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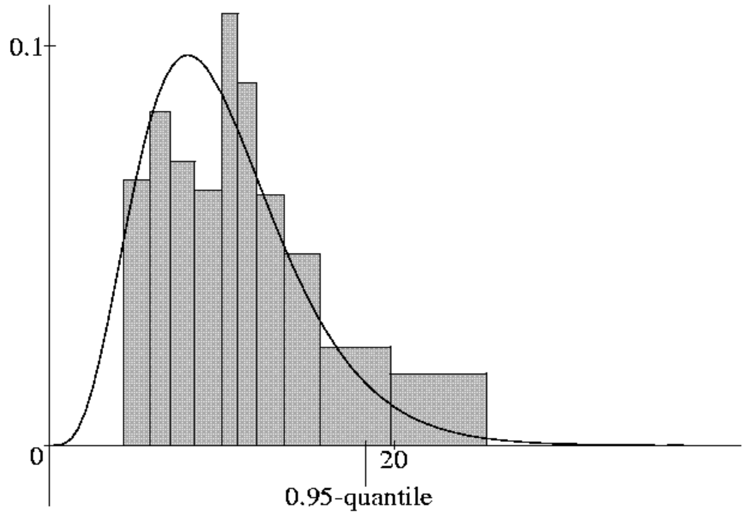

Mathematische Statistik und
Wahrscheinlichkeitstheorie

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Preprint 2005/02

Juli 2005

Impressum

Herausgeber: Mathematische Statistik und Wahrscheinlichkeitstheorie
am Institut für Mathematik

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ISSN 1613-3307

Infinite System of Brownian Balls : Equilibrium measures are canonical Gibbs

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Abstract

We consider a system of infinitely many hard balls in $\mathbb{R}^d$ undergoing Brownian motions and submitted to a smooth pair potential. It is modelized by an infinite-dimensional Stochastic Differential Equation with a local time term. We prove that the set of all equilibrium measures, solution of a Detailed Balance Equation, coincides with the set of canonical Gibbs measures associated to the hard core potential added to the smooth interaction potential.

AMS Classifications: 60H10, 60J60, 60K35.
KEY-WORDS: Stochastic Differential Equation, hard core potential, Canonical Gibbs measure, detailed balance equation, reversible measure.

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1 Introduction

One of the most fundamental problems in Statistical Mechanics is the characterization of the family of all stationary or reversible measures of stochastic dynamics.
Kolmogorov analysed in his pioneer paper [13] the strong connection between time-reversible diffusions and Gibbs measures in the context of finite-dimensional processes. Since that time it has been extended to several stochastic models. Let us refer, among others, to Doss and Royer [4] for infinite-dimensional interacting Brownian diffusions on a lattice (see also [1] for an alternative proof), to Iwata [12] for $P(\varphi)_1$-time diffusions, to Funaki [9] for a multi-dimensional Ginzburg-Landau continuum model or to Sakagawa [26] for a Ginzburg-Landau model of conservative type. Here, we consider the following continuous model : an infinite system of hard balls in $\mathbb{R}^d$, undergoing Brownian motions and submitted to the influence of a smooth finite range pair potential.

On one side, a system of infinite Brownian particles (i.e. balls with radius 0) with smooth pair interaction has first been treated by Lang who constructed it in [14] as solution of an infinite-dimensional stochastic differential equation (see also [8]). Lang also proved in [15] that the canonical Gibbs measures associated to the smooth potential are the unique reversible measures for such dynamics. Georgii obtained in [10] with different techniques a similar result for infinite-dimensional Brownian diffusions associated to more general smooth potentials. On the other side, a system of infinitely many Brownian balls submitted to an external finite range pair potential was constructed by the authors in [5] (only for Gibbsian initial distribution). See also [29] for the case without external potential and [7] for an extension to infinite range pair potentials . The system is the unique solution to an infinite-dimensional Skohorod type equation (see equation $(\mathcal{E})$ stated in section 2) where the hard core situation - balls cannot overlap - appears as a local time term in addition to the basic Brownian motion. We also proved in [5] that Gibbs states are reversible measures but we did not describe the structure of the family of all reversible measures. The goal of this paper is to clarify this last point.

In section 2 we introduce the infinite dimensional equation $(\mathcal{E})$ and state the main results. In section 3 we construct a strong solution for $(\mathcal{E})$ for an explicit set of deterministic initial conditions. We connect in section 4 the time-reversibility of the system with a symmetry property of the associated infinitesimal generator : it is the so-called Detailed Balance equation. In section 5 we show that any measure satisfying the Detailed Balance Equation also obeys to an integral equation exhibiting a symmetry property of the associated Campbell measure. We conclude the proof of the main theorem by proving that such a measure is necessarily canonical Gibbs.

2 Dynamics and main results

2.1 Configuration spaces

The particles we deal with in the present paper move in $\mathbb{R}^d$, for a fixed $d \geqslant 2$, endowed with the Euclidian norm denoted by $|\ |$. $B(y, \rho)$ will denote the closed ball centered in $y \in \mathbb{R}^d$ with radius $\rho \geq 0$ and more generally, for any $A \subset \mathbb{R}^d$, we define

$$B(A, \rho) = \{y \in \mathbb{R}^d \text{ such that } d(y, A) \leqslant \rho\}$$

where $d(y, A)$ denotes the Euclidian distance between y and A. The volume of a subset A in $\mathbb{R}^d$ is also denoted by $|A|$.

The modelization of point configurations may be done in two equivalent ways :
The first possibility is to represent an n points configuration in $\mathbb{R}^d$ as a subset (with multiplicity) of cardinal n in $\mathbb{R}^d$. The second possibility is to modelize it as a point measure $\sum_{i=1}^n \delta_{\xi_i}$ on $\mathbb{R}^d$. More generally, the set of all point configurations in $\mathbb{R}^d$ will be the set $\mathcal{M}$ of all point Radon

measures on $\mathbb{R}^d$:

$$\mathcal{M}=\left\{\xi=\sum_{i\in I}\delta_{\xi_i} \text{ such that } I\subset\mathbb{N},\ \xi_i\in\mathbb{R}^d \text{ and for all } \Lambda \text{ compact in } \mathbb{R}^d,\ \xi(\Lambda)<+\infty\right\}.$$

$\mathcal{M}$ is endowed with the topology of vague convergence. By simplicity, we will identify any point measure $\xi\in\mathcal{M}$ with the subset of $\mathbb{R}^d$ $\{\xi_i, i\in I\}$ corresponding to its support and with the representants of this subset in $(\mathbb{R}^d)^I$, writing for example $\xi_\Lambda=\xi\cap\Lambda$ for the restriction of this configuration to $\Lambda\subset\mathbb{R}^d$, $\xi\eta$ for the concatenation of both configurations ξ and η; in particular, if y is a point in $\mathbb{R}^d$ belonging to the configuration η we write $\eta\diagdown y$ for the configuration η without the point y.
$\mathcal{M}\cap(\mathbb{R}^d)^n$ is the set of all n points configurations.
We introduce the following notations.

- For $\Lambda\subset\mathbb{R}^d$, N_Λ is the counting variable on $\mathcal{M}$: $N_\Lambda(\xi)=\mathrm{Card}\{i\in\mathbb{N},\xi_i\in\Lambda\}$.
- For $\Lambda\subset\mathbb{R}^d$, $\mathcal{B}_\Lambda$ is the σ-algebra on $\mathcal{M}$ generated by the sets $\{N_A=n\}$, $n\in\mathbb{N}$, $A\subset\Lambda$, A bounded.
- π (resp. π_Λ) is the Poisson process on $\mathbb{R}^d$ (resp. on Λ) with intensity measure the Lebesgue measure dy (resp. $dy|_\Lambda$).
- For $z>0$, π^z (resp. π^z_Λ) is the Poisson process on $\mathbb{R}^d$ (resp. on Λ) with activity z, that is with intensity measure $z\ dy$ (resp. $z\ dy|_\Lambda$).

The particles we deal with in this paper are not reduced to points but are hard balls or spheres of diameter r, for a fixed $r>0$. So the set of *allowed configurations* is the following subset of $\mathcal{M}$:

$$\mathcal{A}=\{\xi\in\mathcal{M} \text{ such that } \forall i\neq j\ |\xi_i-\xi_j|\geqslant r\}.$$

We will also use the set $\mathcal{A}_\Lambda=\{\xi\in\mathcal{A} \text{ such that } \forall i\ \xi_i\in\Lambda\}$ of allowed configurations with support in $\Lambda\subset\mathbb{R}^d$.

Remark 2.1 : (i) The number of hard spheres in a unite volume of $\mathbb{R}^d$ is bounded. In particular, if the evolution of the particles is defined by an interaction potential with finite range $R>r$, a fixed particle can interact with at most a finite number $\overline{N}$ of particles, where $\overline{N}$ only depends on d, R/r, and the density of the densest packing of equal spheres.
(ii) Furthermore, a fixed particle of any allowed configuration can touch at most a fixed number $\tau(d)$ of other particles, where $\tau(d)$ is the d-dimensional kissing number.

Proof (i) The sphere packing problem asks for the densest packing of balls of the same size into Euclidean d-space. It is trivial for $d=1$: the maximal density $\Delta(1)$ (that is the proportion of the space which is occupied by the spheres) is equal to one. The answer for $d=2$ has long been known : the standard hexagonal packing is optimal (cf. Figure 1) and $\Delta(2)=\pi/\sqrt{12}=0.9069...$ The famous case $d=3$ was only very recently solved by Hales [11], who proved the old Kepler conjecture : $\Delta(3)=\pi/\sqrt{18}=0.74048$ and the so-called face-centered cubic packing is optimal. See [2] for an extensive study of the state of the art in 1998 and [20] for a recent review of the new proofs.
For $d\geq 4$, the value of $\Delta(d)$ is not exactly known but there exist upper and lower bounds. The function $d\mapsto\Delta(d)$ is decreasing and the bounds which seem to be the best at this day are given by Rogers ([21] page 20) : $2^{-d}\leq\Delta(d)\leq 2^{-0,5990d}$. Thus, let Λ be a convex subset of $\mathbb{R}^d$ and $\xi\in\mathcal{A}$ such that Λ contains at least two points of ξ ($N_\Lambda(\xi)\geq 2$). Then $N_\Lambda(\xi)\leq\Delta(d)\frac{|B(\Lambda,r/2)|}{|B(0,r/2)|}$; in particular, for $\Lambda=B(0,R)$,

$$\overline{N}=\sup_{\xi\in\mathcal{A}}N_{B(0,R)}(\xi)-1\leq\Delta(d)\left(\frac{R+r/2}{r/2}\right)^d-1=\Delta(d)\left(1+\frac{2R}{r}\right)^d-1.$$

(ii) The kissing number $\tau(d)$ (also called Newton number or contact number) is defined as the number of hard spheres that can touch one sphere in dimension d. It is trivial that $\tau(1)=2$ and for $d=2$ $\tau(2)=6$ (see Figure 1). In three dimensions, the value of $\tau(3)$ was the subject of a famous discussion between Newton (who believed the answer was 12) and Gregory (who thought that 13 may be possible) in 1694. The correct answer is 12, and the first complete proof was given in 1953 [27]. Up to now, the exact value of $\tau(d)$ is only known for three dimensions above $d=3$: $\tau(4)=24, \tau(8)=240, \tau(24)=196560$ (see [19] for the most recent progress on this topics).

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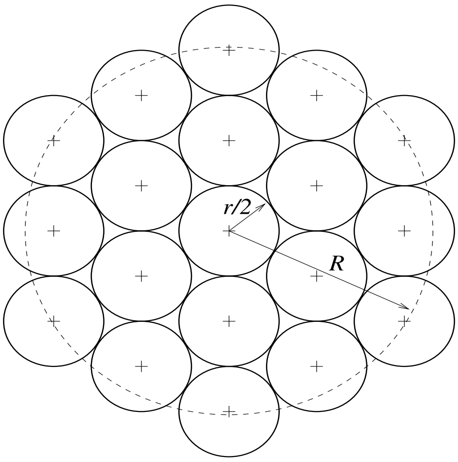

Figure 1: $\overline{N}=18$ if $d=2$ and $R=2r$.

2.2 Interaction potential, associated Gibbs and Canonical Gibbs measures

For a complete description in a general framework of the concepts introduced in this section, we refer the reader to [10].

We are dealing with hard balls with diameter r submitted to the action of a *pair potential*, which is a function on $\mathbb{R}^d$ of class $\mathcal{C}^2$ with finite range $R>r$, i.e. satisfying $\varphi(x)=0$ if $|x|\geqslant R$ and $\varphi(x)=\varphi(-x)$. Due to the hard core situation the values of $\varphi(x)$ may be chosen arbitrarily for $|x|<r$. In particular, one can assume without restriction that φ vanishes in a neighborhood of 0 and that $\nabla\varphi(0)=0$. Since φ has compact support, it is bounded from below : the smallest value of interaction between two particles is given by

$$\underline{\varphi}=\inf_{|x|\geqslant r}\varphi(x) \ \leqslant 0.$$

If this real constant is zero there exists only repulsion between the balls; if it is negative there exists an attraction domain around each ball.

The *energy* of a configuration $\xi\in\mathcal{M}$ submitted to the potential φ in the compact volume $\Lambda\subset\mathbb{R}^d$ with the boundary condition $\eta\in\mathcal{M}$ is given by :

$$E_\Lambda(\xi|\eta)=\begin{cases}\dfrac{1}{2}\displaystyle\sum_{\xi_i,\xi_j\in\Lambda}\varphi(\xi_i-\xi_j)+\sum_{\xi_i\in\Lambda,\eta_j\in\Lambda^c}\varphi(\xi_i-\eta_j) & \text{if } \xi_\Lambda\eta_{\Lambda^c}\in\mathcal{A}\\ +\infty & \text{otherwise.}\end{cases} \tag{1}$$

(the condition $\xi_\Lambda\eta_{\Lambda^c}\in\mathcal{A}$ corresponds to configurations for which $\xi_\Lambda\in\mathcal{A}$, $\eta_{\Lambda^c}\in\mathcal{A}$ and no ball of η_{Λ^c} is overlapping a ball of ξ_Λ). This finite-volume energy is well defined since both sums contain no more than $\frac{|B(\Lambda,r/2)|}{|B(0,r/2)|}\overline{N}$ terms, see Remark 2.1. Moreover, $e^{-E_\Lambda(\xi|\eta)}$ vanishes as soon as the

configuration $\xi_\Lambda \eta_{\Lambda^c}$ is not allowed.
By extension, we can define a *one-point energy* as follows : for $x \in \mathbb{R}^d$ and $\eta \in \mathcal{M}$,

$$E(x|\eta) = \begin{cases} \sum_{\eta_j} \varphi(x - \eta_j) & \text{if } x\eta \in \mathcal{A} \\ +\infty & \text{otherwise.} \end{cases} \tag{2}$$

(this function is finite if and only if η is an allowed configuration for which the configuration $x\eta$ with one extra ball centered in x is still allowed.)

We now define the set $\mathcal{G}(z)$ of Gibbs measures on $\mathcal{A}$ associated to the potential φ with activity parameter $z \in \mathbb{R}^+$. For each compact subset Λ of $\mathbb{R}^d$, let us define a local density function with respect to the Poisson Process π^z_Λ by :

$$f^z_\Lambda(\xi|\eta) = \frac{1}{Z^{\Lambda,\eta}_z} \exp(-E_\Lambda(\xi|\eta)) \tag{3}$$

where the so-called partition function $Z^{\Lambda,\eta}_z$ is the renormalizing constant :

$$Z^{\Lambda,\eta}_z = e^{-z|\Lambda|} \left(1 + \sum_{n=1}^{+\infty} \frac{z^n}{n!} \int_{\Lambda^n} \exp -E_\Lambda(y_1 \cdots y_n|\eta) \, dy_1 \cdots dy_n \right).$$

Due to the hard core, the above series reduces to a finite sum and $0 < Z^{\Lambda,\eta}_z < +\infty$.

Definition 2.2 *A Probability measure μ on $\mathcal{M}$ belongs to the set $\mathcal{G}(z)$ of Gibbs measures on hard balls with activity z and associated potential φ if and only if, for each compact subset $\Lambda \subset \mathbb{R}^d$,*

$$d\mu(\xi|\mathcal{B}_{\Lambda^c})(\eta) = f^z_\Lambda(\xi|\eta) \; d\pi^z_\Lambda(\xi) \quad \textit{for } \mu\textit{-a.e. } \eta.$$

Remark that any Gibbs measure in $\mathcal{G}(z)$ has its support included in $\mathcal{A}$. Dobrushin proved in [3], using compactness arguments, that there exists at least one element in $\mathcal{G}(z)$ when the potential contains a hard core component. Furthermore the set $\mathcal{G}(z)$ is convex and compact. About the cardinality of $\mathcal{G}(z)$, remarking that the sum of the hard core and the smooth potential φ is superstable and lower regular in the sense of Ruelle [23], we do the following remarks :
- If z is small enough, Ruelle proved that uniqueness holds (see [22] Theorem 4.2.3). In our case, a sufficient condition would be : $z \leq e^{\overline{N}\underline{\varphi}-1}(|B(0,r)| + \int \mathbb{1}_{r<|y|<R}|1 - e^{-\varphi(y)}|dy)^{-1}$.
- For z large enough it is conjectured (see [22] and [10]) - but still not proved - that phase transition occurs : Card $\mathcal{G}(z) > 1$.

See also [18] for a construction of a hard core Poisson Process with applications in percolation theory and [30] for the description of such a process as a Gibbs cluster process.

We now define the set $\mathcal{CG}$ of canonical Gibbs states on $\mathcal{A}$ associated to the potential φ.

Definition 2.3 *A Probability measure μ on $\mathcal{A}$ belongs to the set $\mathcal{CG}$ of canonical Gibbs states on $\mathcal{A}$ for the pair potential φ if and only if, for each compact subset $\Lambda \subset \mathbb{R}^d$ and $n \in \mathbb{N}$, for μ-a.e. η,*

$$d\mu(\xi|\mathcal{B}_{\Lambda^c}, N_\Lambda)(\eta, n) = \begin{cases} \frac{1}{Z^{\Lambda,\eta,n}} \mathbb{1}_{N_\Lambda(\xi)=n} \exp(-E_\Lambda(\xi|\eta)) \; d\pi_\Lambda(\xi) & \textit{if } Z^{\Lambda,\eta,n} > 0 \\ 0 & \textit{otherwise,} \end{cases}$$

where the partition function $Z^{\Lambda,\eta,n}$ for the particle number n is the finite renormalizing constant : $Z^{\Lambda,\eta,n} = \frac{e^{-|\Lambda|}}{n!} \int_{\Lambda^n} \exp -E_\Lambda(y_1 \cdots y_n|\eta) \, dy_1 \cdots dy_n$.

Since the potential φ is bounded from below by $\underline{\varphi}$, using Remark 2.1 we deduce that the map $y \mapsto E_\Lambda(y|\eta)$ is bounded from below on $\mathbb{R}^d$ by $\underline{\varphi}\overline{N}$. Thus Georgii's conditions (6.11) and (6.12) from [10] hold, which allows to apply Theorem 6.14 of [10] and to deduce that the set of canonical Gibbs states $\mathcal{CG}$ is obtained by mixing elements of different $\mathcal{G}(z)$, $z \in \mathbb{R}^+$: for any $\mu \in \mathcal{CG}$ there exists a probability measure θ on $\mathbb{R}^+$ such that

$$\mu = \int_{\mathbb{R}^+} \mu_z \, \theta(dz) \text{ with } \mu_z \in \mathcal{G}(z) \text{ for each } z \in \mathbb{R}^+. \tag{4}$$

2.3 The stochastic equation $(\mathcal{E})$ and statement of the main results

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a right continuous filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ such that each $\mathcal{F}_t$ contains all $P-$ negligible sets and let $(W_i(t), t\geqslant 0)_{i\in\mathbb{N}}$ be a family of $\mathcal{F}_t$-adapted independent d-dimensional Brownian motions.

Let us denote $\mathcal{C}(\mathbb{R}^+, \mathcal{M})$ (respectively $\mathcal{C}(\mathbb{R}^+, \mathcal{A})$) the set of continuous $\mathcal{M}$-valued (resp. $\mathcal{A}$-valued) paths on $\mathbb{R}^+$, endowed with the topology of uniform convergence on each compact time interval.

Let φ be the smooth pair potential with finite range R introduced in the previous subsection. We consider the following infinite gradient system of stochastic equations satisfied by the Brownian balls :

$$(\mathcal{E}) \begin{cases} \text{For } i \in \mathbb{N}, t \in \mathbb{R}^+, \\ X_i(t) = X_i(0) + W_i(t) - \dfrac{1}{2} \displaystyle\sum_{j\in\mathbb{N}} \int_0^t \nabla\varphi(X_i(s) - X_j(s)) ds + \sum_{j\in\mathbb{N}} \int_0^t (X_i(s) - X_j(s)) dL_{ij}(s) \end{cases}$$

where

- $(X_i(t), t\geqslant 0)_{i\in\mathbb{N}} \in \mathcal{C}(\mathbb{R}^+, \mathcal{A})$ satisfies $|X_i(t) - X_j(t)| \geqslant r$ for $t\geqslant 0$ and $i \neq j$;
- $(L_{ij}(t), t\geqslant 0)_{i,j\in\mathbb{N}}$ is a family of non-decreasing $\mathbb{R}^+$-valued continuous processes satisfying :

$$L_{ij}(0) = 0, \quad L_{ij} \equiv L_{ji} \quad \text{and} \quad L_{ij}(t) = \int_0^t 1\!\!1_{|X_i(s)-X_j(s)|=r} \, dL_{ij}(s), \quad L_{ii} \equiv 0.$$

A solution of the system $(\mathcal{E})$ with initial condition $x \in \mathcal{A}$ is a family $(X_i^x(t), L_{ij}^x(t), t\geqslant 0, i, j \in \mathbb{N})$ of processes such that equation $(\mathcal{E})$ is satisfied with $X(0) = x$.

Theorem 2.4 *The stochastic equation $(\mathcal{E})$ admits a solution with values in $\mathcal{A}$ for any deterministic initial configuration which belongs to the set $\underline{\underline{\mathcal{A}}} \subset \mathcal{A}$ defined by $\underline{\underline{\mathcal{A}}} = \{x \in \mathcal{A} \ : P(\Omega_0^x \cap \Omega_1^x) = 1\}$ (sets Ω_0^x and Ω_1^x are given in (8) and (9)). Moreover if the initial configuration is random with distribution $\mu \in \mathcal{G}(z)$ for some $z > 0$ and $\mu(\underline{\underline{\mathcal{A}}}) = 1$, then this solution is time-reversible, that is its law is invariant with respect to the time reversal.*

Remark 2.5 : The solution of equation $(\mathcal{E})$ is unique as element of a set of regular paths $\mathcal{C} \subset \mathcal{C}(\mathbb{R}^+, \mathcal{A})$. See proposition 5.4 of [6] for details.

The construction of a solution for $(\mathcal{E})$ when the initial condition is a fixed deterministic configuration is given in section 3, Proposition 3.1. The reversibility for an initial Gibbs measure is proven at the beginning of section 4, in Proposition 4.1.

We are now ready to state the main result of this paper.

Theorem 2.6 *Suppose that μ is a probability measure on $\mathcal{A}$ with $\mu(\underline{\underline{\mathcal{A}}}) = 1$. Furthermore, suppose that for every Λ compact subset of $\mathbb{R}^d$ and μ-almost all η, $\mu(.|\mathcal{B}_{\Lambda^c})(\eta)$ is absolutely continuous with respect to π_Λ and its density $u_\Lambda(.|\eta_{\Lambda^c})$ has the following differentiability property :*

$$\begin{array}{l}\forall \xi \in \mathcal{A}_\Lambda, \textit{ the map } x \mapsto u_\Lambda(x\xi|\eta_{\Lambda^c}) \textit{ is } \mathcal{C}^1 \textit{ on } \Lambda \smallsetminus B(\xi\eta_{\Lambda^c}, r) \textit{ and its derivative} \\ \nabla u_\Lambda(x\xi|\eta_{\Lambda^c}) \textit{ verifies } \displaystyle\int_{\mathcal{A}} \int_{\mathcal{A}_\Lambda} \sup_{x \in \Lambda \smallsetminus B(\xi\eta_{\Lambda^c}, r)} |\nabla u_\Lambda(x\xi|\eta_{\Lambda^c})| \; \pi_\Lambda(d\xi)\mu(d\eta) < +\infty \end{array} \tag{5}$$

If μ is an equilibrium measure for the gradient system $(\mathcal{E})$ in the sense that the Detailed Balance Equation (15) holds under μ, then μ is a canonical Gibbs measure in $\mathcal{CG}$.

Let us remark that measures which are locally absolutely continuous with respect to the Poisson Process, as in the above Theorem, only carry configurations without collisions.

Lemma 2.7 *Let μ be a Probability measure on $\mathcal{M}$. Suppose that for every Λ compact subset of $\mathbb{R}^d$ and μ-almost all η, $\mu(.|\mathcal{B}_{\Lambda^c})(\eta)$ is absolutely continuous with respect to π_Λ with density $u_\Lambda(.|\eta_{\Lambda^c})$. Then*

$$\mu(\{\gamma \in \mathcal{M} : \exists i, j, |\gamma_i - \gamma_j| = r\}) = 0.$$

Proof We first remark that

$$\{\gamma \in \mathcal{M} : \exists i, j, |\gamma_i - \gamma_j| = r\} = \bigcup_{n=1}^{+\infty} \{\gamma \in \mathcal{M} : \exists i, j, |\gamma_i - \gamma_j| = r \text{ and } |\gamma_i| \leqslant n, |\gamma_j| \leqslant n\};$$

so we just have to prove that for any compact set $\Lambda \subset \mathbb{R}^d$ we have $\mu(c(\Lambda)) = 0$ where $c(\Lambda) \subset \mathcal{M}$ is the set of all configurations which contain a collision in Λ : $c(\Lambda) = \{\gamma \in \mathcal{M} : \exists i, j, |\gamma_i - \gamma_j| = r \text{ with } \gamma_i \in \Lambda \text{ and } \gamma_j \in \Lambda\}$. By local absolute continuity of μ with respect to π we have :

$$\mu(c(\Lambda)) = \int_{\mathcal{M}} \int_{\mathcal{M}_\Lambda} 1\!\!1_{c(\Lambda)}(\gamma) \; u_\Lambda(\gamma|\eta_{\Lambda^c}) \; \pi_\Lambda(d\gamma)\mu(d\eta).$$

Thus we only have to prove that $\pi_\Lambda(c(\Lambda)) = 0$. This is straightforward since Lebesgue measure in $\mathbb{R}^d$ does not carry any finite union of hyperplanes :

$$\pi_\Lambda(c(\Lambda)) = e^{-|\Lambda|} \sum_{k=2}^{+\infty} \frac{1}{k!} \int_{\Lambda^k} 1\!\!1_{\{\exists i,j, |x_i - x_j| = r\}}(x_1, \cdots, x_k) \; dx_1 \cdots dx_k = 0$$

■

3 Construction of a strong solution

The solution of $(\mathcal{E})$ will be constructed as a limit of approximating processes $(X^l)_{l \in \mathbb{N}^*}$ by penalization. In [5] and [7] we did it in a reversible framework. Here we need an explicit contruction of the set of allowed initial configurations and a pathwise construction in a non-reversible framework. Since the proofs are very technical, we only present a squetch of the construction and refer the reader who wants more details to [6].
A visualization of the approximating processes moving in $\mathbb{R}^2$ may be found at : `math.univ-lille1.fr/~fradon`

3.1 Approximating processes

In this whole subsection, $l \in \mathbb{N}^*$ is fixed. To simplify we restrict the study of the paths on the time interval $[0, 1]$. It is obvious that all the results in the sequel hold true on any time interval $[0, T]$, $T \geqslant 1$, up to a change of constants.

We construct the approximating process X^l in order that it "essentially" stays in the bounded cube $\Lambda_l = [-l,l]^d$ (in a sense which will be clear soon). To obtain such a behavior, we introduce in the equation $(\mathcal{E})$ a supplementary gradient drift $\nabla\psi^{l,\eta}$ which vanishes in a subset of Λ_l and is repulsive outside of Λ_l.

More precisely, for any allowed configuration $\eta \in \mathcal{A}$ which support is disjoint to Λ_l, we fix a $\mathbb{R}^+$-valued function $\psi^{l,\eta}$ on $\mathbb{R}^d$ which is $\mathcal{C}^2$ with bounded derivatives and vanishes on each (and only on) $y \in \Lambda_l$ such that $y\eta$ is an allowed configuration, that is

$$\psi^{l,\eta}(y) = 0 \quad \Leftrightarrow \quad y \in \Lambda_l = [-l,l]^d \text{ and } y\eta \in \mathcal{A} \quad \Leftrightarrow \quad y \in \Lambda_l = [-l,l]^d \text{ and } d(y,\eta) \geqslant r$$

We extend the definition of $\psi^{l,\eta}$ to any configuration $\eta \in \mathcal{A}$ by : $\psi^{l,\eta} = \psi^{l,\eta_{\Lambda_l^c}}$. We also choose the family $(\psi^{l,\eta})_l$ such that, for every $\eta \in \mathcal{A}$,

$$\sum_{l\in\mathbb{N}^*} \int_{\mathbb{R}^d} \mathbb{1}_{\psi^{l,\eta}(y)>0} \, \exp(-\psi^{l,\eta}(y)) \, dy \quad \leqslant \quad 1. \tag{6}$$

For $\eta \in \mathcal{A}$ and $n \in \mathbb{N}^*$, let us now define the n-dimensional stochastic differential equation :

$$(\mathcal{E}_n^{l,\eta}) \begin{cases} \forall i \in \{1,\ldots,n\}, \text{ for } 0 \le t \le 1, \\ dX_i(t) = dW_i(t) - \dfrac{1}{2}\Big(\displaystyle\sum_{j=1,\ldots,n} \nabla\varphi(X_i(t) - X_j(t)) + \sum_{j:\eta_j\in\Lambda^c} \nabla\varphi(X_i(t) - \eta_j)\Big) dt \\ \qquad\qquad - \dfrac{1}{2}\nabla\psi^{l,\eta}(X_i(t))dt + \displaystyle\sum_{j=1,\ldots,n} (X_i(t) - X_j(t)) dL_{ij}(t) \end{cases}$$

with $L_{ij} \equiv L_{ji}$ for all i and j and $L_{ij}(t) = \int_0^t \mathbb{1}_{|X_i(s)-X_j(s)|=r} \, dL_{ij}(s)$. $(\mathcal{E}_n^{l,\eta})$ is a n-dimensional stochastic differential equation reflected in $\mathcal{A} \cap (\mathbb{R}^d)^n$ with gradient drift $-\frac{1}{2}\nabla\beta_n^{l,\eta}$ where

$$\beta_n^{l,\eta}(x_1,\ldots,x_n) = \sum_{i=1,\ldots,n} \Big(\psi^{l,\eta}(x_i) + \frac{1}{2} \sum_{\substack{j=1,\ldots,n \\ j\neq i}} \varphi(x_i - x_j) + \sum_{j:\eta_j\in\Lambda^c} \varphi(x_i - \eta_j) \Big). \tag{7}$$

Since the drift $-\frac{1}{2}\nabla\beta_n^{l,\eta}$ is bounded and Lipschitz continuous, $(\mathcal{E}_n^{l,\eta})$ admits a unique strong solution for each initial n-point configuration $x \in \mathcal{A} \cap (\mathbb{R}^d)^n$ (see thorem 5.1 of [24]). We denote this solution by $X^{l,\eta,n}(x,\cdot)$. For a general initial configuration $x \in \mathcal{A}$, we extend the above process as follows :

$$X^{l,x}(\cdot) = X^{l,\eta,n}(x_{\Lambda_l},\cdot) x_{\Lambda_l^c}$$

where $\eta = x_{\Lambda_l^c}$ and $n = \text{Card}(x \cap \Lambda_l)$. It is an $\mathcal{M}$-valued (not necessarily $\mathcal{A}$-valued) process with initial configuration x. Particles which are initially in Λ_l move like the $(\mathcal{E}_n^{l,\eta})$-dynamics and the other ones stay fixed outside Λ_l.

3.2 Convergence for a deterministic initial condition

In this section, we construct the limit of $(X^{l,x})_l$ when $l \to +\infty$ for convenient initial configurations x. This is possible except for certain so-called bad paths ω : they are paths such that at least a particle interacts with a great number of other ones, either because it moves very fast, or because it belongs to a large chain of interacting particles. So, for $m \in \mathbb{N}$, $a \ge 1$ and $\varepsilon > 0$, the set of "Bad trajectories" $\mathcal{B}(m,a,\varepsilon)$ is the union of two sets :

$$\mathcal{B}(m,a,\varepsilon) = \tilde{\mathcal{B}}(m,a,\varepsilon) \cup \tilde{\tilde{\mathcal{B}}}(m,\varepsilon).$$

The set $\tilde{\mathcal{B}}(m,a,\varepsilon)$ contains paths for which a particle i has a high modulus of continuity w defined as usual by $w(X_i,\dfrac{1}{m}) = \displaystyle\sup_{0\leqslant s,t\leqslant 1:|t-s|<\frac{1}{m}} |X_i(t) - X_i(s)|$:

$$\tilde{\mathcal{B}}(m,a,\varepsilon) = \Big\{ X \in \mathcal{C}([0,1],\mathcal{A}) : \; \exists i, \; w(X_i,\frac{1}{m}) > \frac{\varepsilon}{4} \text{ and } \exists t \leqslant 1, \; |X_i(t)| \leqslant a + 2m^2 \Big\}.$$

The set $\tilde{\tilde{\mathcal{B}}}(m,\varepsilon)$ contains paths for which at some time a large chain of particles interacts :

$$\tilde{\tilde{\mathcal{B}}}(m,\varepsilon) = \left\{ X \in \mathcal{C}([0,1],\mathcal{A}) : \begin{array}{c} \exists k \in \{0,\dots,m-1\}, \exists i_1,\cdots,i_n \in \mathbb{N}^*, \\ |X_{i_2}(\frac{k}{m}) - X_{i_1}(\frac{k}{m})| \le R+\varepsilon, \cdots, |X_{i_n}(\frac{k}{m}) - X_{i_{n-1}}(\frac{k}{m})| \le R+\varepsilon \\ \text{and } |X_{i_n}(\frac{k}{m}) - X_{i_1}(\frac{k}{m})| > m - R - \varepsilon \end{array} \right\}.$$

For $x \in \mathcal{A}$ let us define the set Ω_0^x as follows :

$$\Omega_0^x = \liminf_{\{\varepsilon:\frac{1}{\varepsilon}\in\mathbb{N}\}} \bigcap_{\rho\in\mathbb{N}^*} \liminf_{l\to+\infty} \left\{X^{l,x} \notin \mathcal{B}\left(m(\rho,l),\rho+m(\rho,l),\varepsilon\right)\right\} \cap \left\{X^{l+1,x} \notin \mathcal{B}\left(m(\rho,l),\rho+m(\rho,l),\varepsilon\right)\right\} \tag{8}$$

where $m(\rho,l) = [\sqrt{l-\rho-r}] - 1$.

We also define the set of paths :

$$\Omega_1^x = \bigcap_{\{\varepsilon:\frac{1}{\varepsilon}\in\mathbb{N}\}} \liminf_{\rho\to+\infty} \limsup_{l\to+\infty} \left\{X^{l,x} \notin \tilde{\mathcal{B}}\left(\rho,R,\varepsilon\right)\right\}. \tag{9}$$

Proposition 3.1 *For every $x \in \mathcal{A}$, for every ω in Ω_0^x and every $i \in \mathbb{N}^*$, the sequence $(X_i^{l,x}(\omega,t), L_{ij}^{l,x}(\omega,t), j \in \mathbb{N}, t \in [0,1])_{l\in\mathbb{N}^*}$ of elements of $\mathcal{C}([0,1],\mathbb{R}^d\times\mathbb{R}_+^{\mathbb{N}})$ converges to a limit denoted by $(X_i^{\infty,x}(\omega,t), L_{ij}^{\infty,x}(\omega,t), j \in \mathbb{N}, t \in [0,1])$.*
Moreover, if $\omega \in \Omega_0^x \cap \Omega_1^x$, $(X^{\infty,x}(\omega,.), L_{ij}^{\infty,x}(\omega,.))$ satisfies equation $(\mathcal{E})$ with $X^{\infty,x}(\omega,0) = x$.
Thus, for any $x \in \underline{\underline{\mathcal{A}}} = \{\xi \in \mathcal{A} : P(\Omega_0^\xi \cap \Omega_1^\xi) = 1\}$, the process $(X^{\infty,x}, L_{ij}^{\infty,x})$ is a solution of $(\mathcal{E})$ with initial condition x.

Proof See sections 4 and 5 of [6]. ■

Remark that for any $x \in \underline{\underline{\mathcal{A}}}$, the convergence of the sequence $(X^{l,x})_l$ towards $X^{\infty,x}$ takes place in $\mathcal{C}(\mathbb{R}^+,\mathcal{M})$.

4 Reversible measures

We first present the already known important fact that Gibbs measures are reversible.

4.1 Canonical Gibbs measures in $\mathcal{G}(z)$ are reversible for $(\mathcal{E})$

Proposition 4.1 *The stochastic equation $(\mathcal{E})$ admits a time-reversible solution with values in $\mathcal{A}$ for any initial Gibbs distribution $\mu \in \mathcal{G}(z)$ with $\mu(\underline{\underline{\mathcal{A}}}) = 1$. Thus any canonical Gibbs measure $\mu \in \mathcal{CG}$ with support included in $\underline{\underline{\mathcal{A}}}$ is reversible too.*

Proof

When the initial measure μ is Gibbsian, the solution of $(\mathcal{E})$ is approximated by reversible finite-dimensional processes solution of $(\mathcal{E}_n^{l,\eta})$. This implies its reversibility (see proposition 5.5 of [6] for a detailed proof).

When the initial measure is canonical Gibbs, it is reversible as a mixing of Gibbs measures, which are reversible. ■

In the next proposition, we claim the existence of Gibbs measures with support included in the space of allowed configurations $\underline{\underline{\mathcal{A}}}$.

Proposition 4.2 *Let z_c be the following value of the activity : $z_c = \dfrac{\exp(2\overline{N}\underline{\varphi})}{(R^d - r^d)|B(0,1)|}$. Any Gibbs measure $\mu \in \mathcal{G}(z)$ with $0 < z < z_c$ has its support included in $\underline{\underline{\mathcal{A}}}$.*

Proof See [6] proposition 5.1. ■

Remark 4.3 : The critical value z_c given here appears for technical reasons in a percolation type estimate. For $z \geq z_c$ Gibbs measures of $\mathcal{G}(z)$ are also reversible (see [7] Proposition 3.1) but we are not able to give an explicit simple description like $\underline{\underline{\mathcal{A}}}$ of their supports .

4.2 The infinitesimal generator A and the spaces of test functions

Let us first introduce some definitions of differentiability for functions defined on the space of configurations $\mathcal{M}$.

Definition 4.4 *A function g on $\mathcal{M}$ is* local *if there exists a compact set $K \subset \mathbb{R}^d$ such that $g(\gamma)$ only depends on $\gamma \cap K$, i.e. $\forall \gamma \in \mathcal{M} \quad g(\gamma) = g(\gamma_K)$. Such a function is called K-local.*
A local function g on $\mathcal{M}$ is called $\mathcal{C}^k$ if for any $n \in \mathbb{N}^*$ the function defined on $(\mathbb{R}^d)^n$ by $(\gamma_1, \cdots, \gamma_n) \longmapsto g(\sum_{i=1}^{n} \delta_{\gamma_i})$ is $\mathcal{C}^k$. For any $\gamma \in \mathcal{M}$, $D_x g(x\gamma)$ and $D^2_{xx} g(x\gamma)$ denote the first and second derivatives of $y \mapsto g(y\gamma)$ at $y = x$.

Remark that any local $\mathcal{C}^0$ function is bounded and that any local $\mathcal{C}^1$ function has a bounded derivative : $\sup_{x \in \mathbb{R}^d} \sup_{\gamma \in \mathcal{M}} |D_x g(x\gamma)| < +\infty$.

Definition 4.5 *$\mathcal{T}$ denotes the set of all local $\mathcal{C}^2$ functions on $\mathcal{M}$.*

Since we study a dynamics with reflection on the boundary of the set of allowed configurations, it is natural to use the following set of test functions.

Definition 4.6 *Let $\mathcal{T}_0 \subset \mathcal{T}$ denote the set of functions on $\mathcal{M}$ whose first derivative is orthogonal to the normal vector on the boundary of the set $\mathcal{A}$ of allowed configurations, that is :*

$$\mathcal{T}_0 = \left\{ \begin{array}{l} f : \mathcal{M} \longrightarrow \mathbb{R} \text{ s.t. } f \text{ is local, } \mathcal{C}^2 \text{ and} \\ \text{for each } \gamma \in \mathcal{M}, \text{ if } \gamma_i, \gamma_j \in \gamma \text{ satisfy } |\gamma_i - \gamma_j| = r \text{ then } D_{\gamma_i} f(\gamma).(\gamma_i - \gamma_j) = 0 \end{array} \right\} \tag{10}$$

Let us consider a fixed test function $f \in \mathcal{T}$ and the strong solution X^∞ of equation $(\mathcal{E})$ constructed in section 3. The Itô Formula holds for $X = X^\infty$ (see e.g. [17], Theorem 27.2) :

$$\left\{ \begin{array}{l} \text{For } t \in \mathbb{R}^+, \\ f(X(t)) = f(X(0)) + \displaystyle\sum_{i \in \mathbb{N}} \int_0^t D_{X_i(s)} f(X(s)) \, dW_i(s) \\ \qquad\qquad\qquad - \displaystyle\frac{1}{2} \sum_{i \in \mathbb{N}} \int_0^t D_{X_i(s)} f(X(s)). \sum_{j \in \mathbb{N}} \nabla\varphi(X_i(s) - X_j(s)) ds \\ \qquad\qquad\qquad + \displaystyle\sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \int_0^t D_{X_i(s)} f(X(s)).(X_i(s) - X_j(s)) dL_{ij}(s) \\ \qquad\qquad\qquad + \displaystyle\frac{1}{2} \sum_{i \in \mathbb{N}} \int_0^t \mathrm{Tr}(D^2_{X_i(s) X_i(s)} f(X(s))) \, ds \end{array} \right.$$

If $f \in \mathcal{T}_0$, the reflection term vanishes :

$$\sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \int_0^t D_{X_i(s)} f(X(s)).(X_i(s) - X_j(s)) dL_{ij}(s) = 0.$$

Since f is local and f's first derivative is bounded, $\sum_{i \in \mathbb{N}} \int_0^t |D_{X_i(s)} f(X(s))|^2 \, ds$ is bounded independently of the initial condition $X(0)$ and thus, $\sum_{i \in \mathbb{N}} \int_0^t D_{X_i(s)} f(X(s)) \, dW_i(s)$ is a square-integrable martingale. Consequently, for each function $f \in \mathcal{T}_0$,

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s)) \, ds \quad \text{is a square-integrable martingale,} \tag{11}$$

where A, called the infinitesimal generator associated to the stochastic differential equation $(\mathcal{E})$, is given by

$$\begin{aligned} Af(\gamma) &= \frac{1}{2}\sum_{i\in\mathbb{N}}\Big(\mathrm{Tr}(D^2_{\gamma_i\gamma_i}f(\gamma)) - D_{\gamma_i}f(\gamma).\sum_{j\in\mathbb{N}}\nabla\varphi(\gamma_i-\gamma_j)\Big) \qquad (12)\\ &= \frac{1}{2}\int_{\mathbb{R}^d}\big(\mathrm{Tr}(D^2_{xx}f(\gamma)) - D_xf(\gamma).(\nabla\varphi*\gamma)(x)\big)\,\gamma(dx) \end{aligned}$$

with $(\nabla\varphi*\gamma)(x) = \int_{\mathbb{R}^d}\nabla\varphi(x-y)\,\gamma(dy)$.

Remark 4.7 : For any $g\in\mathcal{T}$ the function Ag is still local. More precisely, if g is Λ-local then Ag is $B(\Lambda,R)$-local :

$$\begin{aligned} Ag(\eta) &= -\frac{1}{2}\int\int g(\eta).\nabla\varphi(x-y)\eta(dy)\,\eta(dx) + \frac{1}{2}\int\mathrm{Tr}D^2_{xx}g(\eta)\,\eta(dx)\\ &= -\frac{1}{2}\int_\Lambda\int_{B(\Lambda,R)} D_xg(\eta_\Lambda).\nabla\varphi(x-y)\eta(dy)\,\eta(dx) + \frac{1}{2}\int_\Lambda\mathrm{Tr}D^2_{xx}g(\eta_\Lambda)\,\eta(dx)\\ &= \frac{1}{2}\int_\Lambda\Big(-D_xg(\eta_\Lambda).\nabla\varphi*\eta_{B(\Lambda,R)}(x) + \mathrm{Tr}D^2_{xx}g(\eta_\Lambda)\Big)\,\eta(dx)\\ &= Ag(\eta_{B(\Lambda,R)}). \end{aligned}$$

Let us now verify the foundamental symmetry property of the infinitesimal generator A under any measure μ which is reversible for the gradient-system $(\mathcal{E})$.

Proposition 4.8 *Let μ be a Probability measure on $\mathcal{A}$. If the solution of the gradient-system $(\mathcal{E})$ with μ as initial distribution is time-reversible, then the infinitesimal generator A is symmetrical on $\mathcal{T}_0$:*

$$\forall f,g\in\mathcal{T}_0 \qquad \int_{\mathcal{M}} f\ Ag\ d\mu = \int_{\mathcal{M}} g\ Af\ d\mu. \qquad (13)$$

Proof The time-reversibility of the process X solution of $(\mathcal{E})$ implies that, for any time $t>0$ and any $f,g\in\mathcal{T}_0$,

$$E\Big(g(X_0)f(X_t) - g(X_t)f(X_0)\Big) = 0.$$

But, applying the Itô Formula and the martingale property (11) one gets

$$\begin{aligned} E\Big(g(X_0)f(X_t) - g(X_t)f(X_0)\Big) &= E\Big(g(X_0)\int_0^t Af(X_s)ds - f(X_0)\int_0^t Ag(X_s)ds\Big)\\ &= \int_0^t E\big(g(X_0)Af(X_s) - f(X_0)Ag(X_s)\big)ds\\ &= 0. \end{aligned}$$

Since the paths $t\mapsto X_t$ are continuous at time 0 and Af and Ag are bounded $\mathcal{C}^0$ functions when f and g belong to $\mathcal{T}_0$, one obtains

$$\begin{aligned} \lim_{t\to0}\frac{1}{t}\int_0^t E\big(g(X_0)Af(X_s) - f(X_0)Ag(X_s)\big)ds &= E\big(g(X_0)Af(X_0) - f(X_0)Ag(X_0)\big)\\ &= \int_{\mathcal{M}} g\ Af\ d\mu - \int_{\mathcal{M}} f\ Ag\ d\mu\\ &= 0. \end{aligned}$$

■

In a finite-dimensional context, the above Symmetry Property under μ would be strong enough to characterize μ as a Gibbs measure. However, the configuration space we are dealing with is infinite-dimensional, and the space of test functions $\mathcal{T}_0$ is too small to generate all functions on which A is symmetrical. Thus, we introduce in the next subsections a local version of (13), which will be satisfied for the whole set of test functions $\mathcal{T}$.

4.3 The security functions

To localize (13), we define a family of security functions used as "collision detectors" : they vanish for configurations containing, in a bounded region, balls which are too close.

Definition 4.9 *For any fixed compact set* $K \subset \mathbb{R}^d$ *and for* $\varepsilon > 0$, *we define the function* S_K^ε *on* $\mathcal{M}$ *by*

$$S_K^\varepsilon(\gamma) = \tilde{\mathbb{1}}_{]-\infty,0]}\left(\sum_{i\in\mathbb{N}} \tilde{\mathbb{1}}_K(\gamma_i)\ \Big(1 - \prod_{j\in\mathbb{N}} \tilde{\mathbb{1}}_{[2,+\infty[}(\frac{|\gamma_i-\gamma_j|^2-r^2}{\varepsilon^2})\Big)\right) \tag{14}$$

where $\tilde{\mathbb{1}}_{]-\infty,0]}$ *is a* $\mathcal{C}^\infty$ *non-increasing function with value* 1 *on* $]-\infty,0]$ *and* 0 *on* $[1,+\infty[$, *and where* $\tilde{\mathbb{1}}_K$ *is a* $\mathcal{C}^\infty$ *function from* $\mathbb{R}^d$ *to* $[0,1]$ *with value* 1 *on* K *and value* 0 *on the set* $\mathbb{R}^d \setminus B(K,1)$. *Here* $\tilde{\mathbb{1}}_{[2,+\infty[}$ *denotes some fixed* $\mathcal{C}^\infty$ *non-decreasing fonction vanishing on* $]-\infty,1]$ *with value* 1 *on* $[2,+\infty[$.

Remark that the functions S_K^ε are elements of $\mathcal{T}$. Indeed, they are $\overline{K}$-local with $\overline{K} := B(K, 1+\sqrt{r^2+2\varepsilon^2})$ and they are $\mathcal{C}^\infty$ on $\mathcal{M}$. However, S_K^ε does not belong to $\mathcal{T}_0$, since its derivative $D_{\gamma_i} S_K^\varepsilon(\gamma)$ is not orthogonal to $\gamma_i - \gamma_j$ for $\gamma_i \in B(K,1)\setminus K$ and $|\gamma_i - \gamma_j| = r$. Let us now describe some characteristic properties of these functions.

Lemma 4.10 *The function* S_K^ε *vanishes on the following set of configurations :*

$$\{\gamma : \exists \gamma_i \in K, \exists \gamma_j \neq \gamma_i \text{ with } |\gamma_i-\gamma_j|^2 \le r^2+\varepsilon^2\},$$

and is equal to 1 on the set of configurations :

$$\{\gamma : \forall \gamma_i \in B(K,1), \forall \gamma_j \neq \gamma_i, |\gamma_i-\gamma_j|^2 \ge r^2+2\varepsilon^2\}.$$

Moreover, the function S_K^ε *increases as* ε *decreases and*

$$\lim_{\varepsilon \searrow 0} S_K^\varepsilon(\gamma) = 1 \quad \mu\text{-a.s.}$$

for any measure μ *such that* $\mu(\{\gamma : \exists i,j, |\gamma_i-\gamma_j| = r\}) = 0$.

Proof If the center of a particle $\gamma_i \in \gamma$ is in K (hence $\tilde{\mathbb{1}}_K(\gamma_i) = 1$) and at least one other particle of γ is at a distance smaller than $\sqrt{r^2+\varepsilon^2}$ from γ_i (hence $\prod_{j\in\mathbb{N}} \tilde{\mathbb{1}}_{[2,+\infty[}(\frac{|\gamma_i-\gamma_j|^2-r^2}{\varepsilon^2}) = 0$), then in the definition (14) the sum is greater than 1 and $S_K^\varepsilon(\gamma) = 0$.
On the other side, suppose a particle γ_i in γ is at a distance greater than $\sqrt{r^2+2\varepsilon^2}$ from every other particle; then one has $\prod_{j\in\mathbb{N}} \tilde{\mathbb{1}}_{[2,+\infty[}(\frac{|\gamma_i-\gamma_j|^2-r^2}{\varepsilon^2}) = 1$. If this holds for any $\gamma_i \in B(K,1)$, that is for each γ_i such that $\tilde{\mathbb{1}}_K(\gamma_i) \neq 0$, then the sum in (14) vanishes and $S_K^\varepsilon(\gamma) = 1$.

Moreover, for each $\gamma \in \mathcal{A}$, the pseudo-indicator function $\tilde{\mathbb{1}}_{[2,+\infty[}(\frac{|\gamma_i-\gamma_j|^2-r^2}{\varepsilon^2})$ increases as ε decreases to 0 to the indicator function $\mathbb{1}_{|\gamma_i-\gamma_j|\neq r}$, thus

$$S_K^\varepsilon(\gamma) \nearrow \tilde{\mathbb{1}}_{]-\infty,0]}\left(\sum_{i\in\mathbb{N}} \tilde{\mathbb{1}}_K(\gamma_i)\ \mathbb{1}_{\{\exists j, |\gamma_i-\gamma_j|=r\}}\right)$$

For measures μ such that $\mu(\{\gamma : \exists i,j, |\gamma_i-\gamma_j| = r\}) = 0$, this function is μ-a.s. equal to 1 and consequently

$$\lim_{\varepsilon \searrow 0} \uparrow S_K^\varepsilon(\gamma) = 1 \quad \mu\text{-a.s.}$$

■

4.4 The Detailed Balance Equation

For a Λ-local function f defined on $\mathcal{M}$, we say that the compact set $K \subset \mathbb{R}^d$ covers f if $B(\Lambda, R) \subset K$ and denote by f^ε_K the product $f S^\varepsilon_K$. Such functions play from now on the role of test functions.

Definition 4.11 *A Probability measure μ on $\mathcal{A}$ is called an* equilibrium measure *for equation $(\mathcal{E})$ if the infinitesimal generator A is locally μ-symmetric on $\mathcal{T}$ in the following sense :*

$$\forall f, g \in \mathcal{T},\ \forall K \text{ compact set covering } f \text{ and } g,\ \forall \varepsilon > 0 \quad \int_{\mathcal{M}} f^\varepsilon_K\ A g^\varepsilon_K\ d\mu = \int_{\mathcal{M}} g^\varepsilon_K\ A f^\varepsilon_K\ d\mu. \quad (15)$$

Equation (15) is called Detailed Balance Equation.

Notice the important fact that (15) is not equivalent to (13), since $S^\varepsilon_K \notin \mathcal{T}_0$. Anyway, (15) is a reasonable equilibrium condition in the sense that Gibbs measures satisfy it, as the next proposition claims.

Proposition 4.12 *Any canonical Gibbs measure $\mu \in \mathcal{CG}$ with support included in $\underline{\underline{\mathcal{A}}}$ satisfies Detailed Balance Equation (15).*

Proof Let us first recall the property (4) : any canonical Gibbs measure μ has the representation

$$\mu = \int_{\mathbb{R}^+} \mu_z\, \theta(dz) \text{ with } \mu_z \in \mathcal{G}(z) \text{ for each } z \in \mathbb{R}^+,$$

that is, is a mixture of Gibbs measures. As a consequence for $f, g \in \mathcal{T}_0$ and K a compact set covering f and g,

$$\int_{\mathcal{M}} f^\varepsilon_K\ A g^\varepsilon_K\ d\mu = \int_{\mathbb{R}^+} \int_{\mathcal{M}} f^\varepsilon_K\ A g^\varepsilon_K\ d\mu_z\, \theta(dz)$$

and the Detailed Balance Equation will hold for each canonical Gibbs measure as soon as it holds for each $\mu_z \in \mathcal{G}(z)$.

We now fix $\mu \in \mathcal{G}(z)$, $f, g \in \mathcal{T}$, K covering f and g and $\varepsilon > 0$. We want to prove that $\int_{\mathcal{M}} f^\varepsilon_K\ A g^\varepsilon_K\ d\mu$ is symmetric in f and g. By definition (12) of A

$$\begin{aligned}
&\int_{\mathcal{M}} f S^\varepsilon_K\ A(g S^\varepsilon_K)\ d\mu \\
&= \int_{\mathcal{M}} f (S^\varepsilon_K)^2 A g\ d\mu + \int_{\mathcal{M}} f g\ S^\varepsilon_K\ A S^\varepsilon_K\ d\mu + \int_{\mathcal{M}} f(\eta) S^\varepsilon_K(\eta) \int D_x g(\eta) . D_x S^\varepsilon_K(\eta)\, \eta(dx)\ \mu(d\eta) \\
&=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}$$

The second integral $\mathcal{I}_2$ is symmetric in f and g. We now use the assumption $\mu \in \mathcal{G}(z)$ to transform the first integral term $\mathcal{I}_1$. Recall that Ag is K-local, since K covers g.

$$\begin{aligned}
\mathcal{I}_1 &= \int_{\mathcal{A}} f(\eta_K) (S^\varepsilon_K)^2(\eta) A g(\eta_K)\, \mu(d\eta) \\
&= \int_{\mathcal{A}} \int_{\mathcal{A}_K} f(\xi) (S^\varepsilon_K)^2(\xi \eta_{K^c}) A g(\xi) e^{-E_K(\xi | \eta_{K^c})} \pi^z_K(d\xi) \mu(d\eta) \\
&= \frac{1}{2} \int_{\mathcal{A}} \int_{\mathcal{A}_K} \int_K f(\xi) (S^\varepsilon_K)^2(\xi \eta_{K^c})\ \mathrm{Tr} D^2_{xx} g(\xi)\ e^{-E_K(\xi | \eta_{K^c})}\ \xi(dx)\ \pi^z_K(d\xi)\ \mu(d\eta) \\
&\quad - \frac{1}{2} \int_{\mathcal{A}} \int_{\mathcal{A}_K} \int_K f(\xi) (S^\varepsilon_K)^2(\xi \eta_{K^c})\ D_x g(\xi) . (\nabla \varphi * \xi)(x)\ e^{-E_K(\xi | \eta_{K^c})}\ \xi(dx)\ \pi^z_K(d\xi)\ \mu(d\eta) \\
&= \mathcal{J}_1 + \mathcal{J}_2
\end{aligned}$$

To transform $\mathcal{J}_1$ we use the well known fact that the Campbell measure of the Poisson Process π^z_K is equal to the product measure $z\ dx|_K \times \pi^z_K$ (see [16]), that is, for any regular function F on $K \times \mathcal{M}$,

$$\int_{\mathcal{M}} \int_K F(x, \xi) \mathbb{1}_{\mathcal{A}}(\xi)\, \xi(dx)\, \pi^z_K(d\xi) = z \int_{\mathcal{M}} \int_K F(x, x\xi) \mathbb{1}_{\mathcal{A}}(x\xi)\, dx\, \pi^z_K(d\xi).$$

So $\mathcal{J}_1$ becomes :

$$\begin{aligned}
\mathcal{J}_1 &= \frac{1}{2}\int_{\mathcal{A}}\int_{\mathcal{A}_K}\int_K f(x\xi)(S^\varepsilon_K)^2(x\xi\eta_{K^c})\ \mathrm{Tr}D^2_{xx}g(x\xi)\ e^{-E_K(x\xi|\eta_{K^c})}\ \mathbb{1}_{\mathcal{A}}(x\xi)\ z\ dx\ \pi^z_K(d\xi)\ \mu(d\eta)\\
&= \frac{z}{2}\int_{\mathcal{A}}\int_{\mathcal{A}_K}\int_{K\smallsetminus B(\xi,r)} f(x\xi)(S^\varepsilon_K)^2(x\xi\eta_{K^c})\ \mathrm{Tr}D^2_{xx}g(x\xi)\ e^{-E_K(x\xi|\eta_{K^c})}\ dx\ \pi^z_K(d\xi)\ \mu(d\eta)
\end{aligned}$$

Remark that $x \mapsto S^\varepsilon_K(x\xi\eta_{K^c})$ vanishes on the boundary of $B(\xi,r)$ and that $x \mapsto D_x g(x\xi)$ vanishes on the boundary of K; after integrating by parts, $\mathcal{J}_1$ becomes :

$$\mathcal{J}_1 = -\frac{z}{2}\int_{\mathcal{A}}\int_{\mathcal{A}_K}\int_{K\smallsetminus B(\xi,r)} D_x\Big(f(x\xi)(S^\varepsilon_K)^2(x\xi\eta_{K^c})\ e^{-E_K(x\xi|\eta_{K^c})}\Big).D_x g(x\xi)\ dx\ \pi^z_K(d\xi)\ \mu(d\eta).$$

We now expand the derivative. Remark that $D_x E_K(x\xi|\eta_{K^c})$ is equal to $(\nabla\varphi*\xi)(x)+(\nabla\varphi*\eta_{K^c})(x)$. Moreover $(\nabla\varphi*\eta_{K^c})(x)=0$ for $x\notin B(K^c,R)$ and $D_x g(\xi)=0$ for $x\in B(K^c,R)$, so that $\mathcal{J}_1$ is equal to :

$$\begin{aligned}
\mathcal{J}_1 &= -\frac{z}{2}\int_{\mathcal{A}}\int_{\mathcal{A}_K}\int_K \mathbb{1}_{\mathcal{A}}(x\xi)\ (S^\varepsilon_K)^2(x\xi\eta_{K^c})\ D_x f(x\xi).D_x g(x\xi)\ e^{-E_K(x\xi|\eta_{K^c})}\ dx\ \pi^z_K(d\xi)\ \mu(d\eta)\\
&\quad -\frac{z}{2}\int_{\mathcal{A}}\int_{\mathcal{A}_K}\int_K \mathbb{1}_{\mathcal{A}}(x\xi)\ f(x\xi)\ D_x(S^\varepsilon_K)^2(x\xi\eta_{K^c}).D_x g(x\xi)\ e^{-E_K(x\xi|\eta_{K^c})}\ dx\ \pi^z_K(d\xi)\ \mu(d\eta)\\
&\quad +\frac{z}{2}\int_{\mathcal{A}}\int_{\mathcal{A}_K}\int_K \mathbb{1}_{\mathcal{A}}(x\xi)\ f(x\xi)\ (S^\varepsilon_K)^2(x\xi\eta_{K^c})\ (\nabla\varphi*\xi)(x).D_x g(x\xi)\ e^{-E_K(x\xi|\eta_{K^c})}\ dx\ \pi^z_K(d\xi)\ \mu(d\eta)
\end{aligned}$$

Using again the Campbell measure of π^z_K, we remark that the second integral term is exactly the opposite of $\mathcal{I}_3$ and that the last integral term is also the opposite of $\mathcal{J}_2$. So we finally obtain :

$$\begin{aligned}
&\int_{\mathcal{M}} f S^\varepsilon_K\ A(gS^\varepsilon_K)\ d\mu\\
&= -\frac{1}{2}\int_{\mathcal{A}}\int_{\mathcal{A}_K}\int_K (S^\varepsilon_K)^2(\xi\eta_{K^c})\ D_x f(\xi).D_x g(\xi)\ e^{-E_K(\xi|\eta_{K^c})}\ \xi(dx)\ \pi^z_K(d\xi)\ \mu(d\eta) + \int_{\mathcal{M}} fg\ S^\varepsilon_K\ AS^\varepsilon_K\ d\mu\\
&= -\frac{1}{2}\int_{\mathcal{M}}\int_{\mathbb{R}^d} (S^\varepsilon_K)^2(\eta)\ D_x f(\eta).D_x g(\eta)\ \eta(dx)\ \mu(d\eta) + \int_{\mathcal{M}} fg\ S^\varepsilon_K\ AS^\varepsilon_K\ d\mu
\end{aligned}$$

This shows the desired symmetry in (f,g). ■

5 Detailed Balance Equation, Campbell measures and canonical Gibbs measures

In this section we first prove that any measure satisfying the Detailed Balance Equation (15) also obeys an integral equation exhibiting a symmetry property of the associated Campbell measure. In the second subsection we conclude the proof of the main theorem 2.6 by proving that such a measure is necessarily canonical Gibbs associated to the appropriate potential.

5.1 From the Detailed Balance Equation to Campbell measures

Proposition 5.1 *Let μ be a Probability measure on $\mathcal{A}$ with support included in $\underline{\underline{\mathcal{A}}}$. Suppose that for every Δ compact subset of $\mathbb{R}^d$ and μ-almost all η, $\mu(.|\mathcal{B}_{\Delta^c})(\eta)$ has a density $u_\Delta(.|\eta_{\Delta^c})$ with respect to π_Δ which satisfies assumption (5). If the detailed balance equation (15) is satisfied under μ, then for any positive measurable local function F on $\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{A}$, the following symmetry property holds :*

$$\begin{aligned}
&\int_{\mathcal{A}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-E(y|\eta\smallsetminus x)}\,F(x,y,\eta\smallsetminus x)\,dy\,\eta(dx)\,\mu(d\eta)\\
&\qquad = \int_{\mathcal{A}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-E(y|\eta\smallsetminus x)}\,F(y,x,\eta\smallsetminus x)\,dy\,\eta(dx)\,\mu(d\eta). \qquad (16)
\end{aligned}$$

Proof Step 1 : Reduction of (15) to a simpler symmetry property

By definition of $\mathcal{T}$, we can take as elements $f, g \in \mathcal{T}$ the following functions: $f = \tilde{f} f_3$ and $g = \tilde{g} f_3$. Functions $f_3, \tilde{f}, \tilde{g}$ will be precisely described in Step 3. Due to the definition of the infinitesimal generator A, we have :

$$A(gS_K^\varepsilon)(\eta) \quad = \quad (f_3 S_K^\varepsilon)(\eta) A\tilde{g}(\eta) + \tilde{g}(\eta) A(f_3 S_K^\varepsilon)(\eta) + \int D_x \tilde{g}(\eta).D_x(f_3 S_K^\varepsilon)(\eta)\, \eta(dx). \tag{17}$$

So, the left hand side of (15), which is symmetrical in the functions f, g (and thus in the functions $\tilde{f}, \tilde{g}$) is the sum of the three following terms I_1, I_2 and I_3:

$$\begin{aligned}
I_1 &:= \int \tilde{f}(\eta)(f_3 S_K^\varepsilon)^2(\eta) A\tilde{g}(\eta)\, \mu(d\eta) \\
I_2 &:= \int \tilde{g}(\eta)\tilde{f}(\eta)(f_3 S_K^\varepsilon)(\eta) A(f_3 S_K^\varepsilon)(\eta)\, \mu(d\eta) \\
I_3 &:= \int \tilde{f}(\eta)(f_3 S_K^\varepsilon)(\eta) \int D_x(f_3 S_K^\varepsilon)(\eta).D_x\tilde{g}(\eta)\, \eta(dx)\, \mu(d\eta).
\end{aligned}$$

The integral I_2 being symmetric in $\tilde{f}$ and $\tilde{g}$, this implies that the sum $I_1 + I_3$ remains unchanged if $\tilde{f}$ and $\tilde{g}$ are interchanged.

Step 2 : Analysis of $I_1 + I_3$

We consider functions $\tilde{f}$ and $\tilde{g}$ which are Λ-local for some bounded subset Λ with $B(\Lambda, R) \subset \Delta \subset K$ and remark that $D_x\tilde{g}$ and $D^2_{xx}\tilde{g}$ are Λ-local too and $A\tilde{g}$ is Δ-local (since $B(\Lambda, R) \subset \Delta$). We also choose f_3 C-local with $C \cap \Delta = \emptyset$ and $B(C, R) \subset K$. Thus, decomposing η inside and outside of Δ, one has

$$\begin{aligned}
I_1 &= \int_{\mathcal{A}} \tilde{f}(\eta_\Delta) f_3^2(\eta_{\Delta^c})(S_K^\varepsilon)^2(\eta_\Delta\eta_{\Delta^c}) A\tilde{g}(\eta_\Delta)\, \mu(d\eta) \\
&= \int_{\mathcal{A}} f_3^2(\eta_{\Delta^c}) \int_{\mathcal{A}_\Delta} \tilde{f}(\xi)(S_K^\varepsilon)^2(\xi\eta_{\Delta^c}) A\tilde{g}(\xi) u_\Delta(\xi|\eta_{\Delta^c})\pi_\Delta(d\xi)\mu(d\eta) \\
&=: \int_{\mathcal{A}} f_3^2(\eta_{\Delta^c}) J(\eta)\mu(d\eta).
\end{aligned}$$

To transform the integral term $J(\eta)$ we now use the Campbell measure of the Poisson Process π_Δ. Then, since $\nabla\varphi * (x\xi)(x) = \nabla\varphi * \xi(x) + \nabla\varphi(0) = \nabla\varphi * \xi(x)$,

$$\begin{aligned}
J(\eta) &= \frac{1}{2}\int_{\mathcal{A}_\Delta}\int_\Delta \tilde{f}(\xi)(S_K^\varepsilon)^2(\xi\eta_{\Delta^c}) u_\Delta(\xi|\eta_{\Delta^c})\Big(-D_x\tilde{g}(\xi_\Lambda).\nabla\varphi * \xi(x) + \mathrm{Tr}D^2_{xx}\tilde{g}(\xi_\Lambda)\Big)\, \xi(dx)\pi_\Delta(d\xi) \\
&= \frac{1}{2}\int_{\mathcal{M}}\int_\Delta \tilde{f}(x\xi)(S_K^\varepsilon)^2(x\xi\eta_{\Delta^c}) u_\Delta(x\xi|\eta_{\Delta^c})\Big(-D_x\tilde{g}(x\xi_\Lambda).\nabla\varphi * \xi(x) + \mathrm{Tr}D^2_{xx}\tilde{g}(x\xi_\Lambda)\Big)\mathbb{1}_{\mathcal{A}}(x\xi)\, dx\, \pi_\Delta(d\xi).
\end{aligned}$$

We recognize under the Lebesgue integral a divergence term :

$$\forall x \in \Delta, \quad \big(-D_x\tilde{g}(x\xi).\nabla\varphi * \xi(x) + \mathrm{Tr}D^2_{xx}\tilde{g}(x\xi)\big)\mathbb{1}_{\mathcal{A}}(x\xi) = \mathbb{1}_{\Delta\setminus B(\xi,r)}(x)\, e^{\varphi*\xi(x)}\nabla.(D_x\tilde{g}(x\xi)e^{-\varphi*\xi(x)})$$

and then, thanks to the regularity assumptions (5) on the function u_Δ, we get by partial integration

$$J(\eta) = -\frac{1}{2}\int_{\mathcal{M}}\int_{\Delta\setminus B(\xi,r)} \nabla\big(\tilde{f}(x\xi)(S_K^\varepsilon)^2(x\xi\eta_{\Delta^c}) u_\Delta(x\xi|\eta_{\Delta^c}) e^{\varphi*\xi(x)}\big).D_x\tilde{g}(x\xi)e^{-\varphi*\xi(x)}\, \mathbb{1}_{\mathcal{A}}(\xi) dx\, \pi_\Delta(d\xi).$$

(Notice that the boundary terms vanish : on the exterior boundary of Δ, $D_x\tilde{g}(x\xi) = 0$ since $D_x\tilde{g}$ is Λ-local, $\Lambda \subset \Delta$, and on the interior boundary $\partial B(\xi, r) = \{x \in \mathbb{R}^d$ such that $\exists \xi_i : |\xi_i - x| =$

$r\}, S^\varepsilon_K(x\xi\eta_{\Delta^c}) = 0$.) So, $J(\eta) = J_1(\eta) + J_2(\eta) + J_3(\eta)$ where

$$\begin{aligned}
J_1(\eta) &:= -\frac{1}{2}\int_{\mathcal{M}}\int_{\Delta\setminus B(\xi,r)} D_x\tilde{f}(x\xi).D_x\tilde{g}(x\xi)(S^\varepsilon_K)^2(x\xi\eta_{\Delta^c})u_\Delta(x\xi|\eta_{\Delta^c})\mathbb{1}_{\mathcal{A}}(\xi)\,dx\,\pi_\Delta(d\xi)\\
J_2(\eta) &:= -\frac{1}{2}\int_{\mathcal{A}}\int_{\Delta\setminus B(\xi,r)} \tilde{f}(x\xi)(S^\varepsilon_K)^2(x\xi\eta_{\Delta^c})\nabla\big(u_\Delta(x\xi|\eta_{\Delta^c})e^{\varphi*\xi(x)}\big).D_x\tilde{g}(x\xi)e^{-\varphi*\xi(x)}\,dx\,\pi_\Delta(d\xi)\\
J_3(\eta) &:= -\frac{1}{2}\int_{\mathcal{A}}\int_{\Delta\setminus B(\xi,r)} \tilde{f}(x\xi)u_\Delta(x\xi|\eta_{\Delta^c})D_x\big((S^\varepsilon_K)^2\big)(x\xi\eta_{\Delta^c}).D_x\tilde{g}(x\xi)\,dx\,\pi_\Delta(d\xi).
\end{aligned}$$

The integral term J_1 is symmetric in $\tilde{f}$ and $\tilde{g}$.
Using again the Campbell measure of π_Δ, the integral term J_3 becomes

$$J_3(\eta) = -\frac{1}{2}\int_{\mathcal{A}_\Delta}\int_\Delta \tilde{f}(\xi)u_\Delta(\xi|\eta_{\Delta^c})D_x\big((S^\varepsilon_K)^2\big)(\xi\eta_{\Delta^c}).D_x\tilde{g}(\xi)\,\xi(dx)\,\pi_\Delta(d\xi),$$

in such a way that

$$\begin{aligned}
\int_{\mathcal{A}} f_3^2(\eta_{\Delta^c})J_3(\eta)\mu(d\eta) &= -\frac{1}{2}\int_{\mathcal{A}} f_3^2(\eta_{\Delta^c})\tilde{f}(\eta_\Delta)\int_\Delta D_x\big((S^\varepsilon_K)^2\big)(\eta).D_x\tilde{g}(\eta_\Delta)\,\eta_\Delta(dx)\,\mu(d\eta)\\
&= -\int_{\mathcal{A}} f_3^2(\eta_{\Delta^c})\tilde{f}(\eta_\Delta)S^\varepsilon_K(\eta)\int_\Delta D_x(S^\varepsilon_K)(\eta).D_x\tilde{g}(\eta_\Delta)\,\eta_\Delta(dx)\,\mu(d\eta)\\
&= -I_3 + \int_{\mathcal{A}} f_3(\eta_{\Delta^c})\tilde{f}(\eta_\Delta)(S^\varepsilon_K)^2(\eta)\int_\Delta D_x f_3(\eta_{\Delta^c}).D_x\tilde{g}(\eta_\Delta)\,\eta_\Delta(dx)\,\mu(d\eta)\\
&= -I_3
\end{aligned}$$

since $D_x f_3(\eta_{\Delta^c}).D_x\tilde{g}(\eta_\Delta) \equiv 0$.
Thus the symmetry of $I_1 + I_3$ in $\tilde{f}$ and $\tilde{g}$ is equivalent to the symmetry of $\int_{\mathcal{A}} f_3^2(\eta_{\Delta^c})J_2(\eta)\mu(d\eta)$.
Using once more the form of the Campbell measure of π_Δ, this means

$$\begin{aligned}
&\int_{\mathcal{A}} f_3^2(\eta_{\Delta^c})\int_{\mathcal{A}}\int_\Delta \tilde{f}(\xi)(S^\varepsilon_K)^2(\xi\eta_{\Delta^c})\nabla\big(u_\Delta(x(\xi\setminus x)|\eta_{\Delta^c})e^{\varphi*\xi(x)}\big).D_x\tilde{g}(\xi)e^{-\varphi*\xi(x)}\,\xi(dx)\,\pi_\Delta(d\xi)\mu(d\eta)\\
= &\int_{\mathcal{A}} f_3^2(\eta_{\Delta^c})\int_{\mathcal{A}}\int_\Delta \tilde{g}(\xi)(S^\varepsilon_K)^2(\xi\eta_{\Delta^c})\nabla\big(u_\Delta(x(\xi\setminus x)|\eta_{\Delta^c})e^{\varphi*\xi(x)}\big).D_x\tilde{f}(\xi)e^{-\varphi*\xi(x)}\,\xi(dx)\,\pi_\Delta(d\xi)\mu(d\eta).
\end{aligned}$$

For $\tilde{g} = f_4\tilde{f}$, this equality becomes

$$\int_{\mathcal{A}}\int_{\mathcal{A}_\Delta} (S^\varepsilon_K)^2(\xi\eta_{\Delta^c})\,f_3^2(\eta_{\Delta^c})\tilde{f}^2(\xi)\int_\Delta \nabla\big(u_\Delta(x(\xi\setminus x)|\eta_{\Delta^c})e^{\varphi*\xi(x)}\big).D_x f_4(\xi)e^{-\varphi*\xi(x)}\,\xi(dx)\,\pi_\Delta(d\xi)\mu(d\eta) = 0.$$

At this stage, we would like to let disappear the function S^ε_K under the integral by taking the limit for $\varepsilon \to 0$. We may take the limit under the integral, as the following technical lemma claims.

Lemma 5.2 *Let m be a positive measure on some measurable space E. Let g be a real-valued measurable function on E and $(g_n)_n$ be a sequence of positive functions on E. If one of the following assumptions is satisfied*
(A1) *$(g_n)_n$ is bounded increasing and $g \in L^1(m)$*
(A2) *$(g_n)_n$ is decreasing*
then

$$\forall n \in \mathbb{N}, \int_E g_n\,g\,dm = 0 \qquad \Longrightarrow \qquad \int_E \lim_n g_n\,g\,dm = 0.$$

Proof Remark that $\int_E g_n\,g_+\,dm = \int_E g_n\,g_-\,dm$ and apply monotone convergence theorem. ■

Apply Lemma 5.2 to $E = \mathcal{A}_\Delta \times \mathcal{A}_{\Delta^c}$ and to the increasing sequence $g_n = (S_K^{1/n})^2$. One can prove similarly to Lemma 2.7 that $\pi_\Delta \times \mu_{\Delta^c}(\{\gamma : \exists i, j, |\gamma_i - \gamma_j| = r\}) = 0$, and thus by Lemma 4.10 one get $\lim_n S_K^{1/n} = 1$ *a.s.*. Since $f_3, \tilde{f}, D.f_4$ are bounded functions and, by assumption (5), $\int_\Delta \nabla\big(u_\Delta(x(\xi\diagdown x)|\eta_{\Delta^c})e^{\varphi*\xi(x)}\big)\,\xi(dx)$ is $\pi_\Delta \times \mu_{\Delta^c}$-integrable, then (A1) holds and we obtain

$$\int_{\mathcal{A}} f_3^2(\eta_{\Delta^c}) \int_{\mathcal{A}_\Delta} \int_\Delta \tilde{f}^2(\xi)\nabla\big(u_\Delta(x(\xi\diagdown x)|\eta_{\Delta^c})e^{\varphi*\xi(x)}\big).D_x f_4(\xi)e^{-\varphi*\xi(x)}\,\xi(dx)\,\pi_\Delta(d\xi)\mu(d\eta) = 0. \tag{18}$$

Step 3 : Form of the local conditional density $u_\Delta(.|\eta_{\Delta^c})$

To derive informations on the form of u_Δ from the equation (18), we need to particularize the class of test functions. We decompose $\tilde{f}$ and $\tilde{g}$ as follows:

$$\tilde{f} = f_1 f_2 \text{ and } \tilde{g} = f_1 f_2 f_4.$$

The choice we now present for f_1, f_2, f_4 is inspired by [10] Proposition 2.38, in which the author analyzed the case of particles without hard core.

- functions f_2 characterize the configuration inside of Λ: to each f_2 is associated a $\mathcal{C}^\infty$-function φ_2 on $\mathbb{R}^d$ with compact support Λ satisfying $B(\Lambda, R) \subset \Delta$ such that

$$f_2(\xi) = \exp\Big(-\int_\Lambda \varphi_2(x)\xi(dx)\Big) = f_2(\xi_\Lambda).$$

- functions f_3 characterize the configuration outside Δ: to each f_3, is associated a $\mathcal{C}^\infty$-function φ_3 on $\mathbb{R}^d$ with compact support $C \subset \Delta^c$ such that

$$f_3(\eta) = \exp\Big(-\int_C \varphi_3(x)\eta(dx)\Big) = f_3(\eta_{\Delta^c}).$$

- functions f_1 vanish if some ball of the configuration is too close to the boundary of a fixed bounded domain V with $B(V, R) \subset \Lambda$:

$$f_1(\xi) = \psi_1(-\int_\Lambda \varphi_1^\delta(x)\xi(dx)) = f_1(\xi_\Lambda)$$

  where ψ_1 is a $\mathcal{C}^\infty$ non-increasing function on $\mathbb{R}^+$ with values in $[0,1]$ satisfying $\psi_1(0) = 1$ and $\psi_1(u) = 0$ for $u \geq 1$, and φ_1^δ is a $\mathcal{C}^\infty$-function on $\mathbb{R}^d$, δ-approximation of the indicator function of the inner ε_1-boundary of V, $B(V^c, \varepsilon_1) \cap V$: $\varphi_1^\delta(y) = 1$ for $y \in B(V^c, \varepsilon_1) \cap V$, $\varphi_1^\delta(y) = 0$ if $d(y, B(V^c, \varepsilon_1) \cap V) \geq \delta > 0$ and φ_1^δ decreases to $\mathbb{1}_{B(V^c,\varepsilon_1)\cap V}$ when $\delta \searrow 0$.
- functions f_4 have one directional derivative equal to a smooth approximation of the indicator function of a compact interval:

$$f_4(\xi) = \int_\Lambda \varphi_4(x)\xi(dx)$$

  where $\varphi_4(x) = \int_{-\infty}^{x_i} \varphi_4'(u)du$ if $x \in V\backslash B(V, \varepsilon_1)$ and $\varphi_4(x) = 0$ if $x \notin V$ for some $i \in \{1, \dots, d\}$. Moreover φ_4' is a smooth approximation of the indicator function $\mathbb{1}_I$, with $I \subset \mathbb{R}$ a compact interval included in the projection of V along the i-coordinate.

To summerize, $V \subset B(V, R) \subset \Lambda \subset B(\Lambda, R) \subset \Delta \subset \Delta \cup C \subset \Delta \cup B(C, R) \subset K$, f_1 and f_2 are Λ-local, f_4 is V-local and f_3 is C-local. Inserting the value of $D_x f_4$ in the equation (18), we get

$$\int_{\mathcal{A}} f_3^2(\eta_{\Delta^c}) \int_{\mathcal{A}_\Delta} f_1^2(\xi)f_2^2(\xi) \int_V \nabla_i\big(u_\Delta(x(\xi\diagdown x)|\eta_{\Delta^c})e^{\varphi*\xi(x)}\big)\varphi_4'(x_i)e^{-\varphi*\xi(x)}\,\xi(dx)\,\pi_\Delta(d\xi)\mu(d\eta) = 0.$$

Next, we let δ decrease towards 0, which implies that $f_1(\xi)$ decreases towards $\mathbb{1}_{\xi_V \cap B(V^c,\varepsilon_1)=\emptyset}$. Remark that, for any $x \in \xi$, the following equality holds:

$$\varphi_4'(x_i)\mathbb{1}_{\xi_V \cap B(V^c,\varepsilon_1)=\emptyset} = \mathbb{1}_I(x_i)\mathbb{1}_{\xi_V \cap B(V^c,\varepsilon_1)=\emptyset}.$$

Then we let decrease ε_1 towards 0, so that $\mathbb{1}_{\xi_V \cap B(V^c,\varepsilon_1)=\emptyset}$ increases towards 1. Lemma 5.2 justifies the inversion between integrals and limits, and then,

$$\begin{aligned}
&\int_{\mathcal{A}} f_3^2(\eta_{\Delta^c}) \int_{\mathcal{A}_\Delta} f_2^2(\xi) \int_V \mathbb{1}_I(x_i) \nabla_i \big(u_\Delta(x(\xi \smallsetminus x)|\eta_{\Delta^c}) e^{\varphi * \xi(x)} \big) e^{-\varphi * \xi(x)}\, \xi(dx)\, \pi_\Delta(d\xi) \mu(d\eta) \\
= \;& \int_{\mathcal{A}_\Delta} f_2^2(\xi) \int_V \mathbb{1}_I(x_i) \int_{\mathcal{A}} f_3^2(\eta_{\Delta^c}) \nabla_i \big(u_\Delta(x(\xi \smallsetminus x)|\eta_{\Delta^c}) e^{\varphi * \xi(x)} \big) e^{-\varphi * \xi(x)} \mu(d\eta)\, \xi(dx)\, \pi_\Delta(d\xi) \\
= \;& \int_{\mathcal{A}_\Delta} f_2^2(\xi_\Lambda) \int_V \mathbb{1}_I(x_i) \int_{\mathcal{A}} f_3^2(\eta_{\Delta^c}) \nabla_i \big(u_\Delta(x(\xi \smallsetminus x)|\eta_{\Delta^c}) e^{\varphi * \xi(x)} \big) e^{-\varphi * \xi(x)} \mu(d\eta)\, \xi_\Lambda(dx)\, \pi_\Lambda(d\xi) \\
= \;& 0.
\end{aligned}$$

Since this holds for any function f_2 in the class described above, this implies that, for π_Λ-almost all ξ and for any interval I with rational extremities

$$\int_V \mathbb{1}_I(x_i) e^{-\varphi * \xi(x)} \int_{\mathcal{A}} f_3^2(\eta_{\Delta^c}) \nabla_i \big(u_\Delta(x(\xi \smallsetminus x)|\eta_{\Delta^c}) e^{\varphi * \xi(x)} \big) \mu(d\eta)\, \xi(dx) = 0. \tag{19}$$

For a fixed ξ and a fixed x in ξ, let us take I an interval containing only x_i among the i-projections of all points of ξ. Then (19) becomes

$$\int_{\mathcal{A}} f_3^2(\eta_{\Delta^c}) \nabla_i \big(u_\Delta(x(\xi \smallsetminus x)|\eta_{\Delta^c}) e^{\varphi * \xi(x)} \big) \mu(d\eta) = 0,$$

for any $x \in \xi$ and for π_Λ-almost all $\xi \in \mathcal{A}$. We can now let vary f_3 in the class described above, and obtain

$$\nabla_i \big(u_\Delta(x(\xi \smallsetminus x)|\eta_{\Delta^c}) e^{\varphi * \xi(x)} \big) = 0,$$

for any $i \in \{1, \ldots, d\}$ and any $x \in \xi$, for π_Λ-almost all $\xi \in \mathcal{A}$ and for μ-almost all η_{Δ^c}. The last equation means that there exists a function c on $\mathcal{M}$ such that

$$u_\Delta(x\xi|\eta_{\Delta^c}) = c(\xi\eta_{\Delta^c}) e^{-\varphi * \xi(x)} = c(\xi\eta_{\Delta^c}) e^{-E(x|\xi)} \tag{20}$$

for π_Δ-almost all $x\xi \in \mathcal{A}_\Delta$ and μ-almost all η_{Δ^c}.

Step 4 : Symmetry of some integral under the Campbell measure

Let us now consider the left hand side of equation (16) for a function F vanishing for x or y outside a bounded $\Lambda' \subset \mathbb{R}^d$ and Λ-local in η. Taking Δ large enough so that $B(\Lambda \cup \Lambda', R) \subset \Delta$ we obtain

$$\begin{aligned}
&\int_{\mathcal{A}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-E(y|\eta \smallsetminus x)}\, F(x, y, \eta \smallsetminus x)\, dy\, \eta(dx)\, \mu(d\eta) \\
&\quad = \int_{\mathcal{A}} \int_\Delta \int_\Delta e^{-E(y|\eta_\Delta \smallsetminus x)}\, F(x, y, \eta_\Delta \smallsetminus x)\, dy\, \eta_\Delta(dx)\, \mu(d\eta) \\
&\quad = \int_{\mathcal{A}} \int_{\mathcal{A}_\Delta} \int_\Delta \int_\Delta e^{-E(y|\xi \smallsetminus x)}\, F(x, y, \xi \smallsetminus x) u_\Delta(\xi|\eta_{\Delta^c})\, dy\, \xi(dx)\, \pi_\Delta(d\xi)\, \mu(d\eta) \\
&\quad = \int_{\mathcal{A}} \int_\Delta \int_{\mathcal{A}_\Delta} \int_\Delta e^{-E(y|\xi)}\, F(x, y, \xi) u_\Delta(x\xi|\eta_{\Delta^c}) \mathbb{1}_{\mathcal{A}}(x\xi)\, dx\, \pi_\Delta(d\xi)\, dy\, \mu(d\eta) \\
&\quad = \int_{\mathcal{A}} \int_\Delta \int_{\mathcal{A}_\Delta} \int_\Delta e^{-E(y|\xi)}\, F(x, y, \xi) c(\xi\eta_{\Delta^c}) e^{-E(x|\xi)}\, dx\, \pi_\Delta(d\xi)\, dy\, \mu(d\eta) \\
&\quad = \int_{\mathcal{A}} \int_{\mathcal{A}_\Delta} \int_\Delta \int_\Delta e^{-E(y|\xi)}\, e^{-E(x|\xi)}\, F(x, y, \xi) c(\xi\eta_{\Delta^c})\, dx\, dy\, \pi_\Delta(d\xi)\, \mu(d\eta).
\end{aligned}$$

This last expression being symmetric in x and y, it is also equal to the right hand side of equation (16).

■

5.2 Canonical Gibbs measures characterized by their Campbell measures

Proposition 5.3 *Let μ be a Probability measure on $\mathcal{A}$ with support included in $\underline{\underline{\mathcal{A}}}$. Suppose that under the Campbell measure of μ the symmetry property (16) holds. Then μ is a canonical Gibbs measure in $\mathcal{CG}$.*

Proof In [10] Proposition 2.29, the author proved this assertion only for a system without hard core, that is when r, the radius of the balls, vanishes. We adapt here his arguments to the hard core situation. Let F be any positive measurable local function on $\mathbb{R}^d \times \mathbb{R}^d \times \mathcal{A}$. Applying the symmetry equation (16) to the function $(x, y, \eta) \mapsto e^{E(x|\eta)} F(y, x, \eta) \, \mathbb{1}_{\mathcal{A}}(x\eta)$ (with the usual convention $+\infty.0 = 0$) we get

$$\begin{aligned} &\int_{\mathcal{A}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(x, y, \eta \setminus x) \mathbb{1}_{\mathcal{A}}(y(\eta \setminus x)) \, dy \, \eta(dx) \, \mu(d\eta) \\ = &\int_{\mathcal{A}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-E(y|\eta \setminus x) + E(x|\eta \setminus x)} \; F(y, x, \eta \setminus x) \, dy \, \eta(dx) \, \mu(d\eta). \end{aligned}$$

By induction, one proves similarly as in [10] that for any $n \in \mathbb{N}^*$ and any positive measurable local function G on $\mathcal{M} \cap (\mathbb{R}^d)^n \times \mathcal{M} \cap (\mathbb{R}^d)^n \times \mathcal{A}$,

$$\begin{aligned} &\int_{\mathcal{A}} \int_{\mathcal{A} \cap (\mathbb{R}^d)^n} \int_{\mathcal{M} \cap (\mathbb{R}^d)^n} G(\zeta, \xi, \eta \setminus \zeta) \mathbb{1}_{\mathcal{A}}(\xi(\eta \setminus \zeta)) \, d^{(n)}\xi \, \eta^{(n)}(d\zeta) \, \mu(d\eta) \\ = &\int_{\mathcal{A}} \int_{\mathcal{A} \cap (\mathbb{R}^d)^n} \int_{\mathcal{M} \cap (\mathbb{R}^d)^n} e^{-E(\xi|\eta \setminus \zeta) + E(\zeta|\eta \setminus \zeta)} \; G(\xi, \zeta, \eta \setminus \zeta) \, d^{(n)}\xi \, \eta^{(n)}(d\zeta) \, \mu(d\eta), \qquad (21) \end{aligned}$$

where the measure $\eta^{(n)}$ defined on $\mathcal{A} \cap (\mathbb{R}^d)^n$ by $\eta^{(n)} = \sum_{\zeta \subset \eta : \mathrm{Card}\zeta = n} \delta_\zeta$ is the sum of all point measures concentrated on the subsets of η with cardinality n, the measure $d^{(n)}\xi$ is defined on $\mathcal{M} \cap (\mathbb{R}^d)^n$ by $\int G(\xi) d^{(n)}\xi = \frac{1}{n!} \int G(\delta_{x_1} + \cdots + \delta_{x_n}) \, dx_1 \cdots dx_n$ and $E(\xi|\eta) := E_{\mathbb{R}^d}(\xi|\eta)$ is the energy of the n points configuration ξ with respect to the full configuration η as external configuration (which is finite if $\xi\eta \in \mathcal{A}$). It is important to remark that the induction works because

$$\mathbb{1}_{\mathcal{A}}(y(\eta \setminus x)) \mathbb{1}_{\mathcal{A}}(\xi(y\eta \setminus x\zeta)) = \mathbb{1}_{\mathcal{A}}(y(\eta \setminus x)) \mathbb{1}_{\mathcal{A}}(y\xi(\eta \setminus x\zeta)).$$

Therefore, since the symmetry equation (21) is satisfied under μ, using the proof (b) $\Rightarrow$ (a) from Proposition 2.29 [10], we conclude that μ is a canonical Gibbs measure associated to the smooth potential φ on the set of allowed configuration $\mathcal{A}$, that is $\mu \in \mathcal{CG}$. ■

Acknowledgments : For the completion of this work the authors benefited partly from the financial support of the German Academic Exchange Service DAAD and the French Foreign Ministry (Procope agreement Nr. D/0333682) and also from the scientific programme "Phase Transitions and Fluctuation Phenomena for Random Dynamics in Spatially Extended Systems" from the European Science Foundation. All these institutions are here gratefully acknowledged.

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