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Asymptotic staticity and tensor decompositions with fast decay conditions

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Abstract

Corvino, Corvino and Schoen, Chruściel and Delay have shown the existence of a large class of asymptotically flat vacuum initial data for Einstein's field equations which are static or stationary in a neighborhood of space-like infinity, yet quite general in the interior. The proof relies on some abstract, non-constructive arguments which makes it difficult to calculate such data numerically by using similar arguments.

A quasilinear elliptic system of equations is presented of which we expect that it can be used to construct vacuum initial data which are asymptotically flat, time-reflection symmetric, and asymptotic to static data up to a prescribed order at space-like infinity. A perturbation argument is used to show the existence of solutions. It is valid when the order at which the solutions approach staticity is restricted to a certain range.

Difficulties appear when trying to improve this result to show the existence of solutions that are asymptotically static at higher order. The problems arise from the lack of surjectivity of a certain operator.

Some tensor decompositions in asymptotically flat manifolds exhibit some of the difficulties encountered above. The Helmholtz decomposition, which plays a role in the preparation of initial data for the Maxwell equations, is discussed as a model problem. A method to circumvent the difficulties that arise when fast decay rates are required is discussed. This is done in a way that opens the possibility to perform numerical computations.

The insights from the analysis of the Helmholtz decomposition are applied to the York decomposition, which is related to that part of the quasilinear system which gives rise to the difficulties. For this decomposition analogous results are obtained. It turns out, however, that in this case the presence of symmetries of the underlying metric leads to certain complications. The question, whether the results obtained so far can be used again to show by a perturbation argument the existence of vacuum initial data which approach static solutions at infinity at any given order, thus remains open. The answer requires further analysis and perhaps new methods.

Abstrakt

Corvino, Corvino und Schoen als auch Chruściel und Delay haben die Existenz einer grossen Klasse asymptotisch flacher Anfangsdaten für Einstein's Vakuumfeldgleichungen gezeigt, die in einer Umgebung des raumartig Unendlichen statisch oder stationär aber im Inneren der Anfangshyperfläche sehr allgemein sind. Der Beweis beruht zum Teil auf abstrakten, nicht konstruktiven Argumenten, die Schwierigkeiten bereiten, wenn derartige Daten numerisch berechnet werden sollen.

In der Arbeit wird ein quasilineares elliptisches Gleichungssystem vorgestellt, von dem wir annehmen, dass es geeignet ist, asymptotisch flache Vakuumanfangsdaten zu berechnen, die zeitreflektionssymmetrisch sind und im raumartig Unendlichen in einer vorgeschriebenen Ordnung asymptotisch zu statischen Daten sind. Mit einem Störungsargument wird ein Existenzsatz bewiesen, der gilt, solange die Ordnung, in welcher die Lösungen asymptotisch statische Lösungen approximieren, in einem gewissen eingeschränkten Bereich liegt.

Versucht man, den Gültigkeitsbereich des Satzes zu erweitern, treten Schwierigkeiten auf. Diese hängen damit zusammen, dass ein gewisser Operator nicht mehr surjektiv ist.

In einigen Tensorzerlegungen auf asymptotisch flachen Räumen treten ähnliche Probleme auf, wie die oben erwähnten. Die Helmholtzzerlegung, die bei der Bereitstellung von Anfangsdaten für die Maxwellgleichungen eine Rolle spielt, wird als ein Modellfall diskutiert. Es wird eine Methode angegeben, die es erlaubt, die Schwierigkeiten zu umgehen, die auftreten, wenn ein schnelles Abfallverhalten des gesuchten Vektorfeldes im raumartig Unendlichen gefordert wird. Diese Methode gestattet es, solche Felder auch numerisch zu berechnen.

Die Einsichten aus der Analyse der Helmholtzzerlegung werden dann auf die Yorkzerlegung angewandt, die in den Teil des quasilinearen Systems eingeht, der Anlass zu den genannten Schwierigkeiten gibt. Für diese Zerlegung ergeben sich analoge Resultate. Es treten allerdings Schwierigkeiten auf, wenn die zu Grunde liegende Metrik Symmetrien aufweist. Die Frage, ob die Ergebnisse, die soweit erhalten wurden, in einem Störungsargument verwendet werden können um die Existenz von Vakuumdaten zu zeigen, die im räumlich Unendlichen in jeder Ordnung statische Daten approximieren, bleibt daher offen. Die Antwort erfordert eine weitergehende Untersuchung und möglicherweise auch neue Methoden.

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Contents

Chapter 1

Introduction

Gravitational waves are one of the many remarkable predictions of Einstein's general theory of relativity and have stimulated a great number of theoretical, mathematical and experimental developments. A great effort has been underway in the last few decades to observe their effects, yet there is still a large part of the studies of solutions to Einstein's field equations which is concerned with completing the mathematical foundations of the theory.

The current concept of gravitational radiation in the non-linear regime was geometrically idealized by Roger Penrose in the 60's. It is based on the behavior of the field of isolated gravitating systems as one moves along null geodesics to infinity. He proposed a characterization of asymptotic flatness that relies on the assumption that the conformal structure can be extended through null infinity with certain smoothness.

From the mathematical point of view, the *Penrose proposal* raises a number of difficult mathematical questions regarding the long time evolution behavior of gravitational fields. One of the main concerns was whether there exists a sufficiently large class of space-times satisfying these assumptions such that physically relevant scenarios can be studied in this way. In particular, the issue of whether the assumptions made allow for the existence of non-trivial *vacuum* radiative space-times remained unsettled for a long time.

In the late 70's Helmut Friedrich was able to construct a system of equations known as the regular conformal Einstein field equations, which allowed him to prove a semi-global existence result. This result is based on the hyperboloidal initial value problem in which data is given in a hyper-surface in space-time that extends to null infinity.

In this setting Friedrich showed in [Friedrich, 1986] that suitable initial data evolve to have a regular future null infinity locally in time. Furthermore, if such data are sufficiently close to Minkowskian hyperboloidal initial data, then the evolution can be carried out for sufficiently long as to obtain the complete future domain of dependence of the initial data. This was later generalized to the Einstein–Maxwell–Yang–Mills equations in [Friedrich, 1991].

This result opened the door to the possibility of constructing non-trivial complete vacuum space-times for the first time. This could in principle be done by finding Cauchy data such that a suitable hyperboloidal surface was in their future development. A number of complications appear which prevent a straight forward transition from Cauchy data in a space-like hypersurface to null infinity and one needs to come up with a way to deal with them.

There were efforts to surpass these obstacles by building particular types of initial data tailored for the task. In [Cutler and Wald, 1989] time-symmetric initial data was constructed for the Einstein-Maxwell field equations to address this issue. The construction is carried out on a manifold diffeomorphic to $\mathbb{R}^3$ and it is assumed that the metric h is conformally flat. The constraints for the Maxwell field are

$$\operatorname{div}_h E = 0 \tag{1.0.1a}$$
$$\operatorname{div}_h B = 0 \tag{1.0.1b}$$

where div_h is the divergence operator with respect to h. The electric field is assumed to vanish while the magnetic B_a field is compactly supported in a shell and such that $B_a B^a$ is spherically symmetric. The magnetic field is conformally rescaled and the resulting constraint $\operatorname{div}_\delta \tilde{B} = 0$ is solved explicitly, while the hamiltonian contraint is used to solve for the conformal factor afterwards. We want to emphasize that the fact that the divergence operator is under-determined elliptic plays a role in allowing considerable freedom to set up the magnetic field with the required properties, and this is something to which we shall come back in a moment.

The resulting initial data coincides with Schwarzschildian data outside the support of the magnetic field. This provided explicit control over the domain of dependence of the exterior Schwarzschild-like region and thus of the first portion of future null infinity. By making the magnetic field sufficiently small the authors were able to obtain the first global space-time by fitting a hypersurface in the future of this initial data suitable to apply the technique of Friedrich mentioned above.

Though the initial data constructed in [Cutler and Wald, 1989] succeeded in developing a space-time with a smooth null infinity in the sense Penrose proposed, the question of which other types of Cauchy data fall within the same category was still open. One would like to know what features of an initial data set determine whether or not it develops into such a space-time and furthermore if it is possible to construct these data in pure vacuum.

In [Friedrich, 1998] a systematic study was given which exposed the relation between the behavior of Cauchy data at space-like infinity and the smoothness of future null infinity. It was found that one can choose a certain gauge in which space-like infinity is blown-up to a cylinder I which touches $\mathscr{I}^+$ and $\mathscr{I}^-$ at two spheres I^+ and I^- respectively. With respect to Cauchy data for the physical space-time, points at infinity are represented in this picture by a sphere I^0 on the cylinder I which lies in between I^+ and I^-.

The regular conformal field equations reduce on the above mentioned cylinder to interior equations. This implies in particular that certain data can be given at I^0 from which the solution at $I^\pm$ can be obtained. At these sets the conformal field equations show a degeneracy which is a potential source of non-smoothness at null infinity.

It was found that, depending on the Cauchy data, certain logarithmic singularities develop at the sets $I^\pm$ which are expected to propagate along $\mathscr{I}^\pm$ whenever they are present. The internal nature of the equations on the cylinder allowed Friedrich to derive, in the time-symmetric case, a set of necessary conditions on Cauchy data such as to avoid the presence of this class of singularities [Friedrich, 2002]. It was later shown in [Valiente Kroon, 2004], with the aid of computer algebra, that at higher orders these conditions are in fact not sufficient to ensure the smoothness of the fields at $I^\pm$.

More recently, it has been shown that a concept of *asymptotic staticity* (or asymptotic stationarity in the non-time-symmetric case) ensures that the solutions extend smoothly on I to $I^\pm$ at all orders.

It is expected that the requirement that the conformal structure of null infinity be smooth to a certain order p can be related to the imposition on the initial data of being asymptotically static or stationary up to a certain order $q = q(p)$.

1.1 Cauchy data with special asymptotics

As mentioned above, the construction of initial data given in [Cutler and Wald, 1989] relies on finding of rather specific solutions to the Einstein–Maxwell constrains. As is the case with the Einstein constraint equations in vacuum, the underlying under-determinedness of the differential operators involved plays an important role in allowing sufficient room to find such solutions.

More recently, it was shown by Corvino in [Corvino, 2000], and subsequent work in [Chrusciel and Delay, 2003] and [Corvino and Schoen, 2006], that there exist (in an abstract sense) a large class of asymptotically flat vacuum initial data for Einstein's field equations which are static or stationary in a neighborhood of space-like infinity, yet quite general in the interior. These new data are constructed by starting from an arbitrary asymptotically flat initial data set by perturbing it in a transition zone and attaching it to the exterior region of a known asymptotically flat stationary solution. This process is sometimes refered to as 'gluing'.

Initial data sets with such behavior near space-like infinity are therefore relevant in the efforts towards global simulations of isolated systems because their evolution, as was the case for the data of [Cutler and Wald, 1989], can be controlled explicitly in such regions. As was later shown in [Chruściel and Delay, 2002] and [Corvino, 2007], one can make use of these data and the results on the hyperboloidal initial value problem to construct space-times which have a smooth conformal structure at null infinity.

These results provide for the first time a satisfactory argument to say that the Penrose proposal is not overly restrictive concerning the physical scenarios that can be considered. There remains, however, the question of whether there are still more general data which evolve into space-times with smooth asymptotics at null infinity. In fact, the requirement that the data be static or stationary *in a neighborhood* of space-like infinity seems rather strong. As mentioned above, it can be expected to be sufficient for the data to behave at space-like infinity *asymptotically* like static or stationary data.

Unfortunately, from the point of view of numerical computations, the methods discussed in the references above are quite different from the standard methods used so far in the construction of initial data.

1.2 Asymptotic staticity

The initial motivation for the present work was to develop a framework by which initial data with special asymptotics as discussed above could be generated by standard PDE methods. Of particular interest was to obtain a system of equations that incorporated the ideas of asymptotic staticity and which would lend itself to be treated by the techniques currently used in numerical computations.

As a simplifying assumption, we shall consider time-reflection symmetric intial data. The constraint equations reduce in this case to the problem of finding metrics h_{ab} such that

$$\mathrm{R}(h) = 0 \tag{1.2.1}$$

where $\mathrm{R}(h)$ is the scalar curvature of h_{ab}. Equation (1.2.1) (as is the case for the general vacuum Einstein constraints) is largely under-determined. This is manifestly used in what is nowadays the most widely exploited method to construct solutions to Einstein's constraint equations, i.e. the method of *conformal rescalings*. In it, a background metric is prescribed (subject to a certain condition) to which the solution is required to be conformally related. This turns the constraints into an elliptic equation for the conformal factor which can then be solved for. It seems that this method does not allow one to get the kind of finer control on the asymptotics which we have in mind.

The equations we shall study in chapter 4 form a quasilinear elliptic system of equations conceived with the aim of constructing solutions to (1.2.1) which are asymptotically flat and asymptotically static up to a prescribed order at space-like infinity. The system is obtained by altering Einstein's static field equations in a way which we shall briefly describe here for reference. A function v and a Riemannian metric h_{ab} on a manifold $\mathcal{M}$ are said to satisfy Einstein's static vacuum field equations if they are a solution to

$$\Delta_h v = 0 \tag{1.2.2a}$$
$$-v \ \mathrm{Ric}(h) + \mathrm{Hess}_h(v) = 0 \tag{1.2.2b}$$

where Δ_h, Hess_h and $\mathrm{Ric}(h)$ denote the Laplace operator, the Hessian operator and the Ricci tensor associated to the metric h_{ab}.

To begin with, the interest in studying the asymptotics of the solutions makes it reasonable to consider a manifold $\mathcal{M}$ diffeomorphic to $\mathbb{R}^3$. Let σ be a symmetric tensor field defined on $\mathcal{M}$ and consider the system of equations

$$\Delta_h v + \mathrm{tr}_h(\sigma) = 0 \tag{1.2.3a}$$
$$-v \ \mathrm{Ric}(h) + \mathrm{Hess}_h(v) + \sigma = 0 \tag{1.2.3b}$$

where tr_h is the trace operator with respect to h_{ab}. Solutions to this system of equations are also solutions of (1.2.1), and are therefore admissible time-symmetric vacuum initial data.

The objective of the introduction of the tensor σ is twofold. First, if it were possible to solve this system in a way as to guarantee that σ has strong decay at space-like infinity, then one could interpret such solutions as *asymptotically static vacuum initial data*. On the other hand, as was noted very early in [Lichnerowicz, 1955], there do not exist non-trivial asymptotically flat solutions to the static field equations (1.2.2) in $\mathbb{R}^3$, while it is conceivable that the presence of the tensor σ in equations (1.2.3) relaxes this restriction.

One may not, however, freely prescribe the tensor σ. The contracted Bianchi identity gives an integrability condition which reads

$$M_h(\sigma) = 0 \tag{1.2.4}$$

where M_h is the operator given by $M_h(\sigma) = \mathrm{div}_h(\sigma) - d \circ \mathrm{tr}_h(\sigma) = 0$ and will be refered to as the *momentum constraint operator* with respect to h_{ab} for its importance in Einstein's constrains. We shall refer to the system of equations that consists of considering equations (1.2.3) together with (1.2.4) as the *asymptotic staticity equations*.

It is remarkable that, although we are considering time-symmetric initial data (i.e. with vanishing second fundamental form), the momentum constraint operator reappears from taking into account the Bianchi identity. The first order operator M_h is under-determined elliptic and we shall devote considerable attention to some of its properties. As is typical of these situations, one needs a way to split the unknowns into a part which can be prescribed arbitrarily so that the rest of them can be solved for.

A remark is due at this point in support of the expectations that there should exist solutions to the system considered here such that they correspond to a σ with fast decay at infinity.

Consider a manifold $\mathcal{M} \cong \mathbb{R}^3$ and on it an initial data set of the type that can be constructed by gluing. More specifically, let h be an asymptotically flat metric such that it is a solution to the vacuum, time-reflexion symmetric constraints (1.2.1) and let $R > 0$, $v \in C^\infty(\mathcal{M} \setminus \overline{B_R(0)})$ be such that $v > 0$, $v \to 1$ at infinity and the static field equations (1.2.2) are satisfied by (v, h) on $\mathcal{M} \setminus \overline{B_R(0)}$.

It is possible to use such v and h to construct a solution to the asymptotic staticity equations (1.2.3), (1.2.4). To do so, choose a smooth positive extension of v to whole of $\mathcal{M}$ (possibly after choosing a slightly larger value for R) and let $\sigma := v\,\mathrm{Ric}(h) - \mathrm{Hess}_h(v)$ on $\mathcal{M}$. This implies $\Delta_h v + \mathrm{tr}_h(\sigma) = 0$ is satisfied on $\mathcal{M}$, while the contracted Bianchi Identity implies σ satisfies (1.2.4). Then (v, h, σ) are a solution to (1.2.3),(1.2.4) with σ supported on $\overline{B_R(0)}$. One should also note that there remains a rather large freedom to specify the extension of v and this could possibly allow for the imposition of further conditions.

The construction of the previous paragraph suggests that it is not the solvability of the asymptotic staticity equations (1.2.3), (1.2.4) which is a problem but whether the solutions can be obtained constructively, and in particular by methods accessible to numerical computations. The construction also allows for generalizations by considering gluing data which are asymptotically static up to a prescribed order as discussed in [Chrusciel and Delay, 2003].

In chapter 4 an existence result is obtained for the system of equations (1.2.3), (1.2.4), which holds when the order at which the solutions approach staticity is low. The strategy used there to turn equation (1.2.4) into an elliptic equation relies on the standard York decomposition. It consists of prescribing arbitrarily a symmetric tensor ψ and considering the ansatz

$$\sigma = \psi - \mathcal{L}_h X \tag{1.2.5}$$

where X is a 1-form and $\mathcal{L}_h$ is the conformal Killing operator with respect to h_{ab}. The requirement $\sigma \in \ker(M_h)$ turns equation (1.2.4) into the elliptic equation

$$\mathbb{L}_h X - M_h(\psi) = 0 \tag{1.2.6}$$

where $\mathbb{L}_h = M_h \circ \mathcal{L}_h$. This equation is thought of as a condition on X and the tensor ψ is considered as the *free data*. The system of equations consisting of (1.2.6) together with what results of replacing (1.2.5) into (1.2.3) is studied using the implicit function theorem. This allows to show existence of solutions in certain weighted Sobolev spaces under suitable conditions on the weight parameters, which are used to control the decay rate at infinity.

Difficulties appear when trying to improve this result to show the existence of solutions which are asymptotically static to higher order. The problems arise from the lack of surjectivity of the operator $\mathbb{L}_h$ under stronger decay conditions.

1.3 Tensor decompositions with fast decay

The difficulties mentioned above are found in the present work to be related with the possibility of carrying out the York decomposition in an asymptotically flat manifold under the imposition of fast decay conditions. This served as the initial motivation for our study, as a model problem, of what is known in the literature as the Helmholtz decomposition. It turned out to be a problem of interest on its own right and which, though simpler to treat in some respects, retains some of the features present in the York decomposition.

Both the Helmholtz and the York decomposition can be viewed as methods to generate solutions to particular under-determined equations. In the case of the former the operator in question is div_h, while in the latter it is M_h.

As was mentioned above, the construction of [Cutler and Wald, 1989] relies on an explicit solution with compact support to (1.0.1). We were interested, however, in a systematic approach that would allow one to generate a large class of solutions, while considering weaker decay conditions. In this respect weighted Sobolev spaces play a fundamental role by allowing one to rely on the results available in the literature concerning the properties of the differential operators involved as maps between such spaces. A choice of a certain weight parameter determines, roughly speaking, the decay conditions under which one is considering the problem.

The Helmholtz decomposition consists of splitting a 1-form field into a part in the set $\ker(\mathrm{div}_h)$ plus a part which is the gradient of a function. Roughly speaking, the issue of whether one can do this or not depends on the properties of the Laplace operator as a map between certain weighted Sobolev spaces.

As will be discussed in chapter 3, one might not be able to carry out the decomposition in the standard more familiar form on account of a lack of surjectivity of the second order operator in question, namely Δ_h in the case of the Helmholtz decomposition, or $\mathbb{L}_h$ in the case of the York decomposition.

To fix ideas, consider for example the problem of constructing solutions to $\mathrm{div}_h\, \sigma = 0$ such that they behave as

$$o\left(r^{-\frac{5}{2}}\right) \quad \text{for} \quad r \to \infty \tag{1.3.1}$$

at space-like infinity. To do this using the standard Helmholtz decomposition, one prescribes a 1-form ψ satisfying (1.3.1), and then sets $\sigma = \psi - dv$. Requiring σ to have vanishing divergence gives the equation $\Delta_h v = \mathrm{div}_h\, \psi$ which one then solves for v. If the solution v is such that dv also satisfies (1.3.1) then one is done, yet this might not always be the case.

A method was developed to circumvent the problems caused by this lack of surjectivity and is exhibited first in abstract terms in chapter 3. The key property used is that the second order operator is a Fredholm map. This method allows one to find a certain finite number of fields f_ν which serve as an extension to a given decomposition.

In the light of these abstract results one may consider the example above using what we will refer to in the sequel as the *extended Helmholtz decomposition* and which shall be discussed in detail in chapter 3. It is shown that one can choose fields f_1, f_2, f_3 satisfying(1.3.1) such that, given any 1-form ψ satisfying (1.3.1), it can be decomposed as

$$\psi = dv + \sigma + k^1 f_1 + k^2 f_2 + k^3 f_3 \tag{1.3.2}$$

where $\sigma \in \ker(\mathrm{div}_h)$ and dv satisfy (1.3.1), and where k^1, k^2, k^3 are constants.

As will be seen, the number of fields f_ν that are required to extend a decomposition depend on the behavior at infinity one is aiming for. In general, it increases as one looks for solutions with faster decay. The function v and the constants k^ν are unique, once the fields f_ν are fixed. What this leads to is a systematic method to produce, by prescribing a 1-form ψ, a divergence-less 1-form σ with the desired behavior at space-like infinity.

Afterwards it is shown that under certain conditions a given constructed extension enjoys of a type of stability. Let $\{f_\nu\}$ be an extension to the Helmholtz decomposition with respect to a given background metric h_{ab} and let $\widehat{h}_{ab}$ be a another Riemannian metric. It is shown that if $\widehat{h}_{ab}$ is sufficiently close to h_{ab}, then the set $\{f_\nu\}$ also serves as an extension to the Helmholtz decomposition with respect the metric $\widehat{h}_{ab}$. In particular, explicit expressions for extensions with respect to the Euclidean metric are given which, by the above stability, can be used for metrics which are sufficiently small perturbations of Euclidean space.

The abstract discussion from chapter 3 are used in chapter 5 to derive results about the York decomposition. It is shown that given a fixed background, extensions can be constructed for the York decomposition in a way analogous to those obtained for the Helmholtz decomposition in the same circumstances.

It is shown, however, that while in the Helmholtz case the number of fields in an extension depends only on the fall-off considered, in the case of the York decomposition it depends in addition on the number of Killing fields of the background geometry.

This is reflected in the fact that the kind of stability that was obtained for the Helmholtz decomposition only carries through to the York decomposition in a direct way under certain assumptions on the presence of symmetries for the metrics involved. In fact, a stability result is obtained for the York decomposition in which it is assumed that the metrics involved do not have Killing fields.

Explicit expressions can be obtained for the extended York decomposition with respect to Euclidean $\mathbb{R}^3$. One cannot, however, apply the stability result to use this construction for a different background. Loosely speaking, the dimension of the extension with respect to the Euclidean metric falls short of allowing one to solve the issues regarding the lack of surjectivity of the operator $\mathbb{L}_h$.

1.4 Discussion

A system of equations is presented of which we expect that it can be used to construct vacuum initial data which are asymptotically flat, time-reflection symmetric, and asymptotic to static data up to a prescribed order at space-like infinity. A perturbation argument is used to show the existence of solutions. It is valid when the order at which the solutions approach staticity is restricted to a certain range.

The York decomposition, which is part of the system used in the proof, gives rise certain to difficulties when trying to show the existnece of solutions that are asymptotically static at higher order. To explore those difficulties the Helmholtz decomposition, which plays a role in the preparation of initial data for the Maxwell equations, is discussed as a model problem. This decomposition, when considered under the imposition of fast decay conditions, exhibits some of the features encountered above. A method to circumvent those difficulties is discussed in a way that opens the possibility to perform numerical computations.

The insights from the analysis of the Helmholtz decomposition are applied to the York decomposition. For this decomposition analogous results are obtained. It turns out, however, that in this case the presence of symmetries of the underlying metric leads to certain complications.

It remains to be seen whether the results obtained so far can be used again to show by a perturbation argument the existence of vacuum initial data which approach static solutions at infinity at any given order. The expectation is that the system of equations presented here does indeed have solutions with such behavior. A rigorous proof, however, requires further analysis and perhaps new methods.

In this respect, a further possibility to be explored is that one may somehow consider a family of free data $\psi(t)$ to the asymptotic staticity equations with $\psi(0) = 0$ and such that, by imposing a parity condition for example, result in solutions having no Killing fields whenever $t > 0$. One may be able to use this family of free data along with the stability results presented in this work to construct Cauchy data which are asymptotically static up to a higher degree than obtained here.

We would like to point out that as a spin-off of the results of chapter 5, the techniques used there can be used in the context of the construction of solutions to the non-time-symmetric constraint equations by conformal rescalings. There, a background metric is given a priori up to a condition on its conformal class. One can choose this metric so that it has no non-trivial conformal Killing fields and then, using the results of this work, obtain a large class of symmetric tensors K_{ab} with prescribed fast fall-off at infinity which are solutions to the momentum constraint with respect to that metric. These tensors, upon a conformal rescaling which does not alter the fall-off, are admissible as second fundamental forms for initial data sets.

Chapter 2

Operators and Function Spaces

2.1 Notation

$\mathcal{M}$	manifold diffeomorphic to $\mathbb{R}^n$,
$B_R(p)$	ball centered at p of radius R in $\mathbb{R}^n$,
$T^{r,p}\mathcal{M}$	vector bundle of (r,p)-tensors over $\mathcal{M}$,
δ_{ab}	Flat metric in $\mathbb{R}^n$,
h_{ab}	Riemannian metric on $\mathcal{M}$,
$d_h(x_0,x)$	geodesic distance between x_0, x with respect to h_{ab},
$\lvert w\rvert_h$	pointwise norm with respect to h_{ab}, e.g. $\lvert w_a\rvert_h^2 = w_a w_b h^{ab}$,
$\mathrm{d}\mu_h$	volume element associated to h_{ab},
Λ^0	space of scalar functions,
Λ^1	space of 1-forms,
$\mathcal{S}^2$	space of symmetric 2-tensors,
tr_h	trace operator associated to h_{ab},
$\mathring{\nabla}_a$	connection associated with δ_{ab},
∇_a	connection associated with h_{ab},
div_h	divergence operator associated to h_{ab}, i.e. $\mathrm{div}_h\, u = \nabla^a u_a$,
Hess_h	Hessian operator associated to h_{ab}, i.e. $\mathrm{Hess}_h = \nabla_a \nabla_b$,
Sym	Symmetrization operator, e.g. $\mathrm{Sym}\,(u_a v_b) = \frac{1}{2}(u_a v_b + u_b v_a)$,
Δ_h	Laplace operator associated to h_{ab}, i.e. $\Delta_h = \mathrm{tr}_h \circ \mathrm{Hess}_h$,
$\mathcal{X}, \mathcal{Y}, \mathcal{Z}, ..$	Banach space,
$\lVert x\rVert_{\mathcal{X}}$	norm of $x \in \mathcal{X}$,
$\mathcal{O}_x \subset \mathcal{X}$	open set in $\mathcal{X}$ containing x,
$W^{s,p}_\gamma$	Weighted Sobolev space,
$\langle u, v\rangle_h$	pairing of the tensor fields $u \in T^{r,p}\mathcal{M}$ and $v \in (T^{r,p}\mathcal{M})^*$ defined by

$$\langle u, v\rangle_h = \int_M v(u)\mathrm{d}\mu_h$$

whenever the right hand makes sense.

2.2 Abstract Banach Spaces

A normed space $(\mathcal{X}, \| \|_{\mathcal{X}})$ which is complete with respect to its norm is called a *Banach space*. We collect here a number of abstract general statements that will be used in this text.

Definition 2.2.1. The space of all continuous linear forms $f : \mathcal{X} \to \mathbb{R}$ is the *dual space* to $\mathcal{X}$ and it is denoted by $\mathcal{X}^*$. By

$$\|f\| = \sup_{x \in \mathcal{X}} \frac{|f(x)|}{\|x\|_{\mathcal{X}}}. \tag{2.2.1}$$

a norm is defined on $\mathcal{X}^*$ which makes it a Banach space.

We denote by $f(x)$ or alternatively by $\langle f, x \rangle$ the application of $f \in \mathcal{X}^*$ to $x \in \mathcal{X}$.

Definition 2.2.2. [Kato, 1995] Let $T : \mathcal{X} \to \mathcal{Y}$ be a bounded operator. The dual operator of T, denoted by T^*, is a map

$$T^* : \mathcal{Y}^* \to \mathcal{X}^*$$

where $\mathcal{X}^*$ and $\mathcal{Y}^*$ are the dual spaces to $\mathcal{X}$ and $\mathcal{Y}$ respectively. This map is defined by

$$\langle T^* u, v \rangle = \langle u, Tv \rangle, \quad \forall u \in \mathcal{Y}^*, v \in \mathcal{X}$$

Theorem 2.2.3 (Banach's closed range Theorem [Zeidler, 1993])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $T : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator. The following statements are equivalent*

i) $\operatorname{ran}(T)$ *is closed*

ii) $\operatorname{ran}(T^*)$ *is closed*

iii) $\operatorname{ran}(T) = \ker(T^*)^{\perp}$

iv) $\operatorname{ran}(T^*) = {}^{\perp}\ker(T)$

where $\operatorname{ran}(T)$ *and* $\ker(T)$ *denote the range and the kernel of* T *respectively and*

$$\ker(T^*)^{\perp} = \{y \in \mathcal{Y} : \langle y^*, y \rangle = 0 \text{ for all } y^* \in \ker(T^*)\} \tag{2.2.2}$$
$${}^{\perp}\ker(T) = \{x^* \in \mathcal{X}^* : \langle x^*, x \rangle = 0 \text{ for all } x \in \ker(T)\} \tag{2.2.3}$$

Definition 2.2.4 ([Kato, 1995])**.** An bounded linear operator $S : \mathcal{Y} \to \mathcal{Z}$ is said to be

i) *upper semi-Fredholm* if $\operatorname{ran}(S)$ is closed in $\mathcal{Z}$ and $\dim(\ker(S)) < \infty$

ii) *lower semi-Fredholm* if $\operatorname{ran}(S)$ is closed in $\mathcal{Z}$ and $\dim(\operatorname{coker}(S)) < \infty$

iii) *Fredholm* if it is both upper and lower semi-Fredholm, and then its index is given by $\operatorname{index}(S) = \dim(\ker(S)) - \dim(\operatorname{coker}(S))$.

where $\operatorname{coker}(S)$ denotes the cokernel of S defined by $\operatorname{coker}(S) = \mathcal{Z}/\operatorname{ran}(S)$.

Lemma 2.2.5 ([Palais, 1965] section VI, Theorem 7)**.** *Let $\mathcal{Z}$ be a Banach space and let $\mathcal{X} \subset \mathcal{Y}$ be subspaces of $\mathcal{Z}$. If $\mathcal{X}$ is closed and of finite codimension in $\mathcal{Z}$, then the same is true for $\mathcal{Y}$.*

Definition 2.2.6 (Complemented set, [Rudin, 1991])**.** Suppose $\mathcal{X}$ is a closed subspace of a topological vector space $\mathcal{Z}$. If there exists a closed subspace $\mathcal{Y}$ of $\mathcal{Z}$ such that

$$\mathcal{Z} = \mathcal{X} + \mathcal{Y} \qquad \text{and} \qquad \mathcal{X} \cap \mathcal{Y} = \emptyset \tag{2.2.4}$$

then $\mathcal{X}$ is said to be *complemented in* $\mathcal{Z}$. In this case $\mathcal{Z}$ is said to be the direct sum of $\mathcal{X}$ and $\mathcal{Y}$ and the notation $\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y}$ is used[1].

Lemma 2.2.7 ([Rudin, 1991])**.** *Let $\mathcal{X}$ be a closed subspace of a topological vector space $\mathcal{Z}$. If* $\dim(\mathcal{Z}/\mathcal{X}) < \infty$*, then $\mathcal{X}$ is complemented in $\mathcal{Z}$.*

Note that Lemma 2.2.7 implies that the range of any lower semi-Fredholm operator $S : \mathcal{Y} \to \mathcal{Z}$ is complemented in the target space $\mathcal{Z}$, so there exists a closed subspace $\mathcal{I} \subset \mathcal{Z}$ such that

$$\mathcal{Z} = S(\mathcal{Y}) \oplus \mathcal{I} \quad \text{and} \quad S(\mathcal{Y}) \cap \mathcal{I} = \emptyset \ . \tag{2.2.5}$$

We will refer to this by saying $\mathcal{I}$ is a **complement** to the range of S.

Projectors

Two families of s elements $e_1, \dots, e_s \in \mathcal{X}$ and $e_1^*, \dots, e_s^* \in \mathcal{X}^*$ define a *bi-orthogonal system* if and only if they satisfy

$$\langle e_i^*, e_j \rangle = \delta_{ij} \qquad \text{for } i, j = 1, \dots, s \ . \tag{2.2.6}$$

Given s linearly independent elements in $\mathcal{X}$, one can always find s elements in $\mathcal{X}^*$ to construct a bi-orthogonal system. Their existence is guaranteed by the Hahn-Banach theorem. (See [Zeidler, 1993, A.51])

Consider now the subspace $\mathcal{X}_1 = \overline{\{e_i\}_{i=1}^s}$ generated by s linearly independent elements $e_1, \dots, e_s \in \mathcal{X}$. Choosing elements $\{e_j^*\}_{j=1}^s \in \mathcal{X}^*$ such that they form with $\{e_i\}_{i=1}^s$ a bi-orthogonal system, a projection operator $\mathbb{P} : \mathcal{X} \to \mathcal{X}_1$ is defined by setting

$$\mathbb{P}(\,\cdot\,) = \sum_{i=1}^{s} \langle e_i^*, \,\cdot\, \rangle e_i \ . \tag{2.2.7}$$

If $s < \dim(\mathcal{X})$ the space $\mathcal{X}_2 = (\mathbb{I} - \mathbb{P})\mathcal{X}$ depends largely on the choice of $\{e_j^*\}_{j=1}^s$.

2.3 Weighted Sobolev Spaces

In our applications the Banach spaces will mainly the weighted Sobolev spaces that where initially introduced in [Cantor, 1975]. However, we follow the conventions for the weight parameter used in [Bartnik, 1986]. [2]

Consider $\gamma \in \mathbb{R}$, $k \in \mathbb{Z}$ with $k \geq 0$ and $p \in \mathbb{Z}$ with $1 \leq p \leq \infty$. Let x_0 be a point in $\mathbb{R}^n$ and define the function σ_δ by

$$\sigma_\delta(x) = \sqrt{1 + d_\delta(x_0, x)^2} \ . \tag{2.3.1}$$

[1] This is also refered to as the *topological direct sum*, [Zeidler, 1993]

[2] The substitution $\gamma \to -(\gamma + n/p)$ changes from the notation for the weights used by [Cantor, 1975] to the one used in the present work and viceversa.

The weighted Lebesgue spaces of tensor fields L^p_γ and weighted Sobolev spaces $W^{k,p}_\gamma$ over $\mathbb{R}^n$ are defined to be the closure of C^∞_0 in the norm

$$\|u\|_{L^p_\gamma} = \begin{cases} \left(\int_{\mathbb{R}^n} |u|^p_\delta \sigma_\delta^{-\gamma p - n} \mathrm{d}\mu_\delta\right)^{1/p}, & p < \infty \\ \operatorname{ess\,sup}_{\mathbb{R}^n} \left(|u|_\delta \sigma_\delta^{-\gamma}\right), & p = \infty \end{cases} \tag{2.3.2}$$

and

$$\|u\|_{W^{k,p}_\gamma} = \sum_{j=0}^{k} \left\|\mathring{\nabla}^j u\right\|_{L^p_{\gamma-j}} \tag{2.3.3}$$

respectively.

Note: This convention for the weights permits the interpretation of the parameter γ directly as a measure of the growth at infinity for if $\|u\|_{W^{k,p}_\gamma} < \infty$ with $k > \frac{n}{p}$, then $|u(x)| = o\left(r^\gamma\right)$ for $|x| \to \infty$ (see [Bartnik, 1986]).

2.3.1 Properties

Lemma 2.3.1 (Hölder inequality [Bartnik, 1986])**.** *If $u \in L^q_{\gamma_1}$, $v \in L^r_{\gamma_2}$ and $\gamma = \gamma_1 + \gamma_2$, $1 \le p, q \le \infty$, $1/p = 1/q + 1/r$, then*

$$\|uv\|_{L^p_\gamma} \le \|u\|_{L^q_{\gamma_1}} \|v\|_{L^r_{\gamma_2}} \tag{2.3.4}$$

Lemma 2.3.2 (Multiplication Lemma, see [Maxwell, 2005])**.** *If $m \le \min(j,k)$, $p \le q$, $\epsilon > 0$ and $\frac{n}{q} < j + k - m$, then multiplication*

$$W^{j,q}_{\gamma_1} \times W^{k,p}_{\gamma_2} \to W^{m,p}_{\gamma_1+\gamma_2+\epsilon}$$

acting by

$$(u_1, u_2) \to u_1 \otimes u_2 \tag{2.3.5}$$

for tensors $u_1 \in W^{j,q}_{\gamma_1}$, $u_2 \in W^{k,p}_{\gamma_2}$ defines a continuous bilinear map. If furthermore $\frac{n}{p} < k$ and $\gamma < 0$, then $W^{k,p}_\gamma$ is a Banach algebra.

Lemma 2.3.3 (Sobolev inequality, [Bartnik, 1986])**.** *If $u \in W^{k,p}_\gamma$ with $\frac{n}{p} < k$, then there exists a constant C depending only on the dimension such that*

$$\|u\|_{W^{0,\infty}_\gamma} \le C \|u\|_{W^{k,p}_\gamma} \tag{2.3.6}$$

holds.

Lemma 2.3.4 (Poincaré inequality, [Bartnik, 1986])**.** *If $\gamma < 0$ and $u \in W^{0,p}_\gamma$, then there exists a constant C depending only on the dimension such that*

$$\|u\|_{W^{0,p}_\gamma} \le C \|\nabla u\|_{W^{0,p}_{\gamma-1}} \tag{2.3.7}$$

holds.

Proof. A Poincaré inequality for weighted spaces was proven in [Bartnik, 1986] for functions. For tensor fields, it follows from the Cauchy-Schwartz inequality that $|\nabla|w|_h| = 2\frac{|\langle w, \nabla w\rangle_h|}{|w|_h} \le 2|\nabla w|_h$, hence

$$\|w\|_{W^{0,p}_\gamma} = C_1 \||w|_h\|_{W^{0,p}_\gamma} \le C \|\nabla |w|_h\|_{W^{0,p}_{\gamma-1}} \le 2C_2 \|\nabla w\|_{W^{0,p}_{\gamma-1}} \tag{2.3.8}$$

holds. $\square$

2.3.2 Dual Spaces

For given p, we shall always designate the conjugate exponent p' by

$$p' = \begin{cases} 1 & \text{if} \quad p = \infty \\ \frac{p}{p-1} & \text{if} \quad 1 < p < \infty \\ \infty & \text{if} \quad p = 1 \end{cases} \tag{2.3.9}$$

At the level of L^p spaces it is known (see [Adams, 1975]) that if $1 \leq p < \infty$, then $(L^p)^*$ is isometrically isomorphic to $L^{p'}$. It is customary to denote this by

$$(L^p)^* \cong L^{p'} .$$

This statement is sometimes refered to as the Riesz representation theorem for L^p. It is straight forward to carry out an analogous argument to conclude that a similar statement holds for L^p_γ spaces. In fact, if $1 \leq p < \infty$, then $(L^p_\gamma)^*$ is isometrically isomorphic to $L^{p'}_{-\gamma-n}$. Thus if $1 \leq p < \infty$ we write

$$(L^p_\gamma)^* \cong L^{p'}_{-\gamma-n} .$$

In the case of Sobolev spaces involving derivatives, the properties of the dual spaces are most easily obtained by regarding $W^{s,p}$ as a closed subspace of a Cartesian product of L^p spaces, consisting of as many factors as there are multi-indices j satisfying $0 \leq |j| \leq m$. Applying the considerations about duality for L^p spaces to such Cartesian producs, one can conclude that if $1 \leq p < \infty$ then every element of $f \in (W^{s,p})^*$ is an extension of a distribution $T \in \mathcal{D}'$ to a bounded linear functional on $W^{s,p}$. If $1 \leq p < \infty$ and $s \in \mathbb{N}$ it is customary to denote this space by $W^{-s,p'}$, so we write

$$(W^{s,p})^* \cong W^{-s,p'} .$$

The considerations about duality in $W^{s,p}$ carry over to $W^{s,p}_\gamma$ with minor modifications to account for the presence of different powers of the weights in each factor of the Cartesian producs referred to above. The space $(W^{s,p}_\gamma)^*$ is then seen to consist of those elements of $\mathcal{D}'$ which extend to give bounded linear functionals on $W^{s,p}_\gamma$, and is denoted by $W^{-s,p'}_{-\gamma-n}$. Then if $1 \leq p < \infty$ and $s \in \mathbb{N}$ we write

$$(W^{s,p}_\gamma)^* \cong W^{-s,p'}_{-\gamma-n} .$$

2.3.3 Operator Duals

Concerning operators and their duals, it is important to clarify some aspects of the use that will be made of weighted Sobolev spaces. Consider the divergence operator as a map

$$\operatorname{div}_h : W^{s-1,p}_{\gamma-1}(\Lambda^1) \to W^{s-2,p}_{\gamma-2}(\Lambda^0) \tag{2.3.10}$$

for a given weight parameter γ. The operator dual is defined to be the map

$$(\operatorname{div}_h)^* : W^{-s+2,p'}_{-\gamma+2-n}(\Lambda^0) \to W^{-s+1,p'}_{-\gamma+1-n}(\Lambda^1) \tag{2.3.11}$$

such that

$$\langle \operatorname{div}_h \psi, \omega \rangle_h = \langle \psi, (\operatorname{div}_h)^* \omega \rangle_h \tag{2.3.12}$$

for every $\psi \in W^{s-1,p}_{\gamma-1}(\Lambda^1)$ and $\omega \in W^{-s+2,p'}_{-\gamma+2-n}(\Lambda^0)$. A formal expression for $(\mathrm{div}_h)^*$ can be obtained as follows. Let there be a function $\rho \in C^\infty_0$ and a 1-form $\phi \in C^\infty_0$. Then

$$\langle \rho, \mathrm{div}_h \phi \rangle_h = \int \rho \, \mathrm{div}_h \phi \, \mathrm{d}\mu_h = -\int h^{ab} (d\rho)_a (\phi)_b \, \mathrm{d}\mu_h = -\langle d\rho, \phi \rangle_h \tag{2.3.13}$$

on account of the compact support of the integrand. One can now interpret this as a representation (by Riesz's Theorem [Adams, 1975]) of the action of $\mathrm{div}_h \phi$, considered as an element of $W^{s-2,p}_{\gamma-2}$, on an element $\rho \in (W^{s-2,p}_{\gamma-2})^*$. Then, when acting on compactly supported functions, the operator $-d$ coincides with div^*_h. The set C^∞_0 is densely contained in every $W^{s,p}_\gamma$ and therefore this operator extends by continuity to the respective space.

One can therefore regard the operator $-d$ as the formal dual to div_h, as long as the pairing considered is consistent. To be more precise, the operator dual to $\mathrm{div}_h : W^{s-1,p}_{\gamma-1}(\Lambda^1) \to W^{s-2,p}_{\gamma-2}(\Lambda^0)$ acts as the map

$$-d : W^{-s+2,p'}_{-\gamma+2-n}(\Lambda^0) \to W^{-s+1,p'}_{-\gamma+1-n}(\Lambda^1) \, . \tag{2.3.14}$$

In the same sense it can be seen that if one considers the Laplace operator $\Delta_h = h^{ab}\nabla_a\nabla_b$ as the map

$$\Delta_h : W^{s,p}_\gamma \to W^{s-2,p}_{\gamma-2} \tag{2.3.15}$$

then its operator dual Δ^*_h is the map

$$\Delta^*_h : W^{-(s-2),p'}_{-\gamma+2-n} \to W^{-s,p'}_{-\gamma-n} \tag{2.3.16}$$

and it acts as $\Delta^*_h = h^{ab}\nabla_a\nabla_b$ (in the sense of distributions).

Example Application

We now consider the classical Poisson equation in $\left(\mathbb{R}^3, h_{ab}\right)$ as an example. Let $\rho \in W^{s-2,p}_{\gamma-2}$ and consider

$$\Delta_h v = \rho \tag{2.3.17}$$

as an equation for v. Let $Q(\rho)$ be defined by

$$Q(\rho) = \int \rho \, \mathrm{d}\mu_h \, . \tag{2.3.18}$$

Without restricting the weight γ, there is no guarantee 2.3.18 will be finite. However if $\gamma < -1$ then (2.3.18) is finite and $1 \in (W^{s-2,p}_{\gamma-2})^* = W^{-s+2,p'}_{-\gamma-1}$. One can write

$$Q(\rho) = \langle 1, \rho \rangle_h \, .$$

Then it is possible to conclude for example that if $\rho = \mathrm{div}_h \psi$ for a 1-form $\psi \in W^{s-1,p}_{\gamma-1}$ then $Q(\mathrm{div}_h \psi) = -\langle \psi, d(1) \rangle_h = 0$.

2.3.4 The Hilbert space case, $p = 2$

For $p = 2$, the weighted Sobolev spaces acquire the structure of a Hilbert spaces, and it is customary to adopt the notation $W^{s,2}_\gamma = H^s_\gamma$. Let $u_1, u_2 \in H^0_\gamma$. The scalar product of u_1 and u_2 is given by

$$(u_1|u_2)_{H^0_\gamma} = \int_M u_1 \, u_2 \, (\sigma^{-\gamma-n/2})^2 \mathrm{d}\mu_\delta \, .$$

In order to have

$$\|u\|^2_{H^\gamma_k} = (u|u)_{H^k_\gamma}$$

one needs to set

$$(u_1|u_2)_{H^k_\gamma} = \sum_{j=0}^{k} \left(\mathring{\nabla}^j u_1 \middle| \mathring{\nabla}^j u_2 \right)_{H^0_{\gamma-|\alpha|}} .$$

2.3.5 Asymptotic Flatness

Let $x_0 \in \mathcal{M}$ be a given point and for any $R \in \mathbb{R}^+$, let $\{\phi, (1-\phi)\}$ be a partition of unity such that $\phi(x) = 1$ if $d_h(x, x_0) \geq R$.

Definition 2.3.5. Let $(\mathcal{M}, h_{ab})$ be a Riemannian manifold. We call it *asymptotically flat of class* (p, t, τ) if there exists compact sets $\mathcal{K}_1, \mathcal{K}_2$ with $\mathcal{K}_1 \subset \mathring{\mathcal{K}}_2 \subset \mathcal{M}$, a function $\phi : \mathcal{M} \to [0,1]$, $\phi(\mathcal{K}_1) = 0$, $\phi(\mathcal{M} \setminus \mathcal{K}_2) = 1$ and a coordinate system $\{x^j : \mathcal{M} \setminus \mathcal{K}_1 \to \mathbb{R}^n \setminus B_1(0)\}$ (refered to as *end coordinates*) such that in these coordinates one has

$$(1-\phi)h \in W^{t,p}_{-n/p} \tag{2.3.19a}$$

$$\phi(x)(h_{ij} - \delta_{ij}) \in W^{t,p}_\tau \tag{2.3.19b}$$

with $\tau < 0$.

We will be working with metrics h_{ab} which are sufficiently close to the flat metric in the norm used to define asymptotic flatness. Because of this, we will consider the *end coordinates* to be defined globally on $\mathcal{M}$.

Lemma 2.3.6 (Proposition 7.3 in [Cantor, 1981])**.** *Let* $(\mathbb{R}^n, h_{ab})$ *be asymptotically flat of class* (p, t, τ)*. with* $t > \frac{n}{p} + 1$ *and* $\tau < 0$*. Then if* $s \leq t$*, the* $W^{s,p}_\gamma$ *norm with respect to end coordinates is equivalent to the intrinsic norm*

$$\sum_{|\alpha| \leq s} \left(\int |(\nabla^\alpha f)|^p_h \, \sigma_h^{-\gamma p - n} \, \mathrm{d}\mu_h \right)^{1/p} . \tag{2.3.20}$$

Lemma 2.3.7 (Bound for the inverse metric)**.** *Let* $\left(\mathbb{R}^3, h_{ab}\right)$ *be asymptotically flat of class* (p, t, τ)*,* $t > \frac{3}{p}$*. Assume that in (globally defined) end coordinates one has for* $0 < 3\epsilon < 1$

$$\|h_{ij} - \delta_{ij}\|_{W^{t,p}_\tau} < \epsilon . \tag{2.3.21}$$

If $h_{ij} h^{jk} = \delta_i{}^k$*, then in end coordinates it holds*

$$\left\| h^{ij} - \delta^{ij} \right\|_{W^{t,p}_\tau} < \frac{3\epsilon}{1 - 3\epsilon} . \tag{2.3.22}$$

Proof. Defining $\gamma_{ij} = h_{ij} - \delta_{ij}$ one has

$$(\delta^i{}_j + \delta^{ik}\gamma_{kj}) h^{jl} = \delta^{il} \tag{2.3.23}$$

Consider the $(1,1)$-tensor $T^i{}_j = -\delta^{ik}\gamma_{kj}$. One can define the square of T by

$$(T)^2 = T^i{}_k T^k{}_j \tag{2.3.24}$$

and similarly higher powers. By hypothesis $\left\|T^i{}_j\right\|_{W^{t,p}_\tau} < \epsilon$, so now one can compute the weighted norm of $(T)^2$ by first noting

$$\begin{aligned} |T^i{}_k\, T^k{}_j|_\delta = & |\delta^k{}_l T^i{}_k\, T^l{}_j|_\delta \leq |\delta^k{}_l|_\delta |T^i{}_k\, T^l{}_j|_\delta \\ & \leq 3|T^i{}_j\, T^l{}_k|_\delta \end{aligned} \tag{2.3.25}$$

and using the multiplication Lemma 2.3.2

$$\begin{aligned} \left\|(T)^2\right\|_{W^{t,p}_\tau} = & \left\| |T^i{}_k T^k{}_j|_\delta \right\|_{W^{t,p}_\tau} \leq 3 \left\| |T^i{}_k T^l{}_j|_\delta \right\|_{W^{t,p}_\tau} \\ & \leq 3\epsilon^2 \end{aligned} \tag{2.3.26}$$

where $\tau < 0$ and $\frac{3}{p} < t$ where used. Finally one has for $q \geq 1$

$$\|(T)^q\|_{W^{t,p}_\tau} \leq \frac{(3\epsilon)^q}{3} \tag{2.3.27}$$

so now it is possible to consider the series

$$\sum_{q=0}^{\infty}(T)^q = \delta^e{}_i + \sum_{q=1}^{\infty}(-\delta^{ek}\gamma_{ki})^q \tag{2.3.28}$$

is absolutely convergent if $3\epsilon < 1$ and therefore constitutes the inverse to $(\delta^i{}_j + \delta^{ik}\gamma_{kj})$. This gives

$$h^{ij} - \delta^{ij} = \sum_{q=1}^{\infty}\left(-\delta^{ik}\gamma_{ke}\right)^s \delta^{ej} \tag{2.3.29}$$

and consequently

$$\left\|h^{ij} - \delta^{ij}\right\|_{W^{p,t}_\tau} \leq 3\sum_{j=1}^{\infty}\left\|\left(-\delta^{ik}\gamma_{ke}\right)^j\right\|_{W^{p,t}_\tau} \tag{2.3.30}$$

$$\leq \frac{3\epsilon}{1-3\epsilon}\ . \tag{2.3.31}$$

□

2.4 Linear Partial Differential Operators

Suppose $Q = (Q_{ij})$ is a $N \times N$ system of linear partial differential operators in $\mathbb{R}^n$. We use the generalized definition of ellipticity provided by Douglis and Nirenberg [Douglis and Nirenberg, 1955].

It is usual to define ellipticity first at a point, and then to say a system is *elliptic in a region* Ω if it is elliptic at every $x \in \Omega$. In what follows, it is to be understood that *elliptic* refers to the complete manifold.

Definition 2.4.1. Two N-tuples, $\mathbf{t} = (t_1...t_N)$ and $\mathbf{s} = (s_1...s_N)$ of nonnegative integers form a *system of orders* for Q if for $1 \leq i,j \leq N$ we have order$(Q_{ij}) \leq t_j - s_i$ (if $t_j - s_i < 0$ then $Q_{ij} = 0$). Then

- the $(\mathbf{t},\mathbf{s})$-*principal part* of Q, denoted by $\sigma(Q)$, is obtained by replacing each Q_{ij} by its terms which are exactly of order $t_j - s_i$,
- the $(\mathbf{t},\mathbf{s})$-*principal symbol* of Q, denoted by $\sigma_\xi(Q)$, is obtained by replacing each ∂_a in the $(\mathbf{t},\mathbf{s})$-principal part by the covector $\xi_a \in S^{n-1}$,
- the operator Q is *elliptic with respect to* $(\mathbf{t},\mathbf{s})$ if the $(\mathbf{t},\mathbf{s})$-principal symbol of Q has determinant bounded away from zero for $x \in \mathbb{R}^n$ and $\xi_a \in S^{n-1}$.

Assumption 2.4.2. Let Q be a system of linear partial differential operators and let $(\mathbf{t},\mathbf{s})$ be a system of orders for Q. We write

$$Q = Q_\infty + S \tag{2.4.1a}$$

and assume

$$\begin{gathered}Q \text{ is elliptic with respect to } (\mathbf{t},\mathbf{s}) \text{ and}\\ \text{each } (Q_\infty)_{ij} \text{ is either zero or a constant}\\ \text{coefficient operator of order } t_j - s_i,\end{gathered} \tag{2.4.1b}$$

and writing[3] $S = \sum_{|\alpha|\le t_j - s_i} (S_\alpha)_{ij}\partial^\alpha$ we assume

$$(S_\alpha)_{ij} \in C^{s_i}(\mathbb{R}^n)\ , \tag{2.4.1c}$$

$$\lim_{x\to\infty} \left|\sigma^{t_j - s_i - |\alpha| + |\beta|}\partial_\beta (S_\alpha)_{ij}\right| = 0 \tag{2.4.1d}$$

for all $\beta \le s_i \in \mathbb{N}$.

Let $\vec{\gamma}$ represent the N-tuple of weights $\vec{\gamma} = (\gamma_1, ...\gamma_N)$. Define the weighted Sobolev spaces

$$W^{\vec{t},p}_{\vec{\gamma}}(\mathbb{R}^N) = \prod_{j=1}^{N} W^{t_j,p}_{\gamma_j} \tag{2.4.2}$$

In abuse of this notation, we omit the arrow and write simply γ if all the components of $\vec{\gamma}$ are identical and take the value γ. Under the assumptions 2.4.2 the maps

$$Q_\infty : \quad W^{\vec{t},p}_{\gamma+\vec{t}}(\mathbb{R}^N) \to W^{\vec{s},p}_{\gamma+\vec{s}}(\mathbb{R}^N) \tag{2.4.3}$$

$$S : \quad W^{\vec{t},p}_{\gamma+\vec{t}}(\mathbb{R}^N) \to W^{\vec{s},p}_{\gamma+\vec{s}}(\mathbb{R}^N) \tag{2.4.4}$$

define bounded operators.

Let $\mathrm{Poly}(\gamma)$ be the space of polynomials in $x_1, ..., x_n$ of degree $\le \gamma$ and denote by $d_P(\gamma)$ its dimension (note that $\mathrm{Poly}(\gamma) = \{0\}$ if $\gamma < 0$). The number of homogeneous polynomials of degree k is

$$\frac{(k+n-1)!}{k!(n-1)!} \tag{2.4.5}$$

and thus

$$d_P(\gamma) = \sum_{k=0}^{[\gamma]} \frac{(k+n-1)!}{k!(n-1)!} \tag{2.4.6}$$

where $[\gamma]$ represents the largest integer smaller than or equal to γ.

[3]By α we denote a standard multi-index.

Theorem 2.4.3. *Let Q_∞ be a linear partial differential operator satisfying assumption* (2.4.1b)*. Then the map* (2.4.3) *is Fredholm if and only if*

$$\gamma + t_j \notin \mathbb{N} \quad \text{if} \quad 0 \leq \gamma + t_j \tag{2.4.7a}$$
$$-(\gamma + s_j + n) \notin \mathbb{N} \quad \text{if} \quad \gamma + t_j < 0 \tag{2.4.7b}$$

for every $j = 1..N$. Furthermore,

$$\gamma < -t_j \quad \forall j \qquad \Longrightarrow Q_\infty \text{ is injective} \tag{2.4.8a}$$
$$-s_j - n < \gamma < -t_j \quad \forall j \qquad \Longrightarrow Q_\infty \text{ is an isomorphism} \tag{2.4.8b}$$
$$-s_j - n < \gamma \quad \forall j \qquad \Longrightarrow Q_\infty \text{ is surjective} \tag{2.4.8c}$$

The kernel and co-kernel of Q_∞ consist of polynomials. Their dimensions are

$$\dim\left(\ker\left(Q_\infty\right)\right) = \sum_{j=1}^{N} d_P(\gamma + t_j) - d_P(\gamma + s_j) \tag{2.4.9}$$
$$\dim\left(\operatorname{coker}\left(Q_\infty\right)\right) = \sum_{j=1}^{N} d_P(-(\gamma + s_j + n)) - d_P(-(\gamma + t_j + n)). \tag{2.4.10}$$

Proof. This theorem was proven with a different notation for the weights in [Lockhart and McOwen, 1983]. □

If it is the case that $\mathrm{Order}(Q_{ij}) = m$ for all $i, j = 1..N$, then it is convenient to set $\vec{t} = (k+m, ..., k+m)$ and $\vec{s} = (k, ..., k)$. We restate the previous Theorem in this special but useful case.

Theorem 2.4.4. *Let Q_∞ be a linear partial differential operator satisfying assumption* (2.4.1b) *with respect to the system of orders given by $\vec{t} = (k+m, ..., k+m)$ and $\vec{s} = (k, ..., k)$ with $k \in \mathbb{N}$.* [4]*. Then the map*[5]

$$Q_\infty : \quad W^{k+m,p}_{\gamma}(\mathbb{R}^N) \to W^{k,p}_{\gamma-m}(\mathbb{R}^N) \tag{2.4.11}$$

is Fredholm if and only if

$$\gamma \notin \mathbb{N} \quad \text{if} \quad 0 \leq \gamma \tag{2.4.12a}$$
$$-(\gamma + n - m) \notin \mathbb{N} \quad \text{if} \quad \gamma < 0 \tag{2.4.12b}$$

Furthermore,

$$\gamma < 0 \quad \forall j \qquad \Longrightarrow Q_\infty \text{ is injective} \tag{2.4.13a}$$
$$m - n < \gamma < 0 \quad \forall j \qquad \Longrightarrow Q_\infty \text{ is an isomorphism} \tag{2.4.13b}$$
$$m - n < \gamma \quad \forall j \qquad \Longrightarrow Q_\infty \text{ is surjective} \tag{2.4.13c}$$

The kernel and co-kernel of Q consist of polynomials. Their dimensions are

$$\dim\left(\ker\left(Q_\infty\right)\right) = N\left(d_P(\gamma) - d_P(\gamma - m)\right) \tag{2.4.14a}$$
$$\dim\left(\operatorname{coker}\left(Q_\infty\right)\right) = N\left(d_P(-\gamma - n + m) - d_P(-\gamma - n)\right). \tag{2.4.14b}$$

[4] Observe that this implies that $\mathrm{Order}((Q_\infty)_{ij}) = m$ for all i, j

[5] Observe that we have shifted the weight from $\gamma + \vec{t}$ to γ

Theorem 2.4.5. *Let Q be a linear partial differential operator satisfying assumptions* (2.4.1b) *and* (2.4.1c) *with respect to the system of orders given by $\vec{t} = (k+m, ..., k+m)$ and $\vec{s} = (k, ..., k)$ with $k \in \mathbb{N}$. Then the map*

$$Q: \quad W^{k+m,p}_{\gamma}(\mathbb{R}^N) \to W^{k,p}_{\gamma-m}(\mathbb{R}^N) \tag{2.4.15}$$

is Fredholm if and only if conditions (2.4.12) *are satisfied. Moreover*

$$\operatorname{index}(Q) - \operatorname{index}(Q_\infty) \tag{2.4.16}$$

is a constant independent of γ. In particular, if Q is an operator on scalars, then it holds that $\operatorname{index}(Q) = \operatorname{index}(Q_\infty)$.

Remark 2.4.6. Theorem 2.4.5 was proven in [Lockhart and McOwen, 1985, Corollary 9.2], and the statement about scalar operators is from [Lockhart and McOwen, 1983]. See also [Lockhart and McOwen, 1984].

2.5 The Laplace operator

Consider in an asymptotically Euclidean manifold $\left(\mathbb{R}^3, h_{ab}\right)$, the Poisson equation for a potential v with source $\rho \in W^{s-2,p}_{\gamma-2}$

$$\Delta_h \psi = \rho \tag{2.5.1}$$

under the assumptions that $\gamma \notin \mathbb{Z}$ and $\gamma < 0$.

The Laplace operator Δ_h is considered (as described in section 2.3.3) as the mapping

$$\Delta_h : W^{s,p}_{\gamma} \to W^{s-2,p}_{\gamma-2}$$

and Theorem 2.4.4 implies this map is Fredholm, hence $\ker(\Delta_h^*)$ is finite dimensional. Furthermore it will not be empty whenever $\gamma < -1$. By the closed range Theorem one has

$$\begin{aligned} \operatorname{ran}(\Delta_h) &= \ker(\Delta_h^*)^{\perp} \\ &= \left\{ \rho \in W^{s-2,p}_{\gamma-2} : \langle \rho, u \rangle_h = 0 \quad \forall u \in \ker(\Delta_h^*) \right\} . \end{aligned} \tag{2.5.2}$$

After introducing some notation, we will consider in section 2.5.1 the case of Euclidean $\mathbb{R}^3$, both as an illustration of the above discussion and as a matter of convenience for later application.

Notation for basis of $\ker(\Delta_h^*)$

We will denote the elements of a basis for $\ker(\Delta_h^*)$ by a^μ, where the index $\mu = 1, \ldots, M$ enumerates the elements. We use the same notation but with a little circle above (e.g. $\mathring{a}$) when working with the Euclidean metric. Thus we write

$$\ker(\Delta_h^*) = \overline{\{a^\mu\}_{\mu=1}^{M}}, \quad \text{and} \quad \ker(\Delta_\delta^*) = \overline{\{\mathring{a}^\mu\}_{\mu=1}^{M}} \tag{2.5.3}$$

where the overline denotes the linear span of the elements in brackets.

Whenever the sets (2.5.3) are not empty, they contain the constants. As a matter of convenience, the index μ will be chosen so that the first element corresponds to a constant, that is

$$a^1 = \mathring{a}^1 = 1 \, . \tag{2.5.4}$$

2.5.1 The case of Euclidean $\mathbb{R}^3$

To describe in more detail the set (2.5.2) when the manifold under consideration is $\left(\mathbb{R}^3, \delta_{ab}\right)$, one must consider the problem of solving for all the $u \in W_{\theta}^{-s+2,p'}$ such that

$$\Delta_{\delta}^{*}(u) = 0 \ . \tag{2.5.5}$$

The results of Theorem 2.4.4 apply also to this equation, so the solutions to (2.5.5) are polynomial functions of the coordinates.

The set $\ker(\Delta_{\delta}^{*})$ consists of the harmonic polynomials that are contained in the domain. This means they are the ones that have growth at infinity slower than $\left(\sqrt{1+d_{\delta}(x_0,x)^2}\right)^{\gamma}$ where x_0 is any given point in $\mathbb{R}^3$. We shall denote by $\mathcal{H}_{\gamma}$ the set of harmonic polynomials or order $\leq \gamma$.

Counting the harmonic polynomials

By Theorem 2.4.4 the dimension of $\mathcal{H}_{\gamma}$ is given by

$$\begin{aligned} \dim(\mathcal{H}_{\gamma}) &= d_P(\gamma) - d_P(\gamma-2) \\ &= \frac{([\gamma]+n-2)!}{[\gamma]!(n-1)!}\,(2[\gamma]+n-1) \end{aligned} \tag{2.5.6}$$

where (2.4.6) was used. For the case $n = 3$ this gives $\dim(\mathcal{H}_{\gamma}) = ([\gamma]+1)^2$. This coincides with the result of adding the dimensions of the independent spherical harmonics up to $l = [\gamma]$,

$$\sum_{l=0}^{[\gamma]}(2l+1) = ([\gamma]+1)^2 \ . \tag{2.5.7}$$

Basis of harmonic polynomials

We now describe a basis to $\ker(\Delta_{\delta}^{*})$ given explicitly. We start by choosing a set of indices $\mu = 1, \ldots, M$ so that they enumerate the standard spherical harmonics Y_{lm}, where $l = 0, \ldots, l_M$ (for an adequate maximum value l_M) and for each given l, the integer m runs in the range $m = -l, \ldots, l$. Then we set

$$\mathring{a}^{\mu} := r^l Y_{lm}(\theta,\phi) \ . \tag{2.5.8}$$

The way in which the numbering

$$\mu \leftrightarrow (l,m)$$

is constructed need not be specified at this point, except for the condition mentioned earlier that $\mu = 1 \leftrightarrow (l=0, m=0)$ in order to make $\mathring{a}^1$ a constant.

2.5.2 Complement to $\Delta_{\delta}(W_{\gamma}^{s,p})$

In this section we construct explicitly a finite dimensional subspace $\mathcal{I}_{\delta} \subset W_{\gamma-2}^{s-2,p}$ such that

$$W_{\gamma-2}^{s-2,p} = \Delta_{\delta}(W_{\gamma}^{s,p}) \oplus \mathcal{I}_{\delta} \ . \tag{2.5.9}$$

The set $\mathcal{I}_{\delta}$ is a **complement to** $\Delta_{\delta}(W_{\gamma}^{s,p})$.

Lemma 2.5.1. *Let $\gamma \notin \mathbb{Z}$. Let the functions $\mathring{g}_\mu$, $\mu = 1, .., M$, be defined by*

$$\mathring{g}_\mu = \left\{ \begin{array}{ll} -\frac{1}{lr^2}\frac{d}{dr}\left(\frac{r^2\chi(r)}{r^{l-1}}\right) & \mu \neq 1 \\ \chi(r) & \mu = 1 \end{array} \right\} \times \overline{Y}_{lm}(\theta,\phi) \tag{2.5.10}$$

where $\overline{Y}_{lm}$ denotes the complex conjugate of Y_{lm}, the function $\chi(r) \in C_0^\infty$ has support away from the origin and satisfies

$$\int_0^\infty \chi(r) r^2 dr = 1 \ . \tag{2.5.11}$$

Then the two families $\{\mathring{a}^\mu\}_{\mu=1}^M$ and $\{\mathring{g}_\nu\}_{\nu=1}^M$ form a bi-orthogonal system. Furthermore, the set $\mathcal{I}_\delta = \overline{\{\mathring{g}_\mu\}}$ satisfies

$$W^{s-2,p}_{\gamma-2} = \Delta_\delta(W^{s,p}_\gamma) \oplus \mathcal{I}_\delta \ . \tag{2.5.12}$$

Proof. First of all, having $\chi(r) \in C_0^\infty$ supported away from the origin implies that $\mathring{g}_\mu \in W^{s-2,p}_{\gamma-2}$ independently of the weight and differentiability degree under consideration.

Let $f \in W^{s-2,p}_{\gamma-2}$ be arbitrary. The operator Δ_δ is Fredholm by Theorem 2.4.4. Using Theorem 2.2.3 we know there exists a solution v to

$$\Delta_\delta v + \sum_{\nu=1}^M k^\nu \mathring{g}_\nu = f \quad \text{where} \quad k^\nu \in \mathbb{R}^M \tag{2.5.13}$$

if and only if

$$\sum_{\nu=1}^M k^\nu \langle \mathring{g}_\nu, \mathring{a}^\mu \rangle_\delta = \langle f, \mathring{a}^\mu \rangle_\delta \quad \forall \mu = 1, \dots, M \tag{2.5.14}$$

where $\ker(\Delta_\delta^*) = \overline{\{\mathring{a}^\mu\}}$. Under the given assumptions, it holds

$$\langle \mathring{g}_\nu, \mathring{a}^\mu \rangle_\delta = \delta_\nu{}^\mu \tag{2.5.15}$$

which implies $k^\mu = \langle f, \mathring{a}^\mu \rangle_\delta$ and proves the claim. [6] □

Using the bi-orthogonal system constructed above one can build a projection operator

$$\mathbb{J} : W^{k-2,p}_{\gamma-2} \to \Delta_\delta(W^{k,p}_\gamma) \tag{2.5.17}$$

by setting

$$\mathbb{J}(\rho) = \mathbb{I}(\rho) - \sum_{\mu=1}^M \mathring{g}_\mu \langle \mathring{a}^\mu, \rho \rangle_\delta \tag{2.5.18}$$

for every $\rho \in W^{k-2,p}_{\gamma-2}$.

[6] *Remark (simpler basis):* In Lemma 2.5.1, the choice of basis fields that generate $\mathcal{I}_\delta$ was made for convenience in a later application. With this choice, it is possible to write $\mathring{g}_\nu = \mathrm{div}_\delta \mathring{f}_\nu$ for $\nu \neq 1$, as will be shown latter on. A simpler choice for basis fields $\mathring{g}_\nu$ is for example:

$$\mathring{g}_\mu = \chi(r) r^{-l} \overline{Y}_{lm}(\theta,\phi) \ . \tag{2.5.16}$$

with the same conditions on $\chi(r)$ as stated in the lemma.

2.6 The Implicit Function Theorem

Let $\mathcal{O}_{x_0} \in \mathcal{X}$ and $\mathcal{O}_{y_0} \in \mathcal{Y}$ be neighborhoods of $x_0 \in \mathcal{X}$ and $y_0 \in \mathcal{Y}$ respectively. Consider an operator

$$\Psi(x,y) : \mathcal{O}_{x_0} \times \mathcal{O}_{y_0} \to \mathcal{Z}$$

and assume it is C^r, $r \geq 1$ (see [Abraham et al., 1988]). Then we denote the derivative of the map Ψ by $\mathcal{D}\Psi$. Also for $x \in \mathcal{O}_{x_0}$, the derivative of the map $\Psi(x, y_0) : \mathcal{O}_{x_0} \to \mathcal{Z}$ is called *partial derivative of* Ψ in the direction of the first argument and will be denoted by $\mathcal{D}_{\mathcal{X}}\Psi$.

Theorem 2.6.1 (IFT [Abraham et al., 1988]). *Let $\Pi : \mathcal{X} \times \mathcal{Y} \to \mathcal{I}$ be C^r, $r \geq 1$ and consider $x_0 \in \mathcal{O}_{x_0} \subset \mathcal{X}$, $y_0 \in \mathcal{O}_{y_0} \subset \mathcal{Y}$ such that $\Pi(x_0, y_0) = 0$. If*

$$(\mathcal{D}_{\mathcal{X}}\Pi)|_{(x_0,y_0)} : \mathcal{X} \to \mathcal{I} \tag{2.6.1}$$

is an isomorphism, then there exist a neighborhood $\mathcal{O}'_{y_0} \subset \mathcal{O}_{y_0}$ and a unique C^r map $g : \mathcal{O}'_{y_0} \to \mathcal{X}$ satisfying

$$\Pi(g(y), y) = 0 \tag{2.6.2}$$

for all $y \in \mathcal{O}'_{y_0}$.

Corollary 2.6.2. *Let $\Psi : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be C^r with $r \geq 1$ and consider $x_0 \in \mathcal{O}_{x_0} \subset \mathcal{X}$, $y_0 \in \mathcal{O}_{y_0} \subset \mathcal{Y}$ such that $\Psi(x_0, y_0) = 0$. Assume that the operator $\mathcal{D}_{\mathcal{X}}\Psi$ at (x_0, y_0) has closed* $\operatorname{ran}(\mathcal{D}_{\mathcal{X}}\Psi) = \mathcal{I} \subset \mathcal{Z}$ *and that* $\dim(\mathcal{Z}/\mathcal{I})$ *is finite. Let $\mathbb{P} : \mathcal{Z} \to \mathcal{I}$ be a bounded projection operator so that*

$$\mathbb{P}(\mathcal{Z}) = \mathcal{I} \, . \tag{2.6.3}$$

If $(\mathcal{D}_{\mathcal{X}}\Psi)|_{(x_0,y_0)} : \mathcal{X} \to \mathcal{Z}$ is injective then there exists a neighborhood $\mathcal{O}'_{y_0} \subset \mathcal{O}_{y_0} \subset \mathcal{Y}$ and unique C^r map $q : \mathcal{O}'_{y_0} \to \mathcal{X}$ satisfying

$$\mathbb{P} \circ \Psi\,(q(y), y) = 0 \tag{2.6.4}$$

for all $y \in \mathcal{O}'_{y_0}$.

Proof. Define a new operator $\Pi : \mathcal{X} \times \mathcal{Y} \to \mathcal{I}$ by $\Pi(x,y) := \mathbb{P} \circ \Psi(x,y)$ and consider the equation $\Pi(x,y) = 0$. The linearization of Π in the direction of $\mathcal{X}$ is $(\mathcal{D}_{\mathcal{X}}\Pi) = \mathbb{P} \circ (\mathcal{D}_{\mathcal{X}}\Psi)$. It defines a map $(\mathcal{D}_{\mathcal{X}}\Pi)|_{(x_0,y_0)} : \mathcal{X} \to \mathcal{I}$, and is therefore an isomorphism. The Implicit Function Theorem can be applied to Π and this proves the corollary. □

Remark 2.6.3.

i) Observe that by Lemma 2.2.7, the closedness of $\mathcal{I}$ and finite dimensionality of $\mathcal{Z}/\mathcal{I}$ implies $\mathcal{I}$ is complemented.

ii) Boundedness of the projection operator comes as a consequence of the closedness of $\mathcal{I}$. See [Kato, 1995, pag. 156]

2.7 Cokernel Stability for Elliptic Operators

Definition 2.7.1 (Continuous Family)**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $\epsilon \in [0,1]$. Denote by $Q_\epsilon : \mathcal{X} \to \mathcal{Y}$ a family of bounded linear operators with parameter ϵ. We say $Q_\epsilon : \mathcal{X} \to \mathcal{Y}$ is a continuous family of operators if there exists a constant $C > 0$ such that $\|Q_\epsilon - Q_0\|_{Op} \leq C\epsilon$. The operator norm is defined as usual.[7]

Definition 2.7.2 (Uniform Injectivity)**.** Let $\epsilon \in [0,1]$ and denote by $Q_\epsilon : \mathcal{X} \to \mathcal{Y}$ a continuous family of linear elliptic operators between the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. We say the family is uniformly injective if there exists a constant $C > 0$ independent of ϵ such that

$$\|y\|_{\mathcal{X}} \leq C \|Q_\epsilon(y)\|_{\mathcal{Y}} \tag{2.7.2}$$

for all $y \in \mathcal{X}$.

Lemma 2.7.3 (Stability Lemma [Butscher, 2007])**.** *Let $Q_\epsilon : \mathcal{X} \to \mathcal{Y}$, $\epsilon \in [0,1]$ be a continuous family of linear, elliptic operators. Assume that Q_ϵ is uniformly injective. If* $\operatorname{ran}(Q_0) \cap \mathcal{C} = \{0\}$*, where $\mathcal{C}$ is a finite dimensional linear subspace of $\mathcal{Y}$, then there exists* $\epsilon_0 > 0$ *such that* $\operatorname{ran}(Q_\epsilon) \cap \mathcal{C} = \{0\}$ *for all $\epsilon < \epsilon_0$.*

Proof. Assume that for every $\epsilon > 0$ there exists a non-vanishing element $z_\epsilon \in \operatorname{ran}(Q_\epsilon) \cap \mathcal{C}$. Linearity allows for the assumption $\|z_e\|_{\mathcal{Y}} = 1$. Let $\{\epsilon_j\}$ be a sequence converging to zero. The finite dimensionality of $\mathcal{C}$ implies that the unit ball is compact. Thus there is a convergent subsequence which we denote also by $\{z_j\}$ for convenience. The sequence $\{z_j\}$ converges to a non-vanishing element $z \in \mathcal{C}$ with $\|z\|_{\mathcal{Y}} = 1$. Because we have assumed $z_\epsilon \in \operatorname{ran}(Q_\epsilon)$, there exist elements $y_\epsilon \in \mathcal{X}$ for which $Q_\epsilon(y_\epsilon) = z_\epsilon$. We now prove that the sequence y_j converges to an element $y \in \mathcal{X}$. If C denotes the injectivity constant of the family, it holds

$$\begin{aligned}
\|y_i - y_j\|_{\mathcal{X}} &\leq C\Big(\|(Q_0 - Q_i)(y_i)\|_{\mathcal{Y}} + \|(Q_0 - Q_j)(y_j)\|_{\mathcal{Y}} \\
&\quad + \|Q_i(y_i) - Q_j(y_j)\|_{\mathcal{Y}} \Big) \\
&\leq C\Big(\|(Q_0 - Q_i)(y_i)\|_{\mathcal{Y}} + \|(Q_0 - Q_j)(y_j)\|_{\mathcal{Y}} \\
&\quad + \|z_i - z_j\|_{\mathcal{Y}} \Big)
\end{aligned} \tag{2.7.3}$$

The continuity of the operator family implies the first two terms go to zero, while the last term goes to zero because $\{z_i\}$ is Cauchy. This implies $\{y_i\}$ is a Cauchy sequence in the norm $\mathcal{X}$ and therefore converges to an element $y \in \mathcal{X}$. Also we have

$$\begin{aligned}
\|z - Q_0(y)\|_{\mathcal{Y}} &\leq \|z - z_i + Q_i(y_i) - Q_i(y) + Q_i(y) - Q_0(y)\|_{\mathcal{Y}} \\
&\leq \|z - z_i\|_{\mathcal{Y}} + \|Q_i(y_i - y)\|_{\mathcal{Y}} + \|(Q_i - Q_0)(y)\|_{\mathcal{Y}}
\end{aligned} \tag{2.7.4}$$

which implies that $z = Q_0(y)$ and therefore gives $z \in \operatorname{ran}(Q_0) \cap \mathcal{C}$ non-vanishing, contradicting the hypothesis of $\operatorname{ran}(Q_0) \cap \mathcal{C} = \{0\}$. $\square$

[7] The operator norm is defined by

$$\|Q_\epsilon\|_{Op} = \sup_{\|x\|_{\mathcal{X}}=1} \left\{ \|Q_\epsilon(x)\|_{\mathcal{Y}} \right\}. \tag{2.7.1}$$

Chapter 3

Helmholtz decomposition with fast decay

3.1 Introduction

In this chapter we focus our attention on a simple underdetermined model problem, namely the construction of divergenceless 1-form fields. In particular where the solution is required to have fast decay at space-like infinity, this problem exhibits certain difficulties that we will encounter again in chapter 5, where we will use the findings of this case as a guide.

Consider in $(\mathbb{R}^3, h_{ab})$ the problem of finding 1-forms σ that are solutions to the equation

$$\operatorname{div}_h \sigma = 0 \ . \tag{3.1.1}$$

A typical approach to constructing solutions to this equation consists of prescribing a 1-form ψ and using the ansatz

$$\psi = dv + \sigma \tag{3.1.2}$$

where dv is the differential of a function v. The 1-form ψ is thought of as free data. It is usual in some of the literature to refer to this procedure as the *Helmholtz decomposition* of the field ψ, a name we will also use here. Inserting (3.1.2) into (3.1.1) gives $\Delta_h v = \operatorname{div}_h \psi$ and one is now in the position to study whether there exists a function v, with certain fall-off conditions, that solves this equation for arbitrary free data. Weighted Sobolev spaces provide an appropriate setting for analysing this equation.

3.2 Standard Helmholtz decomposition

We state a Theorem of [Cantor, 1981] concerning the Helmholtz decomposition on an asymptotically flat manifold[1].

Theorem 3.2.1 (7.6 in [Cantor, 1981]). *Let $(\mathbb{R}^3, h_{ab})$ be asymptotically flat of class (p, t, τ) with $p > 1$, $t > 3/p + 1$ and $\tau < 0$.*

If $s \in \mathbb{N}$, $3/p + 1 < s \leq t$, $\gamma \notin \mathbb{Z}$ and

$$-2 < \gamma < 1 \tag{3.2.1}$$

[1]While Cantor discusses more general situations, we only consider here the cases in which the number of derivatives controlled for the decomposed field satisfies $s > 3/p + 1$.

then it holds

$$\boxed{W^{s-1,p}_{\gamma-1}(\Lambda^1) = d(W^{s,p}_\gamma) \oplus \ker(\mathrm{div}_h)} \tag{3.2.2}$$

with $d(W^{s,p}_\gamma)$ *being closed.*

In the proof a 1-form ψ is prescribed such that $\psi \in W^{s-1,p}_{\gamma-1}(\Lambda^1)$ and one tries to decompose it as in (3.1.2). The requirement that the 1-form $\sigma \in W^{s-1,p}_{\gamma-1}(\Lambda^1)$ be in $\ker(\mathrm{div}_h)$ gives

$$\begin{cases} \Delta_h : W^{s,p}_\gamma(\Lambda^0) \to W^{s-2,p}_{\gamma-2}(\Lambda^0) \\ \Delta_h v = \mathrm{div}_h\, \psi \end{cases} \tag{3.2.3}$$

which turns the underdetermined elliptic equation (3.1.1) into an elliptic problem. Above and in the following we denote in parenthesis the type of objects of which each space is made of. The details of the proof of this Theorem are posponed to section 3.2.2.

We will refer to the decomposition of Theorem 3.2.1 as **standard Helmholtz decomposition**.

3.2.1 Abstract Banach space decompositions

Before addressing any further particularities of the Helmholtz decomposition, let us discuss the situation from the abstract point of view which generalizes to the situation considered later on. The feasibility of carrying out the Helmholtz decomposition can be understood as of the possibility that a certain Banach space $\mathcal{Y}$ can be split into two pieces, one of which is to be realized as the kernel of the operator $S : \mathcal{Y} \to \mathcal{Z}$ introduced by the problem. To complete the decomposition, an auxiliary Banach space $\mathcal{X}$ is introduced, along with an operator $T : \mathcal{X} \to \mathcal{Y}$. The intention behind this is that $\mathrm{ran}(T)$ should loosely speaking contribute the part of $\mathcal{Y}$ that is not contained in $\ker(S)$ so as to get a spliting

$$\mathcal{Y} = \mathrm{ran}(T) \oplus \ker(S) \ . \tag{3.2.4}$$

Decomposing an arbitrary element $\psi \in \mathcal{Y}$ can be carried out successfully if one can find $v \in \mathcal{X}$ and $\sigma \in \ker(S)$ such that $\psi = T(v) + \sigma$. The condition $\sigma \in \ker(S)$ gives

$$S \circ T(v) = S(\psi) \tag{3.2.5}$$

which is thought of as an equation for v.

In terms of this abstract formulation, the spaces and operators involved in the Helmholtz decomposition are

$$\mathcal{X}(\Lambda^0) \xrightarrow[T]{d} \mathcal{Y}(\Lambda^1) \xrightarrow[S]{\mathrm{div}_h} \mathcal{Z}(\Lambda^0) \ . \tag{3.2.6}$$

The key issue is whether one can choose appropriate norms in each of the spaces involved such that the decomposition can be carried out.

The Helmholtz decomposition and certain other Banach space decompositions can be studied by means of the following Lemma by Cantor (see [Cantor, 1981]).

Lemma 3.2.2 (2.2 in [Cantor, 1981])**.** *If* $T : \mathcal{X} \to \mathcal{Y}$ *and* $S : \mathcal{Y} \to \mathcal{Z}$ *are bounded linear operators between Banach spaces, then the following are equivalent*

i) $\mathrm{ran}(S \circ T) = \mathrm{ran}(S)$ *and* $\ker(S \circ T) = \ker(T)$

ii) $\mathcal{Y} = \mathrm{ran}(T) \oplus \ker(S)$

In particular when i) and ii) hold, $\mathrm{ran}(T)$ *is closed in* $\mathcal{Y}$.

3.2.2 Differential Operators

In the following will shall not discuss the choices of p and t in our assumptions regarding asymptotic flatness, but we keep them for future reference. It is important to note that a choice for a weight parameter γ has to be made depending on the desired decay rate of the prospective decompositions.

The values of γ are taken to be in $\mathbb{R}\setminus\mathbb{Z}$ to deal a technical issue with elliptic operators[2] between weighted Sobolev spaces in $n = 3$, as can already be seen to some extent from Theorems 2.4.3 and 2.4.4.

Assumption 3.2.3. Let $\gamma \in \mathbb{R}$, $p \in \mathbb{N}$, $1 < p < \infty$ and $s \in \mathbb{N}$, $2 \leq s$. The 1-form fields σ and ψ, and the function v, satisfy

$$\sigma, \psi \in W^{s-1,p}_{\gamma-1}(\Lambda^1) \tag{3.2.7a}$$

$$v \in W^{s,p}_{\gamma}(\Lambda^0)\ . \tag{3.2.7b}$$

The differential operators considered here define the following bounded linear maps

$$\mathrm{div}_h : W^{s-1,p}_{\gamma-1}(\Lambda^1) \to W^{s-2,p}_{\gamma-2}(\Lambda^0) \tag{3.2.8a}$$

$$d : W^{s,p}_{\gamma}(\Lambda^0) \to W^{s-1,p}_{\gamma-1}(\Lambda^1) \tag{3.2.8b}$$

$$\mathrm{div}_h \circ d = \Delta_h : W^{s,p}_{\gamma}(\Lambda^0) \to W^{s-2,p}_{\gamma-2}(\Lambda^0) \tag{3.2.8c}$$

and satisfy a number of properties collected in the following lemmas. Throughout the chapter, the operators will be considered as maps acting between the spaces shown above. We shall simply denote them by div_h, d and Δ_h when it is both convenient and clear.

Lemma 3.2.4 ([Cantor, 1981][Lockhart, 1981][Lockhart and McOwen, 1983])**.** *Let $\left(\mathbb{R}^3, h_{ab}\right)$ be asymptotically flat of class (p, t, τ) with $p > 1$, $\tau < 0$, $t > \frac{3}{p} + 1$. If $s \in \mathbb{N}$ is such that $2 \leq s \leq t$, then the operator*

$$\Delta_h : W^{s,p}_{\gamma} \to W^{s-2,p}_{\gamma-2}$$

defines a Fredholm map if and only of $\gamma \notin \mathbb{Z}$. If so, then

i) It is injective iff $\gamma < 0$,

ii) It is surjective iff $-1 < \gamma$.

and

$$\dim\left(\ker\left(\Delta_h\right)\right) = \dim\left(\ker\left(\Delta_\delta\right)\right)\ . \tag{3.2.9}$$

Proof. Follows from Theorem 2.4.5. □

The above Lemma states in particular that for metrics that are sufficiently close to the Euclidean metric in the function spaces that define asymptotic flatness, the dimension of $\ker\left(\Delta_h\right)$ does not depend on the metric.

Lemma 3.2.5. *The operator $d : W^{s,p}_{\gamma} \to W^{s-1,p}_{\gamma-1}$ satisfies*

$$\ker(d) = \begin{cases} \mathbb{R} & \text{if} \quad 0 < \gamma \\ \{0\} & \text{if} \quad \gamma < 0 \end{cases} \tag{3.2.10}$$

[2] Some authors refer to this by saying that γ is *not excepcional* if $\gamma \in \mathbb{R}\setminus\mathbb{Z}$. In dimensions other than $n = 3$, the set of non-excepcional weights takes a slightly different form.

Proof. Clear. □

Lemma 3.2.6. *Let $\left(\mathbb{R}^3, h_{ab}\right)$ be asymptotically flat of class (p,t,τ) with $p>1$, $\tau<0$, $t>\frac{3}{p}+1$. If $s\in\mathbb{N}$ is such that $2\le s\le t$, and if $\gamma\neq\mathbb{Z}$ then the operator*

$$\mathrm{div}_h : W^{s-1,p}_{\gamma-1} \to W^{s-2,p}_{\gamma-2} \tag{3.2.11}$$

is lower semi-Fredholm and satisfies

$$\mathrm{ran}\,(\mathrm{div}_h) = \begin{cases} W^{s-2,p}_{\gamma-2} & \text{if } -1<\gamma \\ \{\mathbb{R}\}^{\perp} & \text{if } \gamma<-1 \end{cases} \tag{3.2.12}$$

Proof. It is clear that $\Delta_h(W^{s,p}_\gamma) = \mathrm{div}_h \circ d(W^{s,p}_\gamma) \subset \mathrm{div}_h(W^{s-1,p}_{\gamma-1})$. By Lemma 3.2.4, the set $\Delta_h(W^{s,p}_\gamma)$ is closed and has finite codimension. Then Lemma 2.2.5 implies that $\mathrm{div}_h(W^{s-1,p}_{\gamma-1})$ is also closed and of finite codimension, thus lower semi-Fredholm. The characterization (3.2.12) follows from the closed range Theorem and the remarks of section 2.3.3. □

Proof of Standard Helmholtz Decomposition

Proof of Theorem 3.2.1. The decomposition (3.2.2) follows from Lemma 3.2.2 and the two following facts.
i) It holds that

$$\ker\left(d : W^{s,p}_\gamma \to W^{s-1,p}_{\gamma-1}\right) = \ker\left(\Delta_h : W^{s,p}_\gamma \to W^{s-2,p}_{\gamma-2}\right) \tag{3.2.13}$$

if and only if $\gamma\notin\mathbb{Z}$ and $\gamma<1$.

It is immediate that $\ker(d)\subset\ker(\Delta_h)$. If $\gamma<0$ then $\ker(\Delta_h)=\emptyset$ and one gets the claim. If $0<\gamma<1$, we know from Lemma 3.2.5 that $\ker(\Delta_h)$ consists of constants only. The constants are in $\ker(d)$ and then with $\ker(\Delta_h)\subset\ker(d)$ the identity follows. For $1<\gamma$ one has that $\dim(\ker(\Delta_h))>1$ so equality cannot hold.
ii) It holds that

$$\mathrm{ran}\left(\mathrm{div}_h : W^{s-1,p}_{\gamma-1} \to W^{s-2,p}_{\gamma-2}\right) = \mathrm{ran}\left(\Delta_h : W^{s,p}_\gamma \to W^{s-2,p}_{\gamma-2}\right) \tag{3.2.14}$$

if and only if $\gamma\notin\mathbb{Z}$ and $-2<\gamma$. This Follows similarly from Lemmas 3.2.4 and 3.2.6. □

3.3 Extended Banach Space Decompositions

As mentioned in the introduction, our main interest is the construction of divergenceless 1-forms with fast decay at space-like infinity. In terms of the weight parameter γ for the Helmholtz decomposition, 'fast decay' translates into the requirement that γ be a large negative number. Difficulties arise because the arguments in the proof of Theorem 3.2.1 do not apply for weights $\gamma<-2$.

To overcome these difficulties we will extend the decompositions (3.2.4) by including a finite dimensional subspace $\mathcal{J}\subset\mathcal{Y}$ such that

$$\mathcal{Y} = \mathrm{ran}\,(T) \oplus \ker\,(S) \oplus \mathcal{J} \tag{3.3.1}$$

holds. In the case of the Helmholtz decomposition, we shall refer to such a set $\mathcal{J}$ as a **complement extending the Helmholtz decomposition**.

The Theorem that follows simplifies and extends earlier work of [Specovius-Neugebauer, 1990] where the Helmholtz decomposition was considered in Euclidean space $(\mathbb{R}^n, \delta_{ab})$.

Theorem 3.3.1 (Abstract decomposition extension). *Let $T : \mathcal{X} \to \mathcal{Y}$ and $S : \mathcal{Y} \to \mathcal{Z}$ be bounded linear operators such that $Q = S \circ T : \mathcal{X} \to \mathcal{Z}$ is Fredholm. Then S is lower semi-Fredholm. Furthermore, there exists a subspace $\mathcal{J} \subset \mathcal{Y}$ of finite dimension $N_+ - N_-$ such that*

$$\mathcal{Y} = \ker(S) \oplus T(\mathcal{X}) \oplus \mathcal{J} \tag{3.3.2}$$

where $N_+ = \dim(\mathcal{Z}/Q(\mathcal{X}))$ and $N_- = \dim(\mathcal{Z}/S(\mathcal{Y}))$.

Proof. Since Q is Fredholm, $Q(\mathcal{X})$ is closed and $\mathcal{Z}/Q(\mathcal{X})$ is finite dimensional. Using $Q(\mathcal{X}) = S \circ T(\mathcal{X}) \subset S(\mathcal{Y})$ and Lemma 2.2.5 it follows that $S(\mathcal{Y})$ is closed and $\mathcal{Z}/S(\mathcal{Y})$ is finite dimensional, making S lower semi-Fredholm (note that this is the same argument of Lemma 3.2.6).

By the closed range Theorem

$$Q(\mathcal{X}) = \{\ker(Q^*)\}^{\perp} \tag{3.3.3a}$$

$$S(\mathcal{Y}) = \{\ker(S^*)\}^{\perp} . \tag{3.3.3b}$$

Given that $\ker(Q^*) = \ker(T^* \circ S^*)$, one knows that $\ker(Q^*) \supset \ker(S^*)$. Let $a^\mu \in \mathcal{Z}^*$, $\mu = 1, \ldots, N_+$ be linearly independent elements such that

$$\ker(Q^*) = \overline{\{a^1, \ldots, a^{N_-}, a^{N_-+1}, \ldots, a^{N_+}\}} \tag{3.3.4a}$$

$$\ker(S^*) = \overline{\{a^1, \ldots, a^{N_-}\}} \tag{3.3.4b}$$

where $N_+ = \dim(\ker(Q^*))$, $N_- = \dim(\ker(S^*))$. As was pointed out in section 2.2, there exists a family of N_+ fields $g_\nu \in \mathcal{Z}$ such that

$$\langle g_\nu, a^\mu \rangle = \delta_\nu{}^\mu \qquad \forall \mu, \nu = 1, \ldots, N_+ \tag{3.3.5}$$

and thus the families $\{a^\mu\}_{\mu=1}^{N_+}$ and $\{g_\nu\}_{\nu=1}^{N_+}$ form a bi-orthogonal system. By equation (3.3.3b) and (3.3.5) one knows that $g_\nu \in \operatorname{ran}(S)$ for $\nu = N_- + 1, \ldots, N_+$. Thus there exist $N_+ - N_-$ elements $f_\nu \in \mathcal{Y}$ such that

$$S(f_\nu) = g_\nu , \qquad \nu = N_- + 1, \ldots, N_+ . \tag{3.3.6}$$

We now let

$$\mathcal{J} = \overline{\left\{f_{N_-+1}, \ldots, f_{N_+}\right\}} . \tag{3.3.7}$$

Let $\psi \in \mathcal{Y}$ be arbitrary. To show that (3.3.2) holds, let us assume ψ is of the form

$$\psi = \sigma + T(x) + \xi \tag{3.3.8}$$

where $x \in \mathcal{X}$, $\xi \in \mathcal{J}$ and $\sigma \in \ker(S)$. This gives the equation

$$Q(x) = S(\psi - \xi) \tag{3.3.9}$$

which we consider as a condition on x and ξ. A solution x to equation (3.3.9) will exist if and only if

$$\langle S(\psi - \xi), a^\mu \rangle = 0 \qquad \forall \mu = 1, \ldots, N_+ . \tag{3.3.10}$$

These conditions are satisfied for the first $\mu = 1, \ldots, N_-$ because of equation (3.3.4b). By writing $\xi \in \mathcal{J}$ as

$$\xi = \sum_{\nu=N_-+1}^{N_+} k^\nu f_\nu \tag{3.3.11}$$

one obtains the linear system

$$\langle S(\psi), a^\mu \rangle = \sum_{\nu=N_-+1}^{N_+} k^\nu \langle S(f_\nu), a^\mu \rangle \tag{3.3.12}$$

which has the unique solution $k^\nu = \langle S(\psi), a^\nu \rangle$ by (3.3.5) and (3.3.6). With the obtained ξ one can solve equation (3.3.9) and the decomposition of ψ is completed as claimed. □

3.3.1 Cokernel Stability

The Lemma that follows will be used in a stability argument later on, but it is convenient that we state and prove it here.

Lemma 3.3.2. *Let $Q_\epsilon : [0,1] \times \mathcal{X} \to \mathcal{Z}$ be a family of elliptic Fredholm operators. Assume Q_ϵ is uniformly injective in the sense of definition 2.7.2 and that $\dim(\ker(Q_\epsilon^*)) = N_+$ is independent of ϵ. Let $\{g_\nu\}_{\nu=1}^{N_+}$ be a set of fields in $\mathcal{Z}$ such that $\mathcal{I}_0 = \overline{\{g_\nu\}}$ satisfies*

$$\mathcal{Z} = Q_0(\mathcal{X}) \oplus \mathcal{I}_0 \, . \tag{3.3.13}$$

Then there exists $\epsilon_0 > 0$ such that

$$\mathcal{Z} = Q_\epsilon(\mathcal{X}) \oplus \mathcal{I}_0 \quad \textit{for} \quad 0 \leq \epsilon < \epsilon_0 \, , \tag{3.3.14}$$

and there exists an ϵ-dependent basis $\{\widehat{a}^\mu\}_{\mu=1}^{N_+}$ of $\ker(Q_\epsilon^*)$ such that the matrix $\langle \widehat{a}^\mu, g_\nu \rangle$ is non-degenerate for $0 \leq \epsilon < \epsilon_0$.

Proof. One wants to show is that it is possible to use, as a complement to $Q_\epsilon(\mathcal{X})$, the set $\mathcal{I}_0$ which is independent of ϵ.

The hypothesis $Q_0(\mathcal{X}) \cap \mathcal{I}_0 = \{0\}$ and Lemma 2.7.3 imply there exists $\epsilon_0 > 0$ such that

$$Q_\epsilon(\mathcal{X}) \cap \mathcal{I}_0 = \{0\} \quad \text{for} \quad 0 \leq \epsilon < \epsilon_0 \, . \tag{3.3.15}$$

By the closed range Theorem 2.2.3, one has $Q_\epsilon(\mathcal{X}) = \{\ker(Q_\epsilon^*)\}^\perp$ and by assumption $\dim(\ker(Q_\epsilon^*)) = N_+$ is independent of ϵ. Let $\{\widehat{a}^\mu\}_{\mu=1}^{N_+}$ denote an ϵ-dependent basis for $\ker(Q_\epsilon^*)$ and consider the square matrix $A^\mu{}_\nu = \langle \widehat{a}^\mu, g_\nu \rangle$. The matrix $A^\mu{}_\nu$ is non-degenerate. To see this, assume there exists $k^\nu \in \mathbb{R}^{N_+} - \{0\}$ such that $A^\mu{}_\nu k^\nu = 0$. Then one would have that

$$\langle \widehat{a}^\mu, k^\nu g_\nu \rangle = 0 \quad \forall \mu \tag{3.3.16}$$

and therefore $0 \neq k^\nu g_\nu \in Q_\epsilon(\mathcal{X}) \cap \mathcal{I}_0$ in contradiction with (3.3.15).

To prove that the decomposition (3.3.14) holds, let $f \in \mathcal{Z}$ be arbitrary. The equation

$$Q_\epsilon(x) + \sum_{\nu=1}^{N_+} g_\nu k^\nu = f \tag{3.3.17}$$

can be solved for the coefficients k^ν by first setting $k^\nu = (A^{-1})^\nu{}_\mu \langle \widehat{a}^\mu, f \rangle$, and then solving for $x \in \mathcal{X}$. □

Remark 3.3.3. Note that one can choose a basis $\{\widehat{a}^\mu\}_{\mu=1}^{N_+}$ such that it satisfies

$$\langle \widehat{a}^\mu, g_\nu \rangle = \delta^\mu{}_\nu \quad \text{for} \quad 0 \leq \epsilon < \epsilon_0 \, . \tag{3.3.18}$$

by a redefinition

$$\widehat{a}^\mu \rightarrow (A^{-1})^\mu{}_\nu \widehat{a}^\nu \, . \tag{3.3.19}$$

3.4 Extended Helmholtz decomposition

The abstract decomposition Theorem 3.3.1 can be applied to the study of divergenceless 1-forms. In what remains of this chapter the symbols n_+ and n_- are used specifically to refer to

$$n_+ = \dim(\operatorname{coker}(\Delta_h)) \tag{3.4.1a}$$
$$n_- = \dim(\operatorname{coker}(\operatorname{div}_h)) \ . \tag{3.4.1b}$$

This numbers depend on the weight parameter γ. By Lemma 3.2.6 it holds that

$$n_- = \begin{cases} 0 & \text{if} \quad -1 < \gamma \\ 1 & \text{if} \quad \gamma < -1 \end{cases} \tag{3.4.2}$$

As for the Laplace operator (see equation (2.5.7)), one can learn from the Euclidean $\mathbb{R}^3$ case that n_+ behaves as

$$n_+ = \begin{cases} 0 & \text{if} \quad -1 < \gamma \\ 1 & \text{if} \quad -2 < \gamma < -1 \\ 4 & \text{if} \quad -3 < \gamma < -2 \\ \dots \end{cases} \tag{3.4.3}$$

and continues to grow the further down the negative values of γ one is looking at.

The following result shows that it is possible to construct a finite dimensional set to extend the Helmholtz decomposition with respect to a given metric h_{ab}.

Theorem 3.4.1. *Let $(\mathbb{R}^3, h_{ab})$ be asymptotically flat of class (p, t, τ) with $\tau < 0$. If $\gamma \notin \mathbb{Z}$ and $\gamma < 1$ then there exist $n_+ - n_-$ fields $f_\nu \in W^{s-1,p}_{\gamma-1}(\Lambda^1)$ such that*

$$\boxed{W^{s-1,p}_{\gamma-1}(\Lambda^1) = d(W^{s,p}_\gamma) \oplus \ker(\operatorname{div}_h) \oplus \mathcal{J}_h} \tag{3.4.4}$$

holds with $\mathcal{J}_h = \overline{\{f_\nu\}_{\nu=1+n_-}^{n_+}}$, and where n_+, n_- are those in (3.4.1a).

Proof. For values of $-2 < \gamma$, the statement coincides with that of Theorem 3.2.1. For $\gamma < -2$ it follows from the abstract version 3.3.1. □

3.4.1 Example Application: Euclidean Case

Fields $\mathring{f}_\nu$ will now be constructed explicitly so that $\mathcal{J}_\delta = \overline{\left\{\mathring{f}_\nu\right\}_{\nu=2}^{n_+}}$ satisfies

$$W^{s-1,p}_{\gamma-1}(\Lambda^1) = d(W^{s,p}_\gamma) \oplus \ker(\operatorname{div}_\delta) \oplus \mathcal{J}_\delta \ . \tag{3.4.5}$$

The conditions that are required of the $\mathring{f}_\nu$ can be stated in terms of explicit expressions for the harmonic polynomials $\mathring{a}^\mu$ and can be transformed into

$$\langle \mathring{a}^\mu, \operatorname{div}_\delta \mathring{f}_\nu \rangle_\delta = -\langle d\mathring{a}^\mu, \mathring{f}_\nu \rangle_\delta \ . \tag{3.4.6}$$

With the choice of basis made in section 2.5.1, one can compute

$$d\mathring{a}^\mu = \begin{cases} 0 \quad \text{for} \quad \mu = 1 \\ r^{q-1}\left(qY_{qm}\, dr + r\, dY_{qm}\right) \quad \text{for} \quad \mu \neq 1 \end{cases} \ . \tag{3.4.7}$$

Consider the 1-form fields $\mathring{f}_\nu \in W^{s-1,p}_{\gamma-1}$ given by

$$\mathring{f}_\nu = -\chi(r)\frac{Y^*_{q'm'}}{q' r^{q'-1}}(dr) \tag{3.4.8}$$

for $\nu = 2, \ldots, n_+$, where $\chi(r)$ satisfies the hypothesis from lemma 2.5.1. This implies

$$\begin{aligned}
\langle d\mathring{a}^\mu, \mathring{f}_\nu\rangle_\delta &= -\int_{\mathbb{R}^3} \chi(r) Y^*_{q'm'} \frac{r^{q-1}}{q' r^{q'-1}} \left(qY_{qm} + r\partial_r Y_{qm}\right) \mathrm{d}\mu_\delta \\
&= -\delta_{q,q'}\delta_{m,m'} C \int_0^\infty \chi(r) r^2 dr \\
&= -\delta^\mu{}_\nu
\end{aligned} \tag{3.4.9}$$

for $\nu = 2, \ldots, n_+$, and where C is the normalization constant for the spherical harmonics. Hence, they are an explicit construction of the fields of of equation (3.3.6) in this particular case.

Remark 3.4.2. It should be noted that by construction, the fields $\mathring{f}_\nu$ satisfy

$$\mathrm{div}_\delta(\mathring{f}_\nu) = \mathring{g}_\nu \quad \text{for} \quad \nu \neq 1 \tag{3.4.10}$$

for the functions $\mathring{g}_\nu$ that where defined in lemma 2.5.1.

3.4.2 Extension Stability

A particular explicit construction of an extension to the Helmholtz decomposition in Euclidean $\mathbb{R}^3$ was shown in (3.4.5). In this section we study the possibility of using the set $\mathcal{J}_\delta$ as a complement for the extended Helmholtz decomposition with respect to another metric h_{ab}. It will be shown that

$$\mathcal{Y} = \mathrm{ran}\,(d) \oplus \ker\,(\mathrm{div}_h) \oplus \mathcal{J}_\delta \tag{3.4.11}$$

holds when h_{ab} is sufficiently close to the Euclidean metric. The proof of this statement will rely on Lemma 3.3.2 and on the following estimate.

Lemma 3.4.3. *Let $\left(\mathbb{R}^3, h_{ab}\right)$ be asymptotically flat of class (p,t,τ), $\tau < 0$, $t > \frac{3}{p}$. If in end coordinates*

$$\|h_{ij} - \delta_{ij}\|_{W^{t,p}_\tau} < \epsilon \tag{3.4.12}$$

for some $0 < \epsilon$, then there exist a constant C such that

$$\|(\mathrm{div}_h - \mathrm{div}_\delta)\, w\|_{W^{0,p}_{\gamma-2}} \leq C\epsilon\, \|w\|_{W^{1,\infty}_{\gamma-1-\tau}} \tag{3.4.13}$$

for every $w \in W^{1,\infty}_{\gamma-1-\tau}(\Lambda^1)$.

Proof. Define γ_{ab} and θ^{ab} by

$$\gamma_{ab} = h_{ab} - \delta_{ab} \tag{3.4.14}$$

$$\theta^{ab} = h^{ab} - \delta^{ab} \tag{3.4.15}$$

and let ∇_a be the derivative operator compatible with h_{ab}, and $\mathring{\nabla}_a$ the derivative operator compatible with δ_{ab}.

By definition,

$$\mathrm{div}_\delta(w_b) = \delta^{ab}\mathring{\nabla}_a w_b \quad , \quad \mathrm{div}_h(w_b) = h^{ab}\nabla_a w_b \ . \tag{3.4.16}$$

Using

$$\nabla_a w_b = \mathring{\nabla}_a w_b - C^c_{ab} f_c \tag{3.4.17}$$

with

$$C^c_{ab} = \frac{1}{2} h^{cd}\left(\mathring{\nabla}_a h_{bd} + \mathring{\nabla}_b h_{ad} - \mathring{\nabla}_d h_{ab}\right) \tag{3.4.18}$$

one obtains

$$\begin{aligned}(\mathrm{div}_h - \mathrm{div}_\delta)\, w_c &= (\delta^{ab} + \theta^{ab})(\mathring{\nabla}_a w_b - C^c_{ab} w_c) - \delta^{ab}\mathring{\nabla}_a w_b && (3.4.19)\\ &= \theta^{ab}\mathring{\nabla}_a w_b - h^{ab} C^c_{ab} w_c \ . && (3.4.20)\end{aligned}$$

This gives

$$\|(\mathrm{div}_h - \mathrm{div}_\delta)\, w_c\|_{W^{0,p}_{\gamma-2}} \le \left\|\theta^{ab}\mathring{\nabla}_a w_b\right\|_{W^{0,p}_{\gamma-2}} + \left\|h^{ab} C^c_{ab} w_c\right\|_{W^{0,p}_{\gamma-2}} \ . \tag{3.4.21}$$

Consider the first term of the right hand side in (3.4.21). By Hölder's inequality

$$\begin{aligned}\left\|\theta^{ab}\mathring{\nabla}_a w_b\right\|_{W^{0,p}_{\gamma-2}} &\le 9\left\|\theta^{ab}\right\|_{W^{0,p}_{\tau}}\left\|\mathring{\nabla}_a w_b\right\|_{W^{0,\infty}_{\gamma-2-\tau}} && (3.4.22)\\ &\le 9\left\|\theta^{ab}\right\|_{W^{1,p}_{\tau}}\|w_b\|_{W^{1,\infty}_{\gamma-1-\tau}} && (3.4.23)\end{aligned}$$

where the definition of the $W^{1,\infty}_{\gamma-1-\tau}$ norm and the $W^{1,p}_{\tau}$ where used and each time a contraction is removed a factor 3 comes out.

For the second term of (3.4.21), first note that one can assume there exists a constant C_1 such that

$$\left\|h^{ab}h^{cd}\right\|_{W^{0,\infty}_0} = \left\|(\delta^{ab} + \theta^{ab})(\delta^{cd} + \theta^{cd})\right\|_{W^{0,\infty}_0} < \frac{C_1}{3^5} \ . \tag{3.4.24}$$

Then

$$\begin{aligned}\left\|h^{ab} C^c_{ab} w_c\right\|_{W^{0,p}_{\gamma-2}} &\le \frac{1}{2}\left\|h^{ab}h^{cd}\left(\mathring{\nabla}_a \gamma_{bd} + \mathring{\nabla}_b \gamma_{ad} - \mathring{\nabla}_d \gamma_{ab}\right) w_c\right\|_{W^{0,p}_{\gamma-2}} && (3.4.25)\\ &\le \frac{C_1}{3}\left\|\left(\mathring{\nabla}_a \gamma_{bd} + \mathring{\nabla}_b \gamma_{ad} - \mathring{\nabla}_d \gamma_{ab}\right) w_c\right\|_{W^{0,p}_{\gamma-2}} && (3.4.26)\\ &\le C_1\left\|\mathring{\nabla}_a \gamma_{bd}\right\|_{W^{0,p}_{\tau-1}}\|w_c\|_{W^{0,\infty}_{\gamma-1-\tau}} && (3.4.27)\end{aligned}$$

where Hölder's inequality was used once more.

By the definition of the $W^{1,p}_{\tau}$ norm of γ_{bc} and the $W^{1,\infty}_{\gamma-1-\tau}$ norm of w_c one obtains then

$$\left\|h^{ab} C^c_{ab} w_c\right\|_{W^{0,p}_{\gamma-2}} < C_1\|\gamma_{bc}\|_{W^{1,p}_{\tau}}\|w_c\|_{W^{1,\infty}_{\gamma-1-\tau}} \ . \tag{3.4.28}$$

Coming back to (3.4.21), one gets

$$\|(\mathrm{div}_h - \mathrm{div}_\delta)\, w_c\|_{W^{0,p}_{\gamma-2}} < \left(9\left\|\theta^{ab}\right\|_{W^{1,p}_{\tau}} + C_1\|\gamma_{bc}\|_{W^{1,p}_{\tau}}\right)\|w_c\|_{W^{1,\infty}_{\gamma-1-\tau}} \ . \tag{3.4.29}$$

Using Lemma 2.3.7 it follows that it is possible to choose ϵ small enough so that the claimed estimate is obtained. $\square$

Theorem 3.4.4. *Let $\gamma \notin \mathbb{Z}$ satisfying $\gamma < 1$. Let $(\mathbb{R}^3, h_{ab})$ be an asymptotically flat metric of class (p, t, τ) with $\tau < 0$, $t > 3/p+1$, and such that in end coordinates $\|h_{ij} - \delta_{ij}\|_{W^{t,p}_\tau}$ is sufficiently small. If $s \in \mathbb{N}$, $\frac{3}{p}+1 \leq s \leq t$, and the set $\mathcal{J}_\delta$ is a complement for the extended Helmholtz decomposition in $(\mathbb{R}^3, \delta_{ab})$, then the decomposition*

$$\boxed{W^{s-1,p}_{\gamma-1}(\Lambda^1) = d(W^{s,p}_\gamma) \oplus \ker(\mathrm{div}_h) \oplus \mathcal{J}_\delta} \tag{3.4.30}$$

holds for $\gamma < 1$.

Proof. The claimed decomposition holds if the matrix

$$A^\mu{}_\nu = \langle a^\mu, \mathrm{div}_h \, \mathring{f}_\nu \rangle_h \quad \text{for} \quad \nu, \mu = 2, \dots, n_+ \tag{3.4.31}$$

is non-degenerate, where the elements $\{a^\mu\}_{\mu=1}^{n_+}$ are a basis of $\ker(\Delta_h^*)$. The fields $\mathring{f}_\nu$ satisfy $\mathrm{div}_\delta \, \mathring{f}_\nu = \mathring{g}_\nu$ for $\nu = 2, \dots, n_+$, so one can write

$$A^\mu{}_\nu = \langle a^\mu, \mathring{g}_\nu \rangle_h + \langle a^\mu, (\mathrm{div}_h - \mathrm{div}_\delta) \mathring{f}_\nu \rangle_h \, . \tag{3.4.32}$$

Using Lemma 3.3.2, one can assume that $\langle a^\mu, \mathring{g}_\nu \rangle_h$ is non-degenerate. Note also that both terms in equation (3.4.32) are linear in a^μ, in $\mathring{f}_\nu$ and in $\mathring{g}_\nu$.

We now show that the first term in (3.4.32) dominates the second term. By the definition of the pairing $\langle \, , \, \rangle_h$ and Hölder's inequality one has

$$\begin{aligned} |\langle a^\mu, \mathring{g}_\nu \rangle_h| &\leq C_1 \, \|a^\mu \mathring{g}_\nu\|_{W^{0,1}_{-3}} \\ &\leq C_1 \, \|a^\mu\|_{W^{0,p'}_{-\gamma-1}} \, \|\mathring{g}_\nu\|_{W^{0,p}_{\gamma-2}} \end{aligned} \tag{3.4.33}$$

where the constant C_1 accounts for the supremum norm of the volume element associated to h_{ab}. Similarly one has

$$\begin{aligned} \left| \langle a^\mu, (\mathrm{div}_h - \mathrm{div}_\delta) \mathring{f}_\nu \rangle_h \right| &\leq C_1 \, \left\| a^\mu (\mathrm{div}_h - \mathrm{div}_\delta) \mathring{f}_\nu \right\|_{W^{0,1}_{-3}} \\ &\leq C_1 \, \|a^\mu\|_{W^{0,p'}_{-\gamma-1}} \, \left\| (\mathrm{div}_h - \mathrm{div}_\delta) \mathring{f}_\nu \right\|_{W^{0,p}_{\gamma-2}} \\ &\leq C C_1 \epsilon \, \|a^\mu\|_{W^{0,p'}_{-\gamma-1}} \, \left\| \mathring{f}_\nu \right\|_{W^{1,\infty}_{\gamma-1-\tau}} \end{aligned} \tag{3.4.34}$$

where Lemma 3.4.3 was used. Comparing (3.4.33) with (3.4.34) one can see that if $\|h_{ij} - \delta_{ij}\|_{W^{t,p}_\tau}$ is sufficiently small then $A^\mu{}_\nu$ is be non-degenerate. □

Chapter 4

Asymptotic Staticity

4.1 Introduction

Consider a 3-dimensional manifold $\mathcal{M}$, a Riemannian metric h_{ab} and a symmetric tensor K_{ab}. One says that $(\mathcal{M}, h_{ab}, K_{ab})$ is an initial data set to Einstein's vacuum field equations if it satisfies the *vacuum constraints*, i.e. the Hamilton constraint

$$\mathrm{R}(h) - |K|_h^2 + (\mathrm{tr}_h K)^2 = 0 \tag{4.1.1a}$$

and the momentum constraint

$$\mathrm{div}_h(K) - d \circ \mathrm{tr}_h(K) = 0 \; . \tag{4.1.1b}$$

Einstein's vacuum constraints constitute an under-determined elliptic system of four equations for twelve unknowns. It is not evident at first sight how one could prescribe a priori some of these unknowns in a way that makes possible solving for the remaining ones.

A well established and widely used method for constructing initial data is what is referred to the literature as the *method of conformal rescalings*. This method provides a way to prescribe some of the unknowns –thought of as *free data*– and turns equations (4.1.1) into a determined elliptic system for the remaining variables.

At the core of this method is the idea that one can specify (up to certain conditions) a background metric $\hat{h}_{ab}$ and require that the solution h_{ab} to (4.1.1) be conformally related to $\hat{h}_{ab}$ so that

$$h_{ab} = \phi^4 \hat{h}_{ab} \tag{4.1.2}$$

with a positive function ϕ on $\mathcal{M}$, which possibly satisfies some boundary conditions. The method also gives a prescription for how to give data for the symmetric tensor K_{ab} which we will discuss later, but in a different context.

The construction of initial data discussed in the present work is specifically intended to address certain aspects in the modeling of isolated gravitating systems, so we restrict our attention to manifolds which are asymptotically flat in the sense of definition 2.3.5.

The method of conformal rescalings does not seem to provide one with the means to get finer control on the behavior of solutions at space-like infinity.

For simplicity, we will focus on the construction of time reflection symmetric initial data. The vacuum constraints reduce in this case (vanishing extrinsic curvature) to the problem of finding metrics h_{ab} that satisfy the equation,

$$\mathrm{R}(h) = 0 \; . \tag{4.1.3}$$

The ansatz (4.1.2) then leads to the equation

$$\Delta_{\hat{h}}\phi = \tfrac{1}{8}\,\mathrm{R}(\hat{h})\phi \tag{4.1.4}$$

and one requires the solution to satisfy the condition $\phi \to 1$ at space-like infinity. In a reasonable function space the solution will then be determined uniquely.

We are interested, however, in constructing initial data with certain special asymptotics which cannot be controlled by this method. A system of equations will be proposed with this specific goal in mind. To motivate the introduction of our system of equations, we consider first Einstein's static vacuum field equations.

4.1.1 Static Vacuum Field Equations

Let $\left(\mathcal{M}^{(4)}, g^{(4)}_{ab}\right)$ be a four dimensional space-time and consider Einstein's vacuum field equations

$$\mathrm{Ric}^{(4)}[g] = 0 \tag{4.1.5}$$

where $\mathrm{Ric}^{(4)}$ denotes the four dimensional Ricci tensor. A space-time is said to be static if there exists a timelike Killing field ξ which is hypersurface orthogonal. In a static space-time it is possible to construct a coordinate t adapted to the integral curves of the Killing field so that $\xi = \partial_t$ and ξ is orthogonal to the hypersurfaces $\{t = \text{const.}\}$. Then $\mathcal{M}^{(4)} \cong \mathbb{R} \times \mathcal{M}$ where $\mathcal{M}$ is a three dimensional manifold and the space-time metric takes the form

$$g^{(4)}_{ab} = -v^2(dt)^2 + h_{ab} \tag{4.1.6}$$

where h_{ab} is a Riemannian metric on $\mathcal{M}$ and $v : \mathcal{M} \to \mathbb{R}$ is a positive function. Equation (4.1.5) reduces in this case to the problem of finding a positive function v and a metric h_{ab} satisfying on $\mathcal{M}$ the *static vacuum field equations*

$$\Delta_h v = 0 \tag{4.1.7a}$$

$$-v\ \mathrm{Ric}(h) + \mathrm{Hess}_h(v) = 0\ . \tag{4.1.7b}$$

A solution (v, h_{ab}) to the static vacuum field equations provides in particular a solution to (4.1.3).

A few important remarks need to be made about the manifold $\mathcal{M}$. To begin with, if $\mathcal{M}$ is a closed manifold then the only solution to (4.1.7) is the trivial one, namely v is constant and h is flat. We consider here open manifolds. There is a classical result of [Lichnerowicz, 1955] which implies that if $(\mathcal{M}, h)$ is an asymptotically flat and complete solution of (4.1.7) with $v \to 1$ at infinity, then $v = 1$ everywhere and the solution corresponds to Euclidean $\mathbb{R}^3$. It was shown in more generality in [Anderson, 2000] that the hypotheses on the asymptotic behavior are not necessary. These results suggest that one must consider equations (4.1.7) on asymptotically flat manifolds having an inner boundary $\partial M \neq \emptyset$ if one is to have non-trivial solutions.

4.2 The system of equations

Let σ be a symmetric tensor field defined on $\mathcal{M}$ and consider the system of equations

$$\Delta_h v + \mathrm{tr}_h(\sigma) = 0 \tag{4.2.1a}$$

$$-v\ \mathrm{Ric}(h) + \mathrm{Hess}_h(v) + \sigma = 0 \tag{4.2.1b}$$

For vanishing σ these equations reduce to (4.1.7) and in this sense they generalize them. Moreover, for suitable non-vanishing fields σ one may expect to find non-trivial solutions on manifolds without boundary. Solutions to this system of equation are then also solutions of (4.1.3), and are therefore admissible time symmetric vacuum initial data.

The idea behind this generalization is that if it were possible to solve this system in a way such that σ has strong decay at space-like infinity, then one could interpret such solutions as *asymptotically static vacuum initial data.*

However, the symmetric tensor field σ cannot be prescribed arbitrarily. To ensure consistency it is necessary satisfy the contracted Bianchi identity

$$\left(\operatorname{div}_h - \tfrac{1}{2} d \circ \operatorname{tr}_h\right) \operatorname{Ric}(h) = 0 \ . \tag{4.2.2}$$

Together with (4.1.3) this leads to the integrability condition

$$\operatorname{div}_h(\sigma) - d \circ \operatorname{tr}_h(\sigma) = 0 \ . \tag{4.2.3}$$

It is remarkable that this integrability condition has the same form as the vacuum momentum constraint equation (4.1.1b). For this reason, we will call the operator $M_h : \mathcal{S}^2 \to \Lambda^1$ defined by

$$M_h = \operatorname{div}_h - d \circ \operatorname{tr}_h \tag{4.2.4}$$

the *momentum constraint operator*.

Rewriting the time-symmetric vacuum Hamilton constraint (4.1.3) in the form of equations (4.2.1a), (4.2.1b) and (4.2.3) allows us to incorporate features of the static vacuum field equations into the system. Moreover, this renders the underdetermined character of (4.1.3) in a new form with may open new ways to exploit it.

The considerations regarding decay properties of the unknowns will now be made more precise by selecting appropriate function spaces. We will use weighted Sobolev spaces as defined in chapter 2.

The first characteristic we want to capture is that of asymptotic flatness. We will require the solution to be *asymptotically flat of class* (p, t, τ) with $\tau < 0$ (see definition 2.3.5). The Positive Mass Theorem further requires that if this metric is to be a non-trivial solution of the constraint equations, then it must approach the flat metric at infinity not faster that $\frac{1}{r}$. This consideration is taken into account by the condition $-1 < \tau$.

We want the function v to approach a constant at space-like infinity, so we require that the function v should satisfy

$$v - 1 \in W^{t,p}_{\tau} \tag{4.2.5}$$

where, for consistency, we use the same weight as that for the metric.

The Laplacian is a second order differential operator and defines a bounded map $\Delta_h : W^{t,p}_{\gamma} \to W^{t-2,p}_{\gamma-2}$. The Ricci tensor is also a differential operator of second order. Under the conditions we have imposed for τ and assuming that the differentiability parameter satisfies $t \geq \frac{3}{p} + 1$, it is possible to use the multiplication Lemma 2.3.2 to conclude that $\operatorname{Ric} : W^{t,p}_{\tau}(\mathcal{S}^2) \to W^{t-2,p}_{\tau-2}(\mathcal{S}^2)$ is likewise a bounded map. This implies

$$\begin{aligned} \Delta_h v &\in W^{t-2,p}_{\tau-2} \\ -v \operatorname{Ric}(h) + \operatorname{Hess}_h(v) &\in W^{t-2,p}_{\tau-2} \ . \end{aligned}$$

We will be interested in constructing solutions for which σ has fast decay at space-like infinity.

Statement of the problem

To summarize the considerations above, we will be studying the *asymptotic staticity equations* which are given by

$$0 = \phi := \Delta_h v + \mathrm{tr}_h(\sigma) \tag{4.2.6a}$$
$$0 = V := M_h(\sigma) \tag{4.2.6b}$$
$$0 = S := -v \;\; \mathrm{Ric}(h) + \mathrm{Hess}_h(v) + \sigma \tag{4.2.6c}$$

and consider the associated map

$$\begin{pmatrix} v-1 & \in & W^{t,p}_{\tau}(\Lambda^0) \\ \sigma & \in & W^{q-1,p}_{\beta-1}(\mathcal{S}^2) \\ h-\delta & \in & W^{t,p}_{\tau}(\mathcal{S}^2) \end{pmatrix} \longrightarrow \begin{pmatrix} \phi & \in & W^{k-2,p}_{\tau-2}(\Lambda^0) \\ V & \in & W^{q-2,p}_{\beta-2}(\Lambda^1) \\ S & \in & W^{k-2,p}_{\tau-2}(\mathcal{S}^2) \end{pmatrix} \tag{4.2.7}$$

under the conditions

$$-1 < \tau < 0 \tag{4.2.8a}$$
$$\beta < \tau - 1 \; . \tag{4.2.8b}$$

where $\beta \in \mathbb{R} \setminus \mathbb{Z}$ to comply with a technical requirement that shall become clear when elliptic operators are analyzed.

Definition 4.2.1 (Asymptotic staticity)**.** [1] Let $\sigma \in W^{q-1,p}_{\beta-1}$ with $\beta < \tau - 1$ and $q = t - 1$. If (v, h_{ab}) is a solution to (4.2.6) corresponding to σ, we say it is *asymptotically static to order* β.

The condition $\beta < \tau - 1$ implies

$$\Delta_h v \in W^{t-2,p}_{\beta-1} \tag{4.2.9}$$
$$-v\,\mathrm{Ric}(h) + \mathrm{Hess}_h(v) \in W^{t-2,p}_{\beta-1} \tag{4.2.10}$$

which, loosely speaking, says that the solution (v, h_{ab}) will behave in an asymptotic expansion at infinity as a solution of the static field equations up to $\mathcal{O}\left(r^{\beta+1}\right)$.

The quasilinear system (4.2.6) has two particular features which one needs to address.

First, $\mathrm{Ric}(h)$ is a geometric operator and coordinate invariance prevents it from being elliptic without further conditions. To obtain an elliptic operator, one can choose a particular coordinate gauge and we will do that in section 4.2.1.

Second, the momentum constraint operator M_h is underdetermined elliptic, it maps symmetric tensors to 1-forms. A method to deal with this operator, which has proven successful in the study of the constraint equations by the method of conformal rescalings (see for example [Bartnik and Isenberg, 2004]), is what is called the *York decomposition*. We will discuss it in section 4.2.2 in the context of the asymptotic staticity equations.

4.2.1 Harmonic Coordinates

Let $\{y^j\}_{j=1}^3$ be a globally defined coordinate system in $\left(\mathbb{R}^3, h\right)$. The Christoffel symbols of h with respect to this coordinate system are given by

$$\Gamma^k_{ij} = \tfrac{1}{2} h^{kl} \left(\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij}\right) \; . \tag{4.2.11}$$

[1] The introduction of $q = t - 1$ is made only for bookkeeping purposes.

The components R_{ij} of the operator $\mathrm{Ric}(h)$ in this coordinate system are given in terms of the Christoffel symbols by

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{kj} + \Gamma^k_{ij}\Gamma^l_{lk} - \Gamma^k_{il}\Gamma^l_{jk} \ . \tag{4.2.12}$$

Consider the quantities $\Gamma^i = h^{jk}\Gamma^i_{jk}$ and $\Gamma_i = h_{ij}\Gamma^j$ as functions of the coordinates $\{y^j\}$. In these coordinates we define the 1-form Γ by

$$\Gamma = \sum_{i=1}^{3} \Gamma_i(y^1, y^2, y^3)\left(dy^i\right) \ . \tag{4.2.13}$$

We note that the coefficients Γ_i of this 1-form do not transform covariantly under coordinate transformations! We set furthermore in this coordinate system

$$\nabla\Gamma = \sum_{i,j=1}^{3} \left(\partial_i \Gamma_j - \Gamma^l_{ij}\Gamma_l\right)\left(dy^i\right)\otimes\left(dy^j\right) \tag{4.2.14}$$

which is the expression which formally looks like the covariant derivative of a 1-form with coefficients Γ_i.

We now define with respect to the coordinate system $\{y^j\}$ the *reduced Ricci tensor* $\mathrm{Ric}^H(h)$

$$\mathrm{Ric}^H(h) = \mathrm{Ric}(h) - \mathrm{Sym}(\nabla\Gamma) \ . \tag{4.2.15}$$

A computation shows that the highest order derivatives of the metric with respect to $\{y^j\}$ that appear in $\mathrm{Ric}^H(h)$ define an elliptic operator. Explicitly this is

$$R^H_{ij} = -\tfrac{1}{2} h^{kl}\partial_k\partial_l h_{ij} + F(\partial h, h) \tag{4.2.16}$$

where $F(\partial h, h)$ is a function of the metric components and their first derivatives.

The coordinate system $\{y^i\}$ is said to be *harmonic* on $(\mathcal{M}, h)$ whenever the coordinate functions y^i satisfy

$$\Delta_h y^i = 0 \ . \tag{4.2.17}$$

If one writes equation (4.2.17) in terms of the coordinate system $\{y^i\}$ one obtains

$$h^{jk}\Gamma^i_{jk} = 0 \ . \tag{4.2.18}$$

Thus in a harmonic coordinate system, the reduced Ricci operator equals the Ricci operator

$$\Gamma = 0 \implies \mathrm{Ric}(h) = \mathrm{Ric}^H(h) \ . \tag{4.2.19}$$

The way in which this property is used is as follows: one assumes that a coordinate system has been fixed and modifies the equations by using instead of $\mathrm{Ric}(h)$, the elliptic operator $\mathrm{Ric}^H(h)$ with respect to those coordinates. The resulting equations are studied in this coordinate system. Then, one must show that the solutions obtained satisfy the harmonic coordinate condition (4.2.17) so that they are also solutions to the original system.

4.2.2 The York decomposition

As was mentioned above, the equation of the system (4.2.6) that involves the momentum constraint operator M_h is underdetermined elliptic. To deal with this operator we shall study now the York decomposition. This will be done in a way which emphasizes the analogy between this decomposition and the Helmholtz decomposition discussed in chapter 3.

Consider the first order differential operator $\mathcal{L}_h : \Lambda^1 \to \mathcal{S}^2$ defined by

$$\mathcal{L}_h X = 2\,\mathrm{Sym}(\nabla X) - \frac{2}{3} h\,\mathrm{div}_h(X)\ . \tag{4.2.20}$$

A 1-form X is called a conformal Killing field of $(\mathcal{M}, h)$ if and only if $X \in \ker(\mathcal{L}_h)$, so it is customary to call $\mathcal{L}_h$ the *conformal Killing operator*. We will also make use of a second order operator $\mathbb{L}_h : \Lambda^1 \to \Lambda^1$ defined by

$$\mathbb{L}_h = M_h \circ \mathcal{L}_h \tag{4.2.21}$$

which will be referred to as the *conformal Killing Laplacian.*

To construct solutions to

$$M_h(\sigma) = 0 \tag{4.2.22}$$

we prescribe an arbitrary symmetric 2-tensor ψ and assume the ansatz

$$\sigma = \psi - \mathcal{L}_h X\ . \tag{4.2.23}$$

In a way analogous to the Helmholtz decomposition, this transforms the underdetermined elliptic problem (4.2.22) into a elliptic equation for the field X, where ψ is considered as *free data.*

In section 4.2 we have stated that we want σ to be in $W^{q-1,p}_{\beta-1}$. For this to happen equation (4.2.23) suggests to choose $\psi \in W^{q-1,p}_{\beta-1}$ and look for solutions $X \in W^{q,p}_{\beta}$. One is therefore looking for solutions X to the following elliptic problem

$$\begin{cases} \mathbb{L}_h : W^{q,p}_{\beta}(\Lambda^1) \to W^{q-2,p}_{\beta-2}(\Lambda^1) \\ \mathbb{L}_h(X) = M_h(\psi) \end{cases} . \tag{4.2.24}$$

One is reminded of the Helmholtz decomposition and equation (3.2.3). To study this decomposition, we shall make use of a Lemma that is the analogue of Lemma 3.2.4 in this context.

Lemma 4.2.2 ([Cantor, 1981][Lockhart, 1981][Lockhart and McOwen, 1983]). *Let $(\mathbb{R}^3, h_{ab})$ be asymptotically flat of class (p, t, τ) with $p > 1$, $\tau < 0$, $t > \frac{3}{p} + 1$. If $s \in \mathbb{N}$ is such that $2 \leq s \leq t$, then the operator*

$$\mathbb{L}_h : W^{s,p}_{\gamma} \to W^{s-2,p}_{\gamma-2} \tag{4.2.25}$$

defines a Fredholm map if and only of $\gamma \notin \mathbb{Z}$. Also

i) $\mathbb{L}_h$ is injective iff $\gamma < 0$,

ii) $\mathbb{L}_h$ is surjective iff $-1 < \gamma$.

Furthermore if $\frac{3}{p} + 3 < s$, then $\mathrm{index}\,\mathbb{L}_h = \mathrm{index}\,\mathbb{L}_\delta$. *In particular this implies*

$$\dim(\mathrm{coker}\,(\mathbb{L}_h)) = \dim(\mathrm{coker}\,(\mathbb{L}_\delta)) = 3\dim(\mathrm{coker}\,(\Delta_\delta)) \tag{4.2.26}$$

where Δ_δ is considered as the map $\Delta_\delta : W^{s,p}_{\gamma} \to W^{s-2,p}_{\gamma-2}$.

Proof. The Fredholm property and parts (i) and (ii) are a consequence of Theorem 2.4.5.

For the equality of the indices, note that $\mathbb{L}_h$ is an isomorphism for $-1 < \gamma < 0$. This was shown in [Christodoulou and O'Murchadha, 1981] for $p = 2$ under the condition $\frac{n}{2} + 3 < s$, and generalizes to $1 < p < \infty$ in a straight forward way. The operator $\mathbb{L}_\delta$ is also seen to be an isomorphism if $-1 < \gamma < 0$, so the indices coincide in this range, and are therefore equal by Theorem 2.4.5. Equation (4.2.26) follows from equation (2.4.14b). □

Under the assumptions made in equation (4.2.8), the operator in (4.2.25) is Fredholm, so $\operatorname{ran}(\mathbb{L}_h) = \{\ker(\mathbb{L}_h^*)\}^\perp$ by the closed range Theorem. The set $\ker(\mathbb{L}_\delta^*)$ is finite dimensional.

Remark 4.2.3. It should be noted that by a similar argument as that of section 2.3.3, it is possible to see that the action of $\mathbb{L}_h^*$ is given formally by the same expression as that for $\mathbb{L}_h$ shown in equation (4.2.21). Likewise, just as was the case discussed in section 2.3.3, one considers them to be operators acting between different spaces.

Notation 4.2.4 (Basis). Let $\{A^\mu\}_{\mu=1}^{N_+}$ denote a basis for $\ker(\mathbb{L}_h^*)$, where $\mathbb{L}_h$ is considered as in the map in equation (4.2.25) and where

$$N_+ = \dim(\ker(\mathbb{L}_h^*)) \ . \tag{4.2.27}$$

Furthermore let $\{G_\nu\}_{\nu=1}^{N_+}$ denote a family of fields in $W^{q-2,p}_{\beta-2}$ which with $\{A^\mu\}_{\mu=1}^{N_+}$ forms a bi-orthogonal system (see section 2.2). The symbols $\mathring{A}^\mu$ and $\mathring{G}_\nu$ refer to such fields in the case of the flat geometry $\left(\mathbb{R}^3, \delta_{ab}\right)$.

Using this notation one can construct a projection operator $\mathbb{P}_h$ such that $\mathbb{L}_h(W^{s,p}_\gamma) = \mathbb{P}_h(W^{s-2,p}_{\gamma-2})$. The projector $\mathbb{P}_h$ is given by

$$\mathbb{P}_h = \mathbb{I} - \sum_{\nu=1}^{N_+} G_\nu \langle A^\nu, \cdot \rangle_h \ . \tag{4.2.28}$$

where N_+ depends on the parameter γ.

It should also be noted that one can give explicit expressions for a basis $\left\{\mathring{A}^\mu\right\}$ of $\ker(\mathbb{L}_\delta^*)$ and in chapter 5 such expressions are given for any value of the weight parameter β.

The necessary conditions for the existence of a solution $X \in W^{q,p}_\beta$ to equation (4.2.24) are given in this notation by

$$\langle A^\mu, M_h(\psi)\rangle_h = 0 \qquad \forall A^\mu \in \ker(\mathbb{L}_h^*) \ . \tag{4.2.29}$$

4.3 A perturbative result

In this section and what remains of the chapter we will consider the manifold to be diffeomorphic to $\mathbb{R}^3$. It will be shown that, under a *small free data* assumption, a solution to the asymptotic staticity equations (4.2.6) exists. The argument consists of two steps, starting first by studying the linearized equations. We will obtain existence of solutions to an auxiliary *projected system* of equations. In the second step, we will show that the solutions so obtained are also solutions of (4.2.6) when a smallness requirement on the free data is assumed and a further restriction on the asymptotic staticity parameter β (on top of (4.2.8)) is met.

Consider the system of equations that results from replacing $\operatorname{Ric}(h)$ with $\operatorname{Ric}^H(h)$ and from plugging the ansatz of the York decomposition (4.2.23) into (4.2.6). Then the set of variables consists of $v \in \Lambda^0$, $X \in \Lambda^1$, $h \in \mathcal{S}^2$ and a tensor $\psi \in \mathcal{S}^2$ which will be thought of as free data. The resulting equations are

$$0 = \phi := \Delta_h v + \operatorname{tr}_h \psi \tag{4.3.1a}$$

$$0 = V := \mathbb{L}_h(X) - M_h(\psi) \tag{4.3.1b}$$

$$0 = S := -v \operatorname{Ric}^H(h) + \operatorname{Hess}_h(v) - \mathcal{L}_h(X) + \psi \tag{4.3.1c}$$

and the associated map is

$$\Psi : \begin{pmatrix} v-1 & \in & W^{t,p}_\tau(\Lambda^0) \\ X & \in & W^{q,p}_\beta(\Lambda^1) \\ h-\delta & \in & W^{t,p}_\tau(\mathcal{S}^2) \\ \psi & \in & W^{q-1,p}_{\beta-1}(\mathcal{S}^2) \end{pmatrix} \to \begin{pmatrix} \phi & \in & W^{t-2,p}_{\tau-2}(\Lambda^0) \\ V & \in & W^{q-2,p}_{\beta-2}(\Lambda^1) \\ S & \in & W^{t-2,p}_{\tau-2}(\mathcal{S}^2) \end{pmatrix} \tag{4.3.2}$$

where τ and β satisfy the conditions (4.2.8). By writing

$$\mathcal{W} = W^{t,p}_\tau(\Lambda^0) \times W^{q,p}_\beta(\Lambda^1) \times W^{t,p}_\tau(\mathcal{S}^2)$$

$$\widehat{\mathcal{W}} = W^{t-2,p}_{\tau-2}(\Lambda^0) \times W^{q-2,p}_{\beta-2}(\Lambda^1) \times W^{t-2,p}_{\tau-2}(\mathcal{S}^2),$$

then (4.3.1) and (4.3.2) then read

$$\Psi(v, X, h\,; \psi) = 0 \tag{4.3.3a}$$

$$\Psi : \mathcal{W} \times W^{q-1,p}_{\beta-1}(\mathcal{S}^2) \to \widehat{\mathcal{W}}\,. \tag{4.3.3b}$$

4.3.1 First Step

We want to study equation (4.3.3) perturbatively in a neighborhood of the trivial solutions corresponding to $\psi = 0$, namely $v = 1$, $X = 0$, and $h = \delta$.

Linearization

The derivative of the map Ψ at the trivial solution defines the map $\mathcal{D}\Psi : \mathcal{W} \times W^{q-1,p}_{\beta-1}(\mathcal{S}^2) \to \widehat{\mathcal{W}}$ given for $(u, Y, g, s) \in \mathcal{W} \times W^{q-1,p}_{\beta-1}(\mathcal{S}^2)$ by

$$(\mathcal{D}\Psi)|_{\psi=0}\,(u, Y, g, s) = \begin{pmatrix} \Delta_\delta(u) + \operatorname{tr}_\delta(s) \\ \mathbb{L}_\delta(Y) - M_\delta(s) \\ \frac{1}{2}\Delta_\delta g + \operatorname{Hess}_\delta(u) - \mathcal{L}_\delta Y + s \end{pmatrix}. \tag{4.3.4}$$

Setting $s = 0$, one obtains the partial derivative of the operator Ψ in the direction of the unknowns $(\mathcal{D}_\mathcal{W}\Psi)|_{\psi=0} : \mathcal{W} \to \widehat{\mathcal{W}}$ and in this section we study its properties. In the following it will be shown that under conditions (4.2.8), the map

$$(\mathcal{D}_\mathcal{W}\Psi)|_{\psi=0} : \mathcal{W} \to \widehat{\mathcal{W}} \tag{4.3.5}$$

defines an injective Fredholm operator with non-trivial cokernel.

Consider for some $(\phi_0, V_0, S_0) \in \widehat{\mathcal{W}}$ the equations

$$\phi_0 = \Delta_\delta u \tag{4.3.6a}$$

$$V_0 = \mathbb{L}_\delta Y \tag{4.3.6b}$$

$$S_0 = \tfrac{1}{2}\Delta_\delta g + \operatorname{Hess}_\delta u - \mathcal{L}_\delta Y \tag{4.3.6c}$$

for the unknowns $(u, Y, g) \in \mathcal{W}$. Equations (4.3.6) define a linear partial differential operator in $\mathbb{R}^3$. A system of orders for this operator is given by $\mathbf{t} = (k+2, k+1, k+2)$ and $\mathbf{s} = (k, k-1, k)$, with $k \in \mathbb{N}$.

Although it will not be the route taken here, one could apply a more general version of Theorem 2.4.3 (with different weights for each field) to the operator of equation (4.3.6). It can be seen that the system satisfies the assumption of 2.4.2 and one can study its Fredholm properties this way. It is actually easier and clearer to take advantage of the particularities of this system and to deduce its properties by discussing its decoupled parts.

First, consider equation (4.3.6a). Lemma 3.2.4 and the condition $-1 < \tau < 0$ imply $\Delta_\delta : W^{t,p}_\tau(\Lambda^0) \to W^{t-2,p}_{\tau-2}(\Lambda^0)$ is a Fredholm map with trivial kernel and cokernel.

Second, from Lemma 4.2.2 and conditions (4.2.8) one learns that the operator $\mathbb{L}_\delta : W^{q,p}_\beta \to W^{q-2,p}_{\beta-2}$ is Fredholm and injective, but not surjective. Using the notation introduced in 4.2.4, one has $\mathbb{L}_\delta(W^{q,p}_\beta) = \mathbb{P}_\delta(W^{q-2,p}_{\beta-2})$. Then, the operator

$$\mathbb{L}_\delta : W^{q,p}_\beta \to \mathbb{P}_\delta(W^{q-2,p}_{\beta-2}) \tag{4.3.7}$$

is an isomorphism. We will postpone a discussion on how to explicitly construct such the projection operator $\mathbb{P}_\delta$ to Lemma 4.3.3.

Finally one considers, for given $u \in W^{t,p}_\tau$ and $Y \in W^{q,p}_\beta$, (4.3.6c) as an equation for g, namely

$$\Delta_\delta g = 2\left(S_0 - \operatorname{Hess}_\delta u - \mathcal{L}_\delta Y\right) . \tag{4.3.8}$$

By assumption one has $S_0 \in W^{t-2,p}_{\tau-2}$. Also u and Y satisfy $\operatorname{Hess}_\delta u \in W^{t-2,p}_{\tau-2}$ and $\mathcal{L}_\delta Y \in W^{q-1,p}_{\beta-1}$ respectively. Conditions (4.2.8), $q = t-1$ and the fact that

$$\gamma_1 \le \gamma_2, \quad s_1 \ge s_2 \implies W^{s_1,p}_{\gamma_1} \subset W^{s_2,p}_{\gamma_2} \tag{4.3.9}$$

holds, imply $S_0 - \operatorname{Hess}_\delta u - \mathcal{L}_\delta Y \in W^{t-2,p}_{\tau-2}$. Then by Lemma 3.2.4 one has that given u and Y, there always exists a unique solution g to (4.3.6c).

To summarize these finding one says that, under conditions (4.2.8), the map (4.3.5) is Fredholm and

$$\ker\left((\mathcal{D}_\mathcal{W}\Psi)|_{\psi=0}\right) = \{0\} \tag{4.3.10a}$$

$$\operatorname{ran}\left((\mathcal{D}_\mathcal{W}\Psi)|_{\psi=0}\right) = \begin{pmatrix} W^{t-2,p}_{\tau-2} \\ \mathbb{P}_\delta\left(W^{q-2,p}_{\beta-2}\right) \\ W^{t-2,p}_{\tau-2} \end{pmatrix} . \tag{4.3.10b}$$

where $\mathbb{P}_\delta\left(W^{q-2,p}_{\beta-2}\right)$ has finite codimension N_+ (see (4.2.27) for the definition).

The projected system

We now return to the non-linear system (4.3.1). The perturbative analysis of the previous section provides a way to come up with a new non-linear system of equations whose linearization is in fact an isomorphism between Banach spaces. To construct this new system we define the projection operator $\mathbb{P} : \widehat{\mathcal{W}} \to \widehat{\mathcal{W}}$ by $\mathbb{P}(\phi, V, S) := (\phi, \mathbb{P}_\delta(V), S)$ where the projector $\mathbb{P}_\delta : W^{q-2,p}_{\beta-2} \to W^{q-2,p}_{\beta-2}$ was introduced in (4.2.28). We consider now a *projected system* given by

$$\mathbb{P} \circ \Psi(v, X, h, \psi) = 0 \tag{4.3.11}$$

where Ψ is the map in equation (4.3.2). Written in terms of v, σ and h, the projected system consists of the equations

$$0 = \Delta_h v + \operatorname{tr}_h \sigma \tag{4.3.12a}$$
$$0 = M_h(\sigma) + \lambda \tag{4.3.12b}$$
$$0 = -v \operatorname{Ric}^H(h) + \operatorname{Hess}_h(v) + \sigma \tag{4.3.12c}$$

where σ is given by $\sigma = \psi - \mathcal{L}_h X$ (see (4.2.23)) and where the *error term* λ is

$$\lambda(\psi) = (\mathbb{I} - \mathbb{P}_\delta) M_h(\sigma) = \sum_{\nu=1}^{N_+} \mathring{G}_\nu \langle \mathring{A}^\nu, M_h(\sigma) \rangle_\delta \ . \tag{4.3.13}$$

Lemma 4.3.1. *If $\psi \in W^{q-1,p}_{\beta-1}$ is sufficiently small, then there exist*

$$v(\psi) \quad : \quad v - 1 \in W^{t,p}_\tau \tag{4.3.14a}$$
$$\sigma(\psi) \quad : \quad \sigma \in W^{q-1,p}_{\beta-1} \tag{4.3.14b}$$
$$h(\psi) \quad : \quad h - \delta \in W^{t,p}_\tau \tag{4.3.14c}$$

satisfying equations (4.3.12) *and depending continuously on the free data ψ.*

Proof. The system (4.3.1), (4.3.2) satisfies the hypotheses of corollary 2.6.2, i.e. the linearization of Ψ at the trivial solution is an injective Fredholm map (see equation (4.3.10)). The projector used in equation (4.3.11) satisfies the condition (2.6.3) of the corollary and therefore there exist $\{v(\psi), X(\psi), h(\psi)\}$ satisfying (4.3.11). From ψ and the resulting $X(\psi)$ we reconstruct σ and write the equations as (4.3.12).

By Corollary 2.6.2 the solution (4.3.14) depends continuously on the free data ψ. □

4.3.2 Second Step

The system in (4.3.12) differs from (4.3.3) in two aspects. The first difference is that the reduced Ricci operator appears in the projected system. The other difference is that in the projected system there is the extra term λ. In order for (4.3.14) to be a solution of the system we started with in (4.3.3), one must make sure that for the solution (4.3.14), the conditions

$$\lambda = 0 \tag{4.3.15a}$$
$$\Gamma = 0 \tag{4.3.15b}$$

are satisfied. If these conditions are satisfied, then $\operatorname{Ric}^H(h) = \operatorname{Ric}(h)$ and (v, h, σ) is in fact a solution to (4.2.6). Using the contracted Bianchi identity (4.2.2) and the projected system (4.3.12), we shall derive an equation relating λ and Γ.

First one replaces the reduced Ricci operator in equation (4.3.12c) using (4.2.15) and obtains the equation

$$\operatorname{Ric}(h) = \frac{\operatorname{Hess}_h(v) + \sigma}{v} + \operatorname{Sym}(\nabla\Gamma) \ . \tag{4.3.16}$$

The contracted Bianchi identity (4.2.2) then implies

$$\left(\operatorname{div}_h - \tfrac{1}{2} d \circ \operatorname{tr}_h\right) \left(\frac{\operatorname{Hess}_h(v) + \sigma}{v} + \operatorname{Sym}(\nabla\Gamma) \right) = 0 \ . \tag{4.3.17}$$

We will compute the action of the differential operator acting on each term separately. First, by (4.3.12a) one has that $\mathrm{tr}_h\left(\mathrm{Hess}_h(v)+\sigma\right)=0$. Then one computes

$$\mathrm{div}_h\left(\frac{\mathrm{Hess}_h(v)+\sigma}{v}\right)=\frac{\Delta_h\circ dv+\mathrm{div}_h\,\sigma}{v}-\frac{\left(\mathrm{Hess}_h(v)+\sigma\right)(dv,\,\cdot\,)}{v^2} \tag{4.3.18}$$

and using the identity

$$\Delta_h\circ dv=d\circ\Delta_h v+\mathrm{Ric}(h)(\nabla v,\,\cdot\,)\ , \tag{4.3.19}$$

and equation (4.3.16) one gets

$$\mathrm{div}_h\left(\frac{\mathrm{Hess}_h(v)+\sigma}{v}\right)=\frac{d\circ\Delta_h v+\mathrm{div}_h\,\sigma+\mathrm{Sym}(\nabla\Gamma)(dv,\,\cdot\,)}{v}\ . \tag{4.3.20}$$

Replacing $\Delta_h v$ using equation (4.3.12a) and then using (4.3.12b) gives

$$\left(\mathrm{div}_h-\tfrac{1}{2}d\circ\mathrm{tr}_h\right)\left(\frac{\mathrm{Hess}_h(v)+\sigma}{v}\right)=\frac{\mathrm{Sym}(\nabla\Gamma)(dv,\,\cdot\,)-\lambda}{v}\ . \tag{4.3.21}$$

A computation shows that

$$\left(\mathrm{div}_h-\tfrac{1}{2}d\circ\mathrm{tr}_h\right)\mathrm{Sym}(\nabla\Gamma)=\frac{\Delta_h\Gamma+\mathrm{Ric}(h)(\Gamma,\,\cdot\,)}{2} \tag{4.3.22}$$

so using (4.3.21) together with (4.3.22) in (4.3.17) one finally gets

$$v\Delta_h\Gamma+2\,\mathrm{Sym}(\nabla\Gamma)(dv,\,\cdot\,)+v\,\mathrm{Ric}(h)(\Gamma,\,\cdot\,)=2\lambda\ . \tag{4.3.23}$$

The left hand side of equation (4.3.23) defines a linear homogeneous second order partial differential operator we shall denote by H_ψ, more specifically

$$\begin{cases} H_\psi : W^{t-1,p}_{\tau-1}(\Lambda^1)\to W^{t-3,p}_{\tau-3}(\Lambda^1) \\ H_\psi(\Gamma):=v\Delta_h(\Gamma)+2\,\mathrm{Sym}(\nabla\Gamma)(dv,\,\cdot\,)+v\,\mathrm{Ric}(h)(\Gamma,\,\cdot\,) \end{cases} \tag{4.3.24}$$

where the choices of weighted Sobolev spaces are such because Γ is defined in terms of first derivatives of the the metric h. The notation H_ψ is meant to emphasize that we want to consider this operator in the following way: associated to given free data ψ there exists a solution as given by Lemma 4.3.1, which in turn gives rise to the corresponding operator H_ψ by equation (4.3.24).

For clarity, let us briefly summarize the steps in the argument that follows. We will use Lemma 2.7.3 to conclude that λ vanishes for sufficiently small free data ψ. To do this, we will first study the range of the operator H_ψ for trivial free data, i.e. when $\psi=0$. The second result we need will take the form of an explicit characterization of the subspace of $W^{q-2,p}_{\beta-2}(\Lambda^1)$ to which the error term λ belongs. The third piece of information we need is the uniform injectivity of H_ψ for sufficiently small ψ. Finally, after concluding that λ vanishes, the injectivity of the operator H_ψ implies $\Gamma=0$.

One should point out at this stage that the characterization of the possible error terms depends on the choice made for the asymptotic staticity parameter, which will limit the applicability of this argument to certain range for β.

In the Lemma that follows it will be shown that if $\psi=0$, then the range of the associated operator $H_{\psi=0}=H_0$ can be explicitly characterized.

Lemma 4.3.2 (Range of H_0)**.** *The 1-form fields*

$$\mathring{A}^{\mu} = (dx^{\mu}) \qquad \text{for} \quad \mu = 1, 2, 3 \tag{4.3.25}$$

are a basis for $\ker(H_0^*)$ *where* $H_0 := H_{\psi=0}$ *and* H_ψ *is the operator defined in equation* (4.3.24).

Proof. When $\psi = 0$, the operator H_ψ reduces to $\Delta_\delta : W^{t-1,p}_{\tau-1}(\Lambda^1) \to W^{t-3,p}_{\tau-3}(\Lambda^1)$ and the result follows from Theorem 2.4.4 and a computation in Cartesian coordinates. □

To give an explicit expression of the error term λ defined in equation (4.3.13), one needs to study the operator $\mathbb{L}_\delta$ in more detail. In the present context, it acts as a map in the spaces shown in equation (4.2.24) where β satisfies the conditions (4.2.8).

As mentioned above, the argument given here applies when the asymptotic staticity parameter β is restricted to certain interval. We therefore strongly emphasize that even if in principle *one would like to avoid any lower bounds on* β, we are only able to carry out this argument when we restrict it to the interval

$$-2 < \beta < -1\,. \tag{4.3.26}$$

This in turn simplifies considerably the description of $\ker(\mathbb{L}_\delta^*)$, for there is a straight forward way to find fields that give a basis for it.

Lemma 4.3.3. *If* $-2 < \beta < -1$ *then the 1-form fields*

$$\mathring{A}^{\mu} = (dx^{\mu}) \qquad \text{for} \quad \mu = 1, 2, 3 \tag{4.3.27}$$

are a basis for $\ker(\mathbb{L}_\delta^*)$ *where* $\mathbb{L}_\delta$ *is considered as in equation* (4.2.24). *Furthermore if* $\chi(r) \in C_0^\infty$ *is such that* $\int \chi(r)\mathrm{d}\mu_\delta = 1$, *then the 1-form fields*

$$\mathring{G}_{\nu} = \chi(r)(dx^{\nu}) \qquad \text{for} \quad \nu = 1, 2, 3 \tag{4.3.28}$$

together with (4.3.27) *form a bi-orthogonal system. This implies*

$$W^{s-2,p}_{\beta-2} = \mathbb{L}_\delta(W^{s,p}_{\beta}) \oplus \overline{\left\{\mathring{G}_1, \mathring{G}_2, \mathring{G}_3\right\}} \tag{4.3.29}$$

Proof. The simplifying observation is that any Killing field of $(\mathbb{R}^3, \delta_{ab})$ in the domain of $\mathbb{L}_\delta^*$ will be in $\ker(\mathbb{L}_\delta^*)$. The assumption on β implies one is considering fields which behave as $o(r)$ at infinity and by Theorem 4.2.2 the dimension of this kernel is 3. The three Killing fields of $(\mathbb{R}^3, \delta_{ab})$ which generate translations have precisely $o(r)$ growth at infinity. □

In particular the error term $\lambda \in \overline{\left\{\mathring{G}_1, \mathring{G}_2, \mathring{G}_3\right\}}$ in equation (4.3.13) is then given by

$$\lambda(\psi) = \sum_{\nu=1}^{3} \mathring{G}_{\nu} \langle \mathring{A}^{\nu}, M_h(\sigma)\rangle_\delta \tag{4.3.30}$$

where the metric h and the tensor σ both depend on the free data.

Lemma 4.3.4 (Injectivity of H_ψ)**.** *If* ψ *is sufficiently small, then the operator* H_ψ *defined in* (4.3.24) *is injective.*

Proof. As was pointed out in Lemma 4.3.1, the solution depends continuously on the free data ψ. This means that if ψ is small in $W^{q-1,p}_{\beta-1}(\mathcal{S}^2)$, then $v-1, X$ and $h-\delta$ are small in the spaces given in equation (4.3.14).

To show that the operator H_ψ is injective, we set $H_\psi(\Gamma) = 0$ and show that this implies $\Gamma = 0$. Under the assumptions made on the differentiability $\frac{3}{p}+1 < t$, the Sobolev inequality (Lemma 2.3.3) implies that $|v-1|$ is bounded, so it is safe to assume that $v > 0$. Thus it suffices to consider the equation $H_\psi(\Gamma)/v = 0$. Furthermore, consider the contraction $\langle H_\psi(\Gamma)/v, \Gamma\rangle_h$, which reads

$$\langle\Gamma, \Delta_h\Gamma\rangle_h + 2\langle d(\ln v)\otimes\Gamma, \mathrm{Sym}(\nabla\Gamma)\rangle_h + \langle \mathrm{Ric}(h), \Gamma\otimes\Gamma\rangle_h = 0 \tag{4.3.31}$$

which will be analyzed term by term.

Using the identity

$$h(\Gamma, \Delta_h\Gamma) = \tfrac{1}{2}\Delta_h|\Gamma|_h^2 - |\nabla\Gamma|_h^2 \tag{4.3.32}$$

one gets for the first term in (4.3.31)

$$\langle\Gamma, \Delta_h\Gamma\rangle_h = \tfrac{1}{2}\langle 1, \Delta_h|\Gamma|_h^2\rangle_h - \langle\nabla\Gamma, \nabla\Gamma\rangle_h\ . \tag{4.3.33}$$

By Lemma 2.3.2, the condition (4.2.8) on τ and the assumption $t > \frac{3}{p}+1$ (as mentioned in section 4.2), one knows that $|\Gamma|_h^2 \in W^{t-1,p}_{\tau-1}$. Given that

$$\mathrm{ran}\left(\Delta_h W^{t-1,p}_{\tau-1} \to W^{t-3,p}_{\tau-3}\right) = \{f \in W^{t-3,p}_{\tau-3} :\ \langle 1, f\rangle_h = 0\}\ , \tag{4.3.34}$$

the first term in (4.3.33) vanishes. As for the second term in (4.3.33), one has

$$C_1\,\|\nabla\Gamma\|^2_{W^{0,2}_{-3/2}} \le \langle\nabla\Gamma, \nabla\Gamma\rangle_h \tag{4.3.35}$$

where C_1 is a constant that accounts for using h instead of δ in the definition of the weighted Sobolev norm and is of the order of 1 in the present circumstances. The first term in (4.3.31) then satisfies

$$\langle\Gamma, \Delta_h\Gamma\rangle_h \le -C_1\,\|\nabla\Gamma\|^2_{W^{0,2}_{-3/2}}\ . \tag{4.3.36}$$

For the second term in (4.3.31) one has

$$\begin{aligned}
2\langle d(\ln v)\otimes\Gamma, \mathrm{Sym}(\nabla\Gamma)\rangle_h &\le 2C_2\int |\mathrm{Sym}(\nabla\Gamma)(d(\ln v), \Gamma)|\,\mathrm{d}\mu_\delta \\
&\le 6C_2\,\|d(\ln v)\|_{W^{0,\infty}_{-2}}\,\|\mathrm{Sym}(\nabla\Gamma)(\,\cdot\,,\Gamma)\|_{W^{0,1}_{-1}} \\
&\le 18C_2\,\|d(\ln v)\|_{W^{0,\infty}_{-2}}\,\|\nabla\Gamma\|_{W^{0,2}_{-1/2}}\,\|\Gamma\|_{W^{0,2}_{-1/2}}
\end{aligned} \tag{4.3.37}$$

where Hölder's inequality 2.3.1 was used twice.[2] The constant $C_2 = \sup\mu(x)$ where $\mathrm{d}\mu_\delta = \mu(x)\,\mathrm{d}\mu_h$ is of the order of 1. Finally one uses $2ab \le a^2+b^2$ and

$$\|\nabla\Gamma\|^2_{W^{0,2}_{-1/2}} \le \|\nabla\Gamma\|^2_{W^{0,2}_{-3/2}} \tag{4.3.38}$$

to obtain

$$\begin{aligned}
&2\langle d(\ln v)\otimes\Gamma, \mathrm{Sym}(\nabla\Gamma)\rangle_h \le \\
&\qquad \le 9C_2\left(\|\nabla\Gamma\|^2_{W^{0,2}_{-3/2}} + \|\Gamma\|^2_{W^{0,2}_{-1/2}}\right)\|d(\ln v)\|_{W^{0,\infty}_{-2}}\ .
\end{aligned} \tag{4.3.39}$$

[2] Each factor of 3 comes from removing a contraction.

For the last term in (4.3.31) we have

$$\begin{aligned}\langle \mathrm{Ric}(h), \Gamma \otimes \Gamma \rangle_h &\leq 9C_2 \left\| \mathrm{Ric}(h) \otimes \Gamma \otimes \Gamma \right\|_{W^{0,1}_{-3}} \\ &\leq 9C_2 \left\| \mathrm{Ric}(h) \right\|_{W^{0,\infty}_{-2}} \left\| \Gamma \right\|^2_{W^{0,2}_{-1/2}} \end{aligned} \tag{4.3.40}$$

as a consequence of using Hölder's inequality twice and C_2 as in (4.3.39).

From (4.3.36), (4.3.39) and (4.3.40) one obtains

$$\begin{aligned}0 \leq 9 &\left(\left\| \mathrm{Ric}(h) \right\|_{W^{0,\infty}_{-2}} + \left\| d(\ln v) \right\|_{W^{0,\infty}_{-2}} \right) \left\| \Gamma \right\|^2_{W^{0,2}_{-1/2}} \\ &+ \left(9 \left\| d(\ln v) \right\|_{W^{0,\infty}_{-2}} - \frac{C_1}{C_2} \right) \left\| \nabla \Gamma \right\|^2_{W^{0,2}_{-3/2}} . \end{aligned} \tag{4.3.41}$$

Using the Poincaré inequality for weighted Sobolev spaces (Lemma 2.3.4) in the form

$$\left\| \Gamma \right\|^2_{W^{0,2}_{-1/2}} \leq C_3 \left\| \nabla \Gamma \right\|^2_{W^{0,2}_{-3/2}} \tag{4.3.42}$$

one arrives at

$$0 \leq \left[\left(C_4 \left\| \mathrm{Ric}(h) \right\|_{W^{0,\infty}_{-2}} + C_5 \left\| d(\ln v) \right\|_{W^{0,\infty}_{-2}} \right) - 1 \right] \left\| \nabla \Gamma \right\|^2_{W^{0,2}_{-3/2}} \tag{4.3.43}$$

for positive constants C_4 and C_5. Letting t be such that $\frac{3}{p} < t - 2$ one knows that small free data ψ implies that $\left\| \mathrm{Ric}(h) \right\|_{W^{0,\infty}_{-2}} + \left\| d(\ln v) \right\|_{W^{0,\infty}_{-2}}$ is small, and therefore

$$\left\| \nabla \Gamma \right\|^2_{W^{0,2}_{-3/2}} = 0 . \tag{4.3.44}$$

Using (4.3.42) this gives $\left\| \Gamma \right\|^2_{W^{0,2}_{-1/2}} = 0$ and the regularity assumed implies that Γ vanishes. $\square$

Consider a family of free data $\psi^{(\epsilon)}$, $\epsilon \in [0, 1]$ such that $\psi^{(0)} = 0$. It will be shown in the following Lemma that assuming $\psi^{(\epsilon)}$ is sufficiently small for every ϵ, there exists $\epsilon_0 \in (0, 1]$ such that the solution (4.3.14) associated to $\psi^{(\epsilon)}$ is also a solution of the asymptotic staticity system (4.2.6).

Lemma 4.3.5 (Satisfying the Harmonicity condition)**.** *Let $\psi^{(\epsilon)}$, $\epsilon \in [0, 1]$ be a family of free data satisfying $\psi^{(0)} = 0$ and $\left\| \psi^{(\epsilon)} \right\|_{W^{q-1,p}_{\beta-1}} \leq \epsilon C_\psi$. If*

$$-2 < \beta < \tau - 1 \tag{4.3.45}$$

and C_ψ is sufficiently small, then there exists $\epsilon_0 > 0$ such that (4.3.14) *is a solution of the system* (4.2.6) *whenever $0 \leq \epsilon < \epsilon_0$.*

Proof. The proof is based on Lemma 2.7.3. The assumptions that $\beta < \tau - 1$ and that C_ψ is small imply there exists, for every $\psi^{(\epsilon)}$, a solution to the projected system as given in equation (4.3.14).

By Lemma 4.3.4 one knows that the resulting operators H_ψ are injective for each ϵ. This can be used to conclude that for any $U \in W^{t-1,p}_{\tau-1}$ we have the estimate

$$\left\| U \right\|_{W^{t-1,p}_{\tau-1}} \leq C^{(\epsilon)} \left\| H_\psi(U) \right\|_{W^{t-3,p}_{\tau-3}} \tag{4.3.46}$$

where $C^{(\epsilon)}$ is a constant (see for example Proposition 1.11 in [Bartnik, 1986]). Taking the largest $C^{(\epsilon)}$ for $\epsilon \in [0,1]$ implies the family can be assumed to be uniformly injective in the sense of 2.7.2.

Thus by Lemmas 4.3.2 and 4.3.3, the family of operators H_ψ associated to $\psi^{(\epsilon)}$ satisfies the hypothesis of the cokernel stability theorem. This implies there exists $\epsilon_0 > 0$ such that $\lambda = 0$ whenever $0 \leq \epsilon < \epsilon_0$. Then the injectivity of H_ψ implies $\Gamma = 0$ for every $0 \leq \epsilon < \epsilon_0$ from which the conclusion of the Lemma follows. □

4.4 Obstacles to obtaining faster decay

The perturbative result of the previous section is valid in a special range of values for the weight parameters. To begin with, they were required to satisfy (4.2.8) when the problem was stated, and no lower bound on β was imposed at that point. However, in the course of the proof of the perturbative result, a lower bound on β was assumed in order to use Lemma 4.3.3. This leaves one with conditions the the weights which can be summarized as

$$-2 < \beta < \tau - 1 < -1 \tag{4.4.1}$$

thereby obstructing the applicability of the proof for lower values of β, i.e. to perturbatively showing the existence of solutions which are asymptotically static up to higher order.

The perturbative result imposes no further restrictions on the free data $\psi \in W^{q-1,p}_{\beta-1}(\mathcal{S}^2)$ other than it being sufficiently small. In this sense it is analogous to finding a solution to a Poisson equation as discussed in section 2.5 where Δ_h is considered as the map in equation (2.5.2). Under the restriction that the weight γ satisfies $-1 < \gamma < 0$, one finds then that there are no further restrictions on the source term ρ other than having bounded norm in $W^{s-2,p}_{\gamma-2}(\Lambda^0)$.

If one is interested, however, in solutions with faster decay at infinity, one needs to choose weights more into the negatives (γ for the example of the Poisson equation, β in the case of the asymptotic staticity equations). One does not expect to find solutions without imposing further conditions on the free data.

In the case of the Poisson equation, the conditions on the right hand side take the form

$$\langle a^\mu, \rho \rangle_h = 0 \qquad \forall \quad a^\mu \in \ker(\Delta_h^*) \tag{4.4.2}$$

where the notation a^μ was introduced in (2.5.3). These conditions are given in terms of functions which depend on the background metric h but not on the source function ρ.

The situation is different in the case of the asymptotic staticity equations. If one allows the asymptotic staticity parameter β to fall below -2, one needs to provide an analog of Lemma 4.3.3 for the resulting situation. The space in which the error term λ can vary is enlarged as a consequence and the argument of Lemma 4.3.5 does not apply.

The problems encountered when trying to extend the perturbative result of the previous section are found to be related to the possibility of constructing solutions to (4.2.22) with fast decay, which is the topic of the next chapter.

Chapter 5

York decomposition with fast decay

5.1 Introduction

Motivated by the result of the previous chapter, we explore now the possibility for improvement in the asymptotic staticity rate for which existence was shown. To do this, we will study the York decomposition in a given asymptotically flat background. We will exploit the analogies between the York decomposition and the Helmholtz decomposition as discussed in chapter 3.

Let $\left(\mathbb{R}^3, h_{ab}\right)$ be an asymptotically flat manifold of class (p, t, τ) with $\tau < 0$. We recall that, as was discussed in chapter 3, the standard Helmholtz decomposition can be viewed as a method by which one can generate solutions to the equation

$$\left\{ \begin{array}{l} \operatorname{div}_h(\sigma) = 0 \\ \operatorname{div}_h : W^{s-1,p}_{\gamma-1}(\Lambda^1) \to W^{s-2,p}_{\gamma-2}(\Lambda^0) \end{array} \right. \tag{5.1.1}$$

for suitable values of γ. This method consists of first prescribing arbitrarily $\psi \in W^{s-1,p}_{\gamma-1}(\Lambda^1)$ and considering the ansatz

$$\sigma = \psi - dv \ , \qquad v \in W^{s,p}_{\gamma}(\Lambda^0) \ . \tag{5.1.2}$$

The under-determined problem (5.1.1) is thereby transformed into the elliptic equation

$$\left\{ \begin{array}{l} \Delta_h v = \operatorname{div}_h \psi \\ \Delta_h : W^{s,p}_{\gamma}(\Lambda^0) \to W^{s-2,p}_{\gamma-2}(\Lambda^0) \end{array} \right. \tag{5.1.3}$$

which we consider as a condition on v. Whenever one can solve this equation for v we say that the 1-form ψ has a standard Helmholtz decomposition into a sum of a divergenceless part σ and the gradient of a function v.

The problem we want to consider now is that of constructing, in $\left(\mathbb{R}^3, h_{ab}\right)$, solutions to

$$\left\{ \begin{array}{l} M_h(\Sigma) = 0 \\ M_h : W^{s-1,p}_{\gamma-1}(\mathcal{S}^2) \to W^{s-2,p}_{\gamma-2}(\Lambda^1) \end{array} \right. \tag{5.1.4}$$

where M_h is the *momentum constraint operator* introduced in equation (4.2.4). To transform this under-determined elliptic problem into an elliptic equation, one prescribes arbitrarily a symmetric 2-tensor $\Psi \in W^{s-1,p}_{\gamma-1}(\mathcal{S}^2)$ and considers the ansatz

$$\Sigma = \Psi - \mathcal{L}_h X \ , \qquad X \in W^{s,p}_{\gamma}(\Lambda^1) \ . \tag{5.1.5}$$

where $\mathcal{L}_h$ is the conformal Killing operator introduced in equation (4.2.20). This gives

$$\begin{cases} \mathbb{L}_h X = M_h \Psi \\ \mathbb{L}_h : W^{s,p}_{\gamma}(\Lambda^1) \to W^{s-2,p}_{\gamma-2}(\Lambda^1) \end{cases} \tag{5.1.6}$$

which we consider as an equation to be solved for X. Note that the symbols for the fields used throughout this chapter are the uppercase format of the symbols for the fields considered in chapter 3, so as to highlight the similarities between the two decompositions.

Some of the notation that will be used was introduced already in chapter 4. Because we need to introduce some more notation, it will be convenient to recall and extend 4.2.4 here. The notation that we introduce here is based on the fact that under suitable assumptions on the weight parameter γ, the operator $\mathbb{L}_h$ is Fredholm and the operator M_h is semi-Fredholm. We will use the notation

$$N_- = \dim\left(\ker\left(M_h^*\right)\right) \tag{5.1.7a}$$
$$N_+ = \dim\left(\ker\left(\mathbb{L}_h^*\right)\right) \tag{5.1.7b}$$

and note that $\mathbb{L}_h = M_h \circ \mathcal{L}_h$ implies $\ker\left(M_h^*\right) \subset \ker\left(\mathbb{L}_h^*\right)$, hence $N_- \leq N_+$. The notation shall be changed accordingly whenever it refers to a different metric, e.g. when considering operators with respect to the metric $\widehat{h}_{ab}$ we write $\widehat{N}_+$.

Lemma 5.1.1. *Assume $Y \in \ker\left(M_h^*\right)$. Then Y is a Killing field of $\left(\mathbb{R}^3, h_{ab}\right)$ and if furtheremore $-1 < \gamma$ then $Y = 0$. It follows that $N_- \leq 6$ and that if $-1 < \gamma$ then $N_- = 0$.*

Proof. A computation shows that the action of M_h^* is given by

$$M_h^* Y = h \operatorname{div}_h(Y) - \operatorname{Sym}(\nabla Y)\ . \tag{5.1.8}$$

If $Y \in \ker\left(M_h^*\right)$ then, by taking $\operatorname{tr}_h \circ M_h^*(Y) = 0$, one finds that $\operatorname{div}_h Y = 0$. Plugging this back in $M_h^* Y = 0$ one gets $\operatorname{Sym} \nabla Y = 0$, therefore Y is Killing. Lemma 4.2.2 implies $\mathbb{L}_h$ is surjective whenever $-1 < \gamma$, thus $N_- = N_+ = 0$. The upper limit on the number of Killing fields in dimension 3 is well known. □

In the Euclidean case $\left(\mathbb{R}^3, \delta_{ab}\right)$, denoting $\mathring{N}_- = \dim\left(\ker\left(M_\delta^*\right)\right)$, it holds

$$\mathring{N}_- = \begin{cases} 6 & \text{for} \quad \gamma < -2 \quad \text{(transl. and rotat.)} \\ 3 & \text{for} \quad -2 < \gamma < -1 \quad \text{(translations)} \\ 0 & \text{for} \quad -1 < \gamma \end{cases}\ . \tag{5.1.9}$$

In general the value of N_- depends on the given metric. In particular if the background metric has no Killing fields, then N_- vanishes.

A crucial difference between the results regarding the Helmholtz decomposition and the results that will be presented in this chapter is the fact that for the former (see Lemma 3.2.6)

$$\begin{cases} n_- = 1 & \text{for} \quad \gamma < -1 \\ n_- = 0 & \text{for} \quad -1 < \gamma \end{cases} \tag{5.1.10}$$

while Lemma 5.1.1 gives in the present situation

$$\begin{cases} 0 \leq N_- \leq 6 & \text{for} \quad \gamma < -1 \\ N_- = 0 & \text{for} \quad -1 < \gamma \end{cases}\ . \tag{5.1.11}$$

5.2 Standard and Extended York decomposition

The Riemannian manifold $\left(\mathbb{R}^3, h_{ab}\right)$ is assumed to be asymptotically flat of class (p, t, τ) with $-1 < \tau < 0$. As was the case for the Helmholtz decomposition (see section 3.2.2), we will not discuss the choices of p and t. The choice for the weight parameter γ has to be made based on the decay rate one desires to achieve. As was mentioned in chapter 3, in dimension 3 this choice must be in $\mathbb{R} \setminus \mathbb{Z}$ for a technical reason.

Assumption 5.2.1. Let $\gamma \in \mathbb{R}$, $p \in \mathbb{N}$, $1 < p < \infty$ and $s \in \mathbb{N}$, $2 \leq s$, and the symmetric tensors Σ and Ψ and the 1-form V satisfy

$$\Sigma, \Psi \in W^{s-1,p}_{\gamma-1}(\mathcal{S}^2) \tag{5.2.1a}$$
$$V \in W^{s,p}_{\gamma}(\Lambda^1)\ . \tag{5.2.1b}$$

The differential operators considered in the York decomposition define the following bounded linear maps

$$M_h : W^{s-1,p}_{\gamma-1}(\mathcal{S}^2) \to W^{s-2,p}_{\gamma-2}(\Lambda^1) \tag{5.2.2a}$$
$$\mathcal{L}_h : W^{s,p}_{\gamma}(\Lambda^1) \to W^{s-1,p}_{\gamma-1}(\mathcal{S}^2) \tag{5.2.2b}$$
$$\mathbb{L}_h : W^{s,p}_{\gamma}(\Lambda^1) \to W^{s-2,p}_{\gamma-2}(\Lambda^1) \tag{5.2.2c}$$

where $\mathcal{L}_h$ and $\mathbb{L}_h$ where introduced in equations (4.2.20) and (4.2.21) respectively. Throughout the chapter the operators M_h, $\mathcal{L}_h$ and $\mathbb{L}_h$ will be considered as maps in the spaces indicated in equations (5.2.2), and we shall omit the reference to these spaces if the choice is clear from the context.

Let $\{A^\mu\}_{\mu=1}^{N_+}$ be a basis for $\ker(\mathbb{L}^*_h)$ such that $\{A^\mu\}_{\mu=1}^{N_-}$ is a basis for $\ker(M^*_h)$. Furthermore let $\{G_\nu\}_{\nu=1}^{N_+}$ denote a family of fields in $W^{s-2,p}_{\gamma-2}(\Lambda^1)$ which with $\{A^\mu\}_{\mu=1}^{N_+}$ forms a bi-orthogonal system (see section 2.2). The symbols $\mathring{A}^\mu$ and $\mathring{G}_\nu$ refer to such fields in the case of the Euclidean background $\left(\mathbb{R}^3, \delta_{ab}\right)$.

Theorem 5.2.2 (Extended York decomposition)**.** *Let $\left(\mathbb{R}^3, h_{ab}\right)$ be asymptotically flat of class (p, t, τ) with $\tau < 0$, $p \in \mathbb{N}$, $1 < p < \infty$ and $t > \frac{3}{p} + 1$. If $s \in \mathbb{N}$, $2 \leq s \leq t$, $\gamma \in \mathbb{R} \setminus \mathbb{Z}$ and*

$$\gamma < 0 \tag{5.2.3}$$

then M_h is lower semi-Fredholm. Furthermore, there exist $N_+ - N_-$ fields $F_\nu \in W^{s-1,p}_{\gamma-1}(\mathcal{S}^2)$ such that

$$M_h(F_\nu) = G_\nu\ , \quad \nu = N_- + 1, \ldots, N_+ \tag{5.2.4}$$

and

$$W^{s-1,p}_{\gamma-1}(\mathcal{S}^2) = \mathcal{L}_h(W^{s,p}_{\gamma}) \oplus \ker(M_h) \oplus \overline{\{F_\nu\}_{\nu=N_-+1}^{N_+}} \tag{5.2.5}$$

holds.

Proof. From Lemma 4.2.2 and the assumption on γ, it follows that $\mathbb{L}_h$ is a Fredholm map when considered as the map in equation (5.2.2c). Then using Theorem 3.3.1 with $S = M_h$, $T = \mathcal{L}_h$, $Q = \mathbb{L}_h$, the result follows. □

We will refer to the set $\overline{\{F_\nu\}_{\nu=N_-+1}^{N_+}}$ as an **extension to the York decomposition for** h_{ab}.

Corollary 5.2.3 (Standard York decomposition)**.** *Under the hypothesis of Theorem 5.2.2, and if $-1 < \gamma < 0$ then*

$$W^{s-1,p}_{\gamma-1}(\mathcal{S}^2) = \mathcal{L}_h(W^{s,p}_{\gamma}) \oplus \ker(M_h) \tag{5.2.6}$$

holds.

Proof. From Lemma 4.2.2 and the assumption on γ, it follows that $\mathbb{L}_h$ is an isomorphism when considered as the map in equation (5.2.2c). Then one finds that $N_+ = 0$ and consequently $N_+ \geq N_- \geq 0$ implies $N_- = 0$. □

5.3 Stability

In this section we consider an asmptotically flat manifold $\left(\mathbb{R}^3, h_{ab}\right)$ having no Killing fields, i.e. $N_- = 0$ by Lemma 5.1.1. It will be shown that if one has an extension for the York decomposition with respect to h_{ab} generated by the set of fields

$$\{F_\nu\}_{\nu=1}^{N_+} \tag{5.3.1}$$

then it is possible to say that the same set of fields constitutes an extension to the York decomposition for another metric $\widehat{h}_{ab}$ if the metrics are sufficiently close in a suitable weighted Sobolev space. Whenever we refer to operators, fields or quantities associated to the metric $\widehat{h}_{ab}$, as described when the notation was introduced in section 5.1, a hat will be added to denote this.

We state the main result of this section in Theorem 5.3.2. The estimate that follows is a generalization of the corresponding Lemma 3.4.3 for the Helmholtz decomposition.

Lemma 5.3.1. *Consider in $\mathbb{R}^3$ metrics h_{ab} and $\widehat{h}_{ab}$ which are asymptotically flat of class (p, t, τ), with $t > \frac{3}{p}$ and $\tau < 0$. If in end coordinates holds*

$$\left\|\widehat{h}_{ij} - h_{ij}\right\|_{W^{t,p}_{\tau}} < \epsilon\ , \tag{5.3.2}$$

for some $0 < \epsilon$, then there exists a constant C such that

$$\left\|\left(M_{\widehat{h}} - M_h\right)\psi\right\|_{W^{0,p}_{\gamma-2}} \leq C\epsilon \left\|\psi\right\|_{W^{1,\infty}_{\gamma-1-\tau}} \tag{5.3.3}$$

holds for every $\psi \in W^{1,\infty}_{\gamma-1-\tau}(\mathcal{S}^2)$.

Proof. Define γ_{ab} and θ^{ab} by

$$\gamma_{ab} = \widehat{h}_{ab} - h_{ab} \tag{5.3.4}$$
$$\theta^{ab} = \widehat{h}^{ab} - h^{ab} \tag{5.3.5}$$

and let ∇ and $\widehat{\nabla}$ be the derivative operators compatible with h_{ab} and $\widehat{h}_{ab}$ respectively.

By definition 4.2.4 we have

$$M_h\psi = \nabla^a\psi_{ab} - \nabla_b(h^{ac}\psi_{ac}) \tag{5.3.6a}$$
$$M_{\widehat{h}}\psi = \widehat{\nabla}^a\psi_{ab} - \widehat{\nabla}_b(\widehat{h}^{ac}\psi_{ac})\ . \tag{5.3.6b}$$

Using

$$\widehat{\nabla}_c\psi_{ab} = \nabla_c\psi_{ab} - C^d_{ca}\psi_{db} - C^d_{cb}\psi_{ad} \tag{5.3.7}$$

with

$$C^c_{ab} = \frac{1}{2}\widehat{h}^{cd}\left(\nabla_a \widehat{h}_{bd} + \nabla_b \widehat{h}_{ad} - \nabla_d \widehat{h}_{ab}\right) \tag{5.3.8}$$

one obtains

$$\begin{aligned}(M_{\widehat{h}}\psi)_b - (M_h\psi)_b &= \theta^{ac}\nabla_c\psi_{ab} - \widehat{h}^{ab}(C^d_{ca}\psi_{db} + C^d_{cb}\psi_{ad}) - \nabla_b(\theta^{ac}\psi_{ac}) \\ &= \theta^{ac}\left(\nabla_c\psi_{ab} - \nabla_b\psi_{ac}\right) - \widehat{h}^{ab}(C^d_{ca}\psi_{db} + C^d_{cb}\psi_{ad}) \\ &\quad - \nabla_b(\theta^{ac})\psi_{ac}\end{aligned} \tag{5.3.9}$$

which gives

$$\begin{aligned}\left\|(M_{\widehat{h}} - M_h)\psi_{ab}\right\|_{W^{0,p}_{\gamma-2}} \leq &\left\|\widehat{h}^{ab}(C^d_{ca}\psi_{db} + C^d_{cb}\psi_{ad}) + \nabla_b(\theta^{ac})\psi_{ac}\right\|_{W^{0,p}_{\gamma-2}} \\ &+ \left\|\theta^{ac}\left(\nabla_c\psi_{ab} - \nabla_b\psi_{ac}\right)\right\|_{W^{0,p}_{\gamma-2}} .\end{aligned} \tag{5.3.10}$$

For the first term of the right hand side in (5.3.10), first note that one can assume there exists a constant C_1 such that

$$\left\|\widehat{h}^{ab}\widehat{h}^{cd}\right\|_{W^{0,\infty}_0} < C_1 . \tag{5.3.11}$$

Then

$$\begin{aligned}\left\|\widehat{h}^{ab}C^d_{ca}\psi_{db}\right\|_{W^{0,p}_{\gamma-2}} &\leq 3^2\left\|\widehat{h}^{ab}\widehat{h}^{de}\left(\nabla_c\gamma_{ae} + \nabla_a\gamma_{ce} - \nabla_e\gamma_{ca}\right)\right\|_{W^{0,p}_{\tau-1}} \left\|\psi_{db}\right\|_{W^{0,\infty}_{\gamma-1-\tau}} \\ &\leq 3^5 C_1 \left\|\nabla_c\gamma_{ae}\right\|_{W^{0,p}_{\tau-1}} \left\|\psi_{db}\right\|_{W^{0,\infty}_{\gamma-1-\tau}}\end{aligned} \tag{5.3.12}$$

where Hölder's inequality was used once more. By the definition of the $W^{1,p}_{\tau}$ norm and Lemma 2.3.6 of γ_{ae} and the $W^{1,\infty}_{\gamma-1-\tau}$ norm of ψ_{db} one obtains

$$\left\|\widehat{h}^{ab}C^d_{ca}\psi_{db}\right\|_{W^{0,p}_{\gamma-2}} < 3^5 C_1 \left\|\gamma_{ae}\right\|_{W^{1,p}_{\tau}} \left\|\psi_{db}\right\|_{W^{1,\infty}_{\gamma-1-\tau}} . \tag{5.3.13}$$

Similarly we have

$$\begin{aligned}\left\|\nabla_b(\theta^{ac})\psi_{ac}\right\|_{W^{0,p}_{\gamma-2}} &\leq 3^2\left\|\nabla_b\theta^{ac}\right\|_{W^{0,p}_{\tau-1}} \left\|\psi_{ac}\right\|_{W^{0,\infty}_{\gamma-1-\tau}} \\ &\leq 3^2\left\|\theta^{ac}\right\|_{W^{1,p}_{\tau}} \left\|\psi_{ac}\right\|_{W^{1,\infty}_{\gamma-1-\tau}} .\end{aligned} \tag{5.3.14}$$

For the second term of the right hand side in (5.3.10), by Hölder's inequality

$$\begin{aligned}\left\|\theta^{ac}\left(\nabla_c\psi_{ab} - \nabla_b\psi_{ac}\right)\right\|_{W^{0,p}_{\gamma-2}} &\leq 2\ 3^2\left\|\theta^{ac}\right\|_{W^{0,p}_{\tau}} \left\|\nabla_c\psi_{ab}\right\|_{W^{0,\infty}_{\gamma-2-\tau}} \\ &\leq 2\ 3^2\left\|\theta^{ab}\right\|_{W^{1,p}_{\tau}} \left\|\psi_{ab}\right\|_{W^{1,\infty}_{\gamma-1-\tau}}\end{aligned} \tag{5.3.15}$$

by Lemma 2.3.6 and the definition of the norms in $W^{1,\infty}_{\gamma-1-\tau}$ and $W^{1,p}_{\tau}$.

Using (5.3.13), (5.3.14) and (5.3.15) in (5.3.10), one gets

$$\left\|(M_{\widehat{h}} - M_h)\psi_{ab}\right\|_{W^{0,p}_{\gamma-2}} < 27\left(18C_1\left\|\gamma_{ae}\right\|_{W^{1,p}_{\tau}} + \left\|\theta^{ac}\right\|_{W^{1,p}_{\tau}}\right)\left\|\psi_{ab}\right\|_{W^{1,\infty}_{\gamma-1-\tau}} . \tag{5.3.16}$$

Finally, by Lemma 2.3.7 it follows that one can find ϵ small enough so that the claimed estimate is obtained. □

Theorem 5.3.2. *Consider in* $\mathbb{R}^3$ *metrics* h_{ab} *and* $\widehat{h}_{ab}$*, both asymptotically flat of class* (p,t,τ) *with* $\tau < 0$*,* $t > 3/p + 1$*, and assume that* h_{ab} *does not have Killing fields. Let* $\gamma \in \mathbb{R} \setminus \mathbb{Z}$*,* $s \in \mathbb{N}$*,* $\frac{3}{p} + 1 \leq s \leq t$*. Assume that* $\overline{\{F_\nu\}_{\nu=1}^{N_+}}$ *is an extension to the York decomposition with respect to* h_{ab}*. There exists* $0 < \epsilon$ *such that if in end coordinates it holds*

$$\left\| h_{ij} - \widehat{h}_{ij} \right\|_{W^{t,p}_\tau} \leq \epsilon \tag{5.3.17}$$

and $\widehat{h}_{ab}$ *has no Killing fields, then*

$$W^{s-1,p}_{\gamma-1}(\mathcal{S}^2) = \mathcal{L}_{\widehat{h}}(W^{s,p}_\gamma) \oplus \ker\left(M_{\widehat{h}}\right) \oplus \overline{\{F_\nu\}_{\nu=1}^{N_+}} \tag{5.3.18}$$

holds.

Proof. Consider an arbitrary element $\psi \in W^{s-1,p}_{\gamma-1}(\mathcal{S}^2)$. The decomposition (5.3.18) holds if one can find $X \in W^{s,p}_\gamma$ and $k^\nu \in \mathbb{R}^{N_+}$ such that

$$\psi = \mathcal{L}_{\widehat{h}} X + \sigma + \sum_{\nu=1}^{N_+} k^\nu F_\nu \ . \tag{5.3.19}$$

By requiring that $\sigma \in \ker\left(M_{\widehat{h}}\right)$ one obtains the equation

$$M_{\widehat{h}}\psi = \mathbb{L}_{\widehat{h}} X + \sum_{\nu=1}^{N_+} k^\nu M_{\widehat{h}} F_\nu \tag{5.3.20}$$

which is considered as a condition on X and k^ν. Letting $\{\widehat{A}^\mu\}_{\mu=1}^{N_+}$ denote a basis of $\ker\left(\mathbb{L}_{\widehat{h}}{}^*\right)$, one has the equation

$$T^\mu{}_\nu k^\nu = \langle \widehat{A}^\mu, M_{\widehat{h}}\psi \rangle_{\widehat{h}} \tag{5.3.21}$$

where the matrix $T^\mu{}_\nu$ is given by

$$T^\mu{}_\nu = \langle \widehat{A}^\mu, M_{\widehat{h}} F_\nu \rangle_{\widehat{h}} \quad \text{for} \quad \nu, \mu = 1, \ldots, N_+ \ . \tag{5.3.22}$$

If the matrix $T^\mu{}_\nu$ is non-degenerate, then one can solve for k^ν and then plugging them back into (5.3.20) solve for X.

The fields F_ν are assumed to be an extension to the York decomposition with respect to h_{ab} which by assumption has no Killing fields, hence $N_- = 0$. They are constructed out of a bi-orthogonal system $\{A^\mu\}_{\mu=1}^{N_+}$, $\{G_\nu\}_{\nu=1}^{N_+}$ by requiring that they satisfy $M_h F_\nu = G_\nu$ for $\nu = 1, \ldots, N_+$. Then one can write

$$T^\mu{}_\nu = \langle \widehat{A}^\mu, G_\nu \rangle_{\widehat{h}} + \langle \widehat{A}^\mu, (M_{\widehat{h}} - M_h) F_\nu \rangle_{\widehat{h}} \ . \tag{5.3.23}$$

Using Lemma 3.3.2, one can assume that $\langle \widehat{A}^\mu, G_\nu \rangle_{\widehat{h}}$ is non-degenerate and the proof follows that of Theorem 3.4.4. □

An important remark about Theorem 5.3.2 is that if the metric $\widehat{h}_{ab}$ has non-trivial Killing fields, one is not able to solve for the coefficients k^ν uniquely. In the notation we have used, one would have that the first $\widehat{N}_-$ of the elements in $\{\widehat{A}^\mu\}_{\mu=1}^{N_+}$ are a basis for

$\ker\left(M^*_{\widehat{h}}\right)$. This implies that the first $\widehat{N}_-$ rows of both the matrix $T^\mu{}_\nu$ and the right hand side of equation (5.3.21) vanish identlically. The equation (5.3.21) is therefore a system of $N_+ - \widehat{N}_-$ non-homogeneous linear equations can be solved for k^ν up to $\widehat{N}_-$ free parameters. If this is the case, the statement (5.3.18) must be replaced with

$$W^{s-1,p}_{\gamma-1}(\mathcal{S}^2) = \mathcal{L}_{\widehat{h}}(W^{s,p}_\gamma) + \ker\left(M_{\widehat{h}}\right) + \overline{\{F_\nu\}^{N_+}_{\nu=1}} \tag{5.3.24}$$

and one could say that there are $\widehat{N}_-$ too many fields in the set $\{F_\nu\}^{N_+}_{\nu=1}$ which make the decomposition not-unique.

One should also note that the set of metrics having no Killing fields is known to be open in the set of asymptotically flat metrics. This implies there allways exist $0 < \epsilon$ such that if h_{ab} has trivial isometry group then the same holds for every $\widehat{h}_{ab}$ in an ϵ-neighborhood around h_{ab}.

5.4 The case of Euclidean $\mathbb{R}^3$

In this section the extensions discussed above are constructed explicitly for a Euclidean background in dimension 3. In particular it will be shown that if one considers a weight parameter $\gamma \in (-2,-1)$ then it happens for $\left(\mathbb{R}^3, \delta_{ab}\right)$ that $N_+ = N_-$ and so Theorem 5.2.2 reduces to the usual York decomposition. This accounts for the fact that whenever $\gamma \in (-2,-1)$, the translation generators are a basis for $\ker\left(M^*_\delta\right)$ as well as for $\ker\left(\mathbb{L}^*_\delta\right)$.

Throughout the section we will use standard spherical coordinates (r,θ,ϕ) in $\mathbb{R}^3$. The spin-weighted spherical harmonics provide a framework to carry out the computations needed, so they will be introduced in section 5.4.1.

In section 5.4.2 a basis $\{\mathring{A}^\mu\}^{N_+}_{\mu=1}$ for the set $\ker\left(\mathbb{L}^*_\delta\right)$ is described. Also a set of fields $\{\mathring{G}_\nu\}^{N_+}_{\nu=1}$ that forms a bi-orthogonal system with $\{\mathring{A}^\mu\}^{N_+}_{\mu=1}$ is given. Some of the fields in the set $\{\mathring{A}^\mu\}^{N_+}_{\mu=1}$ are in $\ker\left(M^*_\delta\right)$, i.e. they are Killing fields. They are identified and labeled conveniently.

One should note that there are other ways to derive explicit expressions for the elements of $\ker\left(\mathbb{L}^*_\delta\right)$. In [Stavrov Allen, 2010] one such computation is given in terms of standard spherical harmonics. The use of spin-weighted spherical harmonics is however much better suited for the purposes of this work.

To construct the extended York decomposition for the flat background one wants to find a set of symmetric tensors

$$\left\{\mathring{F}_\nu\right\}^{N_+}_{\nu=N_-+1} \tag{5.4.1}$$

such that they satisfy

$$M_\delta(\mathring{F}_\nu) = \mathring{G}_\nu \tag{5.4.2}$$

for $\nu = N_- + 1, \ldots, N_+$. To do this, an ansatz for the fields $\mathring{F}_\nu$ is proposed in section 5.4.3 by which one can actually find all the fields $\mathring{F}_\nu$ sought-after.

5.4.1 Spin-weighted spherical harmonics

The spin-weighted spherical harmonics can be thought of as a generalization of the standard spherical harmonics. For details and a thorough discussion we refer the reader to [Goldberg et al., 1967]. Consider an orthonormal triad given by

$$n_a = (dr)_a \tag{5.4.3a}$$
$$m_a = \frac{r}{\sqrt{2}}\left((d\theta)_a + i\sin\theta(d\phi)_a\right) \tag{5.4.3b}$$
$$\overline{m}_a = \frac{r}{\sqrt{2}}\left((d\theta)_a - i\sin\theta(d\phi)_a\right)\ . \tag{5.4.3c}$$

They satisfy $n^a n_a = \overline{m}^a m_a = 1$ and $n^a m_a = n^a \overline{m}_a = m^a m_a = \overline{m}^a \overline{m}_a = 0$. One can write the flat metric in the form $\delta_{ab} = n_a n_b + m_a \overline{m}_b + \overline{m}_a m_b$. The vectors m_a and $\overline{m}_a$ are defined up to a phase transformation

$$m_a \to m'_a = e^{i\varphi} m_a \tag{5.4.4}$$
$$m_a \to \overline{m}'_a = e^{-i\varphi} \overline{m}_a \tag{5.4.5}$$

which leaves the properties of the triad invariant. A function η is said to be of spin-weight s if, under the transformation (5.4.4) it changes by

$$\eta' = e^{is\varphi}\eta\ . \tag{5.4.6}$$

Let η_s denote a function of spin-weight s. We define the operators $\eth$ and $\overline{\eth}$ acting on η_s by

$$\eth\eta_s = -\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi - s\cot\theta\right)\eta_s \tag{5.4.7}$$
$$\overline{\eth}\eta_s = -\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi + s\cot\theta\right)\eta_s\ . \tag{5.4.8}$$

The resulting functions $\eth\eta_s$ and $\overline{\eth}\eta_s$ have spin-weight $s+1$ and $s-1$ respectively.

For integral values of s, j and m, the spin-s spherical harmonics can be defined in terms of the standard spherical harmonics by

$${}_sY_{jm}(\theta,\phi) = \begin{cases} \sqrt{\frac{(j-s)!}{(j+s)!}}\eth^s Y_{jm}\ , & \text{if } 0 \le s \le j \\ \sqrt{\frac{(j+s)!}{(j-s)!}}(-1)^{-s}\overline{\eth}^s Y_{jm}\ , & \text{if } -j \le s \le 0 \\ 0\ , & \text{if } j < |s| \end{cases}\ . \tag{5.4.9}$$

They are eigenfunctions of the operators $\eth\overline{\eth}$ and of $-i\partial_\phi$ with eigenvalues $s(s-1) - j(s+1)$ and m respectively. They satisfy

$$\eth_s Y_{jm} = \sqrt{j(j+1) - s(s+1)}_{s+1}Y_{jm} \tag{5.4.10a}$$
$$\overline{\eth}_s Y_{jm} = -\sqrt{j(j+1) - s(s-1)}_{s-1}Y_{jm}\ . \tag{5.4.10b}$$

Differential operators using spin-weighted components

A 1-form field A_b can be writen in terms of n_a, m_a and $\overline{m}_a$ as

$$A_b = A_0 n_b + A_+ \overline{m}_b + A_- m_b \tag{5.4.11}$$

where $A_0 = A_b n^b$, $A_+ = A_b m^b$, and $A_- = A_b \overline{m}^b$ are of spin weight 0, +1 and −1 respectively. The field A_b is real if and only if $\overline{A_s} = A_{-s}$. Similarly, a symmetric tensor Ψ_{ab} can be writen in the form

$$\begin{aligned}\Psi_{ab} = {}& \Psi_0 n_{(a}n_{b)} + \Psi_{00} m_{(a}\overline{m}_{b)} \\ &+ \Psi_- n_{(a}m_{b)} + \Psi_+ n_{(a}\overline{m}_{b)} + \Psi_{-2} m_{(a}m_{b)} + \Psi_{+2}\overline{m}_{(a}\overline{m}_{b)}\end{aligned} \tag{5.4.12}$$

where Ψ_0 and Ψ_{00} have spin weight 0, and the Ψ_s have spin weight s.

The introduction of the operator $\eth$ allows one to derive expressions for differential equations in terms of the different spin-weight components of the fields involved. After some computation, one finds that the momentum constraint operator is given by

$$\begin{aligned}M_\delta(\Psi) = {}& \frac{n_b}{r}\left\{-\partial_r(r\Psi_{00}) + 2\Psi_0 - \frac{\eth\Psi_- + \bar{\eth}\Psi_+}{2\sqrt{2}}\right\} \\ &+ \frac{m_b}{r}\left\{\frac{\partial_r(r^3\Psi_-)}{2r^2} + \frac{1}{2\sqrt{2}}\bar{\eth}\left(2\Psi_0 + \Psi_{00}\right) - \frac{\eth\Psi_{-2}}{\sqrt{2}}\right\} \\ &+ \frac{\overline{m}_b}{r}\left\{\frac{\partial_r(r^3\Psi_+)}{2r^2} + \frac{1}{2\sqrt{2}}\eth\left(2\Psi_0 + \Psi_{00}\right) - \frac{\bar{\eth}\Psi_{+2}}{\sqrt{2}}\right\} .\end{aligned} \tag{5.4.13}$$

The conformal Killing operator is given by

$$\begin{aligned}\mathcal{L}_\delta A = {}& \frac{2}{3}n_{(a}n_{b)}\left\{2r\partial_r\left(\frac{A_0}{r}\right) + \frac{\eth A_- + \bar{\eth}A_+}{\sqrt{2}r}\right\} \\ &- \frac{2}{3}m_{(a}\overline{m}_{b)}\left\{2r\partial_r\left(\frac{A_0}{r}\right) + \frac{\eth A_- + \bar{\eth}A_+}{\sqrt{2}r}\right\} \\ &+ 2n_{(a}m_{b)}\left\{r\partial_r\left(\frac{A_-}{r}\right) - \frac{\bar{\eth}A_0}{\sqrt{2}r}\right\} \\ &+ 2n_{(a}\overline{m}_{b)}\left\{r\partial_r\left(\frac{A_+}{r}\right) - \frac{\eth A_0}{\sqrt{2}r}\right\} \\ &- \frac{\sqrt{2}}{r}\left\{m_{(a}m_{b)}\bar{\eth}A_- + \overline{m}_{(a}\overline{m}_{b)}\eth A_+\right\} .\end{aligned} \tag{5.4.14}$$

The conformal Killing Laplacian is given by

$$\begin{aligned}\mathbb{L}_\delta(A) = {}& \frac{n_b}{3r^2}\left\{4\left(\partial_r\left(r^2\partial_r A_0\right) - 2A_0\right) + 3\eth\bar{\eth}A_0 + (7 - r\partial_r)\frac{\bar{\eth}A_+ + \eth A_-}{\sqrt{2}}\right\} \\ &+ \frac{\overline{m}_b}{3r^2}\left\{-(r\partial_r + 8)\frac{\eth A_0}{\sqrt{2}} + 3\partial_r\left(r^2\partial_r A_+\right) + \frac{7}{2}\eth\bar{\eth}A_+ + \frac{1}{2}\eth\eth A_-\right\} \\ &+ \frac{m_b}{3r^2}\left\{-(r\partial_r + 8)\frac{\bar{\eth}A_0}{\sqrt{2}} + 3\partial_r\left(r^2\partial_r A_-\right) + \frac{7}{2}\bar{\eth}\eth A_- + \frac{1}{2}\bar{\eth}\bar{\eth}A_+\right\} .\end{aligned} \tag{5.4.15}$$

The advantage of the expression above is that in the present context one can assume that the spin-weighted components of the one form A_b are expanded as

$$A_0(r,\theta,\phi) = \sum_{j=0}^{\infty}\sum_{m=-j}^{j} g_{0jm}(r)\ {}_0Y_{jm}(\theta,\phi) \tag{5.4.16a}$$

$$A_s(r,\theta,\phi) = \sum_{j=1}^{\infty}\sum_{m=-j}^{j} g_{sjm}(r)\sqrt{\frac{2}{j(j+1)}}\,{}_sY_{jm}(\theta,\phi) . \tag{5.4.16b}$$

where $g_{sjm}(r)$ depend only on r. Note that for $A_\pm$ the sum starts from $j = 1$ because by equation (5.4.9) one has ${}_\pm Y_{0m} = 0$.

Conformal Killing fields of Euclidean $\mathbb{R}^3$

One can use the expansion (5.4.16) in equation (5.4.14) to study the conformal Killing equation

$$\mathcal{L}_\delta(A) = 0 \tag{5.4.17}$$

in Euclidean $\mathbb{R}^3$. The result is that a basis for the conformal Killing fields of Euclidean $\mathbb{R}^3$ is given by

$$\begin{aligned} A_b = \sum_{m=-1,0,1} & c_{1m} r^2 \left\{ Y_{1m}\, n_b + ({}_+Y_{1m}\overline{m}_b - {}_-Y_{1m} m_b) \right\} \\ & + c_{2m} r\,({}_+Y_{1m}\overline{m}_b + {}_-Y_{1m} m_b) + c_3 r n_b \\ & + c_{4m} \left\{ Y_{1m}\, n_b - ({}_+Y_{1m}\overline{m}_b - {}_-Y_{1m} m_b) \right\} \end{aligned} \tag{5.4.18}$$

where $m = -1, 0, 1$ and c_{1m}, c_{2m}, c_3 and c_{4m} are 10 constants. If one computes the divergence of (5.4.18) one obtains

$$\mathrm{div}_\delta(A_a) = \sum_{m=-1,0,1} 6\ {}_0Y_{1m} c_{1m} r + 3\ {}_0Y_{00} c_3\ . \tag{5.4.19}$$

Conformal Killing fields whose divergence vanishes, i.e. those with $c_{1m} = c_3 = 0$, are proper Killing fields.

5.4.2 The conformal Killing operator in Euclidean $\mathbb{R}^3$

In this section we start by studying the equation $\mathbb{L}^*_\delta(A) = 0$ in Euclidean $\mathbb{R}^3$ without considering fall-off conditions on the solutions. Using the ansatz (5.4.16) in equation (5.4.15) we obtain a system of ODEs for the functions g_{sjm}. Then, by requiring that the solutions be regular everywhere and that they satisfy certain fall-off conditions it will be seen that the resulting solutions span $\ker(\mathbb{L}^*_\delta)$.

In terms of the unknowns

$$U_0 = g_{0jm} \tag{5.4.20a}$$
$$U_+ = g_{+,jm} + g_{-,jm} \tag{5.4.20b}$$
$$U_- = g_{+,jm} - g_{-,jm} \tag{5.4.20c}$$

the system reads

$$\left(4\,\partial_r\left(r^2\,\partial_r\right) - 6k - 8\right) U_0 + \left(r\,\partial_r - 7\right) U_- = 0 \tag{5.4.21a}$$
$$-2k\left(r\,\partial_r + 8\right) U_0 + \left(3\,\partial_r\left(r^2\,\partial_r\right) - 8k\right) U_- = 0 \tag{5.4.21b}$$
$$\left(\partial_r\left(r^2\,\partial_r\right) - 2k\right) U_+ = 0 \tag{5.4.21c}$$

where we have set $k = \frac{j(j+1)}{2}$. We now set $U_s(r) = u_s r^p$ where u_s are constants and $p \in \mathbb{Z}$. This gives

$$\begin{pmatrix} 4p(p+1) - 6k - 8 & p-7 & 0 \\ -2k(p+8) & 3p(p+1) - 8k & 0 \\ 0 & 0 & p(p+1) - 2k \end{pmatrix} \begin{pmatrix} u_0 \\ u_- \\ u_+ \end{pmatrix} = 0\ . \tag{5.4.22}$$

To find non-trivial solutions one sets the determinant of this matrix to zero. In terms of j, this gives $p = j+1, j, j-1, -j, -j-1, -j-2$ and for each value of p one can find corresponding constants u_s. We are interested in solutions which are regular everywhere, so we consider those having $p \geq 0$. To parametrize all the solutions obtained we consider (in a way similar to what we did for the Helmholtz decomposition) some numbering by indices $\mu = 1, \ldots, N_+$ of the set of triples

$$\mu \leftrightarrow (s, j, m) \tag{5.4.23}$$

where $s = 1, 2, 3$, $j \geq 0$ runs up to a certain maximum value, and for fixed j $m = -j, \ldots, j$. The solutions to $\mathbb{L}^*_\delta(A) = 0$ obtained are given by

$$\begin{aligned} \mathring{A}^\mu &= \delta^1_s r^{j+1} \left\{ \frac{(j-6)}{(j+9)} \, {}_0Y_{jm} n_b - \sqrt{\frac{j}{j+1}} \frac{{}_+Y_{jm}\overline{m}_b - {}_-Y_{jm} m_b}{\sqrt{2}} \right\} \\ &+ \delta^2_s r^j \frac{{}_+Y_{jm}\overline{m}_b + {}_-Y_{jm} m_b}{\sqrt{2}} \\ &+ \delta^3_s r^{j-1} \left\{ \sqrt{\frac{j}{j+1}} \, {}_0Y_{jm} n_b - \frac{{}_+Y_{jm}\overline{m}_b - {}_-Y_{jm} m_b}{\sqrt{2}} \right\} \end{aligned} \tag{5.4.24}$$

where the term corresponding to $s = 3$ vanishes for $j = 0$.

So far we have not said anything regarding which of the solutions found in (5.4.24) are in $\ker(\mathbb{L}^*_\delta)$. A 1-form $\mathring{A}^\mu$ belongs to $\ker(\mathbb{L}^*_\delta)$ if it grows at infinity no faster than $o\left(r^{-\gamma-1}\right)$. This means that in (5.4.24) one must consider for $s = 1$ the solutions with $j \leq -\gamma - 2$, for $s = 2$ the solutions with $j \leq -\gamma - 1$ and for $s = 3$ those with $j \leq -\gamma$. To illustrate the way in which the solutions are enumerated, we have collected some of these properties in table 5.1 for ranges of the weight parameter γ down to -5. Each row indicates how many solutions are in $\ker(\mathbb{L}^*_\delta)$ in addition to all the solutions in rows above it.

γ	$s = 1$	$s = 2$	$s = 3$	N_+	N_-
$\in (-1, 0)$	0	0	0	0	0
$\in (-2, -1)$	0	0	3 ($j = 1$)	3	3
$\in (-3, -2)$	1 ($j = 0$)	3 ($j = 1$)	5 ($j = 2$)	$9 + 3$	6
$\in (-4, -3)$	3 ($j = 1$)	5 ($j = 2$)	7 ($j = 3$)	$15 + 9 + 3$	6
$\in (-5, -4)$	5 ($j = 2$)	7 ($j = 3$)	9 ($j = 4$)	$21 + 15 + 9 + 3$	6

Table 5.1: Ordering of the fields $\mathring{A}^\mu \in \ker(\mathbb{L}^*_\delta)$.

It is also important to stress that the number of solutions (5.4.24) obtained for each $\gamma \in \mathbb{R} \setminus \mathbb{Z}$ coincides with the general result of Theorem 4.2.2 and their linear independence then guarantees that they are in deed a basis for $\ker(\mathbb{L}^*_\delta)$.

The solutions with $s = 3$, $j = 1$ and $m = 1, 0, -1$

$$\mathring{A}_b = \frac{1}{\sqrt{2}} \, {}_0Y_{1m} n_b - \frac{{}_+Y_{1m}\overline{m}_b - {}_-Y_{1m} m_b}{\sqrt{2}} \tag{5.4.25}$$

can be seen to correspond to Killing fields associated to translations while the solutions with $s = 2$, $j = 1$ and $m = 1, 0, -1$

$$\mathring{A}_b = r \frac{{}_+Y_{1m}\overline{m}_b + {}_-Y_{1m} m_b}{\sqrt{2}} \tag{5.4.26}$$

are seen to correspond to be Killing fields generating rotations. They are 6 linearly independent elements of $\ker(M^*_\delta)$ and therefore constitute a basis for it.

The way in which the index μ is assigned by (5.4.23) to each of the fields $\mathring{A}^\mu$ in equation (5.4.24) need not be completely specified at this point, but to be consistent with the notation introduced in section 5.1 and the fact pointed out above, it is convenient to choose

$$\begin{cases} \mu = 1 & \leftrightarrow (2,1,1) \\ \mu = 2 & \leftrightarrow (2,1,0) \\ \mu = 3 & \leftrightarrow (2,1,-1) \end{cases} , \quad \begin{cases} \mu = 4 & \leftrightarrow (3,1,1) \\ \mu = 5 & \leftrightarrow (3,1,0) \\ \mu = 6 & \leftrightarrow (3,1,-1) \end{cases} . \tag{5.4.27}$$

In this sense, the fields $\{\mathring{A}^\mu\}_{\mu=1}^6$ are a basis for $\ker(M^*_\delta)$. This will become important in the next section.

It is convenient at this point to define fields L_0, L_R and L_L by

$$L_0 = {}_0Y_{jm} n_b \tag{5.4.28a}$$

$$L_R = \frac{{}_+Y_{jm}\overline{m}_b + {}_-Y_{jm} m_b}{\sqrt{2}} \tag{5.4.28b}$$

$$L_L = \frac{{}_+Y_{jm}\overline{m}_b - {}_-Y_{jm} m_b}{\sqrt{2}} \tag{5.4.28c}$$

and note that they satisfy $\langle L_0, L_0\rangle_{S^2} = \langle L_R, L_R\rangle_{S^2} = \langle L_L, L_L\rangle_{S^2} = 1$ and $\langle L_0, L_R\rangle_{S^2} = \langle L_0, L_L\rangle_{S^2} = \langle L_R, L_L\rangle_{S^2} = 0$. Let $\chi(r) \in C^\infty_0$ be supported away from $r = 0$ and such that $\int \chi(r) r^2 \mathrm{d}\mu_\delta = 1$. Writing (5.4.24) in terms of L_0, L_R and L_L, one can show that for $\nu = 1, \ldots, N_+$ the set of fields

$$\begin{aligned} \mathring{G}_\nu = {} & \delta^1_{s'} \frac{\chi(r)}{r^{j+1}} \frac{(j+1)(j+9)}{2(7j+3)} \left\{ -L_0 - \sqrt{\frac{j}{j+1}} L_L \right\} \\ & + \delta^2_{s'} \frac{\chi(r)}{r^j} L_R \\ & + \delta^3_{s'} \frac{\chi(r)}{r^{j-1}} \frac{(j+1)(j+9)}{2(7j+3)} \left\{ \sqrt{\frac{j}{j+1}} L_0 + \frac{(j-6)}{j+9} L_L \right\} \end{aligned} \tag{5.4.29}$$

satisfies $\langle \mathring{A}^\mu, \mathring{G}_\nu\rangle_\delta = \delta^\mu{}_\nu$ where ν stands for (s', j', m'). Hence the sets

$$\{\mathring{A}^\mu\}^{N_+}_\mu , \quad \{\mathring{G}_\nu\}^{N_+}_\nu \tag{5.4.30}$$

given by equations (5.4.24) and (5.4.29) form a bi-orthogonal system.

5.4.3 Construction of the complement

One now looks for symmetric tensors $\mathring{F}_\nu$ such that

$$M_\delta(\mathring{F}_\nu) = \mathring{G}_\nu \tag{5.4.31}$$

for $\nu = 7, \ldots, N_+$. There is a large freedom in the way this is done and we now show a construction of such tensors. Consider symmetric tensors of the simple form

$$\mathring{F}_{ab} = \Psi_0 n_{(a} n_{b)} + \Psi_{-2} m_{(a} m_{b)} + \Psi_{+2} \overline{m}_{(a}\overline{m}_{b)} . \tag{5.4.32}$$

Assume the components are of the form

$$\Psi_s = P_s(r)_s Y_{jm}(\theta,\phi) \tag{5.4.33}$$

and change variables to $f_\pm = \frac{1}{\sqrt{2}}(P_+ \pm P_-)$. Using the expression for the momentum constraint operator given in equation (5.4.13), one obtains

$$\begin{aligned} M_\delta(\mathring{F}) &= \frac{L_0}{r} 2P_0 \\ &+ \frac{L_R}{r}\sqrt{\frac{(j-1)(j+2)}{2}} f_{-2} \\ &+ \frac{L_L}{r}\left(\sqrt{\frac{(j-1)(j+2)}{2}} f_{+2} + \sqrt{j(j+1)} P_0\right) . \end{aligned} \tag{5.4.34}$$

By comparing with equation (5.4.29) we obtain relations for the functions $f_s(r)$. These are found to have solutions only for the cases with $\nu = 7, \ldots, N_+$ as expected. The solution obtained reads

$$\begin{aligned} \mathring{F}_\nu &= \delta^1_{s'} \frac{\chi(r)}{r^j} \frac{(j+1)(j+9)}{4(7j+3)} \Bigg(-{}_0Y_{jm} n_{(a}n_{b)} \\ &+ \sqrt{\frac{2j(j-1)}{(j+1)(j+2)}} \frac{{}_{+2}Y_{jm}\overline{m}_{(a}\overline{m}_{b)} + {}_{-2}Y_{jm} m_{(a}m_{b)}}{\sqrt{2}} \Bigg) \\ &+ \delta^2_{s'} r \frac{\chi(r)}{r^j} \sqrt{\frac{2}{(j-1)(j+2)}} \frac{{}_{+2}Y_{jm}\overline{m}_{(a}\overline{m}_{b)} - {}_{-2}Y_{jm} m_{(a}m_{b)}}{\sqrt{2}} \\ &+ \delta^3_{s'} \frac{\chi(r)}{r^{j-2}} \frac{(j+1)(j+9)}{4(7j+3)} \Bigg(\sqrt{\frac{j}{j+1}}\, {}_0Y_{jm} n_{(a}n_{b)} \\ &- \frac{(j+3)(j+4)}{j+9} \sqrt{\frac{2}{(j-1)(j+2)}} \frac{{}_{+2}Y_{jm}\overline{m}_{(a}\overline{m}_{b)} + {}_{-2}Y_{jm} m_{(a}m_{b)}}{\sqrt{2}} \Bigg) . \end{aligned} \tag{5.4.35}$$

Example 5.4.1. To give an example, we consider the first non-trivial elements of this complement which corresponds to considering weights $\gamma \in (-3,-2)$. The complement is generated by 6 fields given by taking in the previous equation the values $(s=1, j=0, m=0)$ and $(s=3, j=2, m=-2,\ldots,2)$. This gives the following 6 fields

$$\begin{aligned} \mathring{F}_\nu &= -\delta^1_{s'} \frac{3\chi(r)}{4\sqrt{4\pi}} n_{(a}n_{b)} \\ &+ \delta^3_{s'} \frac{\chi(r)}{68} \left(11\sqrt{6}\, {}_0Y_{2m} n_{(a}n_{b)} - 45 \left({}_{+2}Y_{2m}\overline{m}_{(a}\overline{m}_{b)} + {}_{-2}Y_{2m} m_{(a}m_{b)} \right) \right) . \end{aligned} \tag{5.4.36}$$

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