On the Riemannian geometry of Seiberg-Witten moduli spaces

Dissertation
zur Erlangung des akademischen Grades
“doctor rerum naturalium” (Dr. rer. nat.)
in der Wissenschaftsdisziplin Geometrie

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät
der Universität Potsdam

von Christian Becker

Institut für Mathematik, Universität Potsdam
CMAT Laurent Schwartz, Ecole Polytechnique

Als Dissertation angenommen von der
Mathematisch-Naturwissenschaftlichen Fakultät der Universität Potsdam
sowie der Ecole Doctorale der Ecole Polytechnique, Palaiseau
aufgrund der Gutachten von:

Prof. Dr. Christian Bär (Potsdam)
Prof. Dr. Olivier Biquard (Strasbourg)
Prof. Dr. Lutz Habermann (Greifswald)

Datum der Disputation: 21. April 2005

Un cygne avance sur l’eau
tout entouré de lui-même,
comme un glissant tableau;
ainsi à certains instants
un être que l’on aime
est un espace mouvant.

Il se rapproche, doublé,
comme ce cygne qui nage,
sur notre âme troublée ...
qui à cet être ajoute
la tremblante image
de bonheur et de doute.

(Rainer Maria Rilke)

Danksagung

Zum Entstehen und Gelingen dieser Arbeit haben viele Menschen beigetragen. Ihnen allen möchte ich an dieser Stelle meinen herzlichen Dank aussprechen. CHRISTIAN BÄR, der mein Interesse auf dieses Thema gelenkt und die Betreuung der Arbeit übernommen hat, danke ich für sein beständiges Interesse an den Ergebnissen, vor allem auch während einiger entmutigender Phasen der Arbeit. Sein klares und weites geometrisches Verständnis und seine Fähigkeit, analytische Probleme geometrisch zu erhellen, haben zur Lösung der in der Arbeit aufgetretenen Probleme oft Wesentliches beigetragen. Ich danke ihm nicht zuletzt auch für die angenehme, freundschaftliche und interessierte Atmosphäre in seiner Arbeitsgruppe.

Ich danke den Gutachtern, CHRISTIAN BÄR, OLIVIER BIQUARD und LUTZ HABERMANN, sowie den weiteren Mitgliedern der Kommission, GILLES COURTOIS, JÜRGEN EHLERS, PAUL GAUDUCHON, MARKUS KLEIN und ANDREI MOROIANU.

Ich habe viel profitiert von Diskussionen mit BERND AMMANN, MATTIAS DAHL, NICOLAS GINOUX, JOCHEN MERKER, SERGIU MOROIANU, FRANK PFÄFFLE, UWE SEMMELMANN, SVEN SCHOPKA und CHRISTOPH SCHWEIGERT. Ihnen allen danke ich für ihr Interesse an der Arbeit und die Ermutigungen, mit denen sie deren Entstehung begleitet haben. Mein Dank gilt auch den Teilnehmern des “Seminars über Quantenphysik und Geometrie”, in dessen Rahmen ich Teile der Arbeit vorstellen und diskutieren durfte. Den (unfreiwilligen) Situationskomikern am Fb. 11 danke ich für manche erheiternde Stunde im Fachbereichsrat.

Schließlich gilt mein besonderer Dank meiner Familie für die Unterstützung während des Studiums und darüber hinaus.

Remerciements

Il y a plein de gens qui ont participé à la genèse et à la réussite de cette thèse. J’aimerais leur exprimer toute ma gratitude. Tout d’abord, je remercie PAUL GAUDUCHON, qui a dirigé mon travail pendant mon séjour au CMAT Laurent Schwartz de l’X, pour toutes ses idées originales et pour son immense engagement pour résoudre des problèmes mathématiques qui sont apparus dans le contexte de ma thèse. Il a toujours essayé de résoudre explicitement des équations non-linéaires que j’avais déja abandonnées. Il a aussi fait de grands efforts pour m’aider à me débarrasser de mon stupide accent allemand.

J’ai beaucoup profité des conversations avec VESTISLAV APOSTOLOV, NICOLE BERLINE, FRÉDÉRIC CHARVE, ISABELLE GALLAGHER, PIERRE GERMAIN, MILDRED HAGER, CHRISTOPH MARGERIN, ANDREI MOROIANU, SYLVIE PAYCHA, BARBARA TUMPACH et CLAUDE VITERBO.

Je les remercie pour leur intérêt et leur encouragement. Puis, je remercie les membres du “séminaire des thésards” de m’avoir donné la possibilité de faire un exposé sur certains aspects de la théorie de jauge.

Mes remerciements vont aussi à MICHELE LAVALLETTE, qui s’est occupée de toutes les affaires administratives de mon séjour au CMAT et de ma bourse et puis de tous les problèmes administratifs de la cotutelle. Sans elle, je n’aurais sûrement pas passé une si bonne période au CMAT, sans aucune perte de temps dans la jungle de la bureaucratie française.

Je remercie ISABELLE GALLAGHER et JEAN-YVE CHEMIN pour la musique qu’on a joué ensemble. Ce fut un grand plaisir de jouer le concerto à deux violons de Bach avec vous!

Abstract

In this thesis, we give two constructions for Riemannian metrics on Seiberg-Witten moduli spaces $\mathfrak{M}$. Both these constructions are naturally induced from the L^2-metric on the configuration space $\mathcal{C}$. The construction of the so called *quotient L^2-metric* is very similar to the one construction of an L^2-metric on Yang-Mills moduli spaces as given by GROISSER and PARKER. To construct a Riemannian metric on the total space $\mathfrak{P}$ of the Seiberg-Witten bundle in a similar way, we define the *reduced gauge group* $\mathcal{G}_\infty$ as a subgroup of the gauge group $\mathcal{G}$. We show, that the bundle $\widetilde{\mathfrak{M}}/\mathcal{G}_\infty \to \mathfrak{M}$ is isomorphic to the Seiberg-Witten bundle $\mathfrak{P} \to \mathfrak{M}$ as represented by the quotient of the premoduli space $\widetilde{\mathfrak{M}}$ by the based gauge group $\mathcal{G}_{x_0}$. The total space $\widetilde{\mathfrak{M}}/\mathcal{G}_\infty$ carries a natural quotient L^2-metric, and the bundle projection $\mathfrak{P} \to \mathfrak{M}$ is a Riemannian submersion with respect to these metrics. We compute explicit formulae for the sectional curvature of the moduli space $\mathfrak{M}$ in terms of Green operators of the elliptic complex $\mathcal{K}_{A,\psi}$ associated with a monopole (A, ψ). Further, we construct a Riemannian metric on the cobordism $\widehat{\mathfrak{M}} = \bigsqcup_{t\in[0,1]} \mathfrak{M}_{\mu_t^+}$ between moduli spaces for different perturbations μ_0^+, μ_1^+, which induces the L^2-metric on the fibre $\mathfrak{M}_{\mu_t^+}$. The second construction of a Riemannian metric on $\mathfrak{M}$ uses a canonical global gauge fixing, which represents the total space $\mathfrak{P}$ of the Seiberg-Witten bundle as a finite dimensional submanifold of the configuration space $\mathcal{C}$.

We consider the Seiberg-Witten moduli space $\mathfrak{M}$ on a simply connected Kähler surface M with $b_2^+ = 1$. We show that $\mathfrak{M}_{\mu^+}$ (when nonempty) is a complex projective space in the irreducible case, and that $\mathfrak{M}_{\mu^+}$ consists of a single point in the reducible case. The Seiberg-Witten bundle $\mathfrak{P} \to \mathfrak{M}$ can then be identified with the Hopf fibration $S^{2m+1} \to \mathbb{CP}^m$. On $M = \mathbb{CP}^2$ with a special Spin$^{\mathbb{C}}$-structure, our Riemannian metrics on the moduli space $\mathfrak{M}$ are Fubini-Study metrics. Correspondingly, the metrics on the total space $\mathfrak{P}$ of the Seiberg-Witten bundle are Berger metrics. We show that the diameter of the moduli space $\mathfrak{M}_{\mu^+}$ shrinks to 0 when $\mathfrak{M}_{\mu^+}$ collapses to a point, i. e. when the perturbation μ^+ approaches the wall Γ_g^+ of "reducible" perturbations. Finally we show, that the quotient L^2-metric on the Seiberg-Witten moduli space $\mathfrak{M}$ on a Kähler surface (M, g) is a Kähler metric.

Contents

Zusammenfassung

In dieser Dissertationsschrift studieren wir die Riemannsche Geometrie von Seiberg-Witten-Modulräumen. Diese Modulräume waren 1994 von den Stringtheoretikern SEIBERG und WITTEN als Modelle zur Erklärung des "quark confinement" eingeführt worden. Zwar ist letzteres ein nach wie vor unverstandenes Phänomen, dennoch haben die Seiberg-Witten-Gleichungen und die zugehörigen Modulräume andere interessante, wenngleich vorwiegend mathematische Anwendungen gefunden.

Der "mathematische Gehalt" der Seiberg-Witten-Gleichungen und der zugehörigen Modulräume war zuerst von WITTEN in [51] umrissen worden. In dieser Arbeit formulierte WITTEN die Vermutung, die Seiberg-Witten-Gleichungen seien hinsichtlich ihres "mathematischen Gehalts" den Yang-Mills-Gleichungen äquivalent, welche ja bereits seit mehr als 20 Jahre intensiv studiert worden waren. Die vage Formulierung der "Äquivalenz des mathematischen Gehalts" hat zwar bislang weder eine befriedigende Präzisierung erfahren, noch wurde der vermutete Zusammenhang der Seiberg-Witten- zur Yang-Mills-Theorie erhellt. Pragmatisch betrachtet, erwies sich WITTENS Vermutung aber insofern als zutreffend, als alle wesentlichen, mittels Yang-Mills-Theorie gewonnenen Resultate über die Geometrie und Topologie von 4-Mannigfaltigkeiten mittels Seiberg-Witten-Theorie neu und einfacher bewiesen werden konnten.

Der Ausgangspunkt sowohl der Yang-Mills- als auch der Seiberg-Witten-Eichtheorie ist das Studium geeigneter nichtlinearer elliptischer partieller Differentialgleichungen und ihrer Lösungsräume auf kompakten, glatten 4-Mannigfaltigkeiten. Die Beziehungen der Lösungsräume linearer elliptischer partieller Differentialgleichungen zur Topologie der zugrundeliegenden Mannigfaltigkeit sind ein klassischer Topos der Differentialgeometrie: die Hodge-Theorie besagt, daß die Dimension des Lösungsraumes $\mathcal{H}^p(M)$ der Laplace-Gleichung $\Delta_p \nu = 0$ auf p-Formen gleich der p-ten Betti-Zahl $b_p(M)$ von M ist. Hier bestimmt also eine topologische Invariante von $\mathcal{H}^p(M)$ – die Dimension – eine topologische Invariante der zugrundeliegenden Mannigfaltigkeit M.

Grob gesagt, kann man die Verwendung von Eichtheorien in der Geometrie als nichtlineares Analogon zur Hodge-Theorie auffassen, vergl. [14]: Die lineare elliptische Gleichung wird hier durch eine nichtlineare ersetzt, der Lösungsraum ist entsprechend ein nichtlinearer Raum, in geeigneten Fällen eine (topologische oder sogar glatte) Mannigfaltigkeit, im allgemeinen ein Orbifold. Wie im linearen Fall gewinnt man aus natürlichen topologischen Invarianten des Lösungsraumes wiederum topologische oder geometrische Invarianten der zugrundeliegenden 4-Mannigfaktigkeit M.

Natürlich greift dieses Programm zur Produktion von Invarianten nicht bei beliebigen nichtlinearen elliptischen Gleichungen. Die bekannten Fälle, in denen diese Konstruktion neue Invarianten liefert, entstammen spezifischen physikalischen Modellen, die eine geometrische Interpretation zulassen (so ist z.B. die Feldstärke F_A eines Vektorpotentials A in der Elektrodynamik bzw. Yang-Mills-Theorie geometrisch als Krümmung eines Zusammenhangs auf einem geeigneten Prinzipalbündel über M zu interpretieren). Diese physikalischen Modelle besitzen "punktweise Symmetrien", d.h. eine kompakte

Liegruppe G operiert punktweise als "Eichsymmetrie". In der physikalischen Literatur wird diese kompakte Liegruppe G oft als Eichgruppe bezeichnet, während derselbe Terminus in der mathematischen Literatur die zugehörige Fréchet-Liegruppe $\mathcal{G}$ bezeichnet, die global auf den Konfigurationen operiert. Die betrachteten physikalischen Gleichungen sind eich-invariant, d.h. die Eichgruppe operiert auf dem Lösungsraum. Der Quotient des Lösungsraumes nach der Operation der Eichgruppe – der sogenannte Modulraum $\mathfrak{M}$ – ist in der Quotienten-Topologie ein endlich-dimensionaler Hausdorffraum. Vermöge geeigneter "kleiner" Störungen der betrachteten Gleichungen läßt sich $\mathfrak{M}$ zu einer glatten Mannigfaltigkeit machen. Der Modulraum $\mathfrak{M}$ ist das Analogon zu dem Lösungsraum einer linearen elliptischen Gleichung, und tatsächlich läßt sich die Linearisierung von $\mathfrak{M}$ als Lösungsraum einer linearen elliptischen Gleichung darstellen. $\mathfrak{M}$ besitzt natürliche topologische Invarianten, die nicht von den in der Konstruktion getroffenen Wahlen (z.B. Riemannscher Metriken oder der zur Glättung verwendeten Störungen), sondern nur von der Topologie und z.B. der differenzierbaren Struktur der unterliegenden Mannigfaltigkeit M abhängen. Somit erweisen sich diese topologischen Invarianten von $\mathfrak{M}$ als (differential-)topologische Invarianten von M, die z.B. die differenzierbare Struktur von M "sehen" und verschiedene differenzierbare Strukturen auf derselben topologischen Mannigfaltigkeit M unterscheidbar machen.

Seiberg-Witten-Theorie ist eine $U(1)$-Eichtheorie, die Eichgruppe $\mathcal{G}$ ist also hier die Fréchet-Liegruppe der $U(1)$-wertigen Funktionen auf M mit punktweiser Multiplikation. Der Seiberg-Witten-Modulraum $\mathfrak{M}$ ist eine kompakte glatte Mannigfaltigkeit, deren Dimension durch den Index eines aus den Gleichungen abgeleiteten linearen elliptischen Operators gegeben ist. Der Modulraum $\mathfrak{M}$ trägt eine natürliche Orientierung, ferner erhält man auf natürliche Weise eine Isomorphieklasse reeller Geradenbündel $\mathfrak{P} \to \mathfrak{M}$. Das Integral über $\mathfrak{M}$ der der höchsten äußeren Potenz der ersten Chern-Klasse von $\mathfrak{P}$ ist eine natürliche topologische Invariante von $\mathfrak{M}$, die nur von der differenzierbaren Struktur von M und der Wahl einer Spin$^{\mathbb{C}}$-Struktur P auf M abhängt. Die so definierte Seiberg-Witten-Invariante $\mathbf{sw}(M,P)$ "sieht" bestimmte topologische und geometrische Eigenschaften von M; so ist $\mathbf{sw}(M,P)$ z.B. eine Obstruktion gegen positive Skalarkrümmung. Eine kompakte 4-Mannigfaltigkeit mit $b_2^+ > 1$, deren sämtliche Seiberg-Witten-Invarianten verschwinden, läßt keine symplektische Form zu. Ein topologisches Beispiel: ein Verklebungssatz für Seiberg-Witten-Invarianten zeigt, daß sich eine kompakte, symplektische 4-Mannigfaltigkeit nicht in die zusammenhängende direkte Summe $M = M_1 \# M_2$ zweier Faktoren mit $b_2^+(M_1), b_2^+(M_2) > 0$ zerlegen läßt.

Die Grundidee der vorliegenden Dissertationsschrift ist die Verfeinerung des Studiums der Seiberg-Witten-Modulräume $\mathfrak{M}$ von der (Differential-)Topologie zur Riemannschen Geometrie. Wir konstruieren dazu Riemannsche Metriken auf $\mathfrak{M}$ und studieren deren Geometrie. Als Motivation mag die Hoffnung dienen, vermöge der Kenntnis der Riemannschen Geometrie der Modulräume verfeinerte Invarianten konstruieren zu können. Eine topologische Verfeinerung der Seiberg-Witten-Invarianten wurde kürzlich von BAUER und FURUTA in [4, 3, 2] eingeführt. Die Autoren beobachten dort, daß die Abbildung $M \mapsto \mathbf{sw}(M,P)$ durch äquivariante stabile Kohomotopiegruppen faktorisiert. Die Werte in diesen stabilen Kohomotopiegruppen dienen dann als verfeinerte Invarianten. BAUER und FURUTA zeigen, daß diese neuen Invarianten berechenbar sind und tatsächlich die Seiberg-Witten-Invarianten verfeinern. In dieser Dissertationsschrift studieren wir die Riemannsche Geometrie der Seiberg-Witten-Modulräume $\mathfrak{M}$ bzgl. zweier natürlicher Riemannscher Metriken. Ob die Kenntnis dieser Geometrien zur Konstruktion neuer Invarianten führen wird, bleibt ein offenes Problem für weiterführende Forschungen.

Im Rahmen der Yang-Mills-Theorie wurden ähnliche Fragen von verschiedenen Forschern in diverse Richtungen verfolgt, vergl. z.B. [19, 20, 17, 1, 21, 22, 18, 32, 33]. Obwohl die formalen Konstruktionen der Metriken auf Seiberg-Witten-Modulräumen denen auf Yang-Mills-Modulräumen ähnlich sind, wirft das Studium der Geometrie dieser Metriken grundverschiedene Fragen auf: Da die Yang-Mills-Modulräume nicht kompakt sind, stellt sich hier zunächst die Frage, ob die natürliche Kompaktifizierung als Vervollständigung des Riemannschen Abstandes realisiert werden kann. Ferner kann man fragen, ob das Volumen des Modulraumes endlich ist, wie sich die Metrik nahe dem Rand verhält etc. Einige schöne Resultate in dieser Richtung findet man in den Arbeiten [19, 20, 17, 1, 21, 22]. In einigen Spezialfällen läßt sich der Diffeomorphietyp des Modulraumes bestimmen. In solchen Fällen kann man fragen, ob die konstruierte L^2-Metrik mit einer natürlichen Metrik auf diesem Modell übereinstimmt. In [19] z.B. identifizieren die Autoren den $k = 1$ Modulraum von $SU(2)$ auf S^4 mit dem 5-dimensionalen hyperbolischen Raum. Die L^2-Metrik auf $\mathfrak{M}$ ist in diesem Fall nicht die natürliche hyperbolische Metrik, sondern hat Punkte mit positiver Krümmung.

Die Seiberg-Witten-Modulräume hingegen sind kompakte glatte Mannigfaltigkeiten, und insofern sind alle konstruierten Riemannschen Metriken vollständig und haben endliches Volumen. Dennoch treten andere, natürliche Fragen auf: Die Konstruktion des Modulraumes $\mathfrak{M}$ hängt von der Wahl eines Störungsparameters μ^+ ab, die Abhängigkeit der Metrik von dem Parameter μ^+ wird also zu untersuchen sein. Die Seiberg-Witten-Modulräume $\mathfrak{M}_{\mu_0^+}, \mathfrak{M}_{\mu_1^+}$ für generische Störungen μ_0^+, μ_1^+ können durch einen glatten Kobordismus $\widehat{\mathfrak{M}}$ verbunden werden. Ein interessantes Resultat der vorliegenden Dissertationsschrift besagt, daß die Konstruktion der L^2-Metrik sich auf den Kobordismus $\widehat{\mathfrak{M}}$ fortsetzen läßt und daß die Einschränkung der Metrik von $\widehat{\mathfrak{M}}$ auf die Faser über $t \in [0, 1]$ die Metrik des Modulraumes $\mathfrak{M}_{\mu^t}$ liefert. Ferner kann man fragen, wie sich die Konstruktion der L^2-Metrik auf dem Modulraum $\mathfrak{M}$ vernünftig auf den Totalraum des Seiberg-Witten-Bündels $\mathfrak{P} \to \mathfrak{M}$ fortsetzen läßt.

Im Kapitel 1 geben wir eine kurze Einführung in Seiberg-Witten-Theorie. Wir zitieren alle wesentlichen, in der Arbeit verwendeten Resultate und erläutern die Notationen. Der Fokus liegt hier auf einer Zusammenstellung aller wichtigen Resultate über positiv-dimensionale Modulräume. Es sind bislang nur wenige Konstruktionen bekannt, aus denen sich auf die Nichttrivialität dieser Räume schließen läßt. Da bislang alle wesentlichen Anwendungen von Seiberg-Witten-Theorie aus dem Studium der 0-dimensionalen Modulräumen geschöpft wurden, ist über Modulräume positiver Dimension überhaupt wenig bekannt. Eine systematische Zusammenstellung dieser Resultate bislang nicht vor, daher referieren wir einige der bekannten Resultate im Abschnitt 1.5. Eine besonders interessante Konstruktion von DEL RIO GUERRA führt zu positiv-dimensionalen, topologisch nicht-trivialen Modulräumen mit verschwindender Seiberg-Witten-Invariante.

In Kapitel 2 geben wir zwei verschiedene Konstruktionen Riemannscher Metriken auf Seiberg-Witten-Modulräumen an. Beide Konstruktionen benutzen die natürliche L^2-Metrik auf dem Konfigurationsraum $\mathcal{C}$, wo die Seiberg-Witten-Gleichungen definiert sind. Die erste Konstruktion folgt dem Zugang von GROISSER und PARKER in [19] zur Konstruktion einer L^2-Metrik auf Yang-Mills-Modulräumen. Hierbei dient der zu einem Monopol (A, ψ) gehörige elliptische Komplex $\mathcal{K}_{(A,\psi)}$ zur Identifikation des Tangentialraumes $T_{[A,\psi]}\mathfrak{M}$ als Unterraum des Konfigurationsraumes. Die resultierende Metrik auf $\mathfrak{M}$ bezeichnen wir als *Quotienten-L^2-Metrik*.

Zur Konstruktion einer ähnlichen L^2-Metrik auf dem Totalraum des Seiberg-Witten-Bündels konstruieren wir in 2.3 einen neuen, geometrischen Repräsentanten der Isomorphieklasse $\mathfrak{P} \to \mathfrak{M}$ von

Geradenbündeln: Wir definieren die *reduzierte Eichgruppe* $\mathcal{G}_\infty \subset \mathcal{G}$ und zeigen, daß das $U(1)$-Bündel $\widetilde{\mathfrak{M}}/\mathcal{G}_\infty \to \mathfrak{M}$ isomorph zu dem Quotienten $\widetilde{\mathfrak{M}}/\mathcal{G}_{x_0} \to \mathfrak{M}$ des Prämodulraumes nach der fixierten Eichgruppe $\mathcal{G}_{x_0}$ ist. Somit repräsentiert das Bündel $\widetilde{\mathfrak{M}}/\mathcal{G}_\infty \to \mathfrak{M}$ die Isomorphieklasse $\mathfrak{P}$. Dieselbe Konstruktion wie in 2.2 liefert nun eine natürliche Riemannsche Metrik auf dem Totalraum des Bündels $\widetilde{\mathfrak{M}}/\mathcal{G}_\infty \to \mathfrak{M}$, und wir können zeigen, daß die Bündelprojektion eine Riemannsche Submersion ist.

In 2.4 und 2.5 berechnen wir explizite Formeln für die Schnittkrümmung de Raumes $\mathcal{B}^*$ der Eichäquivalenzklassen irreduzibler Konfigurationen sowie des Prämodulraumes $\widetilde{\mathfrak{M}}$ und des Modulraumes $\mathfrak{M}$. Diese Formeln geben die Schnittkrümmung dieser Räume in Termen von Green-Operatoren des elliptischen Komplexes $\mathcal{K}_{(A,\psi)}$ an.

In 2.6 konstruieren wir eine natürliche Quotienten-L^2-Metrik auf den Kobordismen $\widehat{\mathfrak{M}} = \bigsqcup_{t\in[0,1]} \mathfrak{M}_{\mu_t^+}$ zwischen Modulräumen für verschiedene Störungen μ_0^+, μ_1^+ und zeigen, daß die Einschränkung dieser Metrik auf die Faser über $t \in [0,1]$ genau die oben konstruierte Quotienten-L^2-Metrik auf dem Modulraum $\mathfrak{M}_{\mu_t^+}$ liefert.

In 2.7 geben wir eine zweite Konstruktion einer Riemannschen Metrik auf dem Modulraum $\mathfrak{M}$ an. Vermöge einer kanonischen globalen Eichfixierung kann der Totalraum des Seiberg-Witten-Bündels $\mathfrak{P}$ direkt als Untermannigfaltigkeit des Konfigurationsraumes $\mathcal{C}$ realisiert werden. Die Einschränkung der L^2-Metrik von $\mathcal{C}$ auf $\mathfrak{P}$ ist $U(1)$-invariant und steigt somit zu einer Metrik auf $\mathfrak{M} = \mathfrak{P}/U(1)$ ab. Wir bezeichnen die so konstruierte Metrik als *kanonisch eichfixierte L^2-Metrik*.

In Kapitel 3 betrachten wir die Seiberg-Witten Modulräume $\mathfrak{M}$ auf einfach zusammenhängenden, kompakten Kählerflächen M mit $b_2^+(M) = 1$. Wir zeigen in 3.1, daß $\mathfrak{M}_{\mu^+}$ (falls nicht leer) diffeomorph zu einem komplex projektiven Raum ist, falls die Störung μ^+ keine reduziblen Lösungen zuläßt; andernfalls besteht $\mathfrak{M}_{\mu^+}$ aus einem einzigen Punkt. Das Seiberg-Witten-Bündel läßt sich mit der Hopf-Faserung $S^{2m+1} \to \mathbb{CP}^m$ identifizieren.

In 3.2 berechnen wir explizite Formeln für das Längenquadrat eines Tangentialvektors in Termen der Lösungen einer Differentialgleichung vom Typ der Kazdan-Warner-Gleichung. Im Fall von $M = \mathbb{CP}^2$ mit einer Spin$^{\mathbb{C}}$-Struktur, für die $\mathfrak{M} \approx \mathbb{CP}^2$ ist, zeigen diese Formeln, daß die oben konstruierten Metriken $U(3)$-invariant und somit Fubini-Study-Metriken sind. Die zugehörigen Metriken auf dem Totalraum $\mathfrak{P}$ des Seiberg-Witten-Bündels sind gleichfalls $U(3)$-invariant und sind also Berger-Metriken. Wir zeigen ferner, daß bei Annäherung der Störung μ^+ an den reduziblen Fall, d.h. bei der Deformation des Modulraumes $\mathfrak{M}_{\mu^+}$ zu einem Punkt, der Durchmesser von $\mathfrak{M}_{\mu^+}$ gegen 0 konvergiert.

In 3.4 schließlich zeigen wir, daß auf jeder Kählerfläche M der Seiberg-Witten-Modulraum $\mathfrak{M}$ als symplektischer Quotient einer unendlichdimensionalen Kähler-Untermannigfaltigkeit des Konfigurationsraumes $\mathcal{C}$ nach der Eichgruppe dargestellt werden kann. Dies impliziert, daß die Quotienten-L^2-Metrik auf $\mathfrak{M}$ eine Kähler-Metrik ist. Unter welchen Bdeingungen diese Kähler-Metrik die Fubini-Study-Metrik ist, bleibt eine offene Frage für weitere Forschungen.

Résumé

Comme le titre l'indique, dans cette thèse nous étudions la géométrie riemanienne des espaces de modules de Seiberg-Witten. Ces espaces de modules ont été introduits en 1994 dans [46, 47] par les physiciens SEIBERG et WITTEN pour expliquer le confinement des quarks dans le contexte de la théorie des cordes.

L'idée fondamentale des théories de jauge est l'étude de certaines équations aux dérivées partielles elliptiques non-linéaires sur une variété lisse compacte de dimension 4. Dans la théorie elliptique linéaire sur des variétés compactes, il est bien connu, que les espaces de solutions des équations elliptiques sont reliés à la géométrie et à la topologie de la variété sous-jacente. La théorie de Hodge nous dit que la dimension de l'espace de solutions $\mathcal{H}^p(M)$ de l'équation de Laplace $\Delta_p \nu = 0$ sur les p-formes $\Omega^p(M)$ est un invariant topologique de la variété M: c'est le nombre de Betti $b_p(M)$. Donc un invariant topologique de l'espace des solutions $\mathcal{H}^p(M)$ – la dimension – nous donne un invariant topologique de la variété sous-jacente. On peut donc regarder les mathématiques de la théorie de jauge comme un analogue non-linéaire de la théorie de Hodge, cf. [14]: L'équation elliptique linéaire sur M est remplacée par une équation non-linéaire, dont on considère l'espace des solutions et son rapport à la topologie et à la géométrie de la variété sous-jacente M. Comme dans la théorie de Hodge, c'est par des invariants topologiques des espaces de solutions qu'on obtient de nouveaux invariants topologiques ou géométriques de M.

La théorie de Seiberg-Witten est une théorie de jauge de groupe de base $U(1)$. C'est-à-dire, les équations de Seiberg-Witten sont invariantes sous l'action du groupe des fonctions sur M à valeurs dans $U(1)$. Une configuration $(A, \psi) \in \mathcal{C}$ consiste en une connexion $A \in \mathcal{A}(\det P)$ et un spineur positif $\psi \in \Gamma(\Sigma^+)$, où $\det P$ dénote le fibré déterminant d'une structure Spin$^{\mathbb{C}}$ P sur M. Un monopole est une solution des équations de Seiberg-Witten. Le quotient de l'espace de solutions $\widetilde{\mathfrak{M}}$ par l'action du groupe de jauge $\mathcal{G} = \Omega^0(M; U(1))$ – c'est ce qu'on appelle l'espace de modules $\mathfrak{M}$ – est un espace topologique séparé, voire une variété topologique compacte. On peut en faire une variété lisse en utilisant des perturbations des équations initiales. L'espace de modules $\mathfrak{M}$ est aussi orientable, et il vient avec un fibré $U(1)$ – ou plutôt une classe d'isomorphisme de fibrés $U(1) \hookrightarrow \mathfrak{P} \twoheadrightarrow \mathfrak{M}$. Donc il y a un invariant naturell associé à $\mathfrak{M}$, c'est le degré maximum de la première classe de Chern évalué sur le cycle fondamental $[\mathfrak{M}]$. Cet invariant ne dépend ni du choix de la perturbation μ^+ ni du choix d'une métrique riemannienne sur M. Elle ne dépend que de la topologie et de la structure différentiable de la variété sous-jacente M.

L'idée de cette thèse est de raffiner l'étude de l'espace de modules de Seiberg-Witten $\mathfrak{M}$ de la topologie de $\mathfrak{M}$ à la géométrie riemannienne. Nous donc construisons des métriques riemaniennes sur l'espace de modules $\mathfrak{M}$ et nous étudions la géométrie de $\mathfrak{M}$ par rapport à ces métriques. Une des motivations pour une telle étude de la géométrie de $\mathfrak{M}$ est l'espoir que les invariants de Seiberg-

Witten pourraient être raffinés par une connaissance plus précise de l'espace de modules. Un raffinement topologique des invariants de Seiberg-Witten a été introduit récemment par BAUER et FURUTA, cf. [4, 3, 2]. Ils observent que l'application, qui à une structure Spin$^{\mathbb{C}}$ P associe l'invariant de Seiberg-Witten de l'espace des modules $\mathfrak{M}(M, P)$ se factorise à travers certains groupes d'homotopie stables équivariants. Dans cette thèse, nous considérons la géométrie riemannienne de $\mathfrak{M}$ par rapports à deux métriques riemanniennes induites naturellement par la métrique L^2 de l'espace de configurations $\mathcal{C}$. La question de savoir si la connaissance de la géométrie de $\mathfrak{M}$ va mener à la construction de nouveaux invariants, reste ouverte pour la recherche à venir.

Dans le cadre de la théorie de Yang-Mills, des recherches similaires sur la géométrie des métriques riemanniennes sur l'espace de modules ont été menées par différents chercheurs dans plusieurs directions, cf. [19, 20, 17, 1, 21, 22, 18, 32, 33]. Même si les constructions des métriques sont identiques, l'intérêt pour la géométrie est différent dans les cadres des espaces de modules de Yang-Mills et de Seiberg-Witten. Comme l'espace de modules de Yang-Mills est non-compact, une question raisonnable est intéressante est de savoir si la compactification naturelle peut être réalisée géométriquement comme la complétion de la distance riemannienne, cf. [19]. En outre, on peut se demander si le volume de $\mathfrak{M}$ est fini ou infini. Au contraire, l'espace de modules de Seiberg-Witten est une variété lisse compacte, donc les métriques riemanniennes là-dessus sont toujours complètes à volume fini.

D'autres questions naturelles apparaissent cependant: par exemple, comme la construction de l'espace de modules comme variété lisse dépend du choix d'une perturbation μ^+, on aimerait comprendre de quelle façon la métrique L^2 sur $\mathfrak{M}_{\mu^+}$ dépend de ce paramètre μ^+. Les espaces de modules $\mathfrak{M}_{\mu_0^+}, \mathfrak{M}_{\mu_1^+}$ pour des perturbations génériques μ_0^+, μ_1^+ peuvent être joints par un cobordisme $\widehat{\mathfrak{M}}$, qui est fibré sur l'intervalle $[0, 1]$ par des espaces de modules. Un des résultats intéressants de cette thèse nous dit qu'on obtient une métrique L^2 naturelle sur ce cobordisme $\widehat{\mathfrak{M}}$, telle que la restriction à la fibre sur $t \in [0, 1]$ coïncide avec la métrique L^2 de $\mathfrak{M}_{\mu_t^+}$. De plus, on peut se demander si la construction des métriques L^2 sur l'espace de modules $\mathfrak{M}$ s'étend de façon naturelle au fibré $\mathfrak{P} \to \mathfrak{M}$ qui définit l'invariant de Seiberg-Witten.

Dans le premier chapitre, nous donnons une brève introduction à la théorie de Seiberg-Witten. Nous y expliquons les notions fondamentales aussi bien que les notations utilisées dans cette thèse. Les linéarisations de l'application de Seiberg-Witten et de l'action du groupe de jauge forment un complexe elliptique $\mathcal{K}_{(A,\psi)}$ qui est à la base de notre construction d'une métrique L^2 naturelle sur l'espace de modules. Jusqu'à présent, presque tous les résultats obtenus par la théorie ont été démontrés en utilisant des espaces de modules de dimension 0. Cependant, les espaces de modules ne sont pas toujours de dimension 0. Comme il n'existe aucune recherche systématique sur les espaces de modules de dimension positive, nous donnons dans la section 2.5 plusieurs résultats connus sur ces espaces de modules. Nous y rappelons quelques constructions des espaces de modules non-triviaux de dimension positive aussi bien que des conditions sous lesquelles tous les espaces de modules de dimension virtuelle positive sont génériquement vides. En particulier, nous décrivons une construction de DEL RIO GUERRA qui produit des espaces de modules topologiquement non-triviaux, dont les invariants de Seiberg-Witten sont triviaux.

Dans le deuxième chapitre, nous construisons deux métriques L^2 sur l'espace de modules de Seiberg-Witten $\mathfrak{M}$ qui sont induites naturellement par la métrique L^2 de l'espace de configurations $\mathcal{C}$. La première construction est similaire à laquelle de GROISSER et PARKER dans [19] pour l'espace de modules de Yang-Mills. Le complexe elliptique $\mathcal{K}_{(A,\psi)}$ associé à un monopole (A,ψ) sert à identifier l'espace tangent de $\mathfrak{M}$ au point base $[A,\psi]$ comme sous-espace de l'espace tangent de $\mathcal{C}$. Cette identification de $T_{[A,\psi]}\mathfrak{M}$ induit une métrique L^2 sur l'espace de modules $\mathfrak{M}$.

Pour construire une métrique sur le fibré $\mathfrak{P} \to \mathfrak{M}$, nous introduisons un nouveau sous-groupe $\mathcal{G}_\infty$ du groupe de jauge $\mathcal{G}$. Le groupe de jauge réduit $\mathcal{G}_\infty$ consiste en les transformations de jauge $u \in \mathcal{G}$, qui satisfont l'équation $\exp\left(\frac{1}{\mathrm{vol}(M)}\int_M \log u \, dv_g\right) = 1$. Pour ce sous-groupe $\mathcal{G}_\infty \subset \mathcal{G}$, on obtient une décomposition orthogonale des algèbres de Lie $\mathfrak{g} = \mathfrak{g}_\infty \oplus i\mathbb{R}$. Nous démontrons dans la section 2.3 que le fibré $U(1) \hookrightarrow \widetilde{\mathfrak{M}}/\mathcal{G}_\infty \twoheadrightarrow \mathfrak{M}$ représente la classe d'isomorphisme des fibrés $\mathfrak{P} \to \mathfrak{M}$, qui définit l'invariant de Seiberg-Witten. Les décompositions elliptiques associées au complexe $\mathcal{K}_{(A,\psi)}$ induisent une métrique L^2 sur le quotient $\widetilde{\mathfrak{M}}/\mathcal{G}_\infty$. Nous démontrons que la projection $\mathfrak{P} \to \mathfrak{M}$ est une submersion riemannienne par rapport aux métriques L^2.

Dans les sections 2.4 et 2.5, nous calculons des formules explicites pour la courbure sectionnelle de l'espace $\mathcal{B}^*$ des classes d'équivalence de configurations irréductibles, aussi bien que de l'espace de prémodules $\widetilde{\mathfrak{M}}$ et de l'espace de modules $\mathfrak{M}$ par rapport aux métriques L^2. Cette formule exprime la courbure sectionnelle de $\mathfrak{M}$ en termes d'opérateurs de Green associés au complexe elliptique $\mathcal{K}_{(A,\psi)}$.

Dans la section 2.6, nous construisons une métrique L^2 naturelle sur le cobordisme $\widehat{\mathfrak{M}} = \bigsqcup_{t\in[0,1]} \mathfrak{M}_{\mu_t^+}$. Nous démontrons que la restriction de cette métrique à la fibre sur $t \in [0,1]$ coïncide avec la métrique L^2 de l'espace des modules $\mathfrak{M}_{\mu_t^+}$.

Une autre métrique L^2 naturelle peut être construite sur $\mathfrak{M}$ à l'aide d'un choix de jauge canonique. En fixant la jauge globalement, on peut représenter l'espace total $\mathfrak{P}$ du fibré $U(1)$ qui définit l'invariant de Seiberg-Witten comme sous-variété lisse de l'espace de configurations $\mathcal{C}$. Par conséquent, il y a une métrique induite sur $\mathfrak{P}$ par la métrique L^2 de $\mathcal{C}$. Donc il existe une métrique unique sur l'espace de modules $\mathfrak{M}$ telle que la projection $\mathfrak{P} \to \mathfrak{M}$ est une submersion riemannienne.

Dans le troisième chapitre, nous étudions le cas particulier des espaces de modules de Seiberg-Witten sur des surfaces kählériennes à $b_2^+ = 1$. Nous démontrons que les espaces de modules à dimension virtuelle positive sur une surface kählérienne simplement connexe M sont difféomorphes à des espaces complexes projectifs, pour autant que la perturbation μ^+ n'admet pas de monopoles réductibles. Pour une perturbation μ^+, qui admet des monopoles réductibles, l'espace de modules consiste en un seul point. Le fibré de Seiberg-Witten $\mathfrak{P} \to \mathfrak{M}$ est isomorphe au fibré de Hopf $S^{2m+1} \to \mathbb{CP}^m$.

Dans la section 3.2, nous calculons des formules (explicites) pour la norme au carré d'un vecteur tangent de $\mathfrak{M}$. Dans le cas particulier de $M = \mathbb{CP}^2$ avec la structure Spin$^{\mathbb{C}}$ telle que l'espace des modules $\mathfrak{M}$ est difféomorphe à $\mathbb{CP}^2$, nous démontrons que les deux métriques L^2 construites au chapitre 2 sont invariantes par l'action canonique de $U(3)$. Par conséquent, ces métriques L^2 coïncident (à reparamétrisation près) avec la métrique de Fubini-Study. Les métriques L^2 sur $\mathfrak{P}$ sont invariantes sous $U(3)$ aussi, donc ce sont des métriques de Berger. Pour la déformation de l'espace de modules le long du paramètre μ^+, nous démontrons que le diamètre de l'espace de module $\mathfrak{M}_{\mu_t^+}$ tend vers zéro, quand la perturbation μ_t^+ s'approche d'une perturbation $\mu_{t_0}^+$ qui admet des monopoles réductibles.

Finalement, dans la section 3.4, nous montrons que l'espace de modules $\mathfrak{M}$ sur une surface kählérienne M est le quotient symplectique d'une certaine sous-variété kählérienne de l'espace de configurations $\mathcal{C}$. Par conséquent, la métrique L^2 sur $\mathfrak{M}$ est une métrique kählérienne. La question reste ouverte de savoir sous quelles conditions supplémentaires cette métrique kählérienne est la métrique de Fubini-Study.

Introduction

As the title indicates, we study in this thesis the Riemannian geometry of Seiberg-Witten moduli spaces. These moduli spaces had been introduced in 1994 in [46, 47] by the string theorists SEIBERG and WITTEN to explain "quark confinement".

The "mathematical content" of the Seiberg-Witten equations and the associated moduli spaces was first outlined by WITTEN in [51]. In this article, WITTEN posed the conjecture, that the mathematical content of the Seiberg-Witten equations is equivalent to the one of the Yang-Mills equations, which had already been studied for more than two decades. The "equivalence of the mathematical content" is clearly not well defined, but pragmatically speaking, WITTEN's guess turned out to be true: all the fundamental results about the topology and geometry of 4-manifolds, which had been obtained from Yang-Mills theory, can be reproven in a far more simple way via Seiberg-Witten theory. Until now, the precise relation between Yang-Mills theory and Seiberg-Witten theory is unclear.

The starting point of both the Yang-Mills and the Seiberg-Witten gauge theory is the study of certain nonlinear elliptic partial differential equations on compact smooth 4-manifolds. As is well known from linear elliptic theory on compact manifolds, the solution spaces of elliptic equations are closely related to the geometry and topology of the manifold. As the cornerstone of linear elliptic theory, Hodge theory says that the dimension of the solution space $\mathcal{H}^p(M)$ of the Laplace equation $\Delta_p \nu = 0$ on p-forms $\Omega^p(M)$ is a topological invariant of M, namely the p-th Betti number $b_p(M)$. Thus a topological invariant – namely the dimension – of the solution space $\mathcal{H}^p(M)$ of the Laplace equation gives a topological invariant of the underlying manifold M.

Roughly speaking, the mathematics of gauge theory can be thought of as a nonlinear analogue of Hodge theory, see [14]: The linear elliptic equation is replaced by a nonlinear one, which then gives a nonlinear solution space. Thus there are additional topological invariants of the solution space than just the dimension, and these lead new topological or geometrical invariants of the underlying manifold M.

The nonlinear equations, for which this programme works, are of course not arbitrary, but they stem from distinct physics models, which admit a geometrical interpretation (e.g. the field strengths F_A of vector potentials A in classical electro-magnetic or Yang-Mills theory are curvatures of connections on certain principle bundles over the spacetime 4-manifold M). These models have "pointwise symmetries", i.e. there is some compact Lie group G acting as "gauge symmetry" on the points of the model. In the physics literature, this group is sometimes called the gauge group. In the mathematics literature, the term "*gauge group*" designates the infinite dimensional Fréchet Lie group $\mathcal{G}$ associated with G, which acts on the configurations of the model. This action preserves the equations and thus induces a natural action on their solution space. The quotient of the solution space by the action of the gauge group – the so called *moduli space* $\mathfrak{M}$ – is a finite dimensional topological Hausdorff space, and it can be made into a smooth manifold via appropriate "small" perturbations of the equations. There are natural topological invariants of this moduli space $\mathfrak{M}$, which are independent of several choices made in the construction (such as the perturbation parameters), but which depend only on the topological, smooth or geometrical structure of the underlying manifold M. These topological invariants of the moduli space $\mathfrak{M}$ thus give new ("secondary") invariants of the underlying manifold M, which make it possible e.g. to distinguish different smooth structures on M.

Seiberg-Witten theory is a $U(1)$ gauge theory, which means that the gauge group $\mathcal{G}$ is the Fréchet Lie group of $U(1)$-valued functions on M with multiplication defined pointwise. The Seiberg-Witten moduli space $\mathfrak{M}$ is a compact smooth manifold, the dimension of which is given by the index of a linear elliptic operator intrisically related to the equations. The moduli space $\mathfrak{M}$ comes together with a natural orientation and a $U(1)$-bundle or rather an isomorphism class of $U(1)$-bundles $\mathfrak{P} \to \mathfrak{M}$. Thus there is a natural numerical invariant of $\mathfrak{M}$, namely top exterior power of the first Chern class $c_1(\mathfrak{P})$, evaluated on the fundamental cycle $[\mathfrak{M}]$. It turns out, that this number is independent of all geometrical choices made in the construction but depends only on the smooth structure of the underlying manifold. This so called Seiberg-Witten invariant $\mathbf{sw}(M, P)$ reflects certain geometrical and topological properties of the manifold M. For example, when $\mathbf{sw}(M, P) \neq 0$, then M does not admit a Riemannian metric of positive scalar curvature. A compact 4-manifold M with $b_2^+ > 1$ all of whose Seiberg-Witten invariants vanish does not admit a Kähler metric, and not even an almost Kähler metric, i.e. it does not admit a symplectic form. As an example for a topological result, a gluing theorem for the Seiberg-Witten invariants implies, that a compact symplectic 4-manifold M cannot be decomposed as a connected sum $M = M_1 \# M_2$ with $b_2^+(M_1), b_2^+(M_2) > 0$.

The idea of this thesis is to refine the study of the Seiberg-Witten moduli space $\mathfrak{M}$ from the (differential) topology to the geometry of $\mathfrak{M}$. Thus we construct Riemannian metrics on the moduli space $\mathfrak{M}$ and analyse the geometry of $\mathfrak{M}$ with respect to these metrics. This study may be motivated by the hope to eventually refine the Seiberg-Witten invariant by geometrical invariants of the moduli space. A topological refinement of the Seiberg-Witten invariant had recently been introduced by BAUER and FURUTA, see [4, 3, 2]. They observe that the map, which associates to a Spin$^{\mathbb{C}}$-structure$P$ on a 4-manifold M the Seiberg-Witten invariant of the moduli space $\mathfrak{M}(M, P)$ factors through certain equivariant stable cohomotopy groups. The values of this map in the stable cohomotopy groups can be taken as refined invariants. BAUER and FURUTA show, that these new invariants are computable and that they indeed refine the Seiberg-Witten invariant. In our thesis, we study the geometry of the Seiberg-Witten moduli space $\mathfrak{M}$ with respect to two different Riemannian metrics induced naturally by the L^2-metric on the configuration space $\mathcal{C}$. Whether the knowledge of the geometry of $\mathfrak{M}$ with respect to these natural metrics leads to interesting new invariants, remains an open question for further research.

In the context of Yang-Mills theory, similar research on the geometry of different (more or less natural) Riemannian metrics on the moduli spaces had been untertaken in several directions by several people, among which we like to mention [19, 20, 17, 1, 21, 22, 18, 32, 33]. Although the constructions of the metrics are similar, the geometrical interests are different for the case of the Yang-Mills and the Seiberg-Witten moduli spaces. Since the Yang-Mills moduli spaces are noncompact, it is particularly interesting, whether the natural compactification, which arises from the analysis of the equations can be realised geometrically, i.e. as the completion with respect to the Riemannian distance. One may also ask whether the volume of the moduli space is finite or infinite, or how the metric behaves near the collar built by the underlying manifold M itself etc. For results in these directions (at least for some interesting and accessible examples), we refer to [19, 20, 17, 1, 21, 22]. In the cases, where the moduli space $\mathfrak{M}$ can be identified explicitly as smooth manifold, one may ask whether the L^2-metric on $\mathfrak{M}$ coincides with some natural Riemannian metric on this manifold (see [19] for the case of the $SU(2)$-bundle with instanton number $k = 1$ on the 4-sphere S^4, the Yang-Mills moduli space $\mathfrak{M}$ of which can naturally be identified with the hyperbolic 5-plane; the L^2-metric on this moduli space $\mathfrak{M}$ is not the natural hyperbolic metric but has positive curvature somewhere).

Since the Seiberg-Witten moduli spaces $\mathfrak{M}$ are compact smooth manifolds, the Riemannian metrics constructed thereon are always complete and have finite volume. However there arise other natural and interesting questions for the geometry of those metrics on $\mathfrak{M}$. For example, since the construction of the moduli space $\mathfrak{M}$ as a smooth manifold depends on a perturbation parameter μ^+, one may ask for the dependence of the metrics on this parameter μ^+. As is well known, the Seiberg-Witten moduli spaces $\mathfrak{M}_{\mu_0^+}, \mathfrak{M}_{\mu_1^+}$ for generic perturbations μ_0^+, μ_1^+ can be joined by a smooth cobordism $\widehat{\mathfrak{M}}$, which fibres over the interval $[0, 1]$ by moduli spaces. It is interesting to see that the construction of the natural Riemannian metric on the moduli spaces extends to this smooth cobordism, and that the restriction of the metric on $\widehat{\mathfrak{M}}$ to the fibre over $t \in [0, 1]$ gives the metric on the moduli space $\mathfrak{M}_{\mu_t^+}$. Furthermore, one may ask whether the construction of the Riemannian metrics on the moduli space $\mathfrak{M}$ can be extended in a reasonable way to the total space of the $U(1)$-bundle $\mathfrak{P} \to \mathfrak{M}$, which defines the Seiberg-Witten invariant.

In chapter 1, we give a brief introduction to Seiberg-Witten theory. We especially cite all the essential results needed in the following chapters and we explain our notation. Until now, almost all results in Seiberg-Witten theory had been obtained from the analysis of Seiberg-Witten moduli spaces of dimension 0. Although it is not the case – as some people are tended to suggest – that the Seiberg-Witten moduli spaces are always 0-dimensional, there are only very few constructions known, which lead to nonvanishing results for the moduli spaces of positive virtual dimension. Since there exists no systematic overview of those constructions, we review some of them in section 1.5. A particularly interesting construction of DEL RIO GUERRA leads to topologically nontrivial moduli spaces with vanishing Seiberg-Witten invariant.

In chapter 2, we construct two different Riemannian metrics on the Seiberg-Witten moduli spaces $\mathfrak{M}$ of positive (virtual) dimension, which are both naturally induced from the L^2-metric on the configuration space $\mathcal{C}$, where the Seiberg-Witten equations live. The first construction follows the approach of GROISSER and PARKER in [19] to define an L^2-metric on the Yang-Mills moduli space. We call this metric the *quotient L^2-metric* on the moduli space $\mathfrak{M}$. To define a Riemannian metric on the total space $\mathfrak{P}$ of the Seiberg-Witten bundle in a similar way, we introduce a new representative of the class of $U(1)$-bundles on $\mathfrak{M}$. We define a new subgroup $\mathcal{G}_\infty$ of the gauge group. We then show, that the bundle $\widetilde{\mathfrak{M}}/\mathcal{G}_\infty \to \mathfrak{M}$ is isomorphic to the Seiberg-Witten bundle as represented by the quotient of the premoduli space $\widetilde{\mathfrak{M}}$ by the based gauge group $\mathcal{G}_{x_0}$. Further we show that the bundle $\widetilde{\mathfrak{M}}/\mathcal{G}_\infty \to \mathfrak{M}$ carries a natural Riemannian metric and that the bundle projection $\mathfrak{P} \to \mathfrak{M}$ is a Riemannian submersion with respect to these metrics. We compute explicit formulae for the sectional curvature of the premoduli space $\widetilde{\mathfrak{M}}$ and the moduli space $\mathfrak{M}$ in terms of Green operators of the elliptic complex $\mathcal{K}_{A,\psi}$ associated with a monopole (A, ψ). Furthermore, we construct a Riemannian metric on the cobordism $\widehat{\mathfrak{M}} = \bigsqcup_{t\in[0,1]} \mathfrak{M}_{\mu_t^+}$ between moduli spaces for different perturbations μ_0^+, μ_1^+, and we show that the restriction of this L^2-metric to the fibre over $t \in [0, 1]$ gives the L^2-metric of the moduli spaces $\mathfrak{M}_{\mu_t^+}$. The second construction of a Riemannian metric uses a canonical global gauge fixing, which realises the total space $\mathfrak{P}$ as a finite dimensional submanifold of the configuration space $\mathcal{C}$. Thus the L^2-metric on $\mathcal{C}$ restricts to a Riemannian metric on $\mathfrak{P}$, and we obtain a Riemannian metric on $\mathfrak{M}$ by requiring the bundle projection $\mathfrak{P} \to \mathfrak{M}$ to be a Riemannian submersion.

In chapter 3, we consider the Seiberg-Witten moduli space $\mathfrak{M}$ on a simply connected Kähler surface M with $b_2^+(M) = 1$. We show that $\mathfrak{M}_{\mu^+}$ (when nonempty) is a complex projective space, if the perturbation μ^+ does not admit reducible monpoles, and that $\mathfrak{M}_{\mu^+}$ consists of a single point otherwise. The Seiberg-Witten bundle $\mathfrak{P} \to \mathfrak{M}$ can then be identified with the Hopf fibration $S^{2m+1} \to \mathbb{CP}^m$. We compute (explicit) formulae for the length square of a tangent vector in terms of the solutions of a Kazdan-Warner type equation obtained from the Seiberg-Witten equations. On $M = \mathbb{CP}^2$ with a special Spin$^{\mathbb{C}}$-structure, for which the moduli space $\mathfrak{M}$ is diffeomorphic to the underlying manifold M, we show that the metrics constructed in chapter 2 are $U(3)$-invariant and thus are Fubini-Study metrics. The metrics on the total space $\mathfrak{P}$ of the Seiberg-Witten bundle are $U(3)$-invariant too and thus are Berger metrics. We show that the diameter of the moduli space $\mathfrak{M}_{\mu^+}$ shrinks to 0 when $\mathfrak{M}_{\mu^+}$ collapses to a point, i.e. when the perturbation μ^+ approaches the wall of those perturbations which admit reducible monopoles. Finally, we show that for any Kähler surface M, the Seiberg-Witten moduli space $\mathfrak{M}$ can be represented as the symplectic quotient of a certain infinite dimensional Kähler submanifold of the configuration space $\mathcal{C}$. Consequently, the quotient L^2-metric on $\mathfrak{M}$ is a Kähler metric. It remains an open question for further research, under which additional conditions this Kähler metric is in fact the Fubini-Study metric.

Chapter 1

Seiberg-Witten theory

In this chapter, we give a brief introduction to Seiberg-Witten theory. We explain the notation used throughout this thesis and recall some of the main results on the moduli spaces and Seiberg-Witten invariants. We state some vanishing and nonvanishing results for the invariants on special manifolds such as Kähler manifolds and almost Kähler manifolds. We also state recent developments resulting from the refinement of the Seiberg-Witten invariant by Bauer-Furuta. For details in Seiberg-Witten theory we refer to the textbooks [38, 43] and to the introductions [37, 12, 36].

1.1 The Seiberg-Witten moduli space

Let M be a compact, connected, oriented smooth Riemannian 4-manifold with a fixed Spin$^{\mathbb{C}}$-structure$P$. We denote by $\mathcal{A}(\det P)$ the space of unitary connections on the determinant line bundle $\det P$, by Σ^+, Σ^- the associated positive resp. negative spinor bundle, by $\mathrm{End}_0(\Sigma^+)$ the bundle of tracefree endomorphisms of the positive spinor bundle and by $\mathcal{D}_A : \Gamma(\Sigma^+) \to \Gamma(\Sigma^-)$ the positive Dirac operator associated with the connection $A \in \mathcal{A}(\det P)$. The Seiberg-Witten equations are the following coupled nonlinear elliptic equations for a unitary connection $A \in \mathcal{A}(\det P)$ and a positive spinor $\psi \in \Gamma(\Sigma^+)$:

$$
\begin{aligned}
F_A^+ &= \frac{1}{2} q(\psi, \psi) := (\psi \otimes \psi^*)_0 && (1.1.1)\\
\mathcal{D}_A(\psi) &= 0\,. && (1.1.2)
\end{aligned}
$$

Here F_A^+ denotes the self-dual part of the curvature F_A of the connection A and q denotes the real bilinear form

$$
\begin{aligned}
q:\ \Gamma(\Sigma^+) \times \Gamma(\Sigma^+) &\to \Gamma(\mathrm{End}_0(\Sigma^+))\\
(\psi, \phi) &\mapsto (\psi^* \otimes \phi + \phi^* \otimes \psi)_0\,.
\end{aligned}
$$

The index $(\cdot)_0$ denotes the trace free part, i.e.

$$
q(\psi, \phi) = (\psi^* \otimes \phi + \phi^* \otimes \psi)_0 = \psi^* \otimes \phi + \phi^* \otimes \psi - \frac{1}{2}(\langle \psi, \phi \rangle + \langle \phi, \psi \rangle) \cdot \mathrm{Id}_{\Sigma^+}\,.
$$

Solutions of the Seiberg-Witten equations are called *Seiberg-Witten monopoles* or *monopoles* for short. Regarded as the zero locus of the Seiberg Witten map

$$\begin{aligned}\mathcal{SW}: \quad \mathcal{A}(\det P)\times\Gamma(\Sigma^+) &\rightarrow \quad \Omega^2_+(M;i\mathbb{R})\times\Gamma(\Sigma^-)\\ \begin{pmatrix} A\\ \psi\end{pmatrix} &\mapsto \begin{pmatrix} F_A^+-\frac{1}{2}q(\psi,\psi)\\ \mathcal{D}_A\psi\end{pmatrix},\end{aligned}$$

the set $\widetilde{\mathfrak{M}}$ of all solutions, which is called the *Seiberg-Witten premoduli space*, may be treated as an infinite dimensional submanifold of the configuration space $\mathcal{C} := \mathcal{A}(\det P)\times\Gamma(\Sigma^+)$. In fact one needs to modify the equations a little in order to acertain that $\widetilde{\mathfrak{M}}$ is in fact a smooth Fréchet manifold, which would follow from an appropriate implicit function, if the linearisation of $\mathcal{SW}$ were surjective. These modifications together with the special requirements for inverse and implicit function theorems in Fréchet manifolds will be discussed below.

The *gauge group* $\mathcal{G} = \mathrm{Aut}(\det P) = \Omega^0(M;U(1))$ of automorphisms of the Spin$^{\mathbb{C}}$-structure$P$ acts on the configuration space $\mathcal{C}$ by

$$\mathcal{G}\ni u:(A,\psi)\mapsto\left(A+2u^{-1}du,u^{-1}\psi\right).$$

The stabilisers of a configuration (A,ψ) are:

$$\mathcal{G}_{(A,\psi)} = \begin{cases}1 & \text{for } \psi\neq 0\\ S^1 & \text{for } \psi = 0\end{cases}$$

A configuration (A,ψ) with $\psi\neq 0$ is called *irreducible*. We denote by $\mathcal{C}^*$ the space of all irreducible configurations and by $\mathcal{B}^*$ its quotient by the gauge group $\mathcal{G}$. Analogously, we denote by $\widetilde{\mathfrak{M}}^*$ the space of all irreducible monopoles. The gauge equivalence class of an irreducible configuration $(A,\psi)\in\mathcal{C}^*$ will be denoted by $[A,\psi]$.

Since the gauge group $\mathcal{G}$ is a Fréchet Lie group and its action on $\mathcal{C}^*$ is a smooth free action, the quotient space $\mathcal{B}^*$ may be expected to be an infinite dimensional smooth Fréchet manifold. To show that this is really the case requires some more work, since the inverse function theorem in Fréchet spaces of Nash and Moser requires stronger assumptions than the usual theorem in Banach spaces.

The Seiberg-Witten equations are invariant under the action of the gauge group $\mathcal{G}$. The curvature F_A and the quadratic form $\frac{1}{2}q(\psi,\psi)$ are invariant under $\mathcal{G}$, since $F_{u\cdot A} = F_A + d(2u^{-1}du) = F_A$, and $(u^{-1}\psi)^*\otimes(u^{-1}\psi) = \psi^*\otimes\psi$. For the Dirac operator, we find:

$$\begin{aligned}\mathcal{D}_{u^*A}(u^{-1}\psi) &= \mathcal{D}_A(u^{-1}\psi)+\frac{1}{2}(2u^{-1}du)\cdot(u^{-1}\psi)\\ &= u^{-1}\mathcal{D}_A\psi + d(u^{-1})\cdot\psi + u^{-2}du\cdot\psi\\ &= u^{-1}\mathcal{D}_A\psi\,.\end{aligned}$$

The quotient of the premoduli space by the gauge action is called the *Seiberg-Witten moduli space* and will be denoted by $\mathfrak{M} := \widetilde{\mathfrak{M}}/\mathcal{G}$. Elements of $\mathfrak{M}$ will be called *monopole classes* and will be denoted by $[A,\psi]$, where (A,ψ) is a monopole. When we assume for the moment that $\widetilde{\mathfrak{M}}$ is in fact a smooth Fréchet manifold, then the irreducible moduli space $\mathfrak{M}^* := \widetilde{\mathfrak{M}}^*/\mathcal{G}$ is a smooth submanifold $\mathfrak{M}^*\hookrightarrow\mathcal{B}^*$ of the space of gauge classes of irreducible configurations.

The elliptic complex $\mathcal{K}_{(A,\psi)}$

Since the premoduli space $\widetilde{\mathfrak{M}}$ is the zero locus of the Seiberg-Witten map $\mathcal{SW}$, its tangent space in a monopole (A,ψ) is the kernel of the linearisation in (A,ψ) of $\mathcal{SW}$. Furthermore, the tangent space in (A,ψ) of the gauge orbit through (A,ψ) is the image of the linearisation in $\mathbf{1}\in\mathcal{G}$ of the orbit map through (A,ψ). The linearisation in (A,ψ) of the Seiberg-Witten map $\mathcal{SW}$ is the map:

$$\begin{aligned} \mathcal{T}_1 :\ \ \Omega^1(M;i\mathbb{R})\times\Gamma(\Sigma^+) &\ \to\ \Omega^2_+(M;i\mathbb{R})\times\Gamma(\Sigma^-) \\ \begin{pmatrix}\nu\\ \phi\end{pmatrix} &\ \mapsto\ \begin{pmatrix} d^+\nu - q(\psi,\phi) \\ \frac{1}{2}\nu\cdot\psi + \mathcal{D}_A\phi\end{pmatrix} \end{aligned} \tag{1.1.3}$$

The linearisation in $\mathbf{1}\in\mathcal{G}$ of the orbit map through (A,ψ) is the map:

$$\begin{aligned} \mathcal{T}_0 :\ \ \Omega^0(M;i\mathbb{R}) &\ \to\ \Omega^1(M;i\mathbb{R})\times\Gamma(\Sigma^+) \\ if &\ \mapsto\ \begin{pmatrix} 2idf \\ -if\cdot\psi\end{pmatrix} \end{aligned} \tag{1.1.4}$$

Both these linearisations depend on a fixed configuration (A,ψ) - the one where we linearise the map $\mathcal{SW}$ resp. where the orbit map is based. We will always drop this dependence in the notation, but we should keep in mind, that all formulae derived from these linearisations will a priori carry this dependence.

These linearisations $\mathcal{T}_0,\mathcal{T}_1$ fit together to the following elliptic complex $\mathcal{K}_{(A,\psi)}$:

$$0\longrightarrow\Omega^0(M;i\mathbb{R})\xrightarrow{\mathcal{T}_0}\Omega^1(M;i\mathbb{R})\times\Gamma(\Sigma^+)\xrightarrow{\mathcal{T}_1}\Omega^2_+(M;i\mathbb{R})\times\Gamma(\Sigma^-)\longrightarrow 0 \qquad \mathcal{K}_{(A,\psi)}$$

It is in reference to this complex, that we denote the linearisations of the orbit map resp. the Seiberg-Witten map by $\mathcal{T}_0$ resp. $\mathcal{T}_1$.

The local structure of the moduli space $\mathfrak{M}$, especially the necessary and sufficient conditions for $\mathfrak{M}$ to be a smooth manifold, can easily be described in terms of this elliptic complex $\mathcal{K}_{(A,\psi)}$: Since the premoduli space $\widetilde{\mathfrak{M}}$ is the zero locus of the Seiberg-Witten map $\mathcal{SW}$, a necessary condition to apply an implicit function theorem is the surjectivity of the map $\mathcal{T}_1$. On the other hand, the moduli space is nonsingular only if it does not contain reducible monopole classes, i.e. if the orbit map resp. its linearisation $\mathcal{T}_0$ is injective. Thus in the above elliptic complex $\mathcal{K}_{(A,\psi)}$ there arise two obstructions for the moduli space $\mathfrak{M}$ to be a smooth manifold near (A,ψ): the kernel $\mathcal{T}_0$ – or the zeroth cohomology $\mathcal{H}^0(\mathcal{K}_{(A,\psi)})$ of the complex – as the obstruction for the gauge action to be free, or the moduli space to be nonsingular, and the cokernel of $\mathcal{T}_1$ – or the second cohomology $\mathcal{H}^2(\mathcal{K}_{(A,\psi)})$ of the complex – as the obstruction for the transversality.

In the discussion of the gauge action above, we called a monopole (A,ψ) *irreducible*, when it has trivial isotropy group, which is equivalent to the vanishing of the zeroth cohomology of $\mathcal{K}_{(A,\psi)}$. Analogously, we say that a monopole (A,ψ) is *regular*, when the second cohomology of $\mathcal{K}_{(A,\psi)}$ vanishes. The gauge class $[A,\psi]$ of an irreducible, regular monopole (A,ψ) will be called a *smooth point* of the moduli space, since $\mathfrak{M}$ turns out to be a smooth manifold in the neighbourhood of any such point.

In the neighbourhood of a smooth point $[A,\psi]$ one might expect the moduli space $\mathfrak{M}$ to be a smooth manifold modelled on the quotient of the kernel of $\mathcal{T}_1$ by the image of $\mathcal{T}_0$, i.e. on the first cohomology

$\mathcal{H}^1(\mathcal{K}_{(A,\psi)})$ of the elliptic complex associated with a monopole $(A,\psi) \in [A,\psi]$. This quotient space is called the *Zariski tangent space* of the moduli space $\mathfrak{M}$ in $[A,\psi]$. The *virtual dimension* d of $\mathfrak{M}$ is defined to be the dimension of the Zariski tangent space in a monopole class $[A,\psi]$ minus the dimension of the obstruction spaces of $\mathcal{K}_{(A,\psi)}$. This is minus the Euler characteristic of the complex $\mathcal{K}_{(A,\psi)}$. Since this complex is elliptic, its Euler characteristic is finite. Using the Atiyah-Singer index theorem, one finds:

$$d = -\chi(\mathcal{K}_{(A,\psi)}) = -b_0 + b_1 - b_2^+ \ + \ \frac{c_1^2 - \sigma}{4} = \frac{1}{4} \cdot \left(c_1^2 - 2\chi - 3\tau\right) \tag{1.1.5}$$

Here $c_1^2 = c_1^2(\det P)[M]$ denotes the first Chern number of the determinant line bundle. It is reasonable to expect, that in the neigbourhood of a smooth point, the moduli space $\mathfrak{M}$ is in fact a smooth manifold with dimension given by this virtual dimension, i.e.

$$d = \dim_{\mathbb{R}}(\mathfrak{M})$$

and with tangent space naturally isomorphic to the Zariski tangent space $\ker \mathcal{T}_1/\mathrm{im}\mathcal{T}_0 = \mathcal{H}^1(\mathcal{K}_{(A,\psi)})$.

The moduli space as a smooth manifold

Now we will discuss how to make the moduli space into a smooth manifold. First of all one needs to get rid of the obstruction spaces. Then one might hope that an appropriate implicit function theorem and slice theorem applies to conclude the smoothness of the premoduli space $\widetilde{\mathfrak{M}}$ and the moduli space $\mathfrak{M}$. The obstruction spaces can be made disappear via the use of perturbations. We fix an imaginary valued selfdual 2-form $\mu^+ \in \Omega^2_+(M; i\mathbb{R})$ as perturbation parameter and replace the equation (1.1.1) by:

$$F_A^+ = \frac{1}{2} q(\psi,\psi) + \mu^+ \ . \tag{1.1.6}$$

Similarly the Seiberg-Witten map $\mathcal{SW}$ is replaced by:

$$\begin{array}{cccc} \mathcal{SW}_{\mu^+}: & \mathcal{A}(\det P) \times \Gamma(\Sigma^+) & \to & \Omega^2_+(M;i\mathbb{R}) \times \Gamma(\Sigma^-) \\ & \begin{pmatrix} A \\ \psi \end{pmatrix} & \mapsto & \begin{pmatrix} F_A^+ - \frac{1}{2}q(\psi,\psi) - \mu^+ \\ \mathcal{D}_A\psi \end{pmatrix} \end{array}$$

We sometimes denote the perturbed moduli space by $\mathfrak{M}_{\mu^+}$ but we will also drop the perturbation parameter in the notation. The zeros of the perturbed Seiberg-Witten map $\mathcal{SW}_{\mu^+}$ will sometimes be called μ^+-monopoles.

It is easy to see that the space $\Gamma_g^+ \subset \Omega^2_+(M; i\mathbb{R})$ of all those perturbations μ^+ for which the perturbed Seiberg-Witten equations admit reducible solutions is an affine subspace of codimension b_2^+: Let $\mu_1^+, \mu_2^+ \in \Gamma_g^+$ be two such perturbations with corresponding reducible monopoles $(A_1, 0)$ and $(A_2, 0)$. Then from (1.1.1) we deduce that the 1-form $\nu := A_2 - A_1$ satisfies $(d\nu)^+ = \mu_2^+ - \mu_1^+$. Thus any two perturbations in Γ_g^+ differ by an exact self-dual 2-form, and hence Γ_g^+ is an affine space with parallel space $\mathrm{im}(d^+)$. This implies that its codimension in $\Omega^2_+(M; i\mathbb{R})$ equals the dimension of the quotient $\Omega^2_+(M; i\mathbb{R})/\mathrm{im}(d^+)$. Now suppose $\omega \in \Omega^2_+(M; i\mathbb{R})$ to be perpendicular to $\mathrm{im}(d^+)$. Then $d^*\omega = 0$ and since ω is self-dual, it is also closed and thus harmonic. Hence the orthogonal complement of $\mathrm{im}(d^+)$ in $\Omega^2_+(M; i\mathbb{R})$ is the space $\mathcal{H}^2_+(M; i\mathbb{R})$ of harmonic self-dual 2-forms and thus has dimension b_2^+ via Hodge theory. With a slight abuse of notation, this space Γ_g^+ will be called the *wall* of reducible perturbations for the metric g.

Note that in the case of a 4-manifold with $b_2^+ > 1$, the space $\Omega^2_+(M; i\mathbb{R}) - \Gamma^+_g$ of perturbations admitting reducible monopoles, is connected, whereas in the case of $b_2^+ = 1$, the wall Γ^+_g separates its complement in $\Omega^2_+(M; i\mathbb{R})$ into two connected components, called *chambers*. In the case of $b_2^+ = 0$ there is no way to avoid reducible solutions. When we assume b_2^+ to be at least 1, then the zeroth cohomology $\mathcal{H}^0(\mathcal{K}_{(A,\psi)})$ can be made disappear by adding an appropriate perturbation in $\Omega^2_+(M; i\mathbb{R})$.

For the vanishing of the second cohomology we use an appropriate infinite dimensional version of Sards theorem to conclude that for a generic perturbation $\mu^+ \in \Omega^2_+(M; i\mathbb{R})$, the linearisation $\mathcal{T}_1$ in (A, ψ) of the Seiberg-Witten map $\mathcal{SW}_{\mu^+}$ is surjective. The original version of such a theorem is Smales generalisation of Sards theorem to Banach spaces in [48], usually referred to as the Sard-Smale theorem. However, since we are dealing here with Fréchet spaces, it might be useful to note that the implicit function theorem of Nash and Moser directly applies to a proof of a Sard-Smale theorem on Fréchet spaces. One must take into account that the assumptions for the Nash-Moser inverse function theorem are much stronger than those for the usual inverse function theorem on Banach spaces, namely they require the spaces and maps to be in the tame smooth category. However, in our case, the Nash-Moser inverse function applies, since all Fréchet manifolds in question are modelled on sections of vector bundles on M, thus are tame Fréchet manifolds, and all maps come from (nonlinear) elliptic operators, and thus are tame smooth maps with tame smooth inverses of their linearisations. For all details on the tame category and the Nash-Moser inverse function theorems we refer to HAMILTONS excellent article [23]; the standard theory of Fréchet manifolds was also treated in [5].

The Nash-Moser-Sard-Smale theorem now states that for a generic perturbation $\mu^+ \in \Omega^2_+(M; i\mathbb{R})$, the linearisation $\mathcal{T}_1$ of $\mathcal{SW}_{\mu^+}$ in a μ^+-monopole (A, ψ) is surjective, i.e. the second cohomology $\mathcal{H}^2(\mathcal{K}_{(A,\psi)})$ vanishes.

To prove that the quotients $\mathcal{B} = \mathcal{C}/\mathcal{G}$ resp. $\mathfrak{M} = \widetilde{\mathfrak{M}}/\mathcal{G}$ are (possibly singular) manifolds, one needs a slice theorem for the action of $\mathcal{G}$. A *local slice* in $(A, \psi) \in \mathcal{C}$ for the action of the gauge group $\mathcal{G}$ on the configuration space $\mathcal{C}$ is a smoothly embedded $\mathcal{G}_{(A,\psi)}$-invariant submanifold $\mathcal{S}_{(A,\psi)} \hookrightarrow \mathcal{C}$ such that the natural map $\mathcal{S}_{(A,\psi)} \times_{\mathcal{G}_{(A,\psi)}} \mathcal{G} \to \mathcal{C}$ is a diffeomorphism onto a neighbourhood of the orbit through (A, ψ). In the context of Fréchet manifolds, such a theorem was proved by SUBRAMANIAM in [49], once again using the Nash-Moser inverse function theorem. We won't state the precise theorem, since this would involve some more notation, but we only mention, that it directly applies to our situation, since the gauge group $\mathcal{G}$ is a tame geometric Fréchet Lie group and the configuration space $\mathcal{C}$ resp. the premoduli space $\widetilde{\mathfrak{M}}$ is a totally proper, elliptic, Finsler $\mathcal{G}$-space in the sense of [49].

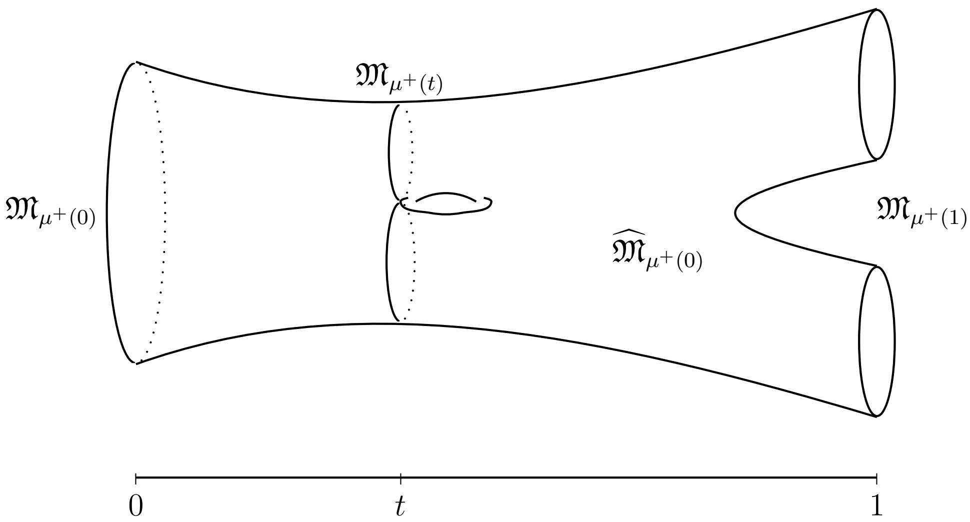
Now the slice theorem together with the implicit function theorem states, that in the neigbourhood of a smooth point $[A, \psi]$, the moduli space $\mathfrak{M}$ is a smooth manifold, modelled on the first cohomology $\mathcal{H}^1(\mathcal{K}_{(A,\psi)}) \cong \ker \mathcal{T}_1/\mathrm{im}\mathcal{T}_0$ associated with a monopole $(A, \psi) \in [A, \psi]$. It also implies that in the neighbourhood of a reducible, regular point $[A, \psi]$, the moduli space is homeomorphic to the quotient of the first cohomology by the action of S^1.

Cobordisms, orientability, Spin structures

Now that we have introduced the perturbed moduli spaces $\mathfrak{M}_{\mu^+}$, we should discuss the dependence on the perturbation parameter μ^+. As stated above, in the case of a 4-manifold M with $b_2^+ > 1$, the space $\Omega^2_+(M; i\mathbb{R}) - \Gamma^+_g$ of all those perturbations, which admit no reducible monopoles, is connected. Thus given two generic perturbations $\mu_0^+, \mu_1^+ \in \Omega^2_+(M; i\mathbb{R}) - \Gamma^+_g$, then we can join them by a path $\mu^+ : [0, 1] \to \Omega^2_+(M; i\mathbb{R}) - \Gamma^+_g$ with $\mu^+(0) = \mu_0^+, \mu^+(1) = \mu_1^+$. We now consider the *parametrised* *moduli space*

$$\widehat{\mathfrak{M}} := \bigsqcup_{t\in[0,1]} \mathfrak{M}_{\mu^+(t)} \ . \tag{1.1.7}$$

Elements of $\widehat{\mathfrak{M}}$ will be denoted by $\widehat{[A,\psi]}$. Another application of a Sard-Smale theorem implies that for a generic path μ^+, the linearisation of the parametrised Seiberg-Witten map is surjective, thus the parametrised moduli space is a smooth manifold. Thus the moduli spaces $\mathfrak{M}_{\mu_0^+}, \mathfrak{M}_{\mu_1^+}$ can be joined by a smooth cobordism. Note that in general not every fibre $\mathfrak{M}_{\mu^+(t)}$ of the cobordism $\widehat{\mathfrak{M}}$ is a smooth manifold.



In the same way, we may start with pairs $(g_0,\mu_0^+), (g_1,\mu_1^+)$ of metrics and perturbations, such that $\mu_i^+ \in (\Omega^2_+(M;i\mathbb{R}))_{g_i} - \Gamma^+_{g_i}$ and join them by a generic path of metrics and perturbations to get a parametrised moduli space, which is a smooth cobordism with boundary components $\mathfrak{M}(g_0,\mu_0^+)$ and $\mathfrak{M}(g_1,\mu_1^+)$.

In the case of a 4-manifold M with $b_2^+ = 1$, we can also join two generic perturbations $\mu_0^+, \mu_1^+ \in \Omega^2_+(M;i\mathbb{R}) - \Gamma^+_g$ by a path $\mu^+ : [0,1] \to \Omega^2_+(M;i\mathbb{R})$ and consider the parametrised moduli space $\widehat{\mathfrak{M}}$. But when μ_0^+, μ_1^+ lie in the different connected components of $\Omega^2_+(M;i\mathbb{R}) - \Gamma^+_g$, then the path μ^+ crosses the wall Γ^+_g at least once, and thus the parametrised moduli space becomes singular. We will come back to this point in the next section, when we discuss the Seiberg-Witten invariant and its change while crossing the wall Γ^+_g.

The Seiberg-Witten moduli spaces carry a natural orientation induced by the choice of orientations of the cohomologies $H^1(M;i\mathbb{R})$ and $H^2_+(M;i\mathbb{R})$. The orientation of $\mathfrak{M}$ is constructed as follows: From the elliptic complex $\mathcal{K}_{(A,\psi)}$ associated with a monopole (A,ψ) we consider the family of operators:

$$\mathcal{T}_0^* \oplus \mathcal{T}_1 : \Omega^1(M;i\mathbb{R}) \times \Gamma(\Sigma^+) \longrightarrow \Omega^0(M;i\mathbb{R}) \oplus \Omega^2_+(M;i\mathbb{R}) \times \Gamma(\Sigma^-) \ .$$

Then we may regard the family of the kernels of these operators as a Fréchet vector bundle over the premoduli space $\widetilde{\mathfrak{M}}$:

$$\ker(\mathcal{T}_0^* \oplus \mathcal{T}_1) \longrightarrow \widetilde{\mathfrak{M}}$$

As pointed out above, this vector bundle has rank $d = \dim_{\mathbb{R}} \mathfrak{M}$ and is the pullback via the natural projection $\widetilde{\mathfrak{M}} \to \mathfrak{M}$ of the tangent bundle $T\mathfrak{M}$ of the moduli space. Thus a $\mathcal{G}$-invariant orientation of the bundle $\ker(\mathcal{T}_0^* \oplus \mathcal{T}_1)$ would induce an orientation of the moduli space. An orientation of a finite rank vector bundle over a finite dimensional base is equivalent to an orientation or trivialisation of its determinant line bundle.

Since the for any (A, ψ), the operator $\mathcal{T}_0^* \oplus \mathcal{T}_1$ is Fredholm, one can construct the determinant line bundle $\det(\mathcal{T}_0^* \oplus \mathcal{T}_1) \to \mathfrak{M}$ on the moduli space in the sense of morphisms of Hilbert space bundles. Then for any monopole class $[A, \psi] \in \mathfrak{M}$ one has

$$\det(\mathcal{T}_0^* \oplus \mathcal{T}_1)_{[A,\psi]} = \det(T\mathfrak{M})_{[A,\psi]} ,$$

where the right hand side is the determinant line bundle in the ordinary finite dimensional sense.

Using a chain homotopy, one deforms the operators $\mathcal{T}_0^* \oplus \mathcal{T}_1$ such that their zeroth-order parts vanish, and one ends up with the family of operators:

$$-2d^* \oplus \mathcal{D}_A \oplus d^+ : \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) \longrightarrow \Omega^0(M; i\mathbb{R}) \oplus \Omega^2_+(M; i\mathbb{R}) \times \Gamma(\Sigma^-) .$$

The summand $\mathcal{D}_A$ is a morphism of complex Fréchet resp. Hilbert bundles, thus its kernel carries a natural orientation. The kernel and cokernel of the operator $(d^+ - 2d^*)$ are the cohomologies

$$\ker(d^+ - 2d^*) = H^1(M; i\mathbb{R}) \qquad \text{and} \qquad \operatorname{coker}(d^+ - 2d^*) = H^2_+(M; i\mathbb{R}) \oplus H^0(M; i\mathbb{R})$$

and thus are independent of (A, ψ). Thus when we fix an orientation of $H^1(M; i\mathbb{R}) \oplus H^2_+(M; i\mathbb{R})$, we obtain an orientation of the determinant line bundle $\det(\ker \mathcal{T}_0^* \oplus \mathcal{T}_1)$ and also an induced orientation of $T\mathfrak{M}$. By carefully identifying these orientations over the parametrised moduli space $\widehat{\mathfrak{M}}$ one eventually finds:

1.1.1 THEOREM. *Let M be a compact, oriented, smooth 4-manifold with a fixed Spin$^{\mathbb{C}}$-structure$P$. If $b_2^+ \geq 1$, then for a fixed Riemannian metric g, there are generic perturbations $\mu^+ \in \Omega^2_+(M; i\mathbb{R}) - \Gamma_g^+$ such that the moduli space $\mathfrak{M}_{\mu^+}$ is a compact smooth manifold of dimension $\dim_{\mathbb{R}}(\mathfrak{M}) = d = \frac{1}{4} \cdot (c_1^2 - 2\chi - 3\tau)$. If $d < 0$, then the moduli space $\mathfrak{M}$ is generically empty. If $b_2^+ > 1$, then for generic pairs $(g_i, \mu_i^+), i = 0, 1$ of metrics and perturbations the moduli spaces $\mathfrak{M}(g_i, \mu_i^+), i = 0, 1$ are oriented cobordant.*

For details of the proof (using the analogue Banach space setting for the implicit function, slice and Sard-Smale theorems) and especially for the proof of the compactness statement, we refer to [38] and [37].

Recently, it was pointed out by SASAHIRA, that using the stable cohomotopy refinement of Seiberg-Witten invariants introduced by BAUER and FURUTA in [4, 3], one can define a canonical Spin structure on the moduli spaces. Furthermore, the moduli spaces for different perturbations are Spin cobordant. The precise statement is the following:

1.1.2 THEOREM (SASAHIRA). *Let M be a compact, oriented, smooth 4-manifold with a fixed Spin$^{\mathbb{C}}$-structure$P$ and determinant line bundle $\det P$, and let $a_1, \ldots, a_{b_1}$ be generators of $H^1(M; \mathbb{Z})$. If $c_{ij} := \frac{1}{2}\langle c_1(\det P) a_i a_j, [M]\rangle$ and the index $\operatorname{ind}\mathcal{D}_A$ of the Dirac operator satisfy:*

$$\begin{aligned} \operatorname{ind}(\mathcal{D}_A) &\equiv 0 \mod 2 \\ c_{ij} &\equiv 0 \mod 2 , \end{aligned}$$

then every choice of an orientation of $H^1(M; i\mathbb{R}) \oplus H^2_+(M; i\mathbb{R})$, of a finite dimensional approximation of the Seiberg-Witten map $\mathcal{SW}$ in the sense of [4] and of a square root of the determinant line bundle of the index bundle $\det(\mathrm{Ind}(\mathcal{D}_A)) \to M$ determines a canonical Spin structure on the Seiberg-Witten moduli space $\mathfrak{M}$. Moreover, the Spin cobordism class of this Spin structure is independent of the choice of the finite dimensional approximation. If $b_2^+ > 1$, then the moduli spaces $\mathfrak{M}(g_i, \mu_i^+), i = 0, 1$ for different generic pairs $(g_i, \mu_i^+), i = 0, 1$ of metrics and perturbations are oriented cobordant.

For the proof we refer to the original paper [45] of SASAHIRA. For a gentle introduction to the Bauer-Furuta refinement of Seiberg-Witten theory we also refer to [28].

1.2 The Seiberg-Witten invariant

The Seiberg-Witten invariant is a numerical invariant depending only on the smooth structure of the 4-manifold M and on the $\mathrm{Spin}^{\mathbb{C}}$-structure$P$. There are several equivalent ways to introduce this invariant. We describe three slightly different approaches in order to see different aspects of how the invariant depends on the cohomology of the moduli space $\mathfrak{M}$ as a submanifold of $\mathcal{B}^*$.

The universal line bundle

The most elucidating way to define the invariant is by introducing a universal line bundle on $M \times \mathcal{B}^*$ and considering the first Chern class of the pullback to $\mathfrak{M}$. Evaluating the top exterior power of this class on the cycle represented by the moduli space $\mathfrak{M} \hookrightarrow \mathcal{B}^*$ gives a numerical invariant, called the *Seiberg-Witten invariant*. Here the top exterior power is meant with respect to $\mathfrak{M}$, since $\mathcal{B}^*$ is infinite dimensional.

The definition of this universal line bundle on $M \times \mathcal{B}^*$ is as follows: On $M \times \mathcal{C}^*$ we consider the trivial complex line bundle

$$\mathcal{C}^* \times M \times \mathbb{C} \to \mathcal{C}^* \times M \ ,$$

with the natural action of the gauge group $\mathcal{G}$ given by:

$$u : \begin{pmatrix} A \\ \psi \\ x \\ z \end{pmatrix} \mapsto \begin{pmatrix} A + 2u^{-1}du \\ u^{-1} \cdot \psi \\ x \\ u(x)^{-1} \cdot z \end{pmatrix} .$$

This equivariance thus induces the so called *universal line bundle* $\mathcal{L}_{\mathcal{B}^* \times M} := (\mathcal{C}^* \times M \times \mathbb{C})/\mathcal{G}$ on the quotient $(\mathcal{C}^* \times M)/\mathcal{G} = \mathcal{B}^* \times M$. The name comes from the fact that the space $\mathcal{B}^*$ of gauge equivalence classes of irreducible configurations factors through $\Gamma(\Sigma^+)/U(1) \sim \mathbb{CP}^\infty$ over the torus $\mathcal{A}(\det(P))/\mathcal{G} \cong H^1(M; U(1)) \cong \mathbb{T}^{b_1}$. Any point $x_0 \in M$ defines a line bundle $\imath_{x_0}^* \mathcal{L}_{\mathcal{B}^* \times M}$ as the pullback via the canonical inclusion $\imath_{x_0} : \mathcal{B}^* \hookrightarrow \mathcal{B}^* \times \{x_0\} \subset \mathcal{B}^* \times M$. Clearly, these line bundles on $\mathcal{B}^*$ are all isomorphic, since the inclusions $\imath_{x_i}$ for different points $x_i, i = 0, 1$ are homotopic. The restriction of the pullback bundle $\imath_{x_0}^* \mathcal{L}_{\mathcal{B}^* \times M}$ to the fibre of the above fibration gives the universal line bundle on $\mathbb{CP}^\infty$.

In the following, we are interested only in the isomorphism class of the pullbacks $\imath_{x_0} \mathcal{L}_{\mathcal{B}^* \times M}$ to $\mathcal{B}^*$. This class does not change, when we replace the trivial line bundle $M \times \mathbb{C} \to M$ in the construction above by any complex line bundle $P \to M$ and take the pullback along $\imath_{x_0}$ of the $\mathcal{G}$-quotient of

$\mathcal{C}^* \times P \to \mathcal{C}^* \times M$, since the induced bundle over $\mathcal{C}^* \times \{x_0\}$ is trivial anyway. This will be helpful later, when we consider natural unitary connections on these bundles, which are induced from connections on the bundle $P \to M$.

The based gauge group

More directly, one can define $U(1)$ bundles over $\mathcal{B}^*$ as the quotient of $\mathcal{C}^*$ by the *based gauge group* $\mathcal{G}_{x_0}$: Fix an arbitrary point $x_0 \in M$ and set:

$$\mathcal{G}_{x_0} := \{u \in \mathcal{G} \mid u(x_0) = 1\} . \tag{1.2.1}$$

Then the quotient $\mathcal{B}^*_{x_0} := \mathcal{C}^*/\mathcal{G}_{x_0} \to \mathcal{B}^*$ yields a principal $U(1)$-bundle, and the associated complex line bundle is canonically isomorphic to $\imath^*_{x_0}\mathcal{L}_{\mathcal{B}^*\times M}$. To construct this isomorphism, we consider the $U(1)$-bundle associated with the line bundle $\imath_{x_0}(\mathcal{C}^* \times M \times \mathbb{C})/\mathcal{G}$. Then the equivariant map

$$\begin{array}{ccc} \mathcal{C}^* \times \{x_0\} \times U(1) & \to & \mathcal{C}^* \\ \begin{pmatrix} A \\ \psi \\ \lambda \end{pmatrix} & \mapsto & \begin{pmatrix} A \\ \lambda^{-1} \cdot \psi \end{pmatrix} \end{array}$$

intertwines the $\mathcal{G}$-action on $\mathcal{C}^* \times \{x_0\}$ with the $\mathcal{G}_{x_0}$-action on $\mathcal{C}^*$ via the natural group homomorphism $\mathcal{G} \to \mathcal{G}_{x_0}, u \mapsto u \cdot u(x_0)^{-1}$. In fact, we have:

$$u \cdot \begin{pmatrix} A \\ \psi \\ \lambda \end{pmatrix} = \begin{pmatrix} A + 2u^{-1}du \\ u^{-1} \cdot \psi \\ u(x_0)^{-1} \cdot \lambda \end{pmatrix} \mapsto \begin{pmatrix} A + 2u^{-1}du \\ u^{-1} \cdot u(x_0) \cdot \lambda^{-1} \cdot \psi \end{pmatrix} = u \cdot u(x_0)^{-1} \cdot \begin{pmatrix} A \\ \lambda^{-1} \cdot \psi \end{pmatrix}$$

Since this map preserves the natural $U(1)$-actions, it induces an isomorphism of the associated $U(1)$-bundles on $\mathcal{B}^*$. The equivalence class of a configuration (A, ψ) in the quotient $\mathcal{B}^*/\mathcal{G}_{x_0}$ will be denoted by $[A, \psi]_{x_0}$.

1.2.1 DEFINITION. *The isomorphism class of the $U(1)$-bundle on the moduli space resp. of its associated complex line bundle as constructed above will be called the Seiberg-Witten bundle and will be denoted by $\mathfrak{P} \to \mathfrak{M}$.*

Another way to construct a $U(1)$-bundle on $\mathfrak{M}$, which represents this isomorphism class is via a canonical global gauge fixing: Once and for all we fix a reference connection $A_0 \in \mathcal{A}(\det P)$. Then it is easy to see, that every monopole (A, ψ) is gauge equivalent to a monopole (A', ψ') such that $d^*(A' - A_0) = 0$. Note that the Laplacian Δ on (imaginary valued) functions is an isomorphism $\Delta : \operatorname{im} d^* \to \operatorname{im} d^*$. Now we set $A' := A - d\Delta^{-1}d^*(A - A_0)$, and we thus have $d^*(A' - A_0) = d^*(A - a_0) - d^*d\Delta^{-1}d^*(A - A_0) = 0$. When we set $u := \exp(-\frac{1}{2}\Delta^{-1}d^*(A - A_0)) \in \mathcal{G}$, then we find:

$$\begin{aligned} u \cdot (A, \psi) = (A + 2u^{-1}du, u^{-1} \cdot \psi) &= \left(A + 2\left(-\frac{1}{2}\right) d(\Delta^{-1}d^*(A - A_0)), u^{-1} \cdot \psi\right) \\ &= (A - d\Delta^{-1}d^*(A - A_0), u^{-1} \cdot \psi) \\ &=: (A', \psi') . \end{aligned}$$

Thus (A', ψ') is gauge equivalent to (A, ψ).

Since the image of d^* is complemented by the constant functions in $\Omega^0(M; i\mathbb{R}) = T_{\mathbf{1}}\mathcal{G}$, the solution space $\widetilde{\mathfrak{M}}_{fix}$ of the globally gauge fixed Seiberg-Witten equations

$$F_A^+ = \frac{1}{2}q(\psi, \psi) + \mu^+ \tag{1.2.2}$$

$$\mathcal{D}_A(\psi) = 0 \tag{1.2.3}$$

$$d^*(A - A_0) = 0 \tag{1.2.4}$$

is the total space of another $U(1)$-bundle over the moduli space $\mathfrak{M}$, the fibres of which come from the constant gauge transformations. We call this solution space $\widetilde{\mathfrak{M}}_{fix}$ the *globally gauge fixed premoduli space*. Since the diagram

$$\begin{array}{ccc} U(1) \times \widetilde{\mathfrak{M}}_{fix} & \xrightarrow{\imath} & \mathcal{G} \times \widetilde{\mathfrak{M}} \\ \downarrow & & \downarrow \\ \widetilde{\mathfrak{M}}_{fix} & \xrightarrow{\imath} & \widetilde{\mathfrak{M}} \\ \downarrow & & \downarrow \\ \mathfrak{M} & \xrightarrow{\mathrm{id}} & \mathfrak{M} \end{array}$$

commutes, the bundle $\widetilde{\mathfrak{M}}_{fix} \to \mathfrak{M}$ represents the isomorphism class $\mathfrak{P}$. Later we will give yet another construction of a line bundle, which represents this isomorphism class, and which is more natural from the point of view of the geometry of the L^2-metric.

We denote by $\Omega := c_1(\mathfrak{P}) \in H^2(\mathfrak{M}; \mathbb{Z})$ the first Chern class of the complex line bundle $\mathfrak{P} \to \mathfrak{M}$. After fixing an orientation of the moduli space $\mathfrak{M}$, we can evaluate the class Ω on the cycle represented by $\mathfrak{M}$, and the resulting number defines the Seiberg-Witten invariant:

1.2.2 DEFINITION. *Let M be a compact, oriented, smooth 4-manifold with a fixed Spin$^{\mathbb{C}}$-structure$P$. When we fix an orientation of $H^1(M) \oplus H^2_+(M)$, then the Seiberg-Witten invariant for a generic Riemannian metric g and a generic perturbation μ^+ is the number*

$$\mathbf{sw}_M(P, g, \mu^+) := \left\langle (1 - \Omega)^{-1}, [\mathfrak{M}(P, g, \mu^+)] \right\rangle , \tag{1.2.5}$$

where $(1 - \Omega)^{-1}$ denotes the formal power series $(1 - \Omega)^{-1} = 1 + \Omega + \Omega^2 + \ldots$

Thus when the dimension $d = \dim(\mathfrak{M})$ of the moduli space is odd, the Seiberg-Witten invariant vanishes, whereas for $d = 2k$ it can be written as

$$\mathbf{sw}_M(P, g, \mu^+) = \int_{\mathfrak{M}(P,g,\mu^+)} \Omega^k \quad . \tag{1.2.6}$$

The topology of $\mathcal{B}^*$

One may also regard the space $\mathcal{B}^* = \mathcal{C}^*/\mathcal{G}$ of gauge equivalence classes of irreducible configurations as a classifying space for the gauge group $\mathcal{G}$, since $\mathcal{G}$ acts freely on the contractible space $\mathcal{C}^*$. Furthermore, we have:

$$\mathcal{G} = \mathcal{C}^\infty(M;U(1)) \sim \mathcal{C}^\infty(M;U(1))_0 \times \pi_0\big(\mathcal{C}^\infty(M;U(1))\big) ,$$

where $\mathcal{C}^\infty(M;U(1))_0$ denotes the space of homotopically constant maps, which may be identified to $U(1)$. Thus $\mathcal{C}^\infty(M;U(1)) \sim U(1) \times H^1(M;\mathbb{Z})$, and the classifying space $\mathcal{B}^* = B\mathcal{G}$ is weakly homotopy equivalent to $\mathbb{CP}^\infty \times \big(H^1(M;\mathbb{R})/H^1(M;\mathbb{Z})\big)$. Hence it has the integer cohomology:

$$H^*(\mathcal{B}^*;\mathbb{Z}) \cong \mathbb{Z}[u] \oplus \Lambda^* H^1(M;\mathbb{Z}) ,$$

where u denotes the generator of $H^*(\mathbb{CP}^\infty, \mathbb{Z})$, which can be represented by the restriction of the class Ω to the fibre $\mathbb{CP}^\infty$ as above. A generator for the $H1(M;\mathbb{Z})$-part will be constructed in section 1.5 below.

From this point of view one might hope to get more information on the moduli space by taking into account the full cohomology of $\mathcal{B}^*$, not only the classes generated by the first Chern class of the universal bundle. One might hope e.g. to get some information on moduli spaces of odd dimension. Explicit examples in this direction, constructed by DEL RIO GUERRA, will be discussed in section 1.5.

Cobordism invariance and wall crossing

In the case of a 4-manifold M with $b_2^+ > 1$, the modulo spaces for different generic metrics g and perturbations μ^+ are oriented corbordant, whereas for $b_2^+ = 1$ the cobordisms become singular when the peturbation parameter μ^+ crosses the wall Γ_g^+. In the former case it follows from Stokes theorem that the Seiberg-Witten invariant is a cobordism invariant: suppose (g_0, μ_0^+) and (g_1, μ_1^+) to be generic pairs of metrics and perturbations such that the moduli spaces $\mathfrak{M}(g_i, \mu_i^+), i = 0, 1$ are smooth manifolds. Then for a generic path μ^+ in $\Omega_+^2(M; i\mathbb{R}) - \Gamma_g^+$ joining μ_0^+ and μ_1^+ the parametrised moduli space $\widehat{\mathfrak{M}}$ is a cobordism with boundary components $\mathfrak{M}(g_0, \mu_0^+)$ and $\mathfrak{M}(g_1, \mu_1^+)$, and we find:

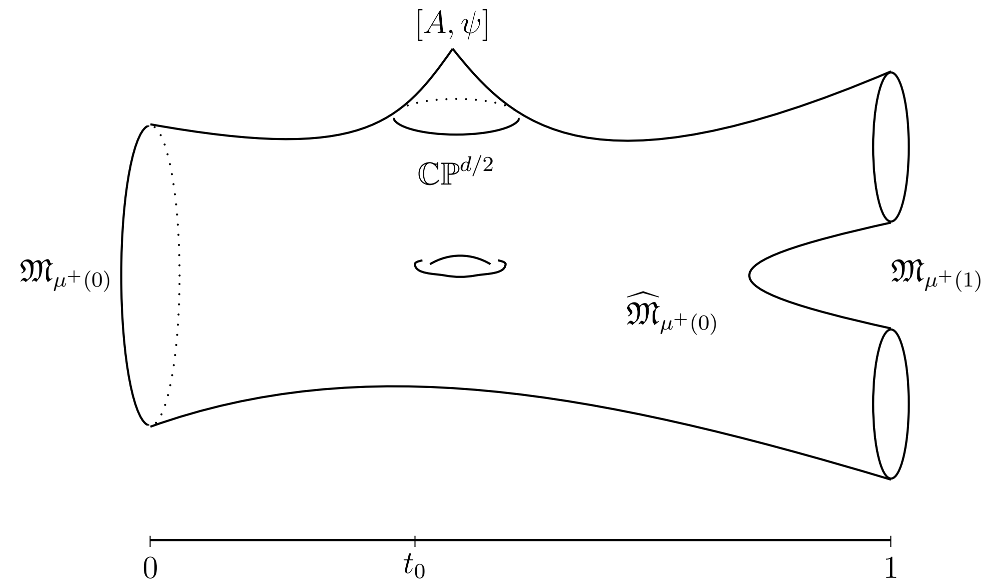
$$\begin{aligned}
\mathbf{sw}_M(g_0,\mu_0^+) - \mathbf{sw}_M(g_1,\mu_1^+) &= \big\langle (1-\Omega)^{-1}, [\mathfrak{M}(g_0,\mu_0^+)]\big\rangle - \big\langle (1-\Omega)^{-1}, [\mathfrak{M}(g_1,\mu_1^+)]\big\rangle \\
&= \big\langle (1-\Omega)^{-1}, [\partial\widehat{\mathfrak{M}}]\big\rangle \\
&= \int_{\widehat{\mathfrak{M}}} d(1-\Omega)^{-1} \\
&= 0 .
\end{aligned}$$

In the case of a 4-manifold M with $b_2^+ = 1$, the moduli spaces for perturbations μ_0^+, μ_1^+ in the different connected components of $\Omega_+^2(M; i\mathbb{R}) - \Gamma_g^+$ can be joined by singular cobordisms only, thus the invariant $\mathbf{sw}_M(P, g, \mu^+)$ depends on the chamber structure. Nevertheless, since the singularities occuring in the cobordisms are of a well known type, one can explicitly compute the change in the invariant. The wall Γ_g^+ was defined to be the space of all those perturbations $\mu^+ \in \Omega_+^2(M; i\mathbb{R})$ for which the perturbed Seiberg-Witten equations admit reducible solutions. When we fix an orthonormal base $\omega \in \mathcal{H}_+^2(M; i\mathbb{R})$, then Γ_g^+ can be written as:

$$\Gamma_g^+ = \{\mu^+ \in \Omega_+^2(M; i\mathbb{R}) \mid (i\mu^+ - 2\pi c_1(\det P), \omega) = 0\} .$$

The two connected components of $\Omega^2_+(M; i\mathbb{R}) - \Gamma^+_g$ will be referred to as the *positive* resp. *negative chamber* according to the sign of $(i\mu^+ - 2\pi c_1(\det P), \omega)$. As above, one can show that the Seiberg-Witten invariant is constant on the $\pm$-chambers. We thus denote by $\mathbf{sw}^\pm_M(P)$ the invariant on the $\pm$-chamber respectively.

When we consider the moduli spaces for a fixed metric g and two perturbations μ^+_0, μ^+_1 which lie in different chambers, then we may chose a path $\mu : [0,1] \to \Omega^2_+(M; i\mathbb{R})$ joining μ^+_0 and μ^+_1, and we may suppose without loss of generality, that μ^+ crosses the separating wall Γ^+_g only once, say at $t = t_0$. It can be shown via a Sard-Smale argument, that the path μ^+ can be chosen such that the linearisation $\widehat{\mathcal{T}}_1(t)$ of the parametrised Seiberg-Witten map $\widehat{\mathcal{SW}}$ is surjective, i.e. that the second cohomology of the parametrised complex vanishes. Thus for a generic path μ^+ all monopoles of the parametrised moduli space $\widehat{\mathfrak{M}}$ are regular. Via the slice theorem one can show, that a neighbourhood of a reducible class $\widehat{[A,\psi]} \in \widehat{\mathfrak{M}}$ is homeomorphic to the quotient of the first cohomology $\mathcal{H}^1(\mathcal{K}_{(A,\psi)})$ by the action of the stabiliser $\mathcal{G}_{(A,\psi)} \cong U(1)$. When M is simply connected, then up to gauge equivalence, there is only one reducible. In this case, the first cohomology $\mathcal{H}^1(\mathcal{K}_{(A,\psi)})$ is isomorphic to the kernel of $\mathcal{D}_A$, and thus it is a complex vector space with the standard action of $U(1)$. Hence the singularity of $\widehat{\mathfrak{M}}$ arising from the reducible monopole class is (topologically) a cone on the complec projective space $\mathbb{CP}^{(\mathrm{ind}(\mathcal{D}_A)-1)}$.



When we substract from $\widehat{\mathfrak{M}}$ a neighbourhood U of the reducible monopole class $[A,\psi]$, then $\widehat{\mathfrak{M}} - U \hookrightarrow \mathcal{B}^*$ is a smooth manifold with boundary components $\mathfrak{M}(P, g, \mu^+_0) \sqcup \mathfrak{M}(P, g, \mu^+_1) \sqcup \mathbb{CP}^{d/2}$. Thus the universal line bundle $\mathcal{L}_{M \times \mathcal{B}^*}$ restricts to $\widehat{\mathfrak{M}} - U$, and by carefully identifying the orientations in question, one finds the following formula for the change of the Seiberg-Witten invariant along the cobordism $\widehat{\mathfrak{M}}$:

1.2.3 Theorem (Wall-crossing formula). *Let (M, g) be a compact, connected and simply connected Riemannian 4-manifold with $b^+_2 = 1$, and let P be a fixed Spin$^{\mathbb{C}}$-structureon M. Then the Seiberg-Witten invariants of P in the $\pm$-chamber are related by the following formula:*

$$\mathbf{sw}^+_M(P) - \mathbf{sw}^-_M(P) = (-1)^{d/2} , \tag{1.2.7}$$

where $d = \dim_{\mathbb{R}}(\mathfrak{M})$ *is the (virtual) dimension of* $\mathfrak{M}$.

For a proof we refer to [38] and the original work [31]. When one drops the assumption $b_1 = 0$, one finds generalised wall-crossing formula, which is described in detail in [40].

1.3 Vanishing theorems and gluing problems

In this section we recall some elementary properties of the Seiberg-Witten invariants, such as the fundamental estimate on the norm of the spinor part ψ of a monopole (A, ψ) and its consequences, as well as some results on Seiberg-Witten invariants of connected sums.

From the Weitzenböck formula

$$\nabla_A^* \nabla_A \psi = \mathcal{D}_A^2 \psi - \frac{\mathrm{scal}}{4} \psi - \frac{1}{2} F_A \cdot \psi$$

together with the equation $\Delta|\psi|^2 = 2\langle \nabla_A^* \nabla_A \psi, \psi \rangle - 2|\psi|^2$ we get the following estimate for a μ^+-monopole (A, ψ):

$$\begin{aligned} \Delta|\psi|^2(x) &\leq 2\langle \nabla_A^* \nabla_A \psi, \psi \rangle_x \\ &= \langle \mathcal{D}_A^2 \psi, \psi \rangle_x - \frac{\mathrm{scal}}{2} |\psi(x)|^2 - \langle F_A^+ \psi, \psi \rangle_x \\ &\overset{(1.1.6)}{=} -\frac{\mathrm{scal}}{2} |\psi(x)|^2 - \langle \frac{1}{4} |\psi|^2 \psi, \psi \rangle_x - \langle \mu^+ \cdot \psi, \psi \rangle_x \\ &\leq -\frac{\mathrm{scal}}{2} |\psi(x)|^2 + 2\|\mu^+\|_\infty \cdot |\psi(x)|^2 - \frac{1}{4} |\psi(x)|^4 \ . \end{aligned} \tag{1.3.1}$$

At the maximum of the function $|\psi|^2$ we have $\Delta|\psi|^2 \leq 0$, and we conclude for the norm of the spinor part ψ of a monopole (A, ψ) the estimate:

$$\|\psi\|_\infty^2 \leq \max_{x \in M}(0, -2 \min_{x \in M} \mathrm{scal}(x) + 8\|\mu^+\|_\infty) \ . \tag{1.3.2}$$

From this estimtate we conclude the following fundamental result on Seiberg-Witten invariants of manifolds admitting positive scalar curvature:

1.3.1 COROLLARY. *Let M be a compact, connected, oriented smooth 4-manifold with $b_2^+ > 1$. If M admits a metric of positive scalar curvature, then for every Spin$^{\mathbb{C}}$-structure$P$, the Seiberg-Witten invariant $\mathbf{sw}_M(P)$ vanishes. If $b_2^+ = 1$ then the Seiberg-Witten invariant vanishes on the chamber, which contains the perturbation $\mu^+ = 0$.*

Proof: The above estimate implies, that the norm of the spinor part ψ of a monopole (A, ψ) for the unperturbed equations satifies $\|\psi\|_\infty \leq 0$, thus any such monopole is reducible. The same holds for μ^+-monopoles with sufficiently small perturbation $\|\mu^+\|_\infty < \frac{1}{4} \min_{x \in M} \mathrm{scal}(x)$. This contradicts the generic smoothness of the monopoles. Consequently in the case $b_2^+ > 1$, the moduli space $\mathfrak{M}_{\mu^+}(P)$ with sufficiently small perturbation μ^+ is generically empty. Thus the invariant $\mathbf{sw}_M(P)$ vanishes for every Spin$^{\mathbb{C}}$-structure$P$. In the case $b_2^+ = 1$ the same holds for the chamber containing the perturbation $\mu^+ = 0$. □

It is a reasonable and interesting question, what happens to the Seiberg-Witten moduli spaces and invariants, when 4-manifolds are glued together along a hypersurface. For an intriguing overview of the relevant gluing techniques we refer to chapter 4 of [38]. The simplest hypersurfaces are embedded 3-spheres, and the gluing of 4-manifolds M and N along embedded 3-spheres is nothing but the connected sum operation $M\#N$. The following proposition on the Seiberg-Witten invariants of connected sums was already announced by WITTEN in [51]. In [12], DONALDSON sketched a proof by indicating a removable singularities theorem and its use for the vanishing argument. The proofs of this theorem and its consequences were finally figured out by SALAMON in [44]. For a much more complicated proof using the full technical machinery of gluing techniques, the ambitious reader may also consult [38].

1.3.2 PROPOSITION (Connected sum theorem). *Let M_1 and M_2 be compact, connected, oriented, smooth 4-manifolds with $b_2^+ > 0$. Then for every Spin$^{\mathbb{C}}$-structure$P$ on the connected sum $M_1\#M_2$ we have:*

$$\mathbf{sw}_{M_1\#M_2}(P) \;=\; 0\;.$$

The above vanishing theorem raises the natural question, what happens with the Seiberg-Witten moduli spaces and invariants, when we glue to a 4-manifold M another one with negative definite intersection form. Due to Freedmans classification this is topologically the same as a connected sum of M with $k\overline{\mathbb{CP}}^2$ for some $k \in \mathbb{N} - \{0\}$, thus the problem reduces to the case of the connected sum $M\#\overline{\mathbb{CP}}^2$. Since on a complex manifold M, the connected sum with $\overline{\mathbb{CP}}^2$ is the same as a blow up in one point, the following result is often referred to as the blow-up formula:

1.3.3 PROPOSITION (Blow-up formula). *Let M be a compact, connected, oriented smooth 4-manifold with a Spin$^{\mathbb{C}}$-structure$P$, and consider on $\overline{\mathbb{CP}}^2$ the Spin$^{\mathbb{C}}$-structureP_n with $c_1(\det P) = (2n+1)$. Then the following formula holds for the Seiberg-Witten invariants of the connected sum $M\#\overline{\mathbb{CP}}^2$ with the Spin$^{\mathbb{C}}$-structure$P\#P_n$ obtained by gluing P and P_n over the neck of the connected sum:*

$$|\mathbf{sw}_{M\#\overline{\mathbb{CP}}^2}(P\#P_n)| = \begin{cases} 0 & : \ \dim_{\mathbb{R}}(\mathfrak{M}(P)) < \pm n(n+1) \\ |\mathbf{sw}_M(P)| & : \ \dim_{\mathbb{R}}(\mathfrak{M}(P)) \geq n(n+1) \end{cases}$$

For a proof, we again refer to [38], p. 466.

1.4 Seiberg-Witten theory on Kähler surfaces

On a Kähler surface, the Seiberg-Witten equations take a very simple form in terms of holomorphic data. It was pointed out by WITTEN in [51] that these equations are a special case of so called vortex equations, which had been studied e.g. by BRADLOW in [8, 9] and by GARCÍA-PRADA in [15]. From an existence theorem for solutions of these vortex equations, WITTEN deduced the nontriviality of the invariants on a Kähler surface. For a detailed discussion of the relation between the Seiberg-Witten equations and vortex equations, we refer to [16].

Throughout this section let (M, g) be a compact, connected Kähler surface with Kähler form ω. Sometimes we will write (M, ω) for the Kähler surface instead of (M, g). The imaginary valued self-dual 2-forms on M are given by:

$$\Omega^2_+(M; i\mathbb{R}) \;=\; \Omega^0(M; i\mathbb{R}) \cdot \omega \,\oplus\, \left\{\vartheta - \bar{\vartheta} \mid \vartheta \in \Omega^{0,2}(M; \mathbb{C})\right\} \;,$$

where the conjugation $\bar{\cdot}$ is described as follows: Choose a local orthonormal base $\{e_1, \mathfrak{J}e_1, e_2, \mathfrak{J}e_2\}$ of TM with an the associated local orthonormal base $\{\eta_1, \mathfrak{J}\eta_1, \eta_2, \mathfrak{J}\eta_2\}$ of T^*M, where $\eta_j = e^*_j$ and $(\mathfrak{J}\eta_j) := \eta_j \circ \mathfrak{J}$. Further denote by $e'_j := \frac{1}{\sqrt{2}}(e_j - i\mathfrak{J}e_j)$ resp. $e''_j := \frac{1}{\sqrt{2}}(e_j + i\mathfrak{J}e_j)$, $j = 0, 1$ the associated unitary base of $T^{1,0}M$ resp. $T^{0,1}M$, and by $\eta'_j := \frac{1}{\sqrt{2}}(\eta_j - i\mathfrak{J}\eta_j)$ resp. $\eta''_j := \frac{1}{\sqrt{2}}(\eta_j + i\mathfrak{J}\eta_j)$, $j = 0, 1$ the associated unitary base of $T^{*\,1,0}M$ resp. $T^{*\,0,1}M$. Then for a $(0, 2)$-form $\vartheta = a \cdot \eta''_1 \wedge \eta''_2$, $a \in \Omega^0(M; \mathbb{C})$ we set $\bar{\vartheta} := \bar{a} \cdot \eta'_1 \wedge \eta'_2$.

The complex structure determines a canonical Spin$^{\mathbb{C}}$-structureP_0, whose determinant line bundle is the dual of the canonical line bundle $K_M = \Lambda^{2,0}T^*M$, i.e. $\det P_0 = K^*_M = \Lambda^{0,2}T^*M$. Any other Spin$^{\mathbb{C}}$-structure$P$ differs from the canonical by a $U(1)$-bundle L, i.e. $P = P_0 \otimes L$, and the determinant line bundle of P is then given by $\det P = K^*_M \otimes L^2$. We will always denote the twisting $U(1)$-bundle L and its associated complex line bundle with the same letter. The positive resp. negative spinor bundles can be identified with complexified differential forms with values in the line bundle L:

$$\Gamma(\Sigma^+) = \Omega^0(M; L) \oplus \Omega^{0,2}(M; L) \qquad \text{and} \qquad \Gamma(\Sigma^-) = \Omega^{0,1}(M; L)\;.$$

The Clifford-multiplication by a real 1-form ν reads:

$$\nu \cdot \psi = \sqrt{2}\nu^{0,1} \wedge \psi \;-\; \sqrt{2}(\nu^{\#})^{0,1} \lrcorner\, \psi\,, \tag{1.4.1}$$

where $\nu^{1,0} := \frac{1}{2}(\nu - i\mathfrak{J}\nu)$ and $\nu^{0,1} := \frac{1}{2}(\nu + i\mathfrak{J}\nu)$. Via Clifford multiplication, the bundle $\Lambda^2_+T^*M \otimes i\mathbb{R}$ of imaginary valued self-dual 2-forms can be identified with the bundle of tracefree hermitean endomorphisms of Σ^+. In the above notation, an imaginary valued self-dual 2-form $\mu^+ = if \cdot \omega + \vartheta - \bar{\vartheta}$ with $\vartheta \in \Omega^{0,2}(M; \mathbb{C})$ operates on a spinor $\psi = (\alpha, \zeta) \in \Omega^0(M; L) \times \Omega^{0,2}(M; L)$ by:

$$\mu^+ \cdot \begin{pmatrix} \alpha \\ \zeta \end{pmatrix} = 2 \begin{pmatrix} f \cdot \alpha + \bar{\vartheta} \lrcorner\, \zeta \\ -f \cdot \zeta + \vartheta \wedge \alpha \end{pmatrix}$$

Conversely, we may identify a tracefree hermitean endomorphism P of Σ^+ with an imaginary valued self-dual 2-form. When we write P with respect to the local basis $\{\mathbf{1}, \eta''_1 \wedge \eta''_2\}$ of $\Sigma^+ = \Lambda^0(M; L) \oplus \Lambda^{0,2}(M; L)$ as the matrix

$$P = \begin{pmatrix} a & \bar{b} \\ b & -a \end{pmatrix}\,,$$

then the associated imaginary valued self-dual 2-form reads:

$$\mu^+ = \frac{i}{2} a \cdot \omega + \vartheta - \bar{\vartheta} \qquad \text{with} \quad \vartheta := \frac{1}{2} b \cdot \eta''_1 \wedge \eta''_2\;.$$

Thus when we consider the endomorphism $\frac{1}{2}q(\psi, \psi) = (\alpha \oplus \zeta)^* \otimes (\alpha \oplus \zeta) - \frac{1}{2}|\alpha \oplus \zeta|^2 \mathrm{Id}_{\Sigma^+}$ in the Seiberg-Witten equation (1.1.1), then we get as associated imaginary valued self-dual 2-form:

$$\frac{1}{2}q(\psi, \psi) = \begin{pmatrix} \frac{1}{2}(|\alpha|^2 - |\zeta|^2) & \alpha\overline{\zeta} \\ \overline{\alpha}\zeta & \frac{1}{2}(|\zeta|^2 - |\alpha|^2) \end{pmatrix} \mapsto \mu^+ = \frac{i}{4}(|\alpha|^2 - |\zeta|^2)\omega + \frac{1}{2}(\overline{\alpha}\zeta - \alpha\overline{\zeta})\;. \tag{1.4.2}$$

These identifications allows us to express the equation (1.1.1) in terms of holomorphic data. Next we express the Dirac operator of the Spin$^{\mathbb{C}}$-structure$P = P_0 \otimes L$ in terms of Cauchy-Riemann operators: Given a connection A_L on the line bundle L together with the canonical hermitean holomorphic (or Chern-) connection A_{can} on the canonical line bundle, the Dirac operator with respect to the product connection $A = A_{can} \otimes A_L^2$ on $\det(P)$ reads:

$$\mathcal{D}_A = \sqrt{2}(\overline{\partial}_{A_L} + \overline{\partial}^*_{A_L}) \ . \tag{1.4.3}$$

Having identified all the data in the definition of the Seiberg-Witten equations (1.1.1–1.1.2), we can now reformulate those equations on a Kähler surface. The equation (1.1.1) splits into two equations according to the above splitting of $\Omega^2_+(M; i\mathbb{R})$. We consider the perturbed Seiberg-Witten equations, and since the multiples of the Kähler form ω are clearly tranversal to the wall Γ^+_g, we will always consider perturbations of the form $\mu^+ = i\lambda\omega, \lambda \in \mathbb{R}$. The reformulated equations thus read:

$$\begin{aligned}
\left(F_A^+\right)^{1,1} &= \frac{i}{4}\left(|\alpha|^2 - |\zeta|^2\right) \cdot \omega \, + \, i\pi\lambda\omega & (1.4.4)\\
\left(F_A^+\right)^{0,2} &= \frac{\overline{\alpha}\zeta}{2} & (1.4.5)\\
\sqrt{2}\left(\overline{\partial}_{A_L}\alpha + \overline{\partial}^*_{A_L}\zeta\right) &= 0 & (1.4.6)
\end{aligned}$$

Finally Seiberg-Witten map on a Kähler surface reads:

$$\mathcal{SW}_{\mu^+} : \ \mathcal{A}(\det P) \times \Omega^0(M; L) \oplus \Omega^{0,2}(M; L) \ \rightarrow \ \Omega^{1,1}_+(M; i\mathbb{R}) \times \Omega^{0,2}_+(M; i\mathbb{R}) \times \Omega^{0,1}(M; L)$$

$$\begin{pmatrix} A \\ \alpha \\ \zeta \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} (F_A^+)^{1,1} - \frac{i}{4}(|\alpha|^2 - |\zeta|^2) - i\pi\lambda\omega \\ (F_A^+)^{0,2} - \frac{1}{2}\overline{\alpha}\zeta \\ \sqrt{2}\overline{\partial}_A\alpha + \sqrt{2}\overline{\partial}^*_A\zeta \end{pmatrix}$$

It was pointed out by WITTEN in [51], that if $(A, \alpha \oplus \zeta)$ is a monopole, then one of the spinor components α, ζ necessarily vanishes. To distinguish which one vanishes, we need to introduce the notion of the *degree* of a complex line bundle on a Kähler surface:

1.4.1 DEFINITION. *The degree of a line bundle L on a Kähler surface (M, g) with Kähler form ω is defined to be the number:*

$$\deg_\omega(L) := \int_M c_1(L) \wedge \omega$$

1.4.2 THEOREM (WITTEN). *Let (M, g) be a compact, connected Kähler surface with Kähler form ω and fixed Spin$^{\mathbb{C}}$-structure$P = P_0 \otimes L$. Let $(A, \alpha \oplus \zeta) \in \mathcal{A}(\det P) \times \Omega^0(M; L) \oplus \Omega^{0,2}(M; L)$ be a μ^+-monopole, where $\mu^+ = i\pi\lambda\omega, \lambda \in \mathbb{R}$. Then the following holds:*

$$\begin{aligned}
\lambda \mathrm{vol}(M) \leq \deg_\omega(\det P) \quad &\Longrightarrow \quad \zeta \equiv 0 \\
\lambda \mathrm{vol}(M) \geq \deg_\omega(\det P) \quad &\Longrightarrow \quad \alpha \equiv 0 \ .
\end{aligned}$$

In either case, the determinant line bundle $\det P = K_M^* \otimes L^2$ *carries the structure of a holomorphic line bundle, and with respect to the induced holomorphic structure on L, the components α resp. $\overline{\zeta}$ are holomorphic sections of L resp. $K_M \otimes L^*$.*

Proof: When we apply the operator $\frac{1}{\sqrt{2}}\overline{\partial}_{A_0}$ to (1.4.6), we get

$$\overline{\partial}_{A_L}\overline{\partial}_{A_L}\alpha \;+\; \overline{\partial}_{A_L}\overline{\partial}^*_{A_L}\zeta \;=\; 0\;.$$

Since $A = A_{can} \otimes A_L^2$, and A_{can} is a holomorphic connection, we get from (1.4.5):

$$\overline{\partial}_{A_L}\overline{\partial}_{A_L}\alpha = (F_{A_L})^{0,2} \wedge \alpha = \frac{1}{2}(F_A)^{0,2} \wedge \alpha = \frac{1}{4}|\alpha|^2 \cdot \zeta\;. \tag{1.4.7}$$

Taking the L^2-product with ζ, we thus find:

$$\frac{1}{4}\int_M |\alpha|^2|\zeta|^2\,dv_g \;+\; \|\overline{\partial}^*_{A_L}\zeta\|^2_{L^2} \;=\; 0 \qquad \Longrightarrow \quad \overline{\alpha}\zeta \equiv 0 \quad \text{and} \quad \overline{\partial}^*_{A_L}\zeta \equiv 0\;, \tag{1.4.8}$$

where $dv_g = \frac{1}{2}\omega \wedge \omega$ is the volume form of the Kähler metric g. From (1.4.5) we get $(F_A)^{0,2} \equiv 0$, thus A is a holomorphic connection, and so is A_L. From (1.4.6) and (1.4.8) we get $\overline{\partial}_{A_L}\alpha \equiv 0$, thus α is a holomorphic section. The equation $\overline{\partial}^*_{A_L}\zeta = 0$ says, that ζ is an antiholomorphic section of $\Omega^{0,2}(M; L)$ or equivalently, that $\overline{\zeta}$ is a holomorphic section with respect to the induced holomorphic structure on $K_M \otimes L^*$. Since both α and ζ are holomorphic resp. anti-holomorphic sections, the equation $\overline{\alpha}\zeta \equiv 0$ implies that at least one of the components α, ζ vanishes identically. Which component vanishes may be detected from the degree of the determinant line bundle by the following computation:

$$\begin{aligned}
\deg_\omega(\det P) \quad &= \quad \int_M c_1(\det P) \wedge \omega \\
&= \quad \int_M \frac{i}{2\pi} F_A \wedge \omega \\
&\overset{(1.4.4)}{=} \quad \int_M \frac{i}{2\pi}\big(\frac{i}{4}(|\alpha|^2 - |\zeta|^2) + i\pi\lambda\big) \cdot \omega \wedge \omega \\
&= \quad -\lambda \mathrm{vol}(M) + \int_M \frac{1}{4\pi}\big(|\zeta|^2 - |\alpha|^2\big)\,dv_g
\end{aligned} \tag{1.4.9}$$

Thus if $\deg_\omega(\det P) \geq \lambda\mathrm{vol}(M)$, then $|\zeta|^2 \equiv 0$ and correspondingly if $\deg_\omega(\det P) \leq \lambda\mathrm{vol}(M)$, then $|\alpha|^2 \equiv 0$. □

Note that if we replace the line bundle L by $K_M \otimes L^*$, then the determinant line bundle $\det(P) = K_M^* \otimes L^2$ will be replaced by its dual. Thus we can always arrange $\deg_\omega(\det P)$ to have a fixed sign. Hence by replacing L if necessary we may always assume the component ζ of a monopole $(A, \alpha \oplus \zeta)$ to vanish:

1.4.3 CONVENTION. *In the following, we always fix a Spin$^{\mathbb{C}}$-structure$P$ and consider only those perturbations, which are sufficiently small with respect to* $\det P$*, i.e. those perturbations $\mu = i\pi\lambda\omega$ such that*

$$\lambda \leq \frac{\deg_\omega(\det P)}{\mathrm{vol}(M)}\;. \tag{1.4.10}$$

Thus from now on, we will treat monopoles as pairs $(A, \alpha) \in \mathcal{A}(\det P) \times \Omega^0(M; L)$.

As was observed by WITTEN in [51], the Seiberg-Witten equations on a Kähler surface are a special case of vortex equations, which had extensively been studied by BRADLOW, GARCÍA-PRADA and others. This identification of monopoles as vortices allows to apply a theorem of KAZDAN and WARNER in [29] on the existence and uniqueness of solutions of a certain nonlinear elliptic partial differential equation. We will explain this method in detail and cite the Kazdan-Warner theorem in section 3.1. Here we only mention, that this identification led to the following first result on the nontriviality of the invariants:

1.4.4 PROPOSITION (Witten). *Let (M, ω) be a compact, connected Kähler surface, equipped with the canonical Spin$^{\mathbb{C}}$-structureP_0. If $b_2^+ > 1$, then the Seiberg-Witten invariant is:*

$$\mathbf{sw}(M, P_0) = 1 \ ,$$

whereas in the case of $b_2^+ = 1$ we have:

$$\mathbf{sw}^+(M, P_0) = 1 \ .$$

Thus if $b_2^+ > 1$, then the associated moduli spaces are generically nonempty. Due to the Hirzebruch signature theorem, we know that these moduli spaces have virtual dimension $d = \dim_{\mathbb{R}} \mathfrak{M} = 0$. (In fact, the canonical and the anticanonical Spin$^{\mathbb{C}}$-structureare the unique Spin$^{\mathbb{C}}$-structureson a Kähler surface for which the moduli spaces have virtual dimension $d = 0$.) Thus these moduli spaces are finite sets of points. When we further assume M to be simply connected, then it is easy to see, that all monopoles are gauge equivalent, i.e. the moduli space $\mathfrak{M}$ consists of a single point.

Similarly, if $b_2^+ = 1$, then the moduli spaces associated with perturbations in the positive chamber are generically nonempty, and they are diffeomorphic to a single point, when M is simply connected.

1.5 Moduli spaces of positive virtual dimension

In this section we review some results on the topology of Seiberg-Witten moduli spaces of positive virtual dimension, as those are in the heart of our interest, when we study the geometry of the moduli spaces in the L^2-metric. We begin with some negative results stating that in special cases the moduli spaces of positive virtual dimension are generically empty. Afterwards we discuss some constructions which lead to nonvanishing results for moduli spaces of positive virtual dimension.

Vanishing results

1.5.1 DEFINITION. *A compact, connected, oriented, smooth 4-manifold M is said to be of SW-simple type if the Seiberg-Witten invariant $\mathbf{sw}_M(P) = 0$ for every Spin$^{\mathbb{C}}$-structure$P$ on M, for which the Seiberg-Witten moduli space $\mathfrak{M}(P)$ has virtual dimension $d > 0$.*

In [51], WITTEN pointed out that any Kähler surface with $b_2^+ > 1$ is of SW-simple type. TAUBES showed in [50], that same holds true, when one weakens the assumption to an almost Kähler surface, i.e. a symplectic manifold (M, ω) with a compatible Riemannian metric g and almost complex structure $\mathfrak{J}$, which needs not be integrable. Thus we have:

1.5.2 THEOREM (WITTEN,TAUBES). *Let (M,ω) be a compact almost Kähler surface with $b_2^+ > 1$. Then the Seiberg-Witten invariant $\mathbf{sw}_M(P)$ vanishes for all but the canonical and anti-canonical Spin$^{\mathbb{C}}$-structureP_0 resp. $\widetilde{P_0} = P_0 \otimes K_M^*$ associated with the almost complex structure $\mathfrak{J}$. Thus M is of SW-simple type.*

In the case of a Kähler surface, the proof is a rather simple consequence of generalised adjunction inequalities, see [43], p. 185f., whereas in the case of an almost Kähler surface, TAUBES deduced the result from his identification of the Seiberg-Witten invariants with Gromov invariants, applied to the same generalised adjunction inequalities. In either case, the proofs state results which are even a little stronger as formulated above, namely that the moduli spaces are (generically) empty.

The above theorem drastically restricts the field of study for the geometry of Seiberg-Witten moduli spaces on Kähler surface: when dealing with a Kähler surface (M, g), we will thus always assume that $b_2^+ = 1$. However, when we blow up a Kähler surface (M, g) in one point, i.e. we take the connected sum $M\#\overline{\mathbb{CP}}$, then we end up with another Kähler surface, which has the same b_2^+ as M. Thus any Kähler surface M with $b_2^+ = 1$ gives an infinite family of Kähler surfaces with $b_2^+ = 1$.

From the blow up formula we further see, that for at least finitely many Spin$^{\mathbb{C}}$-structures$P\#P_n$ on the connected sum $M\#\overline{\mathbb{CP}}^2$, the moduli spaces of $(M\#\overline{CP}^2, P\#P_n)$ are generically nonempty, if those of (M, P) are. In section 3.1, we directly prove a theorem, which states that the Seiberg-Witten moduli spaces $\mathfrak{M}_{\mu^+}(M, P)$ of a Kähler surface (M, g) with Spin$^{\mathbb{C}}$-structure$P$ are generically nonempty for sufficiently large μ^+, if the twisting line bundle L admits holomorphic sections. In this case, we can directly identify the diffeomorphism type of the moduli spaces.

Nonvanishing results

Now we will briefly review another construction which leads to generically nonempty moduli spaces. This method is particularly interesting, since it makes use of the whole cohomology of $\mathcal{B}^*$, not only of its even part.

First we review yet another way to construct the cohomology of the space $\mathcal{B}^*$ of gauge equivalence classes of irreducible configurations. This construction had been introduced by DONALDSON to construct the Donaldson invariants from the cohomology of the Yang-Mills moduli space, see [13]. Recall that the slant product

$$/ : H^k(N_1 \times N_2; \mathbb{Z}) \times H_\ell(N_2; \mathbb{Z}) \to H^{k-\ell}(N_1; \mathbb{Z})$$

is defined to be the adjoint (with respect to the Kronecker product) of the cross product, i.e. $\langle a/y, x\rangle = \langle a, x \times y\rangle$. Applying the slant product to the class $c_1(\mathcal{L}_{\mathcal{B}^* \times M}) \in H^2(\mathcal{B}^* \times M; \mathbb{Z})$, one defines the so called Donaldson μ-map:

$$\begin{array}{rccc} \mu : & H_0(M;\mathbb{Z}) \oplus H_1(M;\mathbb{Z}) & \to & H^*(\mathcal{B}^*;\mathbb{Z}) \\ & y & \mapsto & c_1(\mathcal{L}_{\mathcal{B}^*\times M})/y\ . \end{array}$$

Then one can show, that the cohomology $H^*(\mathcal{B}^*; \mathbb{Z})$ is generated by the image of the μ-map, see [11, 42]. The Seiberg-Witten class $\Omega \in H^2(\mathcal{B}^*; \mathbb{Z})$ as constructed in 1.2 above is obviously given by the restriction to $\mathfrak{M}$ of the class $\mu(\mathbf{1}) = c_1(\mathcal{L}_{\mathcal{B}^* \times M})/\mathbf{1}$.

As mentioned in section 1.2 above, there is an explicit construction for the generator of the $\Lambda^* H^1(M; \mathbb{Z})$-part of the cohomology $H^*(\mathcal{B}^*; \mathbb{Z})$ as well. This generator is given by the so called holonomy class: let $x \in H_1(M;\mathbb{Z})$ be represented by an embedded smooth curve $x : S^1 \to M$ and denote by $\mathrm{hol}_x(A)$ the holonomy of a connection $A \in \mathcal{A}(\det P)$ along x. Then the holonomy may be regarded as a map $\mathrm{hol}_x : \mathcal{C}^* \to U(1) = S^1$. Since $U(1)$ is abelian, the holonomy is gauge invariant, and thus the holonomy map descends to a map $\mathrm{hol}_x : \mathcal{B}^* \to S^1$. When we denote by $d\vartheta$ the standard generator of $H^1(S^1;\mathbb{Z}) \subset H^1(S^1;\mathbb{R})$, then the pullback $d\vartheta$ by the holonomy map is the same as the image of the class $x \in H_1(M;\mathbb{Z})$ under the μ-map:

$$\mu(x) = \mathrm{hol}_x^* d\vartheta \in H^1(\mathcal{B}^*;\mathbb{Z}) \ .$$

This can be seen as follows (see also [11] and [42]): Recall from section 1.2, that the pullback bundle $\imath_{x_0}^*\mathcal{L}_{\mathcal{B}^*\times M}$ remains unchanged, when we replace $\mathcal{L}_{\mathcal{B}^*\times M}$ by the quotient $(\mathcal{C}^* \times \det P)/\mathcal{G}$. Note further that on the bundle $(\mathcal{C}^* \times \det P)/\mathcal{G} \to \mathcal{B}^* \times M$, there is a natural tautological connection $\mathcal{A}$ such that

$$\mathrm{hol}_{[A,\psi]\times x}(\mathcal{A}) = \mathrm{hol}_x(A) \ . \tag{1.5.1}$$

This tautological connection is constructed as follows: The fibres of the bundle $\mathcal{C}^* \times \det P \to \mathcal{C}^* \times M$ are the fibres of $\det P$, thus this bundle admits a tautological connection $\widetilde{\mathcal{A}}$ defined by $\widetilde{\mathcal{A}}_{(A,\psi,p)} := A_p$. Since the connection $\widetilde{\mathcal{A}}$ is $\mathcal{G}$-invariant, it descends to a connection $\mathcal{A}$ on $(\mathcal{C}^* \times \det P)/\mathcal{G} \to \mathcal{B}^* \times M$. The formula (1.5.1) for the holonomy of $\mathcal{A}$ follows directly from the tautological definition of $\widetilde{\mathcal{A}}$.

To show that the image of $x \in H_1(M;\mathbb{Z})$ under the μ-map is indeed the holonomy class, we need to compute $c_1(\mathcal{L}_{\mathcal{B}^*\times M})/x$ on an arbitrary element $y \in H_1(\mathcal{B}^*;\mathbb{Z})$. Let y be represented by an embedded smooth curve $y : S^1 \to \mathcal{B}^*$. When we pull back the bundle $\mathcal{L}_{\mathcal{B}^*\times M}$ along $y \times x$ to the 2-torus $S^1 \times S^1$, then the first Chern number of this bundle is the degree of the map $\vartheta \mapsto \mathrm{hol}_{\vartheta\times S^1}((y\times x)^*\mathcal{L}_{\mathcal{B}^*\times M})$. We thus get:

$$\begin{aligned}
\langle \mu(x), y\rangle &= \langle c_1(\mathcal{L}_{\mathcal{B}^*\times M})/x, y\rangle \\
&= \langle c_1(\mathcal{L}_{\mathcal{B}^*\times M}), y \times x\rangle \\
&= \deg(\vartheta \mapsto \mathrm{hol}_{\vartheta\times S^1}((y\times x)^*\mathcal{L}_{\mathcal{B}^*\times M})) \\
&= \deg(\vartheta \mapsto \mathrm{hol}_{y(\vartheta)\times x}(\mathcal{L}_{\mathcal{B}^*\times M})) \\
&= \deg(\mathrm{hol}_x \circ y) \\
&= \langle \mathrm{hol}_x^* d\vartheta, y\rangle \ .
\end{aligned}$$

Now that we have established an explicit construction of a generator for the degree one part of the cohomology of $\mathcal{B}^*$, we will briefly review a construction of another family of manifolds, for which the Seiberg-Witten moduli spaces are generically nonempty. These examples, constructed by DEL RIO GUERRA in [11], are based on connected sums $M\#S^1 \times S^3$, where M is a Kähler surface with an appropriate Spin$^{\mathbb{C}}$-structure.

Note that for any compact, oriented 4-manifold M, there is a natural projection map $\pi : M\#S^2 \times S^3 \to M$, such that the induced map on cohomology $\pi^* : H^i(M) \to H^i(M\#S^1 \times S^3)$ is injective for $i = 1, 3$ and bijektive for $i = 0, 2, 4$. Since $\pi^* w_2(M) = w_2(M\#S^2 \times S^3)$ this map induces an identification of the Spin$^{\mathbb{C}}$-structuresof M with those of $M\#S^1 \times S^3$. Let P be a Spin$^{\mathbb{C}}$-structureon M, then we denote the corresponding Spin$^{\mathbb{C}}$-structureon $M\#S^1 \times S^3$ by P'.

When the Seiberg-Witten moduli space $\mathfrak{M}(M,P)$ has (virtual) dimension $d = \dim_{\mathbb{R}}(\mathfrak{M})$, then the moduli space $\mathfrak{M}(M\#S^1 \times S^3, P')$ has (virtual) dimension $\dim_{\mathbb{R}}(\mathfrak{M}\#S^1 \times S^3) = d+1$. This increment in the dimension comes from the fact, that there remains an extra degree of freedom, when a monopole

(A,ψ) on M is extended to a monopole (A',ψ') on $M\#S^1\times S^3$, namely the holonomy of A' along the S^1-factor of the connected sum. More precisely, DEL RIO GUERRA showed in [11], that for every $\vartheta\in S^1$ and every $l\gg 0$, there are generic perturbations, such that every monopole (A,ψ) on (M,g) can be extended to a monopole (A',ψ') on $(M\#S^1\times S^3,g_l)$ with $\mathrm{hol}_{S^1}(A')=\vartheta$. Here hol_{S^1} denotes the holonomy along the S^1-factor in $M\#S^1\times S^3$, and g_l denotes the metric obtained from g by gluing a neck of length l. This finally yields the following class of generically nonempty Seiberg-Witten moduli spaces:

1.5.3 PROPOSITION (DEL RIO GUERRA). *Let M be a compact, connected Kähler surface with Spin$^{\mathbb{C}}$-structure$P$. If the Seiberg-Witten invariant $\mathbf{sw}(M,P)$ is nonzero, then Seiberg-Witten moduli space $\mathfrak{M}(M\#S^1\times S^3,P')$ represents a nontrivial cobordism class in $H^*(\mathcal{B}^*;\mathbb{Z})$, i.e. there is cohomology class $a\in H^*(\mathcal{B}^*;\mathbb{Z})$ such that $\langle a,[\mathfrak{M}]\rangle\neq 0$. Further, the moduli space $\mathfrak{M}(M\#2(S^2\times S^3),P'')$ also represents a nontrivial cobordism class but has vanishing Seiberg-Witten invariant.*

For the proof and applications to nonexistence theorems for Einstein metrics see [11]. The proof is based on conformal geometry methods to construct monopoles on manifolds with cylindrical ends and on the observation, that the homology class represented by the S^1-family of monopoles (A',ψ') which extend (A,ψ) is dual to the holonomy class $\mathrm{hol}^*_{S^1}d\vartheta$ along the S^1-factor of the connected sum $M\#S^1\times S^3$.

Chapter 2

The L^2-metric on the moduli space

In this chapter we introduce the concept of our study of Riemannian geometry of Seiberg-Witten moduli spaces. We define two Riemannian metrics on the moduli spaces, which arise naturally from the L^2-metric on the configuration space $\mathcal{C}$. First we consider the metric induced on $\mathfrak{M}$ via the embedding $\mathfrak{M} \hookrightarrow \mathcal{B}^*$ from the quotient metric on $\mathcal{B}^* := \mathcal{C}^*/\mathcal{G}$. This metric can be realised via the identification of the tangent bundle of $\mathfrak{M}$ as the subbundle of $T\mathcal{C}^*$ given by the first cohomologies of the family of elliptic complexes $\mathcal{K}_{(A,\psi)}$. To construct an L^2-metric on the Seiberg-Witten bundle $\mathfrak{P} \to \mathfrak{M}$, we need to replace the elliptic splittings of $T_{(A,\psi)}\mathcal{C}$ induced by the elliptic complex $\mathcal{K}_{(A,\psi)}$ in a way such that the tangent space of $\mathfrak{P}$ splits orthogonally from $T_{(A,\psi)}\mathcal{C}^*$. To this end we introduce a new representative of the bundle $\mathfrak{P} \to \mathfrak{M}$ which amounts to a splitting of $T_{(A,\psi)}\mathcal{C}^*$ appropriate for the construction of a *quotient L^2-metric* on $\mathfrak{P} \to \mathfrak{M}$.

Another way to construct a Riemannian metric on the moduli space $\mathfrak{M}$ from the L^2-metric on the configuration space $\mathcal{C}$ is via the use of the canonical gauge global fixing $d^*(A - A_0) = 0$. The canonically gauge fixed Seiberg-Witten premoduli space $\widetilde{\mathfrak{M}}_{fix}$ is a smooth submanifold of the configuration space, thus it carries a natural induced metric. The induced metrics on the bundle $\widetilde{\mathfrak{M}}_{fix} \to \mathfrak{M}$ will be called *canonically gauge fixed L^2-metrics*.

We calculate explicit formulae for the sectional curvature of the quotient L^2-metric on $\mathfrak{M}$ in terms of the Green operators of the elliptic complex $\mathcal{K}_{(A,\psi)}$ associated with a monopole (A, ψ). Throughout this chapter, we make no further assumptions on the underlying manifolds, while in the following chapter we will discuss the L^2-metrics on moduli spaces of Kähler surfaces.

2.1 The L^2-metric on the configuration space

The configuration space $\mathcal{C} = \mathcal{A}(\det P) \times \Gamma(\Sigma^+)$ is an affine space, thus it carries a natural L^2-metric induced from L^2-metric on its parallel space $\Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+)$. This metric is only a weak Riemannian metric on $\mathcal{C}$ in the sense that the tangent spaces are not complete with respect to the L^2-topology. A priori, it is not clear whether a weak Riemannian metric admits a Levi-Civita connection, because the Kozsul formula gives an element in the cotangent space only, and since the tangent space is only a pre-Hilbert space, it is not clear, whether this element can be represented by a tangent vector, i.e. by a smooth object. However on an affine space, there is a natural candidate for a connection, defined by the directional derivatives:

Let $X, Y \in \mathfrak{X}(\mathcal{C})$ be vector fields on the configuration space, represented by maps $X, Y : \mathcal{C} \to$

$\Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+)$. Then the covariant derivative of Y in (A, ψ) in the direction $X_0 := X_{(A,\psi)}$ is defined by:

$$(\nabla_{X_0} Y)_{(A,\psi)} := \frac{d}{dt}\Big|_0 Y\big((A, \psi) + tX_0 \big) \,. \tag{2.1.1}$$

This connection is clearly torsionfree, since for vector fields $X, Y \in \mathfrak{X}(\mathcal{C})$ we have:

$$\begin{aligned} \big(\nabla_X Y \,-\, \nabla_Y X\big)_{(A,\psi)} &= \frac{d}{dt}\Big|_0 \Big\{ Y\big((A, \psi) + tX_0 \big) \,-\, X\big((A, \psi) + tY_0 \big) \Big\} \\ &= \big(DY.X \,-\, DX.Y\big)_{(A,\psi)} \\ &= [X, Y]_{(A,\psi)} \,. \end{aligned} \tag{2.1.2}$$

Furthermore, since the L^2-metric on the tangent space $T_{(A,\psi)}\mathcal{C}$ does not depend on the point (A, ψ), it is preserved by the connection ∇. Since the Kozsul formula holds even for a weak Riemannian metric, a metric and torsionfree connection is unique. Thus we may call ∇ the ***Levi-Civita connection*** of the L^2-metric. Note that since the metric does not depend on the base point (A, ψ), the metric connection ∇ is flat, namely for vector fields $X, Y, Z \in \mathfrak{X}(\mathcal{C})$ we find:

$$\begin{aligned} &\big(\nabla_X \nabla_Y Z \,-\, \nabla_Y \nabla_X Z\big)_{(A,\psi)} \\ &= \frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 \Big\{ Z\big((A, \psi) + tX_0 + sY_{(A,\psi)+tX_0} \big) \,-\, Z\big((A, \psi) + tY_0 + sX_{(A,\psi)+tY_0} \big) \Big\} \\ &= DZ \,. \frac{d}{dt}\Big|_0 \Big\{ Y_{(A,\psi)+tX_0} - X_{(A,\psi)+tY_0} \Big\} \,=\, \big(DZ \,. [X, Y]\big)_{(A,\psi)} \\ &= \frac{d}{dt}\Big|_0 Z(\, (A, \psi) + t[X, Y]_{(A,\psi)} \,) \,=\, \big(\nabla_{[X,Y]} Z\big)_{(A,\psi)} \,. \end{aligned} \tag{2.1.3}$$

Thus the L^2-metric on $\mathcal{C}$ is the natural flat metric on an affine space.

2.2 The quotient L^2-metric on the moduli space

In this section we define a Riemannian metric on the Seiberg-Witten moduli space $\mathfrak{M}$, which arises naturally from the quotient metric on the space of gauge equivalence classes of irreducible configurations $\mathcal{B}^* := \mathcal{C}^*/\mathcal{G}$ via the embedding $\mathfrak{M} \hookrightarrow \mathcal{B}^*$. We outline how to compute the sectional curvature of $\mathfrak{M}$ in terms of the Green operators of the elliptic complex $\mathcal{K}_{(A,\psi)}$ associated with a monopole (A, ψ). Similar L^2-metrics on the Yang-Mills moduli spaces had intensively been studied by many authors such as GROISSER, HABERMANN, MATSUMOTO, MATUMOTO and PARKER with several approaches to different special cases. Among them we mention the work [19] of GROISSER and PARKER, which gives a detailed introduction to the construction of L^2-metrics on moduli spaces of monopoles.

Since the gauge group $\mathcal{G} = \Omega^0(M; U(1))$ acts on $\mathcal{C}$ by $u : (A, \psi) \mapsto (A + 2u^{-1}du, u^{-1}\psi)$, the induced action on $T\mathcal{C}$ reads $u : (\nu, \phi) \mapsto (\nu, u^{-1}\phi)$, thus the L^2-metric on $\mathcal{C}$ is $\mathcal{G}$-invariant. Hence the quotient space $\mathcal{B}^* := \mathcal{C}^*/\mathcal{G}$ of gauge equivalence classes of irreducible configurations carries a unique (weak) Riemannian metric such that the projection $\mathcal{C}^* \to \mathcal{B}^*$ is a Riemannian submersion. Thus we can use an infinite dimensional analogue of the O'Neill formula for Riemannian submersions to compute the sectional curvature of $\mathcal{B}^*$ in this quotient metric. Since for a generic perturbation $\mu^+ \in \Omega^2_+(M; i\mathbb{R})$ the Seiberg-Witten moduli space $\mathfrak{M}_{\mu^+}$ is a smooth submanifold of this quotient $\mathcal{B}^*$, the Gauss equation for the embedding $\mathfrak{M}_{P,\mu^+} \hookrightarrow \mathcal{B}^*$ enables us to compute the sectional curvature of $\mathfrak{M}_{P,\mu^+}$ from the one of $\mathcal{B}^*$. To this end we locally identify the pullback of the tangent bundle $T\mathfrak{M}$ of the moduli space $\mathfrak{M}$ with the horizontal tangent bundle of the premoduli space $\widetilde{\mathfrak{M}}$.

$$\begin{array}{ccc} \widetilde{\mathfrak{M}}_{\mu^+} & \longrightarrow & \mathcal{C}^* \\ \downarrow & & \downarrow \text{O'Neill} \\ \mathfrak{M}_{\mu^+} & \underset{\text{Gauss}}{\longrightarrow} & \mathcal{C}^*/\mathcal{G} \end{array}$$

Both the O'Neill formula and the Gauss equation involve orthogonal projections onto subspaces of the tangent space. These are given in terms of the Green operators of the elliptic complex

$$0 \longrightarrow \Omega^0(M;i\mathbb{R}) \xrightarrow{\mathcal{T}_0} \Omega^1(M;i\mathbb{R}) \times \Gamma(\Sigma^+) \xrightarrow{\mathcal{T}_1} \Omega^2_+(M;i\mathbb{R}) \times \Gamma(\Sigma^-) \longrightarrow 0 \qquad \mathcal{K}_{(A,\psi)}$$

associated with a monopole (A,ψ). As explained in section 1.1 above, the tangent space in (A,ψ) of the premoduli space $\widetilde{\mathfrak{M}}$ is the kernel of the linearisation $\mathcal{T}_1$ in (A,ψ) of the Seiberg map $\mathcal{SW}$. Correspondingly, the tangent space in (A,ψ) of the gauge orbit through (A,ψ) is the image of the linearisation in $\mathcal{T}_0$ of the orbit map through (A,ψ). The ellipticity of the complex $\mathcal{K}_{(A,\psi)}$ yields the following L^2-orthogonal splittings:

$$\begin{aligned} \Omega^0(M;i\mathbb{R}) &= \ker \mathcal{T}_0 \oplus \mathrm{im}\mathcal{T}_0^* & (2.2.1)\\ \Omega^1(M;i\mathbb{R}) \times \Gamma(\Sigma^+) &= \ker \mathcal{T}_0^* \oplus \mathrm{im}\mathcal{T}_0 \\ &= \ker \mathcal{T}_1 \oplus \mathrm{im}\mathcal{T}_1^* \\ &= (\ker \mathcal{T}_0^* \cap \ker \mathcal{T}_1) \oplus \mathrm{im}\mathcal{T}_0 \oplus \mathrm{im}\mathcal{T}_1^* & (2.2.2)\\ \Omega^2_+(M;i\mathbb{R}) \times \Gamma(\Sigma^-) &= \ker \mathcal{T}_1^* \oplus \mathrm{im}\mathcal{T}_1 & (2.2.3) \end{aligned}$$

Note that all these operators implicitly depend on the configuration (A,ψ) where we do the linearisations. We will drop this dependence in the notation, since this should not lead to confusion.

The adjoint operators are easily computed: When we integrate an element $\mathcal{T}_0(if) = (2idf, -if\cdot\psi)$ of $\mathrm{im}\mathcal{T}_0$ against the linearised configuration $(\nu,\phi) \in \Omega^1(M;i\mathbb{R}) \times \Omega^0(M;L)$, we find:

$$\begin{aligned} \left(\mathcal{T}_0(if)\,,\begin{pmatrix}\nu\\ \phi\end{pmatrix}\right)_{L^2} &= \mathrm{Re}\int_M \langle 2idf,\nu\rangle + \langle -if\cdot\psi,\phi\rangle\, dv_g \\ &= \int_M \langle if, 2d^*\nu\rangle + if\cdot(-i)\cdot\mathrm{Im}\langle\psi,\phi\rangle\, dv_g \\ &= \Big(if\,,\,2d^*\nu + i\mathrm{Im}\langle\psi,\phi\rangle\Big)_{L^2} \end{aligned}$$

Thus the adjoint of $\mathcal{T}_0$ is the operator:

$$\begin{aligned} \mathcal{T}_0^*: \ \Omega^1(M;i\mathbb{R}) \times \Omega^0(M;L) &\to \quad \Omega^0(M;i\mathbb{R}) \\ \begin{pmatrix}\nu\\ \phi\end{pmatrix} &\mapsto \ 2d^*\nu + i\mathrm{Im}\langle\psi,\phi\rangle \end{aligned}$$

Similarly, when we integrate an element $\mathcal{T}_1(\nu,\phi) = (d^+\nu - q(\psi,\phi), \mathcal{D}_A\phi + \frac{1}{2}\nu\cdot\psi)$ of $\mathrm{im}\mathcal{T}_1$ against

$(\mu, \xi) \in \Omega^2_+(M; i\mathbb{R}) \times \Gamma(\Sigma^-)$, we find:

$$\begin{aligned}
\left(\mathcal{T}_1 \begin{pmatrix} \nu \\ \phi \end{pmatrix}, \begin{pmatrix} \mu \\ \xi \end{pmatrix}\right)_{L^2} &= \left(d^+\nu, \mu\right)_{L^2} - \left(q(\psi, \phi), \mu\right)_{L^2} + \left(\mathcal{D}_A, \xi\right)_{L^2} + \frac{1}{2}\left(\nu \cdot \psi, \xi\right)_{L^2} \\
&= \left(\nu, d\mu\right)_{L^2} - \mathrm{Re}\int_M \langle(\psi^* \otimes \phi + \phi^* \otimes \psi)_0, \mu\rangle \, dv_g \\
&\quad + \left(\phi, \mathcal{D}_A\xi\right)_{L^2} + \frac{1}{2}\mathrm{Re}\int_M \langle \nu \cdot \psi, \xi\rangle \, dv_g \\
&= \left(\nu, d\mu\right)_{L^2} - \mathrm{Re}\int_M \langle \phi, \mu \cdot \psi\rangle + \langle \psi, \mu \cdot \phi\rangle \, dv_g \\
&\quad + \left(\phi, \mathcal{D}_A\xi\right)_{L^2} + \frac{1}{2}\mathrm{Re}\int_M \langle \nu, i\mathrm{Im}\langle(\cdot)\cdot\psi, \xi\rangle\rangle \, dv_g \\
&= \left(\nu, d\mu\right)_{L^2} - 2\left(\phi, \mu \cdot \psi\right)_{L^2} + \left(\phi, \mathcal{D}_A\xi\right)_{L^2} + \frac{1}{2}\left(\nu, i\mathrm{Im}\langle(\cdot)\cdot\psi, \xi\rangle\right)_{L^2}
\end{aligned}$$

Thus the adjoint of $\mathcal{T}_1$ is the operator:

$$\begin{array}{rccc}
\mathcal{T}_1^* : & \Omega^2_+(M; i\mathbb{R}) \times \Gamma(\Sigma^-) & \to & \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) \\
& \begin{pmatrix} \mu \\ \xi \end{pmatrix} & \mapsto & \begin{pmatrix} d^*\mu + \frac{i}{2}\mathrm{Im}\langle(\cdot)\cdot\psi, \xi\rangle \\ \mathcal{D}_A\xi - 2\mu\cdot\psi \end{pmatrix}
\end{array} \tag{2.2.4}$$

where $\langle(\cdot)\cdot\psi, \xi\rangle$ denotes the 1-form $\mathfrak{X}(M) \ni X \mapsto \langle X \cdot \psi, \xi\rangle$.

Associated with the complex $\mathcal{K}_{(A,\psi)}$ are three natural differential operators of second order, called the Laplacians of the complex and denoted by $L_j, j = 0, 1, 2$:

$$\begin{array}{rccc}
L_0 = \mathcal{T}_0^* \circ \mathcal{T}_0 : & \Omega^0(M; i\mathbb{R}) & \to & \Omega^0(M; i\mathbb{R}) \\
L_1 = \mathcal{T}_0 \circ \mathcal{T}_0^* \oplus \mathcal{T}_1^* \circ \mathcal{T}_1 : & \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) & \to & \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) \\
L_2 = \mathcal{T}_1 \circ \mathcal{T}_1^* : & \Omega^2_+(M; i\mathbb{R}) \times \Gamma(\Sigma^+) & \to & \Omega^2_+(M; i\mathbb{R}) \times \Gamma(\Sigma^+)
\end{array}$$

Note that via Hodge theory the j-th cohomology of the complex $\mathcal{K}_{(A,\psi)}$ can be expressed as the space of maps, which are harmonic for the Laplacian L_j. Thus the obstruction spaces as defined section 1.1 – the zeroth and the second cohomology of $\mathcal{K}_{(A,\psi)}$ – are the spaces of maps, which are harmonic with respect to L_0 resp. L_2. For a generic perturbation $\mu^+ \in \Omega^2_+(M; i\mathbb{R})$, these obstruction spaces vanish. Thus the kernels of L_0 resp. L_2 vanish generically. However, the ellipticity of $\mathcal{K}_{(A,\psi)}$ implies that the L_j are elliptic operators, so that we may define the Green operators $G_j, j = 0, 1, 2$ as the inverses of the restrictions of the Laplacians L_j to the complements of their kernels:

$$\begin{array}{rcccl}
G_0 : & \Omega^0(M; i\mathbb{R}) & \to & \Omega^0(M; i\mathbb{R}), & G_0 := (L_0|_{\mathrm{im}\mathcal{T}_0^*})^{-1} \oplus 0 \\
G_1 : & \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) & \to & \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+), & G_1 := (L_1|_{\mathrm{im}\mathcal{T}_0 \oplus \mathcal{T}_1^*})^{-1} \oplus 0 \\
G_2 : & \Omega^2_+(M; i\mathbb{R}) \times \Gamma(\Sigma^+) & \to & \Omega^2_+(M; i\mathbb{R}) \times \Gamma(\Sigma^+), & G_2 := (L_2|_{\mathrm{im}\mathcal{T}_0^*})^{-1} \oplus 0
\end{array} \tag{2.2.5}$$

These Green operators are nonlocal elliptic pseudo-differential operators.

Due to the above splittings we can express the orthogonal projectors onto the vertical space $\mathcal{V}_{(A,\psi)} := \mathrm{im}\mathcal{T}_0$ resp. the horizontal space $\mathcal{H}_{(A,\psi)} := \ker T_0^*$ of the gauge action as well as onto the tangent space $T_{(A,\psi)}\widetilde{\mathfrak{M}} = \ker \mathcal{T}_1$ resp. the normal space $N_{(A,\psi)}\widetilde{\mathfrak{M}} = \mathrm{im}\mathcal{T}_1^*$ of the premoduli space in terms of the Green operators $G_j, j = 0, 1, 2$ of the complex $\mathcal{K}_{(A,\psi)}$:

$$\mathrm{vert}_{(A,\psi)} = \mathcal{T}_0 \circ G_0 \circ \mathcal{T}_0^* \qquad \mathrm{hor}_{(A,\psi)} = \mathrm{id}_{\Omega^1(M;i\mathbb{R})\times\Gamma(\Sigma^+)} - \mathrm{vert}_{(A,\psi)} \tag{2.2.6}$$

$$\tan_{(A,\psi)} = \mathrm{id}_{\Omega^1(M;i\mathbb{R})\times\Gamma(\Sigma^+)} - \mathrm{nor}_{(A,\psi)} \qquad \mathrm{nor}_{(A,\psi)} = \mathcal{T}_1^* \circ G_2 \circ \mathcal{T}_1 \tag{2.2.7}$$

That these operators are in fact the orthogonal projections is quite obvious, since e.g. the operator $\mathrm{vert} = \mathcal{T}_0 \circ G_0 \circ \mathcal{T}_0^*$ is the identity on $\mathrm{im}\mathcal{T}_0$ and vanishes on the orthogonal complement $\ker \mathcal{T}_0^*$, thus it is the orthogonal projection from $\Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+)$ to $\mathcal{V}_{(A,\psi)} = \mathrm{im}\mathcal{T}_0$.

We thus have the following natural L^2-orthogonal splitting of $T_{(A,\psi)}\mathcal{C}^*$:

$$\Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) = \underbrace{(\ker \mathcal{T}_0^* \cap \ker \mathcal{T}_1)}_{\cong T_{[A,\psi]}\mathfrak{M}} \oplus \mathrm{im}\mathcal{T}_0 \oplus \mathrm{im}\mathcal{T}_1^* \ . \tag{2.2.8}$$

Thus by restriction to the orthogonal direct summand $T_{[A,\psi]}\mathfrak{M} \cong \ker \mathcal{T}_0^* \cap \ker \mathcal{T}_1$, we get a natural L^2-metric on the moduli space $\mathfrak{M}$, which we call the *quotient L^2-metric*. The curvature of $\mathfrak{M}$ in this L^2-metric can be computed using the O'Neill formula and the Gauss equation together with the above identifications of the orthogonal projectors, see section 2.5 below.

2.3 The quotient L^2-metric on the Seiberg-Witten bundle

In this section, we construct a natural L^2-metric on the total space $\mathfrak{P}$ of the Seiberg-Witten bundle. This L^2-metric is constructed in much the same way as the one on the moduli space by only replacing the orthogonal splitting (2.2.8): here we must split the tangent space of $\mathfrak{P}$ from $\Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+)$. To identify the tangent space of $\mathfrak{P}$ in such a way, that this identification automatically yields an orthogonal splitting, we construct a new representative of the isomorphism class of $U(1)$-bundles $\mathfrak{P} \to \mathfrak{M}$, which is more natural from the point of view of the geometry of the L^2-metric as the representative $\mathfrak{P} = \widetilde{\mathfrak{M}}/\mathcal{G}_{x_0}$ induced by the based gauge group $\mathcal{G}_{x_0}$. For this construction, we need to assume that the underlying manifold M is simply connected.

In the representation of the Seiberg-Witten bundle $\mathfrak{P} \to \mathfrak{M}$ by $\widetilde{\mathfrak{M}}/\mathcal{G}_{x_0} \to \mathfrak{M}$, the $U(1)$-action on $\mathfrak{P} \to \mathfrak{M}$ comes from the action of the constant gauge transformations $U(1) \subset \mathcal{G}$ on $\widetilde{\mathfrak{M}}$. Thus to construct the natural quotient L^2-metric on $\mathfrak{P}$ we need to split the gauge group into constant and non-constant gauge transformations. The splitting provided by the based gauge group is not appropriate in our case, since the Lie algebra of the based gauge group

$$\mathfrak{g}_{x_0} = \mathrm{Lie}\mathcal{G}_{x_0} = \{if \in \Omega^0(M; i\mathbb{R}) \mid f(x_0) = 0\}$$

is L^2-dense in the Lie algebra $\mathfrak{g} = \mathrm{Lie}\mathcal{G} = \Omega^0(M; i\mathbb{R})$ of the full gauge group. Thus it is not topologically complemented with respect to the L^2-topology.

However, the Lie algebra $\mathfrak{g} = \Omega^0(M; i\mathbb{R})$ splits naturally as the orthogonal direct sum of the constant functions and those functions, which integrate to 0 with respect to the volume form induced by the fixed Riemannian metric g:

$$\mathfrak{g} = \Omega^0(M; i\mathbb{R}) = i\mathbb{R} \oplus \left\{ if \in \Omega^0(M; i\mathbb{R}) \;\middle|\; \tfrac{1}{\mathrm{vol}(M)} \int_M if \, dv_g = 0 \right\} \ .$$

This splitting can be realised via a splitting of the gauge group $\mathcal{G}$ itself:

2.3.1 DEFINITION. *The reduced gauge group is the subgroup $\mathcal{G}_\infty \subset \mathcal{G}$ of all gauge transformations $u \in \mathcal{G}$, which satisfy*

$$\exp\left(\frac{1}{\mathrm{vol}(M)}\int_M \log u\, dv_g\right) = 1 \,. \tag{2.3.1}$$

The condition in (2.3.1) actually defines a subgroup of $\mathcal{G}$, since obviously $\mathbf{1} \in \mathcal{G}_\infty$ and for $u_1, u_1 \in \mathcal{G}_\infty$ we find:

$$\begin{aligned}\exp\left(\frac{1}{\mathrm{vol}(M)}\int_M \log(u_1\cdot u_2)\, dv\right) &= \exp\left(\frac{1}{\mathrm{vol}(M)}\int_M \log u\, dv_g + \frac{1}{\mathrm{vol}(M)}\int_M \log u_2\, dv_g\right)\\ &= \exp\left(\frac{1}{\mathrm{vol}(M)}\int_M \log u_1\, dv_g\right)\cdot \exp\left(\frac{1}{\mathrm{vol}(M)}\int_M \log u_2\, dv_g\right) \,.\end{aligned}$$

The gauge group $\mathcal{G}$ splits algebraically as the product $\mathcal{G} = U(1) \times \mathcal{G}_\infty$. The splitting is in fact a topological splitting in the sense of Fréchet-Lie groups. The defining condition of the reduced gauge group $\mathcal{G}_\infty$ is made such that its Lie algebra

$$\mathfrak{g}_\infty = \mathrm{Lie}\mathcal{G}_\infty = \left\{ if \in \Omega^0(M; i\mathbb{R}) \mid \tfrac{1}{\mathrm{vol}(M)}\textstyle\int_M if\, dv_g = 0 \right\} \,.$$

is the orthogonal complement of the Lie algebra $i\mathbb{R}$ of $U(1)$ in $\mathfrak{g}$. Thus the Lie algebra of the full gauge group splits L^2-orthogonally as:

$$\mathfrak{g} = \Omega^0(M; i\mathbb{R}) = i\mathbb{R} \oplus \mathfrak{g}_\infty \,.$$

The quotient of the premoduli space $\widetilde{\mathfrak{M}}$ by the reduced gauge group $\mathcal{G}_\infty$ yields another natural $U(1)$-bundle over the moduli space $\mathfrak{M}$. The $\mathcal{G}_\infty$ equivalence class of a monopole $(A, \psi) \in \widetilde{\mathfrak{M}}$ will be denoted by $[A, \psi]_\infty$. Thus we have constructed another $U(1)$-bundle on $\mathfrak{M}$, and we claim:

2.3.2 LEMMA. *The $U(1)$-bundle $\widetilde{\mathfrak{M}}/\mathcal{G}_\infty \to \mathfrak{M}$ represents the isomorphism class $\mathfrak{P} \to \mathfrak{M}$. Thus for any $x_0 \in M$, the $U(1)$-bundles $\widetilde{\mathfrak{M}}/\mathcal{G}_\infty \to \mathfrak{M}$ and $\widetilde{\mathfrak{M}}/\mathcal{G}_{x_0} \to \mathfrak{M}$ are isomorphic.*

Proof: We define two representations $\varrho_{x_0}, \varrho_\infty : \mathcal{G} \to U(1)$, whose kernels are the subgroups $\mathcal{G}_{x_0}$ resp. $\mathcal{G}_\infty$. We show that the bundles $\widetilde{\mathfrak{M}}/\mathcal{G}_{x_0} \to \mathfrak{M}$ resp. $\widetilde{\mathfrak{M}}/\mathcal{G}_\infty \to \mathfrak{M}$ are associated from the principal $\mathcal{G}$ bundle $\widetilde{\mathfrak{M}} \to \mathfrak{M}$ via the representations $\varrho_{x_0}, \varrho_\infty$. Then a homotopy of representations from ϱ_{x_0} to ϱ_∞ yields a homotopy of the associated principal bundles. This implies that $\widetilde{\mathfrak{M}}/\mathcal{G}_{x_0}$ and $\widetilde{\mathfrak{M}}/\mathcal{G}_\infty$ have the same first Chern class and thus are isomorphic.

We define the representations $\varrho_{x_0}, \varrho_\infty : \mathcal{G} \to U(1)$ by:

$$\begin{aligned}\varrho_{x_0}(u) &:= u(x_0)\\ \varrho_\infty(u) &:= \exp\left(\tfrac{1}{\mathrm{vol}(M)}\textstyle\int_M \log u\, dv_g\right)\end{aligned}$$

Obviously we have $\ker \varrho_{x_0} = \mathcal{G}_{x_0}$ and $\ker \varrho_\infty = \mathcal{G}_\infty$.

In section 1.2, we defined the Seiberg-Witten bundle $\mathfrak{P} \to \mathfrak{M}$ as the quotient $\widetilde{\mathfrak{M}}/\mathcal{G}_{x_0}$ of the premoduli space $\widetilde{\mathfrak{M}}$ by the based gauge group $\mathcal{G}_{x_0}$. It was shown, that this bundle is canonically isomorphic to the restriction to $\mathfrak{M} \subset \mathcal{B}^*$ of the bundle:

$$i_{x_0}^* \mathcal{L}_{\mathcal{B}^*\times M} = \big(\mathcal{C}^* \times \{x_0\} \times U(1)\big)/\mathcal{G} \to \mathcal{B}^* \,.$$

Here $\mathcal{G}$ acts on $U(1)$ via the representation $\varrho_{x_0} : \mathcal{G} \to U(1), u \mapsto u(x_0)$. Thus this bundle is the $U(1)$-bundle associated from the principal $\mathcal{G}$-bundle $\mathcal{C}^* \to \mathcal{B}^*$ via the representation ϱ_{x_0}:

$$\imath_{x_0}^* \mathcal{L}_{\mathcal{B}^* \times M} \cong \mathcal{C}^* \times_{\varrho_{x_0}} U(1) \ .$$

The isomorphism can be made explicit using the equivariant map:

$$\begin{array}{ccc} \mathcal{C}^* \times \{x_0\} \times U(1) & \to & \mathcal{C}^* \\ \big((A,\psi),\lambda\big) & \mapsto & (A, \lambda^{-1} \cdot \psi) \ . \end{array}$$

In the same way, we see that the bundle $\mathcal{C}/\mathcal{G}_\infty \to \mathcal{B}^*$ is the $U(1)$-bundle associated from the principal $\mathcal{G}$-bundle $\mathcal{C}^* \to \mathcal{B}^*$ via the representation ϱ_∞, i.e.:

$$\mathcal{C}^*/\mathcal{G}_\infty \cong \mathcal{C}^* \times_{\varrho_\infty} U(1) \ .$$

Thus our $U(1)$-bundles on the moduli space $\mathfrak{M}$ can by represented as associated $U(1)$-bundles from the principal $\mathcal{G}$-bundle $\widetilde{\mathfrak{M}} \to \mathfrak{M}$ via the representations $\varrho_{x_0}, \varrho_\infty$:

$$\begin{array}{ccc} \widetilde{\mathfrak{M}}/\mathcal{G}_{x_0} & \cong & \widetilde{\mathfrak{M}} \times_{\varrho_{x_0}} U(1) \\ \widetilde{\mathfrak{M}}/\mathcal{G}_\infty & \cong & \widetilde{\mathfrak{M}} \times_{\varrho_\infty} U(1) \ . \end{array}$$

To construct a homotopy of $U(1)$-bundles from $\widetilde{\mathfrak{M}} \times_{\varrho_{x_0}} U(1)$ to $\widetilde{\mathfrak{M}} \times_{\varrho_\infty} U(1)$ it suffices to construct a homotopy of representations from ϱ_{x_0} to ϱ_∞. We define such a homotopy H as follows:

$$\begin{array}{rccl} H: & \mathcal{G} \times [0,1] & \to & U(1) \\ & (u,t) & \mapsto & \begin{cases} u(x_0) & t = 0 \\ \exp\left(\int_M \rho_t \cdot \log u \, dv_g\right) & t \in (0,1] \end{cases} \end{array}$$

The family of functions ρ_t can be constructed explicitly such that for any function f the integral $\int_M \gamma_t \cdot f \, dv_g$ converges to $f(x_0)$ when t tends to 0 and such that $\rho_t \equiv \frac{1}{\mathrm{vol}(M)}$ for t near 1. To make this family ρ_t explicit, we fix an $\epsilon > 0$ with $2\epsilon < \min\{1, \mathrm{inj}(M,g)\}$, where $\mathrm{inj}(M,g)$ denotes the injectivity radius of (M,g). On the interval $(0,\epsilon]$ we smooth out the Dirac distribution δ_{x_0}, on $[\epsilon, 2\epsilon]$ we take the convex combination to the constant function $\frac{1}{\mathrm{vol}(M)}$, and on $[2\epsilon, 1]$ we take the constant family $\rho_t \equiv \frac{1}{\mathrm{vol}(M)}$.

For the smoothing of the Dirac distribution we take $\gamma : \mathbb{R} \to \mathbb{R}$ to be a nonnegative smooth function, constant near 0 with support in $[-1,1]$ such that $\int_{\mathbb{R}^4} \gamma(|x|) dx = 1$. For a small $t > 0$, we can now define the function ρ_t as $\frac{1}{t^4} \cdot \gamma\left(\frac{\mathrm{dist}(x_0,x)}{t}\right)$, where $\mathrm{dist}(x_0,x)$ denotes the Riemannian distance of x_0 and x. Thus the family $\rho_t : M \to \mathbb{R}$ with $t \in (0,1]$ is defined as follows:

$$\rho_t(x) := \begin{cases} \frac{1}{t^4} \cdot \gamma\left(\frac{\mathrm{dist}(x_0,x)}{t}\right) & t \in (0,\epsilon] \\ \left(\frac{-t}{\epsilon} + 2\right) \cdot \frac{1}{t^4} \cdot \gamma\left(\frac{\mathrm{dist}(x_0,x)}{t}\right) + \left(\frac{t}{\epsilon} - 1\right) \cdot \frac{1}{\mathrm{vol}(M)} & t \in [\epsilon, 2\epsilon] \\ \frac{1}{\mathrm{vol}(M)} & t \in [2\epsilon, 1] \end{cases}$$

By construction, the integral $\int_M \rho_t \cdot \log u \, dv_g$ tends to $u(x_0)$ as t tends to zero, thus the homotopy H as defined above is continuous in t and satisfies $H_0 = \varrho_{x_0}$. Since $\rho_t \equiv \frac{1}{\mathrm{vol}(M)}$ near $t = 1$, we also have $H_1 = \varrho_\infty$. Thus H is a homotopy from ϱ_{x_0} to ϱ_∞ as claimed. By construction, for any $t \in [0,1]$, the map $H_t : \mathcal{G} \to U(1)$ is a representation.

The homotopy of representations $H : \mathcal{G} \times [0,1] \to U(1)$ defines a homotopy

$$\widehat{\mathfrak{P}} \to \mathfrak{M} \times [0,1] \, , \qquad \widehat{\mathfrak{P}}_t := \widetilde{\mathfrak{M}} \times_{H_t} U(1)$$

of $U(1)$-bundles over $\mathfrak{M}$ from $\widehat{\mathfrak{P}}_0 = \widetilde{\mathfrak{M}} \times_{\varrho_{x_0}} U(1) \cong \widetilde{\mathfrak{M}}/\mathcal{G}_{x_0}$ to $\widehat{\mathfrak{P}}_1 = \widetilde{\mathfrak{M}} \times_{\varrho_\infty} U(1) \cong \widetilde{\mathfrak{M}}/\mathcal{G}_\infty$. This implies that these bundles have the same first Chern number and are consequently isomorphic. Thus the bundle $\widetilde{\mathfrak{M}}/\mathcal{G}_\infty$ represents the class of $U(1)$-bundles $\mathfrak{P} \to \mathfrak{M}$ as claimed. □

Now we can construct an L^2-metric on the total space $\mathfrak{P}$ in much the same way as we did in section 2.2 for the moduli space $\mathfrak{M}$: We identify the tangent space $T_{[A,\psi]_\infty}\mathfrak{P}$ as the intersection of the kernel of $\mathcal{T}_1$ with the orthogonal complement of the $\mathcal{G}_\infty$-orbit through (A,ψ). Then we split the space $\Omega^1(M;i\mathbb{R}) \times \Gamma(\Sigma^+)$ into $T_{[A,\psi]_\infty}\mathfrak{P}$ and its orthogonal complement.

The reduced gauge group $\mathcal{G}_\infty$ is a Fréchet-Lie subgroup of the full gauge group $\mathcal{G}$ and its Lie algebra $\mathfrak{g}_\infty$ is a tame direct summand of $\mathfrak{g}$. Thus we get a slice theorem for the action of $\mathcal{G}_\infty$ on $\mathcal{C}^*$ in the tame smooth category in the same way as for the Fréchet-Lie group $\mathcal{G}$. When $\mathcal{S}_{(A,\psi)}$ is a local slice for the full gauge group $\mathcal{G}$, then $U(1) \cdot \mathcal{S}_{(A,\psi)}$ is a local slice for the reduced gauge group $\mathcal{G}_\infty$. The linearisation of the orbit map for $\mathcal{G}_\infty$ is the restriction to $\mathfrak{g}_\infty$ of the linearisation $\mathcal{T}_0$ of the orbit map for $\mathcal{G}$. The linearisations of the orbit map and of the Seiberg-Witten map fit together into the complex:

$$0 \longrightarrow \mathfrak{g}_\infty \xrightarrow{\mathcal{T}_0|_{\mathfrak{g}_\infty}} \Omega^1(M;i\mathbb{R}) \times \Gamma(\Sigma^+) \xrightarrow{\mathcal{T}_1} \Omega^2_+(M;i\mathbb{R}) \times \Gamma(\Sigma^-) \longrightarrow 0 \qquad \mathcal{K}^{(\infty)}_{(A,\psi)}$$

The adjoint $\mathcal{T}_0|^*_{\mathfrak{g}_\infty}$ of the restriction $\mathcal{T}_0|_{\mathfrak{g}_\infty}$ is the composition of the adjoint of $\mathcal{T}_0$ with the orthogonal projection to $\mathfrak{g}_\infty$:

$$\mathcal{T}_0|^*_{\mathfrak{g}_\infty} : \ \Omega^1(M;i\mathbb{R}) \times \Gamma(\sigma^+) \to \mathfrak{g}_\infty = \left\{ if \in \Omega^0(M;i\mathbb{R}) \ \middle| \ \int_M if dv_g = 0 \right\}$$

$$\mathcal{T}_0|^*_{\mathfrak{g}_\infty}(\cdot) = \mathcal{T}_0^*(\cdot) - \tfrac{1}{\mathrm{vol}(M)} \int_M \mathcal{T}_0^*(\cdot)\, dv_g$$

Thus the kernel of $\mathcal{T}_0|^*_{\mathfrak{g}_\infty}$ is the set of those linearised configurations, which are mapped to $i\mathbb{R}$ under $\mathcal{T}_0^*$:

$$\ker \mathcal{T}_0|^*_{\mathfrak{g}_\infty} = \left\{ \begin{pmatrix} \nu \\ \phi \end{pmatrix} \ \middle| \ \mathcal{T}_0^* \left(\begin{pmatrix} \nu \\ \phi \end{pmatrix} \right) \in i\mathbb{R} \right\} = (\mathcal{T}_0^*)^{-1}(i\mathbb{R}) \, .$$

As in section 2.2, we derive from the complex $\mathcal{K}^{(\infty)}_{(A,\psi)}$ the following L^2-orthogonal elliptic splitting:

$$\begin{aligned} \Omega^1(M;i\mathbb{R}) \times \Gamma(\Sigma^+) &= (\ker \mathcal{T}_0|^*_{\mathfrak{g}_\infty} \cap \ker \mathcal{T}_1) \oplus \mathrm{im}\mathcal{T}_0|_{\mathfrak{g}_\infty} \oplus \mathrm{im}\mathcal{T}_1^* \\ &= ((\mathcal{T}_0^*)^{-1}(i\mathbb{R}) \cap \ker \mathcal{T}_1) \oplus \mathcal{T}_0(\mathfrak{g}_\infty) \oplus \mathrm{im}\mathcal{T}_1^* \, . \end{aligned} \tag{2.3.2}$$

To define the quotient L^2-metric on $\mathfrak{P}$, we identify the tangent space

$$T_{[A,\psi]}\mathfrak{P} = T_{[A,\psi]}\left(\widetilde{\mathfrak{M}}/\mathcal{G}_\infty \right) = \ker \mathcal{T}_1 / \mathrm{im}\mathcal{T}_0|_{\mathfrak{g}_\infty} = \ker \mathcal{T}_1 / \mathcal{T}_0(\mathfrak{g}_\infty)$$

via the splitting splitting (2.3.2) with the orthogonal complement of $\mathcal{T}_0(\mathfrak{g}_\infty)$ in $\ker \mathcal{T}_1$:

$$T_{[A,\psi]}\mathfrak{P} \cong \big((\mathcal{T}_0(\mathfrak{g}_\infty))^\perp \subset \ker \mathcal{T}_1\big) = \ker \mathcal{T}_0|^*_{\mathfrak{g}_\infty} \cap \ker \mathcal{T}_1 = (\mathcal{T}_0^*)^{-1}(i\mathbb{R}) \cap \ker \mathcal{T}_1 \ .$$

This identification together with the orthogonal splitting (2.3.2) defines a natural Riemannian metric on $\widetilde{\mathfrak{M}}/\mathcal{G}_\infty \cong \mathfrak{P}$, which will be called the *quotient L^2-metric on $\mathfrak{P}$*.

2.3.3 Lemma. *The bundle projection $\mathfrak{P} \to \mathfrak{M}$ is a Riemannian submersion with respect to the quotient L^2-metrics.*

Proof: To show that the bundle projection is a Riemannian submersion, we need to identify the tangent space in $[A,\psi]_\infty$ of the fibre $\mathfrak{P}_{[A,\psi]}$ over $[A,\psi]$ inside the tangent space $T_{[A,\psi]_\infty}\mathfrak{P}$. Then we must show that $T_{[A,\psi]}\mathfrak{P}$ splits orthogonally into the tangent space of $\mathfrak{P}_{[A,\psi]}$ and the tangent space of $\mathfrak{M}$, and that the bundle projection $\pi : T_{[A,\psi]_\infty}\mathfrak{P} \to T_{[A,\psi]}\mathfrak{M}$ is orthogonal with respect to that splitting.

Since $U(1)$ acts on the bundle $\widetilde{\mathfrak{M}}/\mathcal{G}_\infty \to \mathfrak{M}$ via the standard gauge action of $U(1) \subset \mathcal{G}$ on $\widetilde{\mathfrak{M}}$, the tangent space of the $U(1)$-orbit through $[A,\psi]_\infty \in \widetilde{\mathfrak{M}}/\mathcal{G}_\infty$ is the image of $\mathcal{T}_0(i\mathbb{R})$ under the quotient map $\ker \mathcal{T}_1 \to \ker \mathcal{T}_1/\mathcal{T}_0(\mathfrak{g}_\infty)$. Thus in our model $\ker \mathcal{T}_1/\mathcal{T}_0(\mathfrak{g}_\infty) \cong \ker \mathcal{T}_0|^*_{\mathfrak{g}_\infty} \cap \ker \mathcal{T}_1$, the tangent space of the fibre $\mathfrak{P}_{[A,\psi]}$ over $[A,\psi]$ is the image of $\mathcal{T}_0(i\mathbb{R})$ under the orthogonal projection $\pi^\perp : \ker \mathcal{T}_1 \to \ker \mathcal{T}_0|^*_{\mathfrak{g}_\infty} \cap \ker \mathcal{T}_1$. Since $\ker \mathcal{T}_0^* \subset \mathcal{T}_0|^*_{\mathfrak{g}_\infty}$, the projection $\pi^\perp$ is the identity on

$$T_{[A,\psi]}\mathfrak{M} \cong \ker \mathcal{T}_0^* \cap \ker \mathcal{T}_1 \subset \ker \mathcal{T}_0|^*_{\mathfrak{g}_\infty} \cap \ker \mathcal{T}_1 \cong T_{[A,\psi]_\infty}\mathfrak{P} \ .$$

Since $\mathcal{T}_0(i\mathbb{R})$ is orthogonal to $\ker \mathcal{T}_0^* \cap \mathcal{T}_1$, its image under $\pi^\perp$ stays orthogonal to $\ker \mathcal{T}_0^* \cap \ker \mathcal{T}_1$. Thus the tangent space $T_{[A,\psi]_\infty}\mathfrak{P}_\infty$ can be identified with the orthogonal complement of $T_{[A,\psi]}\mathfrak{M} \cong \ker \mathcal{T}_0^* \cap \ker \mathcal{T}_1$ in $T_{[A,\psi]_\infty}\mathfrak{P} \cong \ker \mathcal{T}_0|^*_{\mathfrak{g}_\infty} \cap \ker \mathcal{T}_1$. This orthogonal complement can be made explicit using the 0-th order Green operator G_0 of the elliptic complex $\mathcal{K}_{(A,\psi)}$. Namely, the image of $\mathcal{T}_0$ splits L^2-orthogonally as $\mathrm{im}\mathcal{T}_0 = \mathcal{T}_0 \circ G_0(i\mathbb{R}) \oplus \mathcal{T}_0(\mathfrak{g}_\infty)$, and consequently the orthogonal complement of $\ker \mathcal{T}_0^* \cap \mathcal{T}_1$ in $(\mathcal{T}_0)^{-1}(i\mathbb{R}) \cap \ker \mathcal{T}_1$ is $\mathcal{T}_0 \circ G_0(i\mathbb{R})$. We thus obtain the following L^2-orthogonal splitting:

$$\begin{aligned}\Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) &= ((\mathcal{T}_0^*)^{-1}(i\mathbb{R}) \cap \ker \mathcal{T}_1) \oplus \mathcal{T}_0(\mathfrak{g}_\infty) \oplus \mathrm{im}\mathcal{T}_1^* \\ &= \underbrace{\underbrace{(\ker \mathcal{T}_0^* \cap \ker \mathcal{T}_1)}_{\cong T_{[A,\psi]}\mathfrak{M}} \oplus \underbrace{\mathcal{T}_0 \circ G_0(i\mathbb{R})}_{\cong T_{[A,\psi]_\infty}\mathfrak{P}_{[A,\psi]}}}_{\cong T_{[A,\psi]_\infty}\mathfrak{P}} \oplus \mathcal{T}_0(\mathfrak{g}_\infty) \oplus \mathrm{im}\mathcal{T}_1^* \ .\end{aligned}$$

Thus the tangent space of $\mathfrak{P}$ splits as:

$$T_{[A,\psi]_\infty}\mathfrak{P} = (\ker \mathcal{T}_0^* \cap \mathcal{T}_1) \oplus \mathcal{T}_0 \circ G_0(i\mathbb{R}) \cong T_{[A,\psi]}\mathfrak{M} \oplus T_{[A,\psi]_\infty}\mathfrak{P}_{[A,\psi]} \ .$$

It follows that the restriction to $\ker \mathcal{T}_0^* \cap \ker \mathcal{T}_1$ of the linearisation of the bundle projection $\mathfrak{P} \to \mathfrak{M}$ is an isometry onto $T_{[A,\psi]}\mathfrak{M}$. Hence the bundle projection is a Riemannian submersion as claimed. □

From the identification of the tangent space $T_{[A,\psi]_\infty}\mathfrak{P}_{[A,\psi]}$ of the fibre over $[A,\psi]$ with $\mathcal{T}_0 \circ G_0(i\mathbb{R})$, we deduce that the fundamental vector field $\widetilde{X} \in \mathfrak{X}(\mathfrak{P})$ induced by an element $X \in \mathrm{Lie}U(1) = i\mathbb{R}$ is given by:

$$\widetilde{X}_{[A,\psi]_\infty} = \big(\mathcal{T}_0 \circ G_0\big)_{(A,\psi)}(X) \ ,$$

where the subscript indicates the dependence of the operators $\mathcal{T}_0$ and G_0 on the monopole (A,ψ).

2.4 The curvature of the quotient L^2-metric on $\mathcal{B}^*$

In this section we compute an explicit formula for the sectional curvature of the quotient L^2-metric on the space $\mathcal{B}^*$ of gauge equivalence classes of irreducible configurations in terms of the Green operators of the elliptic complex $\mathcal{K}_{(A,\psi)}$ using the O'Neill formula for the Riemannian submersion $\mathcal{C}^* \to \mathcal{B}^*$.

The O'Neill formula expresses the sectional curvature of the target space $\mathcal{B}^*$ of a Riemannian submersion $\mathcal{C}^* \to \mathcal{B}^*$ as the sectional curvature of the source plus a positive correction term in the commutator of horizontal extensions to the source $\overline{X}, \overline{Y} \in \mathfrak{X}(\mathcal{C})$ of vector fields $X, Y \in \mathfrak{X}(\mathcal{B})$ on the target space. The tangent space in $[A,\psi]$ of the quotient $\mathcal{B}^* = \mathcal{C}^*/\mathcal{G}$ can naturally be identified with the horizontal space $\mathcal{H}_{(A,\psi)} = \ker \mathcal{T}_0^*$. Given two tangent vectors $X_0, Y_0 \in \mathcal{H}_{(A,\psi)}$, represented as linearised configurations by:

$$X_0 = \begin{pmatrix} \nu^X \\ \phi^X \end{pmatrix} \qquad \text{resp.} \qquad Y_0 = \begin{pmatrix} \nu^Y \\ \phi^Y \end{pmatrix} \in \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) \,,$$

there is a rather simple way to extend them to horizontal vector fields $\overline{X}, \overline{Y}$ on $\mathcal{C}$: we first take the constant extension and then project onto the horizontal bundle:

$$\overline{X}_{(A,\psi)} := \mathrm{hor}_{(A,\psi)}(X_0) \qquad \text{resp.} \qquad \overline{Y}_{(A,\psi)} := \mathrm{hor}_{(A,\psi)}(Y_0) \,. \tag{2.4.1}$$

As the proof of the O'Neill formula (see e.g. [10]) relies only on the algebraic properties of the curvature (such as the Koszul formula) and on the submersion properties the formula holds true even in the infinite dimensional case of the Riemannian submersion $\mathcal{C}^* \to \mathcal{B}^*$. For $X_0, Y_0 \in T_{[A,\psi]}\mathcal{B}^* = \ker \mathcal{T}_0^*$, the O'Neill formula reads:

$$\big(R^{\mathcal{B}^*}(X,Y)Y,X\big)_{[A,\psi]} = \big(R^{\mathcal{C}^*}(\overline{X},\overline{Y})\overline{Y},\overline{X}\big)_{(A,\psi)} + \frac{3}{4}\left\|\mathrm{vert}_{(A,\psi)}[\overline{X},\overline{Y}]_{(A,\psi)}\right\|^2 . \tag{2.4.2}$$

We will now compute the terms of this formula using the Levi-Civita connection of the L^2-metric and the formulae (2.2.6) for the orthogonal projectors $\mathrm{vert}_{(A,\psi)}$ and $\mathrm{hor}_{(A,\psi)}$. Since ∇ is torsionfree, we have:

$$[\overline{X},\overline{Y}]_{(A,\psi)} = \nabla_{X_0}\overline{Y} - \nabla_{Y_0}\overline{X} \,.$$

We may thus compute the commutator term in (2.4.2) via the covariant derivatives:

$$\begin{aligned} \big(\nabla_{X_0}\overline{Y}\big)_{(A,\psi)} &= \frac{d}{dt}\Big|_0 \overline{Y}\big(\,(A,\psi) + t\cdot X_0\,\big) \\ &= \frac{d}{dt}\Big|_0 \mathrm{hor}_{(A,\psi)+t\cdot X_0}(Y_0) \\ &= \frac{d}{dt}\Big|_0 \Big(Y - \big\{\mathcal{T}_0(t)\circ G_0(t)\circ \mathcal{T}_0^*(t)\big\}Y_0\Big) \,, \end{aligned}$$

where the variable t indicates, that the linearisations and Green operators are taken in the point $(A,\psi) + t\cdot X_0$. At the initial point $t=0$, we just write $\mathcal{T}_0$ etc. instead of $\mathcal{T}_0(t=0)$. Recall that Y_0 was supposed to be tangent to $\mathcal{B}^*$, i.e. $Y_0 \in \ker \mathcal{T}_0^*$. Hence using the product rule we need only differentiate the operator next to Y_0, and we thus get:

$$\begin{aligned} &= -\frac{d}{dt}\Big|_0 \big\{\mathcal{T}_0\circ G_0\circ \mathcal{T}_0^*(t)\big\}(Y_0) \\ &= -\frac{d}{dt}\Big|_0 \big\{\mathcal{T}_0\circ G_0\big\}\big(2d^*\nu^Y + i\mathrm{Im}\langle \psi + t\cdot\phi^X, \phi^Y\rangle\big) \\ &= -\mathcal{T}_0\circ G_0\; i\mathrm{Im}\langle \phi^X, \phi^Y\rangle \,. \end{aligned}$$

Consequently, the commutator reads:

$$[\overline{X}, \overline{Y}]_{(A,\psi)} = -2\mathcal{T}_0 \circ G_0 \, i\mathrm{Im}\langle \phi^X, \phi^Y \rangle.$$

This term is already vertical, since the vertical bundle is $\mathcal{V} = \mathrm{im}\mathcal{T}_0$. Recall that the L^2-metric on $\mathcal{C}^*$ is flat, so that we find for the sectional curvature of the space $\mathcal{B}^*$ of gauge equivalence classes of irreducible connections:

$$\begin{aligned}\left(R^{\mathcal{B}^*}(X,Y)X,Y\right)_{[A,\psi]} &= \frac{3}{4}\left\| -2\mathcal{T}_0 \circ G_0 \, i\mathrm{Im}\langle \phi^X, \phi^Y \rangle \right\|^2_{L^2} \\ &= 3\left(\mathcal{T}_0 \circ G_0 \, i\mathrm{Im}\langle \phi^X, \phi^Y \rangle \, , \, \mathcal{T}_0 \circ G_0 \, i\mathrm{Im}\langle \phi^X, \phi^Y \rangle \right)_{L^2} \\ &= 3\left(\mathcal{T}_0^* \circ \mathcal{T}_0 \circ G_0 \, i\mathrm{Im}\langle \phi^X, \phi^Y \rangle \, , \, G_0 \, i\mathrm{Im}\langle \phi^X, \phi^Y \rangle \right)_{L^2} \\ &= 3\big(i\mathrm{Im}\langle \phi^X, \phi^Y \rangle \, , \, G_0 \, i\mathrm{Im}\langle \phi^X, \phi^Y \rangle \big)_{L^2} . \end{aligned} \tag{2.4.3}$$

Unfortunately, one knows too little about these nonlocal Green operators, to be able to draw direct consequences out of formulae of this type. The best one can hope for, is that some regularisation techniques allow to compute e.g. regularised traces of these operators or that one can compute the terms more explicitly in special situations. Much the same problem arises, when one considers the L^2-metric on the Yang-Mills-moduli spaces, and there had been some work in these directions:

MAEDA, ROSENBERG and TONDEUR used regularisation techniques to compute an infinite dimensional analogue of the mean curvature of gauge orbits. In [32, 33, 34], they have shown that the regularised mean curvature of the gauge orbit of an irreducible monopole vanishes if and only if the regularised volume of this gauge orbit is extremal among nearby orbits. This is an infinte dimensional version of a theorem of HSIANG, which says that orbits of a compact connected Lie group acting isometrically on a Riemannian manifold are minimal submanifolds if and only if their volume is extremal among nearby orbits.

GROISSER and PARKER used the identification of the Yang-Mills $SU(2)$ moduli space on S^4 with instanton number 1 with the hyperbolic 5-space to compute the curvature of the L^2-metric in the standard instanton A_0 explicitly, see [19, 20]. They found, that the L^2-metric is *not* the standard hyperbolic metric, but that the curvature in the standard instanton A_0 is $\frac{5}{16\pi^2} > 0$.

2.5 The curvature of the (quotient) L^2-metric on the (pre-)moduli space

In this section, we compute explicit formulae for the sectional curvature of the (quotient) L^2-metric on the premoduli space $\widetilde{\mathfrak{M}}$ resp. the moduli space $\mathfrak{M}$ in terms of the Green operators of the elliptic complex $\mathcal{K}_{(A,\psi)}$ using the Gauss equation for the embedding $\widetilde{\mathfrak{M}} \hookrightarrow \mathcal{C}^*$ resp. $\mathfrak{M} \hookrightarrow \mathcal{B}^*$.

The curvature of the premoduli space $\widetilde{\mathfrak{M}}$

The Gauss equation expresses the sectional curvature of a submanifold with the induced metric in terms of the sectional curvature of the ambient space and the second fundamental form of the embedding. Note that the proof of the Gauss equation (see e.g. [30, 41]) relies only on the algebraic properties of the Riemannian curvature tensor and on the definitions of the induced Levi-Civita connection on the submanifold and of the second fundamental form. Those can easily be defined in our case using the orthogonal projections onto the tangent resp. normal space of the submanifolds as discussed in section 2.2 above. Thus, the Gauss equation holds true even for the L^2-metric on the embedding $\widetilde{\mathfrak{M}} \hookrightarrow \mathcal{C}^*$ resp. $\mathfrak{M} \hookrightarrow \mathcal{B}^*$. For the premoduli space $\widetilde{\mathfrak{M}}$ the Gauss equation reads:

$$\begin{aligned}\big(R^{\widetilde{\mathfrak{M}}}(X,Y)Y,X\big)_{(A,\psi)} &= \big(R^{\mathcal{C}^*}(\overline{X},\overline{Y})\overline{Y},\overline{X}\big)_{(A,\psi)} \\ &\quad -\big(\mathrm{II}(X,X),\mathrm{II}(Y,Y)\big)_{(A,\psi)} + \big(\mathrm{II}(X,Y),\mathrm{II}(X,Y)\big)_{(A,\psi)} , \end{aligned} \tag{2.5.1}$$

where the second fundamental form is defined as $\mathrm{II}(X,Y)_{(A,\psi)} := \big(\mathrm{nor}_{(A,\psi)}(\nabla_X \overline{Y})\big)$. In order to compute the terms of (2.5.1), we start with tangent vectors $X_0, Y_0 \in T_{(A,\psi)}\widetilde{\mathfrak{M}} = \ker \mathcal{T}_1$, represented as linearised configurations by

$$X_0 = \begin{pmatrix} \nu^X \\ \phi^X \end{pmatrix} \qquad \text{resp.} \qquad Y_0 = \begin{pmatrix} \nu^Y \\ \phi^Y \end{pmatrix} \in \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+)$$

and locally extend them to vector fields $\overline{X}, \overline{Y}$ on $\mathcal{C}$ via:

$$\overline{X}_{(A,\psi)} := \tan_{(A,\psi)}(X_0) \qquad \text{resp.} \qquad \overline{Y}_{(A,\psi)} := \tan_{(A,\psi)}(Y_0) \, .$$

Note that $\overline{X}, \overline{Y}$ are indeed extensions to $\mathcal{C}^*$ of vector fields on $\widetilde{\mathfrak{M}}$: namely, when (A,ψ) is a monopole, then $\overline{X}_{(A,\psi)} \in T_{(A,\psi)}\widetilde{\mathfrak{M}}$. For the covariant derivative $\nabla_{X_0}\overline{Y}$ we find:

$$\begin{aligned}\big(\nabla_{X_0}\overline{Y}\big)_{(A,\psi)} &= \frac{d}{dt}\Big|_0 \overline{Y}\big((A,\psi) + t \cdot X_0 \big) \\ &= \frac{d}{dt}\Big|_0 \tan_{(A,\psi)+t\cdot X_0}(Y_0) \\ &= \frac{d}{dt}\Big|_0 \Big(Y_0 - \big\{\mathcal{T}_1^*(t) \circ G_2(t) \circ \mathcal{T}_1(t)\big\}Y_0\Big)\end{aligned}$$

Recall that Y_0 was supposed to be tangent to $\widetilde{\mathfrak{M}}$, i.e. $Y_0 \in \ker \mathcal{T}_1$. Hence using the product rule we need only differentiate the operator next to Y_0, and we thus get:

$$\begin{aligned} &= -\frac{d}{dt}\Big|_0 \big\{\mathcal{T}_1^* \circ G_2 \circ \mathcal{T}_1(t)\big\}Y_0 \\ &= -\frac{d}{dt}\Big|_0 \mathcal{T}_1^* \circ G_2 \begin{pmatrix} d^+\nu^Y - q(\psi + t\phi^X, \phi^Y) \\ \frac{1}{2}\nu^Y \cdot (\psi + t\phi^X) + \mathcal{D}_{A+t\nu^X}\phi^Y \end{pmatrix} \\ &= -\,\mathcal{T}_1^* \circ G_2 \begin{pmatrix} -q(\phi^X,\phi^Y) \\ \frac{1}{2}\nu^Y \cdot \phi^X + \frac{1}{2}\nu^X \cdot \phi^Y \end{pmatrix} . \end{aligned}$$

Note that this term is already normal, since the normal space in (A,ψ) is $N_{(A,\psi)}\widetilde{\mathfrak{M}} = \mathrm{im}\mathcal{T}_1^*$. We thus get for the second fundamental form terms in the Gauss equation:

$$\begin{aligned}
\big(\mathrm{II}(X,X),\mathrm{II}(Y,Y)\big)_{(A,\psi)} &= \left(\mathcal{T}_1^*\circ G_2\begin{pmatrix} -q(\phi^X,\phi^X) \\ \mu^X\cdot\phi^X \end{pmatrix}, \mathcal{T}_1^*\circ G_2\begin{pmatrix} -q(\phi^Y,\phi^Y) \\ \mu^Y\cdot\phi^Y \end{pmatrix}\right)_{L^2} \\
&= \left(\mathcal{T}_1\circ\mathcal{T}_1^*\circ G_2\begin{pmatrix} -q(\phi^X,\phi^X) \\ \mu^X\cdot\phi^X \end{pmatrix}, G_2\begin{pmatrix} -q(\phi^Y,\phi^Y \\ \mu^Y\cdot\phi^Y \end{pmatrix}\right)_{L^2} \\
&= \left(\begin{pmatrix} -q(\phi^X,\phi^X) \\ \mu^X\cdot\phi^X \end{pmatrix}, G_2\begin{pmatrix} -q(\phi^X,\phi^X) \\ \mu^X\cdot\phi^X \end{pmatrix}\right)_{L^2}
\end{aligned}$$

and analogously:

$$\big(\mathrm{II}(X,Y),\mathrm{II}(X,Y)\big)_{(A,\psi)} = \left(\begin{pmatrix} -q(\phi^X,\phi^Y) \\ \frac{1}{2}\mu^X\cdot\phi^Y+\frac{1}{2}\mu^Y\cdot\phi^X \end{pmatrix}, G_2\begin{pmatrix} -q(\phi^X,\phi^Y) \\ \frac{1}{2}\mu^Y\cdot\phi^X+\frac{1}{2}\mu^X\cdot\phi^Y \end{pmatrix}\right)_{L^2}$$

By equation (2.1.3), the L^2-metric on the configuration is flat. Hence we find for the sectional curvature of the premoduli space the formula:

$$\begin{aligned}
\big(R^{\widetilde{\mathfrak{M}}}(X,Y)Y,X\big)_{(A,\psi)} &= -\big(\mathrm{II}(X,X),\mathrm{II}(Y,Y)\big)_{(A,\psi)} + \big(\mathrm{II}(X,Y),\mathrm{II}(X,Y)\big)_{(A,\psi)} \\
&= -\left(\begin{pmatrix} -q(\phi^X,\phi^X) \\ \nu^X\cdot\phi^X \end{pmatrix}, G_2\begin{pmatrix} -q(\phi^Y,\phi^Y) \\ \nu^Y\cdot\phi^Y \end{pmatrix}\right)_{L^2} \\
&\quad + \left(\begin{pmatrix} -q(\phi^X,\phi^Y) \\ \frac{1}{2}\nu^Y\cdot\phi^X+\frac{1}{2}\nu^X\cdot\phi^Y \end{pmatrix}, G_2\begin{pmatrix} -q(\phi^X,\phi^Y \\ \frac{1}{2}\nu^Y\cdot\phi^X+\frac{1}{2}\nu^X\cdot\phi^Y \end{pmatrix}\right)_{L^2}
\end{aligned}$$

The curvature of the moduli space $\mathfrak{M}$

Now we can do an analoguous computation for the embedding $\mathfrak{M}\hookrightarrow\mathcal{B}^*$ of the moduli space $\mathfrak{M}$ into the space $\mathcal{B}^*$ of gauge equivalence classes of irreducible configurations. Since for the second fundamental form terms we need to compute the covariant derivatives of vector fields on $\mathfrak{M}$ and since the Levi-Civita connections on $\mathcal{B}^*$ resp. $\mathfrak{M}$ are given by orthogonal projections from the Levi-Civita connection on $\mathcal{C}^*$, we can do our computations on $\mathcal{C}^*$. When we consider the family of vector spaces

$$\mathcal{E} = \ker(\mathcal{T}_0^*\oplus\mathcal{T}_1) = \ker\mathcal{T}_0^*\cap\ker\mathcal{T}_1 \to \mathcal{C}^*\ ,$$

then the restriction of $\mathcal{E}$ to the premoduli space $\widetilde{\mathfrak{M}}$ gives a rank d vector bundle naturally isomorphic to the pullback of the tangent bundle of the moduli space: $\mathcal{E}|_{\widetilde{\mathfrak{M}}}\cong\pi^*T\mathfrak{M}$. Note that $\mathcal{E}$ is not a vector bundle over $\mathcal{C}^*$ neither on the whole configuration space $\mathcal{C}$ nor on $\mathcal{C}^*$: a priori it is not clear, whether the dimension of $\ker\mathcal{T}_0^*\oplus\mathcal{T}_1$ is constant on $\mathcal{C}$. The operator $\mathcal{T}_0^*\oplus\mathcal{T}_1 = \mathcal{T}_0^*\oplus\mathcal{T}_{1(A,\psi)}$ is elliptic for every configuration $(A,\psi)\in\mathcal{C}$, and its index d is independent of (A,ψ). This index equals the dimension of $\ker\big(\mathcal{T}_0^*\oplus\mathcal{T}_1\big)$ minus the dimensions of the obstruction spaces. But the cokernel of $\mathcal{T}_1$ vanishes on $\widetilde{\mathfrak{M}}_{\mu^+}$ for a generic perturbation μ^+, and the cokernel of $\mathcal{T}_0^*$ vanishes on $\mathcal{C}^*$.

Given two tangent vectors $X_0, Y_0\in T_{[A_0,\psi_0]}\mathfrak{M}$, we locally extend them to sections of $\mathcal{E}$ via the projections $\mathrm{hor}_{(A,\psi)}$ and $\tan_{(A,\psi)}$ as introduced in section 2.1 above. When X_0, Y_0 are represented as linearised configurations by

$$X_0 = \begin{pmatrix} \nu^X \\ \phi^X \end{pmatrix} \qquad \text{resp.} \qquad Y_0 = \begin{pmatrix} \nu^Y \\ \phi^Y \end{pmatrix} \in \Omega^1(M;i\mathbb{R})\times\Gamma(\Sigma^+)\ ,$$

then we define the extensions $\overline{X}, \overline{Y}$ to sections of $\mathcal{E}$ as:

$$\overline{X}_{(A,\psi)} := \tan_{(A,\psi)} \circ \mathrm{hor}_{(A,\psi)}(X_0) \qquad \text{resp.} \qquad \overline{Y}_{(A,\psi)} := \tan_{(A,\psi)} \circ \mathrm{hor}_{(A,\psi)}(Y_0)\ . \tag{2.5.2}$$

Note that we could also have chosen the orthogonal projectors $\tan_{(A,\psi)}$ and $\mathrm{hor}_{(A,\psi)}$ in reversed order. Thus we should keep in mind the question whether our formulae depend on the choice of the extension.

Now we proceed as above for the premoduli space to compute the terms of the Gauss equation. For the covariant derivatives $\nabla_{X_0}\overline{Y}$ we find:

$$\begin{aligned}
\big(\nabla_{X_0}\overline{Y}\big)_{[A,\psi]} &= \frac{d}{dt}\Big|_0 \overline{Y}\big(\,(A,\psi) + t\cdot X_0\,\big) \\
&= \frac{d}{dt}\Big|_0 \tan_{(A,\psi)+t\cdot X_0} \circ \mathrm{hor}_{(A,\psi)+t\cdot X_0}(Y_0) \\
&= \frac{d}{dt}\Big|_0 \Big(Y_0 - \big\{\mathcal{T}_1^*(t)\circ G_2(t)\circ \mathcal{T}_1(t)\big\}\, Y_0 - \big\{\mathcal{T}_0(t)\circ G_0(t)\circ \mathcal{T}_0^*(t)\big\}\, Y_0 \\
&\qquad + \big\{\mathcal{T}_1^*(t)\circ G_2(t)\circ \mathcal{T}_1(t)\circ \mathcal{T}_0(t)\circ G_0(t)\circ \mathcal{T}_0^*(t)\big\}\, Y_0 \Big)
\end{aligned}$$

Recall that Y_0 was supposed to be tangent to $\mathfrak{M}$, i.e. $Y_0 \in \ker \mathcal{T}_0^* \cap \ker \mathcal{T}_1$. Hence using the procuct rule, we need only differentiate the operators next to Y_0, and we thus get:

$$\begin{aligned}
&= \frac{d}{dt}\Big|_0 \Big(Y_0 - \big\{\mathcal{T}_1^*\circ G_2\circ \mathcal{T}_1(t)\big\}\, Y_0 - \big\{\mathcal{T}_0\circ G_0\circ \mathcal{T}_0^*(t)\big\}\, Y_0 \\
&\qquad + \big\{\mathcal{T}_1^*\circ G_2\circ \mathcal{T}_1\circ \mathcal{T}_0\circ G_0\circ \mathcal{T}_0^*(t)\big\}\, Y_0 \Big)
\end{aligned}$$

Note that since $\mathcal{K}_{(A,\psi)}$ is an (elliptic) complex, we have $\mathcal{T}_1 \circ \mathcal{T}_0 \equiv 0$, thus the last term vanishes identically. If we would have chosen the operators $\tan_{(A,\psi)}$ and $\mathrm{hor}_{(A,\psi)}$ in reversed order in 2.5.2, then we would have got the term $\frac{d}{dt}\Big|_0 \big\{\mathcal{T}_0\circ G_0\circ \mathcal{T}_0^*\circ \mathcal{T}_1^*\circ G_2\circ \mathcal{T}_1(t)\big\}\, Y_0$ instead. But this vanishes by the same argument, since the adjoint of $\mathcal{K}_{(A,\psi)}$ is an (elliptic) complex too, and thus $\mathcal{T}_0^*\circ\mathcal{T}_1^* \equiv 0$. We thus get:

$$= \frac{d}{dt}\Big|_0 \Big(- \big\{\mathcal{T}_1^*\circ G_2\circ \mathcal{T}_1(t)\big\}\, Y_0 - \big\{\mathcal{T}_0\circ G_0\circ \mathcal{T}_0^*(t)\big\}\, Y_0 \Big)\ . \tag{2.5.3}$$

For the second fundamental form terms we need to take the normal projection $\mathrm{nor}_{(A,\psi)}$ thereof. Since $\big(\mathrm{im}\mathcal{T}_0 \subset \ker \mathcal{T}_1\big) \perp \mathrm{im}\mathcal{T}_1^*$, the last term of 2.5.3 vanishes under $\mathrm{nor}_{(A,\psi)}$ whereas the first term of 2.5.3 – being already normal – stays unaffected. We thus get for the second fundamental form of the embedding $\mathfrak{M} \hookrightarrow \mathcal{B}^*$ exactly the same expression as above for the embedding $\widetilde{\mathfrak{M}} \hookrightarrow \mathcal{C}^*$:

$$\mathrm{II}(X,Y)_{[A,\psi]} = -\,\mathcal{T}_1^*\circ G_2 \begin{pmatrix} -q(\phi^X,\phi^Y) \\ \frac{1}{2}\nu^Y\cdot\phi^X + \frac{1}{2}\nu^X\cdot\phi^Y \end{pmatrix}\ . \tag{2.5.4}$$

Although the formulae for the second fundamental forms of the embedding $\widetilde{\mathfrak{M}} \subset \mathcal{C}^*$ resp. $\mathfrak{M} \subset \mathcal{B}^*$ look exactly the same, one must take into account that the linearised configurations X_0, Y_0 in these two formulae are not the same but lie in the different subspaces $\ker \mathcal{T}_1$ resp. $\ker \mathcal{T}_0^* \cap \ker \mathcal{T}_1$ of $T_{(A,\psi)}\mathcal{C}^*$.

To proceed we need only collect the terms of the Gauss equation as for $\widetilde{\mathfrak{M}}$ above and combine them with the formula (2.4.2) for the sectional curvature of $\mathcal{B}^*$. We finally get the following formula for the sectional curvature of the Seiberg-Witten moduli space with respect to the quotient L^2-metric:

$$\begin{aligned}
\big(R^{\mathfrak{M}}(X,Y)Y,X\big)_{[A,\psi]} &= \big(R^{\mathcal{B}^*}(\overline{X},\overline{Y})\overline{Y},\overline{X}\big)_{[A,\psi]} \\
&\quad -\big(\mathrm{II}(X,X),\mathrm{II}(Y,Y)\big)_{[A,\psi]} + \big(\mathrm{II}(X,Y),\mathrm{II}(X,Y)\big)_{[A,\psi]} \\
&= 3\left(i\mathrm{Im}\langle\phi^X,\phi^Y\rangle\, ,\, G_0 i\mathrm{Im}\langle\phi^X,\phi^Y\rangle \right)_{L^2} \\
&\quad -\left(\begin{pmatrix} -q(\phi^X,\phi^X) \\ \nu^X\cdot\phi^X \end{pmatrix}, G_2\begin{pmatrix} -q(\phi^Y,\phi^Y) \\ \nu^Y\cdot\phi^Y \end{pmatrix}\right)_{L^2} \\
&\quad +\left(\begin{pmatrix} -q(\phi^X,\phi^Y) \\ \frac{1}{2}\nu^Y\cdot\phi^X+\frac{1}{2}\nu^X\cdot\phi^Y \end{pmatrix}, G_2\begin{pmatrix} -q(\phi^X,\phi^Y) \\ \frac{1}{2}\nu^Y\cdot\phi^X+\frac{1}{2}\nu^X\cdot\phi^Y \end{pmatrix}\right)_{L^2}.
\end{aligned}$$

Note that all these formulae for the sectional curvature implicitly depend on the perturbation $\mu^+ \in \Omega^2_+(M;i\mathbb{R})$ used in the construction of the moduli space. This dependence is via the monopoles (A,ψ), where our computations are based. These monopoles clearly change, when the perturbation μ^+ changes.

2.6 The quotient L^2-metric on the parametrised moduli space

In this section, we construct a natural L^2-metric on the parametrised moduli space $\widehat{\mathfrak{M}}$ in the same way as we did for the moduli space $\mathfrak{M}$, namely via appropriate L^2-orthogonal splittings. We show that the restriction of this quotient L^2-metric on $\widehat{\mathfrak{M}}$ to a fibre $\mathfrak{M}_{t_0}$ of the parametrisation $\widehat{\mathfrak{M}} = \bigsqcup_{t\in[0,1]}\mathfrak{M}_t$ is the quotient L^2-metric, at least if $\mathfrak{M}_{t_0}$ is a smooth manifold.

Recall that the parametrised moduli space $\widehat{\mathfrak{M}}$ was defined as the disjoint union of the moduli spaces $\mathfrak{M}_{\mu^+(t)}$ along a curve $\mu^+ : [0,1] \to \Omega^2_+(M;i\mathbb{R})$, and recall further that the curve μ^+ can be chosen generically such that the parametrised moduli space $\widehat{\mathfrak{M}}$ is a smooth manifold. We may further assume that for every $t\in[0,1]$, the derivative $(\mu^+)'_t$ of the curve μ^+ is nontrivial: $(\mu^+)'_t \not\equiv 0$. We can consider the parametrised moduli space as the quotient by $\mathcal{G}$ of the zero locus of the parametrised Seiberg-Witten map:

$$\begin{aligned}
\widehat{\mathcal{SW}}:\ \mathcal{A}(\det P)\times\Gamma(\Sigma^+)\times[0,1] &\to \Omega^2_+(M;i\mathbb{R})\times\Gamma(\Sigma^-) \\
\begin{pmatrix} A \\ \psi \\ t \end{pmatrix} &\mapsto \begin{pmatrix} F_A^+ - \frac{1}{2}q(\psi,\psi) - \mu^+(t) \\ \mathcal{D}_A\psi \end{pmatrix}
\end{aligned}$$

The gauge group $\mathcal{G}$ acts trivially on the $[0,1]$-factor of $\mathcal{C}\times[0,1]$. When we linearise the parametrised Seiberg-Witten map $\widehat{\mathcal{SW}}$ and the orbit map, we end up with the following complex:

$$0 \longrightarrow \Omega^0(M;i\mathbb{R}) \xrightarrow{\widehat{\mathcal{T}}_0} \Omega^1(M;i\mathbb{R})\times\Gamma(\Sigma^+)\times\mathbb{R} \xrightarrow{\widehat{\mathcal{T}}_1} \Omega^2_+(M;i\mathbb{R})\times\Gamma(\Sigma^-) \longrightarrow 0 \qquad \widehat{\mathcal{K}}_{(A,\psi,t_0)}$$

Here $\widehat{\mathcal{T}_1}$ denotes the linearisation in (A, ψ, t_0) of the parametrised Seiberg-Witten map $\widehat{\mathcal{SW}}$:

$$\begin{aligned}\widehat{\mathcal{T}_1}: \quad \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) \times \mathbb{R} \quad &\to \quad \Omega^2_+(M; i\mathbb{R}) \times \Gamma(\Sigma^-) \\ \begin{pmatrix} \nu \\ \phi \\ s \end{pmatrix} \quad &\mapsto \quad \begin{pmatrix} d^+\nu - q(\psi, \phi) - s \cdot (\mu^+)'_{t_0} \\ \frac{1}{2}\nu \cdot \psi + \mathcal{D}_A \phi \end{pmatrix}\end{aligned} \tag{2.6.1}$$

and $\mathcal{T}_0$ denotes the linearisation in $\mathbf{1} \in \mathcal{G}$ of the orbit map through (A, ψ, t_0):

$$\begin{aligned}\widehat{\mathcal{T}_0}: \quad \Omega^0(M; i\mathbb{R}) \quad &\to \quad \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) \times \mathbb{R} \\ if \quad &\mapsto \quad \begin{pmatrix} 2idf \\ -if \cdot \psi \\ 0 \end{pmatrix}\end{aligned} \tag{2.6.2}$$

Note that $\widehat{\mathcal{K}}_{(A,\psi,t_0)}$ is a complex, since $\widehat{\mathcal{T}_1} \circ \widehat{\mathcal{T}_0} \equiv \mathcal{T}_1 \circ \mathcal{T}_0 \equiv 0$ holds trivially, but it is *not* elliptic. However, the splittings which were used in section 2.2 to construct the quotient L^2-metric can also be obtained directly from these operators and their adjoints. The adjoint of $\widehat{\mathcal{T}_0}$ is the operator:

$$\begin{aligned}\widehat{\mathcal{T}_0}^*: \quad \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) \times \mathbb{R} \quad &\to \quad \Omega^0(M; i\mathbb{R}) \\ \begin{pmatrix} \nu \\ \phi \\ s \end{pmatrix} \quad &\mapsto \quad 2d^*\nu \, + \, i\mathrm{Im}\langle \psi, \phi \rangle\end{aligned} \tag{2.6.3}$$

and the adjoint of $\widehat{\mathcal{T}_1}$ is the operator:

$$\begin{aligned}\widehat{\mathcal{T}_1}^*: \quad \Omega^2_+(M; i\mathbb{R}) \times \Gamma(\Sigma^-) \quad &\to \quad \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) \times \mathbb{R} \\ \begin{pmatrix} \mu \\ \xi \\ s \end{pmatrix} \quad &\mapsto \quad \begin{pmatrix} d^*\mu + \frac{i}{2}\mathrm{Im}\langle (\cdot) \cdot \psi, \xi \rangle \\ \mathcal{D}_A \xi - 2\mu \cdot \psi \\ -s \cdot ((\mu^+)'_{t_0}, \mu)_{L^2} \end{pmatrix}\end{aligned} \tag{2.6.4}$$

Although the complex $\widehat{\mathcal{K}}_{(A,\psi)}$ is not elliptic, the operators $\widehat{\mathcal{T}_0}$ and $\widehat{\mathcal{T}_1}^*$ are obviously closed, and we thus have the following L^2-orthogonal splitting:

$$\begin{aligned}\Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) \times \mathbb{R} \quad &= \quad \ker \widehat{\mathcal{T}_0}^* \oplus \mathrm{im}\widehat{\mathcal{T}_0} \\ &= \quad \ker \widehat{\mathcal{T}_1} \oplus \mathrm{im}\widehat{\mathcal{T}_1}^* \\ &= \quad (\ker \widehat{\mathcal{T}_0}^* \cap \widehat{\mathcal{T}_1}) \oplus \mathrm{im}\widehat{\mathcal{T}_0} \oplus \mathrm{im}\widehat{\mathcal{T}_1}^*\end{aligned} \tag{2.6.5}$$

Similar to the case of the moduli space as explained in section 2.2, the intersection of the kernels of $\widehat{\mathcal{T}_0}^*$ and $\widehat{\mathcal{T}_1}$ can be regarded as the Zariski tangent space of the parametrised moduli space $\widehat{\mathfrak{M}}$. Thus in an irreducible point $\widehat{[A, \psi]}$, the parametrised moduli space $\widehat{\mathfrak{M}}$ carries a natural Riemannian metric, induced from the splitting (2.6.5) and the identification $T_{\widehat{[A,\psi]}}\widehat{\mathfrak{M}} \cong \ker \widehat{\mathcal{T}_0}^* \cap \ker \widehat{\mathcal{T}_1}$. As before, we call the metric obtained in this way the *quotient L^2-metric* on the parametrised moduli space $\widehat{\mathfrak{M}}$.

Since by definition of the parametrised moduli space $\widehat{\mathfrak{M}} := \bigsqcup_{t\in[0,1]} \mathfrak{M}_{\mu_t^+}$, the moduli spaces $\mathfrak{M}_{\mu_t^+}$ embed smoothly into $\widehat{\mathfrak{M}}$ except for finitely many $t \in [0,1]$, one might hope, that the restrictions of the quotient L^2-metric on $\widehat{\mathfrak{M}}$ to the submanifolds $\mathfrak{M}_{\mu_t^+}$ yield the quotient L^2-metrics. To see that this is indeed the case, we compare the images and kernels of $\widehat{\mathcal{T}}_j^{(*)}, j = 0,1$ with those of $\mathcal{T}_j^{(*)}, j = 0,1$, and we find:

$$\begin{aligned}
\ker \mathcal{T}_0^* \times \mathbb{R} &= \ker \widehat{\mathcal{T}}_0^* \\
\mathrm{im}\mathcal{T}_0 \times \{0\} &= \mathrm{im}\widehat{\mathcal{T}}_0 \\
\ker \mathcal{T}_1 \times \{0\} &= \ker \widehat{\mathcal{T}}_1 \cap \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) \times \{0\} \\
\mathrm{im}\mathcal{T}_1^* \times \mathbb{R} &= \mathrm{im}\widehat{\mathcal{T}}_1^*
\end{aligned}$$

Thus when we fix the parameter $t_0 \in [0,1]$ in the parametrised configuration space $\mathcal{C}^* \times [0,1]$, and we suppose the fibre $\widehat{\mathfrak{M}}_{t_0}$ over t_0 to be regular, then the tangent space of the moduli space $\mathfrak{M}_{\mu_{t_0}^+}$ can naturally be identified with the intersection of the tangent space of $\widehat{\mathfrak{M}}$ with the tangent space of $\widehat{\mathfrak{M}}_{t_0}$:

$$\begin{aligned}
T_{[A,\psi]}\mathfrak{M}_{\mu^+(t_0)} &= \ker \mathcal{T}_0 \cap \ker \mathcal{T}_1^* \\
&\cong (\ker \widehat{\mathcal{T}}_0 \cap \widehat{\mathcal{T}}_1^*) \cap \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) \times \{0\} \\
&= T_{\widehat{[A,\psi]}}\widehat{\mathfrak{M}} \cap \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) \times \{0\}\ .
\end{aligned}$$

Thus the restriction of the quotient L^2-metric on the parametrised moduli space $\widehat{\mathfrak{M}}$ to a fibre $\mathfrak{M}_{t_0}$ yields the natural quotient L^2-metric, at least when the moduli space $\mathfrak{M}_{\mu_{t_0}^+}$ is a smooth submanifold of $\widehat{\mathfrak{M}}$.

2.7 The canonically gauge fixed L^2-metric

As explained in section 1.2 above, the total space $\mathfrak{P}$ of the Seiberg-Witten bundle can be represented as a subset of the irreducible configuration space $\mathcal{C}^*$ via the canonical global gauge fixing $d^*(A - A_0) = 0$. We denoted the solution space of the gauge fixed Seiberg-Witten equations (1.2.2–1.2.4) by $\widetilde{\mathfrak{M}}_{fix}$. Since $\widetilde{\mathfrak{M}}_{fix}$ is finite dimensional, it is a splitting submanifold of $\mathcal{C}^*$, and the restriction to $\widetilde{\mathfrak{M}}_{fix}$ of the tangent bundle of $\mathcal{C}^*$ splits into the tangent bundle of $\widetilde{\mathfrak{M}}_{fix}$ and its orthogonal complement. (Note that even in Banach spaces, closed subspaces need not have topological complements, thus in general it is not clear that such a splitting exists, whereas for subspaces of finite dimension or codimension, this is clear, see [5]). This splitting can even be realised by an elliptic operator. Thus the natural L^2-metric restricts to a metric on the submanifold $\widetilde{\mathfrak{M}}_{fix} \subset \mathcal{C}^*$.

Since $\widetilde{\mathfrak{M}}_{fix}$ is realised as the solution space of a nonlinear equation, the tangent space in $[A,\psi]_{fix}$ is the solution space of the linearised equation. As we did for the Seiberg-Witten premoduli space, we write $\widetilde{\mathfrak{M}}_{fix}$ as the zero locus of the *canonically gauge fixed Seiberg-Witten map* $\mathcal{SW}_{fix}$:

$$\mathcal{SW}_{fix}:\quad \mathcal{A}(\det P) \times \Gamma(\Sigma^+) \;\to\; \Omega^2_+(M; i\mathbb{R}) \times \Gamma(\Sigma^-) \times \Omega^0(M; i\mathbb{R})$$

$$\begin{pmatrix} A \\ \psi \end{pmatrix} \;\mapsto\; \begin{pmatrix} F_A^+ - \frac{1}{2}q(\psi,\psi) - \mu^+ \\ \mathcal{D}_A\psi \\ d^*(A - A_0) \end{pmatrix}$$

Here A_0 denotes an abritrary fixed connection $A_0 \in \mathcal{A}(\det P)$, which we used to fix the gauge. The linearisation of the gauge fixed Seiberg-Witten map consists of the linearisation $\mathcal{T}_1$ of the ordinary Seiberg-Witten map plus the linearisation of the gauge fixing $A \mapsto d^*(A - A_0)$. Thus $\mathcal{SW}_{fix}$ linearises to the map:

$$\begin{array}{rccc} \mathcal{T}_1 \oplus d^* : & \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) & \rightarrow & \Omega^2_+(M; i\mathbb{R}) \times \Gamma(\Sigma^-) \times \Omega^0(M; i\mathbb{R}) \\ & \begin{pmatrix} \nu \\ \phi \end{pmatrix} & \mapsto & \begin{pmatrix} d^+\nu - q(\psi, \phi) \\ \frac{1}{2}\nu \cdot \psi + \mathcal{D}_A\phi \\ d^*\nu \end{pmatrix} \end{array}$$

The tangent space in $[A, \psi]_{fix}$ of $\widetilde{\mathfrak{M}}_{fix}$ can naturally be identified with the kernel of $\mathcal{T}_1 \oplus d^*$. Although the linearisation $\mathcal{T}_1 \oplus d^*$ does not come from an elliptic complex, it is clear, that it is an elliptic operator, since it has the same symbol as the operator $\mathcal{T}_0^* \oplus \mathcal{T}_1$, which we used in 2.2 to identify the tangent space of $\mathfrak{M}$. Thus the tangent space of $\mathcal{C}^*$ in a gauge fixed monopole (A, ψ) splits as follows:

$$T_{(A,\psi)}\mathcal{C}^* = \Omega^1(M; i\mathbb{R}) \times \Gamma(\Sigma^+) = \ker(\mathcal{T}_1 \oplus d^*) \oplus \operatorname{im}(\mathcal{T}_1 \oplus d^*)^* \cong T_{[A,\psi]_{fix}}\widetilde{\mathfrak{M}}_{fix} \oplus \operatorname{im}(\mathcal{T}_1 \oplus d^*)^*$$

From this splitting, we obtain an L^2-metric on $\widetilde{\mathfrak{M}}_{fix}$ by restriction. Since the $U(1)$-action on $\widetilde{\mathfrak{M}}_{fix}$ preserves the L^2-metric, there is a unique metric on the quotient $\mathfrak{M} = \widetilde{\mathfrak{M}}_{fix}/U(1)$ such that the bundle projection $\pi : \widetilde{\mathfrak{M}}_{fix} \rightarrow \mathfrak{M}$ is a Riemannian submersion. Thus we have constructed another Riemannian metric on the Seiberg-Witten bundle $\mathfrak{P} \rightarrow \mathfrak{M}$. We call this metric the *canonically gauge fixed L^2-metric*.

Chapter 3

Moduli spaces on Kähler surfaces

In this chapter, we study the geometry of Seiberg-Witten moduli spaces on compact Kähler surfaces. Because of theorem 1.5.2, we need to assume that $b_2^+ = 1$, since the moduli spaces of positive virtual dimension on a Kähler surface with $b_2^+ > 1$ are generically empty. We show that the Seiberg-Witten moduli spaces on a simply connected Kähler surface with $b_2^+ = 1$ are generically diffeomorphic to complex projective spaces. For a "reducible" perturbation $\mu^+ \in \Gamma_g^+$, we show that all monopoles are reducible and gauge equivalent, thus the moduli space consists of a single point. Further we compute explicit formulae for the quotient L^2-metric and the canonically gauge fixed L^2-metric. These formulae amount to an identification of the L^2-metrics as Fubini-Study metrics in one special case. Finally we show that the Seiberg-Witten moduli spaces on Kähler surfaces can be realised as symplectic resp. Kähler quotients. Hence the quotient L^2-metric on $\mathfrak{M}$ is a Kähler metric.

3.1 The diffeomorphism type of the moduli spaces

In this section, we identify the Seiberg-Witten moduli space of positive virtual dimension on a compact, connected and simply connected Kähler surface as a complex projective space. We give two alternative proofs for this identification, first a direct one using the continuity method and then a more indirect one using the existence and uniqueness theorem of KAZDAN and WARNER in [29]. When we drop the assumption $b_1 = 0$, then the moduli space fibres over the torus $H^1(M; i\mathbb{R})/H^1(M; 2\pi i\mathbb{Z})$ through complex projective spaces.

As in section 1.4 above, we suppose M to be a compact, connected Kähler surface with a fixed Spin$^{\mathbb{C}}$-structure$P = P_0 \otimes L$. A connection on the determinant line bundle $\det P = K_M^* \otimes L^2$ can be written as $B = A_{can} \otimes A^2$, where A_{can} denotes the canonical hermitean holomorphic (or Chern-) connection on K_M^* and A denotes a unitary connection on L. From section 1.4, we know that a Seiberg-Witten monopole $(B, \alpha \oplus \zeta) \in \mathcal{A}(\det P) \times \Omega^0(M; L) \oplus \Omega^{0,2}(M; L)$ consists of a holomorphic connection B on $\det P$ and a section α of L, which is holomorphic with respect to the Cauchy-Riemann operator $\overline{\partial}_B$ associated with B. According to our convention 1.4.3 we may assume $\zeta \equiv 0$.

The action of the complexification $\mathbb{C}^*$ of $U(1)$ on L induces a natural action of the *complexified gauge group*

$$\mathcal{G}^{\mathbb{C}} := \Omega^0(M; \mathbb{C}^*) \tag{3.1.1}$$

on $\mathcal{A}(L)$, which is given by:

$$u : A \mapsto A + u^{-1}\overline{\partial}u - \bar{u}^{-1}\partial\bar{u} \; .$$

Note that this action extends the standard action of the gauge group $\mathcal{G}$. It also induces an operation on the Cauchy-Riemann operator

$$\overline{\partial}_A := \frac{1}{2}(d_A - i \cdot d_A \circ \mathfrak{J})$$

associated with a connection A given by:

$$u : \overline{\partial}_A \mapsto \overline{\partial}_{u \cdot A} = u^{-1} \circ \overline{\partial} \circ u \ .$$

The complexified gauge group naturally acts on the sections of the line bundle $L \to M$ by $\alpha \mapsto u^{-1} \cdot \alpha$. It is quite obvious from the very definition, that a section $\alpha \in \Omega^0(M; L)$ is holomorphic with respect to $\overline{\partial}_A$ if and only if the section $u^{-1} \cdot \alpha$ is holomorphic with respect to $\overline{\partial}_{u \cdot A}$.

Since a holomorphic structure on L is induced by a Cauchy-Riemann operator and since two holomorphic structures are isomorphic if the corresponding Cauchy-Riemann operators are related by a complex gauge transformation, the space of isomorphism classes of holomorphic structures on L may be regarded as the quotient of the space of holomorphic connections $\mathcal{A}^{0,2}(L)$ by the action of the complexified gauge group $\mathcal{G}^{\mathbb{C}}$. This quotient can be identified with the torus $H^1(M; i\mathbb{R})/H^1(M; 2\pi i\mathbb{Z})$, see [43].

Thus when we assume $b_1 = 0$, then any holomorphic connection A on L can be derived from the hermitean holomorphic (or Chern-) connection A_0 via a complex gauge transformation u. Furthermore, any A-holomorphic section β is deduced via the same complex gauge transformation from an A_0-holomorphic section α. When we write a complex gauge transformation $u \in \mathcal{G}^{\mathbb{C}}$ as $u = e^{-f+ih}$ with real functions f, h, then we find:

$$\begin{aligned} A = u \cdot A_0 &= A_0 + u^{-1}\overline{\partial} u - \bar{u}^{-1}\partial \bar{u} \\ &= A_0 + \overline{\partial}(-f + ih) - \partial(-f - ih) \\ &= A_0 + id^c f + idh \ . \end{aligned}$$

When we carry these identifications over to the determinant line bundle, we will not distinguish in the notation between the hermitean holomorphic connection on L and on $K_M^* \otimes L^2$ but denote both by A_0. This should not lead to confusion, since it will be clear from the context whether A_0 denotes a connection on L or on $K_M^* \otimes L^2$. According to theorem 1.4.2 and our convention 1.4.3, we may now write any monopole (B, β) on a compact, connected and simply connected Kähler surface in the form:

$$\begin{pmatrix} B \\ \beta \end{pmatrix} = \begin{pmatrix} A_0 + 2id^c f + 2idh \\ e^{f-ih}\alpha \end{pmatrix} \ . \tag{3.1.2}$$

We denote by $H^0_{A_0}(M; L)$ resp. $H^0_A(M; L)$ the spaces of sections which are holomorphic with respect to the holomorphic structure induced from the connection A_0 resp. A. The above considerations yield that $H^0_A(M; L) = H^0_{u \cdot A_0}(M; L) = u^{-1} \cdot H^0_{A_0}(M; L)$.

Due to these reformulations we are now able to identify the moduli spaces as complex projective spaces:

3.1.1 THEOREM. *Let M be a compact, connected, and simply connected Kähler surface with $b_2^+ = 1$, and let $P = P_0 \otimes L$ be a fixed Spin$^{\mathbb{C}}$-structureon M, where L is a $U(1)$-bundle, the associated complex line bundle of which admits holomorphic sections. Consider a perturbation $\mu^+ = i\pi\lambda \cdot \omega$ with $\lambda < \deg_\omega(L)$ according to our convention (1.4.3). Then the Seiberg-Witten moduli space $\mathfrak{M}_{\mu^+}$ is diffeomorphic to $\mathbb{CP}^m$, where $m = \frac{d}{2}$. Moreover, there is an equivalence of $U(1)$-bundles between the Seiberg-Witten bundle $\mathfrak{P} \to \mathfrak{M}_{\mu^+}$ and the Hopf fibration $S^{2m+1} \to \mathbb{CP}^m$.*

Proof: According to theorem 1.4.2, a Seiberg-Witten monopole on a Kähler surface is a pair (B, β), where B is a holomorphic connection on $\det P$ and β is a section of the line bundle L. Such a pair (B, β) is a Seiberg-Witten monopole if and only if it satisfies the equation

$$(F_B^+)^{1,1} = \frac{i}{4}|\beta|^2 \cdot \omega + i\pi\lambda \cdot \omega \ . \tag{3.1.3}$$

Due to the above considerations, we may write (B, β) in the form

$$\begin{pmatrix} B \\ \beta \end{pmatrix} = \begin{pmatrix} A_0 + 2id^c f + 2idh \\ e^{f-ih}\alpha \end{pmatrix}$$

with $\alpha \in H^0_{A_0}(M; L)$ and real functions f, h. Since e^{ih} is an ordinary gauge transformation and the Seiberg-Witten equations are gauge invariant, we can reformulate the remaining equation (3.1.3) as an equation in f. When we plug in $(B, \beta) = (A_0 + 2id^c f + 2idh, e^{f-ih}\alpha)$ into (3.1.3) we get:

$$(F_B^+)^{1,1} = F_{A_0}^+ + \left(2idd^c f\right)^+ = \frac{i}{4}|\beta|^2 \cdot \omega + i\pi\lambda \cdot \omega = \frac{i}{4}e^{2f}|\alpha|^2 \cdot \omega + i\pi\lambda \cdot \omega \, .$$

Applying the contraction $i\Lambda_\omega$ to both sides of this equation yields:

$$i\Lambda_\omega F_{A_0}^+ + 2\Delta f \ = \ -\frac{1}{2}e^{2f}|\alpha|^2 - 2\pi\lambda \ .$$

Thus any Seiberg-Witten monopole $(B, \beta) = (A_0 + 2id^c f + 2idh, e^{f-ih}\alpha)$ can be derived from a configuration $(A_0, \alpha), \alpha \in H^0_{A_0}(M; L)$ by solving the following equation in the unknown f:

$$2\Delta f + \frac{1}{2}e^{2f}|\alpha|^2 = -2\pi\lambda - i\Lambda_\omega(F_{A_0}^+) \ . \tag{3.1.4}$$

We refer to this equation as a *Kazdan-Warner type equation*, since nonlinear elliptic equations of this type were intensively studied by KAZDAN and WARNER in their work on scalar curvature functions on surfaces [29]. We cite their result below to deduce another proof of this theorem.

Here we use a continuity method to show the existence and uniqueness of solutions of (3.1.4). The main idea is to deform the (nonlinear) equation (3.1.4) along the parameter interval $[0, 1]$ to a (linear) Laplace equation (for $t = 0$), which obviously admits a solution, and then to show existence and uniqueness of solutions for any of the equations along the deformation. In the context of Seiberg-Witten theory, this method was used by BIQUARD in his work on Seiberg-Witten equations on complex non-Kähler manifolds, see [7].

We consider the following family of equations:

$$2\Delta f + \frac{t}{2}e^{2f}|\alpha|^2 = g - \frac{(1-t)}{\text{vol}(M)}\int_M g \, dv_g \qquad \text{with} \quad t \in [0, 1] \ , \tag{3.1.5}$$

where $g := -2\pi\lambda - i\Lambda_\omega(F_{A_0}^+)$. The solution space of this family of equations may be regarded as the zero locus of the operator:

$$Q : V \times [0, 1] \to W, \qquad (f, t) \mapsto 2\Delta f + \frac{t}{2}e^{2f}|\alpha|^2 - g + \frac{(1-t)}{\text{vol}(M)}\int_M g \, dv_g \ ,$$

where the spaces V resp. W are defined by

$$V := \left\{ f \in \Omega^0(M;\mathbb{R}) \;\middle|\; \tfrac{1}{2}\int_M e^{2f}|\alpha|^2\,dv_g = \int_M g\,dv_g \right\}$$

resp.

$$W := \left\{ u - \tfrac{1}{\mathrm{vol}(M)}\int_M u\,dv_g \;\middle|\; u \in \Omega^0(M;\mathbb{R}) \right\} = \left\{ \tilde{u} \in \Omega^0(M;\mathbb{R}) \;\middle|\; \int_M \tilde{u}\,dv_g = 0 \right\} .$$

For $t = 1$, the family (3.1.5) gives our Kazdan-Warner type equation (3.1.4), whereas for $t = 0$ we obtain the Laplace equation $2\Delta f = g - \frac{1}{\mathrm{vol}(M)}\int_M g\,dv_g$.

In order to use the inverse function theorem, we need to make V resp. W into Banach spaces. Thus we consider instead of V resp. W the corresponding Sobolev spaces V_{k+2} resp. W_k, i.e. the completions of V resp. W in the Sobolev norms $\|\cdot\|_{k+2}$ resp. $\|\cdot\|_k$. Now let $I \subset [0,1]$ be the interval of all those t for which the equation $Q(u,t) = 0$ has a solution. We first show that $I = [0,1]$, which obviously solves the existence problem.

Since the laplacian is an isomorphism $\Delta : V_{k+2} \to W_k$, the equation for $t = 0$

$$Q(f,0) = 0 \quad \Leftrightarrow \quad 2\Delta f = g - \frac{1}{\mathrm{vol}(M)}\int_M g\,dv_g$$

has a solution. Hence I is nonempty and $0 \in I$. To apply the implicit function theorem, we show that the partial derivative $\partial_f Q$ is invertible. This partial derivative is given by:

$$\big(\partial_f Q\big).\varphi = 2\Delta\varphi + t\varphi e^{2f}|\alpha|^2$$

When we set $(\partial_f Q).\varphi = 0$, then by integration against φ we obtain:

$$0 = \big(2\Delta\varphi, \varphi\big) + t\big(\varphi e^{2f}|\alpha|^2, \varphi\big) = 2\|d\varphi\|^2 + t\|\varphi e^f \alpha\|^2 ,$$

which implies $\varphi \equiv 0$. Now we consider the map

$$\Phi : V_{k+2} \times [0,1] \to W_k \times [0,1] , \qquad (f,t) \mapsto (Q(f,t),t) .$$

Since $d\Phi$ is invertible, the inverse function theorem implies that Φ is locally invertible. Thus to any solution $Q(f,t_0) = 0$, there is a neighbourhood of (f,t_0) on which Φ is invertible. Hence there are unique solutions $(f_t,t) = \Phi^{-1}(0,t)$ in a neighbourhood of t_0. We conclude that the interval I is open and that it especially contains a neighbourhood $[0,2\epsilon[$ of 0.

To conclude that the interval I is closed and thus equal to $[0,1]$, we will show that any sequence of solutions $(f_t,t) = \Phi^{-1}(0,t)$ with $t \in I \cap [\epsilon,1]$ admits a subsequence, which converges in $V_{k+2} \times [\epsilon,1]$ to say (f',t'). Since the map $Q : V_{k+2} \to W_k$ is continuous, this would imply that $Q(f',t') = \lim_{t\to t'} Q(f_t,t) = 0$ and thus $t' \in I$, hence that I is closed.

To this end, we use the estimate (1.3.2) for the norm of the spinor part β of a monopole (B,β). We need to derive a similar estimate for solutions of our family (3.1.5). Note that this family corresponds to a family of Seiberg-Witten type equations, where only the equation (1.4.4) is replaced by the family of equations:

$$F_B^+ = \frac{it}{4}|\beta|^2 \cdot \omega + \mu_t^+$$

with $\mu_t^+ = i\pi\big(t\lambda - (1-t)\deg_\omega(\det P)\big)\cdot\omega$. As in the proof of the estimate (1.3.2) in section 1.3, we derive the following estimate for the maximum of the function $t|\beta|^2 = te^{2f_t}|\alpha|^2$:

$$t\|e^{f_t}\alpha\|_\infty^2 \;\le\; 2\max\big((0, -\min_{x\in M}\mathrm{scal}(x) + 4\|\mu_t\|_\infty\big) \le C \qquad \forall t\in[\epsilon,1[\;. \tag{3.1.6}$$

Thus we get a uniform bound C along the interval $[\epsilon,1]$ for the function $e^{2f_t}|\alpha|^2$, and also a uniform bound C' along $[\epsilon,1[$ for the solutions $|f_t|^2$. In order to apply the Rellich theorem, we need to show that the solutions f_t are contained in some bounded set in V_{k+2}. From (3.1.5) together with the uniform bound for $|f_t|^2$, we obtain a uniform bound for $|\Delta f_t|^2$. Now elliptic regularity implies the existence of a uniform bound for the Sobolev norm $\|f_t\|_2$. Further, from (3.1.5) and elliptic regularity for the laplacian we obtain

$$\begin{aligned}\|f_t\|_{k+2}^2 \;&\le\; C''\cdot\big(\,\|2\Delta f_t\|_k^2 \,+\, \|f_t\|_k^2\,\big)\\ &\le\; C''\cdot\left(\frac{t^2}{4}\,\|e^{2f_t}|\alpha|^2\|_k^2 \,+\, \left\|g - \frac{(1-t)}{\mathrm{vol}(M)}\int_M g\,dv_g\right\|_k^2 \,+\, \|\,f_t\,\|_k^2\right)\\ &\le\; C_k\cdot\big(1+\|f_t\|_k^2\big)\end{aligned} \tag{3.1.7}$$

for some appropriate constants C_k. Thus the family of solutions f_t with $t\in I\cap[\epsilon,1]$ is contained in a bounded subset of V_{k+2} for some $k>0$. By the Rellich theorem we may suppose $\{f_t\}$ to be contained in a compact subset of V_{k+1}. Replacing a sequence (f_t,t) of solutions by a subsequence if necessary, we may suppose that it converges to some element $(f',t')\in V_{k+1}\times[\epsilon,1]$. As indicated above, it follows from the continuity of Q that $Q(f',t') = \lim_{t\to t'} Q(f_t,t) = 0$, and thus that f' is a solution.

It remains to show, that the solution f of (3.1.5) for $t=1$ is unique. Suppose we are given two solutions f_1 and f_2 of our initial Kazdan-Warner type equation (3.1.4). Then we integrate the difference (f_1-f_2) against $2\Delta(f_1-f_2)+\frac{1}{2}(e^{2f_1}-e^{2f_2})|\alpha|^2$ to obtain:

$$0 = \Big(2\Delta(f_1-f_2)+\frac{1}{2}(e^{2f_1}-e^{2f_2})|\alpha|^2,(f_1-f_2)\Big)_{L^2} = 2\|d(f_1-f_2)\|^2+\frac{1}{2}\Big(\,(e^{2f_1}-e^{f_2})|\alpha|^2,(f_1-f_2)\,\Big)_{L^2}$$

Since both these terms are nonnegative, we conclude $f_1\equiv f_2$.

So far we have shown that for any holomorphic section $\alpha\in H^0_{A_0}(M;L)$ there is a unique solution of the Kazdan-Warner type equation (3.1.4). Thus every nontrivial holomorphic section α gives a gauge orbit of irreducible Seiberg-Witten monopoles, and to identify the moduli space, we need to consider, under which conditions the monopoles obtained from different holomorphic sections are gauge equivalent. If we rescale the holomorphic section α by a complex number $\lambda\in\mathbb{C}^*$, then it is easy to deduce the corresponding solution of the Kazdan-Warner type equation (3.1.4) from the one for α. We find that the equation

$$2\Delta\tilde f + \frac{1}{2}e^{2\tilde f}|\lambda\cdot\alpha|^2 \;=-2\pi\lambda - i\Lambda_\omega(F^+_{A_0})$$

is solved by $\tilde f = f - \ln|\lambda|$. Thus the corresponding Seiberg-Witten monopole changes to

$$(A_0 + 2id^c\tilde f, e^{\tilde f - ih}\lambda\alpha) = (A_0+2id^cf, \frac{\lambda}{|\lambda|}e^{f-ih}\alpha)\;. \tag{3.1.8}$$

Hence rescaling the holomorphic section α by a real parameter changes the solution f of the Kazdan-Warner type equation (3.1.4) but does not change the corresponding Seiberg-Witten monopole. This establishes a bijection from the sphere $S^{2m+1} \subset H^0_{A_0}(M; L)$ to the total space $\mathfrak{P}$ of the Seiberg-Witten bundle. Correspondingly, rescaling the holomorphic section α by a complex parameter $\lambda \in \mathbb{C}^*$ changes the Seiberg-Witten monopoles according to the constant gauge transformation $u = \frac{\lambda}{|\lambda|}$. This yields an identification of the Seiberg-Witten moduli space $\mathfrak{M}$ with the complex projective space $\mathbb{P}l(H^0_{A_0}(M; L))$ and correspondingly of the Seiberg-Witten bundle $\mathfrak{P} \to \mathfrak{M}$ with the Hopf fibration $H^0_{A_0}(M; L) \supset S^{2m+1} \to \mathbb{CP}^m$. □

We give an alternative proof relying on an existence and uniqueness theorem for solutions of non-linear equations of the type (3.1.4), which was proven by KAZDAN and WARNER in [29].

3.1.2 THEOREM (Kazdan-Warner). *Let M be a compact manifold, $k \in \mathbb{N}_+$ be a positive natural number and w a smooth real valued function on M, which is positive apart from a subset of measure zero. Then the equation*

$$\Delta u + w \cdot e^{ku} = g \in \Omega^0(M; \mathbb{R})$$

has a solution f (which is unique) if and only if g satifies the condition $\int_M g dv_g \geq 0$.

As a corollary of the Kazdan-Warner theorem, we find another proof of our theorem 3.1.1: As in the proof of theorem 3.1.1 above, we derive the equation (3.1.4). Now the Kazdan-Warner theorem 3.1.2 applies with $u = 2f, k = 1, w := |\alpha|^2$ and $g := -2\pi\lambda - i\Lambda_\omega(F^+_{A_0})$, and we also get the existence and uniqueness of a solution f for every nonzero holomorphic section $\alpha \in H^0_{A_0}(M; L)$. Note that the condition $\int_M g dv_g \geq 0$ is satisfied according to our convention on the perturbation $\lambda \mathrm{vol}(M) \leq \deg_\omega(\det P)$. □

Next we consider the moduli spaces for those perturbation parameters $\mu^+ \in \Gamma^+_g$, which admit reducible solutions. The identification of the Seiberg-Witten equations with the Kazdan-Warner type equation (3.1.4) yields:

3.1.3 PROPOSITION. *Let M be a connected, simply connected Kähler surface with $b_2^+ = 1$ with fixed Spin$^{\mathbb{C}}$-structure$P = P_0 \otimes L$. For the perturbation $\mu^+ = i\pi\lambda \cdot \omega$ on the wall Γ^+_g, the moduli space $\mathfrak{M}_{\mu^+}$ consists of a single point.*

Proof: We show that all solutions are reducible and gauge equivalent: According to the above considerations, a monopole (B_1, β_1) can be written in the form $(A_0 + 2id^c f_1 + 2idh_1, e^{f_1 - ih_1}\alpha_1)$, where α_1 is a $\overline{\partial}_{A_0}$-holomorphic section of L. By assumption, the Seiberg-Witten equations admit a reducible solution $(B_2, \beta_2) = (A_0 + 2id^c f_1 + 2idh_1, 0)$. Now the Kazdan-Warner type equation (3.1.4) for these two monopoles reads:

$$2\Delta f_1 + \frac{1}{2} e^{2f_1} |\alpha|^2 = -2\pi\lambda - i\Lambda_\omega(F^+_{A_0}) = 2\Delta f_2 \ . \qquad (3.1.9)$$

This implies $2\Delta(f_1 - f_2) = \frac{1}{2} e^{2f_1} |\alpha_1|^2$. By integration, we find $\|e^{f_1}\alpha_1\|^2 \equiv 0 \implies \alpha_1 \equiv 0$. Thus (B_2, β_2) is a reducible monopole too. The equation (3.1.9) now implies $\Delta(f_1 - f_2) = 0$, thus the functions f_1, f_2, which solve (3.1.9), are unique up to constants. Correspondingly, the monopoles $(A_0 + 2id^c f + 2idh, 0)$ are unique up to gauge equivalence. □

We should mention that the above identifications of the moduli space $\mathfrak{M}$ were already implicitly contained in a more algebraic-geometrical language in WITTENS first article [51], where he identified the monopoles on a Kähler surface with vortices. The Kazdan-Warner theorem yields existence and uniqueness of the corresponding vortex equations. The vortices may then be identified with effective divisors, which is essentially the same as our identification with the projectivisation of the space of holomorphic sections.

3.2 Explicit formulae for the L^2-metrics

In this section we study the different L^2-metrics on the Seiberg-Witten moduli space $\mathfrak{M}$ and on the total space $\mathfrak{P}$ of the Seiberg-Witten bundle on a Kähler surface M, which were introduced in the previous chapter. Due to the identifications of the monopoles in terms of holomorphic data, we are now able to give more explicit formulae for these L^2-metrics. In fact they will only be as explicit as the solutions of the Kazdan-Warner type equation (3.1.4) are. Nevertheless these formulae enable us in the following section to determine the metrics up to isometry in one special case.

The quotient L^2-metrics

In section section 2.2 above, we identified the pullback to $\widetilde{\mathfrak{M}}$ of the tangent bundle of the Seiberg-Witten moduli space $\mathfrak{M}$ with the horizontal tangent bundle of the premoduli space $\widetilde{\mathfrak{M}}$, i.e. $\pi^*T\mathfrak{M} \cong \mathcal{H}\widetilde{\mathfrak{M}} = \ker \mathcal{T}_0^* \cap \ker \mathcal{T}_1$. We used this identifiction to define the quotient L^2-metric on $\mathfrak{M}$ as the metric induced from the splitting (2.2.2). Now we apply this identification of the tangent bundle of $\mathfrak{M}$ to the above identifications of the monopoles on a Kähler surface.

Let thus $(B, \beta) \in \mathcal{A}(\det P) \times \Omega^0(M; L)$ be a monopole. To identify the tangent space $T_{[B,\beta]}\mathfrak{M}$, we compute the linearisation $\mathcal{T}_1$ in (B, β) of the Seiberg-Witten map (1.4). According to theorem 1.4.2 together with our convention 1.4.3 on the degree of the line bundle L, we may drop the second component of the Seiberg-Witten map $\mathcal{SW}_{\mu^+}$. The linearisation $\mathcal{T}_1$ of $\mathcal{SW}_{\mu^+}$ thus reads:

$$\begin{array}{rccc} \mathcal{T}_1 : & \Omega^1(M; i\mathbb{R}) \times \Omega^0(M; L) & \to & \Omega^{1,1}_+(M; i\mathbb{R}) \times \Omega^{0,1}(M; L) \\ & \begin{pmatrix} \nu \\ \phi \end{pmatrix} & \mapsto & \begin{pmatrix} (d\nu^+)^{1,1} - \frac{i}{2}\mathrm{Re}\langle \beta, \phi \rangle \cdot \omega \\ \sqrt{2}\nu^{0,1} \wedge \alpha + \sqrt{2}\overline{\partial}_B \beta \end{pmatrix} \end{array}$$

The linearisation $\mathcal{T}_0$ in $\mathbf{1}$ of the orbit map through (B, β) is given by:

$$\begin{array}{rccc} \mathcal{T}_0 : & \Omega^0(M; i\mathbb{R}) & \to & \Omega^1(M; i\mathbb{R}) \times \Omega^0(M; L) \\ & i\eta & \mapsto & \begin{pmatrix} 2id\eta \\ -i\eta \cdot \beta \end{pmatrix} \end{array}$$

According to theorem 1.4.2, a Seiberg-Witten monopole on a Kähler surface is a pair (B, β), where B is a holomorphic connection on L resp. on $\det P = K_M^* \otimes L^2$ and β is a $\overline{\partial}_B$-holomorphic section of L. As in the previous section, we may write a monopole (B, β) as

$$\begin{pmatrix} B \\ \beta \end{pmatrix} = \begin{pmatrix} A_0 + 2id^c f + 2idh \\ e^{f-ih}\alpha \end{pmatrix}$$

where A_0 is a fixed holomorphic connection on L resp. $\det P = K_M^* \otimes L^2$, e.g. the Chern connection, α is a $\overline{\partial}_{A_0}$-holomorphic section of L and f solves the equation (3.1.4). Due to theorem 3.1.1, we may now replace the kernel of $\mathcal{T}_1$ in the above identification of $T\mathfrak{M}$ by the solution space of the linearisation of equation (3.1.4).

When we linearise the holomorphic section $\alpha \in H^0_{A_0}(M;L)$ to $\sigma \in H^0_{A_0}(M;L)$ and the function $(-f+ih) \in \Omega^0(M;\mathbb{C})$ to $(-\varphi+i\eta) \in \Omega^0(M;\mathbb{C})$, then the configuration $(A_0 + 2id^c f + 2idh, e^{f-ih}\alpha)$ linearises to $(2id^c\varphi + 2id\eta, (\varphi - i\eta)e^{f-ih}\alpha + e^{f-ih}\sigma)$. Hence the linearisation of (3.1.4) gives the following equation in the unknown φ:

$$2\Delta\varphi + \varphi e^{2f}|\alpha|^2 + e^{2f}\mathrm{Re}\langle\sigma,\alpha\rangle \; = 0 \; . \tag{3.2.1}$$

Analogously, we compute the condition for a linearised configuration to lie in the cokernel of $\mathcal{T}_0$, which gives quite a similar equation in the unknown η:

$$\mathcal{T}_0^* \begin{pmatrix} 2id^c\varphi + 2id\eta \\ (\varphi - i\eta)e^{f-ih}\alpha + e^{f-ih}\sigma \end{pmatrix} = 2i\Delta\eta + i\eta e^{2f}|\alpha|^2 - ie^{2f}\mathrm{Im}\langle\sigma,\alpha\rangle \; = 0 \; . \tag{3.2.2}$$

We join these two equations according to their similarity as the real and imaginary part of a single one in the unknown $(-\varphi + i\eta)$. Thus the tangent space $T_{[B,\beta]}\mathfrak{M}$ is the space of linearised configurations

$$\begin{pmatrix} \nu \\ \phi \end{pmatrix} = \begin{pmatrix} 2id^c\varphi + 2id\eta \\ (\varphi - i\eta)e^{f-ih}\alpha + e^{f-ih}\sigma \end{pmatrix} ,$$

which satisfy:

$$\overline{\partial}_{A_0}\sigma = 0 \qquad \text{and} \qquad 2\Delta(\varphi - i\eta) \; + \; (\varphi - i\eta)e^{2f}|\alpha|^2 \; + \; e^{2f}\langle\sigma,\alpha\rangle \; = \; 0 \; . \tag{3.2.3}$$

It is easy to see that for any $\sigma \in H^0_{A_0}(M;L)$, there exists a solution $(\varphi - i\eta)$ of this equation, and that this solution is unique: When we integrate $(\varphi - i\eta)$ against $2\Delta(\varphi - i\eta) + (\varphi - i\eta)e^{2f}|\alpha|^2$, we find:

$$2\Delta(\varphi - i\eta) + (\varphi - i\eta)e^{2f}|\alpha|^2 = 0 \implies (\varphi - i\eta) = 0 \; .$$

Thus the operator $2\Delta + e^{2f}|\alpha|^2 : \Omega^0(M;\mathbb{C}) \to \Omega^0(M;\mathbb{C})$ is selfadjoint and has trivial kernel. Consequently, the equation (3.2.3) has a unique solution $(\varphi - i\eta)$ for any $\sigma \in H^0_{A_0}(M;L)$. Note that for $\sigma = \lambda\alpha$, $\lambda \in \mathbb{C}^*$ we find $(\varphi - i\eta) = -\lambda$, thus the corresponding linearised configuration is:

$$\begin{pmatrix} 2id^c\varphi + 2id\eta \\ (\varphi - i\eta)e^{f-ih}\alpha + e^{f-ih}\sigma \end{pmatrix} = \begin{pmatrix} 2id^c\mathrm{Re}(\lambda) + 2id\mathrm{Im}(\lambda) \\ (-1)e^{f-ih}\alpha + e^{f-ih}\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

This corresponds to our identification of the Seiberg-Witten moduli space $\mathfrak{M}$ with the projectivisation of $H^0_{A_0}(M;L)$: when we vary the connection α in the complex line $\mathbb{C}\cdot\alpha$, the corresponding tangent vector in $\mathbb{CP}(H^0_{A_0}(M;L))$ is zero.

The above identification of the tangent space $T_{[B,\beta]}\mathfrak{M}$ amounts to a computation of a formula for the L^2-metric in terms of the solution $(\varphi - i\eta)$ of the equation (3.2.3). We find for the length square of a tangent vector $X_{[B,\beta]} \in T_{[B,\beta]}\mathfrak{M}$, written in the form of a linearised configuration $X_{[B,\beta]} =$

$(2id^c\varphi + 2id\eta, (\varphi - i\eta)e^{f-ih}\alpha + e^{f-ih}\sigma)$:

$$
\begin{aligned}
\left|X_{[B,\beta]}\right|^2 &= \left\|2id^c\varphi\right\|^2 + \left\|2id\eta\right\|^2 + \left\|(\varphi - i\eta)e^{f-ih}\alpha\right\|^2 + \left\|e^{f-ih}\sigma\right\|^2 \\
&\quad + \mathrm{Re}\Big\{ \Big((\varphi - i\eta)e^{f-ih}\alpha, e^{f-ih}\sigma\Big)_{L^2} + \Big(e^{f-ih}\sigma, (\varphi - i\eta)e^{f-ih}\alpha\Big)_{L^2} \Big\} \\
&= \left\|e^f\sigma\right\|^2 + \Big(4\Delta\varphi + \varphi e^{2f}|\alpha|^2 + 2e^{2f}\mathrm{Re}\langle\sigma,\alpha\rangle\ ,\ \varphi\Big)_{L^2} \\
&\quad + \Big(4i\Delta\eta + i\eta e^{2f}|\alpha|^2 - 2ie^{2f}\mathrm{Im}\langle\sigma,\alpha\rangle\ ,\ i\eta\Big)_{L^2} \\
&= \left\|e^f\sigma\right\|^2 - \left\|(\varphi - i\eta)e^f\alpha\right\|^2 . \qquad (3.2.4)
\end{aligned}
$$

In the same way we can compute a formula for the quotient L^2-metric on the total space $\mathfrak{P}$ of the Seiberg-Witten bundle. Recall that we identified the tangent space in $[B,\beta]_\infty$ of $\mathfrak{P}$ as:

$$
\begin{aligned}
T_{[B,\beta]_\infty}\mathfrak{P} &\cong (\mathcal{T}_0^*)^{-1}(i\mathbb{R}) \cap \ker\mathcal{T}_1 \\
&= (\ker\mathcal{T}_0^* \cap \ker\mathcal{T}_1) \oplus ((\mathcal{T}_0^*)^{-1}(i\mathbb{R}) \cap \mathrm{im}\mathcal{T}_0) \\
&= T_{[B,\beta]}\mathfrak{M} \oplus T_{[B,\beta]_\infty}\mathfrak{P}_{[B,\beta]_\infty} ,
\end{aligned}
$$

where $\mathfrak{P}_{[B,\beta]_\infty}$ denotes the fibre over $[B,\beta]_\infty$. Since in this identification, the tangent space of $\mathfrak{P}$ naturally splits into the tangent spaces of the fibre and the base, it remains only to compute the length square of a vector $X_{[B,\beta]_\infty}$ tangent to the fibre. Corresponding to the identification of the Seiberg-Witten bundle $\mathfrak{P} \to \mathfrak{M}$ with the Hopf fibration $H^0_{A_0}(M;L) \supset S^{2m+1} \to \mathbb{CP}(H^0_{A_0}(M;L))$, the fibre of $\mathfrak{P}$ over $[B,\beta] = [A_0 + 2id^cf + 2idh, e^{f-ih}\alpha]$ corresponds to the set of solutions $f(\alpha')$ of (3.1.4) for the family $\alpha' \in i\mathbb{R}\cdot\alpha$ of holomorphic sections. Thus the tangent space in $[B,\beta]_\infty$ of the fibre over $[B,\beta]$ consists of all those linearised configurations $(\nu,\phi) = (2id^c\varphi + 2id\eta, (\varphi - i\eta)e^{f-ih}\alpha + e^{f-ih}\sigma)$, which solve the equation (3.2.1) with $\sigma = \lambda\cdot\alpha$, $\lambda \in i\mathbb{R}$ – which simply yields $\varphi \equiv 0$ – and which satisfy the additional condition:

$$
\mathcal{T}_0^*\begin{pmatrix}\nu \\ \phi\end{pmatrix} = \mathcal{T}_0^*\begin{pmatrix}2id\eta \\ -i\eta e^{f-ih}\alpha + e^{f-ih}\sigma\end{pmatrix} = 2i\Delta\eta + i\eta e^{2f}|\alpha|^2 - ie^{2f}\mathrm{Im}\langle i\alpha,\alpha\rangle = ir \qquad (3.2.5)
$$

for some $ir \in i\mathbb{R}$. Thus given a vector $X_{[B,\beta]_\infty}$ tangent to the fibre $\mathfrak{P}_{[B,\beta]}$, which is represented by such a linearised configuration, we find for its length square in the quotient L^2-metric:

$$
\begin{aligned}
\left|X_{[B,\beta]}\right|^2 &= \left\|2id\eta\right\|^2 + \left\| - i\eta e^{f-ih}\alpha\right\|^2 + \left\|e^{f-ih}i\alpha\right\|^2 \\
&\quad + \mathrm{Re}\Big\{ \Big(- i\eta e^{f-ih}\alpha, e^{f-ih}i\alpha\Big)_{L^2} + \Big(e^{f-ih}i\alpha, -i\eta e^{f-ih}\alpha\Big)_{L^2} \Big\} \\
&= \left\|e^f\sigma\right\|^2 + \Big(4i\Delta\eta + i\eta e^{2f}|\alpha|^2 - 2ie^{2f}\mathrm{Im}\langle i\alpha,\alpha\rangle\ ,\ i\eta\Big)_{L^2} \\
&= \left\|e^f\alpha\right\|^2 - \left\|i\eta e^f\alpha\right\|^2 + \big(2ir,\eta\big) . \qquad (3.2.6)
\end{aligned}
$$

The canonically gauge fixed L^2-metrics

Now that we have computed explicit formulae for the quotient L^2-metrics on the Seiberg-Witten bundle $\mathfrak{P} \to \mathfrak{M}$, we can do the same for the canonically gauge fixed L^2-metrics. Recall that the gauge fixed premoduli space $\widetilde{\mathfrak{M}}_{fix}$ is the total space of a $U(1)$-bundle on $\mathfrak{M}$, which represents the Seiberg-Witten bundle $\mathfrak{P} \to \mathfrak{M}$. As described in section 2.6, the submanifold $\widetilde{\mathfrak{M}}_{fix} \subset \mathcal{C}^*$ carries an L^2-metric induced by restriction of the L^2-metric on $\mathcal{C}^*$. As above, we compute an explicit formula for the length square of a vector $X_{(B,\beta)_{fix}}$ tangent to $\widetilde{\mathfrak{M}}_{fix}$.

In section 2.6, we identified the tangent space of $\widetilde{\mathfrak{M}}_{fix}$ in $(B,\beta)_{fix}$ with the kernel of the operator $\mathcal{T}_1 \oplus d^*$, where $\mathcal{T}_1$ is the linearisation of the Seiberg-Witten map $\mathcal{SW}$. A tangent vector $X_{(B,\beta)_{fix}} \in T_{(B,\beta)_{fix}}\widetilde{\mathfrak{M}}_{fix}$ is a linearised configuration $X_{(B,\beta)_{fix}} = (2id^c\varphi + 2id\eta, (\varphi - i\eta)e^{f-ih}\alpha + e^{f-ih}\sigma)$, which satifies the equation (3.2.1) and

$$d^*(2id^c\varphi + 2id\eta) = 4\Delta i\eta = 0 \implies i\eta = ir \in \mathbb{R} \, . \tag{3.2.7}$$

We thus find for the length square of a tangent vector $X_{(B,\beta)_{fix}}$, written in the form of a linearised configuration $X_{(B,\beta)_{fix}} = (2id^c\varphi, (\varphi - ir)e^{f-ih}\alpha + e^{f-ih}\sigma)$:

$$\begin{aligned}
\left|X_{(B,\beta)_{fix}}\right|^2 &= \left\|2id^c\varphi\right\|^2 + \left\|(\varphi - ir)e^{f-ih}\alpha\right\|^2 + \left\|e^{f-ih}\sigma\right\|^2 \\
&\quad + \mathrm{Re}\Big\{ \Big((\varphi - ir)e^{f-ih}\alpha, e^{f-ih}\sigma\Big)_{L^2} + \Big(e^{f-ih}\sigma, (\varphi - ir)e^{f-ih}\alpha\Big)_{L^2} \Big\} \\
&= \left\|e^f\sigma\right\|^2 + \Big(4\Delta\varphi + \varphi e^{2f}|\alpha|^2 + 2e^{2f}\mathrm{Re}\langle\sigma,\alpha\rangle \, , \, \varphi\Big)_{L^2} \\
&\quad + \Big(ire^{2f}|\alpha|^2 - 2ie^{2f}\mathrm{Im}\langle\sigma,\alpha\rangle \, , \, ir\Big)_{L^2} \\
&= \left\|e^f\sigma\right\|^2 - \left\|\varphi e^f\alpha\right\|^2 + r^2\left\|e^f\alpha\right\|^2 - 2r\mathrm{Im}\big(e^f\sigma, e^f\alpha\big)_{L^2} \, .
\end{aligned} \tag{3.2.8}$$

In the next section, we use these formulae to identify the quotient L^2-metrics resp. the canonically gauge fixed L^2-metrics on the Seiberg-Witten bundle $\mathfrak{P} \to \mathfrak{M}$ in one special case explicitly up to isometry resp. up to reparametrisation along the fibres.

3.3 The moduli space on $\mathbb{CP}^2$

In this section, we discuss the quotient L^2-metrics and the canonically gauge fixed L^2-metrics on the Seiberg-Witten bundle in a special example. We take as Kähler surface (M,g) the complex projective space $\mathbb{CP}^2$ equipped with the Fubini-Study metric. By $\mathcal{O}(-1)$ we denote the tautological line bundle, i.e. the bundle whose fibre over a point $[x] \in \mathbb{CP}^2$ is the line $[x] \subset \mathbb{C}^3$, and by $\mathcal{O}(1)$ the dual of $\mathcal{O}(-1)$. Further we denote by $\mathcal{O}(\ell)$ resp. $\mathcal{O}(-\ell)$ the ℓ-fold tensor product of $\mathcal{O}(1)$ resp. $\mathcal{O}(-1)$. As is well known, the line bundle $\mathcal{O}(k)$ has holomorphic sections if and only if $k \geq 0$. In this case, the holomorphic sections can be identified with homogeneous polynomials (see e.g. [27]). Explicitly, there is a natural isomorphism of the space of holomorphic sections $H^0_{A_0}(\mathbb{CP}^2; \mathcal{O}(k))$ with the space $\mathbb{C}_k[z^0, z^1, z^2]$ of homogeneous polynomials of degree k in 3 complex variables. A polynomial $\alpha \in \mathbb{C}_1[z^0, z^1, z^2]$ acts on a section σ of $\mathcal{O}(-1)$ by $\langle\alpha,\sigma\rangle_{[x]} := \alpha\big(\sigma([x])\big)$.

The bundles $\mathcal{O}(k)$ carry natural hermitean metrics induced from the natural hermitean metric on $\mathcal{O}(-1)$. The Chern connection A_0 on $\mathcal{O}(k)$ with respect to this metric has the curvature form $F_{A_0} = -2\pi ik \cdot \omega$, where ω denotes the Kähler form of the Fubini-Study metric. The natural $\mathrm{Spin}^{\mathbb{C}}$-structure$P_0$ on $M = \mathbb{CP}^2$ has determinant line bundle $K^*_{\mathbb{CP}^2} = \mathcal{O}(3)$, and the $\mathrm{Spin}^{\mathbb{C}}$-structure$P_k = P_0 \otimes \mathcal{O}(k)$ has determinant line bundle $\det P_k = K^*_M \otimes \mathcal{O}(2k) = \mathcal{O}(3+2k)$. Thus the formula for the virtual dimension of the moduli space of $(\mathbb{CP}^2, P_k)$ yields:

$$\dim_{\mathbb{R}}\big(\mathfrak{M}(\mathbb{CP}^2, P_k)\big) = \frac{1}{4}\big((c_1(\det P_k))^2 - 2\chi - 3\tau\big) = \frac{1}{4}\big((3+2k)^2 - (6+3)\big) = (k^2 + 3k) \, .$$

Correspondingly, the (complex) dimension of the space of holomorphic sections of $\mathcal{O}(k)$ is given by:

$$\dim_{\mathbb{C}}\left(\mathbb{C}_k[z^0, z^1, z^2]\right) = \begin{pmatrix} k+2 \\ 2 \end{pmatrix} = \frac{(k+2)(k+1)}{2} = \frac{k^2+3k}{2} + 1 \,.$$

Due to our theorem 3.1.1, the moduli space $\mathfrak{M}(\mathbb{CP}^2, P_k)$ can be identified with the complex projective space $\mathbb{Pl}(H^0(M; \mathcal{O}(k)) \cong \mathbb{Pl}(\mathbb{C}_k[z^0, z^1, z^2])$. Thus it is a reasonable question, whether the L^2-metrics on $\mathfrak{M}$ have symmetries. In the special choice of $M = \mathbb{CP}^2$ equipped with the Spin$^{\mathbb{C}}$-structure $P_1 = P_0 \otimes \mathcal{O}(1)$ this is indeed the case: Since $M = \mathbb{CP}^2$ is the homogeneous space $U(3)/\big(U(2) \times U(1)\big)$, there is a natural $U(3)$-action on the space of holomorphic sections $H^0_{A_0}(\mathbb{CP}^2; \mathcal{O}(k)) = \mathbb{C}_k[z^0, z^1, z^2]$, which is given by:

$$U(3) \ni A : \alpha \mapsto A \cdot \alpha := \alpha \circ A^{-1}; \quad \text{i.e.} \quad (A \cdot \alpha)(x) := \alpha(A^{-1} \cdot x) \,.$$

We can use this action to identify the L^2-metrics up to isometry:

3.3.1 THEOREM. *The quotient L^2-metrics on the Seiberg-Witten bundle $\mathfrak{P} \to \mathfrak{M}$ on $M = \mathbb{CP}^2$ equipped with the Spin$^{\mathbb{C}}$-structure $P_1 := P_0 \otimes \mathcal{O}(1)$ are $U(3)$-invariant. Consequently, the quotient L^2-metric on the Seiberg-Witten moduli space $\mathfrak{M}(\mathbb{CP}^2, P_1) \approx \mathbb{CP}^2$ is the Fubini-Study metric, whereas the qotient L^2-metric on the total space $\mathfrak{P} \approx S^5$ of the Seiberg-Witten bundle is a Berger metric. The same holds for the gauge fixed L^2-metrics on the Seiberg-Witten bundle.*

Proof: For a polynomial $\alpha \in \mathbb{C}_1[z^0, z^1, z^2]$, the pointwise norm of the corresponding holomorphic section of $\mathcal{O}(1)$ is given by:

$$|\alpha|^2_{[x]} = \big|\alpha(u)\big|^2 \,, \tag{3.3.1}$$

where $u = (u^0, u^1, u^2)$ is a representative of $[x]$ of unit length. When $A \in U(3)$ acts on α, we have $|A \cdot \alpha|^2 = |\alpha|^2 \circ A^{-1}$. Thus when α changes to $\alpha \circ A^{-1}$, then the coefficients in the Kazdan-Warner type equation

$$2\Delta f + \frac{1}{2} e^{2f} |\alpha|^2 = 2\pi\lambda - i\Lambda_\omega(F^+_{A_0}) = 2\pi\lambda - 12\pi$$

all change by composition with A^{-1}. When $f_{(\alpha)}$ is the (unique) solution of the Kazdan-Warner type equation (3.1.4) for the holomorphic section α, then the solution $f_{(\alpha \circ A^{-1})}$ of the equation (3.1.4) for the section $\alpha \circ A^{-1}$ is given by $f_{(\alpha)} \circ A^{-1}$.

The volume form $dv_g = \frac{1}{2}\omega \wedge \omega$ of the Fubini-Study metric is $U(3)$-invariant. According to our formulae (3.2.4), (3.2.6), (3.2.8) for the quotient resp. gauge fixed L^2-metrics, the length square of any tangent vector is $U(3)$-invariant. Thus these metrics are $U(3)$-invariant. Since the only $U(3)$-invariant metrics on the Hopf fibration $S^5 \to \mathbb{CP}^2$ are the Berger metrics on S^5 resp. the Fubini-Study metric on $\mathbb{CP}^2$ (see [6], p. 180), the claim follows from our identification of the Seiberg-Witten bundle $\mathfrak{P} \to \mathfrak{M}$ with the Hopf fibration $S^{2m+1} \to \mathbb{CP}^m$. □

Note that the above argument strongly depends on the symmetry of $M = \mathbb{CP}^2$, and even on the fact that the moduli space $\mathfrak{M}(\mathbb{CP}^2, P_0 \otimes \mathcal{O}(1))$ is diffeomorphic to the underlying manifold $\mathbb{CP}^2$. A priori, there is no reason to hope for the L^2-metric on $\mathfrak{M}(M, P)$ to be $U(m)$-invariant for a manifold M other than $\mathbb{CP}^2$ and not even for other Spin$^{\mathbb{C}}$-structureson $\mathbb{CP}^2$.

The parametrised moduli space

We can now discuss the behaviour of the quotient resp. gauge fixed L^2-metric on the moduli space $M = \mathfrak{M}_{\mu^+}(\mathbb{CP}^2, P_1)$ under changes of the perturbation μ^+. Since $\mathbb{CP}^2$ is simply connected, we know from theorem 3.1.3, that for a perturbation μ^+ on the wall Γ_g^+, the moduli space $\mathfrak{M}$ collapses to a point. We can show that this collaps is indeed a collaps in the quotient resp. gauge fixed L^2-metric, i.e. that the diameter $\operatorname{diam}(\mathfrak{M}_{\mu^+})$ of the moduli space with respect to those metrics shrinks to 0, when the perturbation μ^+ approaches the wall Γ_g^+.

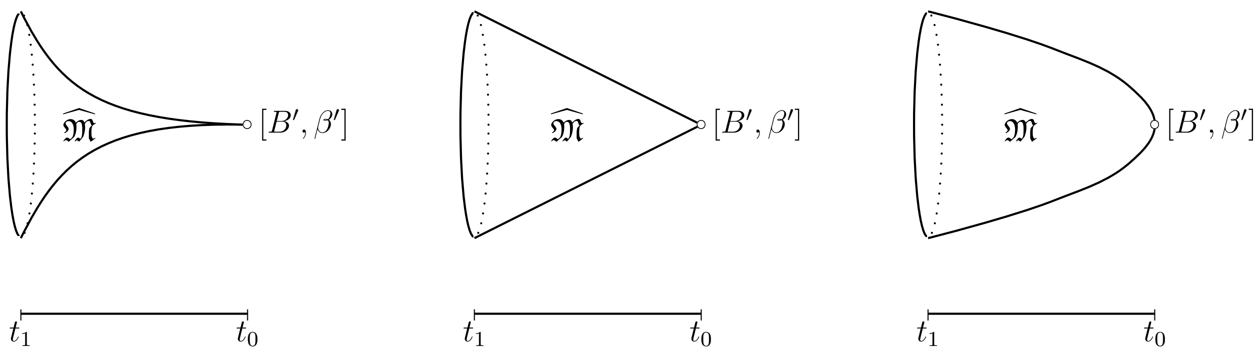
Since the Fubini-Study metric on $\mathbb{CP}^2$ has positive sectional curvature, the moduli spaces $\mathfrak{M}_{\mu^+}$ for perturbations sufficiently close to 0 are empty. By definition, the positive resp. negative chamber of $\Omega_+^2(M; i\mathbb{R}) - \Gamma_g^+$ is the set of those perturbations, for which $(i\mu^+ - 2\pi c_1(\det P), \omega)$ is positive resp. negative. For the family $\mu_t^+ = i\pi t\omega$, we find:

$$\begin{aligned}(i\mu^+ - 2\pi c_1(\det P_1), \omega) &= (-\pi t\omega - 2\pi c_1(\mathcal{O}(5)), \omega)\\ &= (-\pi t\omega - 2\pi \cdot 5\omega), \omega)\\ &= (-(t+10)\pi\omega, \omega)\\ &= -(t+10)\cdot 2\pi \mathrm{vol}(M)\ .\end{aligned}$$

Thus the perturbation $\mu^+ = 0$ lies in the negative chamber, and the family $\mu_t^+ = i\pi t\omega$ crosses the wall Γ_g^+ at $t_0 = -10$. According to our theorems 3.1.1 and 3.3.1, the moduli space $\mathfrak{M}_t := \mathfrak{M}_{\mu_t^+}$ is diffeomorphic to $\mathbb{CP}^2$, if the parameter t in the perturbation is sufficiently small, namely $t < \frac{\deg_\omega(\det P_1)}{\mathrm{vol}(M)} = -10$, and the quotient L^2-metric on $\mathfrak{M}_{\mu_t^+}$ is the Fubini-Study metric. On the wall Γ_g^+, the moduli space $\mathfrak{M}_{t_0}$ consists of a single point. Thus the moduli space $\mathfrak{M}_t$ changes the topological type (at least) two times, when we vary the perturbation $\mu_t^+ = i\pi t\omega$:

$$\mathfrak{M}_t := \mathfrak{M}_{\mu^+ = i\pi t\omega} \approx \begin{cases} \emptyset & : \ |t| \text{near} 0 \quad \text{(theorem 1.4.2)}\\ \{*\} & : \ t = -10 \quad \text{(proposition 3.1.3)}\\ \mathbb{CP}^2 & : \ t < -10 \quad \text{(theorem 3.1.1)}\end{cases}$$

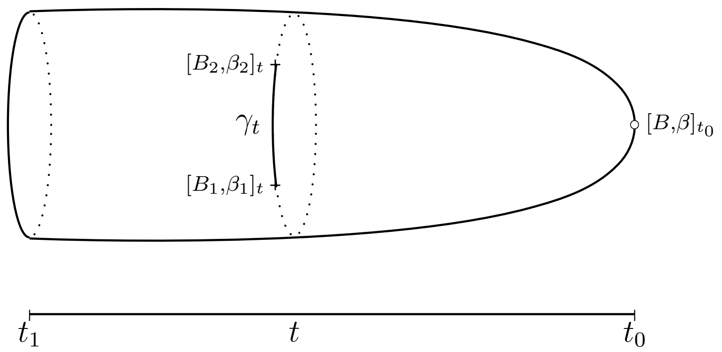
The parametrised moduli space $\widehat{\mathfrak{M}} = \bigsqcup_{t\in[t_1,t_0]} \mathfrak{M}_t$, with $t_1 < t_0$ is compact and has the homeomorphism type of a cone on $\mathfrak{M}_{t_1} \cong \mathbb{CP}^2$. The tip of the cone, i.e. the unique reducible gauge class, will be denoted by $[B', \beta']$. The fibre $\mathfrak{M}_{t_0} = \{[B', \beta']\}$ may or may not be singular in $\widehat{\mathfrak{M}}$.



The quotient L^2-metric on the Zariski tangent spaces of the parametrized moduli space $\widehat{\mathfrak{M}}$ is a Riemannian metric on the nonsingular part $\widehat{\mathfrak{M}}^* = \widehat{\mathfrak{M}} - \mathfrak{M}_{t_0} = \widehat{\mathfrak{M}} - \{[B', \beta']\}$. The Riemannian distance of this Riemannian metric can be extended in the singularity $[B', \beta']$, which then has a finite distance from any other point on $\widehat{\mathfrak{M}}^*$. Thus the Riemannian distance makes the parametrised moduli space into a complete metric space. It is clear, that the diameter of the fibre $\mathfrak{M}_t$ (with respect to this *extrinsic* metric) shrinks to 0, when the parameter t tends to t_0. That the same holds true with respect to the *intrinsic* metric of the fibres, i.e. the quotient L^2-metrics of $\mathfrak{M}_t$, is not a priori clear. Therefor we show:

3.3.2 PROPOSITION. *In the situation as above, the diameter* $\operatorname{diam}(\mathfrak{M}_t)$ *of the fibre* $\mathfrak{M}_t \approx \mathbb{CP}^2$ *shrinks to* 0 *when the perturbation* $\mu_t^+ = i\pi t\omega$ *approaches the wall* Γ_g^+, *i.e. when* t *tends to* t_0.

Proof: Suppose this were not the case. Then there would exist an $\epsilon > 0$ and a sequence of points $[B_1, \beta_1]_t, [B_2, \beta_2]_t \in \mathfrak{M}_t$ such that $\operatorname{dist}([B_1, \beta_1]_t, [B_2, \beta_2]_t) = \epsilon \ \forall t \in [t_1, t_0)$. The points $[B_1, \beta_1]_t, [B_2, \beta_2]_t$ can be joined by geodesics γ_t of length ϵ, and we may take γ_t to be parametrised by arc length. The theorem of Arzela-Ascoli implies that the curves γ_t converge uniformly when t tends to t_0, and it is clear that the limit γ_{t_0} is the constant curve in $\mathfrak{M}_{t_0} = \{[B', \beta']\}$. We show that the length of the limit is bounded from below by $\frac{\epsilon}{2}$:

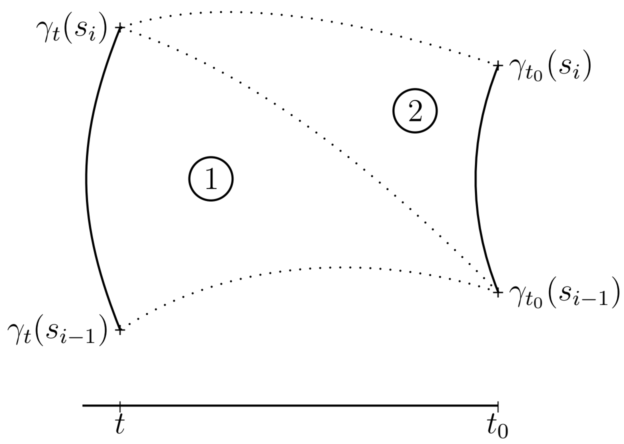


The length of the limit curve γ_{t_0} in the metric space $\widehat{\mathfrak{M}}$ is defined as the supremum over all partitions $0 = s_0 < \ldots < s_n = \epsilon$ of the interval $[0, \epsilon]$ of the length of the polygon through the points $\gamma_{t_0}(s_i)$:

$$\mathrm{L}(\gamma_{t_0}) := \sup_{s_0 < \ldots < s_n} \left(\sum_{i=1}^{n} \operatorname{dist}(\gamma_t(s_{i-1}), \gamma_t(s_i)) \right) .$$

For a given partition $s_0 < \ldots < s_n$, we find a parameter $t' \in [t_1, t_0]$ sufficiently close to t_0 such that

$$\operatorname{dist}(\gamma_t(s_i), \gamma_{t_0}(s_i)) < \delta := \frac{\epsilon}{4n} \qquad \forall t \in [t', t_0], \forall i = 0, \ldots, n \ . \tag{3.3.2}$$



Then the triangle inequalities for the triangles ① and ② read:

$$\begin{aligned} \mathrm{dist}(\gamma_t(s_{i-1}), \gamma_{t_0}(s_{i-1})) \ + \ \mathrm{dist}(\gamma_t(s_{i-1}), \gamma_t(s_i)) \ &\geq \ \mathrm{dist}(\gamma_t(s_i), \gamma_{t_0}(s_{i-1})) && ① \\ \mathrm{dist}(\gamma_t(s_i), \gamma_{t_0}(s_i)) \ + \ \mathrm{dist}(\gamma_{t_0}(s_{i-1}), \gamma_{t_0}(s_i)) \ &\geq \ \mathrm{dist}(\gamma_t(s_i), \gamma_{t_0}(s_{i-1})) && ② \end{aligned}$$

Combining these two inequalities with (3.3.2), we get:

$$\mathrm{dist}(\gamma_{t_0}(s_{i-1}), \gamma_{t_0}(s_i)) > \mathrm{dist}(\gamma_t(s_{i-1}), \gamma_t(s_i)) - 2\delta \qquad \forall t \in [t', t_0], \forall i = 0, \ldots, n \ .$$

We thus obtain the following estimate for the length of the curve γ_{t_0}:

$$\begin{aligned} \mathrm{L}(\gamma_{t_0}) \ &\geq \ \sum_{i=1}^{n} \mathrm{dist}(\gamma_{t_0}(s_{i-1}), \gamma_{t_0}(s_i)) \\ &> \ \sum_{i=1}^{n} \mathrm{dist}(\gamma_t(s_{i-i}), \gamma_t(s_i)) - 2\delta \\ &= \ \mathrm{L}(\gamma_t) - 2n\delta \\ &= \ \epsilon - 2n \cdot \frac{\epsilon}{4n} \\ &= \ \frac{\epsilon}{2} \ . \end{aligned}$$

This contradicts the fact, that the limit γ_{t_0} is the constant curve in $\mathfrak{M}_{t_0} = \{[B', \beta']\}$ and thus has length $L(\gamma_{t_0}) = 0$. □

Obviously, the above argument does not depend on the special case of the complex projective plane $\mathbb{CP}^2$ with the Spin$^{\mathbb{C}}$-structure$P_1 = P_0 \otimes \mathcal{O}(1)$. Thus whenever we take the quotient resp. gauge fixed L^2-metric on the Seiberg-Witten moduli space $\mathfrak{M}_{\mu^+}(M, P)$ of a simply connected Kähler surface M with a Spin$^{\mathbb{C}}$-structureP, such that $\mathfrak{M}(M, P)$ is nonempty, then we know that the diameter $\mathrm{diam}(\mathfrak{M}_{\mu^+})$ shrinks to 0, when the perturbation parameter μ^+ approaches the wall Γ_g^+, for which the moduli space collapses to a point.

3.4 Moduli spaces as symplectic quotients

In this section, we show that the Seiberg-Witten moduli space $\mathfrak{M}$ can be realised as the Marsden-Weinstein reduction or symplectic quotient of a certain submanifold of the irreducible configuration space $\mathcal{C}^*$. Therefor, we identify the Seiberg-Witten equations (1.4.4,1.4.6) as the zero locus equation of a moment map and the defining equation of the submanifold. We show that all manifolds in question are Kähler, which implies that the quotient L^2-metric on $\mathfrak{M}$ is a Kähler metric. Our argumentation follows similar work of HITCHIN on moduli spaces of vortices resp. Higgs bundles in [24, 26, 25]. For further results on moduli spaces of vortices we also refer to [39, 8, 9].

Recall that a moment map on a finite dimensional symplectic manifold (X, ω) with a symplectic action of a Lie group G is a map $\mu : X \to \mathfrak{g}^*$, which satisfies:

(i) $d\mu_Y = \imath_{\widetilde{Y}}\omega$, where $\mu_Y(m) := \langle Y, \mu(m)\rangle$

(ii) $\mu(g \cdot m) = (\mathrm{Ad}_g)^*\mu(m)$.

Here we denote by $\widetilde{Y}$ the fundamental vector field associated with an element $Y \in \mathfrak{g}$ of the Lie algebra, and by $\langle\cdot,\cdot\rangle$ the pairing of $\mathfrak{g}$ with $\mathfrak{g}^*$. Obviously, a moment map is unique only up to addition by constants in the center of $\mathfrak{g}^*$. Because of (ii) above, the zero locus of a moment map is invariant under G. As is well known from symplectic geometry (see e.g. [35, 52]), if 0 is a regular value of the moment map μ, then the quotient $\mu^{-1}(0)/G$ is a symplectic manifold, called symplectic quotient or the Marsden-Weinstein reduction of X with respect to G. Furthermore, if (X, ω) is a Kähler manifold and the group action preserves both the metric and the symplectic form, the symplectic quotient is also a Kähler manifold, which is then called the Kähler quotient of (X, ω) (for a proof, see e.g. [24]). Now we turn to an infinite dimensional analogue, namely the action of the gauge group $\mathcal{G}$ on the configuration space $\mathcal{C}$.

The configuration space $\mathcal{C} = \mathcal{A}(\det P) \times \Omega^0(M; L)$ carries a natural complex structure $\mathfrak{J}^{\mathcal{C}}$ induced by the canonical complex structure $\mathfrak{J}^{T^*M}$ on the first factor and the anticanonical complex structure $(-i)$ on the second:

$$\mathfrak{J}^{\mathcal{C}} = \mathfrak{J}^{T^*M} \oplus (-i) : \quad T_{(B,\beta)}\mathcal{C} = \Omega^1(M; i\mathbb{R}) \times \Omega^0(M; L) \;\to\; \Omega^1(M; i\mathbb{R}) \times \Omega^0(M; L)$$
$$\begin{pmatrix} \nu \\ \phi \end{pmatrix} \;\mapsto\; \begin{pmatrix} \mathfrak{J}^{T^*M}\nu \\ (-i)\cdot\phi \end{pmatrix}.$$

This complex structure is clearly parallel, since its definition does not depend on the base point (B, β). The L^2-metric on $\mathcal{C}$ is $\mathfrak{J}^{\mathcal{C}}$-invariant, and the associated 2-form $\Phi^{\mathcal{C}} = \left(\mathfrak{J}^{\mathcal{C}}\cdot,\cdot\right)_{L^2}$ reads:

$$\Phi^{\mathcal{C}}\left(\begin{pmatrix} \nu_1 \\ \phi_1 \end{pmatrix}, \begin{pmatrix} \nu_2 \\ \phi_2 \end{pmatrix}\right) = \mathrm{Re}\left\{\int_M \nu_1 \wedge \nu_2 \wedge \omega + \frac{1}{2}\int_M \langle(-i)\cdot\phi_1, \phi_2\rangle\omega \wedge \omega\right\} \tag{3.4.1}$$

This form is clearly non degenerate. Since its definition does not depend on the base point (B, β), it is also parallel, and thus the natural L^2-metric on the configuration space $\mathcal{C}$ is a Kähler metric. The action of the gauge group $\mathcal{G}$ on the configuration space $\mathcal{C}$ preserves both the L^2-metric and the symplectic form $\Phi^{\mathcal{C}}$. We identify the Lie algebra $\mathfrak{g} = \Omega^0(M; i\mathbb{R})$ of the gauge group via the L^2-metric as a subset of its dual $\mathfrak{g}^*$, and we denote the pairing of $\mathfrak{g}$ with $\Omega^0(M; i\mathbb{R}) \subset \mathfrak{g}^*$ by square brackets $\left\langle\cdot,\cdot\right\rangle_{L^2}$, although this is a global L^2 scalar product. Now we can show, that the action of the Fréchet Lie group $\mathcal{G}$ on the symplectic Fréchet manifold $(\mathcal{C}, \Phi^{\mathcal{C}})$ admits moment maps:

3.4.1 LEMMA. *Let $(\mathcal{C}, \Phi^{\mathcal{C}})$ be the infinite dimensional symplectic Fréchet manifold given by the 2-form $\Phi^{\mathcal{C}}$ associated with the L^2-metric on the configuration space $\mathcal{C}$. Then the map*

$$\begin{array}{rccl} \mu^{\mathcal{C}}: & \mathcal{C} & \to & \Omega^0(M, i\mathbb{R}) \subset \mathfrak{g}^* \\ & \begin{pmatrix} B \\ \beta \end{pmatrix} & \mapsto & \Lambda_\omega(F_B) - \frac{i}{2}|\beta|^2 \end{array}$$

is a moment map for the action of the gauge group $\mathcal{G}$.

Proof: We show that the map $\mu^{\mathcal{C}}$ satisfies the axioms $(i), (ii)$ for the symplectic manifold $(\mathcal{C}, \Phi^{\mathcal{C}})$ and the gauge group $\mathcal{G}$. For an element $Y = if \in \mathfrak{g} = \Omega^0(M; i\mathbb{R})$ of the Lie algebra of the gauge group, the associated fundamental vector field $\widetilde{Y}$ on $\mathcal{C}$ is given by:

$$\widetilde{Y}_{(B,\beta)} = \begin{pmatrix} 2idf \\ (-if) \cdot \beta \end{pmatrix}$$

Thus for the left hand side of (i) above, we compute:

$$\begin{aligned} \left(d\mu_Y^{\mathcal{C}}\right)_{(B,\beta)} \cdot \begin{pmatrix} \nu \\ \phi \end{pmatrix} &= \left\langle d\mu^{\mathcal{C}}_{(B,\beta)} \cdot \begin{pmatrix} \nu \\ \phi \end{pmatrix}, Y \right\rangle_{L^2} \\ &= \left\langle \Lambda_\omega(2d\nu), if \right\rangle_{L^2} + \left\langle \frac{-i\langle\beta,\phi\rangle - i\langle\phi,\beta\rangle}{2}, if \right\rangle_{L^2} \\ &= -\int_M 2ifd\nu \wedge \omega - \frac{1}{2}\int_M f \mathrm{Re}\langle\beta,\phi\rangle \omega\wedge\omega \\ &= +\int_M 2idf \wedge \nu \wedge \omega - \frac{1}{2}\int_M f \mathrm{Re}\langle\beta,\phi\rangle \omega\wedge\omega \ . \end{aligned}$$

For the right hand side of (i), we must contract the symplectic form Φ with the fundamental vector field $\widetilde{Y}$ and apply the resulting 1-form to $(\nu, \phi) \in T_{(B,\beta)}\mathcal{C}$:

$$\begin{aligned} \left(\imath_{\widetilde{Y}}\Phi^{\mathcal{C}}\right)_{(B,\beta)} \cdot \begin{pmatrix} \nu \\ \phi \end{pmatrix} &= \Phi^{\mathcal{C}}\left(\begin{pmatrix} 2idf \\ (-if)\cdot\beta \end{pmatrix}, \begin{pmatrix} \nu \\ \phi \end{pmatrix} \right) \\ &= \mathrm{Re}\left\{ \int_M 2idf \wedge \nu \wedge \omega + \frac{1}{2}\int_M \langle (-i)\cdot(-if)\cdot\beta, \phi\rangle\, \omega\wedge\omega \right\} \\ &= +\int_M 2idf \wedge \nu\wedge\omega - \frac{1}{2}\int_M f\mathrm{Re}\langle\beta,\phi\rangle\,\omega\wedge\omega \ . \end{aligned}$$

Thus we see that the property (i) in the definition of moment maps holds true. The equivariance property (ii) is easily verified, since the gauge group $\mathcal{G}$ is abelian, thus the coadjoint representation is trivial, and on the other hand, the function $\mu^{\mathcal{C}}(B, \beta) = \Lambda_\omega(F_B) - \frac{i}{2}|\beta|^2$ is gauge invariant. □

Since $\mathfrak{g} = \Omega^0(M; i\mathbb{R})$ is an abelian Lie algebra, we can add any purely imaginary function to $\mu^{\mathcal{C}}$ to get another moment map. Thus for any perturbation $\mu^+ \in \Omega^2_+(M; i\mathbb{R})$, the map

$$(B, \beta) \mapsto \Lambda_\omega(F_B) - \frac{i}{2}|\beta|^2 - \Lambda_\omega(\mu^+)$$

is another moment map. Thus the first Seiberg-Witten equation (1.4.4) appears as the zero locus equation for a moment map on the configuration space. In order show that the Seiberg-Witten moduli space $\mathfrak{M}$ is a symplectic quotient, we only need to impose the second equation (1.4.6). We thus consider the solution space $\mathfrak{N}$ of (1.4.6) as a submanifold of the configuration space:

$$\mathfrak{N} := \left\{ \begin{pmatrix} B \\ \beta \end{pmatrix} \in \mathcal{C} \;\middle|\; \sqrt{2}\,\overline{\partial}_B \beta = 0 \right\}$$

The restriction of the symplectic form $\Phi^{\mathcal{C}}$ to $\mathfrak{N}$ is non degenerate, thus $(\mathfrak{N}, \Phi^{\mathfrak{N}})$ is a symplectic submanifold of $(\mathcal{C}, \Phi^{\mathcal{C}})$. In fact we have:

3.4.2 PROPOSITION. *The solution space $\mathfrak{N}$ of (1.4.6) is a complex submanifold of the configuration space $\mathcal{C}$.*

Proof: We only need to show that the tangent bundle $T\mathfrak{N} \subset T\mathcal{C}$ is a complex subbundle. Note that the tangent space of $\mathfrak{N}$ in (B, β) is given by:

$$T_{(B,\beta)}\mathfrak{N} = \left\{ \begin{pmatrix} \nu \\ \phi \end{pmatrix} \;\middle|\; \sqrt{2}\big(\overline{\partial}^*_B \phi + \nu^{0,1} \wedge \beta\big) = 0 \right\}$$

It is easy to see that $\mathfrak{J}^{\mathcal{C}}$ preserves $T_{(B,\beta)}\mathfrak{N}$. For $(\nu, \phi) \in T_{(B,\beta)}\mathfrak{N}$, we have:

$$\mathfrak{J}^{\mathcal{C}} \begin{pmatrix} \nu \\ \phi \end{pmatrix} = \begin{pmatrix} \mathfrak{J}^{T^*M}\nu \\ (-i)\cdot\phi \end{pmatrix}$$

and we thus find:

$$\sqrt{2}\big(\overline{\partial}^*_B (-i)\cdot\phi + (\mathfrak{J}^{T^*M}\nu)^{0,1} \wedge \beta\big) = -\sqrt{2}i \cdot \big(\overline{\partial}^*_B \phi + \nu^{0,1}\beta)\big) = 0$$

Thus $\mathfrak{J}^{\mathcal{C}}$ preserves $T\mathfrak{N}$, hence $\mathfrak{N}$ is a complex submanifold of the configuration space $\mathcal{C}$. □

Since $\mathfrak{N} \subset \mathcal{C}$ is a complex submanifold, the L^2-metric restricts to a Kähler metric on $\mathfrak{N}$. Since $\mathfrak{N}$ is $\mathcal{G}$-invariant, the induced $\mathcal{G}$-action on $\mathfrak{N}$ preserves the induced metric and symplectic form. It also admits moment maps $\mu^{\mathfrak{N}}$, which are simply given by the restriction to $\mathfrak{N}$ of those on $\mathcal{C}$. According to theorem 1.4.2 and lemma 3.4.1 we know that the Seiberg-Witten moduli space $\mathfrak{M}$ is the quotient of the zero locus of the moment map $\mu^{\mathfrak{N}}$ by the gauge group $\mathcal{G}$. Now we can use standard arguments from symplectic resp. Kähler geometry to show that the induced quotient L^2-metric on $\mathfrak{M}$ is a Kähler metric:

3.4.3 THEOREM. *Let M be a compact connected Kähler surface, and denote by $\mu^{\mathfrak{N}}$ as above the moment map*

$$\begin{array}{rccc} \mu^{\mathfrak{N}}: & \mathfrak{N} & \rightarrow & \Omega^0(M; i\mathbb{R}) \subset \mathfrak{g}^* \\ & \begin{pmatrix} B \\ \beta \end{pmatrix} & \mapsto & \Lambda_\omega(F_B) - \frac{i}{2}|\beta|^2 - \Lambda_\omega(\mu^+) \end{array}$$

for the action of the gauge group $\mathcal{G}$ on $\mathfrak{N}$. Then the induced L^2-metric on the quotient

$$(\mu^{\mathfrak{N}})^{-1}(0)/\mathcal{G} = \mathfrak{M}$$

is a Kähler metric.

Proof: The submanifold $(\mu^{\mathfrak{N}})^{-1}(0) \subset \mathfrak{N} \subset \mathcal{C}$ carries a natural L^2-metric induced from the L^2-metric of the ambient space $\mathcal{C}$. Since the gauge group $\mathcal{G}$ acts by isometries, there is a unique Riemannian metric on the quotient space such that the projection

$$\pi : (\mu^{\mathfrak{N}})^{-1}(0) \to (\mu^{\mathfrak{N}})^{-1}(0)/\mathcal{G} = \mathfrak{M}$$

is a Riemannian submersion. This is of course nothing but the quotient L^2-metric as constructed in the previous chapter. When we denote by $\overline{X}, \overline{Y}$ the horizontal lifts to $(\mu^{\mathfrak{N}})^{-1}(0)$ of vector fields X, Y on $\mathfrak{M}$, then the covariant derivative of the vector fields X, Y is given by the projection of the covariant derivative of the vector fields $\overline{X}, \overline{Y}$:

$$\nabla^{\mathfrak{M}}_X Y = \pi_*(\nabla^{(\mu^{\mathfrak{N}})^{-1}(0)}_{\overline{X}} \overline{Y}) \ . \tag{3.4.2}$$

We need to define a complex structure on the tangent bundle of the moduli space $\mathfrak{M}$. As we did in the construction of the L^2-metric on $\mathfrak{M}$, we identify the pullback to $\widetilde{\mathfrak{M}}$ of the tangent space of $\mathfrak{M}$ to $(\mu^{\mathfrak{N}})^{-1}(0)$ with the intersection of the horizontal tangent bundle of $(\mu^{\mathfrak{N}})^{-1}(0)$ resp. $\mathfrak{N}$:

$$\pi^* T\mathfrak{M} \cong \mathcal{H}(\mu^{\mathfrak{N}})^{-1}(0) = \mathcal{H}\mathfrak{N}|_{(\mu^{\mathfrak{N}})^{-1}(0)} \cap T(\mu^{\mathfrak{N}})^{-1}(0) \ .$$

Note that the restriction to $(\mu^{\mathfrak{N}})^{-1}(0)$ of the tangent bundle of $\mathfrak{N}$ splits L^2-orthogonally as:

$$\begin{aligned} T\mathfrak{N}|_{(\mu^{\mathfrak{N}})^{-1}(0)} &= \mathcal{H}(\mu^{\mathfrak{N}})^{-1}(0) \oplus \big(\mathcal{H}(\mu^{\mathfrak{N}})^{-1}(0)\big)^{\perp} \\ &\cong \pi^* T\mathfrak{M} \oplus \big(\mathcal{H}(\mu^{\mathfrak{N}})^{-1}(0)\big)^{\perp} \\ &= \pi^* T\mathfrak{M} \oplus \mathrm{im}\mathcal{T}_0 \oplus \big(\ker d\mu^{\mathfrak{N}}\big)^{\perp} \ , \end{aligned}$$

where $\mathcal{T}_0$ as always denotes the linearisation of the orbit map. To define a complex structure on $T\mathfrak{M}$ it suffices to show, that the orthogonal complement $\mathrm{im}\mathcal{T}_0 \oplus \big(\ker d\mu^{\mathfrak{N}}\big)^{\perp}$ of $\pi^* T\mathfrak{M}$ as a subbundle of $T\mathfrak{N}_{|_{(\mu^{\mathfrak{N}})^{-1}(0)}}$ is preserved by the induced complex structure $\mathfrak{J}^{\mathfrak{N}}$:

- Note that the image of a function $Y = if \in \Omega^0(M; i\mathbb{R}) = \mathfrak{g}$ under the linearisation $\mathcal{T}_0$ of the orbit map through (B, β) is the same as the fundamental vector field $\widetilde{Y}$ in (B, β). For a vector $\widetilde{Y}_{(B,\beta)} \in \mathrm{im}\mathcal{T}_0$ we thus have:

  $$\big(\mathfrak{J}^{\mathfrak{N}}\widetilde{Y}_{(B,\beta)}, Z\big)_{L^2} = \Phi^{\mathfrak{N}}(\widetilde{Y}_{(B,\beta)}, Z) = d\mu^{\mathfrak{N}}_Y.Z \ ,$$

  thus $\mathfrak{J}^{\mathfrak{N}}\widetilde{Y}_{(B,\beta)}$ is orthogonal to the kernel of $d\mu^{\mathfrak{N}}$.

- For $Z \in \ker(d\mu^{\mathfrak{N}})^{\perp}$ suppose that $\mathfrak{J}^{\mathfrak{N}} Z \perp \mathrm{im}\mathcal{T}_0$. Then for any $Y \in \mathfrak{g} = \Omega^0(M; i\mathbb{R})$, we find:

  $$0 = \big(\widetilde{Y}, \mathfrak{J}^{\mathfrak{N}} Z\big)_{L^2} = -\Phi^{\mathfrak{N}}(\widetilde{Y}, Z) = -d\mu^{\mathfrak{N}}_Y.Z \ .$$

  Thus $Z \in (\ker d\mu^{\mathfrak{N}})^{\perp} \cap (\ker d\mu^{\mathfrak{N}}) = \{0\}$.

Hence the complex structure $\mathfrak{J}^{\mathfrak{N}}$ preserves the splitting

$$T\mathfrak{N}|_{(\mu^{\mathfrak{N}})^{-1}(0)} \cong \pi^* T\mathfrak{M} \oplus \big(\mathcal{H}(\mu^{\mathfrak{N}})^{-1}(0)\big)^{\perp}$$

and thus induces a complex structure $\mathfrak{J}^{\mathfrak{M}}$ on the moduli space $\mathfrak{M}$.

It only remains to show that this induced complex structure $\mathfrak{J}^{\mathfrak{M}}$ is parallel. This follows directly from the fact that the complex structure $\mathfrak{J}^{\mathfrak{N}}$ is parallel. Note that the Levi-Civita connection on $\mathfrak{M}$ is given by projection from the Levi-Civita connection on $\mathfrak{N}$ and that the projection commutes with the complex structure. For vector fields $X, Y \in \mathfrak{X}(\mathfrak{M})$ and horizontal lifts $\overline{X}, \overline{Y} \in \mathfrak{X}((\mu^{\mathfrak{N}})^{-1}(0))$ we find:

$$\begin{aligned}
\nabla^{\mathfrak{M}}_X \mathfrak{J}^{\mathfrak{M}} Y &= \pi_*\big(\mathrm{hor}(\nabla^{(\mu^{\mathfrak{N}})^{-1}(0)}_{\overline{X}} \mathfrak{J}^{\mathfrak{N}}\overline{Y})\big) \\
&= \pi_*\big(\mathrm{hor} \circ \tan(\nabla^{\mathfrak{N}}_{\overline{X}} \mathfrak{J}^{\mathfrak{N}}\overline{Y})\big) \\
&= \pi_*\big(\mathrm{hor} \circ \tan(\mathfrak{J}^{\mathfrak{N}} \nabla^{\mathfrak{N}}_{\overline{X}} \overline{Y})\big) \\
&= \pi_* \mathfrak{J}^{\mathfrak{N}}\big(\mathrm{hor} \circ \tan(\nabla^{\mathfrak{N}}_{\overline{X}} \overline{Y})\big) \\
&= \mathfrak{J}^{\mathfrak{M}} \pi_*\big(\mathrm{hor} \circ \tan(\nabla^{\mathfrak{N}}_{\overline{X}} \overline{Y})\big) \\
&= \mathfrak{J}^{\mathfrak{M}} \nabla^{\mathfrak{M}}_X Y \, .
\end{aligned} \tag{3.4.3}$$

Here we denoted by hor the orthogonal projection onto the horizontal subbundle $\mathcal{H}(\mu^{\mathfrak{N}})^{-1}(0) \subset T(\mu^{\mathfrak{N}})^{-1}(0)$ and by $\tan$ the orthogonal projection onto the subbundle $T(\mu^{\mathfrak{N}})^{-1}(0) \subset T\mathfrak{N}|_{(\mu^{\mathfrak{N}})^{-1}(0)}$. Furthermore, for the horizontal lifts $\overline{X}, \overline{Y}$ we have $\overline{\nabla^{\mathfrak{M}}_X Y} = \mathrm{hor}\nabla^{(\mu^{\mathfrak{N}})^{-1}(0)}_{\overline{X}} \overline{Y}$.

Thus we have shown that the induced complex structure on the moduli space is parallel with respect to the Levi-Civita connection of the induced L^2-metric. Hence the quotient L^2-metric on $\mathfrak{M}$ is a Kähler metric. □

This result together with our identification of the moduli spaces as complex projective spaces raises the question whether the L^2-metric is the Fubini-Study metric in any case, not only in the special situation discussed in section 3.3. To answer this question positively, one would have to show that the L^2-metric satifies one (and hence all) properties, which uniquely characterise the Fubini-Study metric among the Kähler metrics on a complex projective space $\mathbb{CP}^m$, namely, that the isometry group is $U(m)$, that the holomorphic sectional curvature is constant, or that the Kähler metric is also an Einstein metric. To show that the first property holds is hopeless in situations unlike the special one discussed in section 3.3. To compute the holomorphic sectional curvature with respect the complex structure as defined above, one would have to circumvent the Green operators in the formulae for the sectional curvature. Thus the only accessible property which remains, is the third, and we think, that it might be possible to show – under some additional prerequisites – that the L^2-metrics are indeed Kähler-Einstein metrics.

Notation index

Index

Bibliography

[1] Franka Babadshanjan and Lutz Habermann, *A family of metrics on the moduli space of BPST-instantons*, Ann. Global Anal. Geom. **9** (1991), no. 3, 245–252. MR MR1143405 (92i:58030)

[2] Stefan Bauer, *Refined Seiberg-Witten invariants*, preprint math.GT/0312523 v1 (2003), 1–33.

[3] ———, *A stable cohomotopy refinement of Seiberg-Witten invariants. II*, Invent. Math. **155** (2004), no. 1, 21–40. MR 2 025 299

[4] Stefan Bauer and Mikio Furuta, *A stable cohomotopy refinement of Seiberg-Witten invariants. I*, Invent. Math. **155** (2004), no. 1, 1–19. MR 2 025 298

[5] Christian Becker, *Grundbegriffe der Theorie der Fréchet-Mannigfaltigkeiten*, Diplomarbeit, Universität Hamburg, 2001.

[6] Arthur L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987. MR 88f:53087

[7] Olivier Biquard, *Les équations de Seiberg-Witten sur une surface complexe non kählérienne*, Comm. Anal. Geom. **6** (1998), no. 1, 173–197. MR 2000d:57048

[8] Steven B. Bradlow, *Vortices in holomorphic line bundles over closed Kähler manifolds*, Comm. Math. Phys. **135** (1990), no. 1, 1–17. MR 92f:32053

[9] ———, *Special metrics and stability for holomorphic bundles with global sections*, J. Differential Geom. **33** (1991), no. 1, 169–213. MR 91m:32031

[10] Jeff Cheeger and David G. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland Publishing Co., Amsterdam, 1975. MR 56 #16538

[11] Heberto del Rio Guerra, *Seiberg-Witten invariants of nonsimple type and Einstein metrics*, Ann. Global Anal. Geom. **21** (2002), no. 4, 319–339. MR 2003g:53062

[12] S. K. Donaldson, *The Seiberg-Witten equations and* 4*-manifold topology*, Bull. Amer. Math. Soc. (N.S.) **33** (1996), no. 1, 45–70. MR 96k:57033

[13] S. K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1990. MR MR1079726 (92a:57036)

[14] Daniel S. Freed and Karen K. Uhlenbeck, *Instantons and four-manifolds*, Mathematical Sciences Research Institute Publications, vol. 1, Springer-Verlag, New York, 1991. MR MR1081321 (91i:57019)

[15] Oscar García-Prada, *Invariant connections and vortices*, Comm. Math. Phys. **156** (1993), no. 3, 527–546. MR 95d:53024

[16] ______, *Seiberg-Witten invariants and vortex equations*, Symétries quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, pp. 885–934. MR 99k:58190

[17] David Groisser, *The L^2 metric in gauge theory: an introduction and some applications*, Symplectic singularities and geometry of gauge fields (Warsaw, 1995), Banach Center Publ., vol. 39, Polish Acad. Sci., Warsaw, 1997, pp. 317–329. MR MR1458668 (98h:58028)

[18] David Groisser and Michael K. Murray, *Instantons and the information metric*, Ann. Global Anal. Geom. **15** (1997), no. 6, 519–537. MR MR1608663 (99b:58042)

[19] David Groisser and Thomas H. Parker, *The Riemannian geometry of the Yang-Mills moduli space*, Comm. Math. Phys. **112** (1987), no. 4, 663–689. MR 89b:58024

[20] ______, *The geometry of the Yang-Mills moduli space for definite manifolds*, J. Differential Geom. **29** (1989), no. 3, 499–544. MR 90f:58021

[21] Lutz Habermann, *A family of metrics on the moduli space of* CP^2 *instantons*, Comm. Math. Phys. **149** (1992), no. 1, 209–216. MR MR1182418 (93f:58025)

[22] ______, *The L^2-metric on the moduli space of* $\mathrm{SU}(2)$*-instantons with instanton number* 1 *over the Euclidean* 4*-space*, Ann. Global Anal. Geom. **11** (1993), no. 4, 311–322. MR MR1246191 (95f:58019)

[23] Richard S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 1, 65–222. MR MR656198 (83j:58014)

[24] Nigel J. Hitchin, *Metrics on moduli spaces*, The Lefschetz centennial conference, Part I (Mexico City, 1984), Contemp. Math., vol. 58, Amer. Math. Soc., Providence, RI, 1986, pp. 157–178. MR 87k:53106

[25] ______, *The geometry and topology of moduli spaces*, Global geometry and mathematical physics (Montecatini Terme, 1988), Lecture Notes in Math., vol. 1451, Springer, Berlin, 1990, pp. 1–48. MR 92a:58023

[26] ______, *The symplectic geometry of moduli spaces of connections and geometric quantization*, Progr. Theoret. Phys. Suppl. (1990), no. 102, 159–174 (1991). MR 93f:58089

[27] Daniel Huybrechts, *Complex geometry – an introduction*, Universitext, Springer-Verlag, Berlin, 2004.

[28] Masashi Ishida and Claude LeBrun, *Curvature, connected sums, and Seiberg-Witten theory*, Comm. Anal. Geom. **11** (2003), no. 5, 809–836. MR MR2032500

[29] Jerry L. Kazdan and F. W. Warner, *Curvature functions for compact* 2*-manifolds*, Ann. of Math. (2) **99** (1974), 14–47. MR 49 #7949

[30] Shoshichi Kobayashi and Katsumi Nomizu, *Foundations of differential geometry. Vol. I*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1996. MR 97c:53001a

[31] T. J. Li and A. Liu, *General wall crossing formula*, Math. Res. Lett. **2** (1995), no. 6, 797–810. MR 96m:57053

[32] Yoshiaki Maeda, Steven Rosenberg, and Philippe Tondeur, *The mean curvature of gauge orbits*, Global analysis in modern mathematics (Orono, ME, 1991; Waltham, MA, 1992), Publish or Perish, Houston, TX, 1993, pp. 171–217. MR 95k:58006

[33] ______, *Minimal submanifolds in infinite dimensions*, Analysis and geometry in foliated manifolds (Santiago de Compostela, 1994), World Sci. Publishing, River Edge, NJ, 1995, pp. 177–182. MR 97g:58012

[34] ______, *Minimal orbits of metrics*, J. Geom. Phys. **23** (1997), no. 3-4, 319–349. MR 98j:58019

[35] Dusa McDuff and Dietmar Salamon, *Introduction to symplectic topology*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1998. MR 2000g:53098

[36] John Douglas Moore, *Lectures on Seiberg-Witten invariants*, Lecture Notes in Mathematics, vol. 1629, Springer-Verlag, Berlin, 2001. MR 2002a:57043

[37] John W. Morgan, *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, Mathematical Notes, vol. 44, Princeton University Press, Princeton, NJ, 1996. MR 97d:57042

[38] Liviu I. Nicolaescu, *Notes on Seiberg-Witten theory*, Graduate Studies in Mathematics, vol. 28, American Mathematical Society, Providence, RI, 2000. MR 2001k:57037

[39] Christian Okonek and Andrei Teleman, *The coupled Seiberg-Witten equations, vortices, and moduli spaces of stable pairs*, Internat. J. Math. **6** (1995), no. 6, 893–910. MR 97a:32012

[40] ______, *Seiberg-Witten invariants for manifolds with* $b_+ = 1$ *and the universal wall crossing formula*, Internat. J. Math. **7** (1996), no. 6, 811–832. MR 97j:57047

[41] Barrett O'Neill, *Semi-Riemannian geometry*, Pure and Applied Mathematics, vol. 103, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983. MR 85f:53002

[42] Peter Ozsváth and Zoltán Szabó, *Higher type adjunction inequalities in Seiberg-Witten theory*, J. Differential Geom. **55** (2000), no. 3, 385–440. MR 2002j:57061

[43] Dietmar Salamon, *Spin-geometry and Seiberg-Witten invariants*, preprint (Warwick University); to appear in Birkhäuser Verlag, 1994.

[44] ______, *Removable singularities and a vanishing theorem for Seiberg-Witten invariants*, Turkish J. Math. **20** (1996), no. 1, 61–73. MR 97f:57035

[45] Hirofumi Sasahira, *Spin structures on the Seiberg-Witten moduli spaces*, preprint: math.DG/0404275 (2004), 1–10.

[46] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nuclear Phys. B **426** (1994), no. 1, 19–52. MR 95m:81202a

[47] ———, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, Nuclear Phys. B **431** (1994), no. 3, 484–550. MR MR1306869 (95m:81203)

[48] S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861–866. MR MR0185604 (32 #3067)

[49] T. N. Subramaniam, *Slices for actions of infinite-dimensional groups*, Differential analysis in infinite-dimensional spaces (Albany, N.Y., 1983), Contemp. Math., vol. 54, Amer. Math. Soc., Providence, RI, 1986, pp. 65–77. MR MR849796 (87i:58013)

[50] Clifford Henry Taubes, *The Seiberg-Witten and Gromov invariants*, Math. Res. Lett. **2** (1995), no. 2, 221–238. MR 96a:57076

[51] Edward Witten, *Monopoles and four-manifolds*, Math. Res. Lett. **1** (1994), no. 6, 769–796. MR 96d:57035

[52] N. M. J. Woodhouse, *Geometric quantization*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1992. MR MR1183739 (94a:58082)