Minimizers of Generalized Willmore Energies and Applications in General Relativity

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Für meine Mutter

Contents

Chapter 1

Introduction

Goals and Motivations In this thesis we introduce a class of generalized Willmore functionals in order to analyze a wide class of problems from mathematical physics after we develop a general existence and regularity theory.

The Willmore functional, named after T. J. Willmore, is originally defined for a closed, embedded surface $\Sigma \subset \mathbb{R}^3$ with mean curvature H by

$$\mathcal{W}[\Sigma] = \frac{1}{4} \int_\Sigma H^2 \,\mathrm{d}\mu,$$

where μ is the measure on Σ induced by the Euclidean metric.

Our primary inspiration comes from general relativity. S. W. Hawking introduced the Hawking energy as a quasi local energy in [54]. Let (M, g) be a three dimensional Riemannian manifold and let Σ be a spherical surface in M. Then the Hawking energy is given by

$$\mathcal{E}[\Sigma] := \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2 - (\operatorname{tr}_\Sigma K)^2 \,\mathrm{d}\mu\right).$$

Here K is a symmetric two tensor field on M, and ν is the normal vector field of Σ. In general relativity M is embedded in a four dimensional Lorentz manifold N with second fundamental form K. In this way M can be seen as the space at a given instant, contained in the space time N. The motivation for the Hawking energy is the desire to find a (quasi) local notion of energy in general relativity that encompasses the energy of the gravitational field. For example, the famous Schwarzschild spacetime as a model for black holes is a so-called vacuum solution to the Einstein equations. This means there is no classical energy, like matter, present but still it is a curved spacetime and should therefore contain some kind of energy. For more on quasi local energies see for instance the living review [52] by L.B. Szabados.

Unfortunately, the Hawking energy has some undesirable properties. For instance it is not necessarily positive, as can be seen in the case $M = \mathbb{R}^3$, $K = 0$ and Σ any surface but a round sphere, since the round spheres $S_R(a)$ for $R > 0$ and $a \in \mathbb{R}^3$ minimize the Willmore functional. Indeed, $\mathcal{E}$ is normalized to be zero on $S_R(a)$. More importantly, $\mathcal{E}$ is not monotone in the sense that it is possible to find bounded, open domains Ω_1 and Ω_2 with regular boundaries Σ_1 and Σ_2 such that $\overline{\Omega_1} \subset \Omega_2$ but $\mathcal{E}[\Sigma_1] > \mathcal{E}[\Sigma_2]$. In the case $M = \mathbb{R}^3$ and $K = 0$ this occurs if we choose Ω_2 to be any bounded, open domain with regular boundary that is not a ball and compare it to a ball inside of Ω_2. Thus we restrict our focus surfaces adapted to the the Hawking energy shuch as area-constraint maximizers or critical surfaces with large energy. Although this means that we cannot investigate arbitrary regions of M, but only those which contain enough critical surfaces.

Clearly, we can analyze $\mathcal{E}$ subject to an area constraint by investigating

$$\mathcal{H}[\Sigma] = \int_\Sigma H^2 - (\operatorname{tr}_\Sigma K)^2 \,\mathrm{d}\mu$$

subject to an area constraint. Note that $\operatorname{tr}_\Sigma K = \operatorname{tr}_M K - K(\nu,\nu)$ depends on the normal vector field ν.

As it poses little extra difficulty, we study a slight generalization called *Hawking type* functionals. Let $L : TM \to \mathbb{R}$ be a smooth and bounded function, and define

$$\mathcal{H}_L[\Sigma] := \mathcal{W}[\Sigma] + \int_\Sigma L(x,\nu_x)\,\mathrm{d}\mu(x).$$

Apart from Hawking type functionals, we will be interested in another class of functionals. In the beginning of the 19th century S. D. Poisson and S. Germain investigated the roles of principal curvatures for elastic surfaces, see [48] and [12]. Later, in [15] W. Helfrich proposed a model for thin elastic membranes seen as surfaces in $\mathbb{R}^3$ which are critical points of a bending energy under area and volume constraints. Let c be a constant, then his bending energy reads

$$\mathcal{H}_c[\Sigma] = \int_\Sigma (H+c)^2\,\mathrm{d}\mu.$$

Since then many different bending energies have been proposed, see for instance the review [53] by Z.C. Tu and Z.C. Ou-Yang. There they present the following functional for two constants c and b.

$$\mathcal{H}_{c,b}[\Sigma] := \int_\Sigma (H+c)^2\,\mathrm{d}\mu + b\left(\int_\Sigma H\,\mathrm{d}\mu\right)^2$$

Here the last term is added to model nonlocal interactions.

An interesting experimentally observed phenomenon of elastic membranes is budding. It describes the fact that, depending on the external conditions, closed membranes may develop thin necks which pinch, leading to two touching closed membranes. We will see in Chapter 2 that $W^{2,2}$ immersions with bounded Willmore energy can exhibit a similar phenomenon called bubbling.

Let S be a closed Riemann surface and (M,g) Riemannian manifold. In Definition 1.1.7 we present the notion of generalized Willmore functional on the space of conformal $W^{2,2}\cap W^{1,\infty}(S,M)$ immersions that encompasses both $\mathcal{H}_L$ and $\mathcal{H}_{c,b}$. In analogy to Willmore surfaces, the critical points of the Willmore functional, we call the critical points of a generalized Willmore functional, *generalized Willmore surfaces* (or *Hawking type surfaces* when dealing with Hawking type functionals specifically).

Our goal is to prove existence and regularity of generalized Willmore surfaces as well as to analyze Hawking type functionals in the context of general relativity in greater detail.

Existence of Minimizers For the Willmore functional, the question of existence of Willmore surfaces has been thoroughly examined. T. J. Willmore himself showed in [56] that round spheres are minimizers among all surfaces in $\mathbb{R}^3$.

Furthermore, he famously conjectured that the minimizer among all tori in $\mathbb{R}^3$ is the Clifford torus. This remained an open problem for 49 years until F. C. Marques and A. Neves confirmed it in 2014, see [36], using min-max theory of minimal surfaces in S^3. To see the relation of minimal surfaces in S^3 to the Willmore energy note that the Willmore functional is conformally invariant, and the stereographic projection maps $\mathbb{R}^3$ conformally into S^3.

Settling the Willmore conjecture sparked many reviews that examine the historical development and the solution to the problem in much greater detail than we could here, hence we refer to the following two articles by S.T. Yau as well as F. C. Marques and A. Neves [58], [37].

In the following we will briefly review some relatively recent results on Willmore surfaces on which this thesis builds. Let (M, g) be a Rimannian manifold and let $\phi : S \to \Sigma$ be an oriented, immersed surface in M. Here and in the following we employ the notation $\mathcal{W}[\phi]$ and $\mathcal{W}[\Sigma]$ for the Willmore energy of immersions ϕ and their images Σ, interchangeably. The Euler–Lagrange equation of the Willmore functional in (M, g) reads

$$\Delta H + H|\,\mathring{A}\,|^2 + H\,\mathrm{Ric}(\nu, \nu) = 0 \tag{1.0.1}$$

where $\mathring{A} = A - \frac{\gamma}{2}H$ is the trace free part of the second fundamental form A and Ric is the Ricci curvature of M and $\gamma = g|_\Sigma$. It is an elliptic PDE of order four for the immersion, which is quite difficult to solve in general. In order to produce critical points one typically employs variational methods or imposes symmetry assumptions. In this thesis we will construct minimizers of generalized Willmore functionals via direct minimization under constraints and generate critical points by perturbative methods from know solutions.

The idea of direct minimization is to pick a minimizing sequence, prove that it subconverges in a suitable space and show that the limit actually realizes the infimum. Thus a key step is to show compactness for a minimizing sequence.

In his book [14] on harmonic maps F. Hélein discusses the compactness of $W^{2,2}$ immersions $\phi : D \to \mathbb{R}^n$ with a bound on the Willmore energy. Using the method of moving frames he proves that there is a constant C_n such that any sequence ϕ_k of conformal $W^{2,2}(D, \mathbb{R}^n)$ immersions with $\mathcal{W}[\phi_k] \le C_n$ either converges to a constant map or weakly subconverges to a conformal $W^{2,2}$ immersion. In case the sequence ϕ_k converges to a constant, we say it vanishes. An important part in the proof is that the derivatives of the ϕ_k satisfy an elliptic PDE with a Wente type structure on the right hand side which allows for stronger compactness results then expected. Here Wente type structure refers to a distinct algebraic expression. The classic example is the following Dirichlet problem for the Poisson equation on a bounded domain $\Omega \subset \mathbb{R}^2$

$$\begin{cases} \Delta u = \frac{\partial a}{\partial x_1}\frac{\partial b}{\partial x_2} - \frac{\partial a}{\partial x_2}\frac{\partial b}{\partial x_2} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for $a, b \in W^{1,2}(\Omega)$. Solution to this problem exhibit better regularity properties than expected from the standard elliptic regularity theory; see [14, Chapter 3].

Moreover, the above result by F. Hélein already hints at the fact that a sequence of surfaces with bounded Willmore energy can vanish. Indeed, this together with the bubbling of $W^{2,2}$ maps, a phenomenon also known for harmonic maps (see for instance [47]), pose major challenges for the compactness theory. It is not surprising encounter to bubbling in this context since the Willmore energy is related to Dirichlet energy of the Gauss map of the surface.

These difficulties have been overcome independently by J. Chen and Y. Li in [3], as well as A. Mondino and T. Rivière in [42], [43]. The general idea is to use bounds on the area and the Willmore energy to obtain uniform bounds in $W^{2,2}$ and extract a weakly convergent subsequence. After vanishing is ruled out by curvature or topological assumptions on the base manifold (M, g), an in depth analysis is needed to understand the bubbling and show that the sequence converges to a stratified surface in a controlled way. In this context stratified surfaces can be thought of as a collection closed Riemann surfaces that touch at isolated points, see Definition 1.1.3.

It needs to be said that both of the results above rely on the fact that conformal immersions with a bound on the L^2 norm of the second fundamental form and the area can be extended across singularities to $W^{2,2}$ maps. This result is due to E. Kuwert and Y. Li in [23] where they also investigate weak compactness of conformal immersions with bounded Willmore energy. In [25] E. Kuwert and R. Schätzle discuss extensions across point singularities also in the context of Willmore flow, the gradient flow of the Willmore functional.

However, in general these extensions will be branched immersions which again adds complexity in the theory. For more information on the branch points see for instance the works of T. Lamm, H. Nguyen, Y. Bernard, T. Rivière E. Kuwert, R. Schätzle [32], [2], [25].

So far in the literature, the solution to these obstacles for direct minimization has been to impose conditions that prevent vanishing or bubbling altogether.

In [3] J. Chen and Y. Li perform a direct minimization in space of branched, conformal immersions of closed surfaces since they impose energy conditions that prevent bubbling and further stratification altogether.

In [43] A. Mondino and T. Rivière perform a direct minimization in space of branched, conformal immersions of bubble trees since their requirements on the curvature of the ambient manifold rule out vanishing of individual components. Here bubble trees are stratified surfaces whose components are spheres and which exhibit a tree structure; see Definition 1.1.4. Furthermore, we introduce bubble forests as a stratified surface consisting of a closed Riemann surface with finitely many bubble trees attached; see Definition 1.1.4 as well.

In this thesis we do not impose restriction on the ambient manifold or the energy to prevent bubbling and we integrate the possibility of vanishing components into our function space. This way, we obtain a formalism that treats the singular cases on equal footing with the regular ones. To the best knowledge of the author this unified approach is new in the literature.

Inspired by the fact that vanishing components are often called ghosts, we introduce haunted immersions of stratified surface; see Definition 2.0.3. These are maps which are constant on some but not all of the components of the stratified surface and immersions on the rest. a haunted immersion is called irreducible if it does not have any "unnecessary" ghosts; see Definition 2.0.3 and below.

The following is a heuristic version of our compactness theorem. For the proper statement see Theorem 2.0.5 along with the definitions of Section 1.1 and Definition 2.0.3.

Theorem 1.0.1 (heuristic)**.** *Let S^k be a sequence of compact bubble forests. Let $\phi_k \in W^{2,2}(S^k, \mathbb{R}^n)$ be a sequence of irreducible, haunted, branched conformal immersions. Assume ϕ_k, the area $\mathcal{A}[\phi_k]$ and $\mathcal{W}[\phi_k]$ are uniformly bounded.*

Then $\phi_k(S^k)$ either converges to a point or subconverges to an immersed haunted bubble tree $\phi(S)$. In the second case we find

$$\mathcal{A}[\phi] = \lim_{k\to\infty} \mathcal{A}[\phi_k],$$
$$\mathcal{W}[\phi] \leq \lim_{k\to\infty} \mathcal{W}[\phi_k].$$

In order to employ this theorem in the search for minimizers, we need to ensure that a given sequence does not vanish altogether. A straightforward way to establish this is to fix the area and solve a variational problem under constraints. Both of our examples are formulated as constrained variational problems and in fact our definition of generalized

Willmore functionals is carefully chosen so that this approach succeeds. Hence we find the following existence results via direct minimization.

Let $\mathcal{F}(\mathcal{T}, M)$ be the space of haunted, branched immersions of bubble trees, let $\mathcal{A}[\phi]$ be the area of $\phi \in \mathcal{F}(\mathcal{T}, M)$ and for a constant $a > 0$ define

$$\mathcal{F}_a(\mathcal{T}, M) = \{\phi \in \mathcal{F}(\mathcal{T}, M) \mid \mathcal{A}[\phi] = a\}.$$

The notion of a-generalized Willmore functional is introduced in Definition 1.1.7.

Theorem 1.0.2. *Let (M, g) be compact Riemannian manifold and let $\mathcal{H}$ be an a-generalized Willmore functional, then* $\inf \{\mathcal{H}[\phi] \mid \phi \in \mathcal{F}_a(\mathcal{T}, M)\}$ *is attained in $\mathcal{F}_a(\mathcal{T}, M)$.*

Corollary 1.0.3. *Let (M, g) be a non-compact Riemannian manifold with C_B bounded geometry and let $\mathcal{H}$ be an a-generalized Willmore functional. Suppose there exists a transitive group action on M that leaves $\mathcal{H}$ and $\mathcal{A}$ invariant. Then* $\inf \{\mathcal{H}[\phi] \mid \phi \in \mathcal{F}_a(\mathcal{T}, M)\}$ *is attained in $\mathcal{F}_a(\mathcal{T}, M)$.*

Corollary 1.0.3 allows us to treat the existence of minimal membranes with prescribed area and enclosed volume. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Σ and let X be the position vector field. The divergence formula implies

$$\operatorname{Vol}(\Omega) = \int_\Omega 1 \,\mathrm{d}x = \frac{1}{3}\int_\Sigma \operatorname{div} X \,\mathrm{d}\mu = \frac{1}{3}\int_\Sigma \langle X, \nu\rangle \,\mathrm{d}\mu.$$

This motivates the introduction of the functional

$$\mathcal{V}[\phi] := \frac{1}{3}\int_\Sigma \langle X, \nu\rangle \,\mathrm{d}\mu$$

on $\mathcal{F}(\mathcal{T}, \mathbb{R}^3)$. It is well defined and translation invariant, see Section 1.2.2. For $a, v \in \mathbb{R}^+$ define

$$\mathcal{F}_{a,v}(\mathcal{T}, \mathbb{R}^3) := \left\{\phi \in \mathcal{F}(\mathcal{T}, \mathbb{R}^3) \mid \mathcal{A}[\phi] = a,\, \mathcal{V}[\phi] = v\right\},$$

then we have the following theorem.

Theorem 1.0.4. *For any $c, b \in \mathbb{R}$ and $a, v \in \mathbb{R}^+$ such that $3\sqrt{4\pi}v \leq a^{3/2}$ and $-ab \leq 1$ the infimum of $\mathcal{H}_{c,b}$ on $\mathcal{F}_{a,v}(\mathcal{T}, \mathbb{R}^3)$ is attained.*

Note that we treat membranes in the class of bubble trees in a natural manner. This allows us to see budding of membranes as a natural part of the theory and not as an edge case of degenerating surfaces, as it is commonly done in the literature.

Regularity of Generalized Willmore Surfaces While partial results were known before, the question of regularity of Willmore surfaces was settled by A. Mondino and T. Rivière in [43] for arbitrary codimension. They show that the Euler–Lagrange equation of the Willmore functional can be brought into divergence form. In complex coordinates $\{z, \bar{z}\}$, see Section 3.1, we have

$$4e^{-2\lambda} \operatorname{Re}\left(\nabla^M_{\bar{z}}\left[\partial_z H\nu - \frac{1}{2} H H_0 \partial_{\bar{z}}\phi\right]\right) = \Delta H\nu + H|\mathring{A}|^2\nu + 8H \operatorname{Re}\left(\tilde{\operatorname{Rm}}\, e_z\right).$$

Here $e_z = e^{-\lambda}\partial_z\phi$, for a conformal immersion ϕ with conformal factor $e^{2\lambda}$, $H_0 = 4A(e_z, e_z)$ and $\tilde{\operatorname{Rm}} = g\left(\operatorname{Rm}^M(e_{\bar{z}}, e_z)e_z, \nu\right)$ for the Riemann curvature tensor Rm^M of M.

More importantly, they construct potentials for the equation that solve Poissons equation with a right-hand side that exhibits a Wente type structure, thus leading to better regularity results than naively expected. After minor modifications we are able to apply this regularity result for generalized Willmore surfaces as well.

Theorem 1.0.5. *Let $\phi \in W^{2,2}(D, M^3)$ be conformal immersion with conformal factor $e^{2\lambda}$, $\lambda \in L^\infty(D)$. If ϕ solves $\Delta H + H|\mathring{A}|^2 + F(\phi, d\phi, \nabla d\phi) = 0$, where F is as in Definition 1.1.7, then ϕ is smooth.*

In particular, this means that critical points of generalized Willmore functionals in codimension one are smooth, away from finitely many points. Although we prove regularity only in the case of codimension one, we conjecture that it holds in any codimension.

Concentration of Small Hawking Type Surfaces In [28] and [29] T. Lamm and J. Metzger study Willmore surfaces with small area constraint in codimension one. They prove existence as well as regularity of minimizers and they find that small, spherical, area-constrained Willmore surfaces concentrate around critical points of the scalar curvature of the ambient manifold. P. Laurain and A. Mondino investigate the problem in [33] and improve upon the previous results.

Additionally, T. Lamm and J. Metzger and F. Schulze show in [31] that a neighborhood of a non-degenerate critical point of the ambient scalar curvature is foliated by spherical, area-constrained Willmore surfaces. This result was also independently shown by N. Ikoma, A. Malchiodi, A. Mondino in [21].

A similar line of inquiry is the construction of Willmore spheres with prescribed isoperimetric ratios performed J. Schygulla and expanded on by E. Kuwert and Y. Li. If $\Sigma \subset \mathbb{R}^3$ is a closed surface and Ω is the enclosed volume, then the isoperimetric ratio is defined as

$$\sigma(\Sigma) := 6\sqrt{\pi}\frac{\mathrm{Vol}(\Omega)}{|\Sigma|^{3/2}} \in (0, 1].$$

In [49] and [24] they show that in $\mathbb{R}^3$ minimizers exist for any prescribed ratio between 0 and 1 and that the minimizers degenerate to a double round sphere if the ratio approaches 0. This is related to the study of bending energies for membranes since there an area constraint and a constraint on the enclosed volume are common.

In this thesis we investigate Hawking type functionals with small area constraint in greater detail.

In the context of general relativity this constitutes an important generalization to the analysis in [28] and [29]. When equipped with additional constraint equations the tuple (M, g, K) represents the initial data for the Einstein equations as a hyperbolic system. Thus understanding (M, g, K) is the foundation of understanding the spacetime obtained as the solution to the Einstein equations. The case $K = 0$, that was examined previously, corresponds only to static spacetimes.

We calculate expansions on small spheres and characterize concentration points, i.e. points in the ambient manifold around which there exist critical, area-constrained, spherical surfaces Σ_r in any neighborhood $B_r(p)$. The goal is to find an analogue of an energy density as seen by the Hawking energy and to understand which features of the ambient manifold contribute most to the Hawking energy.

In particular, we prove the following result.

Theorem 1.0.6. *Let (M, g) be C_B bounded and let $\mathcal{H}[\Sigma] = \frac{1}{4}\int_\Sigma H^2 - (\mathrm{tr}_\Sigma K)^2 \,\mathrm{d}\mu$. There is a constant $\epsilon_0 > 0$ depending only on C_B and K such that at any concentration point p of $\mathcal{H}$ around which the concentrating surfaces obey $\mathcal{H}[\Sigma_r] \leq 4\pi + \epsilon_0^2$, we have*

$$\nabla^M \left(\mathrm{Sc}_p + \frac{3}{5}\,\mathrm{tr}\, K_p^2 + \frac{1}{5}|K_p|^2 \right) = 0.$$

Under suitable requirements on the ambient manifold, compactness for instance, the existence of the concentration points required in Theorem 1.0.6 is guaranteed by the existence of minimizers with small area.

Furthermore, we find the following expansion.

Theorem 1.0.7. *Let $\Sigma \subset M$ be a spherical surface. Suppose Σ is contained in a normal coordinate neighborhood $B_r(p)$ as in Lemma A.1.11 and that $\| \mathring{A} \|^2_{L^2(\Sigma)} \leq Cr|\Sigma|$. Then $\mathcal{E}$ has the following expansion.*

$$\left| \mathcal{E}[\Sigma] - \frac{1}{12}\left(\frac{|\Sigma|}{4\pi}\right)^{3/2}\left(\mathrm{Sc}_p + \frac{3}{5}\operatorname{tr} K_p^2 + \frac{1}{5}|K_p|^2\right)\right| \leq C|\Sigma|^2$$

Coordinate spheres and small minimizers both fulfill the requirements of the theorem.

Note that Theorem 1.0.7 stands in contrast to the results of G. Horowitz and B. Schmidt in [16]. There they found that the Hawking energy has the following expansion

$$\mathcal{E}[S] \sim \left(\mathrm{Sc}_p + \operatorname{tr} K_p^2 - |K_p|^2\right) R^3 + O(R^4),$$

when calculated on spherical cross sections $S_R(0)$ of the light cone in the tangent space at p.

This discrepancy is very surprising. In general relativity the energy density ρ is given by $16\pi\rho = \mathrm{Sc} + (\operatorname{tr} K)^2 - |K|^2$. As the Hawking energy should serve as a quasi local energy one might think that surfaces with maximal area-constrained Hawking energy would tend to concentrate around critical points of the energy density ρ which is not the case. Similarly, one would expect to find the energy density in the expansion of the Hawking energy.

The fact that the expansion in a space like slice does not capture the energy density, whereas the expansion along a light cone does, is especially vexing since the spheres in the light cone can be though of as lying in a space like slice, belonging to a different time, themselves.

Foliations of Isolated Systems via Hawking Type Surfaces A Riemannian manifold (M, g) is called asymptotically Schwarzschild if there is a compact set $K \subset M$ and a diffeomorphism $x : M\backslash K \to \mathbb{R}^3 \backslash B_\sigma(0)$, for a constant $\sigma > 0$, such that in these coordinates the metric is asymptotic to the Schwarzschild metric g_S; see Definition 5.1.1. The region $M\backslash K$ is called the asymptotically Schwarzschild end of M. This notion is meant to model an isolated system in general relativity, with the additional requirement that, asymptotically, the system exhibits the decay and the rotational symmetry of the Schwarzschild space as apposed to just decaying to the Euclidean space.

Constructing asymptotic foliations for these isolated systems is a means of characterizing their asymptotic behavior and a way of assigning a center of mass to the system.

In [30] T. Lamm, J. Metzger and F. Schulze constructed a foliation of the outer region of asymptotically Schwarzschild manifolds by large, centered, round, area-constrained Willmore spheres, which are minimizing in their class.

Building on these results, T. Koerber showed in [22] that under the area-constrained Willmore flow any sufficiently large, centered and round sphere converges to a leaf of the foliation.

The construction of the foliation above is similar to foliations of the outer region of asymptotically Schwarzschild manifolds by constant mean curvature surfaces obtained by G. Huisken and S.T. Yau [19]. These were generalized by J. Metzger [38] to foliations by surfaces with prescribed mean curvature. This is connected to Mathematical general relativity and the investigation of initial data set of the Einstein equation, which is a very active field, hence we cannot give an overview here. We briefly mention the following articles by M. Eichmair, L.H. Huang J. Metzger, C. Nerz where they directly generalize the results on constant mean curvature foliations [8], [17], [44], [45], [46] .

In this thesis we will proceed along the same lines as [30] to construct a foliation by Hawking type surfaces of the outer region of the asymptotically Schwarzschild end. For the functional corresponding to the Hawking energy this can be interpreted as a notion of center of mass of the ambient manifold as seen by the Hawking energy.

Based on the existence theory developed earlier, we show that asymptotically Schwarzschild manifolds admit area-constrained $\mathcal{H}_L$-minimizing bubble trees, provided the decay of L in the asymptotically Schwarzschild end is fast enough. Moreover, we show that these minimizers are embedded spheres provided they lie in $\mathbb{R}^3 \setminus B_{r_0}(0)$ for a large enough radius r_0.

Relying on the analysis of the Willmore functional in the asymptotically Schwarzschild setting in [30] we construct a foliation by spherical Hawking type surfaces. The following is a heuristic version of our foliation result, see Theorems 5.3.6, 5.3.8 and 5.3.7 for the proper statements.

Theorem 1.0.8 (heuristic)**.** *Let (M, g) be an asymptotically Schwarzschild manifold and suppose $L : TM \to \mathbb{R}$ is smooth and decays fast enough. Then there exist constants $r_0 > 0$, $\lambda_0 > 0$ and a unique foliation $\{\Sigma_\lambda\}_{\lambda\in(0,\lambda_0)}$ of $(\mathbb{R}^3 \setminus B_{r_0}(0), g)$ consisting of centered, spherical, area-constrained Hawking type surfaces with respect to $\mathcal{H}_L$. Along this foliation the energies $\mathcal{H}_L[\Sigma_\lambda]$ are strictly decreasing.*

Moreover, this foliation is obtained via a deformation of the round spheres $S^2_R(0)$ in $(\mathbb{R}^3 \setminus B_{r_0}(0), g_S)$ for $R \in (r_0, \infty)$.

We would like to point out that the results of Section 1.2 as well as Chapters 2, 3 and 4 have been published on the arXiv, see [11] and [10].

Outlook Open questions in the immediate vicinity of this thesis include an analysis of the dynamical stability of the foliation constructed in Chapter 5 analogous to the one carried out by T. Koerber in [22] using the area-constrained Willmore flow.

Furthermore, it should be possible to improve the decay of the geometric quantities in Chapter 5, and thus understand the leaves of the foliation in greater detail.

Another important step would be to replace the condition of asymptotically Schwarzschild manifolds with asymptotically flat manifolds with positive ADM mass since this is a related but more general model for isolated systems in general relativity. The ADM mass is named after R. Arnowitt, S. Deser and C.W. Miser who introduced it in [1]. Similarly, one can investigate asymptotically hyperbolic manifolds.

In the context of membranes it seems worthwhile to analyze precisely under which circumstances bubbling occurs and what shapes are produced. This would constitute a unified discussion of the budding phenomenon for membranes and to the knowledge of the author it would be the first one without symmetry assumptions.

Additionally, it would be interesting to understand cell movement in the context of elastic membranes, either through a gradient flow approach or by constructing minimizers to a changing family energy functionals.

Thesis Structure The remainder of this chapter is divided into two sections. In Section 1.1 we introduce the definitions for branched, conformal immersions, stratified surfaces, bubble trees and their convergence, as well as generalized Willmore functionals, where we took inspiration from [3], [23] and [43].

Section 1.2 discusses our two main examples of generalized Willmore functionals – Hawking type functionals and bending energies for membranes. We prove that Hawking type functionals $\mathcal{H}_L$ are generalized Willmore functionals provided L is smooth and bounded. Additionally, we calculate the Euler–Lagrange equation in the physically interesting cases of $L = (\mathrm{tr}_\Sigma K)^2$ explicitly since we need it for a detailed analysis in Chapter 4. Subsection 1.2.2 is dedicated to the functionals $\mathcal{H}_{c,b}$. We prove that they are generalized Willmore under suitable conditions and show that a volume constraint is well defined for immersed surfaces.

We introduce haunted immersions and prove Theorem 2.0.5, our compactness result for haunted, branched, conformal immersions of bubble forests, in Chapter 2. The main idea is to apply the compactness of [3] on every component of the bubble forest and track the structure of the resulting stratified surface. The existence of minimizers of generalized Willmore functionals in Theorem 2.0.6, Corollary 2.0.7, and Theorem 2.0.8, then follows by direct minimization under constraints.

Chapter 3 is dedicated to the regularity theory for generalized Willmore equations. Here we follow [43] closely, first recalling their notation in codimension one in order to quote the divergence form of Equation (1.0.1). We realize that the crucial [43, Lemma 6.1] remains valid for the more complicated nonlinearities we consider here, see Lemma 3.2.3. This implies that, in order to prove regularity of generalized Willmore surfaces, we only have to adapt the final bootstrap argument, see the proof of Theorem 3.2.6. This chapter also concludes the treatment of generalized Willmore surfaces in their full generality. In the final two chapters we restrict to Hawking type surfaces in order to perform a more detailed analysis.

In Chapter 4, we investigate Hawking type functionals $\mathcal{H}_L$ with small area constraint in the spirit of T. Lamm and J. Metzger in [28] and [29]. First, we establish that small minimizing bubble trees Σ are spheres which have energy $\mathcal{H}_L[\Sigma]$ close to 4π. Then we prove a priori estimates to obtain quantitative control over the extrinsic curvature of critical surfaces with energy close to 4π. This is crucial in order to compare them to round spheres via [5, Theorem 1.1] and [6, Theorem 1.2] by C. De Lellis and S. Müller; see also Theorem A.1.8. This allows us to compute an expansion of the normal variation of $\int_\Sigma L \,\mathrm{d}\mu$, in order to characterize concentration points. Here we rely on the results of [28] and [29] for the expansion the Willmore energy. Additionally, the case $L = (\mathrm{tr}_\Sigma K)^2$ is treated in greater detail, and the expansion of $\mathcal{H}_L$ is readily deduced.

Chapter 5 has the goal of constructing a unique foliation in the asymptotically Schwarzschild end of an asymptotically Schwarzschild manifold by spherical, area-constrained Hawking surfaces. In Section 5.1 we introduce the notation of asymptotically Schwarzschild manifolds as presented in [30] and recall basic results from said paper that relate the geometry of the ambient manifold (M, g) to the model $(\mathbb{R}^3 \setminus \{0\}, g_S)$. We conclude this section with Theorem 5.1.8 were we prove existence of area-constrained minimizing bubble trees for Hawking type functionals in the setting of asymptotically Schwarzschild manifolds. We also show that these minimizers are spheres with Willmore energy close to 4π , provided they are large and lie in the asymptotic Schwarschild end. The next subsection is dedicated to proving a priori estimates in order to obtain quantitative control over Hawking type surfaces similar to Section 4.1. We first show that the mean curvature is bounded from below, then, employing the methods of [30], we obtain the same decay rates as in [30].

This allows us to show that the linearization of the area-constrained Euler–Lagrange equation of the Hawking type functional is invertible; see Section 5.3, Theorem 5.3.4. This proof heavily relies on the corresponding analysis for the area-constrained Willmore equation in [30, Section 7]. Using the invertibility of the linearized equation we construct the unique foliation of the outer region in Theorem 5.3.6, 5.3.7 and 5.3.8 perturbatively, starting from the round Spheres $S^2_R(0)$ in $(\mathbb{R}^3 \setminus \{0\}, g_S)$.

The appendix consists of three chapters. In Chapter A we collect results about C_B bounded Riemannian Manifolds, general facts about surfaces and specialized results for small surfaces. We will refer to this chapter throughout the thesis. In Chapter B we present two lemmas about constructing potentials for vector fields that are used to prove Lemma 3.2.3. Finally, Chapter C houses long calculations needed in Theorem 4.2.4, Corollary 4.2.6 and Lemma 5.1.6. Additionally, we present the code for Mathematica 7 which was used to check said calculations.

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1.1 Preliminaries

Let (M, g) be a Riemanian manifold with Levi–Chivita connection ∇. We denote the Rimannian curvature tensor of by Rm, its Ricci curvature by Ric and the scalar curvature by Sc. Further let, (S, η) be a Riemann surface and let $\phi : S \to \Sigma \subset M$ be an immersion such Σ is an oriented immersed surface. It inherits a metric $\gamma = g|_\Sigma$ and its vector valued second fundamental form is defined as

$$\vec{A}(X, Y) := -\left(\nabla_X Y\right)^\perp$$

for $X, Y \in \Gamma(T\Sigma)$ and $\cdot^\perp$ the projection to the normal bundle of Σ. The mean curvature vector is defined as

$$\vec{H} := \operatorname{tr}_\Sigma A$$

and the vector valued trace free second fundamental form is defined as $\mathring{\vec{A}} := \vec{A} - \frac{\vec{H}}{2}\gamma$. In the case that Σ has codimension one, let ν be the normal vector field. Then the second fundamental form, mean curvature and trace free second fundamental form are defined by

$$\begin{aligned} A(X, Y) &:= g\left(\nabla_X \nu, Y\right), \\ H &:= \operatorname{tr}_\Sigma A = \operatorname{div}_\Sigma \nu, \\ \mathring{\mathrm{A}} &= A - \frac{H}{2}\gamma. \end{aligned}$$

Let μ_γ be the measure on Σ induced by γ. At times we will denote it also by μ_g or simply by μ. Let Σ be closed. Then the area of Σ is defined by $|\Sigma| = \int_\Sigma 1 \, \mathrm{d}\mu$. At times we will denote the area also by $\mathcal{A}[\Sigma]$ if we want to stress its role as a functional. Further, we will regard it and others as functionals on the surface or the immersion interchangeably: $\mathcal{A}[\Sigma] = \mathcal{A}[\phi]$.

Throughout this thesis we use the convention that C denotes a generic constant that can change from line to line.

In the following we present the definitions used in this thesis. In particular, we properly define the notion of generalized Willmore functionals.

Definition 1.1.1 (branched, conformal immersion; cf. [3, Definition 1 and 2]**).** Let (S, η) be a Riemann surface and let (M^n, g) be an n-dimensional, oriented Riemannian manifold which we assume to be isometrically embedded in some $\mathbb{R}^N$.

1. For $k \in \mathbb{Z}$ and $p \in [1, \infty]$ we define the Sobolev spaces as follows:

$$W^{k,p}(S, M) := \left\{ \phi \in W^{k,p}(S, \mathbb{R}^N) \,\middle|\, \phi(S) \subset M \text{ a.e.} \right\}.$$

2. An element $\phi \in W^{2,2}(S, M)$ is called *conformal immersion*, if ϕ is an immersion almost everywhere and if there is a function $e^{2\lambda} : S \to \mathbb{R}$, called the *conformal factor* of ϕ, such that

$$\phi^* g = e^{2\lambda}\eta.$$

3. We say $\phi : S \to M$ is a *branched conformal immersion* with finitely many branch points $B \subset S$, if $\phi \in W^{2,2}_{\mathrm{loc}}(S \setminus B, M)$ is a conformal immersion and if for all $p \in B$ there is an open neighborhood U_p and a constant C such that

$$\int_{U_p \setminus \{p\}} 1 + |\vec{A}|^2 \, \mathrm{d}\mu_g \le C.$$

4. Set

$$\begin{aligned}\mathcal{F}(S,M) := \{\phi \in W^{2,2}(S,M) \mid \phi \text{ is branched, conformal, immersion}\\ \text{with branch points } B; \phi \in W^{1,\infty}_{\text{loc}}(S \setminus B, M)\}\end{aligned}$$

and for $a > 0$ define $\mathcal{F}_a(S,M) := \{\phi \in \mathcal{F}(S,M) \mid |\phi(S)| = a\}$

Note. For the conformal factor $e^{2\lambda}$ of a conformal immersion $\phi \in \mathcal{F}(S,M)$ we have $\lambda \in L^\infty_{\text{loc}}$ away from the branch points.

E. Kuwert and Y. Li showed that branched conformal immersions can be extended to $W^{2,2}$ maps.

Theorem 1.1.2 (see [23, Theorem 3.1])**.** *Let D be the unit disc in $\mathbb{R}^2$ and let $\phi \in W^{2,2}_{loc}(D \setminus \{0\}, \mathbb{R}^n)$, $n \geq 3$, be a conformal immersion, $\phi^* g = e^{2\lambda}\delta$. If ϕ satisfies*

$$\int_{U_p \setminus \{p\}} 1 + |\vec{A}|^2 \, \mathrm{d}\mu_g \leq \infty,$$

then $\phi \in W^{2,2}(D, \mathbb{R}^n)$ and in complex coordinates we have

$$\begin{aligned}\lambda(z) &= m \ln |z| + w(z),\\ -\Delta\lambda &= -2m\pi\delta_0 + K_g e^{2\lambda}.\end{aligned}$$

Here, $m \in \mathbb{N}$, $w \in C^0 \cap W^{1,2}(D)$, K_g is the Gauss curvature of g and δ_0 is the delta distribution at 0. Additionally, the multiplicity of the immersion at $p = \phi(0)$ is given by

$$\theta^2(\phi, p) = \#\phi^{-1}(p) = m + 1.$$

The well known phenomenon of bubbling of $W^{2,2}$ immersions necessitates the introduction of stratified surfaces.

Definition 1.1.3 (**stratified surface**; cf. [3, Definition 3])**.** A compact connected metric space (S, d) is called a *stratified surface with singular points* P, if $P \subset S$ is a finite set such that:

1. the regular part, $S \setminus P$, is a smooth Riemann surface without boundary. It carries a smooth metric η, whose induced distance function agrees with d.

2. Moreover, for each $p \in P$ there is a $\delta > 0$ such that $B_\delta(p) \cap P = \{p\}$ and $B_\delta(p) \setminus \{p\} = \bigcup_{i=1}^{m(p)} \Omega_i$. Here $1 < m(p) < \infty$ and Ω_i are topological discs with one point removed. Additionally, we assume that η can be extended to a smooth metric on each $\Omega_i \cup \{p\}$.

For a stratified surface, the regular part naturally decomposes into finitely many punctured connected Riemann surfaces $S \setminus P = \bigcup_i S^i$. By the second point of the previous definition, we can add finitely many points to each S^i in order to obtain a Riemann surface $\overline{S^i}$. This allows us to interpret a stratified surface as a collection of touching Riemann surfaces.

Consider the stratified torus S in Figure 1.1. It has one singular point p and $S^1 = S \setminus \{p\}$ is a sphere with two punctures. We may add in two points p_1, p_2 such that $\overline{S^1} = S^1 \cup \{p_1\} \cup \{p_2\}$ is a sphere. In this picture we can understand S as the immersion of $\overline{S^1}$.

By abuse of notation we usually denote a stratified surfaces as $S = \bigcup_i S^i$ and refer to Riemannian metrics on S instead of on every $\overline{S}^i$.

The next definition introduces a important structural feature of stratified surfaces.

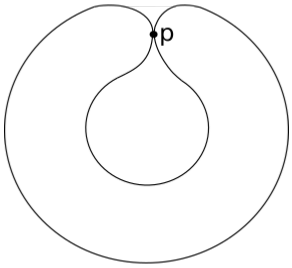

Figure 1.1: A stratified torus with singular point p

Definition 1.1.4 (dual graph, bubble tree, bubble forest).

1. Associate with every stratified surface $S = \bigcup_i S^i$ its *dual graph*, where the vertices correspond to the components S^i and two vertices are joined by an edge whenever the corresponding S^i are joined by a singular point. Note that this construction allows for loops and multiple edges.

2. A stratified surface whose regular part consists of punctured spheres and whose dual graph is a simple tree is called a *bubble tree*. The constituting spheres are called *bubbles*.

3. If S is a stratified surface and $S_1 \subset S$ is a bubble tree, then we say S_1 is *attached* to S at $p \in S$ if $\overline{S \setminus S_1} \cap S_1 = \{p\}$.

4. A stratified surface $S = S_0 \cup \bigcup_{i=1}^m S_i$ consisting of a Riemann surface S_0 with finitely many bubble trees attached at mutually distinct points is called *bubble forest* with *base* S_0. Note that the dual graph of a bubble forest is still a tree.

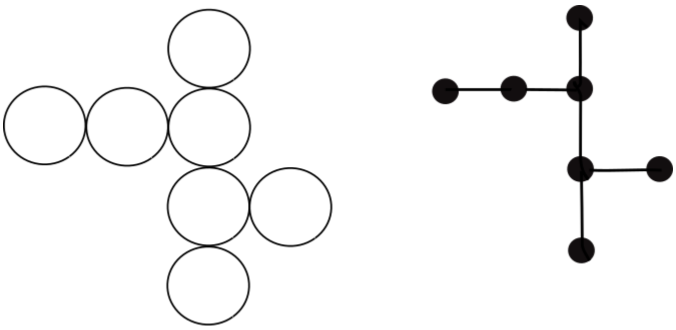

Figure 1.2: A bubble tree and its dual graph

Now we need to combine the notions of branched, conformal immersions and stratified surfaces.

Definition 1.1.5 (branched, conformal immersion of a stratified surface).

1. Let S be a stratified surface with $S \setminus P = \bigcup_{i=1}^m S^i$ and let M be a manifold of dimension three or higher. For $k \in \mathbb{N}$ and $p \in [1, \infty]$ denote by $W^{k,p}(S, M)$ the continuous maps $\phi : S \to M$ for which all $\phi|_{S^i}$ extend to maps in $W^{k,p}(\overline{S^i}, M)$. Additionally, we say that $\phi : S \to M$ is a (branched) immersion if all extensions $\phi|_{\overline{S^i}}$ are (branched) immersions.

2. We extend any functional defined for immersed surfaces componentwise to immersed stratified surfaces. For example for S and ϕ as above we set $\phi^i := \phi|_{\overline{S^i}}$, $\Sigma = \phi(S)$ and $\Sigma^i := \phi^i(\overline{S^i})$; then the area and Willmore energy are given by

$$\mathcal{A}[\phi] := |\Sigma| := \sum_{i=1}^{m} |\Sigma^i| = \sum_{i=1}^{m} \mathcal{A}[\phi^i],$$
$$\mathcal{W}[\phi] := \sum_{i=1}^{m} \mathcal{W}[\phi^i].$$

The following definition is motivated by the treatment of bubbling in [3]. In particular it addresses the fact that the base space may change.

Definition 1.1.6 (**convergence as immersed, stratified surfaces**)**.** Let (S, η_k) be a sequence of compact Riemann surfaces and $\phi_k \in W^{2,2}(S, M)$ a sequence of branched, conformal immersions with conformal factors $e^{2\lambda_k}$. Let (S_∞, η) be a stratified surface with singular set P and let $\phi \in W^{2,2}(S_\infty, M)$ be a branched, conformal immersion with branch points B. We say (S, η_k, ϕ_k) converges to (S_∞, η, ϕ) as immersed, stratified surfaces, if for all $k \in \mathbb{N}$ we can find open sets $U_k \subset S$ and $V_k \subset S_\infty$ such that

1. $V_k \subset V_{k+1}$ and $P = S_\infty \setminus \bigcup_{k=1}^{\infty} V_k$. Moreover, $S_\infty \setminus V_k$ is a union of topological discs with finitely many smaller discs removed.
2. $S \setminus U_k$ is a smooth surface with boundary and $\phi_k(S \setminus U_k)$ converges to $\phi(P)$ in Hausdorff distance.
3. $\phi_k(S)$ converges to $\phi(S_\infty)$ in Hausdorff distance.
4. There is a sequence of diffeomorphisms $\psi_k : V_k \to U_k$ such that $\phi_k \circ \psi_k \rightharpoonup \phi$ weakly in $W^{2,2}(K, M)$.
5. The metrics $\psi_k^* \eta_k$ converge smoothly to η.

Further, let $(S = \bigcup_{i=0}^{m} S_i, \eta_k)$ be a sequence of stratified surfaces and $\phi_k \in W^{2,2}(S, M)$ a sequence of branched, conformal immersions. We say the sequence (S, η_k, ϕ_k) converges to (S_∞, η, ϕ) as immersed, stratified surfaces if $\left(S_i, \eta_k|_{S_i}, \phi_k|_{S_i}\right)$ converges to $(S_i^\infty, \eta_i, \phi^i)$ as immersed, stratified surfaces for all $i \in \{0, ..., m\}$ and $S_\infty = \bigcup_{i=0}^{m} S_i^\infty$, $\eta|_{S_i^\infty} = \eta_i$ and $\phi|_{S_i^\infty} = \phi^i$.

Now we are in the position to introduce the central definition of generalized Willmore surfaces.

Definition 1.1.7 (**generalized Willmore functional**)**.** Let S be a closed stratified surface and let (M, g) be an oriented n-dimensional Riemannian manifold. For $\phi \in \mathcal{F}(S, M)$ with conformal factor $e^{2\lambda}$ denote the Hessian by $\nabla d\phi$. Let $\{e_1, e_2\}$ be a local orthonormal frame on TS and let $\{\nu_i\}_{i=1}^{n-2}$ be a local orthonormal frame of the normal bundle NS.

1. A branched conformal immersion $\phi \in \mathcal{F}(S, M)$ is said to solve a *generalized Willmore equation* (away from the branch points) if

$$\Delta_\perp \vec{H} + \sum_{i,j=1}^{2} g(\vec{A}(e_i, e_j), \vec{H}) \vec{A}(e_i, e_j) - |\vec{H}|^2 \vec{H} + F(\phi, d\phi, \nabla d\phi) = 0. \qquad (1.1.1)$$

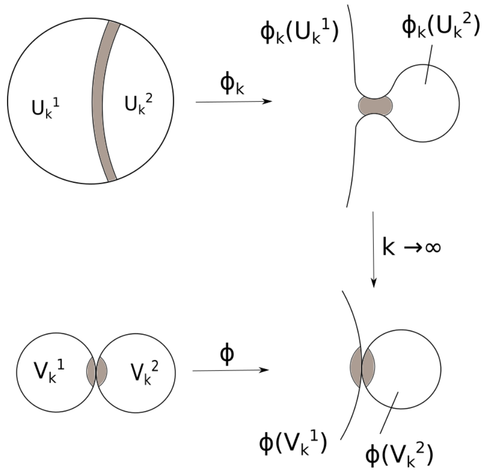

Figure 1.3: Side view of bubbling on a sphere

Here $\Delta_\perp$ denotes the Laplace operator on NS and $F(\phi, d\phi, \nabla d\phi) : S \to NS$ is such that locally in conformal coordinates with $\lambda \in L^\infty$ and $F = F^i \nu_i$ we have

$$e^{2\lambda} F^i(\phi, d\phi, \nabla d\phi) \in L^1 + W^{-1,2}(S) \quad \text{if } \phi \in \mathcal{F}(S, M)$$
$$e^{2\lambda} F(\phi, d\phi, \nabla d\phi) \in W^{k-1,l}(S, NS),\ l := \frac{2p}{2+p} + \epsilon \quad \text{if } \phi \in W^{k+2,p},\ p > 2,\ k \geq 0$$

for some $\epsilon > 0$.

2. A functional $\mathcal{H}$ on $\mathcal{F}(S, M)$ is called an *a-generalized Willmore functional* if
   (a) for any $\phi \in \mathcal{F}_a(S, M)$ a bound $\mathcal{H}[\phi] \leq \Lambda$ implies a bound on the Willmore energy $\mathcal{W}[\phi] \leq C(\Lambda, a, M, \mathcal{H})$.
   (b) $\mathcal{H}$ is bounded from below on $\mathcal{F}_a(S, M)$.
   (c) $\mathcal{H}$ is invariant under diffeomorphisms of S.
   (d) Let $\{\phi_k\}$ be a sequence in $\mathcal{F}(S, M)$ with conformal factors $e^{2\lambda_k}$. For any finite set $\mathfrak{S} \subset S$ the weak convergence $\phi_k \rightharpoonup \phi$ in $W^{2,2}_{\text{loc}}(S \setminus \mathfrak{S}, M)$ together with $\|\lambda_k\|_{L^\infty(K)} \leq C_K$ for any $K \subset\subset S \setminus \mathfrak{S}$ implies $\mathcal{H}[\phi] \leq \lim_{k\to\infty} \mathcal{H}[\phi_k]$.
   (e) $\mathcal{H}$ is differentiable and its Euler–Lagrange equation is a generalized Willmore equation.

   If a functional is an a-generalized Willmore functional for all $a > 0$ or if the area in question is understood we will simply refer to them as generalized Willmore functionals.

Note.

1. The generalized Willmore equation is based on the Euler–Lagrange equation of the Willmore functional which reads

$$\Delta_\perp \vec{H} + \sum_{i,j=1}^{2} g(\vec{A}(e_i, e_j), \vec{H}) \vec{A}(e_i, e_j) - |\vec{H}|^2 \vec{H} - \sum_{i=1}^{2} \left(\mathrm{Rm}^M(\vec{H}, e_i) e_i \right)^\perp = 0.$$

2. If a generalized Willmore equation is induced by a generalized Willmore functional, it will of necessity only depend on invariant quantities, that is $F(\phi, \nabla\phi, \nabla d\phi) =$

$\tilde{F}(y, G, \vec{A})$, where $y \in \phi(S)$ and G is the Gauss map of $\phi(S)$. If $\phi \in W^{k+2,p} \cap W^{1,\infty}$ is a branched, conformal immersion, we have, away from the branch points, $\lambda \in L^\infty$, $e^{2\lambda}, G \in W^{k+1,p} \cap L^\infty$ and $\vec{A} \in W^{k,p}$.

3. The area-constrained variations of any generalized Willmore functional yields a generalized Willmore equation as well since for a normal variation we find $\delta_X \mathcal{A}[\Sigma] = \int_\Sigma g(\vec{H}, X)\,\mathrm{d}\mu$ and $\vec{H}$ obeys the conditions of the first part of Definition 1.1.7.

We will need the following consequence of the uniformisation theorem for Riemann surfaces in order to cite a result from [3]. It serfs to fix preferred metrics on Riemann surfaces.

Theorem 1.1.8 (see [20, Chapter 1])**.** *Let (S, η) be a compact Riemann surface, then S is conformal to a sphere, a torus or a surface of higher genus with constant Gauss curvature 1, 0 or -1 respectively. Moreover, if S is a sphere, then any two metrics are conformal and there is only one with Gauss curvature 1. If (S, η) is a torus, then it is conformal to $\mathbb{C}/(1, a+bi)$ where $-\frac{1}{2} < a \leq \frac{1}{2}$, $b \geq 0$, $a^2 + b^2 \geq 1$ and $a \geq 0$ provided $a^2 + b^2 = 1$.*

1.2 Examples

In this section we prove that the two types of functionals we saw in the introduction are indeed generalized Willmore functionals.

1.2.1 Hawking Type Functionals

General relativity is a quite intricate theory, see for instance [55]. The important part for us is that it is modeled on a foliated Lorentz manifold. Let (N^4, h) be a four dimensional Lorentz manifold and let (M^3_t, g_t) be an oriented, space like foliation of N^4 which we interpret as equal time slices. That is for every $t \in \mathbb{R}$, (M_t, g_t) is a Riemannian manifold, where g_t is the restriction of h to M_t. We will focus on a given leaf and thus drop the t dependence. The second fundamental form K of M in N is given by

$$K(X, Y) := h(\nabla^N_X n, Y),$$

where X and Y are vector fields of M and n is the (time like) normal vector of M. Further, consider an immersed Riemann surface $\phi : S \to \Sigma \subset M$ with induced metric γ.

Analogous to the mean curvature of Σ in M, we define the mean curvature of Σ with respect to K to be

$$P := \operatorname{tr}_\Sigma K = \operatorname{tr}_M K - K(\nu, \nu).$$

Then the mean curvature vector of Σ in N is given by $\vec{H} := H\nu + Pn$.

The Hawking energy of Σ is defined as

$$\begin{aligned}\mathcal{E}[\Sigma] &= \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma |\vec{H}|_h^2\,\mathrm{d}\mu\right)\\ &= \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma H^2 - P^2\,\mathrm{d}\mu\right).\end{aligned}$$

Clearly, minimizing the functional $\int_\Sigma |\vec{H}|_h^2\,\mathrm{d}\mu$ under area constraint amounts to maximizing the Hawking energy under area constraint. Here we take a more general approach and investigate *Hawking type* functionals of the following form. Let $L \in C^\infty(TM)$ be given and define

$$\mathcal{H}[\Sigma] := \mathcal{H}_L[\Sigma] := \mathcal{W}[\Sigma] + \int_\Sigma L(x,\nu)\,\mathrm{d}\mu,$$

where $\mathcal{W}[\Sigma]$ is the the Willmore functional

$$\mathcal{W}[\Sigma] := \frac{1}{4}\int_\Sigma H^2 d\mu.$$

Theorem 1.2.1. *Let (M,g) be a Riemannian manifold and let $\mathcal{H}_L$ be of Hawking type. Suppose that L is smooth and bounded by C_L. Then $\mathcal{H}$ is a generalized Willmore functional.*

Proof. Let $\phi : S \to \Sigma \subset M$ be a closed, branched, immersed Riemann surface with area a and $\mathcal{H}[\Sigma] \leq \Lambda$. We have

$$\mathcal{H}[\Sigma] = \mathcal{W}[\Sigma] + \int_\Sigma L(x,\nu)\,\mathrm{d}\mu > -C_L)a,$$

and

$$\mathcal{W}[\Sigma] \leq \Lambda - \int_\Sigma L(x,\nu)\,\mathrm{d}\mu \leq \Lambda + C_L a.$$

$\mathcal{H}$ is invariant under reparametrisations of Σ as L is defined on TM and $\mathcal{W}[\Sigma]$ is invariant as well.

It is known that the Willmore energy is lower semi continuous in this setting (see for instance [43, Lemma A.8]). Thus we only need to discuss the lower order terms.

The convergence in $W^{2,2}_{\text{loc}}$, $\phi_k \to \phi$ implies local convergence in $W^{1,q}$ for all $1 \leq q < \infty$. Thus we have point wise convergence almost everywhere of $\nabla\phi_k \to \nabla\phi$, $\phi_k \to \phi$ and hence of $\nu_k \to \nu$. Since L is smooth, dominated convergence yields that $\mathcal{H}$ is lower semi continuous. We examine the Euler–Lagrange equation of $\mathcal{H}$ in the subsequent lemmas. $\square$

The variation of the Willmore energy has been calculated in numerous instances, for example in [30]. We will reiterate it here for completeness. Let $\phi : S \to \Sigma \subset M$ be a closed immersed Riemann surfaces in $W^{2,2}$ an immersion such that $\mathcal{H}[\phi] < \infty$.

Set $\phi(S) = \Sigma$ and consider a normal variation of ϕ along the vector field $f\nu$, $f \in C^\infty(\Sigma)$. By abuse of notation we will call this variation ϕ as well, that is

$$\begin{aligned} \phi : I \times S &\to M \\ (s,x) &\mapsto \phi(s,x) = \phi_s(x) \end{aligned}$$

such that for every $s \in I$, $\phi_s(S)$ is an immersed surface in M, $\phi_0(S) = \phi(S) = \Sigma$ and $\left.\frac{\partial\phi}{\partial s}\right|_{s=0} = f\nu$.

Proposition 1.2.2. *Subjected to the variation ϕ_s the geometric quantities behave in the* *following way.*

$$\begin{aligned}
\frac{\partial}{\partial s}\Big|_0 \gamma_{ij} &= 2fA_{ij} \\
\frac{\partial}{\partial s}\Big|_0 \gamma^{ij} &= -2fA^{ij} \\
\frac{\partial}{\partial s}\Big|_0 d\mu &= fHd\mu \\
\frac{\partial}{\partial s}\Big|_0 \nu &= -\nabla^\Sigma f \\
\frac{\partial}{\partial s}\Big|_0 A_{ij} &= -\nabla_i\nabla_j f + f\left(A_iA_i^k - R^M(\partial_i,\nu,\nu,\partial_j)\right) \\
\frac{\partial}{\partial s}\Big|_0 H &= -\Delta f - f\left(|A|^2 + \mathrm{Ric}^M(\nu,\nu)\right) \\
\frac{\partial}{\partial s}\Big|_0 L(\phi_s,\nu_s) &= d_{TM}L(f\nu, -\nabla^\Sigma f) \\
&= fd_ML(\nu) - d_VL(\nabla^\Sigma f)
\end{aligned} \tag{1.2.1}$$

Here d_{TM} denotes the exterior differential of TM, and $d_Mh(X) := d_{TM}h(X,0)$ and $d_Vh(X) := d_{TM}h(0,X)$ for a vector field X on M and $h \in C^1(TM)$.

$$\mathrm{D}_{\lambda,g,L}(\phi) = \Delta H + H|\mathring{A}|^2 + HQ + \gamma(\mathring{A}, S) + 2\lambda H + T = 0. \tag{1.2.2}$$

Here Q, S and T are defined as

$$\begin{aligned}
Q &:= \mathrm{Ric}^M(\nu,\nu) - 2L - \mathrm{tr}_\Sigma \mathrm{Hess}_V L + 2d_VL(\nu) \\
S &:= -2\,\mathrm{Hess}_V L \\
T &:= -2d_ML(\nu) - 2\,\mathrm{div}_\Sigma d_VL,
\end{aligned}$$

where $\mathrm{Hess}_V L$ is the fiber part of the Hessian of L and $\mathrm{div}_\Sigma d_VL = \mathrm{tr}_\Sigma \nabla^M d_VL$ with $\nabla^M d_VL(X,Y) = \nabla^{TM}_{(X,0)} d_{TM}L(0,Y)$.

If L is smooth then 1.2.2 is a generalized Willmore equation.

Proof. The variation of the geometric quantities is widely known, see for instance [18, Theorem 3.2, Section 7] and the variation of L is straightforward.

The variational problem reads $\lambda\delta_f\mathcal{A}(\Sigma) = \delta_f\mathcal{W}(\Sigma) + \delta_f\mathcal{L}(\Sigma)$. We treat all terms separately.

$$\begin{aligned}
\frac{d}{ds}\Big|_0 \mathcal{A}[\Sigma] &= \int_\Sigma fH\,\mathrm{d}\mu \\
\frac{d}{ds}\Big|_0 \mathcal{W}[\Sigma] &= \frac{1}{2}\int_\Sigma H\frac{\partial H}{\partial s}\Big|_0 + f\frac{H^3}{2}\,\mathrm{d}\mu \\
&= \frac{1}{2}\int_\Sigma H(-\Delta f - f\left(|\mathring{A}|^2 + \mathrm{Ric}^M(\nu,\nu)\right))\,\mathrm{d}\mu \\
&= \frac{1}{2}\int_\Sigma -f\left(\Delta H + H|\mathring{A}|^2 + H\,\mathrm{Ric}^M(\nu,\nu)\right)\mathrm{d}\mu \\
\frac{d}{ds}\Big|_0 \mathcal{L}[\Sigma] &= \int_\Sigma \frac{\partial L}{\partial s}\Big|_0 + fLH\,\mathrm{d}\mu \\
\int_\Sigma d_VL(\nabla^\Sigma f)\,\mathrm{d}\mu &= \int_\Sigma \mathrm{div}_\Sigma(fd_VL) - f\,\mathrm{div}_\Sigma\left(d_VL_{(x,\nu)}\right)\mathrm{d}\mu \\
&= -\int_\Sigma f\left((\mathrm{div}_\Sigma d_VL)_{(x,\nu)} + \gamma((\mathrm{Hess}_V L)_{(x,\nu)}, A) - Hd_VL(\nu)\right)\mathrm{d}\mu
\end{aligned}$$

Sorting all the terms yields the desired equation.

$$\Delta H + H|\mathring{A}|^2 + HQ + \gamma(\mathring{A}, S) + 2\lambda H + T = 0$$

In the notation of generalized Willmore equations we have $F = HQ + \gamma(\mathring{A}, S) + 2\lambda H + T \in L^2$, provided $\phi \in \mathcal{F}(S, M)$.

For the higher order regularity note that the worst term of $|\nabla\phi|^2 F\nu$ is of the form $|\nabla\phi|^2 \star A \star \psi(\phi,\nu) \star \nu \star \nu \star \nu$, where the $\star$ denotes a sum of contractions, and $\psi : TM \to \mathbb{R}$ is a smooth and bounded function. If $\phi \in W^{k+2,p} \cap W^{1,\infty}$, $k \geq 0$, $p \geq 2$, then, due to the Sobolev embeddings $W^{k+2,p} \hookrightarrow W^{k+1,q} \hookrightarrow C^{k,\alpha}$, for all $1 \leq q < \infty$ and $\alpha \in (0,1)$, we have $|\nabla\phi|^2 \star \nu \star \nu \star \nu \in W^{k+1,p} \cap L^\infty$. Similarly, $\psi(\phi,\nu) \in W^{k+1,p} \cap L^\infty$. This means, due to the A, component we have $|\nabla\phi|^2 F\nu \in W^{k,p}$, whenever $\phi \in W^{k+2,p} \cap W^{1,\infty}$. □

We return to the Hawking Energy or rather to the associated generalized Willmore functional.

Lemma 1.2.3. *Let X and Y be a vector fields along Σ and introduce the 1-form $\eta := K(\cdot,\nu)$ as well as the musical isomorphism $\# : T^*\Sigma \to T\Sigma$ then the following equations hold.*

$$\begin{aligned}
d_M P^2(X) &= 2P \operatorname{tr}_\Sigma \nabla^M_X K \\
d_V P^2(X) &= -4PK(X,\nu) \\
d_M \left[d_V P^2(X) \right](Y) &= -4 \left(tr_\Sigma \nabla^M_Y K \right) K(X,\nu) - 4P\nabla^M_Y K(X,\nu) \\
\operatorname{Hess}_V P^2 &= 8\eta \otimes \eta - 4PK \\
\operatorname{div}_\Sigma d_V P^2 &= -4 \operatorname{tr}_\Sigma \nabla^M_{\eta^\#} K - 4P \operatorname{div}_\Sigma \eta
\end{aligned}$$

Moreover, the area-constrained Euler–Lagrange equation for the Hawking type functional with $L = -\frac{1}{4}P^2$ reads

$$\Delta H + H|\mathring{A}|^2 + HQ + f\gamma(\mathring{A}, S) + 2\lambda H + T = 0.$$

Q, S and T are given by

$$\begin{aligned}
Q &= \operatorname{Ric}^M(\nu,\nu) - \frac{1}{2}P^2 + 2|\eta|^2 + 2PK(\nu,\nu) \\
S &= -2PK + 4\eta \otimes \eta \\
T &= P \operatorname{tr}_\Sigma \nabla_\nu K - 2 \operatorname{tr}_\Sigma \nabla_{\eta^\#} K - 2P \operatorname{div}_\Sigma \eta.
\end{aligned}$$

Proof. The first equation is clear because the metric is parallel. For the second we use $P = \operatorname{tr} K - K(\nu,\nu)$ and find that

$$d_V P(X) = -2K(X,\nu)$$

as K is bilinear and $\operatorname{tr} K$ does not depend on the fiber. Now the rest follows.

$$\begin{aligned}
d_V P^2(X) &= -4PK(X,\nu) \\
d_M \left[d_V P^2(X) \right](Y) &= -4 \left(\operatorname{tr}_\Sigma \nabla^M_Y K \right) K(X,\nu) - 4P\nabla^M_Y K(X,\nu) \\
\operatorname{Hess}_V P^2 &= 2d_V(-2PK(\cdot,\nu)) \\
&= 8K(\cdot,\nu) \otimes K(\cdot,\nu) - 4PK \\
\operatorname{div}_\Sigma d_V P^2 &= -4 \operatorname{div}_\Sigma P\eta \\
&= -4P \operatorname{div}_\Sigma \eta - 4\gamma^{ij} \operatorname{tr}_\Sigma(\nabla^M_i K)\eta_j \\
&= -4P \operatorname{div}_\Sigma \eta - 4 \operatorname{tr}_\Sigma \nabla^M_{\eta^\#} K
\end{aligned}$$

The Euler–Lagrange equation is obtained from the general formula (1.2.2). □

1.2.2 Thin Membranes

We model a membrane as a branched, immersed, stratified surface $\Sigma \subset (\mathbb{R}^3, \langle\cdot,\cdot\rangle)$ with a parametrization $\phi \in \mathcal{F}(S, \mathbb{R}^3)$ that minimizes the following bending energy under area and volume constraints.

$$\mathcal{H}[\Sigma] := \mathcal{H}_{c,b}[\Sigma] := \int_\Sigma (H + c)^2 \,\mathrm{d}\mu + b \left(\int_\Sigma H \,\mathrm{d}\mu \right)^2$$

Here, c, the spontaneous curvature, is a function on Σ, and b is a constant. The first part of this energy corresponds to the one proposed by Helfrich and the second part is a non local generalization. See [53] for a relatively recent review of membranes as elastic materials.

In the case that Σ is the smooth boundary of a domain Ω, we use the divergence formula to rewrite the volume of Ω. Let x be the position vector field in $\mathbb{R}^3$ and let $x_0 \in \mathbb{R}^3$, then $\mathrm{Vol}(\Omega) = \frac{1}{3}\int_\Sigma \langle x - x_0, \nu\rangle \,\mathrm{d}\mu$, independently of the choice of x_0. This motivates the introduction of the functional

$$\mathcal{V}[\phi] := \frac{1}{3}\int_\Sigma \langle x - x_0, \nu\rangle \,\mathrm{d}\mu$$

on $\mathcal{F}(S, \mathbb{R}^3)$. It is still independent of the base point x_0, which also implies it is invariant under translations. This follows from the fact that any variation of $\tilde{\mathcal{V}}[\phi] = \int_\Sigma \langle x_0, \nu\rangle \,\mathrm{d}\mu$ vanishes. This in turn relies on the divergence formula for vector fields on Σ (see [35, Section 2]). Considering the variation of ϕ induced by scaling, $\phi_t(x) = (1 + t)\phi(x)$, yields

$$0 = \left.\frac{d}{dt}\right|_{t=0} \tilde{\mathcal{V}}[\phi_t] = \left.\frac{d}{dt}\right|_{t=0} (1 + t)^2 \tilde{\mathcal{V}}[\phi] = 2\tilde{\mathcal{V}}[\phi].$$

Hence, a volume constraint for $\phi \in \mathcal{F}(S, \mathbb{R}^3)$ is defined to be a constraint on $\mathcal{V}[\phi]$.

Proposition 1.2.4. *In the setting from above suppose that $c : \mathbb{R}^3 \to \mathbb{R}$ is smooth and bounded. If $-ba < 1$, then $\mathcal{H}$ is an a-generalized Willmore functional. Its Euler–Lagrange equation with area and volume constraints reads*

$$\begin{aligned}\Delta H + H|\mathring{A}|^2 + H \operatorname{Ric}(\nu,\nu) &= \frac{1}{2}Hc^2 + H^2 c + (H + 2c)dc(\nu) - \operatorname{tr}_\Sigma \operatorname{Hess}^M c - \frac{1}{2}\lambda H - \frac{1}{2}p \\ &\quad - \left(c + b\int_\Sigma H \,\mathrm{d}\mu \right)(|\mathring{A}|^2 + \operatorname{Ric}(\nu,\nu)) + \frac{1}{2}bH^2 \int_\Sigma H \,\mathrm{d}\mu.\end{aligned}$$

Here λ and p are the Lagrange parameters for the area and the enclosed volume respectively.

Proof. Let S be a stratified surface with singular set P, let $\phi \in \mathcal{F}(S, \mathbb{R}^3)$ and $\phi(S) = \Sigma$. Suppose that $\mathcal{H}[\Sigma] \le \Lambda$ and $|\Sigma| = a$, then

$$\begin{aligned}4\,\mathcal{W}[\Sigma] &\le \Lambda - \int_\Sigma 2Hc + c^2 \,\mathrm{d}\mu - b\left(\int_\Sigma H \,\mathrm{d}\mu\right)^2 \\ &\le \Lambda + C(c)a^{1/2}\,\mathcal{W}[\Sigma]^{1/2} - b\left(\int_\Sigma H \,\mathrm{d}\mu\right)^2.\end{aligned}$$

If $b \ge 0$, then we omit the last term. Solving the quadratic inequality yields

$$\mathcal{W}[\Sigma] \le C(\Lambda, c, a).$$

If $b < 0$, we estimate further:

$$4\,\mathcal{W}[\Sigma] \le \Lambda + C(c)a^{1/2}\,\mathcal{W}[\Sigma]^{1/2} + 4|b|a\,\mathcal{W}[\Sigma].$$

If $|b|a < 1$ we absorb the last term to the left and find

$$\mathcal{W}[\Sigma] \leq C(\Lambda, c, b, a).$$

Similarly, we obtain an estimate from blow. If $b \geq 0$ then $\mathcal{H}$ is positive, so suppose $b < 0$ and $|b|a \geq 1 - \epsilon$, for an $\epsilon \in (0,1)$.

$$\begin{aligned}
\mathcal{H}[\Sigma] &= \int_\Sigma (H+c)^2 \,\mathrm{d}\mu + b\left(\int_\Sigma H \,\mathrm{d}\mu\right)^2 \\
&\geq \int_\Sigma 2Hc + c^2 \,\mathrm{d}\mu + 4(1-|b|a)\,\mathcal{W}[\Sigma] \\
&\geq \left(1 - \frac{1}{\epsilon}\right)\int_\Sigma c^2 \,\mathrm{d}\mu + 4(1-\epsilon-|b|a)\,\mathcal{W}[\Sigma] \\
&\geq \left(1 - \frac{1}{\epsilon}\right) C(c)a
\end{aligned}$$

Since H is a geometric quantity and c is a function on $\mathbb{R}^3$, $\mathcal{H}$ is invariant under reparametrizations.

To show that $\mathcal{H}$ is lower semi continuous under weak $W^{2,2}_{\mathrm{loc}}(S \setminus P, \mathbb{R}^3)$ convergence, let $\phi_k \in \mathcal{F}(S, \mathbb{R}^3)$ be a sequence of conformal maps with conformal factor $e^{2\lambda_k}$ and let $K \subset\subset S \setminus P$ such that

$$\begin{aligned}
&\phi_k \rightharpoonup \phi \quad \text{weakly in } W^{2,2}(K, \mathbb{R}^3) \quad \text{and} \\
&\|\lambda_k\|_{L^\infty(K)} \leq C_K
\end{aligned} \tag{1.2.3}$$

We already know that the Willmore energy is lower semi continuous under these conditions, see [43, Lemma A.8]. Since the weak $W^{2,2}$ convergence implies strong $W^{1,p}$ convergence, we have that $\phi_k \to \phi$ and $\nabla\phi_k \to \nabla\phi$ pointwise almost everywhere. Dominated convergence and smoothness of c yields

$$\int_K c^2 \circ \phi_k \,\mathrm{d}\mu_{\phi_k} \to \int_K c^2 \circ \phi \,\mathrm{d}\mu_\phi.$$

Weak $W^{2,2}$ convergence and the uniform upper and lower bounds on the conformal factor (1.2.3) imply that $H_k e^{2\lambda_k} \to He^{2\lambda}$ and $H_k e^{2\lambda_k} c \circ \phi_k \to He^{2\lambda} c \circ \phi$ weakly in L^2. Testing with 1 yields claim.

Let $\phi : I \times S \to \mathbb{R}^3$ be a normal variation of Σ with $\frac{\partial}{\partial s}\big|_0 \phi = f\nu$. The behavior of the geometric quantities is widely known, see for instance [18, Section 7]. To derive the Euler–Lagrange equation with area and volume constraints we need to calculate $\delta_f \mathcal{H} = \lambda\delta_f \mathcal{A} + p\delta_f \mathcal{V}$.
In particular, we find $\delta_f \mathcal{V}[\phi] = \int_\Sigma f \,\mathrm{d}\mu$. In terms of generalized Willmore equations we have

$$\Delta H + H|\mathring{A}|^2 + F = 0$$

for

$$\begin{aligned}
F = H \operatorname{Ric}(\nu,\nu) &- \frac{1}{2}Hc^2 - H^2 c - (H+2c)dc(\nu) + \operatorname{tr}_\Sigma \operatorname{Hess}^M c + \frac{1}{2}\lambda H + \frac{1}{2}p \\
&+ \left(c + b\int_\Sigma H \,\mathrm{d}\mu\right)(|\mathring{A}|^2 + \operatorname{Ric}(\nu,\nu)) - \frac{1}{2}bH^2 \int_\Sigma H \,\mathrm{d}\mu
\end{aligned}$$

If $\phi \in W^{2,2} \cap W^{1,\infty}$ is a conformal parametrization of Σ, then $|\nabla\phi|^2 F \in L^1$.

In terms of higher regularity, worst term in F is of the form $|\nabla\phi|^2 c \circ \phi |\mathring{A}|^2 \nu$. If $\phi \in W^{k+2,p} \cap W^{1,\infty}$, $k \geq 0$, $p > 2$, then, due to the Sobolev embeddings $W^{k+2,p} \hookrightarrow W^{k+1,q} \hookrightarrow C^{k,\alpha}$, for all $1 \leq q < \infty$ and some $\alpha \in (0,1)$, we have $|\nabla\phi|^2\nu \in W^{k+1,p}$ and $c \circ \phi \in W^{k+2,p}$. Due to the $|\mathring{A}|^2$ part we find $|\nabla\phi|^2 F\nu \in W^{k,p/2}$. □

Chapter 2

Existence of Generalized Willmore Surfaces

In this chapter we prove the existence of area-constrained generalized Willmore surfaces by constructing area-constrained minimizers via direct minimization. We would like to perform the minimization in the class of branched, conformal immersions of bubble forests but due to the compactness results in [3] we realize the need to introduce haunted immersions; see Definition 2.0.3.

In order to make use of the results on Willmore surfaces in Euclidean space, we briefly recall how the Willmore energy of a closed surface $\Sigma \hookrightarrow M \hookrightarrow \mathbb{R}^N$ with respect to $\mathbb{R}^N$ is controlled by its Willmore energy with respect to M and its area. Here we assume that the target Riemannian manifold manifold (M, g) has been isometrically embedded in some $\mathbb{R}^N$, $\dim M < N$. This can always be achieved via Nash embedding.

Introduce the second fundamental form and the mean curvature vector of Σ in $\mathbb{R}^N$ as $\bar{A}$ and $\bar{H}$ respectively. The second fundamental form of M in $\mathbb{R}^N$ is denoted by K. We have $\bar{A} = \vec{A} + \vec{K}$ as well as $\bar{H} = \vec{H} + \vec{P}$, for $\vec{P} := \operatorname{tr}_\Sigma \vec{K}$. Since Σ is compact, we easily see

$$\int_\Sigma |\bar{H}|^2 \, \mathrm{d}\mu \leq \int_\Sigma |\vec{H}|^2 \, \mathrm{d}\mu + \sup_\Sigma |\vec{P}|^2 |\Sigma|. \tag{2.0.1}$$

In [3] J. Chen and Y. Li proved a Gauss-Bonnet formula for closed branched conformal immersions.

Lemma 2.0.1 (see [3, Lemma 3.2])**.** *If $\phi \in \mathcal{F}(S, \mathbb{R}^n)$ and $\phi(S) = \Sigma$ then*

$$\int_\Sigma \mathrm{Sc}_\Sigma \, \mathrm{d}\mu = 8\pi(1 - q(S)) + 4\pi b.$$

Here $q(S)$ is the genus of S and b is number of branch points counted with multiplicity.

Integrating over the Gauss equation yields

$$2\pi b \leq \frac{1}{4} \int_\Sigma |\bar{H}|^2 \, \mathrm{d}\mu - 4\pi(1 - q(\Sigma)). \tag{2.0.2}$$

In the same paper J. Chen and Y. Li proved a powerful compactness result for $W^{2,2}$ branched, conformal immersions, which is the heart of our existence results.

Theorem 2.0.2 (see [3, Theorem 1])**.** *Let (S, η_k) be a sequence of closed Riemann surfaces with metrics as given by Theorem 1.1.8 and let $\phi_k \in W^{2,2}((S, \eta_k), \mathbb{R}^n)$ be a sequence of* *branched conformal immersions, for some $n > 2$. If $\phi_k(S) \cap B_{R_0} \neq \emptyset$ for some $R_0 > 0$, and if there are positive constants a and Λ such that*

$$\mathcal{A}[\phi_k] \leq a$$
$$\mathcal{W}[\phi_k] \leq \Lambda$$

for all $k \in \mathbb{N}$, then Σ_k either converges to a point or there is a stratified surface (S_∞, η) and a branched, conformal immersion $\phi \in W^{2,2}(S_\infty, \mathbb{R}^n)$ such that a subsequence of (S, η_k, ϕ_k) converges to (S_∞, η, ϕ) in the sense of immersed, stratified surfaces.
Moreover,

$$\mathcal{A}[\phi] = \lim_{k\to\infty} \mathcal{A}[\phi_k],$$
$$\mathcal{W}[\phi] \leq \lim_{k\to\infty} \mathcal{W}[\phi_k].$$

Remark 1. From the proof of the theorem we learn more about the the convergence and the structure of S_∞. It is obtained by attaching finitely many bubble trees to a stratified surface T. This base stratified surface T in turn is formed as the limit of the (S, η_k) as nodal surfaces with possibly some bubbling at the nodal points. Additionally, if the η_k smoothly converge to a smooth metric η on S, then $T = (S, \eta)$ and S_∞ is a bubble forest with base S. In this case, if $U_k \subset S$ are the open sets guaranteed by the convergence as immersed, stratified surfaces, we have $\lim_{k\to 0} |\phi_k(S \setminus U_k)| = 0$.

Furthermore, let P be the singular points of S_∞. We may assume that the branch points of the sequence $\phi_k \circ \psi_k$ converge to a finite set $\tilde{B}$ and that there is a finite set $\mathfrak{S} \subset S_\infty \setminus P$ such that the conformal factors $e^{2\tilde{\lambda}_k}$ corresponding to $\phi_k \circ \psi_k$ obey $\|\tilde{\lambda}_k\|_{L^\infty(K\cap V_k)} \leq C_K$ for all $K \subset\subset S \setminus (P \cup \mathfrak{S} \cup \tilde{B})$; cf. [3, Proof of Theorem 1, Page 30].

Since the convergence as immersed, stratified surfaces leaves the class of surfaces, we need to formulate a compactness theorem for stratified surfaces. Ultimately, our goal is the minimization of a generalized Willmore functional over a surface S_0, hence we restrict ourselves to bubble forests with base S_0. The idea is then to apply Theorem 2.0.2 to every part of the bubble forest. Unfortunately, this means that parts of the forest can vanish, even though the whole forest cannot due to constraints. This would destroy the tree structure of the dual graph and hence leave the class of immersed bubble forests. To remedy this we introduce ghost bubbles and haunted immersions.

Definition 2.0.3 (**haunted immersion**).

1. Let $S = \bigcup_{i=1}^m S_i$ be a stratified surface and let $\phi : S \to M$ be a continuous map into a manifold M. We say ϕ is a *haunted* immersion, if it is constant on some, but not all, components of S and an immersion on the rest. A component S_i is called a *ghost* if $\phi|_{S_i}$ is constant, otherwise it is called *regular*.

2. Let $(S = \bigcup_{i=1}^m S_i, \eta_k)$ be a sequence of compact, stratified surfaces and $\phi_k \in W^{2,2}(S, M)$ a sequence of haunted, branched, conformal immersions. Let (S_∞, η) be a stratified surface and let $\phi \in W^{2,2}(S_\infty, M)$ be a haunted, branched, conformal immersion. We say (S, η_k, ϕ_k) converge to (S_∞, η, ϕ) as haunted, immersed, stratified surfaces if

   (a) $(S_i, \phi_k|_{S_i})$ converges to a point x_i for some but not all i, setting $S_i^\infty = S_i$, $\phi^i = x_i$ and

   (b) the remaining $(S_i, \phi_k|_{S_i})$ converge to $(S_i^\infty, \eta_i, \phi^i)$ as immersed stratified surfaces such that

(c) $S_\infty = \bigcup_{i=0}^m S_i^\infty$, $\eta|_{S_i^\infty} = \eta_i$ and $\phi|_{S_i^\infty} = \phi^i$.

Suppose ϕ is a haunted immersion of a bubble forest S. If a ghost is connected to only one other component then we delete it. If a ghost is connected to two other components, say S_1 and S_2 with common points p_1 and p_2 respectively, we delete it as well and identify p_1 and p_2. Repeating this process until there are no ghosts left or until every ghost is connected to three or more components yields a bubble forest S' (possibly with a different base then S) and a haunted immersion ϕ' which is given by ϕ on every component of S'. A tuple (S', ϕ') obtained that way is called *irreducible*.

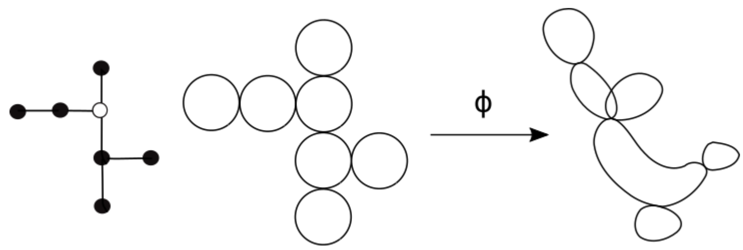

Figure 2.1: An immersed haunted bubble tree and its dual graph, where the ghost is drawn white.

The following lemma on graph coloring asserts that in an irreducible, haunted bubble forest the number of ghosts is bounded by the number of regular components.

Lemma 2.0.4. *Let G be a finite tree, colored in black and white according to the rule: a vertex can be white if its degree is bigger or equal to three. Then there are more black vertices then white ones.*

Proof. For a tree G let $W(G)$ be the number of white vertices and $B(G)$ be the number of black vertices. Note that any endpoint of G has to be black and the claim holds for trees with up to four vertices.

We argue by induction. Suppose the claim holds for trees with n and $n-1$ vertices. Let G be a tree with $n+1$ vertices and let p be a boundary vertex connected to q.

1. Suppose $\deg(q) = 2$, then q has to be black and $W(G) = W(G\setminus\{p\}) \leq B(G\setminus\{p\}) < B(G)$.

2. Suppose $\deg(q) = 3$ and q is black or $\deg(q) \geq 4$, then there is no need to recolor $q \in G \setminus \{p\}$ and we have $W(G) = W(G \setminus \{p\}) \leq B(G \setminus \{p\}) < B(G)$ as before.

3. Suppose $\deg(q) = 3$ and q is white. Let r and s be the other vertices adjacent to q and consider the tree $\tilde{G} = G \setminus \{p, q\}$ where we joined r and s. Then
$$W(G) = W(\tilde{G}) + 1 \leq B(\tilde{G}) + 1 = B(G).$$
□

The following is our central compactness theorem. We use it to prove the existence of area-constrained minimizers in the subsequent theorem.

Theorem 2.0.5. *Let (S^k, η_k) be a sequence of compact bubble forests with base S_0. Suppose η_k is as given by Theorem 1.1.8 on every component of S^k, which fixes η_k on every bubble. Suppose additionally that $\eta_k|_{S_0}$ converges smoothly to a smooth metric η' on S_0.*

Let $\phi_k \in W^{2,2}(S^k, \mathbb{R}^n)$ be a sequence of irreducible, haunted, branched, conformal immersions. Assume $\phi_k(S^k) \cap B_{R_0} \neq \emptyset$ for some R_0 and there are positive constants a and Λ such that

$$\mathcal{A}[\phi_k] \leq a$$
$$\mathcal{W}[\phi_k] \leq \Lambda$$

for all $k \in \mathbb{N}$. Then there exists a bubble forest $S = S_0 \cup \bigcup_{i=1}^m S_i$, a stratified surface $(\tilde{T} = \bigcup_{i=0}^{m'} T_i, \tilde{\eta})$ and a haunted, branched, conformal immersion $\tilde{\phi} \in W^{2,2}(\tilde{T}, \mathbb{R}^n)$, a bubble forest (T, η) (with base S_0 or a sphere) and an irreducible, haunted, branched, conformal immersion $\phi \in W^{2,2}(T, \mathbb{R}^n)$ such that $(\tilde{T}, \tilde{\eta}, \tilde{\phi})$ and (T, η, ϕ) differ only by ghosts and either

1) *ϕ_k converges to a point, or*

2) *there is, a subsequence of $\{\phi_k\}$ defined on (S, η_k) such that (S, η_k, ϕ_k) converges to $(\tilde{T}, \tilde{\eta}, \tilde{\phi})$ as haunted immersed stratified surfaces.*

Moreover, if $U_k \subset S$ are the open sets guaranteed by the convergence as haunted, immersed, stratified surfaces, then $\lim_{k\to 0} |\phi_k(S \setminus U_k)| = 0$ and

$$\mathcal{A}[\phi] = \lim_{k\to\infty} \mathcal{A}[\phi_k],$$
$$\mathcal{W}[\phi] \leq \lim_{k\to\infty} \mathcal{W}[\phi_k].$$

Proof. First, note that the number of regular components of S^k is uniformly bounded as they each have Willmore energy at least 4π and by Lemma 2.0.4 the total number of components is bounded. This means that there are only finitely many possible dual graphs along the sequence S^k and we can choose subsequences of S^k and ϕ_k such that they all agree. This means the S^k agree as topological spaces but differ by their metric and their singular points P^k. Call the underlying topological space $S = S_0 \cup \bigcup_{i=1}^m S_i$ and note that the number of branch points is bounded by (2.0.2).

For an $i \in \{0, 1, ..., m\}$ consider $(S_i, \phi_k|_{S_i})$, we apply Theorem 2.0.2 so that either it becomes a ghost, setting $\tilde{S}_i = S_i$ and $\phi^i(x) = \lim_{k\to\infty} \phi_k(S_i)$, or $(S_i, \phi_k|_{S_i})$ subconverges to $\left(\tilde{S}_i := S_i \cup \bigcup_{j=1}^{m_i} S_{i,j}, \phi^i\right)$ as immersed stratified surfaces, where $\tilde{S}_i$ is a bubble forest and $\phi^i \in W^{2,2}(\tilde{S}_i, \mathbb{R}^n)$ is a branched, conformal immersion.

Next, we track the singular set of points P^k. For any $p \in P^k$ there are two components S_i, S_j such that $p = S_i \cap S_j$, due to the tree structure of S.

We may assume that $\phi_k(p)$ converges to a point $y \in \mathbb{R}^n$. Since $\phi^i(\tilde{S}_i)$ is compact, the distance $d(\phi_k(p), \phi(\tilde{S}_i))$ is attained for a sequence of points $y_k \in \phi^i(\tilde{S}_i)$. Now, $\phi_k(S_i)$ converges to $\phi^i(\tilde{S}_i)$ in Hausdorff distance and we find $y_k \to y \in \phi^i(\tilde{S}_i)$ as $\phi^i(\tilde{S}_i)$ is closed. Choose $x_i \in S_i$ such that $\phi^i(x_i) = y$.

Since the same reasoning holds for $\phi_k|_{S_j}$, we find a $x_j \in \tilde{S}_j$ such that $\phi^j(x_j) = y$. This means we can join the two bubble trees together: define $\tilde{T}_{ij} = \tilde{S}_i \sqcup \tilde{S}_j/(x_i \sim x_j)$ and $\phi^{ij} : \tilde{T}_{ij} \to \mathbb{R}^n$, $\phi^{ij}|_{\tilde{S}_i} := \phi^i$, $\phi^{ij}|_{\tilde{S}_j} := \phi^j$. In this way we join up all the bubble trees to obtain a haunted, branched, immersed, stratified surface $(\tilde{T} = \bigcup_{i=0}^m \tilde{S}_i, \tilde{\phi})$, where $\tilde{\phi}$ is given by ϕ^i on $\tilde{S}_i$ and is continuous throughout. It is then clear that (S, η_k, ϕ_k) converges to $(\tilde{T}, \tilde{\eta}, \tilde{\phi})$ as haunted, immersed, stratified surfaces. Here $\tilde{\eta} = \eta$ on every bubble and $\tilde{\eta} = \eta'$ on S_0, provided $\phi_k(S_0)$ does not shrink to a point.

During this procedure it is possible to loose the tree structure, namely if two or more singular points of $\tilde{T}$ overlap which can only happen at the points constructed above. See Figure 2.2 for illustration.

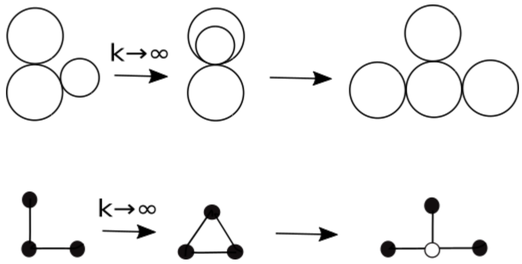

Figure 2.2: Introducing ghosts into a degenerating bubble tree

This is remedied by introducing a ghost. Let p be a singular point of $\tilde{T}$ such that $p \in \bigcap_{j=1}^{l} \tilde{S}_{i_j}$ for $l > 2$. Set $p_j := \{p\} \cap \tilde{S}_{i_j}$, take a sphere S' and l mutually distinct points $\{a_j\}$ on it. Define the stratified surface $\tilde{T}' := \tilde{T} \sqcup S'/\sim$ where we no longer identify the p_j but instead identify p_j with a_j and set $\tilde{\phi}'|_{\tilde{T}} = \tilde{\phi}$ and $\tilde{\phi}'|_{S'} = \tilde{\phi}(p)$.
Employing this method as often as necessary to obtain a tree and then deleting any unnecessary ghost yields the claim.

□

In terms of direct minimization the compactness result for haunted immersions of bubble forest puts the base in competition to the bubbles. Hence, we have to restrict to bubble trees, so as not to loose the base.

Let $\mathcal{T}$ be the class of bubble trees. For a Riemannian manifold (M, g) of dimension three or higher, a positive constant a and an a-generalized Willmore functional $\mathcal{H}$ set

$$\begin{aligned} \mathcal{F}(\mathcal{T}, M) &:= \{\phi \in \mathcal{F}(S, M) \mid S \in \mathcal{T},\, (S, \phi) \text{ is haunted}\}, \\ \mathcal{F}_a(\mathcal{T}, M) &:= \{\phi \in \mathcal{F}(\mathcal{T}, M) \mid \mathcal{A}[\phi] = a\}, \\ \beta(\mathcal{H}, M, a) &:= \inf\{\mathcal{H}[\phi] \mid \phi \in \mathcal{F}_a(\mathcal{T}, M)\}. \end{aligned}$$

Theorem 2.0.6. *Let (M, g) be a compact Riemannian manifold and let $\mathcal{H}$ be an a-generalized Willmore functional, then $\beta(\mathcal{H}, M, a)$ is attained in $\mathcal{F}_a(\mathcal{T}, M)$.*

Proof. Pick a sequence $\phi_k \in \mathcal{F}_a(\mathcal{T}, M)$ realizing $\beta(\mathcal{H}, M, a)$. By definition, $\mathcal{H}$ is bounded from below and along this sequence it is bounded from above. Again, by the definition of a-generalized Willmore functional, we know that the Willmore energy with respect to M is bounded. This implies

$$\mathcal{W}[\phi_k, \mathbb{R}^N] \le \mathcal{W}[\phi_k, M] + Ca \le \Lambda(M, a, \beta, \mathcal{H}).$$

After reducing the ϕ_k, if necessary, we are in the context of Theorem 2.0.5, as M is compact. Since the area is fixed, the sequence ϕ_k cannot shrink to a point. The convergence as haunted, immersed, stratified surfaces yields a limit $\phi \in \mathcal{F}_a(\mathcal{T}, M)$. By Remark 1 and Definition 1.1.7 we know that $\mathcal{H}$ is lower semi continuous with respect to this convergence, so we find

$$\beta(\mathcal{H}, M, a) \le \mathcal{H}[\phi] \le \lim_{k\to\infty} \mathcal{H}[\phi_k] = \beta(\mathcal{H}, M, a).$$

□

Corollary 2.0.7. *Let (M, g) be a non-compact Riemannian manifold with C_B bounded geometry and let $\mathcal{H}$ be an a-generalized Willmore functional. Suppose there exists a transitive group action on M that leaves $\mathcal{H}$ and $\mathcal{A}$ invariant, then $\beta(\mathcal{H}, M, a)$ is attained in $\mathcal{F}_a(\mathcal{T}, M)$.*

Proof. Pick a sequence $\tilde{\phi}_k \in \mathcal{F}_a(\mathcal{T}, M)$ realizing $\beta(\mathcal{H}, M, a)$ and use the transitive action to obtain a sequence $\phi_k \in \mathcal{F}_a(\mathcal{T}, M)$ that still realizes $\beta(\mathcal{H}, M, a)$ and whose images intersect a fixed point. If Σ_k is the image of ϕ_k then [29, Lemma 2.5] asserts

$$\operatorname{diam}_M(\Sigma_k) \leq C(C_B)\left(|\Sigma_k|^{1/2}\,\mathcal{W}[\Sigma_k, M]^{1/2} + |\Sigma_k|\right) \leq C\left(C_B, M, a, \beta, \mathcal{H}\right).$$

Since the diameter in $M \subset \mathbb{R}^N$ is larger then the one in $\mathbb{R}^N$, we conclude with Theorem 2.0.6. □

Theorem 2.0.8. *Let $\mathcal{H}_{c,b}$ and $\mathcal{V}$ be the functionals on $\mathcal{F}(\mathcal{T}, \mathbb{R}^3)$ introduced in Section 1.2.2; representing bending energy and enclosed volume. For $a, v \in \mathbb{R}^+$ define*

$$\mathcal{F}_{a,v}(\mathcal{T}, \mathbb{R}^3) := \left\{\phi \in \mathcal{F}(\mathcal{T}, \mathbb{R}^3) \mid \mathcal{A}[\phi] = a,\, \mathcal{V}[\phi] = v\right\}.$$

For any $c, b \in \mathbb{R}$ and $a, v \in \mathbb{R}^+$ such that $3\sqrt{4\pi}v \leq a^{3/2}$ and $-ab \leq 1$ the infimum of $\mathcal{H}_{c,b}$ on $\mathcal{F}_{a,v}(\mathcal{T}, \mathbb{R}^3)$ is attained.

Proof. Let $\{\tilde{\phi}_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{F}_{a,v}(\mathcal{T}, \mathbb{R}^3)$ realizing the infimum. Let $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of translations such that the image of $\phi_k := T_k \circ \tilde{\phi}_k$ contains the origin. Since $\mathcal{H}_{c,b}$, $\mathcal{A}$ and $\mathcal{V}$ are invariant under translations, $\mathcal{H}_{c,b}[\phi_k]$ still converges to the infimum in $\mathcal{F}_{a,v}(\mathcal{T}, \mathbb{R}^3)$. If $-ab \leq 1$ then Proposition 1.2.4 asserts that $\mathcal{H}_{c,b}$ is an a-generalized Willmore functional. As in the proof of Corollary 2.0.7 we know that $\operatorname{Im}(\phi_k) \subset B_{R_0}(0)$ for a R_0 and all $k \in \mathbb{N}$. By Theorem 2.0.5 we obtain a subsequence, again denoted by $\{\phi_k\}$, $\phi_k \in \mathcal{F}_{a,v}(S, \mathbb{R}^3)$, $S \in \mathcal{T}$, that converges to $\phi \in \mathcal{F}_a(T, \mathbb{R}^3)$ as haunted, immersed, stratified surfaces; where (T, ϕ) differs from a bubble tree only by ghosts. Moreover, we have

$$\mathcal{H}_{c,b}[\phi] \leq \inf\{\mathcal{H}_{c,b}[\psi] \mid \psi \in \mathcal{F}_{a,v}(\mathcal{T}, \mathbb{R}^3)\}.$$

Now we argue that $\mathcal{V}[\phi] = v$.

Since the estimate $\mathcal{V}\left[\phi_k|_{S^i}\right] \leq C \operatorname{diam}\left(\phi_k(S^i)\right)|\phi(S^i)|$ holds on any component S^i of S, the bubbles that shrink to a point do not hold any volume in the limit. Similarly, ghosts do not carry any volume, hence we will disregard them in the following. Let P be the set of singular points of T. The convergence as haunted, immersed stratified surfaces yields, the existence of open sets $U_k \subset S$, $V_k \subset T$, $V_k \subset V_{k+1}$, $T \setminus \bigcup_{k\in\mathbb{N}} V_k \subset P$, and diffeomorphisms $\psi_k : V_k \to U_k$. Furthermore, we know $|\phi_k(S \setminus U_k)| \to 0$, $|\phi_k \circ \psi_k(V_k)| \to a$ and $|\phi(V_k)| \to a$. Let $\bigcup_{i=1}^m S^i$ be the union of all the components of S such that $\phi_k(S^i)$ converges to a point. Set $\phi_k' := \phi_k \circ \psi_k$.

$$\begin{aligned} 3\,|v - \mathcal{V}[\phi]| \leq{}& \left|\int_{\phi_k'(V_k)} \langle x, \nu_k\rangle \,\mathrm{d}\mu_k - \int_{\phi(V_k)} \langle x, \nu\rangle \,\mathrm{d}\mu\right| + \sup_{x\in\phi_k(S)} |x||\phi_k(S \setminus U_k)| \\ &+ \sup_{x\in\phi(T)} |x||\phi(T \setminus V_k)| + \left|\int_{\phi_k(\bigcup_{i=1}^m S^i)} \langle x, \nu_k\rangle \,\mathrm{d}\mu_k\right| \end{aligned}$$

For $j \in \mathbb{N}$ fixed we have

$$\int_{\phi_k'(V_j)} \langle x, \nu_k\rangle \,\mathrm{d}\mu_k \to \int_{\phi(V_j)} \langle x, \nu\rangle \,\mathrm{d}\mu$$

and

$$|\phi_k'(V_j)| \to |\phi(V_j)|$$

for $k \to \infty$ since ϕ'_k converges to ϕ weakly in $\mathcal{W}^{2,2}_{\mathrm{loc}}(T \setminus P, \mathbb{R}^3)$. Additionally, we have

$$\left| \int_{\phi(V_k)} \langle x, \nu \rangle \, \mathrm{d}\mu - \int_{\phi(V_j)} \langle x, \nu \rangle \, \mathrm{d}\mu \right| \leq R_0 |\phi(T) \setminus \phi(V_j)|,$$

as well as

$$\left| \int_{\phi'_k(V_k)} \langle x, \nu_k \rangle \, \mathrm{d}\mu_k - \int_{\phi'_k(V_j)} \langle x, \nu_k \rangle \, \mathrm{d}\mu_k \right| \leq R_0 |\phi'_k(V_k) \setminus \phi'_k(V_j)|.$$

For $\epsilon > 0$ choose $j \in \mathbb{N}$ such that $|\phi(T) \setminus \phi(V_j)| \leq \frac{\epsilon}{4R_0}$ and estimate

$$|\phi'_k(V_k) \setminus \phi'_k(V_j)| \leq \left| |\phi'_k(V_k)| - a \right| + |a - |\phi(V_j)|| + \left| |\phi(V_j)| - |\phi'_k(V_j)| \right| .$$

Now we may choose $k > j$ such that $|v - \mathcal{V}[\phi]| \leq \epsilon$. $\square$

Chapter 3

Regularity of Generalized Willmore Equations

In this chapter we present the regularity theory for immersions of generalized Willmore type analogous to the theory for critical points of the Willmore functional under conformal constraint as it has been developed by A. Mondino and T. Rivière in [43]. It hinges on the fact that the Euler–Lagrange equation in question can be brought into a divergence form and that suitably chosen potentials solve an elliptic PDE system with a Wente type structure; see [14, Chapter 3] for the analogue Poission equation.

We cannot cite the regularity result directly as the function F in Equation (1.1.1) represents a more general nonlinearity then the ones treated in [43], though it turns out that we can follow the same arguments.

3.1 Notation

Consider a conformal embedding $\phi \in W^{2,2} \cap W^{1,\infty}(D, M)$ from the two dimensional open disc $(D, \langle\cdot,\cdot\rangle_E)$ to a three dimensional, oriented Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ with conformal factor $e^{2\lambda}$. For the standard Euclidean coordinates x_1, x_2 on D introduce the complex coordinates $z = x_1 + ix_2$ and $\bar{z}$. Further, complexify the tangent space of M and extend all tensors on it $\mathbb{C}$-linearly. For the remainder of this section we fix the following notation:

$$
\begin{aligned}
&e_i := e^{-\lambda}\partial_{x_i}\phi && \\
&\partial_z := \frac{\partial_{x_1} - i\partial_{x_2}}{2} && \partial_{\bar{z}} = \frac{\partial_{x_1} + i\partial_{x_2}}{2} \\
&e_z := e^{-\lambda}\partial_z\phi = \frac{e_1 - ie_2}{2} && e_{\bar{z}} := e^{-\lambda}\partial_{\bar{z}}\phi = \frac{e_1 + ie_2}{2}.
\end{aligned}
$$

By ∇^M_i, ∇^M_z and $\nabla^M_{\bar{z}}$ we mean $\nabla^M_{\partial_{x_i}\phi}$, $\nabla^M_{\partial_z\phi}$ and $\nabla^M_{\partial_{\bar{z}}\phi}$ respectively, and the following quantity can be seen as a complex version of the trace free second fundamental form.

$$
\begin{aligned}
H_0 &:= 4A(e_z, e_z) = A_{11} - A_{22} - 2iA_{12} \\
\vec{H}_0 &:= H_0\nu
\end{aligned}
$$

Furthermore, we introduce multivector fields on M, denote them by $\Gamma(\Lambda^p TM)$ and extend the covariant derivative of M such that it is a derivation with respect to $\wedge$.

Lemma 3.1.1 (cf. [43, Lemma 3.3]). *The pair $\{e_1, e_2\}$ is an orthonormal frame on Σ and we can choose the orientation of $\{e_1, e_2\}$ such that $\{e_1, e_2, \nu\}$ with*

$$\nu = *e_1 \wedge e_2,$$

is positively oriented. Here $*$, and $\wedge$ are the hogde star operator of (M, g) and the wedge product respectively. Moreover, the following identities hold.

$$|H_0|^2 = 2|\mathring{A}|^2 \qquad \langle e_z, e_{\bar{z}}\rangle = \frac{1}{2}$$

$$\langle e_z, e_z\rangle = 0 \qquad \langle e_{\bar{z}}, e_{\bar{z}}\rangle = 0$$

$$*e_z = -ie_z \wedge \nu \qquad *\nu = -2ie_z \wedge e_{\bar{z}}$$

$$\nabla^M_{\bar{z}}\left(e^{\lambda}e_z\right) = -\frac{e^{2\lambda}}{4}\vec{H} \qquad \nabla^M_z\left(e^{-\lambda}e_z\right) = -\frac{1}{4}\vec{H}_0$$

$$\begin{aligned}\nabla^M_z \nu &= 2A(e_z, e_{\bar{z}})\,\partial_z\,\phi + 2A(e_z, e_z)\,\partial_{\bar{z}}\,\phi \\ &= \frac{1}{2}H\,\partial_z\,\phi + \frac{1}{2}H_0\,\partial_{\bar{z}}\,\phi\end{aligned}$$

Proof. Most of the equations follow directly from the respective definitions and the fact that $\{e_1, e_2, \nu\}$ is an orthonormal frame. We briefly discuss the covariant derivatives.

The first is merely the formula for the mean curvature in conformal coordinates.

$$\begin{aligned}\nabla^M_{\bar{z}}\left(e^{\lambda}e_z\right) &= \nabla^M_{\bar{z}}(\partial_z\phi) \\ &= \frac{1}{4}\Delta_E\phi \\ &= -\frac{e^{2\lambda}}{4}\vec{H}\end{aligned}$$

For the second we first calculate $\nabla^M_z\partial_z\phi$. Let π_n and π_T be the projections normal and tangential to Σ.

$$\pi_n(\nabla^M_z\partial_z\phi) = -e^{2\lambda}A(e_z, e_z)\nu = -\frac{e^{2\lambda}}{4}\vec{H}_0$$

For the tangential part recall that by the definition of the conformal factor we have

$$\langle\nabla^M_j\partial_i\phi, \partial_i\phi\rangle = \frac{1}{2}\partial_j e^{2\lambda}.$$

This leads to

$$\begin{aligned}\langle\nabla^M_z\partial_z\phi, e_1\rangle &= \frac{e^{-\lambda}}{2}\partial_z e^{2\lambda} \\ \langle\nabla^M_z\partial_z\phi, e_2\rangle &= -i\frac{e^{-\lambda}}{2}\partial_z e^{2\lambda} \\ \pi_T\left(\nabla^M_z\partial_z\phi\right) &= e^{-\lambda}\partial_z e^{2\lambda}e_z = 2\partial_z e^{\lambda}e_z.\end{aligned}$$

Hence we arrive at

$$\begin{aligned}\nabla^M_z(e^{-\lambda}e_z) &= \nabla^M_z(e^{-2\lambda}\partial_z\phi) \\ &= -2e^{-\lambda}\partial_z\lambda e_z - \frac{1}{4}\vec{H}_0 + 2e^{-\lambda}\partial_z\lambda e_z \\ &= -\frac{1}{4}\vec{H}_0.\end{aligned}$$

∎

Lemma 3.1.2. *Let $v \in \Gamma(TM)$ and $\alpha \in \Gamma(\Lambda^p TM)$, $p \in \{0,1,2,3\}$ then we have*

$$\nabla_v^M * \alpha = *\nabla_v^M \alpha.$$

Proof. Let $v \in \Gamma(TM)$ and $\alpha, \beta \in \Gamma(\Lambda^p TM)$ and let $\omega := e_1 \wedge e_2 \wedge \nu$, then $*$ is defined by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \omega.$$

Ww calculate

$$\begin{aligned} \alpha \wedge *\nabla_v^M \beta &= \langle \alpha, \nabla_v^M \beta \rangle \omega \\ &= \left(v\langle \alpha, \beta \rangle - \langle \nabla_v^M \alpha, \beta \rangle \right) \omega \\ \alpha \wedge \nabla_v^M * \beta &= \nabla_v^M (\alpha \wedge *\beta) - \left(\nabla_v^M \alpha \right) \wedge *\beta \\ &= v\langle \alpha, \beta \rangle \omega + \langle \alpha, \beta \rangle \nabla_v^M \omega - \langle \nabla_v^M \alpha, \beta \rangle \omega, \end{aligned}$$

but $\nabla_v^M \omega = 0$ since e_z, $e_{\bar{z}}$ and ν are normalized. □

Definition 3.1.3. Define a contraction $\bullet$ of a vector field with a two vector field on pure two vectors as follows. Let $u, v, w \in \Gamma(TM)$ then

$$u \bullet (v \wedge w) := \langle u, v \rangle w - \langle u, v \rangle w.$$

Lemma 3.1.4. *Let $u, v, w, x \in \Gamma(TM)$ then the covariant derivative obeys*

$$\nabla_x^M (u \bullet v \wedge w) = \left(\nabla_v^M u \right) \bullet v \wedge w - u \bullet \nabla_x^M (v \wedge w).$$

Proof. The claim follows by a straightforward calculation and the fact that the connection is metric.

$$\begin{aligned} \nabla_x^M u \bullet v \wedge w &= \langle \nabla_x^M u, v \rangle w + \langle u, \nabla_x^M v \rangle w + \langle u, v \rangle \nabla_x^M w \\ &\quad - \langle \nabla_x^M u, w \rangle v - \langle u, \nabla_x^M w \rangle v - \langle u, w \rangle \nabla_x^M v \\ &= \left(\nabla_x^M u \right) \bullet v \wedge w + u \bullet \left(\nabla_x^M v \wedge w + v \wedge \nabla_x^M w \right) \end{aligned}$$

□

3.2 Divergence Form and System of Conservation Laws

In this section we present the generalized Willmore equation in divergence form and construct auxiliary potentials which allow us to prove regularity for generalized Willmore equations via a bootstrap argument between the immersion ϕ and its mean curvature vector $\vec{H}$.

Lemma 3.2.1 (cf. [43, Lemma 3.4])**.** *In the setting of this chapter we have*

$$e^{-2\lambda} \partial_{\bar{z}} \left(e^{2\lambda} H H_0 \right) = H \partial_z H + H_0 \partial_{\bar{z}} H - 4e^{\lambda} H \tilde{\mathrm{Rm}}, \tag{3.2.1}$$

where $\tilde{\mathrm{Rm}} = g\left(\mathrm{Rm}(e_{\bar{z}}, e_z) e_z, \nu\right)$.

Proof. This is a straightforward consequence of Lemma 3.1.1.

$$\begin{aligned} H\partial_{\overline{z}}H_0 &= g(\nabla^M_{\overline{z}}\vec{H}_0, \vec{H}) = -4g\left(\nabla^M_{\overline{z}}\nabla^M_z(e^{-\lambda}e_z), \vec{H}\right) \\ &= -4g\left(\nabla^M_z\nabla^M_{\overline{z}}(e^{-\lambda}e_z), \vec{H}\right) - 4e^{\lambda}H\tilde{\mathrm{Rm}} \\ &= -4g\left(\nabla^M_z\left[\partial_{\overline{z}}e^{-2\lambda}\partial_z\phi - \frac{1}{4}\vec{H}\right], \vec{H}\right) - 4e^{\lambda}H\tilde{\mathrm{Rm}} \\ &= -2\partial_{\overline{z}}\lambda H_0 H + H\partial_z H - 4e^{\lambda}H\tilde{\mathrm{Rm}} \\ e^{-2\lambda}\partial_{\overline{z}}\left(e^{2\lambda}H_0H\right) &= 2\partial_{\overline{z}}\lambda H_0H + H\partial_{\overline{z}}H_0 + H_0\partial_{\overline{z}}H \\ &= H\partial_z H + H_0\partial_{\overline{z}}H - 4e^{\lambda}H\tilde{\mathrm{Rm}} \end{aligned}$$

□

Theorem 3.2.2 (cf. [43, Theorem 3.1]). *In the setting of this section the following identity holds:*

$$4e^{-2\lambda}\,\mathrm{Re}\left(\nabla^M_{\overline{z}}\left[\partial_z H\nu - \frac{1}{2}HH_0\partial_{\overline{z}}\phi\right]\right) = \Delta H\nu + H|\mathring{A}|^2\nu + 8H\,\mathrm{Re}\left(\tilde{\mathrm{Rm}}\,e_z\right),$$

where $\tilde{\mathrm{Rm}} = g\left(\mathrm{Rm}(e_{\overline{z}}, e_z)e_z, \nu\right)$.
In particular, if ϕ is of generalized Willmore type then we obtain the generalized Willmore equation in divergence form.

$$4\,\mathrm{Re}\left(\nabla^M_{\overline{z}}\left[\partial_z H\nu - \frac{1}{2}HH_0\partial_{\overline{z}}\phi\right]\right) = -e^{2\lambda}F(\phi)\nu + 8e^{2\lambda}H\,\mathrm{Re}\left(\tilde{\mathrm{Rm}}\,e_z\right) \tag{3.2.2}$$

Proof. Start by calculating

$$\begin{aligned} 4e^{-2\lambda}\nabla^M_{\overline{z}}(\partial_z H\nu) &= e^{-2\lambda}\Delta_E H\nu + 4e^{-2\lambda}\partial_z H\nabla^M_{\overline{z}}\nu \\ &= \Delta H\nu + 2e^{-2\lambda}\partial_z H\left(H\,\partial_{\overline{z}}\,\phi + \overline{H_0}\,\partial_z\,\phi\right). \end{aligned}$$

Taking the real part yields

$$4e^{-2\lambda}\,\mathrm{Re}\left(\nabla^M_{\overline{z}}(\partial_z H\nu)\right) = \Delta H\nu + 2e^{-\lambda}\,\mathrm{Re}\left((H\partial_z H + H_0\partial_{\overline{z}}H)e_{\overline{z}}\right).$$

Now we apply Lemma 3.2.1.

$$\begin{aligned} 4e^{-2\lambda}\,\mathrm{Re}\left(\nabla^M_{\overline{z}}(\partial_z H\nu)\right) &= \Delta H\nu + 2e^{-\lambda}\,\mathrm{Re}\left(\left[e^{-2\lambda}\partial_{\overline{z}}(e^{2\lambda}HH_0) + 4e^{\lambda}H\tilde{\mathrm{Rm}}\right]e_{\overline{z}}\right) \\ &= \Delta H\nu + 2e^{-2\lambda}\,\mathrm{Re}\left(\partial_{\overline{z}}(e^{2\lambda}HH_0)e^{-\lambda}e_{\overline{z}}\right) + 8H\,\mathrm{Re}(\tilde{\mathrm{Rm}}\,e_{\overline{z}}) \\ &= \Delta H\nu + 2\,\mathrm{Re}\left(e^{-2\lambda}\nabla^M_{\overline{z}}\left(HH_0\partial_{\overline{z}}\phi\right) - HH_0\overline{\nabla^M_z(e^{-\lambda}e_z)}\right) \\ &\quad + 8H\,\mathrm{Re}(\tilde{\mathrm{Rm}}\,e_{\overline{z}}) \\ &= \Delta H\nu + 2\,\mathrm{Re}\left(e^{-2\lambda}\nabla^M_{\overline{z}}\left(HH_0\partial_{\overline{z}}\phi\right) + \frac{1}{4}H|H_0|^2\nu\right) \\ &\quad + 8H\,\mathrm{Re}(\tilde{\mathrm{Rm}}\,e_{\overline{z}}) \end{aligned}$$

Bringing the term with the $\overline{z}$ derivative to left-hand side, as well as replacing $|H_0|^2 = 2|\mathring{A}|^2$, yields the claim. □

This theorem warrants the investigation of the vector field

$$Y := 2\partial_z H\nu - HH_0\partial_{\bar{z}}\phi.$$

It differs from Y_f, the one used in [43, Equation 5.10], primarily by setting $f = 0$, but its derivative differs by the nonlinearity F. We briefly note that

$$\operatorname{Re}[\langle \partial_{\bar{z}}\phi, Y\rangle] = 0 \tag{3.2.3}$$

$$\operatorname{Re}\left[e_{\bar{z}} \wedge \left(Y - 2\nabla_z^M \vec{H}\right)\right] = 0, \tag{3.2.4}$$

since $\partial_{\bar{z}}\,\phi$ is orthogonal to itself and ν, and we have

$$Y - 2\nabla_z^M \vec{H} = -HH_0\partial_{\bar{z}}\phi - 2H\nabla_z^M\nu,$$

as well as

$$\begin{aligned} e_{\bar{z}} \wedge \nabla_z^M \nu &= \frac{1}{2} e_{\bar{z}} \wedge (H_0\,\partial_{\bar{z}}\,\phi + H\,\partial_z\,\phi) \\ &= -\frac{i}{4} e^{\lambda} H e_1 \wedge e_2. \end{aligned}$$

In the following we assume that $\phi(D)$ is contained in a coordinate patch of M such that we can trivialize $\phi^* TM \cong D \times \mathbb{R}^3$, where $\partial_1\phi \mapsto b_1$, $\partial_2\phi \mapsto b_2$, $\nu \mapsto b_3$ for $\{b_i\}$, the standard basis of $\mathbb{R}^3$. Additionally, we introduce the space $(L^1 + W^{-1,2})(D)$ as the set of functions $f = f_1 + f_2$ with $f_1 \in L^1(D)$ and $f_2 \in W^{-1,2}(D)$. It is equipped with the norm

$$\|f\|_{L^1+W^{-1,2}} = \inf_{f=f_1+f_2}\{\|f_1\|_{L^1} + \|f_2\|_{W^{-1,2}}\}.$$

In the trivialization above Y is a vector field in $(L^1 + W^{-1,2})(D, \mathbb{C}^3)$ since $A \in L^2$.

The next lemma establishes the existence of the auxiliary potentials. It uses the Lorentz space $L^{2,\infty}$. Lorentz spaces can be seen as a refinement of the L^p-spaces, in particular, we have $L^{2,\infty}(D) \hookrightarrow L^2(D)$; see [13, Section 1.4] for an introduction. They play a critical role in the regularity theory of the Laplace operator; see [14, Chapter 3]. By abuse of notation we will write $v \in X$ instead of $X(D, \mathbb{C}^n)$, also for (multi-) vector fields v and function spaces X.

Lemma 3.2.3 (cf. [43, Lemma 6.1])**.** *Let Y be the vector field from above. If ϕ is of generalized Willmore type, then*

1) *there exists a complex vector field $K \in L^{2,\infty}(D)$, with $\operatorname{Im} K \in W^{1,(2,\infty)}(D)$ that is the unique solution of*

$$\begin{cases} \nabla_z^M K = iY & \text{in } D \\ \operatorname{Im} K = 0 & \text{on } \partial D. \end{cases}$$

2) *There exists a complex function $B_0 \in W^{1,(2,\infty)}(D)$, with $\operatorname{Im} B_0 \in W^{2,q}(D)$ for every $q \in (1, 2)$ that solves*

$$\begin{cases} \partial_z B_0 = \langle \partial_z\,\phi, \overline{K}\rangle & \text{in } D \\ \operatorname{Im} B_0 = 0 & \text{on } \partial D. \end{cases}$$

3) *There exits a complex 2-vector field* $B \in W^{1,(2,\infty)}(D)$, *with* $\operatorname{Im} B \in W^{2,q}(D)$ *for every* $q \in (1,2)$ *that solves*

$$\begin{cases} \nabla_z^M B = \partial_z \phi \wedge \overline{K} + 2iH\, \partial_z \phi \wedge \nu & \text{in } D \\ \operatorname{Im} B = 0 & \text{on } \partial D. \end{cases}$$

Proof. This lemma corresponds to [43, Lemma 6.1]. Its proof is a direct application of the general constructions [43, Lemma A.1 and Lemma A.2] (see also B.1.1 and B.1.2) and depends on the fact that Y solves (3.2.3) and (3.2.3). Further, it needs the regularity conditions $Y \in L^1 + W^{-1,2}$ and $\operatorname{Re}(\nabla_{\bar{z}}^M Y) \in L^1 + W^{-1,2}$ both of which are true by the assumptions on ϕ. In particular, it does not depend on the explicit shape of $\operatorname{Re}(\nabla_{\bar{z}}^M Y)$.

Lemma B.1.1 and Lemma B.1.2 deal the existence and regularity of solutions to the following systems

$$\begin{cases} D_z^i u^i = \partial_z u^i + \sum_{k=1}^n \gamma_k^i u^k = F^i & \text{in } D \\ \operatorname{Im} u^i = 0 & \text{on } \partial D. \end{cases}$$

for vector fields $u = (u^1, ..., u^n)$ on $\mathbb{C}$ and functions $\gamma_k^i \in \left(C^0 \cap W^{1,2}(\mathbb{C})\right)$ with small L^∞ norm.

We may assume that $\phi(D)$ is contained in a small normal coordinate neighborhood such that the Christoffel symbols are arbitrarily small and define

$$\gamma_k^j := \Gamma_{ki}^j\, \partial_z \phi^i.$$

Since $\phi \in W^{1,\infty}(D)$, the γ_k^j are small in L^∞ and we have $\gamma_k^j \in \left(C^0 \cap W^{1,2}\right)(D)$. After extending to $\mathbb{C}$ the covariant derivative fulfills the requirements on the differential operator D_z.

1) Extend the vector field Y to all of $\mathbb{C}$ while maintaining $Y \in L^1 + W^{-1,2}(\mathbb{C})$. Lemma B.1.1 further requires $\operatorname{Re}(\nabla_{\bar{z}}^M Y) \in L^1 + W^{-1,2}(\mathbb{C})$. Since ϕ is of generalized Willmore type we have

$$\begin{aligned} \|\operatorname{Re}(\nabla_{\bar{z}}^M Y)\|_{L^1+W^{-1,2}(\mathbb{C})} &\leq C\|\operatorname{Re}(\nabla_{\bar{z}}^M Y)\|_{L^1+W^{-1,2}(D)} \\ &\leq C\|H\|_{L^1(D)} + C\|e^{2\lambda} F(\phi)\nu\|_{L^1+W^{-1,2}(D)}. \end{aligned}$$

2) As $\partial_z \phi$ is bounded and $K \in L^{2,\infty}(D)$ it is clear that $\langle \partial_z \phi, \overline{K}\rangle \in L^1 \cap L^{2,\infty}(D)$. We have to verify that $\operatorname{Im} \partial_z \langle \partial_z \phi, \overline{K}\rangle \in L^q(D)$ for every $q \in (1,2)$. From the definition of Y we get

$$\begin{aligned} 0 &= \operatorname{Im}\langle \partial_{\bar{z}}, iY\rangle = \operatorname{Im}\langle \partial_{\bar{z}}, \nabla_z^M K\rangle \\ &= \operatorname{Im}\left(\partial_z \langle \partial_{\bar{z}} \phi, K\rangle - \langle \nabla_z^M\, \partial_{\bar{z}} \phi, K\rangle\right). \end{aligned}$$

Complex conjugation yields

$$\operatorname{Im} \partial_{\bar{z}} \langle \partial_z \phi, \overline{K}\rangle = -\frac{e^{2\lambda}}{4} H \langle \nu, \operatorname{Im}(K)\rangle.$$

This implies the claim, as $\operatorname{Im} K \in W^{1,(2,\infty)}(D) \hookrightarrow L^p(D)$ for all $p \in [1,\infty)$ and $H \in L^2(D)$. After extending $\langle \partial_z \phi, \overline{K}\rangle$ to all of $\mathbb{C}$ we can apply Lemma B.1.2.

3) Clearly, we have $\partial_z \phi \wedge \overline{K} + 2iH\, \partial_z \phi \wedge \nu \in L^1 \cap L^{2,\infty}(D)$ and (3.2.3) implies

$$\begin{aligned}
0 &= \operatorname{Im}\left[\partial_{\overline{z}} \phi \wedge \left(iY - 2i\nabla_z^M \vec{H}\right)\right] \\
&= \operatorname{Im}\left[\partial_{\overline{z}} \phi \wedge \left(\nabla_z^M K - 2i\nabla_z^M \vec{H}\right)\right] \\
&= -\operatorname{Im}\left[\nabla_{\overline{z}}^M \left(\partial_z \phi \wedge \left(\overline{K} + 2i\vec{H}\right)\right) - \left(\nabla_{\overline{z}}^M \partial_z \phi\right) \wedge \left(\overline{K} + 2i\vec{H}\right)\right] \\
&= -\operatorname{Im}\left[\nabla_{\overline{z}}^M \left(\partial_z \phi \wedge \left(\overline{K} + 2i\vec{H}\right)\right) + \frac{e^{2\lambda}}{4} H\nu \wedge \left(\overline{K} + 2i\vec{H}\right)\right].
\end{aligned}$$

Hence, $\operatorname{Im}\left[\nabla_{\overline{z}}^M \left(\partial_z \phi \wedge \left(\overline{K} + 2i\vec{H}\right)\right)\right] \in L^q(D)$ for all $q \in [1,2)$ and extending the right-hand side to $\mathbb{C}$ allows us to apply Lemma B.1.2.

$\square$

In the next two lemmas we explore the relation between B and B_0 and establish that they solve an elliptic PDE system that exhibits the crucial Wente type structure.

Lemma 3.2.4 (cf. [43, Lemma 6.2]). *The potentials constructed in Lemma 3.2.3 satisfy the following coupled system.*

$$\begin{aligned}
\nabla_z^M B &= i\,(\partial_z B_0)\, e_1 \wedge e_2 - i * \nu \bullet \nabla_z^M B \\
\partial_z B_0 &= -i \left\langle \nabla_z^M B, e_1 \wedge e_2 \right\rangle
\end{aligned}$$

Proof. The first equation follows from expanding K in the basis $\{e_z, e_{\overline{z}}, \nu\}$ and recalling the relations of Lemma 3.1.1.

$$\begin{aligned}
\nabla_z^M B &= \partial_z \phi \wedge \overline{K} + 2iH\, \partial_z \phi \wedge \nu \\
&= 2\langle \partial_z \phi, \overline{K}\rangle e_z \wedge e_{\overline{z}} + \left(\langle \nu, \overline{K}\rangle + 2iH\right) \partial_z \phi \wedge \nu \\
&= i\, \partial_z B_0 e_1 \wedge e_2 - i * \nu \bullet \nabla_z^M B
\end{aligned}$$

The second one is straightforward.

$$\begin{aligned}
\langle \nabla_z^M B, \partial_z \phi \wedge \partial_{\overline{z}} \phi\rangle &= \langle \partial_z \phi \wedge \overline{K}, \partial_z \phi \wedge \partial_{\overline{z}} \phi\rangle \\
&= -\frac{e^{2\lambda}}{2} \langle \overline{K}, \partial_z \phi\rangle \\
&= -\frac{e^{2\lambda}}{2}\, \partial_z B_0
\end{aligned}$$

$\square$

The defining equations of B_0 and B, as well as the last lemma, let us guess that K is actually as regular as $\partial_z B_0$ which shares the regularity with $\nabla_z^M B$. This means H should be as regular as $\nabla_z^M B$. We already have L^p estimates for the imaginary part of $\nabla_z^M B$ and the equations in 3.2.4 relate $(\nabla_z^M B, \partial_z B_0)$ to $i(\nabla_z^M B, \partial_z B_0)$ which allows us to play of the imaginary part against the real part. More specifically, (B, B_0) solves the following Wente type system.

Lemma 3.2.5 (cf. [43, Proposition 6.1]]). *Let ϕ be of generalized Willmore type, then we have*

$$\begin{aligned}\Delta \operatorname{Re} B &= * \left[\left(\nabla_2^M \nu\right) \bullet \nabla_1^M \operatorname{Re} B - \left(\nabla_1^M \nu\right) \bullet \nabla_2^M \operatorname{Re} B\right] \\ &\quad - \left((\partial_1 \operatorname{Re} B_0)\, \nabla_2^M e_1 \wedge e_2 - (\partial_2 \operatorname{Re} B_0)\, \nabla_1^M e_1 \wedge e_2\right) + I \\ \Delta \operatorname{Re} B_0 &= \langle \nabla_1^M \operatorname{Re} B, \nabla_2^M e_1 \wedge e_2 \rangle - \langle \nabla_2^M \operatorname{Re} B, \nabla_1^M e_1 \wedge e_2 \rangle + G\end{aligned}$$

Here I and G are functions in $L^q(D)$, $q \in [1,2)$, which depend on $B, B_0, \Delta \operatorname{Im} B, \Delta \operatorname{Im} B_0$ derivatives of the metric and the second fundamental form A.

Note. In coordinates the system is of the form

$$\Delta U^j = \partial_1 E_k^j\, \partial_2 U^k - \partial_2 E_k^j\, \partial_1 U^k + \tilde{I}^j$$

for $U = (\operatorname{Re}(B_{ij}), \operatorname{Re}(B_0))$, $\partial_i U^j \in L^{2,\infty}(D)$, $\partial_i E_k^j \in L^2(D)$ and where the $\tilde{I}^j \in L^q$, $q \in [1,2)$ are comprised of I or G and terms involving Christoffel symbols, A, U^j and $\partial_i U^j$. This is evident if we recall that the idea of the contraction $\bullet$ is to take scalar product componentwise.

Proof. By abuse of notation we allow the functions I and G to vary from line to line We begin with the scalar potential.

$$\begin{aligned}\Delta \operatorname{Re}(B_0) &= 4 \operatorname{Re} \partial_{\bar z} \partial_z B_0 = 4 \operatorname{Im} \partial_{\bar z} \left\langle \nabla_z^M B, e_1 \wedge e_2 \right\rangle \\ &= 4 \left\langle \operatorname{Im} \nabla_{\bar z}^M \nabla_z^M B, e_1 \wedge e_2 \right\rangle + 4 \operatorname{Im} \left\langle \nabla_z^M B, \nabla_{\bar z}^M e_1 \wedge e_2 \right\rangle \\ &= \operatorname{Im} \left\langle \nabla_1^M B - i \nabla_2^M B, \nabla_1^M e_1 \wedge e_2 + i \nabla_2^M e_1 \wedge e_2 \right\rangle + G \\ &= \left\langle \nabla_1^M \operatorname{Im} B, \nabla_1^M e_1 \wedge e_2 \right\rangle + \left\langle \nabla_2^M \operatorname{Im} B, \nabla_2^M e_1 \wedge e_2 \right\rangle \\ &\quad + \left\langle \nabla_1^M \operatorname{Re} B, \nabla_2^M e_1 \wedge e_2 \right\rangle - \left\langle \nabla_2^M \operatorname{Re} B, \nabla_1^M e_1 \wedge e_2 \right\rangle + G \\ &= \left\langle \nabla_1^M \operatorname{Re} B, \nabla_2^M e_1 \wedge e_2 \right\rangle - \left\langle \nabla_2^M \operatorname{Re} B, \nabla_1^M e_1 \wedge e_2 \right\rangle + G\end{aligned}$$

Here we used extensively that $B \in W^{1,(2,\infty)}$, $\operatorname{Im} B \in W^{2,q}$, $q \in [1,2)$, $e_1 \wedge e_2 \in L^\infty$ and $\nabla_j^M e_1 \wedge e_2 \in L^2$, $j \in \{1,2\}$ as well as the fact that curvature terms of M are bounded on $\phi(D)$. A similar calculation shows the claim for B. First off we have

$$4 \nabla_{\bar z}^M \nabla_z^M B = \Delta B - i[\nabla_1^M, \nabla_2^M] B + I.$$

Note that $[\nabla_1^M, \nabla_2^M]$ is differential operator of order one involving the curvature and hence the term $-i[\nabla_1^M, \nabla_2^M]B$ can be absorbed into the I as well. Using Lemma 3.2.4 we get

$$\begin{aligned}\Delta \operatorname{Re} B &= \operatorname{Re}\left[4i \nabla_{\bar z}^M \left(\partial_z B_0 e_1 \wedge e_2 - *\nu \bullet \nabla_z^M B\right)\right] + I \\ &= -\Delta \operatorname{Im}(B_0) e_1 \wedge e_2 - 4 \operatorname{Im}\left(\partial_z B_0 \nabla_{\bar z}^M e_1 \wedge e_2\right) \\ &\quad + 4 \operatorname{Im}\left[* \left(\nabla_{\bar z}^M \nu\right) \bullet \nabla_z^M B + *\nu \bullet \nabla_{\bar z}^M \nabla_z^M B\right] + I \\ &= -4 \operatorname{Im}\left(\partial_z B_0 \nabla_{\bar z}^M e_1 \wedge e_2\right) + 4 \operatorname{Im}\left[* \left(\nabla_{\bar z}^M \nu\right) \bullet \nabla_z^M B\right] + I \\ &= -\partial_1 \operatorname{Re}(B_0) \nabla_2^M e_1 \wedge e_2 + \partial_2 \operatorname{Re}(B_0) \nabla_1^M e_1 \wedge e_2 \\ &\quad + * \left(\nabla_2^M \nu\right) \bullet \nabla_1^M \operatorname{Re}(B) - * \left(\nabla_1^M \nu\right) \bullet \nabla_2^M \operatorname{Re}(B) + I\end{aligned}$$

□

Now we are in the position to prove our regularity theorem.

Theorem 3.2.6. *Let $\phi \in \mathcal{F}(D, M)$ with conformal factor $e^{2\lambda}$, $\lambda \in L^\infty(D)$. If ϕ is of generalized Willmore type then ϕ is smooth.*

Proof. In the first two steps we briefly sketch how to prove that $(B, B_0) \in W^{1,p}_{\mathrm{loc}}$ for a $p > 2$. This is done completely analogous to the proof of [43, Theorem 6.1]. We include it here for completeness. See also [50] for a comprehensive treatment of the kind of PDE system that (B, B_0) solves.

In the third step we proceed differently. In particular, we adapt the bootstrap procedure between H and ϕ to account for the function F in Equation (3.2.2).

Step 1) We show that $\nabla U \in L^2(B_{1/2})$.

Let $B_r(x)$ be the ball in D of radius r around x, and abbreviate $B_r(0)$ to B_r. By Lemma 3.2.5 we need analyze the regularity the system

$$\Delta U^j = \partial_1 E^j_k \, \partial_2 U^k - \partial_2 E^j_k \, \partial_1 U^k + I^j .$$

We decompose it into three systems of Dirichlet problems. The solution U^j is given as sum of $\tilde{u}^j$, $\tilde{v}^j$, $\tilde{w}^j$, which individually solve

$$\begin{cases} \Delta \tilde{u}^j = \partial_1 E^j_k \, \partial_2 U^k - \partial_2 E^j_k \, \partial_1 U^k & \text{in } D \\ \tilde{u}^j = 0 & \text{on } \partial D \end{cases}$$

$$\begin{cases} \Delta \tilde{v}^j = I^j & \text{in } D \\ \tilde{v}^j = 0 & \text{on } \partial D \end{cases}$$

$$\begin{cases} \Delta \tilde{w}^j = 0 & \text{in } D \\ \tilde{w}^j = U^j & \text{on } \partial D \end{cases}$$

By standard interior elliptic regularity theory and the fact that $I^j \in L^q$, $q \in [1, 2)$ we obtain $\nabla \tilde{v}^j \in L^2(D)$ and $\nabla \tilde{w}^j \in L^2_{\mathrm{loc}}(D)$. As the $\partial_i E^j_k$ are in L^2 and the $\partial_i U^k$ are in $L^{2,\infty}$ we can solve for $\tilde{u}^j$ as well, where $\nabla \tilde{u}^j \in L^2(D)$; see [14, Theorem 3.4.5]. Thus we get $\nabla U^j \in L^2(B_{1/2})$.

Step 2) We show that there exists a constant $p > 2$ such that $\nabla U \in L^p\left(B_{1/2}\right)$.

The goal is to show that

$$\sup_{x \in B_{1/2},\ r \in (0, \rho)} r^\beta \| \Delta U^j \|_{L^1(B_r(x))} \le C$$

for a $\beta \in (0, 1)$. By a paper of Adams [4] this suffices to conclude that $\nabla U^j \in L^p(B_{1/2})$, for a $p > 2$. To achieve this we show that there is an $\alpha \in (0, 1)$ such that

$$\sup_{x \in B_{1/2},\ r < 1/4} r^{-\alpha} \int_{B_r(0)} |\nabla U^j(y)| \, \mathrm{d}y < \infty \tag{3.2.5}$$

since a straightforward estimate reveals

$$\begin{aligned}\int_{B_r(x_0)} |\Delta U^j| \, \mathrm{d}\mu \leq \int_{B_r(x_0)} |\nabla E^j_k||\nabla U^k| \, \mathrm{d}\mu + \int_{B_r(x_0)} |I^j| \, \mathrm{d}\mu \\ \leq \|\nabla E^j_k\|_{L^2_{B_1}} \|\nabla U^k\|_{L^2_{B_r(x_0)}} + |B_r(x_0)| 1/q^* \|I^j\|_{L^q_{B_1}} \\ \leq Cr^\beta.\end{aligned}$$

Here q^* is the Hölder conjugate of q and $\beta = \min(\alpha, 2/q^*)$.

Since the E^j_k are in $L^2(D)$, there is a $\rho > 0$ for any $\epsilon > 0$ such that

$$\sup_{x \in B_{1/2}} \int_{B_\rho(x)} |\nabla E^j_k(y)|^2 \, \mathrm{d}y < \epsilon^2.$$

Similar to the previous step we decompose $U = u + v + w$ on $B_\rho(x_0)$, where u^j, v^j and w^j solve the following equations for an arbitrary but fixed $x_0 \in B_{1/2}$.

$$\begin{cases} \Delta u^j = \partial_1 E^j_k \, \partial_2 U^k - \partial_2 E^j_k \, \partial_1 U^k & \text{in } B_\rho(x_0) \\ u^j = 0 & \text{on } \partial B_\rho(x_0) \end{cases}$$

$$\begin{cases} \Delta v^j = I^j & \text{in } B_\rho(x_0) \\ v^j = 0 & \text{on } \partial B_\rho(x_0) \end{cases}$$

$$\begin{cases} \Delta w^j = 0 & \text{in } B_\rho(x_0) \\ w^j = U^j & \text{on } \partial B_\rho(x_0) \end{cases}$$

As before [14, Theorem 3.4.5] establishes the existence of u^j along with an estimate for the gradient.

$$\|\nabla u^j\|_{L^2(B_\rho(x_0))} \leq C\epsilon \|\nabla U\|_{L^2(B_\rho(x_0))}$$

Since $I \in L^q$, we can employ a scaling argument to control $\|\nabla v^j\|_{L^2}$. Another scaling argument for subharmonic functions establishes a corresponding estimate for w^j. From these three individual estimates (3.2.5) follows; see [43, Proof of Theorem 6.1] for the detailed argument.

Step 3) We show that ϕ is smooth.

The defining equation for B reads

$$2iH\partial_z\phi \wedge \nu = \nabla^M_z B - \partial_z \, \phi \wedge \overline{K}.$$

Projecting this to $\partial_{\bar{z}}\phi \wedge \nu$ and taking the imaginary part yields

$$H = e^{2\lambda} \operatorname{Im}\left(\left\langle \nabla^M_z B, \partial_{\bar{z}}\phi \wedge \nu \right\rangle\right) + \frac{1}{2}\langle \operatorname{Im}(K), \nu \rangle,$$

hence $H \in L^p_{\mathrm{loc}}$, as $\operatorname{Im}(K) \in L^q$ for all $q \in [1, \infty)$. Since ϕ is conformal we have $\Delta_E \phi = e^{2\lambda}\vec{H}$, where Δ_E is the Euclidean Laplace operator, and hence $\phi \in W^{2,p}_{\mathrm{loc}}$. In the following we retreat to $D_{1/2}(0)$, in order to drop the *loc* subscript.

The generalized Willmore equation in divergence form reads

$$4\,\mathrm{Re}\left(\nabla^M_{\bar z}\left[\partial_z\, H\nu - \frac{1}{2}HH_0\,\partial_{\bar z}\,\phi\right]\right) = 8\,\mathrm{Re}\left(e^{2\lambda}H\tilde{R}e_z\right) - e^{2\lambda}F(\phi)\nu.$$

In terms of a local frame $\{b_\alpha\}$ of M with Christoffel symbols Γ, $\nabla^M_{\bar z}\nabla^M_z\vec{H}$ can be expressed as

$$\nabla^M_{\bar z}\nabla^M_z\vec{H} = \frac{1}{4}\Delta_E\vec{H} + \partial_z\,\vec{H}\star\partial_{\bar z}\,\phi\star\Gamma\star b + \partial_{\bar z}(\vec{H}\star\partial_z\,\phi\star\Gamma)\star b + \vec{H}\star\partial_z\,\phi\star\partial_{\bar z}\,\phi\star\Gamma\star\Gamma\star b.$$

Here we employed the $\star$ notation, that is $F\star G$ denotes a sum of contractions of F and G. Combining the last two equations we get an elliptic equation for $\vec{H}$ whose right-hand side we control.

$$\begin{aligned}\Delta_E\vec{H} = 4\,\mathrm{Re}\Big(&\frac{1}{2}\nabla^M_{\bar z}(H^2\,\partial_z\,\phi) + 2e^{2\lambda}H\tilde{R}e_z - \partial_z\,\vec{H}\star\partial_{\bar z}\,\phi\star\Gamma\star b \\ &- \partial_{\bar z}(\vec{H}\star\partial_z\,\phi\star\Gamma)\star b - \vec{H}\star\partial_z\,\phi\star\partial_{\bar z}\,\phi\star\Gamma\star\Gamma\star b\Big) - e^{2\lambda}F(\phi)\nu\end{aligned} \quad (3.2.6)$$

By Definition 1.1.7, there is an $\epsilon > 0$ such that

$$e^{2\lambda}F(\phi)\nu \in W^{k-1,l},\ l = \frac{2p}{2+p} + \epsilon \ \text{ if } \phi \in W^{k+2,p}\cap W^{1,\infty},\ p > 2,\ k \geq 0.$$

Now suppose $\phi \in W^{k+2,p}$ for some $k \geq 0$, $p > 2$ then the right-hand side of (3.2.6) is in $W^{k-1,l'}$, where $l' = \min(l, p/2)$, if $k = 0$ and $l' = \min(l,p)$ if $k > 0$. Hence, $\vec{H} \in W^{k+1,l'}$ and by the equation $\Delta_E\phi = e^{2\lambda}\vec{H}$ we arrive at $\phi \in W^{k+3,l'}$. The following iteration implies the smoothness of ϕ.

Let $p_0 := 2+\delta$, for some $0 < \delta < \epsilon/2$ small enough such that $H \in L^{p_0}$, $\phi \in W^{2,p_0}$. Set $p_i := \frac{2l'_{i-1}}{2-l'_{i-1}}$ for $i \in \mathbb{N}$; $l_i := \frac{2p_i}{2+p_i} + \epsilon$ for $i \in \mathbb{N}_0$ and $l'_i := \min(l_i, p_i/2)$. Since $W^{1,l'_i} \hookrightarrow L^{p_{i+1}}$, we see that $\vec{H} \in W^{1,l'_i}$ for all $i \in \mathbb{N}$. As $p_i \to \infty$ and $l_i \to 2+\epsilon$ we eventually have that $\vec{H} \in W^{1,p_0}$. Now we may iterate again for the higher derivatives.

$\square$

Chapter 4

Concentration of Small Surfaces

In this chapter we analyze area-constrained critical points of Hawking type functionals in order to characterize points in the ambient manifold around which small area-constrained Hawking type surfaces concentrate. This way we are able to identify the analogue of an energy density for Hawking type functionals. We follow [29] closely, where these arguments were developed for the Willmore functional. To that end we fix a C_B-bounded three dimensional ambient manifold (M, g) (see Appendix A) and a Hawking type functional $\mathcal{H} = \mathcal{W} + \mathcal{L}$, where $\mathcal{L}[\Sigma] = \int_\Sigma L(x, \nu)\,\mathrm{d}\mu$, for a smooth $L : TM \to \mathbb{R}$. Throughout Subsection 4.1 we suppose also that L, $d_{TM}L$, $\operatorname{Hess}_V L$ and $\nabla^M d_V L$ are bounded by C_L. Geodesic balls in M are denoted by $\mathcal{B}_r(p)$.

Recall the Euler–Lagrange equation for Hawking type functionals.

$$\Delta H + H|\mathring{A}|^2 + H\operatorname{Ric}(\nu,\nu) + HQ + \gamma(\mathring{A}, S) + 2\lambda H + T = 0, \tag{4.0.1}$$

where Q, S and T are given by

$$\begin{aligned} Q &= -2L - \operatorname{tr}_\Sigma \operatorname{Hess}_V L + 2d_V L(\nu), \\ S &= -2\operatorname{Hess}_V L, \\ T &= -2d_M L(\nu) - 2\operatorname{div}_\Sigma d_V L. \end{aligned}$$

4.1 A Priori Estimates for Small Surfaces

In this section we first characterize small area-constrained minimizers of Hawking type functionals. Subsequently, we establish a priori estimates for Hawking type surfaces with small energy in order to obtain quantitative control of relevant geometric quantities such as the mean curvature. These will be of vital importance on the next section.

Proposition 4.1.1. *There are constants $a_0(C_L, C_B) > 0$ and $C(C_L, C_B) > 0$ such that any $\phi_a \in \mathcal{F}_a(\mathcal{T}, M)$ realizing $\beta(\mathcal{H}_L, M, a) = \inf\{\mathcal{H}_L[\phi] \mid \phi \in \mathcal{F}_a(\mathcal{T}, M)\}$ for $a \le a_0$ is an embedding of a sphere, its image is contained in a normal coordinate neighborhood and ϕ_a satisfies*

$$|\mathcal{H}[\phi_a] - 4\pi| \le C(L, C_B)a.$$

Proof. Let $\phi_a \in \mathcal{F}_a(S, M)$ realize $\beta(\mathcal{H}_L, M, a)$, for a bubble tree S, and set $\Sigma_a := \phi_a(S)$.

In [41] A. Mondino calculated the expansion of the Willmore energy for spheres in coordinates and found

$$\mathcal{W}[S_R, g] \le 4\pi + C(C_B)|S_R|_g.$$

Since L is bounded, we can estimate the Willmore energy of Σ_a by comparing it to spheres S_R, $a = 4\pi R^2$, in coordinates.

$$\mathcal{W}[\Sigma_a, g] \le \mathcal{H}[\Sigma_a] + C_L a \le \mathcal{H}[S_R] + C_L a \le 4\pi + C(C_L, C_B)a$$

Lemma A.1.7 asserts that $\operatorname{diam}_M(\Sigma_a) \le C(L, C_B)\sqrt{a}$. Hence, Σ_a lies in a normal coordinate neighborhood $B_r(0)$, provided a is small enough.

Consider a bubble S^i of S on which ϕ_a is not constant and set $\Sigma^i := \phi_a(S^i)$. We apply corollary A.1.5 as well as the integrated Gauss equation to see

$$\mathcal{W}[\Sigma^i, g] \ge \mathcal{W}[\Sigma^i, g_E] - Cr^2 \ge 4\pi - Cr^2.$$

Hence, there can be only one bubble and by deleting ghosts we may assume that (S, ϕ_a) is not haunted. From the same corollary we gather

$$\mathcal{W}[\Sigma_a, g_E] \le \mathcal{W}[\Sigma_a, g] + Cr^2 \le 4\pi + C(r^2 + a).$$

In order to see embeddedness we employ the Li–Yau inequality [34]; see also Lemma A.1.17. Denote by $\theta^2(\Sigma, p) = \#\phi^{-1}(p)$ the density of Σ at p, then we have

$$\theta^2(\Sigma, p) \le \frac{\mathcal{W}[\Sigma]}{4\pi}.$$

This follows from Simons monotonicity formula, see [25, Appendix A] for a discussion.

Finally, Lemma A.1.11 allows us to choose the normal neighborhood such that r and R are comparable. This yields the final estimate on $\mathcal{H}[\Sigma_a]$.

$$\begin{aligned} \mathcal{H}[\Sigma_a] &= \mathcal{W}[\Sigma_a, g] + \mathcal{L}[\Sigma_a] \le 4\pi + C(L, C_B)a \\ \mathcal{H}[\Sigma_a] &\ge \mathcal{W}[\Sigma_a, g_E] - C(L, C_B)a \ge 4\pi - C(L, C_B)a \end{aligned}$$

□

Proposition 4.1.2. *There are positive constants $r_0(C_B)$ and $C(C_B, C_L)$ such that for all $r \in (0, r_0)$ and $\Sigma \subset \mathcal{B}_r(p)$, immersed, area-constrained, critical surfaces of $\mathcal{H}$, we can estimate the Lagrange multiplier as follows.*

$$|\lambda| \le C|\Sigma|^{-1}\left(|\Sigma|^{1/2}\mathcal{W}[\Sigma]^{1/2} + |\Sigma| + r\int_\Sigma |A|^2 \mathrm{d}\mu\right)$$

Proof. As in [29, Proposition 5.3] the idea is to consider an area-constrained normal variation of $\mathcal{H}$ in direction $f = g(x, \nu)$, for the position vector field x in B_r in normal coordinates to obtain

$$\delta_f\,\mathcal{H}[\Sigma] = \lambda\delta_f\,\mathcal{A}[\Sigma].$$

If the variation of the area is non zero, we calculate the Lagrange parameter λ as the quotient

$$\lambda = \frac{\delta_f\,\mathcal{H}[\Sigma]}{\delta_f\,\mathcal{A}[\Sigma]}.$$

We choose $f = g(x, \nu)$, where x is the position vector field in B_r in normal coordinates and estimate $\delta_f\,\mathcal{L}$ directly.

$$\begin{aligned} |\nabla^\Sigma f| &\le \sum_i |\gamma^{ij} g(\nabla^M_j x, \nu)| + \left|\gamma^{ij} g\left(x, \frac{\partial\phi}{\partial x^k}\right) A^k_j\right| \\ &\le C(C_B) + C(C_B)r|A| \\ \delta_f\,\mathcal{L}[\Sigma] &= \int_\Sigma f d_M L(\nu) + d_V L(-\nabla^\Sigma f) + fLH \,\mathrm{d}\mu \\ &\le C(C_L, C_B)|\Sigma| + C(C_L, C_B)r|\Sigma|^{1/2}\left(\int_\Sigma |A|^2 \,\mathrm{d}\mu\right)^{1/2} \end{aligned}$$

For $\delta_f \mathcal{W}$ and $\delta_f \mathcal{A}$ we use the results from [29, Proposition 5.3]

$$
\begin{aligned}
&|\delta_f \mathcal{A}[\Sigma]| \geq |\Sigma|, \\
&|\delta_f \mathcal{W}[\Sigma]| \leq C|\Sigma|^{1/2} \mathcal{W}[\Sigma]^{1/2} + Cr \int_\Sigma |A|^2 \,\mathrm{d}\mu.
\end{aligned}
$$

□

Theorem 4.1.3. *There are positive constants ϵ_0 and C depending only on C_B and C_L such that any spherical immersed surface Σ that*

1. *solves equation (1.2.2), satisfies*
2. *$\mathcal{H}(\Sigma) \leq 4\pi + \epsilon^2$ and*
3. *$|\Sigma| \leq \epsilon^2$*

for an $\epsilon \in (0, \epsilon_0)$, obeys the following estimate.

$$
\int_\Sigma |\nabla^2 H|^2 + H^2 |\nabla H|^2 + H^2 |\nabla \mathring{A}|^2 + H^4 |\mathring{A}|^2 d\mu \leq C
$$

Proof. We start by integrating the Gauss equation over Σ, to obtain

$$
2\,\mathcal{W}[\Sigma] = 8\pi + \int_\Sigma |\mathring{A}|^2 d\mu + 2\int_\Sigma \mathrm{Ric}(\nu,\nu) - \frac{1}{2}\,\mathrm{Sc}\,\mathrm{d}\mu.
$$

Since the curvature is bounded and $\mathcal{H}$ is close to 4π, we can estimate $\|\mathring{A}\|^2_{L^2(\Sigma)}$.

$$
\begin{aligned}
\|\mathring{A}\|^2_{L^2(\Sigma)} &= 2\,\mathcal{H}[\Sigma] - 8\pi - 2\,\mathcal{L}[\Sigma] - 2\int_\Sigma G(\nu,\nu)\,\mathrm{d}\mu \\
&\leq C(C_L, C_B)\epsilon^2
\end{aligned}
$$

Moreover, Lemma A.1.7 and Lemma A.1.11 assert that we can operate in a normal coordinate neighborhood $B_r(0)$ adapted to Σ such that $r \leq C|\Sigma|^{1/2}$. This simplifies the estimate for the Lagrange multiplier to

$$
|\lambda| \leq C(C_L, C_B)|\Sigma|^{-1}\epsilon.
$$

It also enables us to use the Michael-Simon-Sobolev inequality (see Lemma A.1.13).

We multiply equation (1.2.2) by ΔH and integrate over Σ. Through integration by parts and Young's inequality we obtain

$$
\begin{aligned}
\int_\Sigma (\Delta H)^2\, d\mu &= -\int_\Sigma \Delta H H |\mathring{A}|^2 + \Delta H H Q + \Delta H \gamma(\mathring{A}, S) + \Delta H T - 2|\nabla H|^2 \lambda\,\mathrm{d}\mu \\
&\leq \int_\Sigma \frac{(\Delta H)^2}{2} + 2H^2|\mathring{A}|^4 + 2H^2 Q^2 + 2\gamma(\mathring{A}, S)^2 + 2T^2 + \frac{C\epsilon}{|\Sigma|}|\nabla H|^2\,\mathrm{d}\mu,
\end{aligned}
$$

and

$$
\int_\Sigma (\Delta H)^2\, d\mu \leq C(C_L, C_B) + C(C_L, C_B)\frac{\epsilon}{|\Sigma|}\int_\Sigma |\nabla H|^2\, d\mu + 4\int_\Sigma H^2 |\mathring{A}|^4\,\mathrm{d}\mu. \tag{4.1.1}
$$

From here on the proof proceeds exactly as the one of [29, Proposition 5.1]. We preset it here for completeness.

We use the Bochner identity (see Lemma A.1.12) on $\nabla^2 H$, the Hessian of H.

$$\int_\Sigma |\nabla^2 H|^2 + H^2|\nabla H|^2 \, d\mu \leq \int_\Sigma (\Delta H)^2 + C|\nabla H|^2 \left(|\mathring{A}|^2 + 1\right) \mathrm{d}\mu \tag{4.1.2}$$

The Micheal–Simon–Sobolev inequality implies

$$\begin{aligned}\int_\Sigma |\nabla H|^2 \, \mathrm{d}\mu &\leq C \left(\int_\Sigma |\nabla|\nabla H|| + |H||\nabla H| \, \mathrm{d}\mu\right)^2 \\ &\leq C|\Sigma| \int_\Sigma |\nabla^2 H|^2 + H^2 |\nabla H|^2 \, \mathrm{d}\mu \end{aligned} \tag{4.1.3}$$

Inserting (4.1.1) and (4.1.3) into (4.1.2), as well as absorbing several terms yields

$$\int_\Sigma |\nabla^2 H|^2 + H^2|\nabla H|^2 \, \mathrm{d}\mu \leq C + C\int_\Sigma H^2 |\mathring{A}|^4 d\mu + |\nabla H|^2 |\mathring{A}|^2 \, \mathrm{d}\mu. \tag{4.1.4}$$

We continue with the Simons identity for immersed surfaces; see [51].

$$\begin{aligned} -\mathring{A}^{ij} \Delta \mathring{A}_{ij} + \frac{1}{2} H^2 |\mathring{A}|^2 = &-\gamma(\mathring{A}, \nabla^2 H) + |\mathring{A}|^4 + |\mathring{A}|^2 \operatorname{Ric}(\nu,\nu) \\ &- \mathring{A}^{ij} \mathring{A}_j^k \operatorname{Ric}_{ik} - 2\gamma(\mathring{A}, \nabla\omega) \end{aligned} \tag{4.1.5}$$

Here we introduced a new 1-form, $\omega := \operatorname{Ric}(\nu, \cdot)^T$, where $\cdot^T$ denotes the restriction to the tangent space of Σ. We multiply (4.1.5) with H^2 and integate over Σ. We will estimate every term seperately, using integration by parts and the Codazzi equation

$\operatorname{div} \mathring{A} = \frac{1}{2}\nabla H + \omega.$

$$
\begin{aligned}
-\int_\Sigma H^2 \mathring{A}_{ij} \Delta \mathring{A}^{ij} \,\mathrm{d}\mu &= \int_\Sigma \gamma(\nabla(H^2 \mathring{A}^{ij}), \nabla \mathring{A}^{ij}) \,\mathrm{d}\mu \\
&= \int_\Sigma 2H \mathring{A}^{ij} \gamma(\nabla H, \nabla \mathring{A}_{ij}) + H^2 |\nabla \mathring{A}|^2 \,\mathrm{d}\mu \\
\left| \int_\Sigma 2H \mathring{A}^{ij} \gamma(\nabla H, \nabla \mathring{A}_{ij}) \,\mathrm{d}\mu \right| &\le 2 \int_\Sigma |\mathring{A}||H||\nabla H||\nabla \mathring{A}| \,\mathrm{d}\mu \\
&\le \int_\Sigma 1/2 H^2 |\nabla \mathring{A}|^2 + 2|\mathring{A}|^2 |\nabla H|^2 \,\mathrm{d}\mu \\
|\mathring{A}^{ij} \mathring{A}_i^k \operatorname{Ric}_{kj}| &\le C|\mathring{A}|^2 \\
\int_\Sigma H^2 |\mathring{A}|^2 \,\mathrm{d}\mu &\le \int_\Sigma H^2 + H^2 |\mathring{A}|^4 \,\mathrm{d}\mu \\
&\le \mathcal{W}[\Sigma] + \int_\Sigma H^2 |\mathring{A}|^4 \,\mathrm{d}\mu \\
&\le C + \int_\Sigma H^2 |\mathring{A}|^4 \,\mathrm{d}\mu \\
-\int_\Sigma H^2 \gamma(\mathring{A}, \nabla(\nabla H - 2\omega)) \,\mathrm{d}\mu &= \int_\Sigma \gamma(\operatorname{div}(H^2 \mathring{A}), \nabla H - 2\omega) \,\mathrm{d}\mu \\
&= \int_\Sigma 2H \mathring{A}(\nabla H, \nabla H - 2\omega^\#) \\
&\qquad + H^2 \gamma(\operatorname{div} \mathring{A}, \nabla H + 2\omega) \,\mathrm{d}\mu \\
&= \int_\Sigma 2H \mathring{A}(\nabla H, \nabla H) - 4H \mathring{A}(\nabla H, \omega^\#) \\
&\qquad + H^2 \left(\frac{1}{2} |\nabla H|^2 + 2|\omega|^2 + 2\gamma(\nabla H, \omega) \right) \,\mathrm{d}\mu \\
&\le C \int_\Sigma |H||\mathring{A}||\nabla H|^2 + |H||\mathring{A}||\nabla H| \\
&\qquad + H^2 |\nabla H|^2 + H^2 |\nabla H| + H^2 \,\mathrm{d}\mu \\
&\le C \int_\Sigma H^2 |\nabla H|^2 + |\mathring{A}|^2 |\nabla H|^2 + H^2 \,\mathrm{d}\mu \\
&\le C + C \int_\Sigma H^2 |\nabla H|^2 + |\mathring{A}|^2 |\nabla H|^2 \,\mathrm{d}\mu
\end{aligned}
$$

Piecing all this together yields

$$
\int_\Sigma H^2 |\nabla \mathring{A}|^2 + H^4 |\mathring{A}|^2 \,\mathrm{d}\mu \le C(C_B) \left(1 + \int_\Sigma H^2 |\nabla H|^2 + |\mathring{A}|^2 |\nabla H|^2 + H^2 |\mathring{A}|^4 \,\mathrm{d}\mu \right),
$$

and in view of (4.1.4) we arrive at

$$
\begin{aligned}
&\int_\Sigma |\nabla^2 H|^2 + H^2 |\nabla H|^2 + H^2 |\nabla \mathring{A}|^2 + H^4 |\mathring{A}|^2 \,\mathrm{d}\mu \\
&\qquad \le C(C_B, C_L) \left(1 + \int_\Sigma |\mathring{A}|^2 |\nabla H|^2 + H^2 |\mathring{A}|^4 \,\mathrm{d}\mu \right).
\end{aligned} \tag{4.1.6}
$$

Now we treat the terms on the right-hand side with the Michael-Simon-Sobolev inequality.

$$
\begin{aligned}
\int_\Sigma H^2 |\mathring{A}|^4 \,\mathrm{d}\mu &\le C \left(\int_\Sigma |\nabla H |\mathring{A}|^2| + H^2 |\mathring{A}|^2 \,\mathrm{d}\mu \right)^2 \\
&\le C \left(\int_\Sigma |\nabla H||\mathring{A}|^2 + |H||\mathring{A}||\nabla \mathring{A}| + H^2 |\mathring{A}|^2 \,\mathrm{d}\mu \right)^2 \\
&\le C \int_\Sigma |\mathring{A}|^2 \,\mathrm{d}\mu \int_\Sigma |\mathring{A}|^2 |\nabla H|^2 + |H|^2 |\nabla \mathring{A}|^2 + H^4 |\mathring{A}|^2 \,\mathrm{d}\mu
\end{aligned}
$$

Since $\|\mathring{A}\|_{L^2}$ is bounded by $C\epsilon$ we may absorb the last two terms on the right to the left of (4.1.6). We treat the term $|\mathring{A}|^2|\nabla H|^2$ similarly.

$$\begin{aligned}\int_\Sigma |\mathring{A}|^2|\nabla H|^2\,\mathrm{d}\mu &\leq C\left(\int_\Sigma |\nabla\mathring{A}||\nabla H| + |\mathring{A}||\nabla^2 H| + |H||\mathring{A}||\nabla H|\,\mathrm{d}\mu\right)^2\\ &\leq C\left(\int_\Sigma |\mathring{A}||\nabla^2 H| + |H||\mathring{A}||\nabla H|\,\mathrm{d}\mu\right)^2 + C\left(\int_\Sigma |\nabla\mathring{A}||\nabla H|\,\mathrm{d}\mu\right)^2\\ &\leq C\int_\Sigma |\mathring{A}|^2\,\mathrm{d}\mu\int_\Sigma |\nabla^2 H|^2 + H^2|\nabla H|^2\,\mathrm{d}\mu\\ &\quad + C\left(\int_\Sigma |\nabla\mathring{A}|^2 + |\nabla H|^2\,\mathrm{d}\mu\right)^2 \qquad (4.1.7)\end{aligned}$$

The first two terms on the right-hand side can be absorbed again. For the last one we recall the Codazzi equation $\nabla H = 2\operatorname{div}\mathring{A} - 2\omega$. It implies

$$|\nabla H|^2 \leq C + C|\nabla\mathring{A}|^2.$$

Thus we are left with one final term. To treat it, we employ the integrated Simons identity; compare (4.1.5).

$$\int_\Sigma |\nabla\mathring{A}|^2 + \frac{1}{2}H^2|\mathring{A}|^2\,\mathrm{d}\mu \leq C|\Sigma| + \int_\Sigma -\gamma(\mathring{A},\nabla^2 H) - 2\gamma(\mathring{A},\nabla\omega) + 2|\mathring{A}^4|\,\mathrm{d}\mu$$

Using the Michael-Simon-Sobolev inequality, we find

$$\begin{aligned}\int_\Sigma |\mathring{A}|^4\,\mathrm{d}\mu &\leq C\left(\int_\Sigma |\mathring{A}||\nabla\mathring{A}| + |H||\mathring{A}|^2\,\mathrm{d}\mu\right)^2\\ &\leq C\int_\Sigma |\mathring{A}|^2\,\mathrm{d}\mu\int_\Sigma |\nabla\mathring{A}|^2 + |H|^2|\mathring{A}|^2\,\mathrm{d}\mu,\end{aligned}$$

which can be absorbed. Next we observe

$$\begin{aligned}\int_\Sigma -2\gamma(\mathring{A},\nabla\omega)\,\mathrm{d}\mu &\leq C\int_\Sigma |\mathring{A}||\nabla\operatorname{Ric}| + |\mathring{A}|^2 + |\mathring{A}||H|\,\mathrm{d}\mu\\ &\leq C\int_\Sigma |\mathring{A}|^2\,\mathrm{d}\mu + C\left(\int_\Sigma |\mathring{A}|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Finally, we use the Hölder inequality to infer

$$\int_\Sigma |\nabla\mathring{A}|^2\,\mathrm{d}\mu \leq C(C_B, C_L)\left(|\Sigma| + \|\mathring{A}\|_{L^2(\Sigma)}\right) + \|\mathring{A}\|_{L^2(\Sigma)}\|\nabla^2 H\|_{L^2(\Sigma)}. \qquad (4.1.8)$$

Since this term enters quadratic in (4.1.7), we have shown that we can absorb every term but the constant one on the right-hand side of (4.1.6) to the left.

□

The next corollary is in some ways the heart of this section as it establishes the roundness of small surfaces of generalized Willmore type.

Corollary 4.1.4. *Assume Σ is a surface as in Theorem 4.1.3 and define R via $|\Sigma| = 4\pi R^2$. If $|\Sigma|$ is small enough then there exists a constant $C = C(C_L, C_B)$ such that the following estimates hold.*

$$\|\mathring{A}\|_{L^2(\Sigma)} \leq C|\Sigma|$$
$$\|H - 2/R\|_{L^\infty(\Sigma)} \leq C|\Sigma|^{1/2}$$

In particular, the mean curvature has to be positive and the inverse of the mean curvature has to be bounded.

$$\|H^{-1}\|_{L^\infty(\Sigma)} \le C|\Sigma|^{1/2}$$
$$\|H^{-1} - R/2\|_{L^\infty(\Sigma)} \le C|\Sigma|^{3/2}$$

Proof. For the first part we will apply the Michael-Simon-Sobolev inequality to $|\mathring{A}|^2$ and $H^2|\mathring{A}|^2$.

$$\begin{aligned}\int_\Sigma |\mathring{A}|^2\,\mathrm{d}\mu &\le C\left(\int_\Sigma |\nabla \mathring{A}| + H|\mathring{A}|\,\mathrm{d}\mu\right)^2\\ &\le C|\Sigma|\int_\Sigma |\nabla \mathring{A}|^2 + H^2|\mathring{A}|^2\,\mathrm{d}\mu\\ \int_\Sigma H^2|\mathring{A}|^2\,\mathrm{d}\mu &\le C\left(\int_\Sigma |\nabla H||\mathring{A}| + |H||\nabla \mathring{A}| + H^2|\mathring{A}|\,\mathrm{d}\mu\right)^2\\ &\le C|\Sigma|\int_\Sigma |\mathring{A}|^2|\nabla H|^2 + |H|^2|\nabla \mathring{A}|^2 + H^4|\mathring{A}|^2\,\mathrm{d}\mu\\ &\le C|\Sigma| + C|\Sigma|\int_\Sigma |\mathring{A}|^2|\nabla H|^2\,\mathrm{d}\mu\end{aligned}$$

We may use estimates (4.1.8) and (4.1.7) from the proof of Theorem 4.1.3 to deal with $\|\nabla \mathring{A}\|_{L^2}^2$ and $\||\mathring{A}||\nabla H|\|_{L^2}^2$.

$$\begin{aligned}\int_\Sigma |\mathring{A}|^2|\nabla H|^2\,\mathrm{d}\mu &\le C\int_\Sigma |\mathring{A}|^2\,\mathrm{d}\mu + C\left(\int_\Sigma |\nabla \mathring{A}|^2\,d\mu + |\Sigma|\,\mathrm{d}\mu\right)^2\\ &\le C\|\mathring{A}\|_{L^2(\Sigma)}^2 + C\left(|\Sigma| + \|\mathring{A}\|_{L^2(\Sigma)}^2 + \|\mathring{A}\|_{L^2(\Sigma)}\right)^2\\ &\le C\|\mathring{A}\|_{L^2(\Sigma)}^2 + C|\Sigma|^2\\ \int_\Sigma |\nabla \mathring{A}|^2\,\mathrm{d}\mu &\le C\left(|\Sigma| + \|\mathring{A}\|_{L^2(\Sigma)}\right)\end{aligned}$$

Thus we get

$$\|\mathring{A}\|_{L^2(\Sigma)}^2 \le C|\Sigma|^2 + C|\Sigma|\|\mathring{A}\|_{L^2(\Sigma)} + C|\Sigma|^2\|\mathring{A}\|_{L^2(\Sigma)}^2 + C|\Sigma|^4$$

We may absorb two terms to obtain

$$\|\mathring{A}\|_{L^2(\Sigma)} \le C|\Sigma|.$$

For the second part we use the estimate of De Lellis and Müller [5], see also Theorem A.1.8, in conjunction with Lemma A.1.3 and corollary A.1.5 to estimate

$$\|H - 2/R\|_{L^2(\Sigma,\gamma)} \le C\|\mathring{A}\|_{L^2(\Sigma,\gamma)} + C|\Sigma|^2 \le C|\Sigma|.$$

From Lemma A.1.15 we gather that

$$\|H - 2/R\|_{L^\infty(\Sigma)}^4 \le \|H - 2/R\|_{L^2(\Sigma)}^2\int_\Sigma |\nabla^2 H|^2 + H^4(H - 2/R)^2\,d\mu.$$

The first term on the right can be estimated with Theorem 4.1.3 and the second one can be absorbed to the left. To see that, let $a := 2/R$ and note

$$H^4 \le 4(H-a)^4 + 8a^2(H-a)^2 + 4a^4.$$

This implies

$$\|H-a\|^4_{L^\infty(\Sigma)} \le C|\Sigma|^2 + C|\Sigma|^2\|H-a\|^4_{L^\infty(\Sigma)} + C|\Sigma|\|H-a\|^2_{L^\infty(\Sigma)}$$

After applying Young's inequality to the last term on the right-hand side, we can absorb all but the constant term to the left.

□

4.2 Surface Concentration

In this section we characterize the points around which surfaces of generalized Willmore type concentrate. First we present a result for general Hawking type functionals, then we perform detailed calculation for the physically interesting case $L = -1/4\, P^2$.

Definition 4.2.1 (concentration point). A Point $p \in M$ is called a concentration point of $\mathcal{H}$ if there is a constants $r_0 > 0$ and an $A_0 > 0$ such that for every $r \in (0, r_0)$ there is an $A \in (0, A_0)$ and a spherical, area-constrained, critical surface Σ_r of $\mathcal{H}$ with $|\Sigma_r| = A$ contained in the geodesic ball $\mathcal{B}_r(p)$.

Definition 4.2.2. Let $S := S^2_1(a)$ be the two sphere around $a \in \mathbb{R}^3$ with outer normal vector field ν and fet $F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be bounded. For a multi index $(\alpha_1, ..., \alpha_k)$, $k \in \mathbb{N}$ introduce

$$c^{(\alpha_1,\dots,\alpha_k)}(F,a) := \int_{S^2_1(a)} F(a,\nu)(x-a)^{\alpha_1}...(x-a)^{\alpha_k}\,\mathrm{d}\mu,$$
$$c(F,a) := \int_{S^2_1(a)} F(a,\nu)\,\mathrm{d}\mu.$$

Theorem 4.2.3. *Let (M,g) be C_B bounded and let $\mathcal{H} = \mathcal{W} + \mathcal{L}$ be a Hawking type functional with $\mathcal{L}[\Sigma] = \int_\Sigma L(x,\nu)\,\mathrm{d}\mu$ for a smooth L.*

1. *Let M be compact, then there exits at least one concentration point of $\mathcal{H}$. The concentrating surfaces Σ_r at that point are area-constrained minimizers of $\mathcal{H}$ and obey $\mathcal{H}[\Sigma_r] \le 4\pi + \epsilon_0^2$, where ϵ_0 is the constant from Proposition 4.1.3.*

2. *Let $p \in M$ be a concentration point of $\mathcal{H}$ such that the concentrating surfaces Σ_r have energy $\mathcal{H}[\Sigma_r] \le 4\pi + \epsilon_0^2$.*

   Then, in Riemannian normal coordinates around p, the vector V_p with components

   $$V_p^\alpha = -c\left(d_V L_\alpha, p\right) + 2c^\alpha\left(L,p\right) + c^{(\alpha,\beta)}\left(d_V L_\beta, p\right)$$

   vanishes.
   Moreover, if V_{a_i} vanishes identically for a sequence of points $\{a_i\}$ converging to p, as constructed in the proof, then we have that

   $$\nabla^M \operatorname{Sc}_p - W_p = 0.$$

   Here W_p is a vector whose components read

   $$W_p^\alpha = \frac{3}{2\pi}\left(-c^\beta\left(\nabla^M_\beta d_V L_\alpha, p\right) + 3c^{(\alpha,\beta)}\left(d_M L_\beta, p\right) + c^{(\alpha,\beta,\gamma)}\left(\nabla^M_\gamma d_V L_\beta, p\right)\right).$$

Remark 2.

a) If $L(x, \nu)$ is even in ν, then the V_a vanish as all the involved integrals vanish.

b) W_p involves only terms with a d_M. We can therefore see it as the gradient of some function w at p. This leads to the interpretation that, provided V vanishes, a concentration point of $\mathcal{H}$ is a critical point for $\operatorname{Sc} - w$.

Proof. For the first part, we know by Proposition 4.1.1 that there is a minimizing area-constrained embedded sphere with $\mathcal{H}[\Sigma_A] \leq 4\pi + \epsilon_0^2$, for any small enough area A. Moreover, they are contained in normal neighborhoods $\mathcal{B}_{r_A}(p_A)$, where r_A and $\sqrt{A}$ are comparable. For $A \to 0$ the points p_A will subconverge to a point p which is a concentration point by construction.

For the second part, let r_0 and A_0 be as in the definition of concentration point. Suppose $r \in (0, r_0)$ and $r_0 \leq \operatorname{inj}(M, g)$. Let Σ be a spherical, area-constrained, critical point of $\mathcal{H}$ contained in $\mathcal{B}_r(p)$ with area $|\Sigma| = 4\pi R^2$ and $\mathcal{H}[\Sigma_r] \leq 4\pi + \epsilon_0^2$. Since L is smooth and we work in $\mathcal{B}_r(p)$, the results of Section 4.1 apply. In Appendix A we discuss that, by choosing r_0 smaller if necessary, we have the estimates $d := \operatorname{diam}_g \Sigma \leq CR \leq Cr$. Since there is at least one such Σ for any $r \in (0, r_0)$, we may suppose that $2d < r$. This allows us to use Lemma A.1.11 in order to find normal coordinates ψ adapted to Σ around $p_\Sigma \in M$ such that $\Sigma \subset \mathcal{B}_{2d}(p_\Sigma) \subset \mathcal{B}_r(p_\Sigma)$, $d_g(p, p_\Sigma) \leq r$ and

$$\int_{\psi(\Sigma)} y \, \mathrm{d}\mu_g(y) = 0.$$

Additionally, in these adapted normal coordinates we have

$$\max_{x \in \Sigma} |x|_E \leq CR.$$

We will operate in these coordinates from now on.

Consider the area-constrained variation of $\mathcal{H}[\Sigma]$ with respect to the vector field $f\nu = g(b, \nu)/H\nu$, where b is a constant vector field to be chosen later. Recalling the traced Gauss equation

$$\operatorname{Sc}^\Sigma = \operatorname{Sc} - 2\operatorname{Ric}(\nu, \nu) + \frac{1}{2}H^2 - |\mathring{A}|^2,$$

we may split the Willmore functional into two new functionals

$$\begin{aligned}
\mathcal{U}[\Sigma] &= \frac{1}{2}\int_\Sigma |\mathring{A}|^2 \, d\mu \\
\mathcal{V}[\Sigma] &= \int_\Sigma \operatorname{Ric}(\nu, \nu) - \frac{1}{2}\operatorname{Sc} \, d\mu \\
\mathcal{W}[\Sigma] &= 4\pi(1 - q(\Sigma)) + \mathcal{U}[\Sigma] + \mathcal{V}[\Sigma]
\end{aligned}$$

and arrive at

$$\lambda \delta_f \mathcal{A} = \delta_f \mathcal{H} = \delta_f \mathcal{U} + \delta_f \mathcal{V} + \delta_f \mathcal{L}.$$

Let Ω be the region enclosed by Σ and let $\langle \cdot, \cdot \rangle$ denote the Euclidean scalar product on $\mathbb{R}^3$.

Estimating $\operatorname{Vol}(\Omega)$, $\delta_f \mathcal{A}[\Sigma]$ and $\delta_f \mathcal{U}[\Sigma]$, as well as the better part of $\delta_f \mathcal{V}[\Sigma]$ as in [28, Section 4] yields

$$\begin{aligned}\left|\mathrm{Vol}(\Omega)-\frac{4\pi}{3}R^3\right| &\leq CR^5,\\ |\lambda\delta_f\,\mathcal{A}[\Sigma]| &\leq CR^4,\\ |\delta_f\,\mathcal{U}[\Sigma]| &\leq CR^4,\\ \left|\delta_f\,\mathcal{V}[\Sigma]+\frac{1}{4}\,\mathrm{Vol}(\Omega)\langle\nabla^M\,\mathrm{Sc}_{p_\Sigma},b\rangle\right| &\leq CR^4,\end{aligned}$$

and hence

$$\left|\frac{\pi}{3}R^3\langle b,\nabla^M\,\mathrm{Sc}^M_{p_\Sigma}\rangle-\delta_f\,\mathcal{L}[\Sigma]\right|\leq CR^4. \tag{4.2.1}$$

Thus we need to estimate the variation of $\mathcal{L}$.

$$\delta_f\,\mathcal{L}[\Sigma]=\int_\Sigma f\,d_M L(\nu)-d_V L(\nabla f)+LHf\,d\mu \tag{4.2.2}$$

We start with the second term on the right-hand side, using $e_i:=\frac{\partial\phi}{\partial x_i}$.

$$\begin{aligned}\int_\Sigma d_V L(\nabla f)\,d\mu &= \int_\Sigma \frac{1}{H}d_V L(e_j)\gamma^{ij}\left(g(\nabla^M_i b,\nu)+A(b^T,e_i)-\frac{1}{H}\frac{\partial H}{\partial x^i}g(b,\nu)\right)\,d\mu\\ &= \int_\Sigma \frac{1}{H}d_V L(e_j)\gamma^{ij}\left(\mathring{A}(b^T,e_i)+g(\nabla^M_i b,\nu)\right)-\frac{1}{H^2}d_V L(\nabla H)\\ &\quad+\frac{1}{2}d_V L(b^T)\,d\mu\end{aligned}$$

The first three terms on the right-hand side can be estimated rather easily, using the results of Section 4.1 as well as the fact that we use Riemann normal coordinates on M.

$$\int_\Sigma \frac{1}{H}d_V L(e_j)\gamma^{ij}\left(\mathring{A}(b^T,e_i)+g(\nabla^M_i b,\nu)\right)-\frac{1}{H^2}d_V L(\nabla H)\,d\mu\leq CR^4$$

The other terms, that is $\int_\Sigma 1/2\,d_V L(b^T)+f\,d_M L(\nu)+LHf\,d\mu$, need to be treated in more detail. We will pull them back to an approximating sphere to perform explicit calculations. In Lemma A.1.3, Theorem A.1.8 and corollary A.1.10 we detailed how this is possible. The estimates derived there in conjunction with corollary 4.1.4 imply that, up to order $O(R^4)$, we have to estimate

$$\int_S -\frac{1}{2}\,d_V L(b^T)+\frac{R_E}{2}\langle b,\nu\rangle d_M L(\nu)+\langle b,\nu\rangle L\,\mathrm{d}\mu.$$

Here R_E is the Euclidean radius of Σ, $|\Sigma|_E=4\pi R_E^2$, which is comparable to R, and S is the round sphere of radius R_E, centered at $a=\int_\Sigma x\,\mathrm{d}\mu_E$, the Euclidean center of mass of Σ. The outer normal to S is given by $\nu=(x-a)/R_E$, where x is the position vector field.

Note that in this construction $d_g(p_\Sigma,a)\leq Cd^3$ and hence

$$d_g(p,a)\leq Cr. \tag{4.2.3}$$

Note also that the term $1/2d_V L(b^T)+\langle b,\nu\rangle L$ is of order one, whereas $R_E/2\langle b,\nu\rangle d_M L(\nu)$ is of order R. This means, unless $\int_\Sigma 1/2d_V L(b^T)+\langle b,\nu\rangle L$ vanishes up to $O(R^2)$, it will dominate the concentration point p. We will perform a Taylor expansion in the first variable around a in order to separate the orders of magnitude.

$$\begin{aligned} d_V L(b^T)_{(x,\nu)} &= d_V L(b^T)_{(a,\nu)} + \nabla^M_{x-a} d_V L(b^T)_{(a,\nu)} + O(R^2) \\ \langle b,\nu\rangle L(x,\nu) &= \langle b,\nu\rangle L(a,\nu) + \langle b,\nu\rangle d_M L(x-a)_{(a,\nu)} + O(R^2) \\ \frac{R_E}{2}\langle b,\nu\rangle d_M L(\nu)_{(x,\nu)} &= \frac{R_E}{2}\langle b,\nu\rangle d_M L(\nu)_{(a,\nu)} + O(R^2) \end{aligned}$$

Integrating over S and separating by powers of R_E yields

$$\begin{aligned} &\int_S -\frac{1}{2} d_V L(b^T) + \frac{R_E}{2}\langle b,\nu\rangle d_M L(\nu) + \langle b,\nu\rangle L \,\mathrm{d}\mu, \qquad (4.2.4)\\ &= R_E^2\left(-\frac{b^\alpha}{2}c(d_V L_\alpha) + b^\alpha c^\alpha(L) + \frac{b^\alpha}{2}c^{(\alpha,\beta)}(d_V L_\beta)\right), \\ &+ R_E^3\left(-\frac{b^\alpha}{2}c^\beta(\nabla^M_\beta d_V L_\alpha) + \frac{3b^\alpha}{2}c^{(\alpha,\beta)}(d_M L_\beta) + \frac{b^\alpha}{2}c^{(\alpha,\beta,\gamma)}(\nabla^M_\gamma d_V L_\beta)\right), \\ &+ O(R^4). \end{aligned}$$

Define the components of two vectors V_a and W_a by

$$\begin{aligned} V_a^\alpha &= (-c(d_V L_\alpha) + 2c^\alpha(L) + c^{(\alpha,\beta)}(d_V L_\beta)), \\ W_a^\alpha &= \frac{3}{2\pi}\left(-c^\beta(\nabla^M_\beta d_V L_\alpha) + 3c^{(\alpha,\beta)}(d_M L_\beta) + c^{(\alpha,\beta,\gamma)}(\nabla^M_\gamma d_V L_\beta)\right). \end{aligned}$$

If V_a is not zero, then equation (4.2.1) implies that $\langle b, V_a\rangle \to 0$ for $r \to 0$ and any constant vector b. Moreover, by equation (4.2.3) we get that $a \to p$ as $r \to 0$. Choosing $b = V_p$ yields that p is characterized by the vanishing of V_p.
If V_a vanishes, we get

$$\nabla^M \operatorname{Sc}_p - W_p = 0,$$

using equation (4.2.1) and $b = \nabla^M \operatorname{Sc}_p - W_p$.

□

Now we apply the previous result to the Hawking energy $\mathcal{E}$. Recall

$$\mathcal{E}[\Sigma] = \frac{|\Sigma|^{1/2}}{16\pi^{3/2}}\left(4\pi - \mathcal{H}[\Sigma]\right)$$

for $\mathcal{H}[\Sigma] = \mathcal{W}[\Sigma] - \frac{1}{4}\int_\Sigma (\operatorname{tr}_\Sigma K)^2 \,\mathrm{d}\mu$, where K is a smooth symmetric 2-tensor on M. Clearly the area-constrained minimizers of $\mathcal{H}$ are the area-constrained maximizers of $\mathcal{E}$.

Theorem 4.2.4. *Let $\mathcal{H}$ be as above. At any concentration point p of $\mathcal{H}$ around which the concentrating surfaces obey $\mathcal{H}[\Sigma_r] \leq 4\pi + \epsilon_0^2$, where ϵ_0 is the constant from Proposition 4.1.3 we have*

$$\nabla^M\left(\operatorname{Sc}_p + \frac{3}{5}\operatorname{tr}_M K_p^2 + \frac{1}{5}|K_p|^2\right) = 0.$$

Proof. We apply Theorem 4.2.3. First note that the function $L = -\frac{1}{4}(\operatorname{tr}_\Sigma K)^2$ is even in ν and hence the vectors V_a vanish. Thus we need to compute the vector W_p with components

$$W_p^\alpha = \frac{3}{2\pi}\left(-c^\beta(\nabla^M_\beta d_V L_\alpha, p) + 3c^{(\alpha,\beta)}(d_M L_\beta, p) + c^{(\alpha,\beta,\gamma)}(\nabla^M_\gamma d_V L_\beta, p)\right).$$

Recall the following derivatives of $P^2 = (\operatorname{tr}_\Sigma K)^2$ from Lemma 1.2.3.

$$\begin{aligned} d_M P^2(X) &= 2P \operatorname{tr}_\Sigma \nabla^M_X K \\ d_V P^2(X) &= -4PK(X,\nu) \\ \nabla^M_Y d_V P^2(X) &= -4\left(\operatorname{tr}_\Sigma \nabla^M_Y K\right) K(X,\nu) - 4P\nabla^M_Y K(X,\nu) \end{aligned}$$

We will calculate the three terms of W separately. Since we choose normal coordinates around p, we have $p = 0$. The relevant integrals are presented in Lemma C.2.2. For better readability we drop the subscript from tr_M.

1)

$$\begin{aligned} c^{(\alpha,\beta)}\left((d_M P^2)_\beta, p\right) &= \int_{S_1} 2P \operatorname{tr}_{S_1} \nabla^M_\beta K x^\alpha x^\beta \,\mathrm{d}\mu \\ &= 2\int_{S_1} (\operatorname{tr} K - K(\nu,\nu))\left(\operatorname{tr} \nabla^M_\beta K - \nabla^M_\beta K(\nu,\nu)\right) x^\alpha x^\beta \,\mathrm{d}\mu \end{aligned}$$

$$\begin{aligned} \int_{S_1} 2\operatorname{tr} K \operatorname{tr} \nabla^M_\beta K x^\alpha x^\beta \,\mathrm{d}\mu &= 2\operatorname{tr} K \operatorname{tr} \nabla^M_\beta K \int_{S_1} x^\alpha x^\beta \,\mathrm{d}\mu \\ &= \frac{8\pi}{3} \operatorname{tr} K \operatorname{tr} \nabla^M_\alpha K \\ \int_{S_1} -2K(\nu,\nu) \operatorname{tr} \nabla^M_\beta K x^\alpha x^\beta \,\mathrm{d}\mu &= -2K_{\gamma\delta} \operatorname{tr} \nabla^M_\beta K \int_{S_1} x^\alpha x^\beta x^\gamma x^\delta \,\mathrm{d}\mu \\ &= -\frac{8\pi}{15}\left(\operatorname{tr} K \operatorname{tr} \nabla^M_\alpha K + 2\langle K(e_\alpha,\cdot), \operatorname{tr} \nabla^M_\cdot K\rangle\right) \\ \int_{S_1} -2\operatorname{tr} K \nabla^M_\beta K(\nu,\nu) x^\alpha x^\beta \,\mathrm{d}\mu &= -2\operatorname{tr} K \nabla^M_\beta K_{\gamma\delta} \int_{S_1} x^\alpha x^\beta x^\gamma x^\delta \,\mathrm{d}\mu \\ &= -\frac{8\pi}{15}\left(\operatorname{tr} K \operatorname{tr} \nabla^M_\alpha K + 2\operatorname{tr} K \operatorname{div}_E K(e_\alpha)\right) \end{aligned}$$

$$\begin{aligned} \int_{S_1} 2K(\nu,\nu)\nabla^M_\beta K(\nu,\nu) x^\alpha x^\beta \,\mathrm{d}\mu &= 2K_{\gamma\delta}\nabla^M_\beta K_{\mu\nu} \int_{S_1} x^\alpha x^\beta x^\gamma x^\delta x^\mu x^\nu \,\mathrm{d}\mu \\ &= \frac{8\pi}{105}\big(\operatorname{tr} K \nabla_\alpha \operatorname{tr} K + 2\operatorname{tr} K \operatorname{div} K(e_\alpha,\cdot) + 2\langle K(e_\alpha,\cdot), \nabla \operatorname{tr} K\rangle \\ &\quad + 4\langle K(e_\alpha,\cdot), \operatorname{div} K\rangle + 2\langle K, \nabla_\alpha K\rangle + 4\langle K, \nabla K(e_\alpha,\cdot)\rangle\big) \\ &=: \frac{8\pi}{105} I_\alpha \end{aligned}$$

2)

$$\begin{aligned} c^\beta\left((d_M[d_V P^2]_\alpha)_\beta, p\right) &= -4\int_{S_1}\left(\left(\operatorname{tr}_{S_1} \nabla^M_{e_\beta} K\right) K(e_\alpha,\nu) + P\nabla^M_{e_\beta} K(e_\alpha,\nu)\right) x^\beta \,\mathrm{d}\mu \\ &= -4\int_{S_1}\Big(\left(\operatorname{tr} \nabla^M_{e_\beta} K - \nabla^M_{e_\beta} K(\nu,\nu)\right) K(e_\alpha,\nu) \\ &\quad + (\operatorname{tr} K - K(\nu,\nu))\, \nabla^M_{e_\beta} K(e_\alpha,\nu)\Big) x^\beta \,\mathrm{d}\mu \end{aligned}$$

$$\begin{aligned}
\int_{S_1} -4\operatorname{tr}\nabla^M_\beta K K(e_\alpha,\nu)x^\beta \,\mathrm{d}\mu &= -4\operatorname{tr} K\nabla^M_\beta K_{\beta,\gamma}\int_{S_1} x^\beta x^\gamma \,\mathrm{d}\mu \\
&= -\frac{16\pi}{3}\operatorname{tr}\nabla^M_\beta K K_{\alpha\beta} \\
&= -\frac{16\pi}{3}\langle \operatorname{tr}\nabla^M_\cdot K, K(e_\alpha,\cdot)\rangle \\
\int_{S_1} 4\nabla^M_\beta K(\nu,\nu)K(e_\alpha,\nu)x^\beta \,\mathrm{d}\mu &= 4\nabla^M_\beta K_{\gamma\delta}K_{\alpha\epsilon}\int_{S_1} x^\beta x^\gamma x^\delta x^\epsilon \,\mathrm{d}\mu \\
&= \frac{16\pi}{15}\Big(2\langle \operatorname{div}_E K, K(e_\alpha,\cdot)\rangle + \langle \operatorname{tr}\nabla^M_\cdot K, K(e_\alpha,\cdot)\rangle\Big) \\
\int_{S_1} -4\operatorname{tr} K\nabla^M_\beta K(e_\alpha,\nu)x^\beta \,\mathrm{d}\mu &= -4\operatorname{tr} K\nabla^M_\beta K_{\alpha\gamma}\int_{S_1} x^\beta x^\gamma \,\mathrm{d}\mu \\
&= -\frac{16\pi}{3}\operatorname{tr} K\operatorname{div}_E K(e_\alpha) \\
\int_{S_1} 4K(\nu,\nu)\nabla^M_\beta K(e_\alpha,\nu)x^\beta \,\mathrm{d}\mu &= 4K_{\gamma\delta}\nabla^M_\beta K_{\alpha\epsilon}\int_{S_1} x^\beta x^\gamma x^\delta x^\epsilon \,\mathrm{d}\mu \\
&= \frac{16\pi}{15}\Big(\operatorname{tr} K\operatorname{div}_E K(e_\alpha) + 2\langle K, \nabla^M_\cdot K(e_\alpha,\cdot)\rangle\Big)
\end{aligned}$$

3)

$$\begin{aligned}
c^{(\alpha,\beta,\gamma)}\Big((d_M[d_V P^2]_\beta)_\gamma, p\Big) = -4\int_{S_1} \Big(&(\operatorname{tr}\nabla^M_\gamma K - \nabla^M_\gamma K(\nu,\nu))K(e_\beta,\nu) \\
&+(\operatorname{tr} K - K(\nu,\nu))\nabla^M_\gamma K(e_\beta,\nu)\Big)\, x^\alpha x^\beta x^\gamma \,\mathrm{d}\mu
\end{aligned}$$

$$\begin{aligned}
\int_{S_1} -4\operatorname{tr}\nabla^M_\gamma K K(e_\beta,\nu)x^\alpha x^\beta x^\gamma \,\mathrm{d}\mu &= -4\operatorname{tr}\nabla^M_\gamma K K_{\beta\delta}\int_{S_1} x^\alpha x^\beta x^\gamma x^\delta \,\mathrm{d}\mu \\
&= -\frac{16\pi}{15}\Big(\operatorname{tr} K\operatorname{tr}\nabla^M_\alpha K + 2\langle \operatorname{tr}\nabla^M_\cdot K, K(e_\alpha,\cdot)\rangle\Big) \\
\int_{S_1} 4\nabla^M_\gamma K(\nu,\nu)K(e_\beta,\nu)x^\alpha x^\beta x^\gamma \,\mathrm{d}\mu &= 2\frac{8\pi}{105}I_\alpha \\
\int_{S_1} -4\operatorname{tr} K\nabla^M_\gamma K(e_\beta,\nu)x^\alpha x^\beta x^\gamma \,\mathrm{d}\mu &= -4\operatorname{tr} K\nabla^M_\gamma K_{\beta\delta}\int_{S_1} x^\alpha x^\beta x^\gamma x^\delta \,\mathrm{d}\mu \\
&= -\frac{16\pi}{15}\Big(\operatorname{tr} K\operatorname{tr}\nabla^M_\alpha K + 2\operatorname{tr} K\operatorname{div}_E K(e_\alpha)\Big) \\
\int_{S_1} 4K(\nu,\nu)\nabla^M_\gamma K(e_\beta,\nu)x^\alpha x^\beta x^\gamma \,\mathrm{d}\mu &= 2\frac{8\pi}{105}I_\alpha
\end{aligned}$$

Adding all these terms up yields

$$\begin{aligned}
W^\alpha_p[P^2] = &-\frac{8}{5}\langle \operatorname{tr}\nabla^M_\cdot K, K(e_\alpha,\cdot)\rangle - \frac{8}{5}\operatorname{tr} K\operatorname{div}_E K(e_\alpha) + 4\operatorname{tr} K\operatorname{tr}\nabla_\alpha K \\
&- \frac{16}{5}\langle K(e_\alpha,\cdot), \operatorname{div}_E K\rangle - \frac{16}{5}\langle K, \nabla^M_\cdot K(e_\alpha,\cdot)K\rangle + \frac{4}{5}I_\alpha, \\
= &\ \frac{4}{5}\partial_\alpha\left(3\operatorname{tr} K^2 + |K|^2\right).
\end{aligned}$$

□

Corollary 4.2.5. *Using the the various integrals calculated in the proof above we find*

$$\begin{aligned} W^\alpha[\operatorname{tr} K^2] &= 6\partial_\alpha \operatorname{tr} K^2 \\ W^\alpha[K(\nu,\nu)^2] &= \frac{2}{5}\partial_\alpha \left(\operatorname{tr} K^2 + 2|K|^2\right) \\ W^\alpha[\operatorname{tr} K K(\nu,\nu)] &= 2\partial_\alpha \operatorname{tr} K^2. \end{aligned}$$

In particular, this allows us to determine functions $L(x,\nu)$ such that $\mathcal{H}_L$ concentrates at points with $\partial_\alpha\left(\mathrm{Sc} + \operatorname{tr} K^2 - |K|^2\right) = 0$. For instance

$$\begin{aligned} L &= -\frac{3}{4}P^2 + 2K(\nu,\nu)^2, \\ L &= -\frac{1}{4}\operatorname{tr} K^2 + \frac{5}{4}K(\nu,\nu)^2. \end{aligned}$$

Moreover, we can determine the expansion of $\mathcal{H}$ on small spheres.

Corollary 4.2.6. *Let $\Sigma \subset M$ be a spherical surfaces, with $|\Sigma| = 4\pi R^2$. Suppose Σ is contained in a normal coordinate neighborhood $B_r(p)$ as in Lemma A.1.11 and that $\|\mathring{A}\|^2_{L^2(\Sigma)} \leq C_1 r R^2$ for a constant C_1. Let $\mathcal{H}_L$ be of Hawking type for an $L \in C^1$, then there is a constant $r_0 > 0$ such that for all $r \in (0, r_0)$ we have the expansion*

$$\left|\mathcal{H}_L[\Sigma] - 4\pi + \frac{2\pi}{3}R^2 \operatorname{Sc}_p - R^2 c(L,p)\right| \leq CR^3.$$

Where C depends only on r_0, C_1, L and C_B. Additionally, we calculate

$$\begin{aligned} c(\operatorname{tr} K^2, p) &= 4\pi \operatorname{tr} K_p^2 \\ c(K(\nu,\nu)^2, p) &= \frac{4\pi}{15}\left(\operatorname{tr} K_p^2 + 2|K_p|^2\right) \\ c(\operatorname{tr} K K(\nu,\nu), p) &= \frac{4\pi}{3}\operatorname{tr} K_p^2. \end{aligned}$$

Hence we obtain

$$\left|\mathcal{H}[\Sigma] - 4\pi + \frac{2\pi}{3}R^2 \operatorname{Sc}_p + \frac{2\pi}{15}R^2(3\operatorname{tr} K_p^2 + |K_p|^2)\right| \leq CR^3$$

for the functional corresponding to the Hawking energy.

Proof. From [28, Theorem 5.1] we get the expansion

$$\left|\mathcal{W}[\Sigma] - 4\pi + \frac{2\pi}{3}R^2 \operatorname{Sc}_p\right| \leq CR^3.$$

Using the coordinates of Lemma A.1.11 and Lemma A.1.10 we have $\mathcal{L}[\Sigma] = R_E^2 c(L,p) + O(R^3)$. This implies the expansion since R and R_E are all comparable. The explicit calculation of c is straightforward, using the results of Appendix C.2. ☐

Corollary 4.2.7. *Let $L = \alpha \operatorname{tr} K^2 + \beta K(\nu,\nu)^2 + \gamma \operatorname{tr} K K(\nu,\nu)$ for α, β, $\gamma \in \mathbb{R}$, then we have*

$$W_p[L] = \nabla^M \frac{3}{2\pi} c(L,p),$$

i.e. a concentration point of $\mathcal{H}_L$ is a critical point of its second order expansion.

We would like to point out that we have checked the calculations in Theorem 4.2.4 and Corollary 4.2.6 using Mathematica 7, see Remark 4 in Appendix C.2.

Chapter 5

Foliations of Asymptotically Schwarzschild Manifolds by Hawking Type Surfaces

The goal of this chapter is to construct a foliation by spherical, area-constrained Hawking type surfaces of the outer regions of asymptotically Schwarzschild manifolds. From the perspective of general relativity these manifold represent isolated systems. Here the idea is that in the absents of classical energy the spacetime should become asymptotically flat. Foliations can be used to describe the asymptotic behavior of the ambient manifold and provide a notion of center of mass.

The key step in constructing the foliation is to prove that the linearization of the Euler-Lagrange equation 1.2.2 is invertible since this allows us to employ the implicit function theorem in order to construct the foliation perturbatively from the known foliation of the Schwarzschild space by the spheres $S_R(0)$. This is done in the beginning of Section 5.3. The next two sections introduce the necessary notation and a priori estimates.

5.1 Notation and Existence of Minimizers

We work in the setting of asymptotically Schwarzschild manifolds as introduced in [30], briefly recalling the notation and preliminary results here. Let g_S be the (Euclidean) Schwarzschild metric on $\mathbb{R}^3 \setminus \{0\}$. It is conformally flat and can be written as $g_S = \phi^4 g_E$ where g_E is the Euclidean metric, $\phi = 1 + \frac{m}{2r}$ for a strictly positive mass parameter m. In case we need to specify the mass we will also write g^S_m for g_S We refer to the position vector field by x, its norm is denoted by $r = |x|_E$ and we denote the radial vector field by $\rho = \frac{x}{r}$. For a compact surface $\Sigma \subset \mathbb{R}^3$ we set $r_{\min} := \min_{x\in\Sigma} |x|_E$. We use the indices S and E to refer to geometric quantities depending on g_S or g_E respectively. When we compare Willmore functionals with respect to different metrics, we employ the notation $\mathcal{W}[\Sigma, g]$ to specify the metric.

Definition 5.1.1 (see [30, Defintion 1])**.** A Riemannian manifold (M, g) is called (m, σ, η)–asymptotically Schwarzschild if there is a compact $K \subset M$ and a diffeomorphism $x : M\backslash K \to \mathbb{R}^3 \setminus B_\sigma(0)$ such that in these coordinates we have the following estimate with respect to g_S.

$$\sup_{\mathbb{R}^3 \setminus B_\sigma(0)} \left(r^2|g - g_S| + r^3|\nabla - \nabla^S| + r^4|\operatorname{Ric} - \operatorname{Ric}_S| + r^5|\nabla \operatorname{Ric} - \nabla^S \operatorname{Ric}_S|\right) \le \eta$$

Here the norms are taken with respect to g_s, but the estimate implies that the norms with respect to g_S and g are equivalent with a factor depending only on (m, σ, η).

The region $M \backslash K \cong \mathbb{R}^3 \backslash B_\sigma(0)$ is called the asymptotically Schwarzschild end of M.

Lemma 5.1.2 (see [30, Lemma 1]).

1. *The Ricci curvature of g_S is given by*

$$\mathrm{Ric}_S = \frac{m}{r^3}\phi^{-2}\left(g_E - 3\rho^\flat \otimes \rho^\flat\right)$$

here $\cdot^\flat$ is the musical isomorphism with respect to g_E.

2. *If $\Sigma \subset \mathbb{R}^3 \backslash \{0\}$ is a surface, then the following relations hold.*

$$\begin{aligned}
\nu_S &= \phi^{-2}\nu_E \\
\mathrm{d}\mu_S &= \phi^4 \mathrm{d}\mu_E \\
\mathring{A}_S &= \phi^2\, \mathring{A}_E \\
H_S &= \phi^{-2}H_E + 4\phi^{-3}\partial_{\nu_E}\phi
\end{aligned}$$

Lemma 5.1.3 (see [30, Lemma 2]). *Let (M, g) be (m, σ, η)–asymptotically Schwarzschild. Let $\Sigma \subset B_\sigma(0)$ a surface and let μ and μ_S be the the measures on it induced by g and g_S. To compare the area elements write $\mathrm{d}\mu - \mathrm{d}\mu_S = h\mathrm{d}\mu$, then we have the following estimates.*

$$r^2|\nu - \nu_S| \leq C\eta r^2|h| \leq C\eta$$

For the second fundamental form we find

$$\begin{aligned}
|A - A_S| &\leq C\eta(r^{-3} + r^{-2}|A|), \\
|\nabla A - \nabla A_S| &\leq C\eta(r^{-4} + r^{-3}|A| + r^{-2}|\nabla A|).
\end{aligned}$$

The following lemma indirectly relates the different scales $r_{\min}^2$ and $|\Sigma|$. It is a variant of [19, Lemma 5.3].

Lemma 5.1.4 (see [30, Lemma 3]). *For each $\alpha_0 > 2$ there exist constants $r_0(m, \sigma, \eta, \alpha_0) > 0$ and $C(m, \sigma, \eta, \alpha_0) > 0$, such that for all $\alpha > \alpha_0$ and for all surfaces $\Sigma \subset \mathbb{R}^3 \backslash B_{r_0}(0)$ for which the divergence formula holds the following inequality is satisfied.*

$$\int_\Sigma r^{-\alpha}\,\mathrm{d}\mu \leq C r_{\min}^{\alpha-2}\,\mathcal{W}[\Sigma]$$

Proof. We apply the divergence formula to the scaled position vector field. To that end, note that

$$\begin{aligned}
\mathrm{div}_\Sigma(xr^{-\alpha}) &= r^{-\alpha}\,\mathrm{div}_\Sigma\, x - \alpha r^{-\alpha-2} g(x^T, x), \\
g(x^T, x) &= |x|_g^2 - g(x, \nu)^2,
\end{aligned}$$

and

$$|\,\mathrm{div}_\Sigma\, x - 2| \leq Cr^{-1},$$

hence

$$\left|\int_\Sigma (2-\alpha)r^{-\alpha} + \alpha r^{-\alpha-2}g(x,\nu)^2 - Hr^{-\alpha}g(x,\nu)\,\mathrm{d}\mu\right| \leq C\int_\Sigma r^{-\alpha-1}\,\mathrm{d}\mu. \tag{5.1.1}$$

Now choose $\alpha = 2$ to obtain

$$\int_\Sigma r^{-4} g(x,\nu)^2 \,\mathrm{d}\mu \le C \int_\Sigma r^{-3} \,\mathrm{d}\mu + \int_\Sigma H^2 + \frac{1}{4} g(x,\nu)^2 r^{-4} \,\mathrm{d}\mu,$$

and therefore

$$\int_\Sigma r^{-4} g(x,\nu)^2 \,\mathrm{d}\mu \le C \int_\Sigma r^{-3} + H^2 \,\mathrm{d}\mu.$$

For $\alpha = 3$ we find

$$\begin{aligned}\int_\Sigma r^{-3} \,\mathrm{d}\mu &\le C \int_\Sigma r^{-4} + 3r^{-5} g(x,\nu)^2 - r^{-3} H g(x,\nu) \,\mathrm{d}\mu,\\ &\le C r_{\min}^{-1} \int_\Sigma r^{-3} \,\mathrm{d}\mu + C r_{\min}^{-1} \int_\Sigma r^{-3} + H^2 \,\mathrm{d}\mu,\end{aligned}$$

and thus

$$\int_\Sigma r^{-3} \,\mathrm{d}\mu \le C r_{\min}^{-1} \int_\Sigma H^2 \,\mathrm{d}\mu.$$

Inserting this back into (5.1.1)yields the claim for $\alpha > 2$.

$$\begin{aligned}\int_\Sigma r^{-\alpha} \,\mathrm{d}\mu &\le \frac{C}{\alpha - 2} \int_\Sigma r^{-\alpha-1} + \alpha r^{-\alpha-2} g(x,\nu)^2 - r^{-\alpha-1} H g(x,\nu) \,\mathrm{d}\mu\\ &\le \frac{C}{\alpha-2} r_{\min}^{-1} \int_\Sigma r^{-\alpha} \,\mathrm{d}\mu + \frac{C}{\alpha-2} r_{\min}^{-\alpha+2} \int_\Sigma r^{-4} g(x,\nu)^2 - r^{-3} |H||g(x,\nu)| \,\mathrm{d}\mu\\ &\le \frac{C}{\alpha-2} r_{\min}^{-1} \int_\Sigma r^{-\alpha} \,\mathrm{d}\mu + \frac{C}{\alpha-2} r_{\min}^{-\alpha+2} \int_\Sigma H^2 \,\mathrm{d}\mu\end{aligned}$$

□

Now we can relate the Willmore energy in the asymptotically Schwarzschild end of (M, g) to the one in the Schwarzschild space and to the Euclidean one. Then next lemma follows directly form Lemma 5.1.3.

Lemma 5.1.5 (see [30, Lemma 5])**.** *Let (M, g) be (m, σ, η)–asymtotically Schwarzschild then there exist constantsr$_0(m,\sigma,\eta) > 0$ and $C(m,\sigma,\eta) > 0$ such that for every surface Σ in $\mathbb{R}^3 \setminus B_{r_0}(0)$ we have*

$$|\mathcal{W}[\Sigma, g] - \mathcal{W}[\Sigma, g_S]| \le C\eta r_{\min}^{-2} \left(\|A\|^2_{L^2(\Sigma,g)} + \eta r_{\min}^{-2} \|A\|^2_{L^2(\Sigma,g)} \right)$$

$$\begin{aligned}\left| \| \mathring{A} \|^2_{L^2(\Sigma,g)} - \| \mathring{A}_E \|^2_{L^2(\Sigma,g_E)} \right| \le C\eta r_{\min}^{-2} \Big(&\| \mathring{A} \|^2_{L^2(\Sigma,g)} + \|H\|_{L^2(\Sigma,g)} \| \mathring{A} \|_{L^2(\Sigma,g)}\\ &+ \eta r_{\min}^{-2} \|H\|^2_{L^2(\Sigma,g)} \Big).\end{aligned}$$

In the next lemma we compute the expansion of the Willmore energy of round sphere explicitly. This is used in Theorem 5.1.8 to compare the energy of a sequence minimizing a Hawking type functionals to the energy of spheres in order to establish that the sequence does not drift of to infinity. This in turn ensures the existence of minimizers.

Lemma 5.1.6. *Consider $(\mathbb{R}^3 \setminus \{0\}, g_S)$, then the Willmore energy of a surface Σ is given by*

$$\mathcal{W}[\Sigma, g_S] = \mathcal{W}[\Sigma, g_E] - 2m \int_\Sigma H_E \frac{\langle x, \nu_E \rangle_E}{r^2(2r+m)} \,\mathrm{d}\mu_E + 4m^2 \int_\Sigma \frac{\langle x, \nu_E \rangle_E^2}{r^4(2r+m)^2} \,\mathrm{d}\mu_E.$$

For $\Sigma = S_R(a)$ we find the following expansions.

1. *For* $|a| \to 0$

$$\mathcal{W}[S_R(a), g_S] = 4\pi - \frac{32\pi m R}{(2R+m)^2} + O(|a|^2). \tag{5.1.2}$$

2. *For* $|a| \to \infty$*, set* $|a| = r_0 + R$

$$\mathcal{W}[S_R(a), g_S] = 4\pi + \frac{8\pi m^2 R^4}{5 r_0^6} + O(r_0^7). \tag{5.1.3}$$

3. *Similarly, for* $|a| = r_0 + R$ *and* $R \to 0$

$$\mathcal{W}[S_R(a), g_S] = 4\pi + \frac{128\pi m^2}{5 r_0^2 (2r_0 + m)^4} R^4 + O(R^5). \tag{5.1.4}$$

Proof. The formula for the Willmore energy is a direct consequence of Lemma 5.1.2. The expansions follow by straightforward calculation, see Lemma C.3.1. □

The following lemma will be used to control the position of surfaces in $\mathbb{R}^3 \setminus B_\sigma$.

Lemma 5.1.7. *Let* M *be* (m, σ, η)*–asymptotically Schwarzschild. For a closed surface* $\Sigma \subset M$ *define* $\Sigma_\sigma := \Sigma \cap \left(\mathbb{R}^3 \setminus B_\sigma\right)$*. There exists a constant* $C(m, \sigma, \eta) > 0$ *such that all closed surfaces* Σ *with* $\Sigma_\sigma \neq \emptyset$ *satisfy*

$$\max_{x \in \Sigma_\sigma} |x|_E \leq \min_{x \in \Sigma_\sigma} |x|_E + C\left(|\Sigma|^{1/2} \mathcal{W}[\Sigma]^{1/2} + |\Sigma|\right)$$

Proof. Lemma A.1.7 asserts

$$\mathrm{diam}_M(\Sigma) \leq C(m, \sigma, \eta)\left(|\Sigma|^{1/2} \mathcal{W}[\Sigma]^{1/2} + |\Sigma|\right),$$

and we can relate the diameter in M to the Euclidean one in $\mathbb{R}^3$ as follows.

Let x and y, be two distinct points in $\mathbb{R}^3 \setminus B_\sigma(0)$, let $\gamma : [0, 1] \to \mathbb{R}^3 \setminus B_\sigma(0)$ be a length minimizing geodesic in M such that $\gamma(0) = x$, $\gamma(1) = y$. Then we have

$$\begin{aligned} d_E(x, y) \leq L_E(\gamma) &= \int_0^1 \frac{\phi^2}{\phi^2} \sqrt{g_E(\gamma', \gamma')}\, \mathrm{d}s \\ &\leq \int_0^1 \sqrt{g_S(\gamma', \gamma')}\, \mathrm{d}s \\ &\leq \int_0^1 (1 + C\sqrt{\eta}\sigma^{-1})\sqrt{g(\gamma', \gamma')}\, \mathrm{d}s \\ &= C d_M(x, y). \end{aligned}$$

For a closed surface $\Sigma \subset M$ such that $\Sigma_\sigma \neq \emptyset$, define $r_{\max} := \max_{x \in \Sigma_\sigma} |x|_E$ and $r_{\min} := \min_{x \in \Sigma_\sigma} |x|_E$. Let $x_{\min}$ and $x_{\max}$ be two points in Σ_σ such that $|x_{\min}|_E = r_{\min}$ and $|x_{\max}|_E = r_{\max}$. Let γ be a length minimizing geodesic in M from $x_{\min}$ to $x_{\max}$. Let $x_1 \in B_\sigma(0)$ be the first point where γ leaves $\mathbb{R}^3 \setminus B_\sigma(0)$ (if at all) and let $x_2 \in B_\sigma(0)$ be the last point where γ reenters $\mathbb{R}^3 \setminus B_\sigma(0)$. Now we can estimate $r_{\max}$.

$$\begin{aligned} r_{\max} &\leq d_E(x_{\min}, x_{\max}) + r_{\min} \\ &\leq d_E(x_{\min}, x_1) + d_E(x_{\max}, x_2) + 2\sigma + r_{\min} \\ &\leq C d_M(x_{\min}, x_1) + C d_M(x_{\max}, x_2) + r_{\min} \\ &\leq C d_M(x_{\min}, x_{\max}) + r_{\min} \\ &\leq C\, \mathrm{diam}_M(\Sigma) + + r_{\min} \end{aligned}$$

□

Now we are in the position to prove the existence of minimizers in asymptotically Schwarzschild manifolds and to characterize them asymptotically.

Theorem 5.1.8. *Let (M, g) be (m, σ, η)–asymptotically Schwarzschild, let $\mathcal{H}[\Sigma] = \mathcal{W}[\Sigma] + \mathcal{L}[\Sigma]$ be a Hawing type functional and suppose $\mathcal{L}[\Sigma] = \int_\Sigma L(x, \nu)\, \mathrm{d}\mu_g$ with $|L| \leq C_L r^{-\alpha}$, $\alpha > 2$, in $\mathbb{R}^3 \setminus B_\sigma(0)$.*

1. *If $\alpha > 3$, then there exits a constant $A_0(C_L, m, \eta, \sigma) > 0$ such that for all $A \geq A_0$ there exists an area-constrained, $\mathcal{H}$ minimizing, haunted, branched, immersed bubble tree Σ_A with area $|\Sigma_A| = A$.*

2. *Suppose there are constants $r_1 > 0$, $\beta \in (0, 4)$ and $\tilde{C}_L > 0$ such that $\mathcal{L}[S_R(a)] \leq -\tilde{C}_L r_{\min}^{-\beta}$ for all $S_R(a) \subset \mathbb{R}^3 \setminus B_{r_1}$ with $|a| > R$. Then there exists an area-constrained $\mathcal{H}$ minimizing haunted, branched, immersed bubble tree Σ_A for any area $A > 0$.*

3. *There exists a constant $r_0(C_L, m, \sigma, \eta) > \sigma$ such that any area-constrained $\mathcal{H}$ minimizer Σ_A is an embedded sphere provided it is contained in $\mathbb{R}^3 \setminus B_{r_0}(0)$. Additionally, for every $\epsilon > 0$ the there exits a constant $r_1(C_L, m, \sigma, \eta) \geq r_0$ such that $\Sigma_A \subset \mathbb{R}^3 \setminus B_{r_1}(0)$ satisfies*
$$\mathcal{W}[\Sigma] \leq 4\pi + \epsilon.$$

Proof. In order to apply the compactness result of Theorem 2.0.5 we need to show that a minimizing sequence does not drift off to infinity, i.e. that it is contained in a compact region of the ambient space. By Lemma 5.1.7 it is enough to obtain a bound on $r_{\min}$.

Consider the spheres $S_R(a) \subset \mathbb{R}^3 \setminus B_\sigma(0)$ with $|a| \to \infty$. By Lemma 5.1.5 and 5.1.6, as well as the requirement on L we see

$$\beta(\mathcal{H}, M, 4\pi R^2) \leq \lim_{|a| \to \infty} \mathcal{W}[S_R(a), g] + \mathcal{L}[S_R(a)] = 4\pi.$$

Let Σ_k be a sequence of haunted, branched, immersed bubble trees with area A realizing $\beta(\mathcal{H}, M, A)$. If the $\Sigma_k \subset K$, where K is compact such that $M \setminus K \hat{=} \mathbb{R}^3 \setminus B_\sigma(0)$ we are done. So suppose that $\Sigma_k \subset \mathbb{R}^3 \setminus B_\sigma(0)$ and that $r_{k,min} := \inf_{\Sigma_k} |x|_E$ diverges. We show that this implies that the Σ_k have to be spheres and that $\mathcal{H}[\Sigma_k] \to 4\pi$. Since the Σ_k are minimizing, we have $\mathcal{H}[\Sigma_k] = \beta(\mathcal{H}, M, A) + \delta_k \leq 4\pi + \delta_k$, where $\delta_k \to 0$ as $k \to \infty$ and hence

$$\mathcal{W}[\Sigma_k, g] \leq \mathcal{H}[\Sigma_k] - \mathcal{L}[\Sigma_k] \leq 4\pi + C\delta_k$$

As we have argued in Theorem 2.0.5, we may assume all Σ_k are parametrized by the same topological bubble tree $S = \bigcup_i S^i$, $\phi_k(S) = \Sigma_k$. If S^i is not a ghost, then $0 < \mathcal{W}[\phi_k|_{S^i}, g] \leq \mathcal{W}[\Sigma_k, g] \leq 4\pi + \delta_k$. Integrating the Gauss equation over $\Sigma_k^i = \phi_k(S^i)$ and estimating the curvature terms yields that

$$\| \mathring{A}_k^i \|^2_{L^2(\Sigma_k^i, g)} \leq C\delta_k + C(m, \eta) r_{k,min}^{-1} \leq C\delta_k.$$

Using Lemma 5.1.5 we see that

$$|\mathcal{W}[\Sigma_k^i, g] - \mathcal{W}[\Sigma_k^i, g_S]| \leq C\delta_k,$$
$$\| \mathring{A}_{E,k}^i \|^2_{L^2(\Sigma_k^i, g_E)} \leq C\delta_k.$$

Therefore, Theorem A.1.8 implies

$$|\mathcal{W}[\Sigma_k^i, g_E] - 4\pi| \leq C\delta_k.$$

Now, Lemma 5.1.6 shows that $\mathcal{W}[\Sigma_k^i, g] \to 4\pi$ and hence $\mathcal{H}[\Sigma_k^i] \to 4\pi$. This means, asymptotically there can be only one regular bubble without any ghosts and Lemma A.1.17 implies that Σ_k has to be embedded. This discussion already proves assertion number three.

Next consider a centered sphere $S_R(0)$. Since its Euclidean trace free second fundamental form vanishes, we infer the following from Lemma 5.1.5, provided $r_{\min}(m, \sigma, \eta) = R$ is large enough

$$\begin{aligned} \| \mathring{A} \|^2_{L^2(S_R(0),g)} &\leq C\eta R^{-4}\, \mathcal{W}[S_R(0), g], \\ \mathcal{W}[S_R(0), g] &\leq \mathcal{W}[S_R(0), g_S] + C\eta^2 R^{-4}. \end{aligned}$$

From equation (5.1.2) we see that

$$\mathcal{H}[S_R(0)] \leq 4\pi - \frac{32\pi m R}{(2R+m)^2} + C(C_L, m, \eta)(R^{-\alpha+2} + R^{-4}) < 4\pi,$$

provided R is large enough. As $\alpha > 3$, this shows that the minimizing sequence cannot drift off to infinity altogether.

For the second statement consider the off-center spheres $S_R(a)$ for $|a|_E = r_0 + R$, $r_0(R, m, \sigma, \eta) \geq r_1$. Their generalized Willmore energies can be estimated as follows.

$$\begin{aligned} \mathcal{H}[S_R(a)] &= \mathcal{W}[S_R(a), g] + \mathcal{L}[S_R(a)] \\ &\leq \mathcal{W}[S_R(a), g_S] + C(\sigma, \eta)\eta^2 r_0^{-4} - C_L r_0^{-\beta} \\ &\leq 4\pi - \tilde{C}_L r_0^{-\beta} + C(\sigma, \eta)\eta^2 r_0^{-4} \end{aligned}$$

For $r_0(R, \tilde{C}_L, m, \sigma, \eta)$ large enough we have $\mathcal{H}[S_R(a)] < 4\pi$, which implies again that the minimizing sequence cannot drift off to infinity. □

5.2 Integral Estimates

Let $\mathcal{H}[\Sigma] = \mathcal{W}[\Sigma] + \int_\Sigma L(x, \nu)\, \mathrm{d}\mu$ be a Hawking type functional. Suppose L is smooth and decays like $|L(x, \nu)| + |d_V L(x, \nu)| + r|d_M L(x, \nu)| \leq C_L r^{-\alpha}$. We have the functional related to the Hawking energy in mind, $L = -\operatorname{tr}_\Sigma K^2$. In this case $K = O(r^{-2})$ and $\alpha = 4$ is a natural assumption. We would like to perform a similar analysis as carried out in [30], that is multiply the Euler–Lagrange equation by H^{-1} and integrate. But unlike in the Willmore case we have to treat terms of the form $r^{-\alpha} H^{-1}$. Therefore we first need quantitative control over H^{-1}. We proceed as we did for small surfaces in Section 4.1 in order produce the integral and pointwise estimates. Due to the third assertion of Theorem 5.1.8 we will consider only critical surfaces, in the next two sections.

Lemma 5.2.1. *Let $\Sigma \subset \mathbb{R}^3 \setminus B_{r_0}$ be an area-constrained critical surface of $\mathcal{H}$. Then there is an $r_0 > 0$ and a constant $C > 0$ depending on m, η and C_L, such that the Lagrange multiplier obeys*

$$|\lambda| \leq \frac{C}{|\Sigma|}\left(r_{\min}^{-1} + r_{\min}^{-\alpha+2}\right) \|A\|^2_{L^2} + C r_{\min}^{-2} |\Sigma|^{-1/2}\, \mathcal{W}[\Sigma]^{1/2}.$$

Proof. The proof is analogous to Proposition 4.1.2(see also [29, Proposition 5.3]) where we examined small surfaces. For any normal variation δ_f we have

$$\delta_f\, \mathcal{H}[\Sigma] = \lambda \delta_f\, \mathcal{A},$$

and thus

$$|\lambda| = \left| \frac{\delta_f \mathcal{W}[\Sigma] + \delta_f \mathcal{L}[\Sigma]}{2\delta_f \mathcal{A}[\Sigma]} \right|,$$

provided $\delta_f \mathcal{A} \neq 0$. We choose $f = g(x, \nu)$, where x is the position vector field.

$$\begin{aligned}
\delta_f \mathcal{A}[\Sigma] &= \int_\Sigma g(x,\nu) H \,\mathrm{d}\mu \\
&= \int_\Sigma \operatorname{div}_\Sigma x \,\mathrm{d}\mu \\
&= 2|\Sigma| + |\Sigma| O(r_{\min}^{-2}) \\
&\geq |\Sigma| \\
\delta_f \mathcal{L}[\Sigma] &= \int_\Sigma f d_M L(\nu) - d_V L(\nabla^\Sigma f) + f L H \,\mathrm{d}\mu \\
&\leq C \int_\Sigma r^{-\alpha} + r^{-\alpha}(r^{-1} + r|A|) + r^{-\alpha+1} H \,\mathrm{d}\mu \\
&\leq C r_{\min}^{-\alpha+2} \mathcal{W}[\Sigma] + C \int_\Sigma r^{-\alpha} + r^{-\alpha+2}|A|^2 \,\mathrm{d}\mu \\
&\leq C r_{\min}^{-\alpha+2} \|A\|_{L^2}^2 \\
\delta_f \mathcal{W}[\Sigma] &= \int_\Sigma H \Delta_\Sigma f + f H |\mathring{A}|^2 + f H \operatorname{Ric}(\nu,\nu) \,\mathrm{d}\mu
\end{aligned}$$

Here we calculate $\Delta_\Sigma f$ more carefully as we need to cancel the term $fH|\mathring{A}|^2$. Introduce normal coordinates and the $\star$ notation, that is $F \star G$ denotes a sum of contractions of F and G. We proceed with calculation in local coordinates.

$$\begin{aligned}
\nabla^\Sigma f^i &= \gamma^{ij} \partial_j g(x,\nu) \\
&= \gamma^{ij} g(\partial_j + x^\beta \frac{\partial \phi^\gamma}{\partial y^j} \Gamma^\delta_{\beta\gamma} e_\delta, \nu) + \gamma^{ij} g(x, \nabla^M \nu) \\
&= \gamma^{-1} \star g \star x \star d\phi \star \Gamma \star \nu + \gamma^{ij} A(x^T, \partial_j) \\
\Delta_\Sigma f &= \gamma^{-1} \star g \star (d\phi \star d\phi \star \Gamma \star \nu + x \star d\phi \star d\phi \star d\Gamma \star \nu) + \gamma^{-1} \star g \star x \star d\phi \star \Gamma \star A \\
&\quad + \gamma^{ij} \nabla^\Sigma_i A(x^T, \partial_j) + \gamma^{ij} A(\nabla^\Sigma_i x^T, \partial_j) \\
\left(\nabla^\Sigma_i x^T\right)^l &= \partial_i g\left(x, \frac{\partial \phi}{\partial y^k}\right) \gamma^{kl} \\
&= \gamma^{kl} g\left(x, \nabla^M_i \frac{\partial \phi}{\partial y^k}\right) + \gamma^{kl} g\left(\nabla^M_i x, \frac{\partial \phi}{\partial y^k}\right) \\
&= -g(x,\nu)\gamma^{kl} A_{ik} + \gamma^{kl}\gamma_{ki} + \gamma^{-1} \star x \star d\phi \star d\phi \star \Gamma \star g
\end{aligned}$$

Moreover, the Codazzi equations imply

$$\gamma^{ij} \nabla^\Sigma_i A(x^T, \partial_j) = \gamma(\nabla^\Sigma H, x^T) + \operatorname{Ric}(x^T, \nu).$$

Thus we have

$$\Delta_\Sigma f = -g(x,\nu)|A|^2 + H + \gamma(\nabla^\Sigma H, x) + \operatorname{Ric}(x^T, \nu) + O(r^{-2}) + A \star O(r^{-1}).$$

Note that

$$\begin{aligned}
2 \int_\Sigma H \gamma(\nabla^\Sigma H, x) \,\mathrm{d}\mu &= \int_\Sigma \operatorname{div}_\Sigma H^2 x - H^2 \operatorname{div}_\Sigma x \,\mathrm{d}\mu \\
&= \int_\Sigma H^3 g(x,\nu) - 2H^2 + H^2 O(r^{-2}) \,\mathrm{d}\mu.
\end{aligned}$$

Now we can compute the variation of $\mathcal{W}$.

$$\begin{aligned}
\delta_f \mathcal{W}[\Sigma] &= \int_\Sigma H\Delta_\Sigma f + fH|\mathring{A}|^2 + fH\operatorname{Ric}(\nu,\nu)\,\mathrm{d}\mu \\
&= \int_\Sigma -g(x,\nu)H|A|^2 + H^2 + \frac{1}{2}H^3 g(x,\nu) - H^2 + g(x,\nu)H|\mathring{A}|^2 + H\operatorname{Ric}(x,\nu) \\
&\quad + H^2 O(r^{-2}) + HO(r^{-2}) + HA * O(r^{-1})\,\mathrm{d}\mu \\
&= \int_\Sigma H\operatorname{Ric}(x,\nu) + H^2 O(r^{-2}) + HO(r^{-2}) + HA * O(r^{-1})\,\mathrm{d}\mu \\
&\le Cr_{\min}^{-1}\|A\|_{L^2}^2 + Cr_{\min}^{-2}|\Sigma|^{1/2}\mathcal{W}[\Sigma]^{1/2}
\end{aligned}$$

□

Theorem 5.2.2. *There are positive constants ϵ, r_0 and $C(m,\sigma,\eta,C_L)$ such that any spherical surface $\Sigma \subset \mathbb{R}^3 \setminus B_{r_0}$ with*

1. $\mathcal{W}[\Sigma] \le 4\pi + \epsilon$ *and*
2. $\mathrm{D}_\lambda(\Sigma) = 0$ *for a given* λ

satisfies

$$\int_\Sigma |\nabla^2 H|^2 + H^2|\nabla H|^2 + H^2|\nabla\mathring{A}|^2 + H^4|\mathring{A}|^2\,\mathrm{d}\mu \le Cd. \tag{5.2.1}$$

Here $d := r_{\min}^{-2\alpha} + r_{\min}^{-6} + \left(r_{\min}^{-2} + r_{\min}^{-2\alpha+4}\right)|\Sigma|^{-2} + r_{\min}^{-4}|\Sigma|^{-1}$.

Proof. This proof proceeds along the same lines as the one of Theorem 4.1.3. First we note that the integrated Gauss equation yields an estimate for $\mathring{A}$.

$$\begin{aligned}
\|\mathring{A}\|_{L^2}^2 &= 2\mathcal{W}[\Sigma] - 8\pi - 2\int_\Sigma \operatorname{Sc}^M - 2\operatorname{Ric}^M(\nu,\nu)\,\mathrm{d}\mu \\
&\le 2\epsilon + Cr_{\min}^{-1}\mathcal{W}[\Sigma] \\
&\le C\epsilon
\end{aligned}$$

Here we used the fact that the curvature is of order r^{-3} and employed Lemma 5.1.4. Recall the Euler–Lagrange equation (1.2.2) satisfied by Σ.

$$\Delta H + H|\mathring{A}|^2 + H\operatorname{Ric}(\nu,\nu) + HQ + \gamma(\mathring{A},S) + 2\lambda H + T = 0$$

Here Q, S and T depend on the normal of Σ and L. They scale like $r^{-\alpha}$, $r^{-\alpha}$ and $r^{-\alpha-1}$ respectively. Multiply the equation by ΔH and integrate over Σ.

$$\begin{aligned}
\int_\Sigma (\Delta H)^2\,\mathrm{d}\mu &\le \int_\Sigma \frac{1}{8}(\Delta H)^2 + 8H^2|\mathring{A}|^4 + C(r^{-3} + r^{-\alpha} + |\lambda|)|H\Delta H| \\
&\quad + Cr^{-\alpha}|\mathring{A}||\Delta H| + Cr^{-\alpha-1}|\Delta H|\,\mathrm{d}\mu \\
&\le \int_\Sigma \frac{1}{8}(\Delta H)^2 + 8H^2|\mathring{A}|^4 + C(r^{-3} + r^{-\alpha} + |\lambda|)^2 H^2 + \frac{1}{8}(\Delta H)^2 \\
&\quad + Cr^{-2\alpha}|\mathring{A}|^2 + \frac{1}{8}(\Delta H)^2 + Cr^{-2\alpha-2} + \frac{1}{8}(\Delta H)^2\,\mathrm{d}\mu
\end{aligned}$$

After absorbing we find

$$\int_\Sigma (\Delta H)^2\,\mathrm{d}\mu \le Cd + C\int_\Sigma H^2|\mathring{A}|^4\,\mathrm{d}\mu,$$

Next we use the integrated Bochner identity for H, see A.1.12.

$$\int_\Sigma |\nabla^2 H|^2 + H^2|\nabla H|^2 \,\mathrm{d}\mu = \int_\Sigma (\Delta H)^2 + \frac{1}{2}|\nabla H|^2 \left(|\mathring{A}|^2 - \frac{1}{2}H^2 + \mathrm{Sc}^M + \mathrm{Ric}^M(\nu,\nu)\right) + H^2|\nabla H|^2 \,\mathrm{d}\mu$$

We may absorb a term and use integration by parts.

$$\begin{aligned}\int_\Sigma |\nabla H|^2 r^{-3} \,\mathrm{d}\mu &= \int_\Sigma -r^3 H\Delta H + 3r^{-4} H\gamma(\nabla r, \nabla H)\,\mathrm{d}\mu \\ &\le C r_{\min}^{-6} + \int_\Sigma (\Delta H)^2 + \frac{1}{2}H^2|\nabla H|^2 \,\mathrm{d}\mu\end{aligned}$$

Absorbing into the left-hand side yields

$$\int_\Sigma |\nabla^2 H|^2 + H^2|\nabla H|^2 \,\mathrm{d}\mu \le C r_{\min}^{-6} + \int_\Sigma C(\Delta H)^2 + C|\nabla H|^2|\mathring{A}|^2 \,\mathrm{d}\mu.$$

We continue by integrating Simons' identity, see [51].

$$\begin{aligned}-\mathring{A}^{ij}\Delta\mathring{A}_{ij} + \frac{1}{2}H^2|\mathring{A}|^2 &= -\gamma(\mathring{A}, \nabla^2 H) + |\mathring{A}|^4 + |\mathring{A}|^2\,\mathrm{Ric}(\nu,\nu) \\ &\quad - \mathring{A}^{ij}\mathring{A}_j^k\,\mathrm{Ric}_{ik} - 2\gamma(\mathring{A}, \nabla\omega),\end{aligned}$$

where $\omega := \mathrm{Ric}^M(\nu,\cdot)$, multiply it by H^2 and integrate over Σ. We integrate twice by parts

$$\begin{aligned}\int_\Sigma -H^2\mathring{A}^{ij}\Delta\mathring{A}_{ij}\,\mathrm{d}\mu &= \int_\Sigma H^2|\nabla\mathring{A}|^2 + HA^{ij}\gamma(\nabla H, \nabla A_{ij})\,\mathrm{d}\mu \\ \int_\Sigma -H^2\gamma(\mathring{A}, \nabla^2 H - 2\nabla\omega)\,\mathrm{d}\mu &= \int_\Sigma 2H\,\mathring{A}(\nabla H, \nabla H - 2\omega^\#) \\ &\quad + H^2\,\mathrm{div}_\Sigma\,\mathring{A}(\cdot, \nabla H - 2\omega^\#)\,\mathrm{d}\mu\end{aligned}$$

and continue with the following estimate.

$$\begin{aligned}\int_\Sigma H^2|\nabla\mathring{A}|^2 + \frac{1}{2}H^4|\mathring{A}|^2\,\mathrm{d}\mu &\le 2\int_\Sigma |H||\mathring{A}||\nabla H||\nabla\mathring{A}| + H^2|\nabla\mathring{A}||\nabla H| + |H||\mathring{A}||\nabla H|^2 \\ &\quad + Cr^{-3}(H^2|\mathring{A}|^2 + |H||\mathring{A}||\nabla H| + H^2|\nabla\mathring{A}|) \\ &\quad + H^2|\mathring{A}|^4\,\mathrm{d}\mu \\ &\le \int_\Sigma \frac{1}{8}H^2|\nabla\mathring{A}|^2 + 8|\mathring{A}|^2|\nabla H|^2 \\ &\quad + \frac{1}{8}H^2|\nabla\mathring{A}|^2 + 8H^2|\nabla H|^2 + |H|^2|\nabla H|^2 + |\mathring{A}|^2|\nabla H|^2 \\ &\quad + Cr^{-6}H^2 + H^2|\mathring{A}|^4 + |\mathring{A}|^2|\nabla H|^2 + \frac{1}{8}H^2|\nabla\mathring{A}|^2 \\ &\quad + H^2|\mathring{A}|^4\,\mathrm{d}\mu\end{aligned}$$

$$\int_\Sigma H^2|\nabla\mathring{A}|^2 + H^4|\mathring{A}|^2\,\mathrm{d}\mu \le Cr_{\min}^{-6} + C\int_\Sigma |\mathring{A}|^2|\nabla H|^2 + H^2|\nabla H|^2 + H^2|\mathring{A}|^4\,\mathrm{d}\mu$$

So far we have shown

$$\begin{aligned}\int_\Sigma |\nabla^2 H|^2 + H^2|\nabla H|^2 + H^2|\nabla\mathring{A}|^2 + H^4|\mathring{A}|^2\,\mathrm{d}\mu &\le Cr_{\min}^{-6} \\ &\quad + C\int_\Sigma (\Delta H)^2 + |\mathring{A}|^2|\nabla H|^2 + H^2|\mathring{A}|^4\,\mathrm{d}\mu \\ &\le Cd + C\int_\Sigma |\mathring{A}|^2|\nabla H|^2 + H^2|\mathring{A}|^4\,\mathrm{d}\mu. \qquad (5.2.2)\end{aligned}$$

To deal with the last two terms we employ the Michael–Simon–Sobolev inequality, see Lemma A.1.14 or [30, Proposition 1]. The next part works exactly as in case of small surfaces, see (4.1.7) and above.

$$\begin{aligned}\int_\Sigma H^2|\mathring{A}|^4\,\mathrm{d}\mu &\le C\int_\Sigma|\mathring{A}|^2\,\mathrm{d}\mu\int_\Sigma|\mathring{A}|^2|\nabla H|^2+|H|^2|\nabla\mathring{A}|^2+H^4|\mathring{A}|^2\,\mathrm{d}\mu\\ \int_\Sigma|\mathring{A}|^2|\nabla H|^2\,\mathrm{d}\mu &\le C\int_\Sigma|\mathring{A}|^2\,\mathrm{d}\mu\int_\Sigma|\nabla^2H|^2+H^2|\nabla H|^2\,\mathrm{d}\mu\\ &\quad+C\left(\int_\Sigma|\nabla\mathring{A}|^2+|\nabla H|^2\,\mathrm{d}\mu\right)^2\end{aligned}$$

Since $\|\mathring{A}\|_{L^2}$ is bounded by $C\epsilon$, we may absorb all but the last two terms of the second inequality. Due to the Codazzi equation $\nabla H = 2\operatorname{div}\mathring{A}+2\omega$, we only need to find an estimate for $|\nabla\mathring{A}|^2$. To this end we integrate Simons' identity over Σ.

$$\begin{aligned}\int_\Sigma|\nabla\mathring{A}|^2+\frac{1}{2}H^2|\mathring{A}|^2\,\mathrm{d}\mu &\le C\int_\Sigma|\mathring{A}||\nabla^2H|+|\mathring{A}|^4+r^{-6}+|\mathring{A}||\nabla\omega|\,\mathrm{d}\mu\\ &\le C(\int_\Sigma|\mathring{A}||\nabla^2H|+|\mathring{A}|^4+r^{-6}\\ &\qquad+|\mathring{A}||\nabla\operatorname{Ric}|+|\mathring{A}||A||\operatorname{Ric}|\,\mathrm{d}\mu\\ \int_\Sigma|\nabla\mathring{A}|^2+H^2|\mathring{A}|^2\,\mathrm{d}\mu &\le C\left(r_{\min}^{-4}+\|\mathring{A}\|_{L^2}\|\nabla^2H\|_{L^2}+\int_\Sigma|\mathring{A}|^4\,\mathrm{d}\mu\right)\end{aligned}\tag{5.2.3}$$

The last term can be treated with the Michael–Simon–Sobolev inequality.

$$\begin{aligned}\int_\Sigma|\mathring{A}|^4\,\mathrm{d}\mu &\le C\left(\int_\Sigma|\mathring{A}||\nabla\mathring{A}|+H|\mathring{A}|^2\,\mathrm{d}\mu\right)^2\\ &\le C\|\mathring{A}\|_{L^2}\int_\Sigma|\nabla\mathring{A}|^2+H^2|\mathring{A}|^2\,\mathrm{d}\mu\end{aligned}\tag{5.2.4}$$

Finally we see that all terms in (5.2.2) except the constant one can be absorbed into the left-hand side. □

Corollary 5.2.3. *Under the assumptions of Theorem 5.2.2 we have the following estimate.*

$$\int_\Sigma|\mathring{A}|^2\,\mathrm{d}\mu\le C(m,\eta,C_L)|\Sigma|\left(r_{\min}^{-4}+|\Sigma|d\right)\tag{5.2.5}$$

Proof. We start with the Michael-Simon-Sobolev inequality and carry on with (5.2.3) and (5.2.4).

$$\begin{aligned}\int_\Sigma|\mathring{A}|^2\,\mathrm{d}\mu &\le C|\Sigma|\int_\Sigma|\nabla\mathring{A}|^2+H^2|\mathring{A}|^2\,\mathrm{d}\mu\\ &\le|\Sigma|\left(r_{\min}^{-4}+\|\mathring{A}\|_{L^2}\|\nabla^2H\|_{L^2}\right)\\ &\le|\Sigma|\left(r_{\min}^{-4}+\|\mathring{A}\|_{L^2}d^{1/2}\right)\end{aligned}$$

Solving the quadratic inequality yields the estimate. □

We would like to use these estimates to produce a more quantitative estimate of $\|\mathring{A}\|_{L^2}$, which would yield control of H in L^∞. To achieve this, we need to be able to compare the scales $r_{\min}^2$ and $|\Sigma|$. Define the Euclidean radius and the Euclidean center of mass of a closed surface Σ via $|\Sigma|_E =: 4\pi R_E^2$ and $a_E := |\Sigma|_E^{-1}\int_\Sigma x\,\mathrm{d}\mu_E$. As the next lemma establishes, we need to control the ratio

$$\tau := \frac{|a_E|_E}{R_E}.$$

Lemma 5.2.4. *Let $\epsilon \in (0,1)$. There exist positive constants constants $r_0(m,\sigma,\eta,C_L,\epsilon)$, $C(m,\sigma,\eta,C_L,\epsilon)$ and a constant $C_1(m,\sigma,\eta,C_L) > 0$ such that for any surface Σ as in Theorem 5.2.2 which satisfies*

1. *$\tau \le 1-\epsilon$ and*
2. *$|\Sigma| \le \frac{\epsilon}{16C_1} r_{\min}^{\min(3,\alpha)}$*

we have

$$C^{-1} r_{\min} \le R_E \le C r_{\min}.$$

Due to Lemma 5.1.3 and Lemma 5.1.2 this also implies that $|\Sigma|$and $r_{\min}^2$ are uniformly comparable.

Proof. We employ A.1.8 to relate Σ to an approximating sphere. Let $\psi : S_R(0) \to \Sigma$ be the map from that theorem and note that we can estimate $\| \mathring{A}_E \|_{L^2(\Sigma,g_E)} \le C\| \mathring{A} \|_{L^2(\Sigma,g)} + C\eta r_{\min}^2$. Let $x_{\min} \in \Sigma$ such that $|x_{\min}| = r_{\min}$ and $\tilde{x}_{\min}$ such that $\psi(\tilde{x}_{\min}) = x_{\min}$. Then the triangle inequality reveals

$$\begin{aligned} R_E = |\tilde{x}_{\min}|_E &\le |\psi(\tilde{x}_{\min}) - (\tilde{x}_{\min} + a)|_E + |a_E|_E + |\psi(\tilde{x}_{\min})|_E, \\ &\le C_1 R_E \left(|\Sigma| r_{\min}^{-4} + |\Sigma|^2 d\right)^{1/2} + C R_E \eta r_{\min}^{-2} + |a_E|_E + r_{\min}. \end{aligned}$$

Rearranging shows

$$\begin{aligned} r_{\min} &\ge R_E(1-\tau) - C_1 R_E \left(|\Sigma| r_{\min}^{-4} + |\Sigma|^2 (r_{\min}^{-6} + r_{\min}^{-2\alpha}) + r_{\min}^{-2} + r_{\min}^{-2\alpha+4}\right)^{1/2} - C R_E \eta r_{\min}^{-2}, \\ &\ge \frac{\epsilon}{2} R_E. \end{aligned}$$

Estimating $r_{\min}$ from above is more straightforward.

$$\begin{aligned} r_{\min} = |\psi(\tilde{x}_{\min})|_E &\le |\psi(\tilde{x}_{\min}) - (\tilde{x}_{\min} + a)|_E + |a_E|_E + R_E \\ &\le R_E(1+\tau) + 2C_1 R_E \left(|\Sigma|^2 (r_{\min}^{-6} + r_{\min}^{-2\alpha}) + r_{\min}^{-2} + r_{\min}^{-2\alpha+4}\right)^{1/2} \\ &\le 4R_E \end{aligned}$$

□

Definition 5.2.5. Let Σ be a surface in the asymptotically Schwarzschild end of M. As before, define R_E via $|\Sigma|_E = 4\pi R_E^2$ and set

$$\begin{aligned} \overline{\phi} &:= 1 + \frac{m}{2R_E}, \\ R_S &:= \overline{\phi}^2 R_E, \\ \overline{H}_S &:= \overline{\phi}^{-2} \frac{2}{R_E} - \overline{\phi}^{-3} \frac{2m}{R_E^2}. \end{aligned}$$

In corollary 4.1.4 we already detailed how to obtain L^∞ bounds on H. Using (5.2.5) and (5.2.1) we can prove an analogous estimate.

Proposition 5.2.6. *Let Σ be as in Lemma 5.2.4. Then we have the following estimates for a constant $C(m,\sigma,\eta,C_L)$.*

$$\int_\Sigma |\mathring{A}|^2 \, d\mu \le \left(r_{\min}^{-2} + r_{\min}^{-2\alpha+4}\right) \tag{5.2.6}$$

$$\|H - \overline{H}_S\|_{L^\infty} \le C(m,\eta) r_{\min}^{-2} \tag{5.2.7}$$

$$\|H - 2/R_E\|_{L^\infty} \le C r_{\min}^{-2}$$

In particular, the mean curvature is positive and its inverse is bounded, provided $r_{\min}$ is large enough.

$$\|H^{-1}\|_{L^\infty} \leq C r_{\min}$$

Proof. The first estimates follows directly from (5.2.5) and Lemma 5.2.4.

For the second one we first estimate$\|H - \overline{H}_S\|_{L^2}$ like in [30, Proposition 7]; using Theorem A.1.8 again as well as Lemma 5.1.3.

$$\begin{aligned}
\|H - \overline{H}_S\|_{L^2} \leq C \Bigg(&\|H - H_S\|_{L^2} \left\| \phi^{-2} \left(H_E - \frac{2}{R_E} \right) \right\|_{L^2} + \left\| \frac{2}{R_E} (\phi^{-2} - \overline{\phi}^{-2}) \right\|_{L^2} \\
&+ \left\| \frac{2m}{R_E} (\phi^{-3} - \overline{\phi}^{-3}) \right\|_{L^2} + \left\| \phi^{-3} \left(\frac{2m}{r^2} g_E(\rho, \nu_E) - \frac{2m}{R_E^2} \right) \right\|_{L^2} \Bigg) \\
\leq C &\left(\| \mathring{A} \|_{L^2} + r_{\min}^{-1} \right) \\
\leq C &r_{\min}^{-1}
\end{aligned}$$

The L^∞ estimate follows from Lemma A.1.15:

$$\|H - \overline{H}_S\|^4_{L^\infty(\Sigma)} \leq \|H - \overline{H}_S\|^2_{L^2(\Sigma)} \int_\Sigma |\nabla^2 H|^2 + H^4 (H - \overline{H}_S)^2 \, d\mu.$$

The first term has the right decay and the second one can be estimated as follows.

$$\begin{aligned}
\int_\Sigma H^4 (H - \overline{H}_S)^2 \, d\mu &\leq \int_\Sigma 2H^2 (H - \overline{H}_S)^4 + 2H^2 \overline{H}_S^2 (H - \overline{H}_S)^2 \, d\mu \\
&\leq \int_\Sigma 4 (H - \overline{H}_S)^6 + 4\overline{H}_S^4 (H - \overline{H}_S)^2 + 8 \overline{H}_S^2 (H - \overline{H}_S)^4 \, d\mu \\
&\leq \left(C \|H - \overline{H}_S\|^4_{L^\infty} + \overline{H}_S^4 \|H - \overline{H}_S\|^2_{L^2} \right)
\end{aligned}$$

After absorbing we get

$$\|H - \overline{H}_S\|^4_{L^\infty} \leq C(M, L) r_{\min}^{-8}. \qquad \square$$

The estimate on the inverse of the mean curvature in Proposition 5.2.6 allows to derive additional integral and L^∞ estimates. The following theorem combines results analogous those in [30, Section 4 to 6].

Definition 5.2.7. In analogy to the definition of (m, σ, η)–asymptotic Schwarzschild manifolds we introduce the following decay conditions for smooth $L : TM \to \mathbb{R}$. Let α and η be positive, $k \in \mathbb{N}$, we define $L \in O^k_\eta(r^{-\alpha})$ recursively. We say $L \in O^0_\eta(r^{-\alpha})$ if in $\mathbb{R}^3 \setminus B_\sigma(0)$ we have $|L| \leq \eta r^{-\alpha}$ and $L \in O^{k+1}_\eta(r^{-\alpha})$ if $|L| \leq \eta r^{-\alpha}$, $d_M L \in O^k_\eta(r^{-\alpha-1})$, $d_V L \in O^k_\eta(r^{-\alpha})$.

Theorem 5.2.8. *For every positive m, η and σ there exits positive constants r_0, ϵ_0 and C, depending only on (m, η, σ), such that if (M, g) is (m, η, σ)–asymptotically Schwarzschild with $|\,\mathrm{Sc}\,| \leq \eta r_{\min}^{-5}$ and $\Sigma \subset B_{r_0}$ is spherical and has the following properties:*

1. *Σ satisfies (1.2.2) for $\lambda > 0$ and $L \in O^3_\eta(r^{-4})$*
2. *$|\mathcal{W}[\Sigma] - 4\pi| \leq \epsilon_0$*
3. *there are $\epsilon \in (0, 1)$ and $C > 0$, s.t. $\tau \leq 1 - \epsilon$ and $|\Sigma| \leq \frac{\epsilon}{16 C_1} r_{\min}^3$;*

then Σ satisfies

$$\tau \leq C\sqrt{\eta}r_{\min}^{-1}.$$

Moreover, we have the following estimates.

$$\|H - \overline{H^S}\|_{L^\infty} + \|\mathring{A}\|_{L^\infty} + r_{\min}\|\nabla H\|_{L^\infty} \leq C\sqrt{\eta}r_{\min}^{-3} \tag{5.2.8}$$

$$\|\nu - \phi^{-2}\rho\|_{L^\infty} \leq C\sqrt{\eta}r_{\min}^{-1} \tag{5.2.9}$$

$$\|\omega\|_{L^\infty} + r_{\min}\|\nabla\omega\|_{L^\infty} \leq C\sqrt{\eta}r_{\min}^{-4}$$

$$\|\lambda + \operatorname{Ric}(\nu,\nu)\|_{L^\infty} + \|\operatorname{Ric}(\nu,\nu) + 2mR_S^{-3}\|_{L^\infty} \leq C\sqrt{\eta}r_{\min}^{-4}$$

$$\int_\Sigma \frac{|\Delta H|^2}{H^2} + |\nabla A|^2 + |\nabla \ln H|^4 + |A|^2|\mathring{A}|^2 \,\mathrm{d}\mu \leq C\eta r_{\min}^{-6} \tag{5.2.10}$$

Proof. We present just the general idea of the proof. The calculations are completely analogous to [30, Lemma 6 to Theorem 8] with minor variations to account for the terms coming from $\mathcal{L}$. Suppose for now that $L = O_\eta^3(r^{-\alpha})$ for $\alpha > 3$.

First, we derive integral estimates from the Euler–Lagrange equation and the Gauss equation.

$$\begin{aligned}\int_\Sigma \frac{|\Delta H|^2}{H^2} + |\nabla A|^2 + |\nabla \ln H|^4 + |A|^2|\mathring{A}|^2 \,\mathrm{d}\mu \\ \leq \int_\Sigma |\omega|^2 + (\operatorname{Ric}(\nu,\nu)+\lambda)^2 \,\mathrm{d}\mu + C\eta^2 r_{\min}^{-2\alpha+2}\end{aligned}$$

$$\int_\Sigma |\nabla \ln H|^2 + |\mathring{A}|^2 \,\mathrm{d}\mu \leq Cr_{\min}^2 \int_\Sigma |\omega|^2 + (\operatorname{Ric}(\nu,\nu)+\lambda)^2 \,\mathrm{d}\mu + C\eta^2 r_{\min}^{-2\alpha+4}$$

Using the explicit expression of Ric_S we find

$$\begin{aligned}\left|\lambda - \frac{2m}{R_S^3}\right| &\leq C\int_\Sigma |\omega|^2 + (\operatorname{Ric}(\nu,\nu)+\lambda)^2 \,\mathrm{d}\mu \\ &\quad + Cr_{\min}^{-4}\left(\tau + r_{\min}\|\mathring{A}\|_{L^2} + \eta r_{\min}^{-1} + \eta r_{\min}^{-\alpha+4}\right),\end{aligned}$$

as well as

$$\begin{aligned}\|\lambda + \operatorname{Ric}(\nu,\nu)\|_{L^2} &\leq Cr_{\min}\left(\|\omega\|_{L^2}^2 + \|\lambda + \operatorname{Ric}(\nu,\nu)\|_{L^2}^2\right) \\ &\quad + Cr_{\min}^{-3}\left(\tau + r_{\min}\|\mathring{A}\|_{L^2} + \sqrt{\eta}r_{\min}^{-1} + \eta r_{\min}^{-\alpha+4}\right).\end{aligned}$$

The next step is to show the decay with respect to the Schwarzschild background is good enough.

$$\|\nu - \phi^{-2}\rho\|_{L^2}^2 \leq Cr_{\min}^2\left(\tau^2 + \|\mathring{A}\|_{L^2}^2 + \eta r_{\min}^{-2}\right)$$

$$\|\operatorname{Ric}(\nu,\nu) - \phi^{-4}\operatorname{Ric}^S(\rho,\rho)\|_{L^2}^2 \leq Cr_{\min}^{-4}\left(\tau^2 + \|\mathring{A}\|_{L^2}^2 + \eta r_{\min}^{-2}\right)$$

$$\|\omega\|_{L^2}^2 \leq Cr_{\min}^{-4}\left(\tau^2 + \|\mathring{A}\|_{L^2}^2 + \eta r_{\min}^{-2}\right)$$

$$\|\operatorname{Ric}^T - P^S_{\phi^{-2}\rho}\operatorname{Ric}^S\|_{L^2}^2 \leq Cr_{\min}^{-4}\left(\tau^2 + \|\mathring{A}\|_{L^2}^2 + \eta r_{\min}^{-2}\right)$$

Using these approximations and absorption we arrive at the following estimates.

$$\begin{aligned}
\|\lambda + \operatorname{Ric}(\nu,\nu)\|_{L^2} &\leq C r_{\min}^{-3}\left(\tau + \sqrt{\eta} r_{\min}^{-1} + \eta r_{\min}^{-\alpha+4}\right) \\
\|\omega\|_{L^2} &\leq C r_{\min}^{-2}\left(\tau + \sqrt{\eta} r_{\min}^{-1} + \eta r_{\min}^{-\alpha+2}\right) \\
\|\nabla\omega\|_{L^2} + \|\nabla \operatorname{Ric}(\nu,\nu)\|_{L^2} &\leq C r_{\min}^{-3}\left(\tau + \sqrt{\eta} r_{\min}^{-1} + \eta r_{\min}^{-\alpha+2}\right) \\
\|\nabla \ln H\|_{L^2} + \| \mathring{A} \|_{L^2} &\leq C r_{\min}^{-1}\left(\tau + \sqrt{\eta} r_{\min}^{-1} + \eta r_{\min}^{-\alpha+3}\right) \\
\int_\Sigma \frac{|\Delta H|^2}{H^2} + |\nabla A|^2 + &|\nabla \ln H|^4 + |A|^2 |\mathring{A}|^2 \,\mathrm{d}\mu \\
&\leq C r_{\min}^{-4}\left(\tau^2 + \eta r_{\min}^{-2} + \eta^2 r_{\min}^{-2\alpha+6}\right)
\end{aligned} \tag{5.2.11}$$

The L^∞ estimates again follow from A.1.15.

$$\|H - H^S\|_{L^\infty} + \| \mathring{A} \|_{L^\infty} + r_{\min}\|\nabla H\|_{L^\infty} \leq C r_{\min}^{-2}\left(\tau + \sqrt{\eta} r_{\min}^{-1} + \eta r_{\min}^{-\alpha+3}\right)$$

The final step is to show the decay of τ. This is done analogous to [30, Chapter 6], there they perform an intricate expansion of the variation of $\mathcal{W}$ for which the variation of $\mathcal{L}$ is a lower order term. For $\alpha = 4$, the estimates above have the same decay as the ones in [30], hence we obtain the same decay for τ as well. □

Remark 3. Suppose $L \in O_\eta(r^{-\alpha})$ for $\alpha \in (3,4)$. Based on the proof of Theorem 5.2.8 we conjecture the decay $\tau \leq C\sqrt{\eta} r_{\min}^{-\beta}$ for a $\beta \in (0,1)$ and a corresponding one for the other quantities.

Lemma 5.2.9. *There exists a $r_0(m,\sigma,\eta) > 0$ such that any closed surface $\Sigma \subset \mathbb{R}^3 \setminus B_{r_0}$ satisfying*

$$\|\nu - \phi^{-2}\rho\|_{L^\infty(\Sigma)} \leq C r_{\min}^{-1}$$

is a graph over $S^2_\sigma(0)$.

Proof. Let $\rho = \frac{y}{|y|}$, for $y \in \Sigma$ and note that (5.2.9) and Lemma 5.1.3 imply

$$\begin{aligned}
|\rho^T|^2 &= |\rho - \langle \rho, \nu_E\rangle \nu_E|^2 \\
&= |1 - g_S(\phi^{-2}\rho, \nu_E)^2| \\
&\leq C r_{\min}^{-1}.
\end{aligned}$$

Consider $f : \Sigma \to S : y \mapsto \frac{y}{|y|}\sigma = \rho\sigma$ and calculate

$$\begin{aligned}
df_y(v) &= \frac{\sigma v}{|y|} - \frac{\sigma}{|y|^3}\langle y, v\rangle \\
&= \frac{\sigma}{|y|}\left(v - \langle \rho, v\rangle \rho\right) \\
(df_y)_i^j &\geq \frac{\sigma}{|y|}\left(\delta_i^j - |\rho^T| - |\langle \nu, e_i\rangle|\right) \\
&\geq \frac{\sigma}{|y|}\left(\delta_i^j - C r_{\min}^{-1/2}\right)
\end{aligned}$$

This means that f is regular and thus locally invertible. Consider the partition

$$S = \dot{\bigcup}_{j\in\mathbb{N}} M_j$$

for $M_j := \{p \in S \mid |f^{-1}(p)| = j\}$. All M_j are open as f is locally invertible and they are closed as well, since $M_j = S \backslash \dot{\bigcup}_{i\in\mathbb{N},\, i\neq j} M_j$. Thus every M_j is either empty or all of S. Therefore Σ is a $1-$covering of S, and hence f is a diffeomorphism. We can write every point $y = \rho(y)r(y) \in \Sigma$ as $y = p + u(p)\rho(p) = \sigma^{-1}(\sigma + u)p$ for $p = f(y)$ and $u(p) = r(f^{-1}(p)) - \sigma$. $\square$

5.3 Foliation

In order to construct a foliation by critical surfaces via the implicit function theorem, we need to show that the linearization of the generalized Willmore equation (1.2.2) is invertible on critical surfaces. This is achieved in the same way as in [30]. In particular, using the estimates of Theorem 5.2.8, we may cite their discussion of the Jacobi operator.

Let $J := -\Delta - |A|^2 - \mathrm{Ric}(\nu,\nu)$ and decompose $W^{2,2}(\Sigma) = V_0 \oplus V_1 \oplus V_2$ into L^2 orthogonal eigenspaces of J. Here $V_0 = \mathrm{span}\{\varphi_0\}$, $V_1 = \mathrm{span}\{\varphi_1, \varphi_2, \varphi_3\}$ for φ_i eigenfunctions of J, and V_2 the L^2 orthogonal complement. For a function f the projections to V_i are denoted by f_i, $i \in \{0,1,2\}$. The eigenvalues of J are denoted by μ_i. The linearization of the area-constrained Willmore operator $LH + \frac{1}{2}H^3 - 2\lambda H$ is denoted by W_λ.

Lemma 5.3.1. *Let Σ be as in Theorem 5.2.8. Write equation (1.2.2) as*

$$\Delta H + H|\mathring{A}|^2 + H\,\mathrm{Ric}^M(\nu,\nu) + \gamma(A,S) + 2\lambda H + T = 0,$$

with S and T defined as

$$\begin{aligned} S &:= -2\,\mathrm{Hess}_V L - 2L\gamma + 2d_V L(\nu)\gamma, \\ T &:= -2d_M L(\nu) - 2\,\mathrm{div}_\Sigma d_V L. \end{aligned}$$

and denote the linearization of $\gamma(A,S) + T$ by E_L then we have the following estimate for all $f \in W^{4,2}(\Sigma)$.

$$\int_\Sigma f E_L f \,\mathrm{d}\mu \leq C\eta r_{\min}^{-4}\left(r_{\min}^{-2}\|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2\right) \tag{5.3.1}$$

Proof. Calculate the variation of $\gamma(A,S) + T$ in direction $f\nu$.

$$\begin{aligned} \left.\frac{\partial}{\partial s}\right|_0 T(x,\nu) &= d_m T(f\nu) + d_V T(-\nabla f) \\ \left.\frac{\partial}{\partial s}\right|_0 \gamma(A,S) &= \gamma\left(\mathrm{Hess}\, f, S\right) + \gamma(\nabla^V_{-\nabla f} S, A) + f\Big(\gamma(\nabla^M_\nu S, A) \\ &\quad -\gamma(S, A^2) + \gamma(S, \mathrm{Rm}(\cdot,\nu,\nu,\cdot))\Big) \end{aligned}$$

The estimate follow from Theorem 5.2.8 and the decay of L via integration by parts. $\square$

Proposition 5.3.2. *Let Σ be as in Theorem 5.2.8. There exists η_0 and r_0 depending on m and σ and ϵ such that the following estimate holds for all $f \in V_0^\perp$.*

$$\int_\Sigma f E_\lambda f \,\mathrm{d}\mu \geq 6m^2 r_{\min}^{-6} \int_\Sigma f^2 \,\mathrm{d}\mu$$

Proof. In [30, Chapter 7] it is shown how to estimate $\|\nabla f_1\|_{L^2}$ by a constant times $r_{\min}^{-2}\|f_1\|_{L^2}$ and that W_λ obeys the estimate

$$\int_\Sigma f W_\lambda f \,\mathrm{d}\mu \geq 12m^2 r_{\min}^{-6}\|f\|_{L^2}^2 + \frac{1}{2} r_{\min}^{-2}\|\nabla f_2\|_{L^2}^2.$$

The arguments used there relay solely on the estimates in Theorem 5.2.8 and facts about the eigenvalues of J. Hence we will use the results of [30, Chapter 7] here as well.

The estimate (5.3.1) shows that we can absorb part coming from E_L.

$$\int_\Sigma fE_\lambda f \,\mathrm{d}\mu \geq 12m^2 r_{\min}^{-6}\|f\|_{L^2}^2 + \frac{1}{2}r_{\min}^{-2}\|\nabla f_2\|_{L^2}^2 - C\eta r_{\min}^{-4}\left(r_{\min}^{-2}\|f\|_{L^2}^2 + \|\nabla f_2\|_{L^2}^2\right)$$

□

In order to show that E_λ is invertible we need to have control over functions in V_0. The necessary results are collected in the next proposition. It summarizes the lemmas 20 to 24 of [30] and can be proven in a completely analogous fashion, as we have the same decay estimates in Theorem 5.2.8 and E_L is a lower order perturbation.

Proposition 5.3.3. *Let Σ be as in Theorem 5.2.8. For a function $u \in L^1(\Sigma)$ denote its mean by $\overline{u} = |\Sigma|^{-1}\int_\Sigma u\,\mathrm{d}\mu$.*

1. *Let $u \in W^{4,2}(\Sigma)$ then we have the following estimates.*

$$\left|\mu_0 + |A|^2 + \mathrm{Ric}(\nu,\nu)\right| \leq C\sqrt{\eta}r_{\min}^{-4} \tag{5.3.2}$$

$$\|\nabla^2 u\|_{L^2(\Sigma)}^2 + r_{\min}^{-2}\|\nabla u\|_{L^2(\Sigma)}^2 \leq Cr_{\min}^{-4}\|u\|_{L^2(\Sigma)}^2 + Cr_{\min}\left|\int_\Sigma uE_\lambda u\,\mathrm{d}\mu\right| \tag{5.3.3}$$

2. *If, additionally, $u \in C^\infty(\Sigma)$ is a solution to $Ju = \mu_0 u$, then we find*

$$\|u-\overline{u}\|_{L^2(\Sigma)}^2 + r_{\min}^2\|\nabla u\|_{L^2(\Sigma)}^2 + r_{\min}^6\|\nabla u\|_{L^2(\Sigma)}^2\,\mathrm{d}\mu \leq C\sqrt{\eta}r_{\min}^{-2}\|u\|_{L^2(\Sigma)}^2, \tag{5.3.4}$$

$$\|u-\overline{u}\|_{L^\infty(\Sigma)} \leq C\eta^{1/4}r_{\min}^{-2}\|u\|_{L^2(\Sigma)}. \tag{5.3.5}$$

3. *Let $\delta > 0$ and let $u \in W^{4,2}(\Sigma)$ be a solution to $E_\lambda u = f$ with $\int_\Sigma (f-f_0)^2\,\mathrm{d}\mu \leq \delta r_{\min}^{-12}\|u\|_{L^2}^2$.*

$$\|u-u_0\|_{L^2(\Sigma)} \leq C\left(\sqrt{\delta} + \sqrt{\eta} + r_{\min}^{-1}\right)\|u\|_{L^2}. \tag{5.3.6}$$

4. *There exists a $\delta_0 > 0$ such that for all $\delta \in (0,\delta_0)$ and all solutions $u \in W^{4,2}(\Sigma)$ of $E_\lambda u = f$ with $\int_\Sigma (u-u_0)f\,\mathrm{d}\mu \leq \delta r_{\min}^{-6}\|u\|_{L^2(\Sigma)}\|u-u_0\|_{L^2(\Sigma)}$ we have*

$$\|u-\overline{u_0}\|_{L^\infty(\Sigma)} \leq C\left(\sqrt{\delta} + \eta^{1/4} + r_{\min}^{-1}\right)|\overline{u_0}|. \tag{5.3.7}$$

Now we are in the position to discuss the invertibility of E_λ.

Theorem 5.3.4. *There exists positive constants $\delta_0 > 0$, η_0 and r_0 such that for all $\eta \in (0,\eta_0)$ and all surface Σ as in Theorem 5.2.8 the operator*

$$E_\lambda : W^{4,2}(\Sigma) \subset L^2(\Sigma) \to L^2(\Sigma)$$

is invertible and satisfies the following estimates.

$$\|u\|_{L^2(\Sigma)} \leq \frac{r_{\min}^6}{\delta_0}\|E_\lambda\|_{L^2(\Sigma)}$$

$$r_{\min}^{-1}\|\nabla u\|_{L^2(\Sigma)} + \|\nabla^2 u\|_{L^2(\Sigma)} \leq C(m,\sigma,\eta)\frac{r_{\min}^4}{\delta_0}\|E_\lambda u\|_{L^2}$$

Proof. We argue by contradiction. Assume that E_λ is not injective, then there is a $u \in W^{4,2}(\Sigma)\backslash\{0\}$ such that $E_\lambda u = 0$ and $\|u\|_{L^2(\Sigma)} = 1$. In particular we have

$$\sup_{v\in L^2(\Sigma),\, \|v\|_{L^2}=1} \left| \int_\Sigma v E_\lambda u \right| \le \delta r_{\min}^{-6} \tag{5.3.8}$$

for any $\delta \in (0, \delta_0)$. Choose $v = \frac{u-u_0}{\|u-u_0\|_{L^2(\Sigma)}}$, δ and η small, and $r_{\min}$ large enough then (5.3.7) implies $\overline{u_0} \neq 0$, and we may assume $\overline{u_0} > 0$, as well as $\frac{\overline{u_0}}{2} \le u(x) \le 2\overline{u_0}$ for all $x \in \Sigma$. By orthogonality and (5.3.6) we also have $\frac{1}{2} \le \|u_0\|_{L^2(\Sigma)} \le 1$. This yields

$$\frac{1}{2}|\Sigma|^{-1/2} \le |\Sigma|^{-1/2}\|u_0\|_{L^2} \le 2\overline{u_0} \le 2|\Sigma|^{-1/2}\|u_0\|_{L^2} \le 2|\Sigma|^{-1/2}. \tag{5.3.9}$$

Now we show the decay of $\|H\|_{L^2}$ which leads to a contradiction to the assumption $|\mathcal{W}[\Sigma] - 4\pi| \le \epsilon_0$. Progressing as in the proof of [30, Theorem 11], using the estimates above, Proposition 5.3.3, Theorem 5.2.8 and equation (5.2.11), it can be shown that

$$-\int_\Sigma uH^2 \operatorname{Ric}(\nu,\nu)\,\mathrm{d}\mu \le C\left|\int_\Sigma W_\lambda u\,\mathrm{d}\mu\right| + Cr_{\min}^{-5}.$$

Hence we get

$$\begin{aligned}
2mr_{\min}^{-3}\int_\Sigma H^2\,\mathrm{d}\mu &\le -\int_\Sigma H^2 \operatorname{Ric}(\nu,\nu)\,\mathrm{d}\mu + Cr_{\min}^{-4} \\
&\le -\frac{1}{2\overline{u_0}}\int_\Sigma uH^2 \operatorname{Ric}(\nu,\nu)\,\mathrm{d}\mu + Cr_{\min}^{-4} \\
&\le Cr_{\min}\left|\int_\Sigma W_\lambda u\,\mathrm{d}\mu\right| + Cr_{\min}^{-4} \\
&\le Cr_{\min}^{-4} + Cr_{\min}\left|\int_\Sigma E_L u\,\mathrm{d}\mu\right| \\
&\le Cr_{\min}^{-4},
\end{aligned}$$

where we used the expansion of $\operatorname{Ric}(\nu,\nu)$, Lemma 5.3.1 and the estimates in (5.3.9). This establishes the injectivity. Since E_λ is an elliptic operator it is Fredholm and by the Fredholm alternative, it is surjective as well. The L^2 estimate follows from the negation of (5.3.8) and (5.3.3) this yields the $W^{2,2}$ estimate. □

The following theorem asserts that the round spheres in Schwarzschild space are the unique critical surfaces in the class we consider. They will serve as the starting point for the perturbation in Theorem 5.3.6

Theorem 5.3.5 (see [30, Theorem 12])**.** *For all $m > 0$ there exit constants $r_0 > 0$, $\tau_0 > 0$ and $\epsilon_0 > 0$ with the following properties.*
Assume that $(M, g) = (\mathbb{R}^3\backslash\{0\}, g_m^S)$ and let Σ be a surface satisfying (1.2.2), for $\lambda > 0$ and $L = 0$, with

1. $\left|\int_\Sigma H^2\,\mathrm{d}\mu - 16\pi\right| \le \epsilon_0$,
2. $r_{\min} > r_0$,
3. $\tau \le \tau_0$ *and*
4. $R_e \le \epsilon_0 r_{\min}^2$.

Then $\Sigma = S_{R_e}(0)$.

Now we are in the position to prove the three main statements of this chapter. The idea of their proofs is the same as the ones for [30, Theorem 13, 14 and 15] which in turn are modeled after [38, Section 6]. In Theorem 5.3.6 we construct a deformation of the spheres $S_R(0)$ in $(\mathbb{R}^3 \setminus B_\sigma(0), g_S)$, for large radii, to Hawking type surfaces in $(\mathbb{R}^3 \setminus B_\sigma(0), g)$. In Theorem 5.3.7 we show that this deformation already yields a foliation and Theorem 5.3.8 asserts that it is unique in its class.

Theorem 5.3.6. *For all $m > 0$ and $\sigma > 0$ there exists an $\eta_0 > 0$, $\lambda_0 > 0$ and a $C > 0$ depending only on m and σ with the following properties.*

Assume (M, g) is a (m, η, σ)–asymptotically Schwarzschild manifold and is $L : TM \to \mathbb{R}$ smooth and of order $O^4_\eta(r^{-4})$, satisfying

1. $\eta < \eta_0$,
2. $|\,\mathrm{Sc}\,| \le \eta r^{-5}$.

Additionally, suppose that we have $g \in C^{4,\alpha}(\mathbb{R}^3 \setminus B_\sigma(0), g_S)$ and $L \in C^{4,\alpha}(\mathbb{R}^3 \setminus B_\sigma(0), g_S)$ with bounds on the norms that depend only on (m, σ, η_0), then for all $0 < \lambda < \lambda_0$ there exits a surface Σ_λ which solves (1.2.2) for L and a given λ.

Moreover, let $g_t = (1-t)g^S + tg$ and $L_t = tL$ then there exists a differentiable deformation $G : S^2 \times (0, \lambda_0) \times [0, 1] \to M$ such that

1. $\mathrm{D}(\lambda, g_t, L_t)(G(S^2, \lambda, t)) = 0$ *for all* $\lambda \in (0, \lambda_0)$ *and* $t \in [0, 1]$,
2. $G(S^2, \lambda, 0) = S^2_{r(\lambda)}(0)$,
3. $G(S^2, \lambda, 1) = \Sigma_\lambda$.

Proof. For $\delta_0 \in (0, 1)$ and $t \in [-\delta_0, 1 + \delta_0]$ define $g_t = (1-t)g_S + tg$ as well as $L_t = tL$. Choose δ_0 so small that g_t is (m, σ, η_0)–asymptotically Schwarzschild. We introduce δ_0 to ensure that g and g_S are not a boundary cases of g_t.

A calculation reveals that S_r solves (1.2.2) for $\lambda(r) = \frac{2m}{r^3}\left(1 + \frac{m}{2r}\right)^{-6}$. For large r this is invertible, hence for any small enough λ there exits a $r(\lambda)$ such that $S_{r(\lambda)}(0)$ solves (1.2.2) for that λ. This yields an upper bound for λ_0. Furthermore, we define the following conditions.

A1) $\left|\int_\Sigma H^2 \,\mathrm{d}\mu - 16\pi\right| \le \epsilon_0$

A2) $\tau \le \tau_0$

A3) $|\Sigma| \le \epsilon_0 r^3_{\min}$

Here τ_0 and ϵ_0 are chosen such that we can apply the results of this section. If Σ satisfies (1.2.2) and A1)–A3) we gather from Lemma 5.2.4, Proposition 5.2.6, Theorem 5.2.8 that the following conditions hold.

B1) $\left|H - \overline{H^S}\right| \le C\sqrt{\eta_0}\, r^{-3}_{\min}$

B2) $\tau \le C\sqrt{\eta_0}\, r^{-1}_{\min}$

B3) $C^{-1} r_{\min} \le R_E \le C r_{\min}$

In this case Lemma 5.2.9 implies that Σ is a graph over $S_\sigma(0)$. Further, we may assume that η_0 and $r_{\min}$ are such that the conditions B1)–B3) imply Theorem 5.3.4, provided Σ satisfies (1.2.2).

For a graphical surface $\Sigma = \phi(S_\sigma)$, we may regard the operator $\mathrm{D}_{\lambda,g,L}$ as an operator on the graph function. Provided Σ is as in Theorem 5.3.4, it also allows us to see E_λ as an invertible operator on $C^{4,\alpha}(S_\sigma, g_S)$ satisfying the estimate

$$\|u\|_{C^{4,\alpha}(S,g_S)} \le C(m,\eta_0,\sigma,\alpha)\frac{r_{\min}^p}{\delta_0}\|E_\lambda\|_{C^{0,\alpha}(S,g_S)}$$

for a positive p. First, switch the metric in the estimate of Theorem 5.3.4 to g_S and pull it back to S to obtain a $W^{2,2}(S,\phi^* g_S)$ estimate.

Using the position estimate in Theorem A.1.8 and the estimate on τ in Theorem 5.2.8 we control ϕ. The estimate on the normal in (5.2.9) controls $d\phi$. Thus we may switch from $\phi^* g_S$ to g_S and obtain

$$\|u\|_{W^{2,2}(S,g_S)} \le C(m,\sigma,\eta_0)\frac{r_{\min}^p}{\delta_0}\|E_\lambda u\|_{L^2(S,g_S)},$$

for a positive p. The Sobolev embedding implies $\|u\|_{C^{0,\alpha}(S,g_S)} \le C(S,g_S)\|u\|_{W^{2,2}(S,g_S)}$ and the Schauder estimates for E_λ assert

$$\|u\|_{C^{4,\alpha}(S,g_S)} \le C_S(E_\lambda,\alpha,S,g_S)\left(\|u\|_{C^{0,\alpha}(S,g_S)} + \|E_\lambda u\|_{C^{0,\alpha}(S,g_S)}\right).$$

The constant C_S depends on E_λ only through a bound of the $C^{0,\alpha}$ norms of its coefficients, which in turn only depends on (m,σ,η_0) by our assumptions on g and L.

Define the function space

$$X^t = \{u \in C^{4,\alpha}(S) \mid r_{\min} > 2r_0,\ \text{B1), B2), and B3) hold for } \operatorname{Graph}(u) \text{ and } g_t\}.$$

For $\lambda_1 \in (0,\lambda_0)$ and a smooth curve

$$\begin{aligned} \kappa : [0,1] &\to (\lambda_1,\lambda_0)\times[0,1] \\ s &\mapsto (\lambda(s), t(s)) \end{aligned}$$

with $t(0) = 0$ and $t(1) = 1$ define

$$I_\kappa := \left\{ s \in [0,1] \,\middle|\, \exists u \in X_2^{t(s)} \text{ such that } \mathrm{D}_s(u) := \mathrm{D}(\lambda(s), g_{t(s)}, L_{t(s)})(u) = 0 \right\}.$$

Now we show that $I_\kappa = [0,1]$, the graph function corresponding to $t = 1$ will describe the sought after surface. We choose λ_0 so small that for all $\lambda \in (0,\lambda_0)$ the $S_{r(\lambda)}(0)$ are in X^0, hence $0 \in I_\kappa$, and it is enough to show that I_κ is open and closed.

To show that I_κ is open, fix $(\lambda_2, t_0) = (\lambda(s_0), t(s_0)) \in (\lambda_1,\lambda_0)\times[0,1]$ and consider a solution $u_0 \in C^{4,\alpha}(S)$ to $\mathrm{D}_{\lambda_2,t_0} u_0 := \mathrm{D}(\lambda_2, g_{t_0}, L_{t_0})(u_0) = 0$ and set $\phi_0 = \mathrm{id}_S + u_0\rho$. There is constant $c(u_0) > 0$ such that $\tilde\phi_f = \phi_0 + f\nu$ is a normal variation of ϕ_0, for any $f \in C^{4,\alpha}(S)$, $\|f\|_{C^0} \le c$. Define $Y := \{f \in C^{4,\alpha} \mid \|f\|_{C^0} \le c\}$ and

$$\begin{aligned} F : Y \times (0,\lambda_0) \times (-\delta_0, 1] &\to C^{0,\alpha}(S) \\ (f,\lambda,t) &\mapsto \mathrm{D}_{\lambda,t}(\tilde\phi_f) \end{aligned}$$

By the choice of (λ_2, t_0) we have $F(0,\lambda_2,t_0) = 0$ as well as $dF_{(0,\lambda_2,t_0)}(f,0,0) = E_{\lambda_2,t_0} f$ and we know from Theorem 5.3.4 that E_{λ_2,t_0} is invertible, hence we can apply the implicit function theorem to F. There exits a $\delta > 0$ and a differentiable map $\xi : (\lambda_2 - \delta, \lambda_2 + \delta) \times (t_0 - \delta, t_0 + \delta) \to C^{4,\alpha}(S)$, such that $\xi(\lambda_2, t_0) = 0$ and

$$\mathrm{D}_{\lambda,t}(\tilde{\phi}_{\xi(\lambda,t)}) = F(\xi(\lambda, t), \lambda, t) = 0 \tag{5.3.10}$$

for all $(\lambda, t) \in (\lambda_2 - \delta, \lambda_2 + \delta) \times (t_0 - \delta, t_0 + \delta)$.

Now we consider $\tilde{\xi} := \xi \circ \kappa : (s_0 - \tilde{\delta}, s_0 + \tilde{\delta}) \to C^{4,\alpha}$, for an appropriate $\tilde{\delta}$. Since $\xi(\lambda, t)$ is $C^{4,\alpha}$ close to 0 we can use the conditions B1)–B3) for ϕ_0 to see that $\tilde{\phi}_{\tilde{\xi}(s)}$ satisfies A1)–A3) for all $s \in (s_0 - \tilde{\delta}, s_0 + \tilde{\delta})$; where we decrease the $\tilde{\delta}$ if necessary. This implies that $\tilde{\phi}_{\xi(s)}(S)$ is graphical with a graph function $\tilde{u}_s \in X^{t(s)}$ for all $s \in (s_0 - \tilde{\delta}, s_0 + \tilde{\delta})$, and hence I_κ is open.

To show that I_κ is closed, let $\{s_i\}$ be a sequence in I_κ converging to s. Then the sequence $\{g_{t(s_i)}\}$ a converges smoothly to $g_{t(s)}$. Moreover, there is a sequence of functions $u_i \in X_2^{t(s_i)}$ which is uniformly bounded in $W^{4,2}(S_\sigma(0))$.

The bounds on the u_i follows from the position estimate in Theorem A.1.8, the estimate on $\tau(u_i)$ in Theorem 5.2.8 and a bound $R_{E,i} \leq C\lambda_1^{-1/3}$ from Lemma 5.2.1. The gradient estimate follows from the estimate on the normal in (5.2.9) since it can be expressed as

$$\nu_i = \frac{-\nabla u_i + (1 + u/\sigma)\,\rho}{\left||\nabla u_i|^2 + (1 + u_i/\sigma)^2\right|^{1/2}}.$$

The estimates on the mean curvature and its derivatives in (5.2.8) and (5.2.10) together with elliptic regularity theory imply bounds on the higher derivatives of u_i. Thus we may assume that u_i converges to a function u weakly in $W^{4,2}$, strongly in $W^{3,2}$ and in $C^{2,\alpha}$. These three kinds of convergence imply that u solves $\mathrm{D}(\lambda(s), t(s))(u) = 0$ weakly. By the regularity theory for generalized Willmore equations we know that u is smooth. As the conditions B1)–B3) hold along the sequence u_i the $C^{2,\alpha}$ convergence implies that u satisfies A1)–A3) and we conclude $u \in X^{t(s)}$. □

Theorem 5.3.7. *There exits a constant $r_0 > 0$ such that the family of surfaces $\{\Sigma_\lambda\}_{(0,\lambda_0)}$ constructed in Theorem 5.3.6 form a foliation of $M \setminus B_{r_0}(0)$.*

Moreover, the Hawking type functional is strictly decreasing along the foliation.

Proof. Consider the map $G : S \times (0, \lambda_0) \times [0, 1] \to M$ from Theorem 5.3.6. From the construction of Σ_λ we know that G is differentiable in λ. First we show that $\alpha_\lambda(x, t) := g\left(\frac{\partial G}{\partial \lambda}(x, \lambda, t), \nu_\lambda(x, t)\right)$ is non-zero everywhere, as this implies that $G(\cdot, \cdot, t)$ is a local diffeomorphism for all $t \in [0, 1]$.

To this end take a $\lambda_1 \in (0, \lambda_0)$ and consider the curve $\lambda(s) = (1 - s)\lambda_1 + s\lambda_2$. We can regard the family of surfaces $\Sigma_{\lambda(s)}$ as a variation of Σ_{λ_1} with normal variational vector field $\alpha_\lambda \nu_\lambda$. Let $G^\perp$ be that normal variation and calculate

$$\begin{aligned} E_{\lambda(s)}\alpha_{\lambda(s)}(\lambda_1 - \lambda_0) &= \frac{d}{dt}\mathrm{D}_{\lambda(s)}(F^\perp(\cdot, \lambda(s))) - 2H\frac{d}{ds}\lambda(s) \\ &= -2H(\lambda_1 - \lambda_0). \end{aligned}$$

This shows that α_λ solves $E_\lambda \alpha_\lambda = -2H$.

Analogous to the proof of [30, Theorem 15], this allows us to prove the estimate

$$\int_{\Sigma_\lambda} (\alpha_\lambda - (\alpha_\lambda)_0)\, H \, \mathrm{d}\mu \leq \frac{C\eta^{1/4}}{r_{\min}^6} \|\alpha_\lambda\|_{L^2(\Sigma_\lambda)} \|\alpha_\lambda - (\alpha_\lambda)_0\|_{L^2(\Sigma_\lambda)}.$$

Now estimate (5.3.7) implies that $\alpha_{\lambda(s)}$ is non-zero everywhere, for all $s \in (0,1)$, since α cannot be identically 0, due to the linearity of E_λ.

Furthermore, we know that all the surfaces $G(S,\lambda,t)$ are graphical over S. Thus we have

$$G(x,\lambda,t) = u(x,\lambda,t)\rho(x)$$

and

$$\alpha_\lambda = \frac{du}{d\lambda} g(\rho,\nu_\lambda).$$

Now $\alpha_\lambda(x,t)$ is continuous in t and the expansion (5.1.2) of the Willmore functional in $\mathbb{R}^3 \setminus \{0\}, g_S)$ shows that $\alpha_\lambda(x,0) = \frac{du}{d\lambda}(x,\lambda,0)$ is negative. Thus α is negative for all $(x,\lambda,t) \in S \times (0,\lambda_0) \times [0,1]$. Together with the estimate on the normal (5.2.9) this shows that $u(x,\lambda,t)$ is strictly decreasing in λ, and hence bijective as a map

$$u(x,\cdot,t) : (0,\lambda_0) \to (u(x,\lambda_0,t),\infty).$$

This implies that $G(\cdot,\cdot,1) : S \times (0,\lambda_0) \to \mathbb{R}^3 \setminus \Omega_{\lambda_0}$ is bijective, where Ω_{λ_0} is the closure of the region bounded by $G(S,\lambda_0,1)$. This in turn shows that Σ_λ is a foliation.

To see that $H_L(\Sigma_\lambda)$ is strictly decreasing in λ, simply calculate

$$\frac{d}{d\lambda}\mathcal{H}_L[\Sigma_\lambda] = \lambda \int_{\Sigma_\lambda} \alpha H \,\mathrm{d}\mu < 0.$$

□

Theorem 5.3.8. *The surfaces Σ_λ constructed in Theorem 5.3.6 and therefore the foliation of Theorem 5.3.7 are unique in the following sense.*

There exits positive constants η_0, λ_0 and r_0 such that any two surfaces Σ_1 and Σ_2 in $\mathbb{R}^3 \setminus B_{r_0}$ satisfying $B1) - B3)$ and solving $\mathrm{D}(\lambda,g,L)(\Sigma) = 0$ for $L \in O^4_\eta(r^{-4})$, $\eta \in (0,\eta_0)$, and $\lambda \in (0,\lambda_0)$ have to agree.

Proof. Let Σ_1 and Σ_2 be as above. Consider the curve $\kappa : [0,1] \to (0,\lambda_0) \times [0,1+\delta_0)$, $\kappa(t) = (\lambda,t)$, along with $g_t = (1-t)g_S + tg$ and $L_t = tL$ for a small $\delta_0 > 0$. Starting from each Σ_i, $i \in \{1,2\}$, we construct deformations $F_i : S^2 \times (0,1+\delta_0) \to M$ as in Theorem 5.3.6 such that $F_i(S^2,1) = \Sigma_i$ and such that $S_i = F(S^2,0)$ solve $\mathrm{D}_{\lambda,g_S,0}(S_i) = 0$ as well as satisfy $B1) - B3)$. These surfaces are centered constant mean curvature spheres by [30, Theorem 12] (see also Theorem 5.3.5 above) and they are unique by [19, Section 5]. Since the deformations are locally constructed via the implicit function theorem which asserts uniqueness, and since the $C^{4,\alpha}$ estimate on E_λ is uniform in the proof of Theorem 5.3.6 the deformations F_i agree.

□

Appendices

Appendix A

Bounded Geometry and Surfaces

In this section we will briefly introduce bounded geometry, as presented in [28] and [29]. For a more comprehensive treatment see for instance [9, Chapter 2].

Definition A.1.1. Let (M, g) be a complete Riemannian manifold with injectivity radius $r_{\text{inj}}(M, g, p)$ at $p \in M$ and Riemannian curvature tensor Rm. We say M has C_B bounded geometry if there exists a constant $C_B > 0$ such that for each $p \in M$ we have

$$r_{\text{inj}}(M, g, p) \geq C_B^{-1}$$

and

$$|\operatorname{Rm}| + |\nabla \operatorname{Rm}| \leq C_B.$$

We may combine the well known results on normal coordinates with the uniform bound on the injectivity radius to obtain the following lemma.

Lemma A.1.2 (cf. [28, Section 2.1])**.** *Let (M, g) be a manifold of C_B bounded geometry, let $B_r(y)$ be the Euclidean ball at $y \in T_pM$ of radius r and $\mathcal{B}_r(p)$ the geodesic ball at $p \in M$ with radius r. There exists constants h_0 and r_0, depending only on C_B, such that in normal coordinates $\exp_p : B_{r_0}(0) \to \mathcal{B}_{r_0}(p)$ the metric satisfies*

$$g = g_E + h,$$

where g_E is the Euclidean metric and h obeys

$$\sup_{B_{r_0}(0)} \left(|x|^2 |h| + |x||\nabla^E h| + \left| \left(\nabla^E\right)^2 h \right| \right) \leq h_0.$$

Here x is the position vector field in $B_r(0)$, $|\cdot|$ is the Euclidean norm and ∇^E is the Euclidean connection.

Next we consider small surfaces (Σ, γ) that are isometrically immersed in a three dimensional, C_B bounded manifold (M, g). That is we deal with closed surfaces contained in geodesic balls $\Sigma \subset \mathcal{B}_{r_0}(y)$ for some point $y \in M$ and $r_0 \leq \min(r_{\text{inj}}, 1)$. With our previous result in mind, we regard them as immersed in $B_{r_0}(0)$ equipped with the metric $g = g_E + h$ as above. We fix this setting for now, unless stated otherwise. Additionally, we will denote all geometric quantities computed with respect to the Euclidean metric by an index E.

Lemma A.1.3 (see [28, Lemma 2.1]). *There exists a constant C, depending only on r_0 and h_0, such that for all surfaces $\Sigma \subset B_r(0) = B_r$, $r \leq r_0$, we have*

$$
\begin{aligned}
|\gamma - \gamma_E|_E &\leq C|x|_E^2 \\
|\sqrt{|\det \gamma|} - \sqrt{|\det \gamma_E|}| &\leq C\sqrt{\det \gamma_E}|x|_E^2 \\
|\sqrt{|\det \gamma|} - \sqrt{|\det \gamma_E|}| &\leq C\sqrt{\det \gamma}|x|_E^2 \\
|\nu - \nu_E| &\leq C|x|_E^2 \\
|A - A_E|_E &\leq C(|x|_E + |x|_E^2|A_E|_E)
\end{aligned}
$$

Proof. The first claim is obvious since γ is the restriction of g. For the second one we need to calculate the determinant of $\gamma = \gamma_E + \eta$, with $\eta = h|_\Sigma$. Using the estimate on h and the expansion of the determinant we have that

$$
|\det \gamma - \det \gamma_E| = |\det \gamma_E \left[\operatorname{tr}((\gamma_E)^{-1}h) + O(|x|_E^4)| \right] \leq C \det(\gamma_E)|x|_E^2
$$

and hence

$$
\begin{aligned}
\left| \sqrt{\det \gamma} - \sqrt{\det \gamma_E} \right| &\leq \frac{|\det \gamma - \det \gamma_E|}{\sqrt{\det \gamma} + \sqrt{\det \gamma_E}} \\
&\leq \frac{C \det(\gamma_E)|x|_E^2}{\sqrt{\det \gamma} + \sqrt{\det \gamma_E}} \\
&\leq C\sqrt{\det \gamma_E}|x|_E^2.
\end{aligned}
$$

To get the estimate on normal vectors, we first note that the projection to the tangent space of Σ is small. Let $\{e_i\}$ be a local frame of the tangent space of Σ. We have $g(e_i, \nu) = 0 = g_E(e_i, \nu_E)$ and hence

$$
\begin{aligned}
|g_E(e_i, \nu - \nu_E)| &= |g_E(e_i, \nu)| \\
&= |h(e_i, \nu)| \\
&\leq C|x|_E^2.
\end{aligned}
$$

Next we consider the projection of $\nu - \nu_E$ in the ν_E direction, but we expand $\nu - \nu_E$ in the basis $\{e_i\} \cup \nu$.

$$
\begin{aligned}
g_E(\nu_E, \nu) - 1 &= g_E(\nu_E, \nu - \nu_E) \\
&= g(\nu - \nu_E, \nu)g_E(\nu_E, \nu) + O(|x|_E^4) \\
&= g_E(\nu_E, \nu) - g_E(\nu_E, \nu)^2 + O(|x|_E^2)
\end{aligned}
$$

This yields

$$
|g_E(\nu_E, \nu)^2 - 1| = O(|x|_E^2).
$$

Since the normal vector depends continuously on the metric, we may assume that $g_E(\nu_E, \nu)$ is positive. Hence we have

$$
\begin{aligned}
|g_E(\nu_E, \nu - \nu_E)| &= |g_E(\nu_E, \nu) - 1|, \\
&\leq |g_E(\nu_E, \nu) - 1||g_E(\nu_E, \nu) + 1|, \\
&= O(|x|_E^2)
\end{aligned}
$$

The last computation is straightforward. We denote by $\{y_\alpha\}$ the normal coordinates on $(B_r(0), g^E)$ and recall that the Christoffel symbols are basically a linear combination of the first derivatives of the metric.

$$\begin{aligned}
|A_{ij} - (A_E)_{ij}| &= |g_E(\nabla_i^E e_j, \nu^E) - g(\nabla_i e_j, \nu)| \\
&\leq |g_E(\nabla_i^E e_j - \nabla_i e_j, \nu^E)| + |g_E(\nabla_i e_j, \nu_E - \nu)| + |h(\nabla_i e_j, \nu)| \\
&\leq |g_E(e_i^\alpha e_j^\beta \Gamma_{\alpha\beta}^\gamma \partial_\gamma, \nu)| + C|x|_E^2 |A_{ij}| \\
&\leq C|x|_E + C|x|_E^2 |A_{ij}|
\end{aligned}$$

Using this to estimate A_{ij} again yields

$$|A_{ij} - (A_E)_{ij}| \leq C|x|_E + C|x|_E^2 |(A_E)_{ij}|. \qquad \square$$

Definition A.1.4. We define the radius R of Σ with respect to γ by the relation $|\Sigma| =: 4\pi R^2$. Analogously, the corresponding Euclidean radius R_E is given by $|\Sigma|_E =: 4\pi R_E^2$, where $|\Sigma|_E := \int_\Sigma \mathrm{d}\mu_{\gamma_E}$.

Corollary A.1.5 (cf. [28, Lemma 2.5])**.** *In the setting of Lemma A.1.3 the following estimates hold.*

1.

$$\begin{aligned}
||\Sigma| - |\Sigma|_E| &\leq Cr^2 |\Sigma| \\
||\Sigma| - |\Sigma|_E| &\leq Cr^2 |\Sigma|_E \\
|R - R_E| &\leq Cr^2 R \\
|R - R_E| &\leq Cr^2 R_E
\end{aligned}$$

In particular, the areas $|\Sigma|$ and $|\Sigma|_E$ are comparable, as are the corresponding radii R and R_E.

2.

$$\begin{aligned}
|\mathcal{W}[\Sigma, g] - \mathcal{W}[\Sigma, g_E]| &\leq C(C_B) r^2 \left(|\Sigma| + \mathcal{W}[\Sigma, \gamma] + r^2 \| \mathring{A} \|^2_{L^2(\Sigma,\gamma)} \right) \\
\| \mathring{A}^E \|^2_{L^2(\Sigma, \gamma_E)} &\leq C(C_B) \| \mathring{A} \|^2_{L^2(\Sigma,\gamma)} + Cr^4 \mathcal{W}[\Sigma, g]
\end{aligned}$$

Lemma A.1.6 (see [28, Lemma 2.2])**.** *There exists $0 < r_1 \leq r_0$ and a purely numerical constant C such that for all $\Sigma \subset B_r$, $r \leq r_1$, we have*

$$|\Sigma| \leq Cr^2 \mathcal{W}[\Sigma].$$

Lemma A.1.7 (see [29, Lemma 2.5])**.** *There exists a constant C, depending only on C_B, such that all connected surfaces $\Sigma \subset M$ obey*

$$\operatorname{diam}_M(\Sigma) \leq C \left(|\Sigma|^{1/2} \mathcal{W}[\Sigma]^{1/2} + |\Sigma| \right).$$

Clearly, the previous two lemmas also hold for stratified surfaces if we apply them to every component.

In Section 4.2 it is necessary to approximate a surfaces by a spheres, hence we state a scaled version of the results of De Lellis and Müller on that topic together with an estimate on the normal vectors.

Theorem A.1.8 (cf [5, Theorem 1.1] and [6, Theorem 1.2])**.** *Let $\Sigma \subset \mathbb{R}^3$ be a surfaces with induced metric γ_E and $\| \mathring{A}_E \|^2_{L^2(\Sigma,\gamma_E)} < 8\pi$ and consider its Euclidean radius R_E as well as its Euclidean center of gravity $a_E := |\Sigma|_E^{-1} \int_\Sigma x \, d\mu_{\gamma_E}$. Then there exists a universal constant C and a conformal map $\psi : S := S^2_{R_E}(a_E) \to \Sigma$ with the following properties. Let σ be the round metric on S, ν_S its unit normal vector field and let α be the conformal factor of ψ, i.e. $\psi^* \gamma_E = \alpha^2 \sigma$. Then the following estimates hold.*

$$\begin{aligned}
\|A_E - R_E \, \mathrm{id}\|_{L^2(\Sigma,\gamma_E)} &\le C \| \mathring{A}_E \|_{L^2(\Sigma,\gamma_E)} \\
\left\| H_E - \frac{2}{R_E} \right\|_{L^2(\Sigma,\gamma_E)} &\le C \| \mathring{A}_E \|_{L^2(\Sigma,\gamma_E)} \\
\|\psi - \mathrm{id}_S \|_{L^2(S)} &\le C R_E^2 \| \mathring{A}_E \|_{L^2(\Sigma,\gamma_E)} \\
\|\psi - \mathrm{id}_S \|_{L^\infty(S)} &\le C R_E \| \mathring{A}_E \|_{L^2(\Sigma,\gamma_E)} \\
\|\mathrm{d}\psi - \mathrm{id}_{TS} \|_{L^2(S)} &\le C R_E \| \mathring{A}_E \|_{L^2(\Sigma,\gamma_E)} \\
\|\alpha - 1\|_{L^\infty(S)} &\le C \| \mathring{A}_E \|_{L^2(\Sigma,\gamma_E)}
\end{aligned}$$

Corollary A.1.9. *Assume additionally that $\| \mathring{A}_E \|_{L^2(\Sigma,\gamma_E)}$ is so small that $\|\alpha - 1\|_{L^\infty(S)} \le 1/2$. Then there is a universal constant C such that*

$$\|\nu_S - \nu_E \circ \psi\|_{L^2(S)} \le C R_E \| \mathring{A}_E \|_{L^2(\Sigma,\gamma_E)}.$$

Proof. Let $\{e_i\}$ be a local orthonormal frame on S and define $\{f_i := d\psi(e_i)\}$, an orthogonal frame on Σ. Further, denote by g the Euclidean metric on $\mathbb{R}^3$, by $\wedge$ the wedge product and by $*_g$ the Hodge star operator induced by g and the orientation $\{e_i\} \cup \nu_S$. Then, locally, we may express the normal vectors as

$$\begin{aligned}
\nu_S &= *_g (e_1 \wedge e_2) \\
\nu_E &= \frac{*_g (f_1 \wedge f_2)}{| *_g (f_1 \wedge f_2) |_g} = \frac{*_g (f_1 \wedge f_2)}{\alpha^2}.
\end{aligned}$$

This allows us to derive pointwise estimates.

$$\begin{aligned}
|\nu_S - \nu_E|_g &= \left| e_1 \wedge e_2 - \frac{1}{\alpha^2} f_1 \wedge f_2 \right|_g \\
&\le \left| 1 - \frac{1}{\alpha^2} \right|_g + \frac{1}{\alpha^2} |e_1 \wedge e_2 - e_1 \wedge f_2|_g + |e_1 \wedge f_2 - f_1 \wedge f_2|_g \\
&\le \left| 1 - \frac{1}{\alpha^2} \right|_g + C \left(\frac{1}{\alpha^2} + \frac{1}{\alpha} \right) |d\psi - \mathrm{id}\,|_g
\end{aligned}$$

Here C is a numerical constant. The supremum estimate on α implies that $1/\alpha \le 2$ and that

$$\left| \frac{1}{\alpha^2} - 1 \right| \le 2C \| \mathring{A}_E \|_{L^2(\Sigma,\gamma_E)}.$$

Combining these two estimates and integrating yields the claim. □

We may combine the results of Lemma A.1.3, Theorem A.1.8 and the previous corollary in order to approximate a small surface (Σ, γ) with (S, σ).

Corollary A.1.10. *Let $\Sigma \subset B_{r_0}$ be a small surface, and assume that $\| \mathring{A}_E \|_{L(\Sigma,\gamma_E)} \le 8\pi$ is small enough that corollary A.1.9 holds. Assume further that H^{-1} and H_E^{-1} are uniformly bounded. Then we have the following estimates for a constant C dependent only on C_B.*

$$\begin{aligned}|\mathrm{d}\,\mu_\gamma - \mathrm{d}\,\mu_\sigma| &\le C\left(r^2 + \| \mathring{A}_E \|_{L^2(\Sigma,\gamma_E)}\right)\\ \|\nu\circ\psi - \nu_S\|_{L^2(S,\sigma)} &\le C\left(r^2 R_E + R_E \| \mathring{A}_E \|_{L^2(\Sigma,\gamma_E)}\right)\\ \|H^{-1} - R_E/2\|_{L^2(\Sigma,\gamma_E)} &\le C\left(\sup_\Sigma \frac{r}{HH_E}\left(1 + r^2\|A_E\|^2_{L^2(\Sigma,\gamma_E)}\right) + \sup_\Sigma \frac{R_E}{H_E}\| \mathring{A}_E \|_{L^2(\Sigma,\gamma_E)}\right)\end{aligned}$$

Moreover, we may transport any bounded Lipschitz function $F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ from Σ to S.

$$\left|\int_\Sigma F(y,\nu)\mathrm{d}\,\mu_\gamma - \int_S F(x,\nu_S)\mathrm{d}\,\mu_\sigma\right| \le C_1 R_E^2\left(r^2 + \| \mathring{A}_E \|_{L^2(\Sigma,\gamma_E)}\right)$$

Here C_1 is a constant that depends on C_B and F.

It is possible to choose normal coordinates of (M,g) well suited for a given closed surface Σ.

Lemma A.1.11 (see [39, Lemma 3.1])**.** *Let $\Sigma \subset M$ be a surface with extrinsic diameter d such that $2d \le \operatorname{inj}(M,g)$. Then there exists a point $p_0 \in M$ with $\operatorname{dist}(p_0,\Sigma) \le d$ and such that in normal coordinates ψ centered at p_0 we have that*

$$a = \frac{1}{|\Sigma|}\int_{\psi(\Sigma)} y\,\mathrm{d}\mu_g = 0$$

and

$$|a_E|_E = \frac{1}{|\Sigma|_E}\left|\int_{\psi(\Sigma)} y\,\mathrm{d}\mu_E\right|_E \le C(C_B)d^3.$$

Additionally, if Σ obeys $\| \mathring{A}^E \|^2_{L^2(\Sigma,\gamma_E)} \le 8\pi$, then we have

$$\max_{x\in\Sigma} |x|_E \le C(C_B)R_E.$$

We close the Chapter by recalling several useful lemmas for deriving Integral estimates.

Lemma A.1.12 (The Bochner identity)**.** *Let (M,g) be a Riemannian manifold and $u \in \mathbb{C}^\infty(M)$, then we have*

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + g(\nabla u, \nabla\Delta u) + \mathrm{Ric}(\nabla u,\nabla u),$$

where $\nabla^2 u$ denote the Hessian of u.

If M is compact and without boundary or u has compact support away from the boundary, we may integrate the above relation to obtain

$$\int_M |\nabla^2 u|^2\,d\mu = \int_M (\Delta u)^2 - \mathrm{Ric}(\nabla u,\nabla u)\,d\mu.$$

Additionally, if M is a surface isometrically immersed in a three dimensional manifold N, we may use the Gauss equation to infer

$$\int_M |\nabla^2 u|^2\,d\mu = \int_M (\Delta u)^2 + |\nabla u|^2(\frac{1}{2}| \mathring{A} |^2 - \frac{1}{4}H^2 - \frac{1}{2}\,\mathrm{Sc}^N + \mathrm{Ric}^N(\nu,\nu))\,d\mu.$$

Th following two lemma are variants of the Michael–Simon–Sobolev inequality for perturbed metrics, see [40] or [7, Section 3.5] the Euclidean version. See also [19, Proposition 5.4]

Lemma A.1.13. *Let $g = g_E + h$, $|h| \leq |x|^2 h_0$, be given on B_ρ. Then there exists a $0 < r_0 \leq \rho$ and a constant $C(r_0, h_0)$ such that for all closed surfaces $\Sigma \subset B_{r_0}$ with $H \in L^2(\Sigma)$ and all $f \in C^\infty(\Sigma)$ we have*

$$\left(\int_\Sigma f^2 \,\mathrm{d}\mu_g\right)^2 \leq C \int_\Sigma |\nabla f| + |Hf| \,\mathrm{d}\mu_g.$$

Lemma A.1.14. *Let (M, g) be (m, σ, η)– asymptotically Schwarzschild. Then there exits constants $r_0 > 0$ and C depending on (m, σ, η) such that for all closed surfaces $\Sigma \subset \mathbb{R}^3 \setminus B_{r_0}$ with $H \in L^2(\Sigma)$ and all $f \in C^\infty(\Sigma)$ we have*

$$\left(\int_\Sigma f^2 \,\mathrm{d}\mu_g\right)^2 \leq C \int_\Sigma |\nabla f| + |Hf| \,\mathrm{d}\mu_g.$$

The next two lemma are a consequence of the Michael–Simon–Sobolev inequality. See [27, Theorem 5.6] and [26, Lemma 2.8] for the proof in the Euclidean case. Proving the statement with the perturbed background metric just requires minor changes.

Lemma A.1.15. *Let $g = g_E + h$, $|h| \leq |x|^2 h_0$, be given on B_ρ. Then there exists a $0 < r_0 \leq \rho$ and a constant $C(r_0, h_0)$ such that for all closed surfaces $\Sigma \subset B_{r_0}$ with $H \in L^2(\Sigma)$ and all smooth forms ϕ we have*

$$\|\phi\|^4_{L^\infty(\Sigma)} \leq C \|\phi\|^2_{L^2(\Sigma, g)} \int_\Sigma |\nabla^2 \phi|^2_g + |H|^4 |\phi|^2_g \,\mathrm{d}\mu_g.$$

Lemma A.1.16. *Let (M, g) be (m, σ, η)– asymptotically Schwarzschild. Then there exits constants $r_0 > 0$ and C depending on (m, σ, η) such that for all closed surfaces $\Sigma \subset \mathbb{R}^3 \setminus B_{r_0}$ with $H \in L^2(\Sigma)$ and all smooth forms ϕ we have*

$$\|\phi\|^4_{L^\infty(\Sigma)} \leq C \|\phi\|^2_{L^2(\Sigma, g)} \int_\Sigma |\nabla^2 \phi|^2_g + |H|^4 |\phi|^2_g \,\mathrm{d}\mu_g.$$

The final lemma is perhaps the most useful tool to show that a given surface has to be embedded. It follows from Simons monotonicity formula, see [25, Appendix A] for a discussion.

Lemma A.1.17 (Li–Yau inequality; see [34])**.** *Let $\phi : S \to \Sigma \subset \mathbb{R}^3$ be a branched, immersed stratified surface with mean curvature $H \in L^2(\Sigma)$. Denote by $\theta^2(\Sigma, p)$ the density of Σ at p, i.e. $\theta^2(\Sigma, p) = \#\phi^{-1}(p)$, then we have*

$$\theta^2(\Sigma, p) \leq \frac{\mathcal{W}[\Sigma]}{4\pi}.$$

Appendix B

Vector Potentials

In this chapter we briefly present the existence and regularity of certain systems of partial differential equations of first order which we will use to construct potentials of generalized Willmore immersions. These results can be found in the appendix of [43].
Let $\gamma_i^j \in \left(C^0 \cap W^{1,2}\right)(\mathbb{C})$ for $i, j \in \{1, ..., n\}$ with $\operatorname{supp} \gamma_i^j \subset B_2(0)$. For any $U \in L^1_{\text{loc}}(\mathbb{C}, \mathbb{R}^n)$ and $\{b_i\}_{i=1}^n$ the standard basis of $\mathbb{R}^n$ define the differential operator

$$D_z U := \partial_z U + \sum_{k=1}^n \gamma_i^j U^i b_j$$

in the distributional sense.

Lemma B.1.1 (see [43, Lemma A.1])**.** *Let $Y \in \left(L^1 + H^{-1}\right)(\mathbb{C}, \mathbb{R}^n)$ with $\operatorname{Im}(D_{\overline{z}} Y) \in \left(L^1 + H^{-1}\right)(\mathbb{C}, \mathbb{R}^n)$. There is an $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ and γ_k^i satisfying $\|\gamma_k^i\|_{L^\infty} \leq \epsilon$ there exists a unique $U \in L^{2,\infty}(D, \mathbb{R}^n)$ with $\operatorname{Im}(U) \in W^{1,(2\infty)}(D, \mathbb{R}^n)$ solving*

$$\begin{cases} D_z U = Y & \text{in } D \\ \operatorname{Im}(U) = 0 & \text{on } \partial D \end{cases}.$$

Moreover, we have the estimate

$$\|U\|_{L^{2,\infty}} + \|\nabla \operatorname{Im}(U)\|_{L^{2,\infty}} \leq C\left(\|Y\|_{L^1 + H^{-1}} + \|\operatorname{Im}(D_{\overline{z}} Y)\|_{L^+ H^{-1}}\right).$$

Lemma B.1.2 (see [43, Lemma A.2])**.** *Let $Y \in \left(L^1 \cap L^{2,\infty}\right)(\mathbb{C}, \mathbb{R}^n)$ with $\operatorname{Im}(D_{\overline{z}} Y) \in L^q$ for some $1 < q < 2$. There is an $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ and γ_k^i satisfying $\|\gamma_k^i\|_{L^\infty} \leq \epsilon$ there exists a unique $U \in W^{1,(2,\infty)}(D, \mathbb{R}^n)$ with $\operatorname{Im}(U) \in W^{2,q}(D, \mathbb{R}^n)$ solving*

$$\begin{cases} D_z U = Y & \text{in } D \\ \operatorname{Im}(U) = 0 & \text{on } \partial D \end{cases}$$

Moreover, we have the estimate

$$\|U\|_{W^{1,(2,\infty)}} + \|\nabla^2 \operatorname{Im}(U)\|_{L^q} \leq C\left(\|Y\|_{L^1 \cap L^{2,\infty}} + \|\operatorname{Im}(D_{\overline{z}} Y)\|_{L^q}\right).$$

Appendix C

Useful Calculations

C.2 Integrals Over the Sphere

Consider the 2-sphere $S := S^2_1(0)$ in $\mathbb{R}^3$ and let $\{x^\alpha\}$ be the standard Euclidean coordinates on it. We aim the calculate integrals of the form

$$\int_S x^{\alpha_1} \dots x^{\alpha_n} \, d\mu^S,$$

where n is an integer up to 6. If n is an odd number, the integral always vanishes.

Since we treat the integral in spherical coordinates we state relevant trigonometric identities to facilitate the calculation.

Lemma C.2.1. *Let $a, b \in \mathbb{R}$, then the following identities hold.*

- $2\cos(a)\sin(b) = \sin(a+b) - \sin(a-b)$
- $2\cos(a)\cos(b) = \cos(a+b) + \cos(a-b)$
- $2\sin(a)\sin(b) = -\cos(a+b) + \cos(a-b)$

In particular, we have

- $2\cos(a)^2 = \cos(2a) + 1$ *and*
- $2\sin(a)^2 = -\cos(2a) + 1$

Moreover, the substitution $u(x) = \cos x$, $\frac{du}{dx} = -\sin x$ will be very useful.

We will always arrange our coordinates $\{x = \sin\theta\cos\phi, y = \sin\theta\sin\phi, z = \cos\theta\}$, $\phi \in (0, 2\pi)$, $\theta \in (0, \pi)$, such that the coordinate in the integrand with the highest exponent is given by z, the second highest by x and the third by y. In assuming this we may break the orientation but it irrelevant for these integrals. Moreover, we will number the cases by the triple (a, b, c), where a, b, c denote the exponents of x, y, z respectively.

$(0, 0, 2)$

$$\begin{aligned} \int_0^{2\pi}\int_0^{\pi} \cos\theta^2 \sin\theta \, d\phi d\theta &= 2\pi \int_{-1}^{1} u^2 \, du \\ &= \frac{4}{3}\pi \end{aligned}$$

$(1,0,1)$

$$\int_0^{2\pi}\int_0^{\pi}\cos\theta\sin\theta^2\cos\phi\,d\phi d\theta = 0$$

$(0,0,4)$

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\cos\theta^4\sin\theta\,d\phi d\theta &= 2\pi\int_{-1}^{1}u^4\,du\\ &= \frac{4}{5}\pi\end{aligned}$$

$(1,0,3)$

$$\int_0^{2\pi}\int_0^{\pi}\cos\theta^3\sin\theta^2\cos\phi\,d\phi d\theta = 0$$

$(2,0,2)$

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\cos\theta^2\sin\theta^3\cos\phi^2\,d\phi d\theta &= \pi\int_0^{\pi}\cos\theta^2\sin\theta-\cos\theta^4\sin\theta\,d\theta\\ &= \pi\left(\frac{2}{3}-\frac{2}{5}\right)\\ &= \frac{4}{15}\pi\end{aligned}$$

$(1,1,2)$

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\cos\theta^2\sin\theta^3\cos\phi\sin\phi\,d\phi d\theta &= \frac{4}{15}\int_0^{2\pi}\sin 2\phi\,d\phi\\ &= 0\end{aligned}$$

$(0,0,6)$

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\cos\theta^6\sin\theta\,d\phi d\theta &= 2\pi\int_{-1}^{1}u^6\,du\\ &= \frac{4}{7}\pi\end{aligned}$$

$(1,0,5)$

$$\int_0^{2\pi}\int_0^{\pi}\cos\theta^5\sin\theta\cos\phi\,d\phi d\theta = 0$$

$(2,0,4)$

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\cos\theta^4\sin\theta^3\cos\phi^2\,d\phi d\theta &= \pi\int_0^{\pi}\cos\theta^4\sin\theta-\cos\theta^6\sin\theta\,d\theta\\ &= \pi\left(\frac{2}{5}-\frac{2}{7}\right)\\ &= \frac{4}{35}\pi\end{aligned}$$

$(1, 1, 4)$

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi} \cos\theta^4 \sin\theta^2 \cos\phi \sin\phi \, d\phi d\theta &= \frac{4}{35}\int_0^{2\pi} \sin 2\phi \, d\phi \\ &= 0\end{aligned}$$

$(3, 0, 3)$ Consider

$$\begin{aligned}\int_0^{2\pi} \cos\phi^3 \, d\phi &= \frac{1}{2}\int_0^{2\pi} \cos 2\phi \cos\phi + \cos\phi \, d\phi \\ &= \frac{1}{4}\int_0^{2\pi} \cos 3\phi + \cos\phi + 2\cos\phi \, d\phi \\ &= 0\end{aligned}$$

This leads to

$$\int_0^{2\pi}\int_0^{\pi} \cos\theta^3 \sin\theta^4 \cos\phi^3 \, d\phi d\theta = 0$$

$(2, 1, 3)$ Consider

$$\begin{aligned}\int_0^{2\pi} \cos\phi^2 \sin\phi \, d\phi &= \frac{1}{2}\int_0^{2\pi} \cos 2\phi \sin\phi + \sin\phi \, d\phi \\ &= \frac{1}{4}\int_0^{2\pi} \sin 3\phi - \sin\phi + 2\sin\phi \, d\phi \\ &= 0\end{aligned}$$

This leads to

$$\int_0^{2\pi}\int_0^{\pi} \cos\theta^3 \sin\theta^4 \cos\phi^2 \sin\phi \, d\phi d\theta = 0$$

$(2, 2, 2)$ Consider

$$\begin{aligned}\int_0^{2\pi} \cos\phi^2 \sin\phi^2 \, d\phi &= \int_0^{2\pi} \cos\phi^2 - \cos\phi^4 \, d\phi \\ &= \pi - \frac{1}{4}\int_0^{2\pi} \cos(2\phi)^2 - \cos 2\phi + 1 \, d\phi \\ &= \frac{\pi}{4}\end{aligned}$$

This leads to

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi} \cos\theta^2 \sin\theta^5 \cos\phi^2 \sin\phi^2 \, d\phi d\theta &= \frac{\pi}{4}\int_0^{\pi} \cos\theta^2 \sin\theta^3 - \cos\theta^4 \sin\theta^3 \, d\theta \\ &= \frac{\pi}{4}\int_0^{\pi} \cos\theta^2 \sin\theta - 2\cos\theta^4 \sin\theta + \cos\theta^6 \sin\theta \, d\theta \\ &= \frac{\pi}{4}\left(\frac{2}{3} - \frac{4}{5} + \frac{2}{7}\right) \\ &= \frac{4}{105}\pi\end{aligned}$$

Lemma C.2.2. *We may condense these calculations to the following three formulas, employing the Kronecker delta.*

$$\begin{aligned}
\int_S x^\alpha x^\beta \, d\mu &= \frac{4\pi}{3}\delta^{\alpha\beta} \\
\int_S x^\alpha x^\beta x^\gamma x^\delta \, d\mu &= \frac{4\pi}{15}(\delta^{\alpha\beta}\delta^{\gamma\delta} + \delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}) \\
\int_S x^\alpha x^\beta x^\gamma x^\delta x^\epsilon x^\rho \, d\mu &= \frac{4\pi}{105}(\delta^{\alpha\beta}\delta^{\gamma\delta}\delta^{\epsilon\rho} + \delta^{\alpha\beta}\delta^{\gamma\epsilon}\delta^{\delta\rho} + \delta^{\alpha\beta}\delta^{\gamma\rho}\delta^{\epsilon\delta} \\
&\quad + \delta^{\alpha\gamma}\delta^{\beta\delta}\delta^{\epsilon\rho} + \delta^{\alpha\gamma}\delta^{\beta\epsilon}\delta^{\delta\rho} + \delta^{\alpha\gamma}\delta^{\beta\rho}\delta^{\epsilon\delta} \\
&\quad + \delta^{\alpha\delta}\delta^{\gamma\beta}\delta^{\epsilon\rho} + \delta^{\alpha\delta}\delta^{\gamma\epsilon}\delta^{\beta\rho} + \delta^{\alpha\delta}\delta^{\gamma\rho}\delta^{\epsilon\beta} \\
&\quad + \delta^{\alpha\epsilon}\delta^{\gamma\delta}\delta^{\beta\rho} + \delta^{\alpha\epsilon}\delta^{\gamma\beta}\delta^{\delta\rho} + \delta^{\alpha\epsilon}\delta^{\gamma\rho}\delta^{\beta\delta} \\
&\quad + \delta^{\alpha\rho}\delta^{\gamma\delta}\delta^{\beta\epsilon} + \delta^{\alpha\rho}\delta^{\gamma\beta}\delta^{\delta\epsilon} + \delta^{\alpha\rho}\delta^{\gamma\epsilon}\delta^{\beta\delta})
\end{aligned}$$

Remark 4. We checked the calculations in the proof of Theorem 4.2.4 using the following code for Mathematica 7 by calculating (4.2.4) for $L = -P^2$ directly. Here U represents ∇K.

```
K = {{K11, K21, K31}, {K21, K22, K32}, {K31, K32, K33}};
U1 = {{U111, U121, U131}, {U121, U221, U321}, {U131, U321, U331}};
U2 = {{U112, U122, U132}, {U122, U222, U322}, {U132, U322, U332}};
U3 = {{U113, U123, U133}, {U123, U223, U323}, {U133, U323, U333}};
b = {b1, b2, b3};
x1 = Cos[phi] Sin[theta];
x2 = Sin[phi] Sin[theta];
x3 = Cos[theta];
x = {x1, x2, x3};

bT[x] = b - b.x x; (* tangential projection of b*)
P0[x] = Tr[K] - x.K.x; (* P evaluated at 0 *)
U[x, x, x] = {x.U1.x, x.U2.x, x.U3.x}.x;
Ub[b, x, x] = {b.U1.x, b.U2.x, b.U3.x}.x; (* nabla_x K (x,b)*)
UU[bT[x], x, x] = bT[x].U1.x*x1 + bT[x].U2.x*x2 + bT[x].U3.x*x3;
trU[x] = {Tr[U1] ,  Tr[U2],  Tr[U3]}.x;
P[x] = P0[x] + trU[x] - U[x, x, x];

I1 = Integrate[(P [x] (bT[x].K.x + UU[bT[x], x, x])) Sin[theta], {theta, 0,
   Pi}, {phi, 0, 2 Pi}]
I2 = Integrate[(b.x/2 P0[x] (trU[x] - U[x, x, x])) Sin[theta], {theta, 0,
  Pi}, {phi, 0, 2 Pi}]
I3 = Integrate[(P[x]^2 b.x/2) Sin[theta], {theta, 0, Pi}, {phi, 0,
  2 Pi}]

FullSimplify[I1+I2+I3]
```

The result of the last line yields $\langle b, W\rangle$ up to a constant.

To support Corollary 4.2.6 we also calculate the following with the same variables as above.

```
Integrate[(x.K.x )^2 Sin[theta], {theta, 0, Pi}, {phi, 0, 2 Pi}]
```

```
Integrate[x.K.x  Sin[theta], {theta, 0, Pi}, {phi, 0, 2 Pi}]
Integrate[(P0[x] )^2 Sin[theta], {theta, 0, Pi}, {phi, 0, 2 Pi}]
```

C.3 Expansion of the Willmore Functional

In Section 5.1 we postponed the proof of the following lemma.

Lemma C.3.1. *Consider* $(\mathbb{R}^3\setminus\{0\}, g_S)$*, then the Willmore energy of an embedded surface is given by*

$$\mathcal{W}[\Sigma, g_S] = \mathcal{W}[\Sigma, g_E] - 2m\int_\Sigma H_E \frac{\langle x,\nu_E\rangle_E}{r^2(2r+m)}\,\mathrm{d}\mu_E + 4m^2\int_\Sigma \frac{\langle x,\nu_E\rangle_E^2}{r^4(2r+m)^2}\,\mathrm{d}\mu_E.$$

Moreover, for $\Sigma = S_R(a)$ *we find the following expansions.*

1. *For* $|a|\to 0$

$$\mathcal{W}[S_R(a), g_S] = 4\pi - \frac{32\pi m R}{(2R+m)^2} + O(|a|^2).$$

2. *For* $|a|\to\infty$*, set* $|a| = r_0 + R$

$$\mathcal{W}[S_R(a), g_S] = 4\pi + \frac{8\pi m^2 R^4}{5r_0^6} + O(r_0^7).$$

3. *Similarly, for* $|a| = r_0 + R$ *and* $R\to 0$

$$\mathcal{W}[S_R(a), g_S] = 4\pi + \frac{128\pi m^2}{5r_0^2(2r_0+m)^4}R^4 + O(R^5).$$

Proof. The formula for the Willmore energy is a direct consequence of the formulas for the mean curvature $H_S = \phi^{-2}H_E + 4\phi^{-3}\partial_{\nu_E}\phi$, $\partial_{\nu_E}\phi = -\frac{m}{2r^3}\langle x,\nu_E\rangle_E$ and the measure $\mathrm{d}\mu_S = \phi^4\mathrm{d}\mu_E$.

The expansions follow by straightforward calculation. For any sphere $S_R(a)$ the Euclidean mean curvature is $H_E = 2/R$, hence we need to calculate

$$I := \int_{S_R(a)} \frac{\langle x,\nu_E\rangle_E}{r^2(2r+m)}\,\mathrm{d}\mu_E$$
$$II := \int_{S_R(a)} \frac{\langle x,\nu_E\rangle_E^2}{r^4(2r+m)^2}\,\mathrm{d}\mu_E.$$

We parametrize $S_R(a)$ as $x = a + R\tilde{x}$, where $\tilde{x}\in S_1(0)$. Of course we have $\tilde{x} = \nu_E(x)$. Then we introduce spherical coordinates such that $\langle a,\nu_E\rangle_E = |a|\cos\theta$ and abbreviate $\alpha = |a|^2 + R^2$.

$$I = R^2\int_0^\pi\int_0^{2\pi} \frac{|a|\cos\theta + R}{(\alpha + 2R|a|\cos\theta)(2(\alpha+2R|a|\cos\theta)^{1/2}+m)}\sin\theta\,\mathrm{d}\phi\mathrm{d}\theta$$
$$II = R^2\int_0^\pi\int_0^{2\pi} \frac{|a^2|\cos^2\theta + 2|a|R\cos\theta + R^2}{(\alpha + 2R|a|\cos\theta)^2(2(\alpha+2R|a|\cos\theta)^{1/2}+m)^2}\sin\theta\,\mathrm{d}\phi\mathrm{d}\theta.$$

Substituting $y = \cos\theta$ yields

$$I = 2\pi R^2 \int_{-1}^{1} \frac{|a|y + R}{(\alpha + 2R|a|y)(2(\alpha + 2R|a|y)^{1/2} + m)}\,\mathrm{d}y$$
$$II = 2\pi R^2 \int_{-1}^{1} \frac{|a^2|y^2 + 2|a|Ry + R^2}{(\alpha + 2R|a|y)^2(2(\alpha + 2R|a|y)^{1/2} + m)^2}\,\mathrm{d}y.$$

If $|a| = 0$ we obtain

$$I = \frac{4\pi R}{2R + m}$$
$$II = \frac{4\pi}{(2R + m)^2},$$

which is consistent with the expansion at $|a| = 0$.
If $|a| \neq 0$ then we substitute $z = \alpha + 2|a|Ry$.

$$I = \frac{\pi}{2|a|} \int_{(R-|a|)^2}^{(R+|a|)^2} \frac{z + R^2 - |a|^2}{z(2\sqrt{z} + m)}\,\mathrm{d}z$$
$$II = \frac{\pi}{4R|a|} \int_{(R-|a|)^2}^{(R+|a|)^2} \frac{z^2 + 2(R^2 - |a|^2)z + (R^2 - |a|^2)^2}{z^2(2\sqrt{z} + m)^2}\,\mathrm{d}z$$

The integration is not quite obvious but still elementary. We obtain

$$I = \frac{\pi}{2|a|}\Bigg[|a| + R - ||a| - R| - \frac{m}{2}\ln\left(\frac{2(|a| + R) + m}{2||a| - R| + m}\right) + \frac{2(R^2 - a^2)}{m}\left(\ln\left(\frac{|a| + R}{||a| - R|}\right) - \ln\left(\frac{2(|a| + R) + m}{2||a| - R| + m}\right)\right)\Bigg],$$
$$II = \frac{\pi}{4R|a|}\left[A + 2(R^2 - |a|^2)B + (R^2 - |a|^2)^2 C\right],$$

where

$$A = \frac{1}{2}\left(\frac{m}{2(R + |a|) + m} - \frac{m}{2|R - |a|| + m} + \ln\left(\frac{2(|a| + R) + m}{2||a| - R| + m}\right)\right)$$
$$B = \frac{2}{m^2}\left(\frac{m}{2(R + |a|) + m} - \frac{m}{2|R - |a|| + m} - \ln\left(\frac{2(|a| + R) + m}{2||a| - R| + m}\right) + \ln\left(\frac{|a| + R}{||a| - R|}\right)\right)$$
$$C = \frac{1}{m^4}\Bigg(\frac{8m}{2(R + |a|) + m} - \frac{8m}{2|R - |a|| + m} + \frac{8m}{R + |a|} - \frac{8m}{|R - |a||} - \frac{m^2}{(R + |a|)^2} + \frac{m^2}{(R - |a|)^2} - 24\ln\left(\frac{2(|a| + R) + m}{2||a| - R| + m}\right) + 24\ln\left(\frac{|a| + R}{||a| - R|}\right)\Bigg)$$

The expansions follow after a long and tedious calculation best left to a computer algebra system. We used Mathematica 7 to obtain the formulas in the statement, see the code below.

```
I1[a, R, m] = - 2 Pi m/(R a)(2 a - m/2 Log[(2(a + R) + m)/(2(R - a) +m)]
               + 2/m (a^2 - R^2) ( Log[(2 (a + R) + m)/(2 (R - a) + m)]
               - Log[(R + a)/(R - a)]))

A[a, R, m] =  1/2 (m/(m + 2 (R + a)) - m/(2 (R - a) + m)
```

```
               + Log[(2 (a + R) + m)/(2 (R - a) + m)])

B[a, R, m] = 2/m^2 (m/(m + 2 (R + a)) - m/(2 (R - a) + m)
              - Log[(2 (a + R) + m)/(2 (R - a) + m)]
              + Log[(R + a)/(R - a)])

C[a, R, m] =  1/m^4 (8 m (1/(m + 2 (R + a)) - 1/(2 (R - a) + m)
                + 1/(R + a) -1/(R - a))- m^2/(R + a)^2
                + m^2/(R - a)^2-24 Log[(2 (a + R)+ m)/(2 (R - a)+ m)]
                +24 Log[(R + a)/(R - a)])

I2[a, R, m] =  Pi m^2/(R a) (A[a, R, m] + 2 (R^2 - a^2) B[a, R, m]
               + ( R^2 - a^2)^2 C[a, R, m])

Series[I1[a, R, m] + I2[a, R, m], {a, 0, 3}]
Series[I1[r0, R, m] + I2[r0, R, m], {r0, Infinity, 6}]
Series[I1[r0, R, m] + I2[r0, R, m], {R, 0, 4}]
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□

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