=n+1}^{m}g_i^{-1}\right)g_{m+1}\cdots g_{N-1}\left(\sum_{i=m+1}^{N-1}g_i^{-1}\right)\\
&\quad -g_{N-1}^2 g_0\cdots g_n\left(\sum_{i=0}^{n}g_i^{-1}\right)g_{n+1}\cdots g_m\left(\sum_{i=n+1}^{m}g_i^{-1}\right)g_{m+1}\cdots g_{N-2}\left(\sum_{i=m+1}^{N-2}g_i^{-1}\right),
\end{aligned}
$$

which by a tedious but straightforward calculation can be further simplified to

$$= g_0 \dots g_{N-1} \sum_{0\le i<j\le N-1} (j-i)[N-(j-i)] g_i^{-1} g_j^{-1}.$$

Again, the case where the first or the last row and column get deleted have to be considered in a separate calculation or treated by symmetry arguments. The same result is obtained in these cases. □

From the last proposition we can read off

$$\begin{aligned}
w[0,0] &= 0,\\
w[0,1] &= w[0,3] = w[0,5] = \ldots = 0,\\
w[0,2] &= g_0 \dots g_{N-1},\\
w[0,4] &= -\frac{1}{12} g_0 \dots g_{N-1},\\
w[1,0] &= N g_0 \dots g_{N-1} \sum_{i=0}^{N-1} g_i^{-1},\\
w[1,1] &= w[1,2] = w[1,3] = \ldots = 0,\\
w[2,0] &= g_0 \dots g_{N-1} \sum_{0\le i<j\le N-1} (j-i)[N-(j-i)] g_i^{-1} g_j^{-1},\\
w[2,1] &= w[2,2] = w[2,3] = \ldots = 0.
\end{aligned}$$

Combining lemma 6.1.1 with the above we can now write down the beginning of the expansion of $\omega_{\mathrm{ac}}^2(k)$ in k. We use the shorthands $c_{ij} \doteq (j-i)[N-(j-i)]$ and

$$\langle f \rangle \doteq \frac{1}{N} \sum_{i=0}^{N-1} f_n$$

for the average of some quantity over the period of the lattice. Then the expansion of $\omega_{\mathrm{ac}}^2(k)$ reads

$$\omega_{\mathrm{ac}}^2(k) = \frac{1}{N^2} \frac{1}{\langle g^{-1} \rangle} k^2 + \left(\frac{1}{N^6} \frac{\sum_{i<j} c_{ij} g_i^{-1} g_j^{-1}}{\langle g^{-1} \rangle^3} - \frac{1}{12N^2} \frac{1}{\langle g^{-1} \rangle} \right) k^4 + O\left(k^6\right).$$

If we interprete the system as a KG field on a lattice (see (6.6)) it is instructive to replace k by Lk in the above formula where L is the sum of the lengths l_i, thus k becomes a dimensionful quantity. We get

$$\omega_{\mathrm{ac}}^2(k) = \frac{\langle l \rangle^2}{\langle l^2 \rangle} |k|^2 + \left(\frac{1}{L^2} \frac{\langle l \rangle^6}{\langle l^2 \rangle^3} \sum_{i<j} c_{ij} l_i^2 l_j^2 - \frac{L^2}{12} \frac{\langle l \rangle^2}{\langle l^2 \rangle} \right) |k|^4 + O\left(|k|^6\right). \tag{6.15}$$

We will finish this section by discussing (6.15) in several limiting cases.

The case $l_i = l$ for all i:

As a consistency check we consider the case where all l_i are equal to l: As for the lowest order term, obviously $\langle l \rangle^2 / \langle l^2 \rangle = 1$. For the higher order term we note that

$$\sum_{0\le i<j\le N-1} c_{ij} = \frac{1}{12} N^2 (N^2 - 1), \tag{6.16}$$

whence

$$\frac{1}{L^2}\frac{\langle l\rangle^6}{\langle l^2\rangle^3}\sum_{i<j} c_{ij} l_i^2 l_j^2 - \frac{L^2}{12}\frac{\langle l\rangle^2}{\langle l^2\rangle} = \frac{1}{12}\frac{l^4}{L^2}N^2(N^2-1) - \frac{L^2}{12} = -\frac{l^2}{12}.$$

So we get back precisely the two lowest order terms of the dispersion relation (6.7) of the regular lattice.

The $N \to \infty$, $L = const.$ limit:

We now want to discuss the question whether our model system reproduces the continuum behavior in the limit in which the lattice gets finer and finer with L staying constant. We will just look at the dispersion relation for the acoustic mode (6.15), but caution the reader that the limit of the whole theory is more complex, the number of modes growing with N.
For L constant, $\langle l\rangle$ scales as $\langle l\rangle = L/N$. $\langle l^2\rangle$ will also tend to zero for $N \to \infty$ but it could do so less fast than L^2/N^2. We set

$$\lim_{N\to\infty}\frac{\langle l\rangle^2}{\langle l^2\rangle} \doteq c$$

bearing in mind that c might be anything in $[0,1]$ depending on the distribution assumed for the l_i.
Now we consider the limit of the higher order terms. The second one is simply independent of N and therefore stays constant in the limit. A rough estimate of the c_{ij}-term shows:

$$\sum_{i<j} c_{ij} l_i^2 l_j^2 \leq \frac{N^4}{12}\sum_{i,j} l_i^2 l_j^2 = \frac{N^4}{12}\langle l^2\rangle^2 = \frac{1}{12}\frac{\langle l^2\rangle^2}{\langle l\rangle^4}L^4 \tag{6.17}$$

so it does not diverge but it does not necessarily go to zero, either. So what we are left with in the limit is not the simple continuum dispersion relation but one with correction terms and a changed velocity of light as an imprint of the micro-structure of the lattice before taking the limit. We should not fail to point out that both correction terms are of the same order of magnitude

$$\sum_{i<j} c_{ij} l_i^2 l_j^2 \approx \frac{N^4}{12}\langle l^2\rangle^2 \quad\Longrightarrow\quad \frac{1}{L^2}\frac{\langle l\rangle^6}{\langle l^2\rangle^3}\sum_{i<j} c_{ij} l_i^2 l_j^2 \approx \frac{L^2}{12}\frac{\langle l\rangle^2}{\langle l^2\rangle} \tag{6.18}$$

but have different signs, so cancellations are possible. Thus the limit will crucially depend on the distribution assumed for the l_i

The $L \to \infty$, $\langle l\rangle = const.$ limit:

We now come to the limit where L gets large while $\langle l\rangle$ is kept fixed. This is the limit one is interested in if the goal was to compute corrections to the continuum dispersion relation for a field propagating on a random lattice.
Again we do only look at the dispersion relation of the acoustic branch (6.15) and caution the reader that the limit of the whole system is more complex: In the limit we are considering, N will grow proportionally to L and hence there might be more and more optical branches reaching to lower and lower frequencies. Thus in the limit, the overall picture might look more like figure 6.1 than like a well defined dispersion relation.
Again

$$\lim_{L\to\infty}\frac{\langle l\rangle^2}{\langle l^2\rangle} \doteq c.$$

will be something in $[0,1]$ so there is no problem with the first order term. The second order terms do however diverge: The second one obviously so, the first one as the estimate (6.17) gives at least the correct order of magnitude of the term. At first sight this looks as if the limit is not well defined. Because of cancellation effects (see (6.18)) this might however not be the case for specific choices of distributions of the length l_i and clearly deserves further study.

We want to finish this section with some brief remarks.

- The model which we have dealt with should be further examined. Specifically, the limiting behavior should be studied for some reasonable choice of distribution of the l_i.
  Also, it would be very interesting to investigate the nature of the eigen*vectors* $\underline{c}$, at least in low order in k. For example: Does the one for the acoustic branch really look like a plane wave, at least for large N? We hope to come back to these tasks in future work.

- One should note that quite some computational efforts were needed to obtain the result (6.15) even in this simple model system. On the other hand, the details of (6.15) (such as the c_{ij} or the "velocity of light" $\langle l\rangle^2/\langle l^2\rangle$) would be very hard to guess before actually doing the calculation. Thus a lot of work will be needed to establish reliable results for the much more complicated models obtained from LQG.

6.2. Dispersion relations from LQG

As we have seen in the preceeding chapters, the Hamiltonians for matter coupled to loop quantum gravity are very complicated. Therefore, analytical treatment of the equations of motion (or the spectra of the corresponding operators in the QFT case) is out of the question. Still, the arguments given in the introduction to this chapter show that a dispersion relation should describe these systems at least approximately in the low energy regime. How can one "guess" this dispersion relation?

The idea which we would like to advocate in this section will be most easily explained at hand of the scalar field Hamiltonian (6.3): One could try to replace (6.3) by a simpler Hamiltonian which

- is a good approximation of (6.3) for slowly varying π and ϕ and
- is simple enough such that the EOM can be solved exactly.

The resulting theory should still contain some remnants of the underlying microscopic theory, but most of the information contained in (6.3) will be "integrated out".

This idea underlies also the work [27] and, at a rather simple level, is the basis for the recovery of continuum elasticity theory from the atomic description in solid state physics.

The advantage of this strategy as compared to detailed calculations in the spirit of the last section is that it can be easily applied to the models obtained from LQG, the disadvantage is that we do not have good control on the validity of the results.

To implement the strategy, we will proceed in several steps: The first step consists in replacing the discrete fields by fields in the continuum, evaluated at the lattice points, and replacing discrete derivatives at a given vertex by Taylor-expansion around that vertex. This is is just an equivalent description of the original theory.
The second step consists in replacing the vertex dependent coefficients in the Hamiltonian by suitable

averages. This is a key step, and it is here that the assumption about the scales on which the fields vary will enter.
The third step is to go over to a true continuum theory by replacing sums by integrals. The resulting theory can then be treated as an ordinary continuum field theory, and equations of motion and dispersion relations can be computed.
Let us explain these steps in more detail and apply them to the simplified model

$$H(\phi,\pi)=\frac{1}{2}\sum_{\underline{z}}\left(\pi_v^2+F(\underline{z})\sum_{I=1}^{3}\partial_I^+\phi(\underline{z})\partial_I^+\phi(\underline{z})\right)$$

on a graph of cubic topology (for the related notation see chapter 2).

First step.

To make the transition to a continuum theory possible, we reformulate the lattice Hamiltonian in terms of continuous fields, evaluated at the vertices, i.e.

$$\phi_v \longrightarrow \phi\left(\vec{x}(\underline{v})\right), \qquad \pi_v \longrightarrow \pi\left(\vec{x}(\underline{v})\right).$$

A peculiarity of the lattice Hamiltonians is that their contribution at a given vertex v contains, for example via the lattice derivatives, the value of the field at neighboring points. In the continuous case, the contribution at a given point $\vec{x}$ usually only contains the fields and their derivatives at $\vec{x}$. So again, in order to make the transition to the continuum case possible, we trade contributions from neighboring vertices against derivatives of the field at the given vertex via a Taylor expansion. For the lattice derivative we get for example

$$\partial_e^+\phi\left(\vec{x}(v)\right)=b_e^i\partial_i\phi\left(\vec{x}(v)\right)+\frac{1}{2}b_e^i(v)b_e^j(v)\partial_i\partial_j\phi\left(\vec{x}(v)\right)+\frac{1}{6}b_e^i(v)b_e^j(v)b_e^k(v)\partial_i\partial_j\partial_k\phi\left(\vec{x}(v)\right)+\ldots.$$

Note that as long as we really keep all the terms in this expansion, we will not have changed the theory we are considering. However, in the applications we have in mind, the norm of the Euclidean vectors $\vec{b}_e$ connecting adjacent vertices (see figure 2.1) will be tiny, so truncation at a finite order in $\vec{b}$ will lead to a very good approximation to the original theory.

Let us apply these transformations to the cubic graph Hamiltonian as an example. We get:

$$\begin{aligned}
H(\phi,\pi)=\frac{1}{2}\sum_{\underline{z}\in\mathbb{Z}^d}\Bigg[&\pi^2\left(\vec{x}(\underline{z})\right)+F_{\underline{z}}\sum_I\Big(b_I^ib_I^j\partial_i\phi\partial_j\phi\left(\vec{x}(\underline{z})\right)\\
&+2\frac{1}{2}b_I^ib_I^jb_I^k\partial_i\partial_j\phi\partial_k\phi\left(\vec{x}(\underline{z})\right)\\
&+\frac{1}{4}b_I^ib_I^jb_I^kb_I^l\partial_i\partial_j\phi\partial_k\partial_l\phi\left(\vec{x}(\underline{z})\right)+2\frac{1}{6}b_I^ib_I^jb_I^kb_I^l\partial_i\partial_j\partial_k\phi\partial_l\phi\left(\vec{x}(\underline{z})\right)+O(b_I^5)\Big)\Bigg]\\
=:\frac{1}{2}\sum_{\underline{z}\in\mathbb{Z}^d}\Bigg[&\pi^2\left(\vec{x}(\underline{z})\right)V_{\underline{z}}+\sum_I\Big(B_I^{(i)(j)}(v)\partial_i\phi\partial_j\phi\left(\vec{x}(\underline{z})\right)\\
&+B_I^{(ij)(k)}(v)\partial_i\partial_j\phi\partial_k\phi\left(\vec{x}(\underline{z})\right)\\
&+\frac{1}{4}B_I^{(ij)(kl)}(v)\partial_i\partial_j\phi\partial_k\partial_l\phi\left(\vec{x}(\underline{z})\right)+\frac{1}{3}B_I^{(ijk)(l)}(v)\partial_i\partial_j\partial_k\phi\partial_l\phi\left(\vec{x}(\underline{z})\right)+O(b_I^5)\Big)\Bigg]
\end{aligned}$$

where in the second step we have introduced some notation which will be convenient for the more complicated models treated later.

Second step.

Now we will replace the vertex dependent coefficients in the Hamiltonian by suitable averages. To this end we introduce the following notation: Let γ be a graph and $\mathcal{R}$ a region in Euclidean space. Then denote

$$\langle O\rangle_{\mathcal{R}} \doteq \frac{1}{\mathrm{Vol}(\mathcal{R})} \sum_{v\in V(\gamma):\vec{x}(v)\in\mathcal{R}} O_v$$

where O is some vertex dependent quantity and the volume is measured with respect to the flat background metric. Also, we will use the notation $\mathcal{R}(\gamma)$ for the region contained in the cell complex dual to γ and write $\langle O\rangle$ as a shorthand for $\langle O\rangle_{\mathcal{R}(\gamma)}$.

Now we will consider a semiclassical state with its underlying graph γ. The vertex dependent functions in the Hamiltonian are expectation values of geometric operators in the semiclassical state and little can be said about them a priori. We can however safely assume that they depend on the local geometric properties of the underlying graph in some way, otherwise it is hard to imagine how the state could have semiclassical properties. Consequently, we will invoke assumption 4.1.1 with respect to these expectation values. The first part of this assumption then states that there is a length scale L (much bigger than the typical edge-length of the graph such that averages $\langle\,\cdot\,\rangle_{\mathcal{R}}$ of these coefficients over regions $\mathcal{R}$ of diameter L and bigger equal their averages $\langle\,\cdot\,\rangle_{\mathcal{R}(\gamma)}$ over the whole graph.
Let now $\{\mathcal{R}_i\}$ be a partition of $\mathcal{R}(\gamma)$ such that the dimensions of the $\mathcal{R}_i$ are $\simeq L$. We look at a term $O_v f(\phi(v),\pi(v),\partial\phi(v),\ldots)$ in the Hamiltonian:

$$\begin{aligned}\sum_{v\in V(\gamma)} O_v f(\phi(v),\pi(v),\ldots) &= \sum_i \sum_{v\in\mathcal{R}_i} O_v f(\phi(v),\pi(v),\ldots)\\ &= \sum_i \left[\sum_{v\in\mathcal{R}_i} OV_v f(\phi(v),\pi(v),\ldots) + \sum_{v\in\mathcal{R}_i} (O_v - OV_v) f(\phi(v),\pi(v),\ldots)\right].\end{aligned}$$

We now want to choose the constant O such that we can drop the second term in the last line. Because we assume that the fields vary only very little on and below the scale L, $f(\phi(v),\pi(v),\ldots)$ is approximately constant, we disregard it and compute

$$\sum_{v\in\mathcal{R}_i} (O_v - OV_v) \stackrel{!}{=} 0 \quad\Longrightarrow\quad O = \langle O\rangle_{\mathcal{R}_i}.$$

Again because of the assumption, $\langle O\rangle_{\mathcal{R}_i} \simeq \langle O\rangle_{\mathcal{R}(\gamma)}$, so that we finally end up with the approximation

$$\sum_{v\in V(\gamma)} O_v f(\phi(v),\pi(v),\ldots) \simeq \langle O\rangle_{\mathcal{R}(\gamma)} \sum V_v f(\phi(v),\pi(v),\ldots).$$

This approximation certainly changes the theory we are considering to a certain amount. The hope is that the change is small in a sense, i.e. the location of the extrema of $H[\phi,\pi]$ only change a little bit (with respect to some natural topology on the space of smooth fields), and that the change mostly concerns the high energy behavior of the theory, something we are not interested in.
The virtue is that due to this change, we will eventually end up with a theory that does admit plane wave solutions but still contains some information about the microstructure of the graph.

For the cubic graph model, this averaging leads to

$$\begin{aligned} H(\phi,\pi) = \frac{1}{2}\sum_{\underline{z}\in\mathbb{Z}^d} V_v \Big[\pi^2\left(\vec{x}(\underline{z})\right) + \sum_I \Big(&\langle B_I^{(i)(j)}\rangle \partial_i\phi\partial_j\phi\left(\vec{x}(\underline{z})\right) \\ &+ \langle B_I^{(ij)(k)}\rangle \partial_i\partial_j\phi\partial_k\phi\left(\vec{x}(\underline{z})\right) \\ &+ \frac{1}{4}\langle B_I^{(ij)(kl)}\rangle \partial_i\partial_j\phi\partial_k\partial_l\phi\left(\vec{x}(\underline{z})\right) + \frac{1}{3}\langle B_I^{(ijk)(l)}\rangle \partial_i\partial_j\partial_k\phi\partial_l\phi\left(\vec{x}(\underline{z})\right) + O(b_I^5)\Big)\Big]. \end{aligned}$$

Note that, since the graph γ underlying the semiclassical state will in general not be invariant under rotations, neither will be the tensors $B^{(i)(j)}, B^{(i)(jk)}, \ldots$. However, the second part of assumption 4.1.1 states that the averages $\langle B^{(i)(j)}\rangle, \langle B^{(i)(jk)}\rangle, \ldots$ will be. So, although the exact values of the $\langle B\rangle$ can only be computed when an exact specification of the construction of the underlying graph is given, information on their structure can already be obtained from this requirement of "rotation invariance at large scales".

Third step.

Now we go over to a continuum theory. That is, we simply make the replacement

$$\sum_{v\in V(\gamma)} V_v \Big(\ldots \Big)(\vec{x}(v)) \quad \longrightarrow \int_{\mathcal{R}(\gamma)} \Big(\ldots \Big)(\vec{x})\, dx.$$

The Hamiltonian obtained in that way can now be treated in the usual way to obtain the equations of motion.

In the case of our toy model, we obtain the continuum Hamiltonian

$$\begin{aligned} H[\phi,\pi] = \frac{1}{2}\int_{\mathcal{R}(\gamma)} \Big[\pi^2\left(\vec{x}\right)) + \sum_I \Big(&\langle B_I^{(i)(j)}\rangle \partial_i\phi\partial_j\phi\left(\vec{x}\right) + \langle B_I^{(ij)(k)}\rangle \partial_i\partial_j\phi\partial_k\phi\left(\vec{x}\right) \\ &+ \frac{1}{4}\langle B_I^{(ij)(kl)}\rangle \partial_i\partial_j\phi\partial_k\partial_l\phi\left(\vec{x}\right) + \frac{1}{3}\langle B_I^{(ijk)(l)}\rangle \partial_i\partial_j\partial_k\phi\partial_l\phi\left(\vec{x}\right) + O(b_I^5)\Big)\Big] \end{aligned}$$

Using the formula

$$\frac{\delta}{\delta\phi(\vec{y})}\int \left(\partial_{i_1}\ldots\partial_{i_m}\phi\right)(\vec{x})\left(\partial_{j_1}\ldots\partial_{j_n}\phi\right)(\vec{x})\, dx = \left[(-1)^m + (-1)^n\right]\partial_{i_1}\ldots\partial_{i_m}\partial_{j_1}\ldots\partial_{j_n}\phi(\vec{y}),$$

we arrive at the equation of motion

$$\ddot{\phi} = \sum_I \langle B_I^{(i)(j)}\rangle \partial_i\partial_j\phi + \left(\frac{1}{3}\langle B_I^{(ijk)(l)}\rangle - \frac{1}{4}\langle B_I^{(ij)(kl)}\rangle\right)\partial_i\partial_j\partial_k\partial_l\phi + \ldots. \tag{6.19}$$

Thus we have achieved our goal: (6.19) admits plane waves as solutions, with the dispersion relation

$$\omega^2(\vec{k}) = \sum_I \langle B_I^{(i)(j)}\rangle k_i k_j + \left(\frac{1}{3}\langle B_I^{(ijk)(l)}\rangle - \frac{1}{4}\langle B_I^{(ij)(kl)}\rangle\right) k_i k_j k_k k_l + \ldots.$$

Furthermore, upon setting $b_I^j = \delta_I^j$, $F(v) = 1$ the above dispersion relation simplifies to

$$\omega^2(\vec{k}) = \sum_i \left[k_i^2 - \frac{1}{12}k_i^4 + \ldots\right],$$

thus we recover, order by order, the dispersion relation for "lattice plane waves" on a regular cubic lattice. This shows that our procedure, albeit yielding a field theory in the continuum, preserves some information about the lattice we started with.

6.2.1. The Scalar field

Application of the procedure presented in the last section to the case of a scalar field on a random graph, coupled to the expectation values of the gravity degrees of freedom is not much more complicated than the application to the simplified model discussed above. Let us write the Hamiltonian (6.3) in the form

$$H(\phi,\pi) = \frac{1}{2}\sum_v \left[A_v^{(1)}\pi_v^2 + \sum_{e,e'\in E(v)} A^{(2)}_{vee'}(\partial_e^+\phi)_v(\partial_{e'}^+\phi)_v + A_v^{(3)}\phi_v^2 \right].$$

Going through the three steps, we arrive at

$$\begin{aligned} H[\phi,\pi] = \frac{1}{2}\int_{\mathcal{R}(\gamma)} \Big[&\langle A^{(1)}V^2\rangle \pi^2(\vec{x}) + \langle A^{(3)}\rangle \phi^2(\vec{x}) + \langle B^{(i)(j)}\rangle \partial_i\phi\partial_j\phi(\vec{x}) + \langle B^{(ij)(k)}\rangle \partial_i\partial_j\phi\partial_k\phi(\vec{x}) \\ &+ \frac{1}{4}\langle B^{(ij)(kl)}\rangle \partial_i\partial_j\phi\partial_k\partial_l\phi(\vec{x}) + \frac{1}{3}\langle B^{(ijk)(l)}\rangle \partial_i\partial_j\partial_k\phi\partial_l\phi(\vec{x}) + \ldots \Big]\, dx, \end{aligned}$$

where now

$$\langle B^{(i)(j)}\rangle = \langle \sum_{e,e'\in E(v)} b_e^i(v) b_{e'}^j(v) A^{(2)}_{vee'}\rangle, \qquad \langle B^{(ij)(j)}\rangle = \langle \sum_{e,e'\in E(v)} b_e^i b_e^j b_{e'}^k(v) A^{(2)}_{vee'}\rangle, \tag{6.20}$$

$$\langle B^{(ij)(kl)}\rangle = \langle \sum_{e,e'\in E(v)} b_e^i b_e^j b_{e'}^k b_{e'}^l(v) A^{(2)}_{vee'}\rangle, \qquad \langle B^{(ijk)(k)}\rangle = \langle \sum_{e,e'\in E(v)} b_e^i b_e^j b_e^k b_{e'}^l(v) A^{(2)}_{vee'}\rangle, \tag{6.21}$$

and so on for the higher order terms. This leads to the equations of motion

$$\ddot{\phi} = \langle A^{(1)}V^2\rangle \left[-\langle B^{(i)(j)}\rangle \partial_i\partial_j\phi + \left(\frac{1}{4}\langle B^{(ij)(kl)}\rangle - \frac{1}{3}\langle B^{(ijk)(l)}\rangle \right) \partial_i\partial_j\partial_k\partial_l\phi + \ldots \right]$$

which have plane wave solutions with the dispersion relation

$$\omega^2(\vec{k}) = \langle A^{(1)}V^2\rangle \left[\langle B^{(i)(j)}\rangle k_ik_j + \left(\frac{1}{3}\langle B^{(ijk)(l)}\rangle - \frac{1}{4}\langle B^{(ij)(kl)}\rangle \right) k_ik_jk_kk_l + \ldots \right].$$

Despite the fact that we haven not specified a random graph prescription, a little bit more can be said, if we invoke part 2 of assumption 4.1.1:
The space of tensors of second rank in three dimensions which are rotationally invariant is one dimensional and spanned by δ^{ij}. Thus for a random graph prescription being invariant under rotations *on average* in the sense of assumption 4.1.1, we have $B^{(i)(j)} \sim \delta^{ij}$.
For the tensors of fourth rank the situation is slightly more complicated:

$$\delta^{ij}\delta^{kl}, \quad \delta^{ik}\delta^{jl}, \quad \delta^{il}\delta^{jk}$$

span the space of rotation invariant tensors. But contraction of any of them with $k_ik_jk_kk_l$ leads to $|k|^4$, thus

$$\left(\frac{1}{3}\langle B^{(ijk)(l)}\rangle - \frac{1}{4}\langle B^{(ij)(kl)}\rangle \right) k_ik_jk_kk_l \sim |k|^4 .$$

Putting everything together, we get

$$\omega^2(\vec{k}) = \langle A^{(1)}V^2\rangle \left[\frac{1}{3}\sum_i \langle B^{(i)(i)}\rangle \left|\vec{k}\right|^2 + \frac{1}{3}\sum_i \left(\frac{1}{3}\langle B^{(iii)(i)}\rangle - \frac{1}{4}\langle B^{(ii)(ii)}\rangle \right) \left|\vec{k}\right|^4 + \ldots \right].$$

6.2.2. The Maxwell field

In this section, we will treat the Maxwell field according to the procedure outlined above. Let us write the Hamiltonian to be approximated as

$$H(E,A) = \frac{1}{2}\sum_v \left[\sum_{e,e'\in E(v)} G^{(1)}_{vee'} E^e E^{e'} + \sum_{\alpha,\alpha' \text{ based at } v} G^{(2)}_{v\alpha\alpha'} A^\alpha A^{\alpha'} \right],$$

where α and α' are minimal loops based at the respective vertex and the coefficients $G^{(1)}, G^{(2)}$ derive from the expectation values $\langle F_4\rangle_{\Psi_{\text{flat}}}, \langle F_5\rangle_{\Psi_{\text{flat}}}$ in an obvious way.

Let us consider the $G^{(1)}$ term first: Again we want to rewrite it in terms of a continuous field $E(\vec{x})$ and its derivatives at the locations of the vertices. For that, we Taylor expand in the definition of the E^e. We remind the reader, however, of the discussion of the validity of this step, given above.

$$\begin{aligned} E^e_v &= \int_{S^e_v} *E = \int_{S^e_v} n_i(\vec{y}) E^i(\vec{y})\, dy \\ &= \int_{S^e_v} n_i(\vec{y}) \Big(E^i\left(\vec{x}(v)\right) + (\vec{y}-\vec{x}(v))_j \partial^j E^i(\vec{x}(v)) \\ &\qquad\qquad + (\vec{y}-\vec{x}(v))_j (\vec{y}-\vec{x}(v))_k \partial^j\partial^k E^i(\vec{x}(v)) + \ldots \Big)\, dy. \end{aligned}$$

This suggests the definitions

$$s^e_i(v) \doteq \int_{S^e_v} n_i(\vec{y})\, dy, \qquad s^e_{ij}(v) \doteq \int_{S^e_v} n_i(\vec{y})(\vec{y}-\vec{x}(v))_j\, dy, \tag{6.22}$$

$$s^e_{ijk}(v) \doteq \int_{S^e_v} n_i(\vec{y})(\vec{y}-\vec{x}(v))_j(\vec{y}-\vec{x}(v))_k\, dy, \tag{6.23}$$

etc. Furthermore we introduce the shorthands

$$S_{(i)(i')}(v) \doteq \sum_{e,e'} G^{(1)}_{vee'} s^e_i(v) s^{e'}_{i'}(v), \qquad S_{(i)(i'j')}(v) \doteq \sum_{e,e'} G^{(1)}_{vee'} s^e_i(v) s^{e'}_{i'j'}(v),$$

$$S_{(ij)(i'j')}(v) \doteq \sum_{e,e'} G^{(1)}_{vee'} s^e_{ij}(v) s^{e'}_{i'j'}(v), \qquad S_{(i)(i'j'k')}(v) \doteq \sum_{e,e'} G^{(1)}_{vee'} s^e_i(v) s^{e'}_{i'j'k'}(v).$$

With this notation at hand we can write:

$$\begin{aligned} \sum_{e,e'\in E(v)} G^{(1)}_{vee'} E^e E^{e'} &= S_{(i)(i')}(v) E^i E^{i'}(\vec{x}(v)) + 2S_{(i)(i'j')}(v) E^i \partial^{j'} E^{i'}(\vec{x}(v)) \\ &\quad + S_{(ij)(i'j')}(v) \partial^j E^i \partial^{j'} E^{i'}(\vec{x}(v)) + S_{(i)(i'j'k')}(v) E^i \partial^{j'}\partial^{k'} E^{i'}(\vec{x}(v)) + \ldots. \end{aligned}$$

We now turn to the $G^{(2)}$ term: For a start, let α be any loop and v any vertex. Then

$$\begin{aligned} A^\alpha &= \sum_{e\in\alpha} A^e = \sum_{e\in\alpha} \int_0^1 A_i(\vec{e}(s)) \dot{e}^i(s)\, ds \\ &= \sum_{e\in\alpha} \int_0^1 \dot{e}^i(s) \Big[A_i(\vec{x}(v)) + (\vec{e}(s)-\vec{x}(v))^j \partial_j A_i(\vec{x}(v)) \\ &\qquad\qquad + \frac{1}{2}(\vec{e}(s)-\vec{x}(v))^j (\vec{e}(s)-\vec{x}(v))^k \partial_j\partial_k A_i(\vec{x}(v)) + \ldots \Big]. \end{aligned}$$

The first term does not contribute since

$$\sum_{e\in\alpha}\int_0^1 \dot{e}^i(s) = \sum_{e\in\alpha}\big(e^i(1)-e^i(0)\big) = 0,$$

and for the rest we introduce the shorthands

$$b_e^{ij}(v) \doteq \int_0^1 \dot{e}^i(s)(\vec{e}(s)-\vec{x}(v))^j, \qquad b_e^{ijk}(v) \doteq \int_0^1 \dot{e}^i(s)(\vec{e}(s)-\vec{x}(v))^j(\vec{e}(s)-\vec{x}(v))^k \tag{6.24}$$

etc. With

$$\begin{aligned} B^{(ij)(i'j')}(v) &= \sum_{\alpha,\alpha' \text{ based at } v} G^{(2)}_{v\alpha\alpha'} \sum_{e\in\alpha, e'\in\alpha'} b_e^{ij} b_{e'}^{i'j'}, \\ B^{(ij)(i'j'k')}(v) &= \sum_{\alpha,\alpha' \text{ based at } v} G^{(2)}_{v\alpha\alpha'} \sum_{e\in\alpha, e'\in\alpha'} b_e^{ij} b_{e'}^{i'j'k'} \end{aligned}$$

etc., the $G^{(2)}$ term reads

$$\begin{aligned} \sum_{\alpha,\alpha'} G^{(2)}_{v\alpha\alpha'} A^\alpha A^{\alpha'} = B^{(ij)(i'j')}(v)\partial_j A_i \partial_{j'} A_{i'}(\vec{x}(v)) &+ B^{(ij)(i'j'k')}(v)\partial_j A_i \partial_{j'}\partial_{k'} A_{i'}(\vec{x}(v)) \\ &+ \frac{1}{2} B^{(ijk)(i'j'k')}(v)\partial_j\partial_k A_i \partial_{j'}\partial_{k'} A_{i'}(\vec{x}(v)) \\ &+ \frac{1}{3} B^{(ij)(i'j'k'l')}(v)\partial_j A_i \partial_{j'}\partial_{k'}\partial_{l'} A_{i'}(\vec{x}(v)) + \ldots . \end{aligned}$$

We are now ready to write down the continuum Hamiltonian. For brevity we display only the lowest correction terms:

$$\begin{aligned} H[\vec{E},\vec{A}] = \frac{1}{2}\int & \langle S_{(i)(i')}\rangle E^i E^{i'}(\vec{x}) + 2\langle S_{(i)(i'j')}\rangle E^i \partial^{j'} E^{i'}(\vec{x}) \\ & + \langle B^{(ij)(i'j')}\rangle \partial_j A_i \partial_{j'} A_{i'} + \langle B^{(ij)(i'j'k')}\rangle \partial_j A_i \partial_{j'}\partial_{k'} A_{i'} + \ldots \, dx. \end{aligned}$$

From this Hamiltonian we get the equations of motion

$$\begin{aligned} \ddot{A}_l(\vec{x}) = \frac{1}{4}\mathrm{I}_{il}\left[\mathrm{III}^{imjj'}\partial_j\partial_{j'}A_m(\vec{x}) - \mathrm{IV}^{imjj'k'}\partial_j\partial'_j\partial_{k'}A_m(\vec{x})\right] & \\ + \frac{1}{2}\mathrm{II}_{ilk'}\mathrm{III}^{imjj'}\partial^{k'}\partial_j\partial_{j'}A_m(\vec{x}) + \ldots, & \end{aligned} \tag{6.25}$$

where we have only written down terms that contain derivatives of $\vec{A}$ up to third order and used the shorthands

$$\begin{aligned} \mathrm{I}_{ij} &= \langle S_{(i)(j)}\rangle + \langle S_{(j)(i)}\rangle, & \mathrm{II}_{ijk} &= \langle S_{(i)(jk)}\rangle - \langle S_{(j)(ik)}\rangle, \\ \mathrm{III}^{ijkl} &= \langle B^{(jk)(il)}\rangle + \langle B^{(ik)(jl)}\rangle, & \mathrm{IV}^{ijklm} &= \langle B^{(jk)(ilm)}\rangle + \langle B^{(k)(ijlm)}\rangle. \end{aligned}$$

We now invoke our symmetry assumption about the semiclassical states: The above tensors should be rotationally invariant. For I we can write

$$\mathrm{I}_{ij} = 2c_1\delta_{ij}, \text{ with } c_1 = \frac{1}{3}\sum_i \langle S_{(i)(i)}\rangle.$$

There is also only one invariant third order tensor, whence

$$\mathrm{II}_{ijk} = 2c_2\epsilon_{ikj}, \text{ with } c_2 = \frac{1}{6}\epsilon^{ijk}\langle S_{(i)(kj)}\rangle.$$

The space of invariant rank 4 tensors is three dimensional. If symmetry under exchange of two of the indices is required, this reduces to two dimensions. We write

$$\mathrm{III}^{ijkl}\partial_k\partial_l A_j = \left(2c_3\delta^{ij}\delta^{kl} + c_4(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{kj})\right)\partial_k\partial_l A_j,$$

whence c_3, c_4 are given by

$$c_3 = \frac{1}{3}\left(\sum_{ij}\langle B^{(ij)(ij)}\rangle - \sum_i\langle B^{(ii)(ii)}\rangle\right), \qquad c_4 = \frac{1}{3}\left(3\sum_i\langle B^{(ii)(ii)}\rangle - \sum_{ij}\langle B^{(ij)(ij)}\rangle\right).$$

So we can write the whole expression containing III as

$$\mathrm{III}^{ijkl}\partial_k\partial_l A_j = c_3\Delta A_i + c_4\partial_i \operatorname{div}\vec{A}.$$

Finally we consider IV: The space of invariant rank 5 tensors is ten dimensional. But if we take into consideration the symmetry of the term IV gets contracted with, there is only one tensor left. We set

$$\mathrm{IV}^{ijklm}\partial_k\partial_l\partial_m A_j = c_5\epsilon^{jik}\delta^{lm}\partial_k\partial_l\partial_m A_j = c_5\left(\Delta \operatorname{rot}\vec{A}\right)_i$$

and obtain

$$c_5 = \frac{1}{3}\langle B^{(jn)(inn)}\rangle\epsilon_{jin}.$$

Taking these symmetry considerations into account, the equations of motion (6.25) take the form

$$\ddot{\vec{A}}(t,\vec{x}) = \frac{1}{2}c_1c_3\Delta\vec{A} + \frac{1}{2}c_1c_4 \operatorname{grad}\operatorname{div}\vec{A} + \frac{1}{2}(c_2c_3 - c_1c_5)\Delta\operatorname{rot}\vec{A} + \dots.$$

The equation has the form of the wave equation for the electromagnetic field in a birefringent medium. We can compute the dispersion relation: As a first step, note that we can drop the grad $\vec{A}$ term upon imposing the gauge condition for A. Then it is not hard to see that for example a circularly polarized wave propagating in z direction

$$\vec{A}_\pm(\vec{x},t) = \begin{pmatrix} \cos(\omega_\pm t - k_3x_3) \\ \pm\sin(\omega_\pm t - k_3x_3) \\ 0 \end{pmatrix}$$

solves this equation, provided

$$\omega_\pm(\vec{k}) = |k|\sqrt{\frac{1}{2}c_1c_3 \pm \frac{1}{2}(c_2c_3 - c_1c_5)k_3}. \tag{6.26}$$

Thus we find a chiral modification of the standard dispersion relation for the electromagnetic field.

7. A simplified example

In the present chapter we fuse together the different results of the thesis: We will compute the expectation values of the matter Hamiltonians (chapter 5) in a coherent state for loop quantum gravity (LQC, chapter 4), and derive from them a low energy dispersion relation with the methods developed in chapter 6. To make the computations manageable, we have to introduce some simplifications described below. Also some questions concerning semiclassical states in general, as well as ambiguities in the definition of the LQC remain to be settled in the future. Finally, the method we use to compute the dispersion relation will have to be investigated more thoroughly. For all these reasons, the formulae given below should not be read as a ready-to-use prediction for quantum gravity effects but as demonstrating what can be done with the methods at hand and what qualitative features are to be expected in a more complete calculation. We will give a discussion of our results in section 7.2.

The difficult step in the determination of dispersion relations of the matter fields along the lines of chapter 6 is the computation of the expectation values of the quadratic forms $F_1, \ldots, F_5$ in a LQC. The gravity parts of these forms are all constructed out of the operator $\widehat{Q}(v,e,r)$ for various v,e,r, a complicated nonlinear function of the basic variables $h_e, \widehat{P}_e$. As we do not know a basis of eigenvectors for this operator, determination of the expectation values in a coherent state is a very difficult task. Since we are at this stage only interested in qualitative features, we do not want to spend too much technical effort. Therefore we make our life simple by replacing SU(2) by $\mathrm{U}(1)^3$. Thereby we dispose of the complications coming from a non-Abelian gauge group[i] and obtain a basis of eigenvectors: The $\mathrm{U}(1)^3$ spin networks diagonalize the volume operator and consequently $\widehat{Q}$. As a heuristic justification for this simplification, we note that SU(2) gets replaced by $\mathrm{U}(1)^3$ in the $G_{\text{Newton}} \to 0$ limit if one rescales the gravitational connection A by A/G_{Newton} (Iönü-Wigner contraction). But $G_{Newton} \to 0$ also implies $l_P \to 0$ and this is precisely the regime we are interested in. Ultimately however, the computation must be carried out for full SU(2) to lead to definite results. Despite this simplifications, the computation remains messy. We will therefore carry it out in an appendix only cite the results in the next section.

The other simplifying assumption we will make is that the random graph the LQC is based on is of cubic topology. This simplifies the calculation tremendously because we can choose the charts used in the definition of the quadratic forms $F_1, \ldots F_5$ in such a way that the related coefficients $(\mu(v), \nu(v), \omega_I(e,v), \ldots)$ become edge- and vertex-independent. For details we refer to the appendix.

7.1. Dispersion relations for the matter fields

In the present section we will use the notation in connection with graphs of cubic topology introduced in section 2.1. Let us also briefly mention the consequences of the simplifying replacement SU(2) $\to$ U(1) which is used in the appendix:

[i]See [22, 23] for how non-Abelian gauge groups blow up the computational effort by an order of magnitude.

When we replace SU(2) by U(1), the connection one form A_a^I now takes values in $\mathrm{u}(1)^3$, were I labels the u(1)-copies. Consequently the variables P_e^I introduced in chapter 4 now take values in U(1), I labelling the three U(1)-copies. The other operators are changed accordingly. For details we refer to the appendix.

We are now ready to spell out the low energy dispersion relations for the Klein-Gordon and the Maxwell field. We consider flat space and fix a global Euclidean coordinate system that we will use throughout. In the $\mathrm{U}(1)^3$-setting, we can model the flat space situation by choosing the classical values

$$A_a^I(x) = 0, \qquad E_I^a(x) = \delta_I^a \qquad \text{for all } x \in \Sigma$$

with respect to our global coordinates. Therefore all holonomies are trivial and for the fluxes we find

$$P_e^i(v) = \int_{S_e} dn^i.$$

From the appendix we recall the definitions

$$\begin{aligned} P_I^J(v) &\doteq \frac{1}{2}\left(P_{e_I}^J(v) + P_{e_I}^J(v - a_I)\right) \\ P_{II'}^2(v) &\doteq \sum_J P_I^J(v) P_{I'}^J(v). \end{aligned}$$

We can now state the results of the expectation value computations. We display only the leading terms and first order corrections to the expectation values, so our use of "=" in the following will be slightly imprecise. The quadratic forms relevant for the scalar field yield

$$\begin{aligned} \langle F_1(\phi)\rangle_{\Psi\mathrm{grav}} &= \sum_v \sqrt{\det P(v)}\left[1 + \frac{l_P^7}{\sqrt{t}}\frac{1}{32}\operatorname{Tr} P^{-2}(v)\right]\phi_v^2, \\ \langle F_2(\phi)\rangle_{\Psi\mathrm{grav}} &= \sum_v \sum_{I\sigma I'\sigma'} \left[\frac{\sigma\sigma' P_{II'}^2(v)}{\sqrt{\det P(v)}}\right. \\ &\qquad \left. + \frac{l_P^4}{t}\frac{\sigma\sigma'}{\sqrt{\det P(v)}}\left(\frac{1173}{128}\operatorname{Tr}(P^{-2})P_{II'}^2(v) + \frac{19}{32}\delta_{II'}\right)\right]\partial_{e_{I\sigma}}^+\phi_v \partial_{e_{I'\sigma'}}^+\phi_v, \\ \langle F_3(\pi)\rangle_{\Psi\mathrm{grav}} &= \sum_v \frac{1}{\sqrt{\det P(v)}}\left[1 + \frac{l_P^4}{t}\frac{1707}{512}\operatorname{Tr} P^{-2}(v)\right]\pi_v^2. \end{aligned}$$

The terms relevant for the electromagnetic field are

$$\begin{aligned} \langle F_4(E)\rangle_{\Psi\mathrm{grav}} &= \sum_v \sum_{I\sigma I'\sigma'}\left[\sqrt{\det P(v)}P_{II'}^{-2} + \frac{l_P^4}{t}\left(\frac{763}{512}P_{II'}^{-2}\operatorname{Tr} P^{-2} - \frac{13}{16}P_{II'}^{-4}\right)\right]\sigma\sigma' E_{e_{\sigma I}}(v) E_{e_{\sigma' I'}}(v), \\ \langle F_5(B)\rangle_{\Psi\mathrm{grav}} &= \sum_v \sum_{II'}\left[\sqrt{\det P(v)}P_{II'}^{-2} + \frac{l_P^4}{t}\left(\frac{763}{512}P_{II'}^{-2}\operatorname{Tr} P^{-2} - \frac{13}{16}P_{II'}^{-4}\right)\right]A_{\tilde{\alpha}_I} A_{\tilde{\alpha}_{I'}} \end{aligned}$$

Let us now display the dispersion relations resulting from these expectation values.

Dispersion relation for the scalar field.

To write down the dispersion relation for the scalar field, we have to form the graph averages of the coefficients of the forms F_1,F_2,F_3 according to section 6.2. We separate into the leading order term and the first order correction. For the mass term we define

$$\langle C_0 \rangle \doteq \frac{1}{N} \sum_v \frac{\sqrt{\det P(v)}}{V_v},$$
$$\langle C_1 \rangle \doteq \frac{1}{32} \frac{l_P^7}{\sqrt{t}} \frac{1}{N} \sum_v \frac{\sqrt{\det P(v)}\,\mathrm{Tr}\, P^{-2}(v)}{V_v}.$$

For the derivative term, we introduce the Euclidean vectors

$$\widetilde{b}_I^i(v) \doteq b_{e_I}^i(v) - b_{e_I}^i(v - \underline{e}_I),$$

joining the two vertices adjacent to v in direction I. These come into play because of the interplay of the derivatives $\partial^+_{e_{\sigma I}}$ with the signs σ in the result for $\langle F_2(\phi)\rangle_{\Psi\text{grav}}$. We now compute the graph averages defined in (6.20), separated into leading order and first order correction:

$$\langle B_0^{(i)(i')} \rangle = \frac{1}{N} \sum_v \sum_{I,I'} \frac{\sqrt{\det P(v)}}{V_v} P_{II'}^2 \widetilde{b}_I^i \widetilde{b}_{I'}^{i'},$$
$$\langle B_1^{(i)(i')} \rangle = \frac{l_P^4}{t} \frac{1}{N} \sum_v \sum_{I,I'} \frac{\sqrt{\det P(v)}}{V_v} \left(\frac{1173}{128} \mathrm{Tr}(P^{-2}) P_{II'}^2(v) + \frac{19}{32} \delta_{II'} \right) \widetilde{b}_I^i \widetilde{b}_{I'}^{i'},$$
$$\langle B_0^{(ijk)(i')} \rangle = \frac{1}{N} \sum_v \sum_{I,I'} \frac{\sqrt{\det P(v)}}{V_v} P_{II'}^2 \widetilde{b}_I^i \widetilde{b}_I^j \widetilde{b}_I^k \widetilde{b}_{I'}^{i'},$$
$$\langle B_0^{(ij)(i'j')} \rangle = \frac{1}{N} \sum_v \sum_{I,I'} \frac{\sqrt{\det P(v)}}{V_v} P_{II'}^2 \widetilde{b}_I^i \widetilde{b}_I^j \widetilde{b}_{I'}^{i'} \widetilde{b}_{I'}^{j'}.$$

Finally we come to the kinetic term:

$$\langle A_0 \rangle = \frac{1}{N} \sum_v \frac{V_v}{\sqrt{\det P(v)}},$$
$$\langle A_1 \rangle = \frac{1707}{512} \frac{l_P^4}{t} \frac{1}{N} \sum_v \frac{\mathrm{Tr}\, P(v)}{V_v \sqrt{\det P(v)}}.$$

We assume rotation invariance and introduce the shortcuts

$$\langle B_0 \rangle \doteq \frac{1}{3} \sum_i \langle B_0^{(i)(i')} \rangle,$$
$$\langle B_1 \rangle \doteq \frac{1}{3} \sum_i \langle B_1^{(i)(i')} \rangle,$$
$$\langle B_1' \rangle \doteq \frac{1}{3} \sum_i \left(\frac{1}{3} \langle B_0^{(iii)(i)} \rangle - \frac{1}{4} \langle B_0^{(ii)(ii)} \rangle \right).$$

We can now write down the dispersion relation for the scalar field:

$$\begin{aligned} \omega^2(\vec{k}) = & K^2 \left[\langle A_0 \rangle \langle C_0 \rangle + \langle A_0 \rangle \langle C_1 \rangle + \langle A_1 \rangle \langle C_0 \rangle \right] \\ & + |k|^2 \left[\langle A_0 \rangle \langle B_1 \rangle + \langle A_1 \rangle \langle B_0 \rangle + \langle B_0 \rangle \langle A_0 \rangle \right] \\ & + |k|^4 \langle A_0 \rangle \langle B_1' \rangle + \ldots, \end{aligned}$$

where we have only displayed leading order terms and corrections of first order.

Dispersion relation for the Maxwell field.

To obtain the dispersion relation for the electromagnetic field we have to write down certain graph averages of the coefficients of the forms F_4 and F_5 obtained in the appendix.
Let us start with the electric field term F_4. Again, in order to deal with the summation over σ, σ', we introduce

$$\widetilde{s}_i^I(v) \doteq s_i^{e_I}(v) - s_i^{e_{-I}}(v), \qquad \widetilde{s}_{ij}^I(v) \doteq s_{ij}^{e_I}(v) - s_{ij}^{e_{-I}}(v), \ldots,$$

where the $s_i^e, s_{ij}^e, \ldots$ were defined in (6.22). Then we can make contact with the notation of chapter 6:

$$\begin{aligned}
\langle S_{(i)(i')}^{(0)} \rangle &= \frac{1}{N} \sum_v \sum_{II'} \frac{\sqrt{\det P}}{V_v} P_{II'}^{-2}(v) \widetilde{s}_i^I(v) \widetilde{s}_{i'}^{I'}(v), \\
\langle S_{(i)(i')}^{(1)} \rangle &= \frac{l_P^4}{t} \frac{1}{N} \sum_v \sum_{II'} \frac{1}{V_v} \Big(\frac{763}{512} P_{II'}^{-2} \operatorname{Tr} P^{-2} - \frac{13}{16} P_{II'}^{-4} \Big) \widetilde{s}_i^I(v) \widetilde{s}_{i'}^{I'}(v), \\
\langle S_{(i)(i'j')}^{(0)} \rangle &= \frac{1}{N} \sum_v \sum_{II'} \frac{\sqrt{\det P}}{V_v} P_{II'}^{-2}(v) \widetilde{s}_i^I(v) \widetilde{s}_{i'j'}^{I'}(v),
\end{aligned}$$

the only difference to chapter 6 being that we have separated into leading order (superscript $^{(0)}$) and correction (superscript $^{(1)}$).
Analogously we have

$$\begin{aligned}
\langle B_0^{(ij)(i'j')} \rangle &= \frac{1}{N} \sum_v \sum_{II'} \frac{\sqrt{\det P}}{P}{}^{-2}_{II'}(v) b_{\widetilde{\alpha}_I}^{ij}(v) b_{\widetilde{\alpha}_{I'}}^{i'j'}(v), \\
\langle B_1^{(ij)(i'j')} \rangle &= \frac{l_P^4}{t} \frac{1}{N} \sum_v \sum_{II'} \frac{1}{V_v} \Big(\frac{763}{512} P_{II'}^{-2} \operatorname{Tr} P^{-2} - \frac{13}{16} P_{II'}^{-4} \Big) b_{\widetilde{\alpha}_I}^{ij}(v) b_{\widetilde{\alpha}_{I'}}^{i'j'}(v), \\
\langle B_0^{(ij)(i'j'k')} \rangle &= \frac{1}{N} \sum_v \sum_{II'} \frac{\sqrt{\det P}}{V_v} P_{II'}^{-2}(v) b_{\widetilde{\alpha}_I}^{ij}(v) b_{\widetilde{\alpha}_{I'}}^{i'j'k'}(v),
\end{aligned}$$

where the $b_{\widetilde{\alpha}}^{ij}, b_{\widetilde{\alpha}}^{ijk}$ etc. were defined in (6.24). Proceeding further along the lines of chapter 6 we find

$$\begin{aligned}
c_1^{(0/1)} &= \frac{1}{3} \sum_i \langle S_{(i)(i)}^{(0/1)} \rangle, & c_3^{(0/1)} &= \frac{1}{6} \left(\sum_{ij} \langle B_{0/1}^{(ij)(ij)} \rangle - \sum_i \langle B_{0/1}^{(ii)(ii)} \rangle \right), \\
c_2 &= \frac{1}{6} \sum_{ijk} \epsilon^{ijk} \langle S_{(i)(jk)}^{(0)} \rangle, & c_5 &= \frac{1}{6} \sum_{ijk} \epsilon_{jik} \langle B_0^{(jk)(ikk)} \rangle.
\end{aligned}$$

Thus we can write the dispersion relation (6.26) for a wave positive/negative helicity as

$$\omega_\pm(\vec{k}) = |k| \sqrt{\Big(c_1^{(0)} c_3^{(0)} + c_1^{(0)} c_3^{(1)} + c_1^{(1)} c_3^{(0)} \Big) \pm (c_2 c_3^{(0)} - c_1^{(0)} c_5) k_3}.$$

Note that in the above expression we have just kept leading order and first order corrections.

Let finish this section by making a few remarks concerning units and orders of magnitude. We will consider F_2 as an example – similar considerations apply to the other terms.

The classical term corresponding to $(\langle B_0\rangle + \langle B_1\rangle)$ is $\sqrt{\det q}q^{ab}$. The latter is dimensionless, since q is. $\langle B_0\rangle$ has the structure

$$\langle B_0\rangle \sim \frac{1}{\mathrm{Vol}}\frac{P^2}{\sqrt{\det P}}bb. \tag{7.1}$$

Since $[P] = \text{meter}^2$, $[P^2/\sqrt{\det P}] = \text{meter}$. b is also a length, unit-wise, so $\langle B_0\rangle$ is indeed dimensionless. $\langle B_1\rangle$ has the structure

$$\langle B_1\rangle \sim \frac{l_P^4}{t\mathrm{Vol}}\frac{P^2\,\mathrm{Tr}\,P^{-2}-1}{\sqrt{\det P}}bb, \tag{7.2}$$

so it is again dimensionless as it should be. The structure of $\langle B_1'\rangle$ is

$$\langle B_1'\rangle \sim \frac{1}{\mathrm{Vol}}\frac{P^2}{\sqrt{\det P}}bbbb, \tag{7.3}$$

so its unit is meter^2 which is the correct one for a term proportional to $|k|^4$ in the dispersion relation. As for orders of magnitude, we remark the following. Assume $q_{ab} = O(1)$ in the chosen coordinate system. Then

$$P = O(\epsilon^2), \qquad \mathrm{Vol} = O(\epsilon^3) \quad \text{and} \quad b = O(\epsilon). \tag{7.4}$$

Using (7.1) it follows that $\langle B_0\rangle = O(1)$, so the leading order term has the right order of magnitude. As for the order of magnitude of $\langle B_1\rangle$, we use (7.2) and (7.4) to conclude that

$$\langle B_1\rangle = O\left(\frac{1}{t}\frac{l_P^4}{\epsilon^4}\right) = O\left(\left(\frac{l_P}{L}\right)^{2-4\alpha}\right) = O(t^{1-2\alpha})$$

which is very small since $\alpha < 1/2$.
Consider finally $\langle B_1'\rangle$: From (7.3) and (7.4) we see that$\langle B_1'\rangle = O(\epsilon^2)$.
As for the other terms in the dispersion relation, similar results can be seen to hold: The leading order term has same unit and order of magnitude as the corresponding classical term and the ratio of leading order to first order correction is of order $t^{1-2\alpha}$. We will discuss the implications of these results in the next section.

7.2. Discussion

The first remark that we want to make, is that in order to really discuss the implications of the results of the last section, one would have to fix a random graph prescription and actually compute the relevant graph averages. The computation will be hard to do analytically, but one could make a computer do the necessary work rather easily so this does not present a principal difficulty. The more serious issue here is that there are certainly many random graph prescriptions, all leading to different graph averages and hence different predictions, and it is hard to see how one should single out the "right" one. We note however that the different graph averages showing up in the dispersion relations will be related in a not too pathological random graph prescription. For example, a good guess would be that

$$\langle\sqrt{\det P}\rangle \approx \left(\langle\frac{1}{\sqrt{\det P}}\rangle\right)^{-1}$$

and that their difference would not depend very strongly on the chosen prescription. Thus there will be approximate relations between the different coefficients in the dispersion relations which are

not affected by the choice of a specific prescription.
Moreover we note that even the leading order terms in the coefficients depend on the random graph prescription. This might at first seem to be a problem as well, since it means that we will have to tune the random graph prescription in such a way that the leading order terms assume their classical values. On the other hand, this might be a blessing: Fixing the leading order term means to fix one "moment" of the distribution associated to the random graph prescription. Via the relations conjectured above, this will also approximately fix other moments, independently of the specific distribution assumed, and thereby maybe the higher order corrections.
Investigations in this direction are worthwhile but beyond the scope of the present work. Let us for the rest of this section assume that a prescription is fixed and the graph averages have been computed.

Next we observe that two different sorts of corrections appear in the dispersion relations: The first sort of correction is simply a correction to the leading order term. Its relative magnitude was found to be $t^{1-2\alpha}$. We will call this sort of correction a *fluctuation correction*.
The other sort of correction is a term containing a higher power of $|k|$ as compared to the standard dispersion relation. We will call this kind of correction a *lattice correction*. We have demonstrated for the example of $\langle B_1' \rangle$ that the terms proportional to $|k|^4$ are of the order ϵ^2, therefore the relative magnitude of the lattice corrections is of the order

$$O\left(\frac{\epsilon^2}{\lambda^2}\right) = \frac{L^2}{\lambda^2} O\left(t^\alpha\right).$$

Similarly the terms proportional to $|k|^3$ in the dispersion relation for the electromagnetic field are of the order $t^\alpha L/\lambda$.

When comparing our results for the electromagnetic field with the ones of [26, 27] we find the following: The result of Pullin and Gambini [26] does not contain any fluctuation corrections. This is however not result of the calculation but rather assumed from the beginning. As for the lattice corrections, they find a chiral modification to the dispersion relation as we do here. The relative magnitude of the correction is however l_P/λ.
Alfaro et al. [27] also do not have fluctuation corrections by assumption. They find the helicity dependent correction of [26] and the present work, again of the order l_P/λ. They also get higher order corrections the precise structure of which depends on a parameter which is not fixed.
Thus our results agree with that of [26, 27] as far as the structure of the dispersion relation is concerned. We additionally have fluctuation corrections and, perhaps most importantly, the corrections found do not scale with an integer power of l_P, counter to their finding.

Finally we should make a few remarks concerning a possible detection of the corrections in experiments. The fluctuation corrections will not show up in an experiment testing for a frequency dependence of the velocity c of light, since they merely correspond to a frequency independent shift of c. Also, these corrections are certainly not measurable by measuring the flight-time of photons since their velocity would already be the "bare" leading order term plus the fluctuation correction. Fluctuation corrections may however be measurable by comparing flight-times of photons in different geometries, since the corrections will change when the calculations presented in this chapter are repeated with LQC approximating a non-flat spacetime. To discuss how this could be done in practice is however beyond the scope of the present work.
Whether the lattice corrections are big enough to be detectable in the data from current or planned γ-ray burst observations crucially depends on the values of α and L. For the value $\alpha = 1/3$ which renders fluctuation and lattice corrections equal in magnitude, and L of the order of a γ-ray

wavelength, a rough estimate shows that the lattice corrections would indeed be detectable in the foreseeable future.

So, to conclude this chapter, we should repeat that no deep significance should be attached to the coefficients in the dispersion relations obtained: Too many ambiguities are present in the LQC, the quantization of the Hamiltonians, in the procedure to obtain the dispersion relations from the expectation values and, as a consequence, in the coefficients themselves. Also the replacement $SU(2) \to U(1)$ will certainly affect the precise numerical outcome. The orders of magnitude t^{α}, $t^{1-2\alpha}$ of the two sorts of corrections are rather robust, however, and the approximate relations between the different graph averages conjectured above might make the predictions of a more complete calculation much less dependent on the random graph prescription chosen, then one might at first fear.

8. Discussion and a lot of questions

In this final chapter, we want to gather the results obtained in the present work as well as their shortcomings, and list problems for future research.

The basic assumption of the thesis was that the complicated dynamics of a full theory of LQG coupled to matter fields could be simplified by using kinematical semiclassical states for the gravity part of the theory and treating the matter parts in the Hamilton constraint of the full theory as Hamiltonians generating the matter dynamics. This amounts to approximating the dynamics of the gravitational field coming from its self interaction and neglecting the back-reaction of the matter fields on the gravitational field completely.
Using this assumption we have obtained the following results:

1. We have proposed a quantum theory of the scalar field coupled to LQG. In this theory, the scalar field is represented by operators on a Fock space and the dynamics is generated by a Hamiltonian just as in ordinary QFT. In other respects, the theory is very different from ordinary QFT, thus reflecting basic properties of loop quantum gravity:

   - The basic excitations of the gravitational field in LQG live on graphs. The requirement of diffeomorphism invariance forces the matter degrees of freedom to be confined to the same graph as the gravitational field. The matter fields are therefore bound to become quantum fields propagating on a discrete structure.
   - In ordinary QFT, the background metric enters the definition of the ground state and the commutation relations of the fields. In LQG on the other hand, the geometry is a dynamical variable, represented by suitable operators A QFT coupled to LQG will therefore have to contain these operators in its very definition. This is reflected in the theory presented in this thesis by the fact that the Hilbert space is not the Fock space over the one particle space of the scalar field but over its tensor product with the kinematical Hilbert space of LQG.

   We also discussed how a “QFT on curved space-time limit” can be obtained from this theory, using a semiclassical state of the gravitational field.

2. We have discussed how modified dispersion relations for the matter fields arise in the context of LQG and motivated a method for computing them from the (partial) expectation values of the quantum matter Hamiltonians in a semiclassical state.
   To shed some light on the issues associated with that method, we have begun the investigation of simple model systems in which the method may be tested and, possibly, refined.

3. We have demonstrated the use of LQC for the computation of dispersion relations for matter fields coupled to LQG. We obtained expressions which just depend on a macroscopic length scale L and on certain graph averages which have to be calculated for specific random graph prescriptions.
   Unlike the specific expressions for the coefficients in the dispersion relations, the order of

magnitude estimate of the resulting corrections is rather robust: It will apply to any graph based semiclassical that is a product of states on the edges in which the fluctuation of the momentum degrees of freedom is inversely proportional to that of the configuration degrees of freedom. The only scale that enters these estimates is the macroscopic scale L.
The dispersion relation that we find for the electromagnetic field has the same chiral correction term that shows up in the results of [26, 27]. The main difference to the cited works is that the order of magnitude does not scale as an integer power in the Planck length in our calculation.

But we also have to say that these results are by no means completely satisfying: The matter-gravity quantum theory is not mathematically rigorous but rather formal. Furthermore it is extremely complicated such that concrete calculations are out of reach.
The procedure presented for obtaining the dispersion relations lacks a mathematical proof and the investigations of the issues in simple model system have only been started.
Finally, in the computation of the dispersion relations, ambiguities in the quantization of the Hamiltonians, the definition of the semiclassical states and the uncertainties about the $\mathrm{SU}(2) \to \mathrm{U}(1)^3$ replacement and the method used to obtain the dispersion relations add and render the concrete expressions for the coefficients highly untrustworthy.
For these reasons we deem it much more important than the results itselve, that we have tried to state our assumptions, methods and intuitions clearly, thus making them available to criticism and revision. It is therefore very much in the spirit of this work if we close it with a list of questions that require further investigation. We will start with problems that are rather closely related to the material presented in the thesis and end with more general questions about the dynamics of quantum gravity coupled to matter.

- Can the method used to obtain the dispersion relations from the expectation values of the gravitational operators be made rigorous? How does it have to be refined?
- How big are the ambiguities in the definition of semiclassical states, and how do they affect the results? What are the random graph prescriptions to be used? Are there semiclassical states fundamentally different from the LQC used in this work?
- How strong are the back-reaction effects of the matter on the gravitational field that we have neglected? What kind of correction to the dispersion relation do they yield?
- What about other observable effects like “distance fuzziness”? Are they present in LQG? How could they be computed?
- Is there a better (i.e. more fundamental) way to couple gravity and matter fields then to take the classical Hamiltonians, regulate and quantize (whether as constraints or as Hamiltonians)? Or, posed in a more pronounced way: Are there expressions, much simpler or at least “more natural” than the Hamiltonian operators obtained in chapter 5, that nevertheless yield the classical Hamiltonians in some “low-energy classical limit”?
- How big are the errors that we make by using kinematical semiclassical states for the gravitational field instead of states from a (yet to be defined) dynamical Hilbert space? How can the dynamics of the gravitational field be taken into account more directly? Will fundamentally new effects arise?

One should be able to make progress on the questions at the beginning of the list rather easily, answers to them will however be of limited use as long as the big problems towards the end of the list have not been addressed. The latter on the other hand are very difficult and will require new methods and most likely a thorough revision of the ideas about quantum gravity that have been gathered so far. In any case we are very curious about the answers that will be given in the future!

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Finally I thank whatever it is that paints the Golm skies and clouds, in forms and colors you have never seen before, at times with whirling leafs and water and in all shades of gray, every day anew. Sometimes that was quite life saving.

A. Coherent state expectation values

The purpose of this chapter is to present the calculation of the expectation values of the operator valued forms $\widehat{F}_1, \ldots, \widehat{F}_5$ in the coherent states for loop quantum gravity of [21, 22, 23]. In the first section we will explain the simplifying assumptions used for the computation and introduce the neccessary notation. Section A.2 is devoted to the computation of the expectation values of $\widehat{Q}_I(v, e, r)$, $\widehat{V}_v$, and in section A.3 the results are used to give the expectation values of $\widehat{F}_1, \ldots, \widehat{F}_5$.

A.1. Implementation of the simplifying assumptions

The cubic lattice:

The first simplification that we will make concerns the random graphs: In the following we will exclusively work with states based on graphs of cubic topology. This simplifies both the notation and the c-number coefficients in $\widehat{F}_1, \ldots, \widehat{F}_5$.
A random cubic graph has been depicted in in figure 2.1: Each vertex is six-valent with three edges ingoing and three outgoing. We denote the outgoing edges by e_I, $I = 1, 2, 3$ and choose an ordering, such that the tangents of e_1, e_2, e_3 form a right handed triple wrt. the given orientation of Σ. The vertices can be labeled by elements n of $\mathbb{Z}^3$. We denote by b_I the three basis vectors in the $\mathbb{Z}^3$ lattice and write $e_I^+(n) := e_I(n)$, $e_I^-(n) := e_I(n - b_I)$.
As for the c-number coefficients in $\widehat{F}_1, \ldots, \widehat{F}_5$, since the graph topologically looks the same in a neighbourhood of any vertex, we can choose the charts of section 5.1.4 such that the edges and the respective dual surfaces are always given as the image of the canonical $\vec{e}_I(t) = t\vec{e}_I$, $S_I = \{\vec{x}|x_I = 1/2,\ x_J, x_k \in [-1/2, 1/2]\}$ of euclidean space, under the charts. Doing so results in having the coefficients

$$\mu_L(e_I(n)) = \delta_{LI} \qquad \omega^I(e_J(n)) = 4\delta^I_J$$
$$\nu(v) = 1 \qquad \mu(v) = 8$$

independent of the vertices. Next notice that for each vertex v there are twelve minimal loops based at v, namely the obvious plaquette loops. We may label them as $\alpha_I(n) := e_J(n) \circ e_K(n+b_J) \circ e_J(n+b_K)^{-1} \circ e_K(n)^{-1}$, $\epsilon_{IJK} = 1$ and the ones based at a given vertex n are $\alpha_I(n), \alpha_I(n - b_J), \alpha_I(n - b_J - b_K), \alpha_I(n - b_K)$ making use of the fact that the theory is Abelian so that the starting point of the loop is actually irrelevant.
$\rho_I(\alpha_J(n)) = \delta_{IJ}$ and $\epsilon_{\alpha_I(n), \alpha_J(n'), \alpha_K(n'')} = \epsilon_{IJK}$ where n, n', n'' are such that the three loops are based at v. It follows that $\rho(\alpha, \alpha', \alpha'') = |\epsilon_{\alpha,\alpha',\alpha''}|$. Finally, there are 4^3 triples of loops such that $\rho(\alpha, \alpha', \alpha'') = 1$ implying that $\rho(v) = 4^3$ and there are 4^2 pair of loops α', α'' such that $\epsilon_{\alpha,\alpha',\alpha''} \neq 0$ for given α implying that $\xi_I(\alpha_J(n)) = 4^2\delta_{IJ}$.
Furthermore, using the above results one finds that

$$\sum_\alpha \xi^I(\alpha) B_\alpha = 8 A_{\tilde{\alpha}_I}$$

where $\tilde{\alpha}_I(v)$ is the loop in the I plane depicted in figure 2.2, and $A_{\tilde{\alpha}_I}$ is the line integral of the connection around that loop.
Finally we note that, as a consequence of the values of the coefficients computed above, the volume operator (5.7) becomes

$$\widehat{V}_{\gamma,n} = l_P^3 \sqrt{|\epsilon^{jkl}[\frac{X_j^{e_1^+(n)} - X_j^{(e_1^-(n))^{-1}}}{2}]\,[\frac{X_k^{e_2^+(n)} - X_k^{(e_2^-(n))^{-1}}}{2}]\,[\frac{X_l^{e_3^+(n)} - X_l^{(e_3^-(n))^{-1}}}{2}]|}.$$

Replacing $\mathrm{SU}(2)$ **by** $\mathrm{U}(1)^3$**:**

As already mentioned, for technical reasons we will assume that the outcomes of our calculation are not qualitatively affected when replacing $\mathrm{SU}(2)$ by $\mathrm{U}(1)^3$. Consequently we will replace $\widehat{Q}$ as well as the volume operator itselve by appropriate $\mathrm{U}(1)^3$ counterparts. For $\mathrm{U}(1)^3$ each edge is not labelled by a single, non-negative, half-integral spin degree of freedom but rather by three integers $n_j \in \mathbb{Z},\ j = 1, 2, 3$ and we have three kinds of holonomies h_e^j. The generators τ_j of $\mathrm{U}(1)^3$ are simply i (imaginary unit). The canonical commutation relations on $L^2(\mathrm{U}(1)^3, d^3\mu_H)$ are replaced by

$$\begin{aligned}
[\widehat{h}^j, \widehat{h}^k] &= 0 \\
[\widehat{P}_j, \widehat{h}^k] &= i l_P^2 \delta_j^k \widehat{h}^j \\
[\widehat{P}_j, \widehat{P}_k] &= 0
\end{aligned}$$

(cf. (4.11)) with adjointness relations $(\widehat{h}^j)^\dagger = (\widehat{h}^j)^{-1}$, $(\widehat{P}_j)^\dagger = \widehat{P}_j$. It follows that (5.16) gets replaced by

$$\widehat{Q}_I(v, e, \alpha) = 2i\hbar \operatorname{Tr}\left[\tau_I h_e^{-1}\left[h_e, \widehat{V}_v^\alpha\right]\right],$$

which is easily seen to be essentially self-adjoint. The factor of 2 is due to the normalization $\mathrm{Tr}(\tau_j, \tau_k) = -2\delta_{jk}$. Finally (A.1) is replaced by

$$\widehat{V}_{\gamma,n} = l_p^3 \sqrt{|\epsilon^{jkl}[\frac{\widehat{P}_j^{e_1^+(n)} - \widehat{P}_j^{(e_1^-(n))^{-1}}}{2}]\,[\frac{\widehat{P}_k^{e_2^+(n)} - \widehat{P}_k^{(e_2^-(n))^{-1}}}{2}]\,[\frac{\widehat{P}_l^{e_3^+(n)} - \widehat{P}_l^{(e_3^-(n))^{-1}}}{2}]|}$$

with $\widehat{P}_j^e = i l_P^2 (ih\partial/\partial h^j)$.

The $\mathrm{U}(1)^3$ coherent states over any graph γ are given by (see [22])

$$\psi_{\gamma,m}^t = \otimes_{e \in E(\gamma)} \otimes_{j=1}^3 \psi_{g_e^j(m)}^t$$

where

$$\psi_g^t = \sum_{n \in \mathbb{Z}} e^{-tn^2/2} (gh^{-1})^n$$

and $g_e^j(m) = e^{t/l_P^2 P_j^e(m)} h_e^j(m) \in \mathbb{C} - \{0\} = \mathrm{U}(1)^{\mathbb{C}}$. Here m is a point in the gravitational phase space and

$$\begin{aligned}
h_e^j(m) &\doteq \mathcal{P}\exp(i \int_e A^j) \\
P_j^e(m) &\doteq \int_{S_e} (*E)_j
\end{aligned}$$

that is, due to the Abelian nature of our simplified gauge group the path system in S_e is no longer needed.

As is obvious from the explicit form of the $\widehat{F}_1, \ldots, \widehat{F}_5$, our calculation can be done vertex by vertex since there is no inter-gravitational interaction between the associated operators. We can therefore concentrate on a single vertex for the remainder of this section and drop the label v or n in what follows.

For the sake of the computation to follow, we will introduce the shorthands

$$h_{J\sigma j} \doteq (h^j_{e^\sigma_J(n)})^\sigma, \qquad p_{J\sigma j} \doteq \frac{t}{a^2} P_j^{(e^\sigma_J(n))^\sigma},$$
$$g_{J\sigma j} \doteq e^{p_{J\sigma j}} h_{J\sigma j}, \qquad \widehat{\mho} \doteq \frac{1}{l_P^3}\widehat{V}$$

and similarly the operators $\widehat{p}_{J\sigma j}$ corresponding to $p_{J\sigma j}$. The parameter $a \doteq \sqrt{t} l_p$ was introduced to render the p dimensionless. The advantage is that we can more easily discuss orders of magnitude in the computations below. In the same spirit set

$$\widehat{q}_{J\sigma j}(r) \doteq 2i\widehat{h}_{J\sigma j}\frac{[\widehat{h}^{-1}_{J\sigma j}, \widehat{\mho}^r]}{it}.$$

Note that

$$\widehat{\mho} = \sqrt{|\epsilon^{jkl}[\frac{\widehat{p}_{1,+,j} - \widehat{p}_{1,-,j}}{2}]\,[\frac{\widehat{p}_{2,+,k} - \widehat{p}_{2,-,k}}{2}]\,[\frac{\widehat{p}_{3,+,l} - \widehat{p}_{3,-,l}}{2}]|}.$$

The huge advantage of $\mathrm{U}(1)^3$ over SU(2) is that the "spin-network functions"

$$T_{\{n_{J\sigma j}\}}(\{h_{J\sigma j}\}) = \prod_{J\sigma j} h_{J\sigma j}^{-n_{J\sigma j}}$$

are simultaneous eigenfunctions of all the $\widehat{p}_{J\sigma j}$ with respective eigenvalue $itn_{J\sigma j}$. Even better, the operator $\widehat{q}_{J_0\sigma_0 j_0}(r)$ is also diagonal with eigenvalue

$$\lambda^r_{J_0\sigma_0 j_0}(\{n_{J\sigma j}\}) = 2\frac{\lambda^r(\{n_{J\sigma j}\}) - \lambda^r(\{n_{J\sigma j} + \delta_{(J_0\sigma_0 j_0),(J\sigma j)}\})}{t}$$

where

$$\lambda^r(\{n_{J\sigma j}\}) = t^{3r/2}\left(\sqrt{|\epsilon^{jkl}[\frac{n_{1,+,j} - n_{1,-,j}}{2}]\,[\frac{n_{2,+,k} - n_{2,-,k}}{2}]\,[\frac{n_{3,+,l} - n_{3,-,l}}{2}]|}\right)^r$$

A.2. The expectation values of $\widehat{q}$

Now we will explicitly calculate the expectation values of the operator $\widehat{q}$ and $\widehat{\mho}$. The quadratic forms $\widehat{F}_1, \ldots \widehat{F}_5$ are all sums over these operators which act only on the edges of a specific vertex, therefore we can restrict consideration to a single vertex and consequently to a part

$$\psi^t_{\{g_{J\sigma j}\}}(\{h_{J\sigma j}\}) \doteq \prod_{J\sigma j} \psi^t_{g_{J\sigma j}}(h_{J\sigma j})$$

of the coherent state which just contains the factors corresponding to the edges of a single vertex. What we are looking for is the expectation value of an arbitrary polynomial of the $\widehat{q}$:

$$\begin{aligned}\langle\,\cdot\,\rangle &\doteq \frac{\langle\psi^t_{\{g_{J\sigma j}\}}, \prod_{k=1}^N \widehat{q}_{J_k\sigma_k j_k}(r_k)\psi^t_{\{g_{J\sigma j}\}}\rangle}{||\psi^t_{\{g_{J\sigma j}\}}||^2} \\ &= \frac{\sum_{\{n_{J\sigma j}\}} e^{-t\sum_{J,\sigma,j} n^2_{J\sigma j}} e^{2\sum_{J\sigma j} p_{J\sigma j} n_{J\sigma j}} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{n_{J\sigma j}\})}{\prod_{J,\sigma,j} ||\psi^t_{g_{J\sigma j}}||^2}\end{aligned} \tag{A.1}$$

where (see [22])

$$||\psi^t_g||^2 = \sqrt{\frac{\pi}{t}} e^{p^2/t}[1+K_t(p)],\ g = e^p e^{i\varphi},\ |K_t(p)| \le K_t = O(t^\infty). \tag{A.2}$$

As in [22], in order to extract useful information out of the formula (A.1) it is of outmost importance to perform a Poisson transformation on it because we are interested in tiny values of t for which (A.1) converges rather slowly while the transformed series converges rapidly since then t gets replaced by $1/t$. To that end, let us introduce $T \doteq \sqrt{t}, x_{J\sigma j} \doteq T n_{J\sigma j}$, whereupon

$$\langle\,\cdot\,\rangle = \frac{\sum_{\{x_{J\sigma j}\}} e^{-\sum_{J,\sigma,j} x^2_{J\sigma j}} e^{2\sum_{J\sigma j} x_{J\sigma j} p_{J\sigma j}/T} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{J\sigma j}\})}{\prod_{J,\sigma,j} ||\psi^t_{g_{J\sigma j}}||^2} \tag{A.3}$$

where

$$\begin{aligned}\lambda^r_{J_0\sigma_0 j_0}(\{x_{J\sigma j}\}) &= 2\frac{\lambda^r(\{x_{J\sigma j}\}) - \lambda^r(\{x_{J\sigma j} + T\delta_{(J_0\sigma_0 j_0),(J\sigma j)}\})}{t} \\ \lambda^r(\{x_{J\sigma j}\}) &= t^{3r/4}\sqrt{|\epsilon^{jkl}[\frac{x_{1,+,j} - x_{1,-,j}}{2}]\,[\frac{x_{2,+,k} - x_{2,-,k}}{2}]\,[\frac{x_{3,+,l} - x_{3,-,l}}{2}]|}^{\,r}\end{aligned} \tag{A.4}$$

Then Poisson's theorem gives

$$\langle\,\cdot\,\rangle = \frac{\frac{1}{T^{18}}\sum_{\{n_{J\sigma j}\}}\int_{\mathbb{R}^{18}} d^{18}x e^{\sum_{J,\sigma,j}[-x^2_{J\sigma j} + 2x_{J\sigma j}(p_{J\sigma j} - i\pi n_{J\sigma j})/T]} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{J\sigma j}\})}{\prod_{J,\sigma,j} ||\psi^t_{g_{J\sigma j}}||^2} \tag{A.5}$$

An observation that reduces the eighteen dimensional integral to a nine dimensional one is that the integrand in (A.5) only depends on $x_{Jj} \doteq x^-_{Jj} \doteq [x_{J,+,j} - x_{J,-,j}]/2$ and not on $x^+_{Jj} \doteq [x_{J,+,j} + x_{J,-,j}]/2$. Consider also the analogous quantities $p^\pm_{Jj} \doteq [p_{J,+,j} \pm p_{J,-,j}]/2$, $n^\pm_{Jj} \doteq [n_{J,+,j} \pm n_{J,-,j}]/2$ and let $p_{Jm} \doteq p^-_{Jj}$, $n_{Jm} \doteq p^-_{Jj}$. Switching to the coordinates $x^\pm_{Jj}$, noticing that $|\det(\partial\{x_{J\sigma j}\}/\partial\{x^+_{Jj}, x^-_{Jj}\}| = 2^9$ we obtain

$$\begin{aligned}\langle\,\cdot\,\rangle = &\frac{(\frac{2}{t})^9 \sum_{\{n_{J\sigma j}\}}[\int_{\mathbb{R}^9} d^9x^+ e^{2\sum_{Jj}[-(x^+_{Jj})^2 + 2x^+_{Jj}(p^+_{Jj} - i\pi n^+_{Jj})/T]}]}{\prod_{J,\sigma,j} ||\psi^t_{g_{J\sigma j}}||^2} \times \\ &\times \left[\int_{\mathbb{R}^9} d^9x e^{2\sum_{Jj}[-x^2_{Jj} + 2x_{Jj}(p_{Jj} - i\pi n_{Jj})/T]} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{Jj}\})\right]\end{aligned} \tag{A.6}$$

where

$$\begin{aligned}\lambda^r_{J_0\sigma_0 j_0}(\{x_{Jj}\}) =& 2\frac{\lambda^r(\{x_{Jj}\}) - \lambda^r(\{x_{Jj} + T\delta_{(J_0 j_0),(Jj)}/2\})}{t} =: \lambda^r_{J_0 j_0}(\{x_{Jj}\}) \\ \lambda^r(\{x_{Jj}\}) =& t^{3r/4}\,(|\det(\{x_{Jj}\})^{r/2}\end{aligned} \tag{A.7}$$

actually *no longer depends on* σ_0! The integral over x^+_{Jj} in (A.7) can be immediately performed by using a contour argument with the result

$$\langle \cdot \rangle = \frac{(\frac{\sqrt{2\pi}}{t})^9 \sum_{\{n_{J\sigma j}\}} e^{\frac{2}{t}\sum_{Jj}(p^+_{Jj}-in^+_{Jj})^2}[\int_{\mathbb{R}^9} d^9x e^{2\sum_{Jj}[-x^2_{Jj}+2x_{Jj}(p_{Jj}-i\pi n_{Jj})/T]} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{Jj}\})]}{\prod_{J,\sigma,j} ||\psi^t_{g_{J\sigma j}}||^2} \tag{A.8}$$

Finally, using (A.2) we can further simplify to

$$\langle \cdot \rangle = \frac{\sqrt{\frac{2}{\pi}}^9}{[(1-K_t)^{18},(1+K_t)^{18}]} \sum_{\{n_{J\sigma j}\}} e^{\frac{2}{t}\sum_{Jj}[(p^+_{Jj}-i\pi n^+_{Jj})^2-(p^+)^2_{Jj}-p^2_{Jj}]} \times$$
$$\times \int_{\mathbb{R}^9} d^9x e^{2\sum_{Jj}[-x^2_{Jj}+2x_{Jj}(p_{Jj}-i\pi n_{Jj})/T]} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{Jj}\}) \tag{A.9}$$

where the notation for the denominator means that its value ranges at most in the interval indicated. Its precise value will be irrelevant for what follows since its departure from unity is $O(\infty)$.

Only the $n_{J\sigma,j}=0$ terms matter:

The remaining integral in (A.9) cannot be computed in closed form so that we must confine ourselves to a judicious estimate. We wish to show that the only term in the infinite sum of (A.9) which contributes corrections to the classical result of finite order in t is the one with $n_{J\sigma j}=0$ for all $J,\sigma j$. In order to do that, we must demonstrate that all the other terms can be estimated in such a way that the series of their estimates converges to an $O(t^\infty)$ number. This would be easy if we could complete the square in the exponent of the integrand but since for $r/2$ not being an even positive integer the function λ^r is not analytic in $\mathbb{C}^9$ we cannot immediately use a contour argument in order to estimate the remaining integral. In order to proceed and to complete the square anyway we expand the product $\prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{Jj}\})$ into monomials of the form $\prod_{k=1}^N \frac{\lambda^r(\{x_{Jj}+c^k_{Jj}\})}{t}$ with $c^k_{Jj}=T\delta_{J_kj_k,Jj}/2$ or $c^k_{Jj}=0$ and estimate the integrals over the latter. We trivially have

$$\lambda^r(\{x_{Jj}+c^k_{Jj}\}) = t^{3r/4}([\det(\{x_{Jj}+c^k_{Jj}\})]^2)^{r/4} = t^{3r/4}\exp\left(\frac{r}{4}\ln([\det(\{x_{Jj}+c^k_{Jj}\})]^2)\right) \tag{A.10}$$

where we must use the branch of the logarithm with $\ln(z)=\ln(|z|)+i\varphi$ for any complex number $z=|z|e^{i\varphi}$ with $\varphi\in[0,2\pi)$. With this branch understood, in the form (A.10) the integrand of (A.9) becomes univalent on the entire complex manifold $\mathbb{C}^9$ except at the points where $\det(\{x_{Jj}+c^k_{Jj}\})=0$. Now a labourious contour argument can be given tho the extent that we can move the path of integration away from the real hyperplane in $\mathbb{C}^9$ without changing the result. Therefore we can indeed complete the square in the exponent.

It remains to estimate (A.9) from above. Isolating the term with $n_{J\sigma,j} = 0$ for all J, σ, j we have

$$\left| \langle \cdot \rangle - \frac{\sqrt{\frac{2}{\pi}}^9}{[(1-K_t)^{18}, (1+K_t)^{18}]} \int_{\mathbb{R}^9} d^9x e^{-2\sum_{Jj} x_{Jj}^2} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{Jj} + p_{Jj}/T\}) \right|$$
$$= |\frac{\sqrt{\frac{2}{\pi}}^9}{[(1-K_t)^{18}, (1+K_t)^{18}]} \sum_{\{n_{J\sigma j}\}\neq\{0\}} e^{\frac{2}{t}\sum_{Jj}[(p^+_{Jj} - i\pi n^+_{Jj})^2 + (p_{Jj} - i\pi n_{Jj})^2 - (p^+)^2_{Jj} - p^2_{Jj}]} \times$$
$$\times \int_{\mathbb{R}^9} d^9x e^{-2\sum_{Jj} x_{Jj}^2} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{Jj} + (p_{Jj} - i\pi n_{Jj})/T\})$$
$$\leq (\frac{2}{t})^N \left| \frac{\sqrt{\frac{2}{\pi}}^9}{(1-K_t)^{18}} \sum_{\{n_{J\sigma j}\}\neq\{0\}} e^{-\frac{\pi^2}{t}\sum_{J\sigma j} n^2_{J\sigma j}} \int_{\mathbb{R}^9} d^9x e^{-2\sum_{Jj} x^2_{Jj}} \times \right.$$
$$\left. \times \prod_{k=1}^N [e^{\frac{r}{2}\ln(|\det(\{Tx_{Jj} + (p_{Jj} - i\pi n_{Jj})\})|)} + e^{\frac{r}{2}\ln(|\det(\{Tx_{Jj} + t\delta_{(Jj),(J_kj_k)} + (p_{Jj} - i\pi n_{Jj})\})|)}] \right| \tag{A.11}$$

Let w_{Jj} be a matrix of complex numbers and define the norm $\|w\|^2 \doteq \sum_{Jj} |w_{Jj}|^2$ so that in particular $||w_1 + w_2|| \leq ||w_1|| + ||w_2||$ and $|w_{Jj}| \leq \|w\|$ for all J, j. Now $\det(\{w_{Jj}\})$ is a linear combination of six monomials of the form $w_{J_1j_1}w_{J_2j_2}w_{J_3j_3}$ so that $|\det(\{w_{Jj}\})| \leq 6\|w\|^3$. In particular, $|\det(\{Tx_{Jj} + (p_{Jj} - i\pi n_{Jj})\})| \leq 6(T\|x\| + \|p\| + \pi\|n\|)^3$ and $|\det(\{Tx_{Jj} + t\delta_{(Jj),(J_kj_k)}/2 + (p_{Jj} - i\pi n_{Jj})\})| \leq 6(T\|x\| + t + \|p\| + \pi\|n\|)^3$. Invoking this result into (A.11) we find

$$\leq (\frac{4}{t})^N \left| \frac{\sqrt{\frac{2}{\pi}}^9}{(1-K_t)^{18}} \sum_{\{n_{J\sigma j}\}\neq\{0\}} e^{-\frac{\pi^2}{t}\sum_{J\sigma j} n^2_{J\sigma j}} \int_{\mathbb{R}^9} d^9x e^{-2\|x\|^2} e^{\frac{Nr}{2}\ln(6[T\|x\| + t + \|p\| + \pi\|n\|]^3)} \right|$$
$$\leq (\frac{4\,6^{r/2}}{t})^N \left| \frac{\sqrt{\frac{2}{\pi}}^9}{(1-K_t)^{18}} \sum_{\{n_{J\sigma j}\}\neq\{0\}} e^{-\frac{\pi^2}{t}\sum_{J\sigma j} n^2_{J\sigma j}} \times \right.$$
$$\left. \times \int_{\mathbb{R}^9} d^9x e^{-2\|x\|^2} [\frac{1}{4} + t\|x\|^2 + t + \|p\| + \pi\|n\|]^{[\frac{3Nr}{2}]+1} \right| \tag{A.12}$$

where $[3Nr/2]$ is the Gauss bracket of a real number (largest integer smaller than or equal to $3Nr/2$) and in the last step we have used the elementary estimate $x \leq x^2 + 1/4$ valid for any real number x. The integral in the last line of (A.12) can be evaluated exactly by invoking the binomial theorem. Consider the integrals of the form

$$I_k \doteq \sqrt{\frac{2}{\pi}}^m \int_{\mathbb{R}^m} d^m x e^{-2\|x\|^2} \|x\|^{2k} \tag{A.13}$$

for any positive integer m. Switching to polar coordinates one easily proves the recursion formula

$$I_k = \frac{m + 2(k-1)}{4} I_{k-1} \tag{A.14}$$

and since $I_0 = 1$ we find

$$I_k = \frac{(\frac{m}{2}+k-1)!}{2^k(\frac{m}{2})!} \text{ if } m \text{ even}$$
$$I_k = \frac{(m-1+2k)!\,(\frac{m-1}{2})!}{8^k(m-1)!(\frac{m-1}{2}+k)!} \text{ if } m \text{ odd} \tag{A.15}$$

Using the elementary estimate $e(n/e)^n \le n! \le e((n+1)/e)^{n+1}$ we find for $0 \le k \le n$ and $n \ge 2$ that

$$I_k \le e(\frac{m+2n}{2e})^{m/2}(\frac{m+2n}{4e})^k \doteq C_{m,n}(\frac{m+2n}{4e})^k \text{ if } m \text{ even}$$
$$I_k \le \frac{m-1}{2e}\frac{(\frac{m-1}{2})!}{(m-1)!}(\frac{m+2n}{m-1})^m(\frac{m+2n}{4(m-1)})^k =: C_{m,n}(\frac{m+2n}{4(m-1)})^k \text{ if } m \text{ odd} \tag{A.16}$$

In our case $m = 9$ and $n = [\frac{3Nr}{2}]+1$. Thus, we can finish the estimate of (A.12) with

$$| < . > - \frac{\sqrt{\frac{2}{\pi}}^9}{[(1-K_t)^{18},(1+K_t)^{18}]}\int_{\mathbb{R}^9} d^9x e^{-2\sum_{Jj} x_{Jj}^2}\prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{Jj}+p_{Jj}/T\})|$$
$$\le \frac{(\frac{4\,6^{r/2}}{t})^N C_{9,[\frac{3Nr}{2}]+1}}{(1-K_t)^{18}} \sum_{\{n_{J\sigma j}\}\neq\{0\}} e^{-\frac{\pi^2}{t}\sum_{J\sigma j} n_{J\sigma j}^2} \times$$
$$\times [\frac{1}{4}+t\frac{9+2([\frac{3Nr}{2}]+1)}{32}+t+\|p\|+\pi\,\|n\|]^{[\frac{3Nr}{2}]+1} \tag{A.17}$$

which is obviously of order $O(t^\infty)$. We can give a bound independent of p since in our applications $\|p\|$ can be bounded by a constant of the order of t^α.

Let us summarize our findings in the form of a theorem.

Theorem A.2.1. *Let $\|p(v)\|^2 \doteq \sum_{Jj} p_{Jj}(v)^2$. Suppose that there exists a positive constant K such that $\sup_{v\in V(\gamma),m\in\mathcal{M}} \|p(v)\| =: \|p\| \le K$ is uniformly bounded. Then for small t*

$$\langle \cdot \rangle = \frac{\sqrt{\frac{2}{\pi}}^9}{[(1-K_t)^{18},(1+K_t)^{18}]}\int_{\mathbb{R}^9} d^9x e^{-2\sum_{Jj} x_{Jj}^2}\prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{Jj}+p_{Jj}/T\}) + O(t^\infty) \tag{A.18}$$

independently of $m \in \mathcal{M}, v \in V(\gamma)$.

Expansion of the remaining integral:

It remains to compute the power expansion (in T) of the remaining integral in (A.18) and to show that at each order the remainder is smaller than the given order. We will see that only even powers of T contribute so that this expansion is actually an expansion in t. The basic reason is that the expansion of the integrand in powers of T is at the same time an expansion in powers of x_{Jj} as is obvious from the explicit form of the functions $\lambda^r(\{x_{Jj}\})$. These powers of x_{Jj} are integrated against the Gaussian $e^{-2\|x\|^2}$ which is an even function under the reflection $x_{Jj} \to -x_{Jj}$ whence the integral for odd powers (an odd function under reflection) must vanish. We will not be able to show that the integral in (A.18), which certainly converges for any p_{Jj}, t (just set $\|n\| = 0$ in above estimate), can be expanded into an infinite series in powers of t, rather our estimates will be only good enough in order to show that there is a maximal order n_0 (which becomes infinite as $t \to 0$) in the sense that the remainder at order n is smaller than the given order for all $n \leq n_0$. We will use rather coarse estimates which could possibly be much improved in order to raise the value of n_0 derived here but for all practical purposes the analysis described below will be sufficient since n_0 is anyway a rather large positive integer.

Consider once more the function $\lambda^r_{J\sigma j}(x + p/T)$: Let us introduce $q \doteq pt^{-\alpha}$ which is of order unity and $s = t^{1/2-\alpha}$. Then

$$\lambda^r_{J\sigma j}(x + p/T) = 2|\det(p)|^{r/2} \frac{|\det(1 + q^{-1}xs)|^{r/2} - |\det(1 + q^{-1}xs + q^{-1}\delta_{Jj}sT/2)|^{r/2}}{t} \tag{A.19}$$

Now for any matrix A we have $\det(1+A) = 1+\mathrm{Tr}(A)+\frac{1}{2}[(\mathrm{Tr}(A))^2-\mathrm{Tr}(A^2)]+\det(A) =: 1+z'_A$ and so $\det(1+A)^2 = 1+2z'_A+(z'_A)^2 =: 1+z_A =: y_A \geq 0$. Let $y \doteq 1+z_{q^{-1}xs}$ and $y_1 \doteq 1+z_{q^{-1}[xs+\sigma\delta_{Jj}sT]}$. Then (A.19) becomes

$$\lambda^r_{J\sigma j}(x + p/T)| = \frac{2|\det(p)|^{r/2}}{t}[y^{r/4} - y_1^{r/4}] \tag{A.20}$$

and we should expand $y^{r/4}, y_1^{r/4}$ around $y = y_1 = 1$. We now invoke our knowledge that $0 < r \leq 1$ is a rational number, so we find positive integers $M > L > 0$ without common prime factor such that $r/4 = L/M$. Let us define recursively

$$\begin{aligned} f^{(0)}_{L/M}(y) &\doteq y^{L/M} \\ f^{(n+1)}_{L/M}(y) &\doteq \frac{f^{(n)}_{L/M}(y) - f^{(n)}_{L/M}(1)}{y-1} \end{aligned} \tag{A.21}$$

It follows from this definition that

$$f^{(0)}_{L/M}(y) = \sum_{k=0}^{n} f^{(k)}_{L/M}(1)[y-1]^k + f^{(n+1)}_{L/M}(y)[y-1]^{n+1} \tag{A.22}$$

Lemma A.2.2. *We have*

$$f^{(k)}_{L/M}(1) = (L/M, k) \tag{A.23}$$

where

$$(L/M, k) \doteq \frac{(L/M)(L/M-1)\ldots(L/M-k+1)}{k!} = (-1)^{k+1}\frac{L}{M}\frac{M-L}{2M}\frac{2M-L}{3M}\ldots\frac{(k-1)M-L}{kM}$$

and the following recursion holds for all $n \geq 1$

$$f^{(n+1)}_{L/M}(y) = \frac{\sum_{k=1}^{L-1} f^{(n)}_{k/M}(y) - \sum_{l=1}^{n} f^{(l)}_{L/M}(1)\sum_{k=1}^{M-1} f^{(n-l+1)}_{k/M}(y)}{\sum_{k=0}^{M-1} f^{(0)}_{k/M}(y)} \tag{A.24}$$

The proof of the lemma consists in a straightforward taylor expansion (first part) and an induction (second part) and will not be reproduced here.

The motivation for the derivation of this recursion is that it allows us to estimate $|f^{(n+1)}_{L/M}(y)|$ once we have an estimate for all the $|f^{(l)}_{k/M}(y)|$ with $0 \leq k \leq M-1, 0 \leq l \leq n$.

Lemma A.2.3. *For all* $0 < L \leq M,\ n \geq 0$ *we have*

$$|f^{(n)}_{L/M}(y)| \leq (1+y)(\beta M)^n \tag{A.25}$$

where $\beta > 1$ *is any positive number satisfying* $\beta \geq 1 + \frac{\beta}{\beta-1}$, *e.g.* $\beta = 3$.

This lemma can be proven by induction, using the results of the previous one.

Using the expansion (A.22) and the fact that y is a polynomial in the x_{Jj} it is possible evaluate the Gaussian integrals over the first n terms the last one of which is obviously at least of order s^n. We would like to know at which order n_0 the remaining term in (A.22) is no longer of order at least s^{n_0+1}.

To that end recall that $y = 1 + 2z + z^2$ where $z = \mathrm{Tr}(A) + \frac{1}{2}[(\mathrm{Tr}(A))^2 - \mathrm{Tr}(A^2)] + \det(A)$ and $A_{jk} = s\sum_J (q^{-1})_{Jj} x_{Jk}$. We now have the following basic estimates

$$\begin{aligned}
|\,\mathrm{Tr}(A)| &= s|\sum_{Jj} q^{-1}_{Jj} x_{Jj}| \leq s||q^{-1}||\ \ \|x\| \\
|(q^{-1}x)_{jk}| &= |\sum_J q^{-1}_{Jj} x_{Jk}| \leq \sqrt{\sum_J [q^{-1}_{Jj}]^2}\sqrt{\sum_J [x_{Jk}]^2} \\
|\,\mathrm{Tr}(A^2)| &= s^2|\sum_{jk} (q^{-1}x)_{jk}(q^{-1}x)_{kj}| \leq s^2|\sum_{jk} |(q^{-1}x)_{jk}|\ |(q^{-1}x)_{kj}| \\
&\leq s^2[\sum_j \sqrt{\sum_J [q^{-1}_{Jj}]^2}\sqrt{\sum_J [x_{Jj}]^2}][\sum_k \sqrt{\sum_J [q^{-1}_{Jk}]^2}\sqrt{\sum_J [x_{Jk}]^2}] \\
&\leq s^2[\sqrt{\sum_j \sqrt{\sum_J [q^{-1}_{Jj}]^2}^2}\sqrt{\sum_j \sqrt{\sum_J [x_{Jj}]^2}^2}\ ]^2 \\
&\leq s^2||q^{-1}||^2\ \ \|x\|^2 \\
|\det(A)| &\leq 6s^3||q^{-1}x||^3 \leq 6s^3||q^{-1}||^3\ \ \|x\|^3
\end{aligned}$$

where in the first line we have made use of the Cauchy-Schwarz inequality for the inner product $< x, x' >= \sum_{Jj} x_{Jj}x'_{Jj}$, in the second for the inner product $< x, x' >= \sum_J x_J x'_J$, in the fourth line for the inner product $< x, x' >= \sum_j x_j x'_j$ and finally in the last line we have used the estimate derived between equations (A.11) and (A.12). These estimates imply that

$$\begin{aligned}
|z| &\leq s\left\|q^{-1}\right\|\ \ \|x\| + s^2\left\|q^{-1}\right\|^2\ \ \|x\|^2 + 6|\det(q^{-1})|\ \ \|x\|^3 =: u(\|x\|), \\
|y-1| &\leq 2u + u^2 =: P(\|x\|)
\end{aligned}$$

and $P(\|x\|)$ is a polynomial of sixth order in $\|x\|$.

We are now ready to estimate the Gaussian integral over the remainder:

$$\begin{aligned}
E_n &\doteq \left|\sqrt{\frac{2}{\pi}}^9 \int_{\mathbb{R}^9} d^9x e^{-2\|x\|^2} f^{(n+1)}_{L/M}(y)[y-1]^{n+1}\right| \\
&\leq \sqrt{\frac{2}{\pi}}^9 (3M)^{n+1} \int_{\mathbb{R}^9} d^9x e^{-2\|x\|^2}[(P(\|x\|))^{n+2} + 2(P(\|x\|))^{n+1}]
\end{aligned} \tag{A.26}$$

Consider an arbitrary polynomial in $\|x\|$ of the form

$$P(x) = \sum_{k=0}^{l} a_k \,\|x\|^k$$

By the multinomial theorem

$$(P(x))^n = \sum_{n_0+..+n_l=n} \frac{n!}{(n_0!)..(n_l)!}[\prod_{k=0}^{l} a_k^{n_k}] \;\|x\|^{\sum_{k=0}^{l} k n_k}$$

Let us consider Gaussian integrals of the form

$$\sqrt{\frac{2}{\pi}}^m \int_{\mathbb{R}^m} d^m x e^{-2\|x\|^2} \,\|x\|^n = V_{m-1}\sqrt{\frac{2}{\pi}}^m \int_0^\infty dr e^{-2r^2} r^{n+m-1} =: V_{m-1}\sqrt{\frac{2}{\pi}}^m J_{n+m-1}$$

where $V_m = 2\pi^{m/2}/\Gamma(m/2)$ is the volume of S^m. Now

$$J_n = \frac{\sqrt{2\pi}}{4} 2^{-3n/2} \frac{n!}{\frac{n}{2}!} \text{ for } n \text{ even}$$
$$J_n = \frac{1}{4} 2^{-(n-1)/2} (\frac{n-1}{2}!) \text{ for } n \text{ odd} \tag{A.27}$$

and one immediately checks that

$$J_n \le \frac{\sqrt{2\pi}}{4} \frac{[\frac{n}{2}]!}{2^{[\frac{n}{2}]}}$$

where [.] again denotes the Gauss bracket. Using the above used estimate for the factorial $n! \le e(\frac{(n+1)}{e})^{n+1}$ we may further estimate

$$J_n \le \frac{e\sqrt{2\pi}}{4} \frac{(\frac{n+1}{2e})^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}}} = \frac{e\sqrt{2\pi}}{4} 2^{-n} (\frac{n+1}{e})^{\frac{n+1}{2}}$$

where we used $\frac{n-1}{2} \le [\frac{n}{2}] \le \frac{n}{2}$. Finally, if $n \le n_M$ then

$$J_n \le \frac{e\sqrt{2\pi}}{4} 2^{-n} (\frac{n_M+1}{e})^{\frac{n+1}{2}} \tag{A.28}$$

Combining these results we obtain the final estimate

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}^m \int_{\mathbb{R}^m} d^m x e^{-2\|x\|^2} P(x)^n = V_{m-1}\sqrt{\frac{2}{\pi}}^m \frac{e\sqrt{2\pi}}{4} \sum_{n_0+..+n_l=n} \frac{n!}{(n_0!)..(n_l)!}[\prod_{k=0}^{l} a_k^{n_k}] J_{\sum_{k=0}^{l} k n_k + m-1} \\
&\le V_{m-1}\sqrt{\frac{2}{\pi}}^m \frac{e\sqrt{2\pi}}{4} \sum_{n_0+..+n_l=n} \frac{n!}{(n_0!)..(n_l)!}[\prod_{k=0}^{l} a_k^{n_k}] 2^{-(\sum_{k=0}^{l} k n_k + m-1)} (\frac{m+ln}{e})^{\frac{\sum_{k=0}^{l} k n_k + m - 1 + 1}{2}} \\
&= V_{m-1}\sqrt{\frac{2}{\pi}}^m \frac{e\sqrt{2\pi}}{2} (\frac{m+ln}{4e})^{\frac{m}{2}} \sum_{n_0+..+n_l=n} \frac{n!}{(n_0!)..(n_l)!}[\prod_{k=0}^{l} (a_k \sqrt{\frac{m+ln}{4e}}^k)^{n_k}] \\
&= V_{m-1}\sqrt{\frac{2}{\pi}}^m \frac{e\sqrt{2\pi}}{2} (\frac{m+ln}{4e})^{\frac{m}{2}} [\sum_{k=0}^{l} a_k \sqrt{\frac{1+ln}{4e}}^k]^n \\
&=: K_{m,l} (\frac{m+ln}{4e})^{\frac{m}{2}} P(\sqrt{\frac{m+ln}{4e}})
\end{aligned} \tag{A.29}$$

since $\sum_{k=0}^{l} kn_k \leq ln = n_M - m$ for any configuration of the n_k subject to the constraint $n_0+..+n_l = n$.

In our case we have $m = 9, l = 6$ and thus we can bound the remainder (A.26) from above:

$$E_n \leq K_{9,6}(3M)^{n+1}\Big[(\frac{9+6(n+2)}{4e})^{\frac{9}{2}}(P(\sqrt{\frac{9+6(n+2)}{4e}})^{n+2} + 2(\frac{9+6(n+1)}{4e})^{\frac{9}{2}}(P(\sqrt{\frac{1+6(n+1)}{4e}})^{n+1}\Big] \tag{A.30}$$

For small n the error E_n is the number s^{n+1} times a constant of order unity. For large n, however, the error becomes comparable to the order of accuracy (in powers of s) that we are interested in itself. The value $n = n_0$ from where onwards it does not make sense any longer to compute corrections can be estimated from the condition

$$E_{n+1}/E_n \geq 1 \tag{A.31}$$

Due to the complicated structure of (A.30) the precise value of n_0 cannot be computed analytically but its order of magnitude can be obtained under the self-consistency assumption that n_0 is quite large so that the change of $P(\sqrt{(9+6(n_0+2))/(4e)})$ as we change n_0 by 1 is much smaller than its value. A tedious but straightforward estimate shows that under this assumption

$$n_0 = \frac{4e(\frac{\tau_0(M)}{s||q^{-1}||})^2 - 9}{6} - 3 \tag{A.32}$$

where $\tau_0(M)$ is of order unity. Thus n_0 is a very large number if $||q^{-1}||$ is of order unity and s is tiny. Moreover,

$$\delta P = 2(u+1)(1+2\tau+18\tau^2)\delta\tau = 6(u+1)u\delta\tau/\tau \leq 6P\frac{\delta\tau}{\tau} \tag{A.33}$$

But under the change $\delta n = 1$

$$\delta\tau \approx \frac{d\tau}{dn}\delta n = \frac{\tau}{9(9+2n)} \tag{A.34}$$

whence

$$(\frac{\delta P}{P})_{n=n_0} \leq \frac{2}{3(9+2n_0)} \ll 1 \tag{A.35}$$

as desired since n_0 is a large number.

Let us now finally go back to our desired expectation value (A.18) which we would like to compute up to some order $n < n_0$ in s. Let again $y \doteq 1+z_{q^{-1}xs} = 1+z$ and $y_{J\sigma j} \doteq 1+z_{q^{-1}[xs+\delta_{Jj}sT/2]} = 1+z_{J\sigma j}$ with $z_A = (z'_A)^2 + 2z'_A,\ z'_A = \mathrm{Tr}(A) + \frac{1}{2}[(\mathrm{Tr}(A))^2 - \mathrm{Tr}(A^2)] + \det(A)$ for any matrix A and recall our convention $r/4 = L/M$. Thus (A.20) becomes up to order n

$$\begin{aligned}\lambda^r_{J\sigma j}(x+p/T) &= \frac{2|\det(p)|^{2L/M}}{t}[y^{L/M} - y^{L/M}_{J\sigma j}] \\ &= \frac{2|\det(q)|^{2L/M}t^{6L/M\alpha}}{t}\{[(y-y_{J\sigma j})\sum_{k=1}^{n} f^{(k)}_{L/M}(1)\sum_{l=0}^{k-1}(y-1)^l(y_{J\sigma j}-1)^{k-1-l}] \\ &\quad + [f^{(n+1)}_{L/M}(y)(y-1)^{n+1} - f^{(n+1)}_{L/M}(y_{J\sigma j})(y_{J\sigma j}-1)^{n+1}]\}\end{aligned} \tag{A.36}$$

In order to compute (A.18) up to order n with respect to s we have to plug the expansions (A.36) into formula (A.18) and to collect all the contributions up to order s^n. The integral over the remainder is then still smaller as long as $n < n_0$ as shown above. In the present work we are interested only in the leading order (classical limit) and next to leading order (first quantum correction) as well as in an estimate of the error at the next to leading order.

A laborious but straightforward power counting reveals that

$$\lambda^r_{J\sigma j} = \frac{sT}{t}(1 + sx + (sx)^2 + O(sT)) \tag{A.37}$$

where the notation just means that $\lambda^r_{J\sigma j}$ is a polynomial in x_{Jj} of order two where the monoms of order $0, 1, 2$ come with a power of s of the order indicated *or higher*. We thus see that if we wish to keep only terms up to order $(sT/t)^N$ and $(sT/t)^N s^2$ in $\prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(x + p/T)$ it will be sufficient to do the following: For the term of order $(sT/t)^N$ keep only the terms proportional to x^0 in each of the factors of the form (A.37) and for term of order $(sT/t)^N s^2$ keep either 1. only the terms proportional to x^2 in one of the factors of the form (A.37) and only the terms of order x^0 in the others or 2. only the terms proportional to x^1 in two of the factors of the form (A.37) and only the terms of order x^0 in the others. Clearly terms of order $(sT/t)^N s$ do not survive since they are linear in x and integrate to zero against the Gaussian.

In estimating the error that we make notice that there are two errors, one coming from neglecting all higher orders in (A.37) and one from the remainder in the expansion (A.36). As for the first error, notice that all Gaussian integrals are of order unity so that the indicated powers of t correctly display the error (compared to $(sT/t)^N s^2$) as of higher order in s. As for the second error we can use the expansion (A.36) up to some order $n' > 2$ until $s^{n'+1} \ll sTs^2$ in view of the estimate (A.30). The minimal value of n' depends on the value of α. For instance, if $\alpha = 1/6$ as indicated by [30] then $s = t^{1/3}$ so that $s^{n'-2} = t^{(n'-2)/3} \ll T = t^{1/2}$ means $n' > 2 + 3/2$ so the minimal value would be $n' = 4$ in this case. This value is well below $n_0 \gg 1$ so that the error is indeed of higher order in s as compared to $(sT/t)^N s^2$.

With these things said we can now actually compute the first contributing correction to the classical limit. We will not bother with the higher order corrections since we just showed that they can be bounded by terms of sub-leading order as compared to $(sT/t)^N s^2$. In particular, we will replace the $O(t^\infty)$ corrections by zero in (A.18). We then have

$$\begin{aligned}
\langle\,\cdot\,\rangle = \sqrt{\frac{2}{\pi}}^9 \int_{\mathbb{R}^9} d^9x e^{-2\|x\|^2} \Big\{ &[\prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(x + p/T)_{|x^0}] \\
&+ [\sum_{l=1}^N \lambda^r_{J_l\sigma_l j_l}(x + p/T)_{|x^2}] \prod_{k\neq l} \lambda^r_{J_k\sigma_k j_k}(x + p/T)_{|x^0}] \\
&+ [\sum_{1\leq l<m\leq N}^N \lambda^r_{J_l\sigma_l j_l}(x + p/T)_{|x^1}] \lambda^r_{J_m\sigma_m j_m}(x + p/T)_{|x^1}] \prod_{k\neq l,m} \lambda^r_{J_k\sigma_k j_k}(x + p/T)_{|x^0}] \Big\}
\end{aligned} \tag{A.38}$$

$$+ O(t^{(N[3r/2-1]\alpha} sT) \tag{A.39}$$

where the restrictions mean the ones to the appropriate powers of x as derived above. It remains to explicitly compute these restrictions and to do the Gaussian integrals. According to what we have said above we write

$$\lambda^r_{J\sigma j}(x+p/T) = O(t^{[3r/2-1]\alpha}sT) + 2|\det(q)|^{r/2}t^{[3r/2-1]\alpha}\{[f^{(1)}_{r/4}(1)(\frac{y-y_{J\sigma j}}{sT})_{|x^0}]$$
$$+ [f^{(1)}_{r/4}(1)(\frac{y-y_{J\sigma j}}{sT})_{|x^1}] + f^{(2)}_{r/4}(1)(\frac{y-y_{J\sigma j}}{sT})_{|x^0}((y-1)_{|x^1} + (y_{J\sigma j}-1)_{|x^1})]$$
$$+ [f^{(1)}_{r/4}(1)(\frac{y-y_{J\sigma j}}{sT})_{|x^2}] + f^{(2)}_{r/4}(1)(\frac{y-y_{J\sigma j}}{sT})_{|x^0}((y-1)_{|x^2} + (y_{J\sigma j}-1)_{|x^2})$$
$$+ f^{(3)}_{r/4}(1)(\frac{y-y_{J\sigma j}}{sT})_{|x^0}(((y-1)_{|x^1})^2 + ((y_{J\sigma j}-1)_{|x^1})^2 + (y-1)_{|x^1}(y_{J\sigma j}-1)_{|x^1})]\} \tag{A.40}$$

And furthermore

$$\begin{aligned}
y-1 &= 2sq^{-1}_{Mm}x_{Mm} + s^2(2q^{-1}_{Mm}q^{-1}_{Nn} - q^{-1}_{Mn}q^{-1}_{Nm})x_{Mm}x_{Nn} + O(s^3)\\
&=: sC^{Mm}x_{Mm} + s^2C^{Mm,Nn}x_{Mm}x_{Nn} + O(s^3)\\
y_{J\sigma j}-1 &= 2s\,\mathrm{Tr}(q^{-1}x) + s^2[2\,\mathrm{Tr}(q^{-1}x)^2 - \mathrm{Tr}(q^{-1}xq^{-1}x)] + O(sT)\\
&=: sC^{Mm}x_{Mm} + s^2C^{Mm,Nn}x_{Mm}x_{Nn} + O(sT)\\
\frac{y_{J\sigma j}-y}{sT} &= q^{-1}_{Jj} + s(2q^{-1}_{Jj}q^{-1}_{Mm} - q^{-1}_{Jm}q^{-1}_{Mj})x_{Mm} + \frac{s^2}{2}[\det(q^{-1})\epsilon_{jmn}\epsilon_{JMN} + q^{-1}_{Jj}(q^{-1}_{Mm}q^{-1}_{Nn} - q^{-1}_{Mn}q^{-1}_{Nm})\\
&\quad + 2q^{-1}_{Mm}(q^{-1}_{Jj}q^{-1}_{Nn} - q^{-1}_{Jn}q^{-1}_{Nj})]x_{Mm}x_{Nn}\\
&=: C_{J\sigma j} + sC^{Mm}_{J\sigma j}x_{Mm} + s^2C^{Mm,Nn}_{J\sigma j}x_{Mm}x_{Nn}
\end{aligned} \tag{A.41}$$

We can therefore simplify (A.40) to

$$\begin{aligned}
\lambda^r_{J\sigma j}(x+p/T) &= O(t^{[3r/2-1]\alpha}sT) + 2|\det(q)|^{r/2}t^{[3r/2-1]\alpha}\{[f^{(1)}_{r/4}(1)C_{J\sigma j}]\\
&\quad + s[f^{(1)}_{r/4}(1)C^{Mm}_{J\sigma j} + 2f^{(2)}_{r/4}(1)C_{J\sigma j}C^{Mm}]x_{Mm}\\
&\quad + s^2[f^{(1)}_{r/4}(1)C^{Mm,Nn}_{J\sigma j} + 2f^{(2)}_{r/4}(1)C_{J\sigma j}C^{Mm,Nn} + 3f^{(3)}_{r/4}(1)C_{J\sigma j}C^{Mm}C^{Nn}]x_{Mm}x_{Nn}\}\\
&\quad =: O(t^{[3r/2-1]\alpha}sT) + 2|\det(q)|^{r/2}t^{[3r/2-1]\alpha}\{D_{J\sigma j}(r) + sD^{Mm}_{J\sigma j}(r)x_{Mm}\\
&\quad + s^2D^{Mm,Nn}_{J\sigma j}(r)x_{Mm}x_{Nn}\}
\end{aligned} \tag{A.42}$$

Putting everything together now yields the following theorem.

Theorem A.2.4. *For the classical limit and lowest order quantum corrections of expectation values of monomials of the operators $\widehat{q}_{J\sigma j}(r)$ for topologically cubic graphs we have*

$$\begin{aligned}
\frac{\langle\psi^t_{\{g_{J\sigma j}\}}, \prod_{k=1}^N \widehat{q}_{J_k\sigma_k j_k}(r_k)\psi^t_{\{g_{J\sigma j}\}}\rangle}{||\psi^t_{\{g_{J\sigma j}\}}||^2} &= (2|\det(q)|^{r/2}t^{[3r/2-1]\alpha})^N \times\\
&\times\{[\prod_{k=1}^N D_{J_k\sigma_k j_k}(r)] + \frac{s^2}{4}\sum_{M,m}[\sum_{l=1}^N D^{Mm,Mm}_{J_l\sigma_l j_l}(r)\prod_{k\neq l}D_{J_k\sigma_k j_k}(r))\\
&+ \sum_{1\le i<l\le N} D^{Mm}_{J_i\sigma_i j_i}(r)D^{Mm}_{J_l\sigma_l j_l}(r)\prod_{k\neq l,i}D_{J_k\sigma_k j_k}(r)]\}
\end{aligned} \tag{A.43}$$

where the constants are given by

$$
\begin{aligned}
C^{Mm} &= 2q^{-1}_{Mm} \\
C^{Mm,Nn} &= 2q^{-1}_{Mm}q^{-1}_{Nn} - q^{-1}_{Mn}q^{-1}_{Nm} \\
C_{J\sigma j} &= q^{-1}_{Jj} \\
C^{Mm}_{J\sigma j} &= (2q^{-1}_{Jj}q^{-1}_{Mm} - q^{-1}_{Jm}q^{-1}_{Mj}) \\
C^{Mm,Nn}_{J\sigma j} &= \frac{1}{2}[\det(q^{-1})\epsilon_{jmn}\epsilon_{JMN} + q^{-1}_{Jj}(q^{-1}_{Mm}q^{-1}_{Nn} - q^{-1}_{Mn}q^{-1}_{Nm}) + 2q^{-1}_{Mm}(q^{-1}_{Jj}q^{-1}_{Nn} - q^{-1}_{Jn}q^{-1}_{Nj})] \\
D_{J\sigma j}(r) &= f^{(1)}_{r/4}(1)C_{J\sigma j} \\
D^{Mm}_{J\sigma j}(r) &= f^{(1)}_{r/4}(1)C^{Mm}_{J\sigma j} + 2f^{(2)}_{r/4}(1)C_{J\sigma j}C^{Mm} \\
D^{Mm,Nn}_{J\sigma j}(r) &= f^{(1)}_{r/4}(1)C^{Mm,Nn}_{J\sigma j} + 2f^{(2)}_{r/4}(1)C_{J\sigma j}C^{Mm,Nn} + 3f^{(3)}_{r/4}(1)C_{J\sigma j}C^{Mm}C^{Nn}
\end{aligned}
$$

and the $f^{(k)}_{r/4}(1) = (r/4, k)$ are simply the binomial coefficients.
The first correction is small as long as $\alpha < 1/2$. The error as compared to the first quantum correction of order $O(t^{(N[3r/2-1]\alpha}s^2)$ is a constant of order unity times $t^{(N[3r/2-1]\alpha}sT$ and thus small as long as $0 < \alpha$.

So far we did not look at the classical limit and the first quantum corrections of (powers of) the volume operator itself but it is clear that it can be analyzed by similar methods, in fact, the analysis is even much simpler because we just need to expand $\lambda^r(x + p/T)$ in powers of s without dividing by t and thus will have to do an expansion in terms of $y-1$ of one order less than for $\lambda^r_{J\sigma j}(x+p/T)$. Clearly the classical order will be that of $|\det(p)|^{r/2} = |\det(q)|^{r/2}t^{3r\alpha/2} = O(t^{3r\alpha/2})$ and the first quantum correction will be of order $O(t^{3r\alpha/2}s^2)$. We thus have, in expanding up to second order in $y-1$, where $y = \det(1 + sq^{-1}x)^2$ as before

$$
\lambda^r(x + p/T) = |\det(q)|^{r/2}t^{3r\alpha/2}\Big\{1 + sf^{(1)}_{r/4}(1)C^{Mm}x_{Mm} \tag{A.44}
$$

$$
+ s^2[f^{(2)}_{r/4}(1)C^{Mm,Nn} + f^{(1)}_{r/4}(1)C^{Mm}C^{Nn}]x_{Mm}x_{Nn}\Big\} + O(t^{3r\alpha/2}s^3). \tag{A.45}
$$

Thus we obtain an analogue of theorem A.2.4 above:

Theorem A.2.5. *For the classical limit and lowest order quantum corrections of expectation values of powers of the volume operators $\widehat{\square}^r_v$ for topologically cubic graphs we have*

$$
\frac{\langle\psi^t_{\{g_{J\sigma j}\}}, \widehat{\square}^r_v\psi^t_{\{g_{J\sigma j}\}}\rangle}{||\psi^t_{\{g_{J\sigma j}\}}||^2} = |det(q)|^{r/2}t^{3r\alpha/2}\{1 + \frac{s^2}{4}\sum_{M,m}[f^{(2)}_{r/4}(1)C^{Mm,Nn} + f^{(1)}_{r/4}(1)C^{Mm}C^{Nn}\} \tag{A.46}
$$

The first correction is small as long as $\alpha < 1/2$. The error as compared to the first quantum correction of order $O(t^{(N[3r/2-1]\alpha}s^2)$ is a constant of order unity times $t^{(N[3r/2-1]\alpha}s^3$ and thus small as long as $0 < \alpha$.

A.3. The expectation values of $F_1 \dots F_5$

So far our considerations were completely general and model independent and we see that our coherent states indeed predict small quantum predictions as long as $0 < \alpha < 1/2$ and $l_P/\Lambda \ll 1$ with controllable error. However, now we will specialize to the case of the scalar and the electromagnetic field coupled to gravity and compute the expectation values of the gravitational operators occuring in the quadratic forms $F_1 \ldots F_5$. We will use the formulae given in theorems A.2.4 and A.2.5 with the appropriate values of r, N, J_k, σ_k, j_k inserted, and and perform the additional computations neccessary.

The form F_3:

Reconsidering formula (5.20) we see that we have $N = 6, r = 1/2$ so that and using (A.43) we find

$$\begin{aligned}\langle\widehat{F}_3(v)\rangle = &\frac{1}{a^3}(\frac{4}{3})^2\ (2|\det(q)|^{1/4}t^{[3/4-1]\alpha})^6\ \epsilon^{J_1J_2J_3}\epsilon_{j_1j_2j_3}\epsilon^{J_4J_5J_6}\epsilon_{j_4j_5j_6}\times\\ &\times\sum_{\sigma_1,..,\sigma_6=\pm}\{[\prod_{k=1}^{6}D_{J_k\sigma_kj_k}(1/2)]+\frac{s^2}{4}\sum_{M,m}[\sum_{l=1}^{6}D^{Mm,Mm}_{J_l\sigma_lj_l}(1/2)\prod_{k\neq l}D_{J_k\sigma_kj_k}(1/2))\\ &+\sum_{1\leq i<l\leq 6}D^{Mm}_{J_i\sigma_ij_i}(1/2)D^{Mm}_{J_l\sigma_lj_l}(1/2)\prod_{k\neq l,i}D_{J_k\sigma_kj_k}(1/2)]\}\end{aligned}\tag{A.47}$$

For $r = 1/2$ we have

$$a_1 \doteq f^{(1)}_{1/8}(1) = \frac{1}{8},\ a_2 \doteq f^{(2)}_{1/8}(1) = -\frac{1}{8}\frac{7}{16} = -\frac{7}{128},\ a_3 \doteq f^{(3)}_{1/8}(1) = \frac{7}{128}\frac{15}{24} = \frac{35}{1024}\tag{A.48}$$

and consequently

$$\sum_{M,m}D^{Mm,Mm}_{J\sigma j}(1/2) = [a_1+3a_3]q^{-1}_{Jj}\operatorname{Tr}(q^{-2}) - \frac{a_1}{2}q^{-3}_{Jj}\tag{A.49}$$

$$\begin{aligned}\sum_{Mm}D^{Mm}_{J_1\sigma_1j_1}(1/2)D^{Mm}_{J_2\sigma_2j_2}(1/2) = &4[a_1+a_2]^2q^{-1}_{J_1j_1}q^{-1}_{J_2j_2}\operatorname{Tr}(q^{-2})\\ &-2a_1[a_1+a_2](q^{-1}_{J_1j_1}q^{-3}_{J_2j_2}+q^{-1}_{J_2j_2}q^{-3}_{J_1j_1})+a_1^2q^{-2}_{J_1J_2}q^{-2}_{j_1j_2}\end{aligned}\tag{A.50}$$

Now we have to deal with the contractions in (A.47). It is easy to see that

$$\begin{aligned}&\epsilon^{J_1J_2J_3}\epsilon_{j_1j_2j_3}\epsilon^{J_4J_5J_6}\epsilon_{j_4j_5j_6}\prod_{k=1}^{6}q^{-1}_{J_kj_k}] = \frac{36}{\det(q)^2}\\ &\epsilon^{J_1J_2J_3}\epsilon_{j_1j_2j_3}\epsilon^{J_4J_5J_6}\epsilon_{j_4j_5j_6}q^{-3}_{J_lj_l})\prod_{k\neq l}q^{-1}_{J_kj_k} = \frac{12\operatorname{Tr}(q^{-2})}{\det(q)^2}\\ &\epsilon^{J_1J_2J_3}\epsilon_{j_1j_2j_3}\epsilon^{J_4J_5J_6}\epsilon_{j_4j_5j_6}q^{-2}_{J_iJ_l}q^{-2}_{j_ij_l})\prod_{k\neq l,i}q^{-1}_{J_kj_k} = 0 \text{ if } l,i\in\{1,2,3\} \text{ or } l,i\in\{4,5,6\}\\ &\epsilon^{J_1J_2J_3}\epsilon_{j_1j_2j_3}\epsilon^{J_4J_5J_6}\epsilon_{j_4j_5j_6}q^{-2}_{J_iJ_l}q^{-2}_{j_ij_l})\prod_{k\neq l,i}q^{-1}_{J_kj_k} = \frac{4\operatorname{Tr}(q^{-2})}{\det(q)^2}\text{ otherwise.}\end{aligned}\tag{A.51}$$

Using the above together with (A.49) in (A.47) yields

$$\begin{aligned}
\langle \widehat{F}_3(v)\rangle &= \frac{1}{a^3}(\frac{4}{3})^2 \,\frac{(4|\det(q)|^{1/4}t^{[3/4-1]\alpha})^6}{\det(q)^2}\, \{36[a_1^6] + \frac{s^2}{4}\operatorname{Tr}(q^{-2})[6a_1^5(36[a_1+3a_3] - 12\frac{a_1}{2}) \\
&+ a_1^4(15(4[a_1+a_2]^2(36) - 2a_1[a_1+a_2](12+12)) + 9a_1^2]\} \\
&= \frac{1}{a^3\sqcup_v}\,\{1 + \frac{t}{4}\operatorname{Tr}(p^{-2})[(5+24a_3) + 15(4[1+8a_2]^2 - \frac{4}{3}[1+8a_2]) + \frac{1}{4}]\} \\
&= \frac{1}{a^3\sqcup_v}\,\{1 + t\frac{1707}{512}\operatorname{Tr}(p^{-2})\}. \qquad \text{(A.52)}
\end{aligned}$$

Let us finally transform back to the dimensionfull quantities used in the main text: Using that $a^3\sqcup_v = \sqrt{\det P}$ and $ta^2 = l_P^2$ we find

$$\widehat{F}_3(v) = \frac{1}{\sqrt{\det P(v)}}\left[1 + \frac{l_P^4}{t}\frac{1707}{512}\operatorname{Tr} P^{-2}(v)\right].$$

The form F_2:

Reviewing the definition (5.19) of $\widehat{F}_2$ we see that we can write

$$\widehat{F}_2(\phi) = \sum_v \sum_{e,e'\in E(v)} \widehat{F}_{2ee'}(v)\partial_e^+\phi(v)\partial_{e'}^+\phi(v)$$

where $\widehat{F}_{2ee'}$ is a term that requires $N=4, r=3/4$. More explicitely, on the cubic lattice

$$\begin{aligned}
\langle \widehat{F}_{2J\sigma J'\sigma'}\rangle &= a(\frac{2}{9})^2 \sum_j < \{4\sigma \sum_{M,N} \frac{\epsilon^{JMN}\epsilon_{jmn}}{2} \sum_{\sigma_1,\sigma_2} [4\widehat{q}_{M,\sigma_1,m}(3/4)][4\widehat{q}_{N,\sigma_2 n}(3/4)]\}^\dagger \times \\
&\times \{4\sigma' \sum_{M,N} \frac{\epsilon^{J'MN}\epsilon_{jmn}}{2} \sum_{\sigma_1,\sigma_2} [4\widehat{q}^m_{M,\sigma_1,m}(3/4)][4\widehat{q}^n_{N,\sigma_2,n}(3/4)]\} > \\
&= a\sigma\sigma' 4^5 (\frac{2}{9})^2 \,(2|\det(q)|^{3/8}t^{[9/8-1]\alpha})^4 \sum_j \epsilon^{JJ_1J_1}\epsilon_{jj_1j_2}\epsilon^{J'J_3J_4}\epsilon_{jj_3j_4} \sum_{\sigma_1,..\sigma_4} \times \\
&\times \{[\prod_{k=1}^4 D_{J_k\sigma_k j_k}(3/4)] + \frac{s^2}{4}\sum_{M,m}[\sum_{l=1}^4 D^{Mm,Mm}_{J_l\sigma_l j_l}(3/4)\prod_{k\neq l} D_{J_k\sigma_k j_k}(3/4)) \\
&+ \sum_{1\leq i<l\leq 4} D^{Mm}_{J_i\sigma_i j_i}(3/4) D^{Mm}_{J_l\sigma_l j_l}(3/4)\prod_{k\neq l,i} D_{J_k\sigma_k j_k}(3/4)]\} \qquad \text{(A.53)}
\end{aligned}$$

For $r = 3/4$ we have

$$a_1 \doteq f^{(1)}_{3/16}(1) = \frac{3}{16},\; a_2 \doteq f^{(2)}_{3/16}(1) = -\frac{3}{16}\frac{29}{32} = -\frac{3\cdot 29}{2^9},\; a_3 \doteq f^{(3)}_{3/16}(1) = \frac{3\cdot 29}{2^9}\frac{45}{48} = \frac{3^2\cdot 5\cdot 29}{2^{13}} \qquad \text{(A.54)}$$

Furthermore the reader may verify that

$$
\begin{aligned}
\sum_j \epsilon^{JJ_1J_2}\epsilon_{jj_1j_2}\epsilon^{J'J_3J_4}\epsilon_{jj_3j_4}\prod_{k=1}^{4} q^{-1}_{J_kj_k}] &= \frac{4q^2_{JJ'}}{\det(q)^2}\\
\sum_j \epsilon^{JJ_1J_2}\epsilon_{jj_1j_2}\epsilon^{J'J_3J_4}\epsilon_{jj_3j_4}q^{-3}_{J_lj_l})\prod_{k\neq l} q^{-1}_{J_kj_k} &= \frac{2[q^2_{JJ'}\operatorname{Tr}(q^{-2})-\delta_{JJ'}]}{\det(q)^2}\\
\sum_j \epsilon^{JJ_1J_2}\epsilon_{jj_1j_2}\epsilon^{J'J_3J_4}\epsilon_{jj_3j_4}q^{-2}_{J_iJ_l}q^{-2}_{j_ij_l}\prod_{k\neq l,i} q^{-1}_{J_kj_k} &= 0 \text{ if } l,i\in\{1,2\} \text{ or } l,i\in\{3,4\}\\
\sum_j \epsilon^{JJ_1J_2}\epsilon_{jj_1j_2}\epsilon^{J'J_3J_4}\epsilon_{jj_3j_4}q^{-2}_{J_iJ_l}q^{-2}_{j_ij_l}\prod_{k\neq l,i} q^{-1}_{J_kj_k} &= \frac{q^2_{JJ'}\operatorname{Tr}(q^{-2})+\delta_{JJ'}}{\det(q)^2} \text{ otherwise}
\end{aligned} \tag{A.55}
$$

Thus we can finish with a tedious but straightforward computation:

$$
\begin{aligned}
\langle\widehat{F}_{2J\sigma J'\sigma'}\rangle &= a\sigma\sigma' 4^7(\frac{2}{9})^2(2|\det(q)|^{3/8}t^{[9/8-1]\alpha})^4 \times\\
&\quad\times\Big\{[a_1^4\frac{4q^2_{JJ'}}{\det(q)^2}]+\frac{s^2}{4}[4a_1^3([a_1+3a_3]\frac{4q^2_{JJ'}}{\det(q)^2}\operatorname{Tr}(q^{-2})-\frac{a_1}{2}\frac{2[q^2_{JJ'}\operatorname{Tr}(q^{-2})-\delta_{JJ'}]}{\det(q)^2})\\
&\quad\quad+a_1^2(4[a_1+a_2]^2 6\frac{4q^2_{JJ'}}{\det(q)^2}\operatorname{Tr}(q^{-2})-2a_1[a_1+a_2]12\frac{2[q^2_{JJ'}\operatorname{Tr}(q^{-2})-\delta_{JJ'}]}{\det(q)^2}\\
&\quad\quad+4a_1^2\frac{q^2_{JJ'}\operatorname{Tr}(q^{-2})+\delta_{JJ'}}{\det(q)^2})]\Big\}\\
&= a\sigma\sigma'\frac{1}{\sqrt{|\det(p)|}}\times\\
&\quad\times\Big\{p^2_{JJ'}+\frac{t}{4}[p^2_{JJ'}\operatorname{Tr}(p^{-2})[4(1+16a_3)-\frac{1}{4}+\frac{8}{3}(3+16a_2)^2-4(3+16a_2)+1]\\
&\quad\quad+\delta_{JJ'}[\frac{1}{4}+4(3+16a_2)+1]]\Big\}\\
&= a\sigma\sigma'\frac{1}{\sqrt{|\det(p)|}}\Big\{p^2_{JJ'}+t\Big[\frac{1173}{128}p^2_{JJ'}\operatorname{Tr}(p^{-2})+\frac{19}{32}\delta_{JJ'}\Big]\Big\}.
\end{aligned} \tag{A.56}
$$

Again as a last step we transform back to the quantities used in the main text and find

$$
\langle\widehat{F}_{2I\sigma I'\sigma'}\rangle = \frac{\sigma\sigma' P^2_{II'}(v)}{\sqrt{\det P(v)}}+\frac{l_P^4}{t}\frac{\sigma\sigma'}{\sqrt{\det P(v)}}\left(\frac{1173}{128}\operatorname{Tr}(P^{-2})P^2_{II'}(v)+\frac{19}{32}\delta_{II'}\right).
$$

The form F_1:

We now consider the operator valued form $\widehat{F}_1$. Its basic building block is the volume operator itselve, so we can apply theorem A.2.5 with $r=1$. In the by now familiar way we find

$$\begin{aligned}
\langle \widehat{V}_v \rangle &= a^3 |det(q)|^{1/2} t^{3\alpha/2} \{1 + \frac{s^2}{4} \sum_{M,m} [f^{(2)}_{1/4}(1) C^{Mm,Nn} + f^{(1)}_{1/4}(1) C^{Mm} C^{Nn}\} \\
&= a^3 |det(q)|^{1/2} t^{3\alpha/2} \{1 + \frac{s^2}{4} \operatorname{Tr}(q^{-2}) [\frac{1}{4} - 4\frac{3}{32}]\} \\
&= a^3 |det(p)|^{1/2} \{1 - \frac{t}{32} \operatorname{Tr}(p^{-2})\} \\
&= \sqrt{\det P(v)} \left[1 + \frac{l_P^7}{\sqrt{t}} \frac{1}{32} \operatorname{Tr} P^{-2}(v)\right]. \qquad \text{(A.57)}
\end{aligned}$$

The forms F_4 and F_5:

The operator valued forms F_4 and F_5 differ by their c-number coefficients, but the gravitational operator at the heart of both is the same, corresponding to $N=2$ and $r=1/2$.In both cases we have to compute $\langle \widehat{q}_{J_1 j}(1/2) \widehat{q}_{J_2 j}(1/2) \rangle$.
Let us use the definitions of a_1, a_2, a_3 given in (A.48) and equations (A.49), (A.50). We find

$$\begin{aligned}
&< \widehat{q}_{J_1 j}(1/2) \widehat{q}_{J_2 j}(1/2) >= \delta_{j_1 j_2} (2 |\det(q)|^{1/4} t^{[3/4-1]\alpha})^2 \times \\
&\quad \times \Big\{ [\prod_{k=1}^{2} D_{J_k \sigma_k j_k}(1/2)] + \frac{s^2}{4} \sum_{M,m} \Big[\sum_{l=1}^{2} D^{Mm,Mm}_{J_l \sigma_l j_l}(1/2) \prod_{k \neq l} D_{J_k \sigma_k j_k}(1/2) \\
&\quad + \sum_{1 \le i < l \le 2} D^{Mm}_{J_i \sigma_i j_i}(1/2) D^{Mm}_{J_l \sigma_l j_l}(1/2) \prod_{k \neq l,i} D_{J_k \sigma_k j_k}(1/2) \Big] \Big\} \\
&= (2a_1 |\det(q)|^{1/4} t^{[3/4-1]\alpha})^2 \Big\{ q^{-2}_{J_1 J_2} + \frac{s^2}{4} [2([1 + 3\frac{a_3}{a_1}] q^{-2}_{J_1 J_2} \operatorname{Tr}(q^{-2}) - \frac{1}{2} q^{-4}_{J_1 J_2}) \\
&\quad + 4[1 + \frac{a_2}{a_1}]^2 q^{-2}_{J_1 J_2} \operatorname{Tr}(q^{-2}) - 4[1 + \frac{a_2}{a_1}] q^{-4}_{J_1 J_2} + q^{-2}_{J_1 J_2} \operatorname{Tr}(q^{-2})] \Big\} \\
&= (|\det(q)|^{1/4} t^{[3/4-1]\alpha}/4)^2 \{ q^{-2}_{J_1 J_2} + \frac{s^2}{4} [q^{-2}_{J_1 J_2} \operatorname{Tr}(q^{-2})(7 + 3\frac{35}{2^7} - \frac{7}{2} + \frac{3^2 \cdot 5^2}{2^6}) - q^{-4}_{J_1 J_2}(5 - \frac{7}{4})]\} \\
&= \frac{\sqrt{|\det(p)|}}{16} \{ p^{-2}_{J_1 J_2} + t[\frac{763}{512} q^{-2}_{J_1 J_2} \operatorname{Tr}(p^{-2}) - \frac{13}{16} p^{-4}_{J_1 J_2}]\} \qquad \text{(A.58)}
\end{aligned}$$

We can now employ this result to give the explicit expressions for $\langle \widehat{F}_4 \rangle$ and $\langle \widehat{F}_5 \rangle$. To this end let us write

$$\widehat{F}_4(E) = \sum_v \sum_{ee'} \widehat{F}_{4ee'} E^e(v) E^{e'}(v), \qquad \widehat{F}_5(B) = \sum_v \sum_{ee'} \widehat{F}_{5ee'} A_e(v) A_{e'}(v)$$

Upon specializing to the cubic graph and using the above expectation value, we find that

$$\langle \widehat{F}_{4I\sigma I'\sigma'} \rangle = \sigma\sigma' \big[\sqrt{\det P(v)} P^{-2}_{II'} + \frac{l_P^4}{t} \Big(\frac{763}{512} P^{-2}_{II'} \operatorname{Tr} P^{-2} - \frac{13}{16} P^{-4}_{II'} \Big) \big].$$

On a cubic graph, many terms cancel in F_5, leaving us with

$$\langle \widehat{F}_5 \rangle(B) = \sum_v \sum_{II'} \left[\sqrt{\det P(v)} P^{-2}_{II'} + \frac{l_P^4}{t} \Big(\frac{763}{512} P^{-2}_{II'} \operatorname{Tr} P^{-2} - \frac{13}{16} P^{-4}_{II'} \Big) \right] A_{\tilde{\alpha}_I} A_{\tilde{\alpha}_{I'}}.$$

The loops $\tilde{\alpha}_I$ were defined in 7.1.

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