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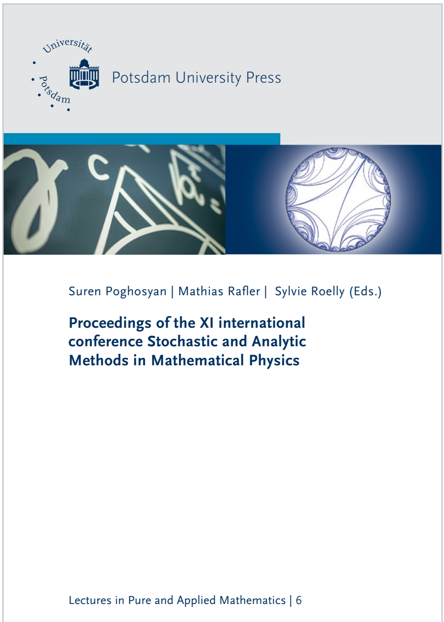

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17

Zero-range hamiltonians for three quantum particles

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Abstract. *Characterisation of the confined states of quantum systems made of many particles interacting via short range forces was the main goal for theoretical physicists investigating the structure of nuclei in the early years of Quantum Mechanics. A rigorous formulation of the problem was given at the beginning of the sixties by the Russian school of mathematical physics. The analysis of the three-body problem already revealed intriguing pathologies opening at the same time promising prospects for the future. We summarise the history and recent attempts of this line of research.*

1 Introduction

The three quantum particle problem is a line of research that Robert A. Minlos has been following for most of his scientific career. Together with Berezin and Faddeev he framed the problem of zero-range interactions in Quantum Mechanics inside the theory of self-adjoint extensions of symmetric operators. He was able to formulate in a rigorous way the

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unboundedness problem for three-particle zero-range Hamiltonians and he also suggested possible ways out of such a difficulty.

Following his suggestions, resumed later by Albeverio, Hoegh-Krohn and Wu [2], we attempted to work out partial solutions to the problem. It is worth mentioning that nowadays the interest in the problem has shifted toward many other research fields, e. g. it is actively investigated by physicists and applied mathematicians working in low temperature physics of quantum many particle systems (see e. g. [7] and reference therein). We want first to give an outline of the way zero-range interactions and the quantum three-body problem appeared in the physical literature.

Heuristically, point interactions are quantum interactions supported on points or "thin sets" (e. g. low dimensional hypersurfaces). They are also called zero-range interactions or contact interactions. They are used whenever the range of interparticle interactions is much shorter than other relevant length scales. They have the advantage of permitting better insight allowing for "explicit computations": for this reason they are used in the mathematical modeling of many natural phenomena.

Let $\mathscr{M}$ be a submanifold of $\mathbb{R}^d$ of dimension $s < d$. Consider the operator

$$H_{0,0} := -\Delta \restriction C_0^\infty\big(\mathbb{R}^d \setminus \mathscr{M}\big)$$

As a restriction of a self-adjoint operator $H_{0,0}$ is symmetric but not self-adjoint. In fact, denoting with $(\,\cdot\,,\,\cdot\,)$ the inner product in $L^2(\mathbb{R}^d)$,

$$D(H_{0,0}^*) = \Big\{\psi \in L^2(\mathbb{R}^d) \Big| \, |(\psi, -\Delta\phi)| < C\|\phi\| \ \forall \phi \in C_0^\infty(\mathbb{R}^d \setminus \mathscr{M})\Big\}$$

includes any function in $D(-\Delta) = H^2(\mathbb{R}^d)$ as well as any function $\psi \in L^2(\mathbb{R}^d)$ such that

$$-\Delta\psi = \xi + T, \qquad \xi \in L^2(\mathbb{R}^d),\, T \in D'(\mathbb{R}^d) \text{ with } \operatorname{supp} T \subseteq \mathscr{M}$$

where $D'(\mathbb{R}^d)$ is the vector space of distributions in $\mathbb{R}^d$.

Definition 17.1 Any (non-trivial) self-adjoint extension of $H_{0,0}$ (if any) will be denoted as a Hamiltonian with zero-range interaction on $\mathscr{M}$.

The simplest case is when $\mathscr{M} = \underline{y} \equiv \{y_1, \dots, y_N\} \in \mathbb{R}^{Nd}$, i. e. a discrete set of points of $\mathbb{R}^d$.

Take $\psi = G^z(\cdot - y_i)$ where $G^z = \mathscr{F}^{-1}(k^2 - z)^{-1}$ for any $z \in \mathbb{C} \setminus \mathbb{R}^+$. It belongs to $L^2(\mathbb{R}^d)$ for $d = 1,2,3$ and

$$\big(G^z(\cdot - y_i), -\Delta_x \phi\big) = \big([-\Delta_x - z]G^z(\cdot - y_i), \phi\big) + \big(zG^z(\cdot - y_i), \phi\big) = \big(zG^z(\cdot - y_i), \phi\big)$$

for all $\phi \in C_0^\infty\big(\mathbb{R}^d \setminus \{y_1, \dots, y_N\}\big)$, which means that $G^z \in D(H^*_{0,0})$ (but it does not belong to $H^2(\mathbb{R}^d)$) and that $G^z(\cdot - y_i)$ is an eigenvector of $H^*_{0,0}$ relative to the eigenvalue z. The same result holds true for any partial derivative of G^z belonging to $L^2(\mathbb{R}^d)$ (which is true only for the first derivatives of G^z in $d = 1$).

It is possible to classify the entire family of self-adjoint extensions of $H_{0,0}$ for $d = 1,2$ and 3. It turns out that in each dimension the family of self-adjoint extensions shows peculiar properties. We will be interested in particular in the following operators that can be proved (see [1]) to be a subset of the family of self-adjoint extensions of $H_{0,0}$ in $L^2(\mathbb{R}^3)$.

For any $\underline{\alpha} = \{\alpha_1, \dots, \alpha_n\} \in \mathbb{R}^n$ and $\underline{y} = \{y_1, \dots, y_n\} \in \mathbb{R}^{3n}$, the operator $H_{\underline{\alpha},\underline{y}}$ defined by

$$\begin{aligned} D(H_{\underline{\alpha},\underline{y}}) = \Big\{ u \in L^2(\mathbb{R}^3) \Big| \, u = \phi_\lambda + \sum_{k=1}^n q_k G_\lambda(\cdot - y_k)\phi_\lambda \in H^2(\mathbb{R}^3), \\ \phi_\lambda(y_j) = \sum_{k=1}^n \big[\Gamma_{\underline{\alpha},\underline{y}}(\lambda)\big]_{jk} q_k, j = 1, \dots, n \Big\} \end{aligned} \tag{17.1}$$

$$\big(H_{\underline{\alpha},\underline{y}} + \lambda\big)u = \big(-\Delta + \lambda\big)\phi_\lambda \tag{17.2}$$

where $G_\lambda \equiv G^z\big|_{z=-\lambda}$ and

$$\big[\Gamma_{\underline{\alpha},\underline{y}}(\lambda)\big]_{jk} = \left(\alpha_j + \frac{\sqrt{\lambda}}{4\pi}\right)\delta_{jk} - G_\lambda(y_j - y_k)(1 - \delta_{jk}) \tag{17.3}$$

vanishing at $y_1, \dots, y_n$ one has, see (17.1), $q_k = 0, \forall k$, and then, from (17.2), $H_{\underline{\alpha},\underline{y}} u = -\Delta u$.

At each point $y_j \in \mathbb{R}^3$ the elements of the domain satisfy a boundary condition expressed by the last equality in (17.1). If we define $r_j = |x - y_j|$ it is easy to see that the boundary condition satisfied by functions $u \in D(H_{\underline{\alpha},\underline{y}})$ can be equivalently written as

$$\lim_{r_j \to 0} \left[\frac{\partial(r_j u)}{\partial r_j} - 4\pi\alpha_j(r_j u)\right] = 0, \qquad j = 1, \dots, n. \tag{17.4}$$

This explains the term “local” given to this class of extensions.

The spectral structure of local point interaction Hamiltonians is not at all trivial and it is easily investigated. In fact, $-\lambda$ is a negative eigenvalue of the Hamiltonian $H_{\underline{\alpha},\underline{y}}$ if and only if $\det \Gamma_{\underline{\alpha},\underline{y}}(\lambda) = 0$ and the generalised eigenfunctions are non-trivial and explicitly known. Details can be found in [1]. Here, we want only to point out that if two scatterer positions come close one to the other the off-diagonal terms of the matrix (17.3) become very large with respect to any value of the strength parameters $\underline{\alpha}$. It is easy to check that in the limit of zero distance the ground state eigenvalue of the Hamiltonian is approaching $-\infty$ (for details when n=2, see [1]).

Let us now consider the much more difficult case of many particles. The Hamiltonians for a system of N particles interacting via zero-range forces will be defined as any self-adjoint extension of

$$-\sum_{i=1}^{N} \Delta_{x_i} \upharpoonright C_0^\infty\Big(\mathbb{R}^{dN} \setminus \bigcup_{i<j} \sigma_{ij}\Big)$$

$$\sigma_{ij} = \big\{x = (x_1, \ldots, x_N) \in \mathbb{R}^{dN} \big| x_i = x_j\big\}$$

acting on state vectors with symmetry properties which will depend on the type of particles under investigation.

In the following, we will consider the case of $N = 3$ identical bosons in $\mathbb{R}^3$ with masses $1/2$, in the center of mass reference frame. Expressed in terms of the Jacobi coordinates (see e. g. [16])

$$\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_3, \quad \mathbf{y} = \frac{1}{2}(\mathbf{x}_2 + \mathbf{x}_3) - \mathbf{x}_1, \qquad \mathbf{x}_i \in \mathbb{R}^3, i = 1,2,3 \tag{17.5}$$

the space of square integrable functions completely symmetric in the exchange of particle coordinates is

$$L^2_s(\mathbb{R}^6) = \Big\{\psi \in L^2(\mathbb{R}^6) \,\Big|\, \psi(\mathbf{x},\mathbf{y}) = \psi(-\mathbf{x},\mathbf{y}) = \psi\Big(\tfrac{1}{2}\mathbf{x} + \mathbf{y}, \tfrac{3}{4}\mathbf{x} - \tfrac{1}{2}\mathbf{y}\Big)\Big\}. \tag{17.6}$$

Zero-range interactions among particles will be confined on the three-dimensional hyperplanes

$$\Sigma = \{\mathbf{x} = 0\} \cup \{\mathbf{y} - \mathbf{x}/2 = 0\} \cup \{\mathbf{y} + \mathbf{x}/2 = 0\}. \tag{17.7}$$

As we pointed out already, this means that we are looking for Hamiltonians in $L^2_s(\mathbb{R}^6)$ which are non-trivial s.a. extension of the operator

$$\tilde{H}_0 = -\Delta_{\mathbf{x}} - \frac{3}{4}\Delta_{\mathbf{y}}, \qquad D(\tilde{H}_0) = \left\{ \psi \in L^2_s(\mathbb{R}^6) \,\middle|\, \psi \in H^2(\mathbb{R}^6), \psi\big|_\Sigma = 0 \right\}. \tag{17.8}$$

The defect spaces of $\tilde{H}_0$ are now of infinite dimensions. This makes the examination of classes of self-adjoint extensions much more difficult and their physical interpretation more complicated.

Ter-Martirosian and Skorniakov [17], on the basis of the analogy with the point interaction potentials, proposed to define an operator H_α acting as the free Hamiltonian outside the hyperplanes and satisfying a boundary condition close to the hyperplanes. Specifically, they impose for the functions in the domain of the Hamiltonian the boundary condition

$$\psi(\mathbf{x},\mathbf{y}) = \frac{\xi(\mathbf{y})}{|\mathbf{x}|} + \alpha\,\xi(\mathbf{y}) + o(1), \qquad \text{for } |\mathbf{x}| \to 0 \text{ and } y \neq 0 \tag{17.9}$$

where ξ is a function depending on ψ. The same behaviour must hold close to the other coincidence hyperplanes for symmetry reasons. Being the singular part in (17.9), the behaviour of the potential of a charge ξ distributed on the hyperplane, the operators H_α and the boundary condition were expressed in terms of charge distribution potentials, i. e. imposing that functions in the domain of H_α were the sum of a regular and a singular part in the following way:

$$\psi = w^\lambda + \mathscr{G}^\lambda \xi, \qquad w^\lambda \in H^2(\mathbb{R}^6), \tag{17.10}$$

where $\lambda > 0$ and

$$\widehat{\mathscr{G}^\lambda \xi}(\varkappa,\mathbf{p}) = \sqrt{\frac{2}{\pi}} \cdot \frac{\hat{\xi}(\mathbf{p}) + \hat{\xi}(\varkappa - \frac{1}{2}\mathbf{p}) + \hat{\xi}(-\varkappa - \frac{1}{2}\mathbf{p})}{|\varkappa|^2 + \frac{3}{4}|\mathbf{p}|^2 + \lambda}. \tag{17.11}$$

is the ($\lambda-$) potential of a charge density ξ identically distributed on each coincidence plane. The behaviour of the function $\mathscr{G}^\lambda \xi(\mathbf{x},\mathbf{y})$ close to the planes is easily computed:

$$\mathscr{G}^\lambda \xi(\mathbf{x},\mathbf{y}) = \frac{\xi(\mathbf{y})}{|\mathbf{x}|} - \frac{1}{(2\pi)^{3/2}} \int d\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{y}} \big(T^\lambda \hat{\xi}\big)(\mathbf{p}) + o(1), \tag{17.12}$$

where

$$\big(T^\lambda \hat{\xi}\,\big)(\mathbf{p}) := \sqrt{\frac{3}{4}|\mathbf{p}|^2+\lambda}\cdot \hat{\xi}(\mathbf{p}) - \frac{1}{\pi^2}\int d\mathbf{p}'\,\frac{\hat{\xi}(\mathbf{p}')}{|\mathbf{p}|^2+|\mathbf{p}'|^2+\mathbf{p}\cdot\mathbf{p}'+\lambda}. \tag{17.13}$$

In this way the boundary condition (17.9) can be rephrased as an integral equation for the "charges" ξ (for details see [5] and references therein). As noticed by Danilov [9], the operators constructed in this way are not self-adjoint and admit a continuum set of eigenvalues tending to minus infinity.

2 Minlos and Faddeev seminal papers

In two fundamental papers [13, 14] on the subject Minlos and Faddeev succeeded, in 1962, in rigorously translating the attempts of Ter-Martirosian and Skornyakov in terms of Birman's theory of self-adjoint extensions of positive symmetric operators. They proved that the boundary condition (17.9) about the behaviour of functions in the domain of the Hamiltonians close to the coincidence planes was not enough to guarantee their self-adjointness.

The final result can be summarised in the following characterisation, written in momentum space, of a two-parameter family of self-adjoint Hamiltonians:

$$\begin{aligned} D(H_{\alpha,\beta}) = \Big\{ \psi \in L^2_s(\mathbb{R}^6) \,\Big|\, \psi = w^\lambda + \mathscr{G}^\lambda \xi,\ w^\lambda \in H^2(\mathbb{R}^6),\ \hat{\xi} \in D(T^\lambda_\beta), \\ \alpha\,\hat{\xi}(\mathbf{p}) + \big(T^\lambda \hat{\xi}\big)(\mathbf{p}) = \widehat{w^\lambda(0,\cdot)}(\mathbf{p}) \Big\}, \end{aligned} \tag{17.14}$$

$$(H_{\alpha,\beta}+\lambda)\psi = (H_0+\lambda)w^\lambda, \tag{17.15}$$

where

$$H_0 = -\Delta_{\mathbf{x}} - \frac{3}{4}\Delta_{\mathbf{y}}, \qquad D(H_0) = H^2(\mathbb{R}^6), \tag{17.16}$$

with

$$\begin{aligned} D(T^\lambda_\beta) = \Big\{ \hat{\xi} \in L^2(\mathbb{R}^3) \,\Big|\, \hat{\xi} = \hat{\xi}_1 + \hat{\xi}_2,\ \hat{\xi}_1 \in D(T^\lambda) \text{ and} \\ \hat{\xi}_2(\varkappa) = \frac{c}{|\varkappa|^2+1}\Big(\beta \sin\big(s_0 \log|\varkappa|\big) + \cos\big(s_0 \log|\varkappa|\big)\Big) \Big\} \end{aligned} \tag{17.17}$$

where c is an arbitrary constant, s_0 is the positive solution of the equation

$$1 - \frac{8}{\sqrt{3}} \cdot \frac{\sinh \frac{\pi s}{6}}{s \cosh \frac{\pi s}{2}} = 0. \tag{17.18}$$

Apart from technical complications due to the self-adjointness requirement, one should notice the similarity between (17.14)–(17.15) and (17.1). Each function in the domain of the Hamiltonians is the sum of a regular part and the potential of some charge density distributed on the coincidence planes, the Hamiltonians operate as the free Hamiltonian acting on the regular part and the boundary condition can be expressed as an equation on the charges.

The Hamiltonians defined in the way described above were finally self-adjoint, but Minlos and Faddeev realised that their spectral structure made those Hamiltonians unphysical models for a three-body quantum system. In fact, the authors found that their point spectrum contains an infinite sequence of negative eigenvalues unbounded from below (see [10] for an alternative proof). The authors also suggest a possible way out of this unboudedness pathology. In short, their hint amounts to substitute the constant α in (17.14) with the operator A defined, in Fourier space, by

$$\big(A\hat{\xi}\big)(\mathbf{p}) = \alpha\hat{\xi}(\mathbf{p}) + \big(K\hat{\xi}\big)(\mathbf{p}) \tag{17.19}$$

with $\alpha \in \mathbb{R}$ and K the convolution operator with kernel $K(p)$ behaving for large $|\mathbf{p}|$ as

$$K(\mathbf{p}) \sim \frac{\gamma}{|\mathbf{p}|^2} \qquad \text{for } |\mathbf{p}| \to \infty.$$

3 On the negative eigenvalues

In a private communication happened years ago between one of us and L. D. Faddeev, he appeared absolutely confident that zero-range Hamiltonians bounded from below for the three-body quantum system would exist. He renewed the suggestion that he and Minlos gave in their 1962 papers, mentioning that, with regret, they were no longer involved. On the other hand, Minlos, in the rest of his scientific career, went back occasionally to zero-range Hamiltonians for many-particle quantum systems approaching the interesting case of N, $N \geq 2$, identical fermions interacting, via zero-range forces, with a different particle, making important contributions to the stability problem (see e. g. [11, 12]; for more recent developments see [15] and references therein). Recently, we showed that at least in the case $\alpha = 0$ the strategy works very well. For details of the proof see [10].

Considering the Ter-Martirosian, Skorniakov boundary condition (17.9) for $\alpha = 0$ and adding the term suggested by Minlos and Faddeev, we have that $-\lambda$, $\lambda > 0$, is a negative eigenvalue of the Hamiltonian if

$$\frac{\delta}{2\pi^2}\int d\mathbf{p}'\,\frac{\hat{\xi}(\mathbf{p}')}{|\mathbf{p}-\mathbf{p}'|^2} + \sqrt{\frac{3}{4}|\mathbf{p}|^2+\lambda}\cdot\hat{\xi}(\mathbf{p}) - \frac{1}{\pi^2}\int d\mathbf{p}'\,\frac{\hat{\xi}(\mathbf{p}')}{|\mathbf{p}|^2+|\mathbf{p}'|^2+\mathbf{p}\cdot\mathbf{p}'+\lambda} = 0, \tag{17.20}$$

where δ is a real parameter.

In the rotationally invariant case $\hat{\xi} = \hat{\xi}(|\mathbf{p}|)$, integrating out the angular variables one gets

$$\frac{\delta}{\pi}\int_0^\infty dp'\,p'\hat{\xi}(p')\log\frac{p+p'}{|p-p'|} + \sqrt{\frac{3}{4}p^2+\lambda}\cdot p\hat{\xi}(p) - \frac{2}{\pi}\int_0^\infty dp'\,p'\hat{\xi}(p')\log\frac{p^2+p'^2+pp'+\lambda}{p^2+p'^2-pp'+\lambda} = 0. \tag{17.21}$$

The following statement holds true:

Proposition 17.2 Let

$$\delta_0 = \frac{\sqrt{3}}{\pi}\left(\frac{4\pi}{3\sqrt{3}} - 1\right). \tag{17.22}$$

Then for $\delta > \delta_0$, Equation (17.21) has only the trivial solution.

The main technical tool used in the proof is the following change of variable (see [8])

$$p = \frac{2\sqrt{\lambda}}{\sqrt{3}}\sinh x, \qquad x = \log\left(\frac{\sqrt{3}p}{2\sqrt{\lambda}} + \sqrt{\frac{3p^2}{4\lambda}+1}\right) \tag{17.23}$$

which allows to diagonalise Equation (17.21) for the new function

$$\theta(x) = \begin{cases} \lambda\sinh x\cdot\cosh x\cdot\hat{\xi}\left(\frac{2\sqrt{\lambda}}{\sqrt{3}}\sinh x\right) & \text{for } x \geq 0 \\ -\theta(-x) & \text{for } x < 0 \end{cases} \tag{17.24}$$

giving the following equation for the Fourier transform of the function θ

$$\left(1+2\frac{\delta\sinh\frac{\pi}{2}s-4\sinh\frac{\pi}{6}s}{\sqrt{3}s\cosh\frac{\pi}{2}s}\right)\hat{\theta}(s)=0. \tag{17.25}$$

It is then easy to conclude the proof showing that

$$1+2\frac{\delta\sinh\frac{\pi}{2}s-4\sinh\frac{\pi}{6}s}{\sqrt{3}s\cosh\frac{\pi}{2}s}>0 \qquad \text{for } \delta>\delta_0.$$

Other recent attempts to obtain zero-range three-body Hamiltonians bounded from below can be found in [3] and [4].

Dedication. The authors want to dedicate this contribution to the memory of Robert A. Minlos, a leading mind of mathematical physics and a wonderful human being.

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18

3-D incompressible Navier-Stokes equations: Complex blow-up and related real flows

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Abstract. *In the framework of the Global Regularity Problem for the Navier-Stokes (NS) Equations in* $\mathbb{R}^3$ *Li and Sinai proved the existence of singular complex solutions ("blow-up"). We give an outline of their approach and discuss the perspectives of its extension to real solutions. We also illustrate, with the help of computer simulations, the behaviour of a real solution related to the complex blow-up. It does not blow up, due to its approximate axial symmetry, but it shows a remarkable tornado-like behaviour, with a rapid concentration and increase of speed and vorticity.*

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1 Introduction

The problem whether there are smooth solutions of the incompressible Navier-Stokes equations in $\mathbb{R}^3$ that become singular at a finite time ("blow-up") dates back to Jean Leray [6] who first proved a global weak existence theorem for all times and a uniqueness and regularity theorem only for finite times. It goes under the name "Global Regularity Problem" (GRP), and is still open, in spite of many brilliant contributions.

Leray thought that singularities exist and are related to turbulence. We now describe turbulence as a chaotic flow with no relation to singularities, but the singular solutions, if they exist, could provide a model for phenomena such as tornadoes, which exhibit a rapid increase of speed in a limited region of space, and for which there is at present no effective model. In fact we know [9] that a loss of smoothness for the NS equations implies divergence of the velocity at some point. Proofs of a finite-time blow-up were obtained for some variants of the dyadic model [3], a discrete model of the NS equations which preserves energy conservation. Moreover T. Tao [10] proved a finite-time blow-up for a NS system with a modified bilinear term satisfying the energy identity .

The evidence from computer simulations is inconclusive: a theoretical guideline on the behaviour of singularities is needed in order to control the difficulties arising in computing solutions of the 3-D NS equations for high values of the vorticity [4].

In 2008 Li and Sinai [7] proposed a negative answer to the GRP, i. e., a plan to construct explicit singular solutions. As a first step they proved that there are complex singular solutions following from initial data such that the support of the Fourier transform $\mathbf{v}(\mathbf{k},t)$ of the velocity field $\mathbf{u}(\mathbf{x},t)$ (see below) extends rapidly to high $|\mathbf{k}|$-values. The proof relies on Renormalisation Group methods, and their approach can be applied to other models [8] as well. The extension of their methods to real solutions requires more work. In the meantime important indications can come from computer simulations, which, as we explain below, if implemented in Fourier $\mathbf{k}$-space, are made easier for the class of initial data under consideration, by the fact that the extension of the support to the high $|\mathbf{k}|$-region is confined to a rather small region around a fixed axis.

The plan of the paper is as follows. We first describe the main features of the Li-Sinai approach, also with the help of some simple new results, and discuss its extension to real solutions. We then report results of recent simulations describing the behaviour of a real solution related to the complex blow-up in [7]. The solution behaves very much tornado-like, but, as we discuss in the concluding remarks, does not blow up because of axial symmetry and we need to consider non-symmetric solutions.

2 The Li-Sinai approach

2.1 NS in k-space

Passing to a precise formulation, we consider the incompressible Navier-Stokes equations in the whole space $\mathbb{R}^3$ with no boundary conditions and external forces:

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j}\mathbf{u} = \Delta \mathbf{u} - \nabla p, \qquad \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \tag{18.1}$$

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}(\cdot, 0) = \mathbf{u}_0.$$

Here $\mathbf{u} : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ is the velocity field, p is the pressure and we assume for the viscosity $\nu = 1$, which is always possible by rescaling. Two important physical quantities are the total energy $E(t)$ and the enstrophy $S(t)$, which is the integral of the square vorticity:

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{u}(\mathbf{x},t)|^2 d\mathbf{x}, \qquad S(t) = \int_{\mathbb{R}^3} |\omega(\mathbf{x},t)|^2 d\mathbf{x} \tag{18.2}$$

where $\omega(\mathbf{x},t) = \nabla \times \mathbf{u}(\mathbf{x},t)$ is the vorticity. They are related by the energy equality

$$E(t) + \int_0^t S(s) ds = E(0), \tag{18.3}$$

which implies that $E(t)$ cannot increase. If the enstrophy is bounded, it can be shown by an "enstrophy inequality" that global regularity holds [11], so that for a blow-up the enstrophy must diverge in an integrable way as we approach a critical time. A divergence of the enstrophy implies that the support is shifting to the high $\mathbf{k}$-region in Fourier space, i. e., to the fine scale structure in the physical space.

As we work in $\mathbf{k}$ space, we write the NS system (18.1) in terms of a modified Fourier transform of the velocity field $\mathbf{u}(\mathbf{x},t)$

$$\mathbf{v}(\mathbf{k},t) = \frac{i}{(2\pi)^3}\int_{\mathbb{R}^3} \mathbf{u}(\mathbf{x},t) e^{-i\langle \mathbf{k},\mathbf{x}\rangle} d\mathbf{x}, \qquad \mathbf{k} = (k_1,k_2,k_3) \in \mathbb{R}^3, \tag{18.4}$$

where $\langle \cdot , \cdot \rangle$ denotes the scalar product in $\mathbb{R}^3$. By a Duhamel formula the system (18.1) is written as a single integral equation:

$$\mathbf{v}(\mathbf{k},t) = e^{-t\mathbf{k}^2}\mathbf{v}_0(\mathbf{k}) + \int_0^t e^{-(t-s)|\mathbf{k}|^2} \int_{\mathbb{R}^3} \left\langle \mathbf{v}(k-k',s),\mathbf{k} \right\rangle P_{\mathbf{k}}\mathbf{v}(\mathbf{k}',s)\,d\mathbf{k}'\,ds, \qquad (18.5)$$

where $P_{\mathbf{k}}\mathbf{v} := \mathbf{v} - \frac{\langle \mathbf{v},\mathbf{k}\rangle}{|\mathbf{k}|^2}\mathbf{k}$ denotes the solenoidal projector and $\mathbf{v}_0$ is the transform of $\mathbf{u}_0$. In general $\mathbf{v}(\mathbf{k},t)$ is a complex function. Li and Sinai consider only real solutions of (18.5), which in general correspond to complex solutions of (18.1). However if $\mathbf{v}_0(\mathbf{k})$ (and hence $\mathbf{v}(\mathbf{k},t)$ for $t>0$) is antisymmetric, the solution $\mathbf{u}(\mathbf{x},t)$ is also real and antisymmetric in $\mathbf{x}$.

Taking $\mathbf{v}_0(\mathbf{k}) = A\bar{\mathbf{v}}(\mathbf{k})$, where A is a real parameter which controls the initial energy, and iterating the Duhamel formula, the solution of (18.5) is written as a power series:

$$\mathbf{v}_A(\mathbf{k},t) = A\mathbf{g}^{(1)}(\mathbf{k},t) + \sum_{p=2}^{\infty} A^p \int_0^t e^{-\mathbf{k}^2(t-s)}\mathbf{g}^{(p)}(\mathbf{k},s)\,ds, \qquad (18.6)$$

where $\mathbf{g}^{(1)}(\mathbf{k},s) = e^{-s\mathbf{k}^2}\bar{\mathbf{v}}(\mathbf{k})$, $\mathbf{g}^{(2)}(\mathbf{k},s) = \int_{\mathbb{R}^3} \left\langle \mathbf{g}^{(1)}(\mathbf{k}-\mathbf{k}',s),\mathbf{k} \right\rangle P_{\mathbf{k}}\mathbf{g}^{(1)}(\mathbf{k}',s)\,d\mathbf{k}'$ and

$$\mathbf{g}^{(p)}(\mathbf{k},s) = \sum_{\substack{p_1+p_2=p \\ p_1,p_2>1}} \int_0^s ds_1 \int_0^s ds_2\, \mathbf{g}^{(p_1,p_2)}(\mathbf{k},s_1,s_2) + \textbf{boundary terms}, \qquad p>2 \qquad (18.7)$$

$$\mathbf{g}^{(p_1,p_2)}(\mathbf{k},s_1,s_2) = \int_{\mathbb{R}^3} \left\langle \mathbf{g}^{(p_1)}(\mathbf{k}-\mathbf{k}',s_1),\mathbf{k} \right\rangle P_{\mathbf{k}}\mathbf{g}^{(p_2)}(\mathbf{k}',s_2)\, e^{-(s-s_1)(\mathbf{k}-\mathbf{k}')^2-(s-s_2)(\mathbf{k}')^2} d\mathbf{k}'$$

The boundary terms involve $\mathbf{g}^{(1)}$ and have a slightly different form [7]. The following lemma shows that the functions $\mathbf{g}^{(p)}$ satisfy, as $p\to\infty$, a Gaussian bound.

Lemma 18.1 If $\bar{\mathbf{v}} \in L_2(\mathbb{R}^3)$ is a bounded function, then the following inequalities hold

$$\begin{aligned} |\mathbf{g}^{(p)}(\mathbf{k},t)| &\le K^{p-1}p^{1/2}t^{p-3/2}\phi_0^{(p)}(\mathbf{k}), \\ \phi_0^{(p)}(\mathbf{k}) = (\underbrace{\phi_0 * \ldots * \phi_0}_{p \text{ times}})(\mathbf{k}), &\qquad \phi_0(\mathbf{k}) = |\bar{\mathbf{v}}(\mathbf{k})|, \qquad p\ge 2, \end{aligned} \qquad (18.8)$$

where $*$ denotes convolution and K is a positive constant.

Proof. Let $\sup_{\alpha\ge 0} \alpha^{1/2}e^{-\alpha} = c_1 := (2e)^{-1/2}$. It follows that

$$|\mathbf{g}^{(2)}(\mathbf{k},s)| \le c_1A^2s^{-1/2} \int_{\mathbb{R}^3} \phi_0(\mathbf{k}-\mathbf{k}\mathbb{P})\phi_0(\mathbf{k}\mathbb{P})\,d\mathbf{k}' = c_1A^2s^{-1/2}\,\phi_0^{(2)}(\mathbf{k}).$$

For the terms of the sum in (18.7), using Inequality (18.8) as an ansatz, we have

$$\begin{aligned}\int_0^s ds_1 \int_0^s ds_2 \left|\mathbf{g}^{(p_1,p_2)}(\mathbf{k},s_1,s_2)\right| \\ &\leq c_1 \int_0^s ds_1 \int_0^s \frac{ds_2}{\sqrt{s-s_2}} \int_{\mathbb{R}^3} |\mathbf{g}^{(p_1)}(\mathbf{k}-\mathbf{k}\mathbb{P},s_1)| \cdot |\mathbf{g}^{(p_2)}(\mathbf{k}\mathbb{P},s_2)|\, d\mathbf{k}\mathbb{P} \\ &\leq c_1 K^{p-2}(p_1 p_2)^{\frac{1}{2}} \int_0^s s_1^{\frac{p_1-3}{2}} ds_1\, I_{\frac{p_2-3}{2}}(s)\, \phi_0^{(p)}(\mathbf{k}),\end{aligned}$$

where $I_\alpha(s) := \int_0^s \frac{u^\alpha}{\sqrt{s-u}} du$ satisfies for a semiinteger $\alpha \geq -\frac{1}{2}$ the inequality $I_\alpha(s) \leq D\frac{s^{\alpha+\frac{1}{2}}}{\alpha+1}$, for some constant $D > 0$. Hence we have $|\mathbf{g}^{(p_1,p_2)}(\mathbf{k},s)| \leq c_1 D \frac{K^{p-2}}{\sqrt{p_1}} s^{\frac{p-3}{2}}$. The boundary terms give a similar inequality. The conclusion now comes, for a suitable choice of K, by observing that $\sum_{n=1}^{p} \frac{1}{\sqrt{n}} \leq c_2\sqrt{p}$, for some constant $c_2 > 0$. □

By Lemma 18.1 the series (18.6) converges absolutely for small t. Moreover if the initial enstrophy is bounded, i. e., $\int_{\mathbb{R}^3} \mathbf{k}^2 |\mathbf{v}_0(\mathbf{k})|^2 d\mathbf{k} < \infty$, the local variant of the central limit theorem holds for the distribution with density $\widehat{\phi}_0(\mathbf{k}) = \frac{\phi_0(\mathbf{k})}{N}$, where $N = \int_{\mathbb{R}^3} \phi_0(\mathbf{k}) d\mathbf{k}$. Hence as $p \to \infty$ the convolution on the right of (18.8) tends to the Gaussian density with average $p\,\mathbf{m}$ and covariance matrix $\sqrt{p}\mathscr{C}$, where $\mathbf{m} = \int_{\mathbb{R}^3} \mathbf{k}\widehat{\phi}_0(\mathbf{k})\, d\mathbf{k}$ and $\mathscr{C} = (C_{ij})_{i,j=1,\ldots,3}$, $C_{ij} = \int_{\mathbb{R}^3} (k_i - m_i)(k_j - m_j)\widehat{\phi}_0(\mathbf{k})\, d\mathbf{k}$.

2.2 Blow-up for complex solutions and behaviour of related real solutions

In the paper [7] Li and Sinai choose initial data with support inside a sphere K_R of radius R centered around a point $\mathbf{k}^{(0)}$ with $|\mathbf{k}^{(0)}| \gg R$. By Lemma 18.1 the support of $\mathbf{g}^{(p)}$ is centered around $p\mathbf{k}^{(0)}$ with a diameter of the order $\mathscr{O}(\sqrt{p})$, so that as terms $\mathbf{g}^{(p)}$ with growing p are excited, if $|\mathbf{k}^{(0)}|$ is large, the support of $\mathbf{v}(\mathbf{k},t)$ quickly extends to the high $|\mathbf{k}|$ region, causing a strong increase of the enstrophy. The key for the proof of a blow-up is the asymptotic behaviour of $\mathbf{g}^{(p)}$ as $p \to \infty$. In view of the Gaussian dominance, Li and Sinai introduce new functions $\tilde{\mathbf{g}}^{(p)}(\bar{Y},s) = \mathbf{g}^{(p)}(p\mathbf{k}^{(0)} + \sqrt{p}\bar{Y},s)$ of the new variables $\bar{Y} = \frac{\mathbf{k}-\mathbf{k}_0}{\sqrt{p}}$, and look for initial values $\mathbf{v}_0$ which lead to the asymptotics

$$\tilde{\mathbf{g}}^{(p)}(\bar{Y},s) \sim p(\Lambda(s))^p \prod_{i=1}^{3} g(Y_i)\big(\mathbf{H}(\bar{Y}) + \delta^{(p)}(\bar{Y},s)\big). \tag{18.9}$$

Here $\mathbf{H}$ is a solution of a fixed point equation of the map $L^\infty := \lim_{p\to\infty} L^{(p)}$ where $L^{(p)}$: $\tilde{\mathbf{g}}^{(p)} \to \tilde{\mathbf{g}}^{(p+1)}$, $g(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$ is the standard Gaussian, Λ is a strictly increasing smooth function and $\delta^{(p)}(\bar{Y},s) \to 0$ as $s \to \infty$. (The Gaussian can always be made standard by a change of variables.) The functions $\mathbf{H}$ and Λ control the excitation of the high $\mathbf{k}$-modes. As shown in [7], there are infinitely many solutions of the fixed point equation. Assuming $\mathbf{k}^{(0)} = (0,0,a)$, with $a > 0$, the ansatz (18.9) is proved in [7] for $\mathbf{H} = c\,(Y_1,Y_2,0) =: \mathbf{H}^{(0)}(\bar{Y})$, with $c > 0$, and for a monotonic increasing function Λ. The linearisation of L^∞ at $\mathbf{H}^{(0)}$ has a 6-dimensional unstable subspace, a 4-dimensional neutral subspace, and an infinite-dimensional stable one.

The main result of [7] can be formulated as follows.

Theorem 18.2 Let $a > b \gg 1$, and consider, as for (18.6), initial data $\mathbf{v}_0 = A\bar{\mathbf{v}}$ with

$$\bar{\mathbf{v}}(\mathbf{k}) = \left[\left(k_1,k_2,-\frac{k_1^2+k_2^2}{k_3}\right) + \Phi(k_1,k_2,k_3)\right]\prod_{i=1}^{2} g(k_i)g(k_3-a)\,\chi_b(|\mathbf{k}-\mathbf{k}^{(0)}|) \quad (18.10)$$

where $\mathbf{k}^{(0)} = (0,0,a)$, $a > 0$, g is the standard Gaussian, $\chi_b(\mathbf{k})$ is a smooth function with $\chi_b(\mathbf{k}) = 0$ if $|\mathbf{k}| \geq b$, $\chi_b(\mathbf{k}) = 1$ if $|\mathbf{k}| \leq b-\varepsilon$, for ε small enough, $\Phi = \Phi^{(1)} + \Phi^{(2)}$, $\Phi^{(1)}$ is a linear combination of the unstable and neutral eigenfunctions of the linearised map at $\mathbf{H}^{(0)}$, and $\Phi^{(2)}$ is in the stable subspace. Then if $\Phi^{(2)}$ is small enough, there is a time interval $(S_- \leq s \leq S_+)$ and an open set of the parameters defining $\Phi^{(1)}$ for which the ansatz (18.9) with $\mathbf{H} = \mathbf{H}^{(0)}(\bar{Y})$ holds.

The blow-up is an easy consequence of Theorem 18.2. Taking $A = \pm\frac{1}{\Lambda(\tau)}$, $\tau \in (S_-,S_+)$, and replacing $\mathbf{g}^{(p)}$ by the asymptotics (18.9), it is easy to see that the series (18.6) diverges as $s \uparrow \tau$. As the initial data (18.10) are not antisymmetric the solution $\mathbf{u}(\mathbf{x},t)$ in the physical $\mathbf{x}$-space is, as we said above, a complex function, and at the critical time τ the energy $E(t)$ diverges along with the enstrophy $S(t)$ (for complex solutions the energy equality holds but it is not coercive).

Coming to real solutions, it is natural to consider initial data obtained by antisymmetrising the data (18.10) associated to the solutions that blow-up, i. e., of the type $\mathbf{v}_0(\mathbf{k}) = A\big(\mathbf{v}_+(\mathbf{k}) + \mathbf{v}_-(\mathbf{k})\big)$, where $\mathbf{v}_+ = \bar{\mathbf{v}}$ and $\mathbf{v}_-(\mathbf{k}) = -\bar{\mathbf{v}}(-\mathbf{k})$. The functions $\mathbf{g}^{(p)}$ are now a sum of terms centered around the points $(0,0,\ell\,\mathbf{k}^{(0)})$, $\ell = -p,\ldots,p$. In fact, substituting in (18.7) the expressions of $\mathbf{g}^{(p_i)}, i = 1,2$, in terms of functions with lower indices, down to $\mathbf{g}^{(1)}$, we see that $\mathbf{g}^{(p)}(\mathbf{k},s) = \mathscr{L}_s^{(p)}\big(\mathbf{v}_0,\mathbf{v}_0,\ldots,\mathbf{v}_0\big)(\mathbf{k})$, where $\mathscr{L}_s^{(p)}$ is a p-linear functional. The expression for $\mathbf{g}^{(p)}(\mathbf{k},s)$ breaks into 2^p terms $\mathscr{L}_s^{(p)}\big(\mathbf{v}_{i_1},\mathbf{v}_{i_2},\ldots,\mathbf{v}_{i_p}\big)(\mathbf{k})$, $i_q \in \{\pm\}$, $q = 1,\ldots,p$, of which $\binom{p}{\frac{p-\ell}{2}}$ are centered around the point $\ell\mathbf{k}^{(0)}$. For large p the main contributions comes from values $\ell = \mathscr{O}(\sqrt{p})$, and we again have a shift of the support to the high $|\mathbf{k}|$-modes.

The analysis of the fixed points for the real solutions is more difficult. In absence of theoretical results on the behaviour of the functions $\mathbf{g}^{(p)}$ for the real antisymmetric solutions, important information can be obtained by computer simulations, which can also reveal physically relevant details.

3 Results of computer simulations

Computer simulations, with a new program for the numerical study of solutions of the integral equation (18.5), were first performed for the complex functions proposed in [7] in order to find out explicit values of the parameters leading to the blow-up and the most relevant details of its development [1], [2]. As shown by Lemma 18.1, the solutions with initial data of the type (18.10), extend their support in $\mathbf{k}$-space inside a thin region around the direction of $\mathbf{k}^{(0)} = (0,0,a)$, which for large k_3 has a transverse diameter $\mathscr{O}(\sqrt{k_3 a^{-1}})$. We could then compute for values of $|\mathbf{k}|$ up to a few thousand, and could follow the solutions up to times close to the blow-up.

The simulations showed that if $20 \leq a \leq 40$ and the initial energy E_0 is of the order of 10^5, all initial values of the type (18.10) with Φ small, as prescribed in [7], lead to a blow up with a critical time t_c of the order of 10^{-4} time units. For smaller values of E_0 it is also possible that the critical time t_c is larger than the available computer time. The function Φ in (18.10) does not have much influence on the behaviour of the solution, except that it increases the critical time t_c. Therefore most simulations were done with $\Phi = 0$.

As discussed in [1, 2], there are two types of singular complex solutions, depending on the sign of the constant A. In both cases the divergence of the total energy and enstrophy goes as an inverse power of $t - t_c$, but the rate of divergence is slower for $A < 0$, as there are cancellations between neighbouring terms in the series (18.6).

The initial data for the computer simulations of the real flow are obtained by antisymmetrising the function (18.10) with $\Phi \equiv 0$. The general implementation of the numerical approach was obtained from previous simulations of the complex blow up [1, 2]. We only report results obtained for $a = 30$ and $E_0 = 62 \cdot 10^6$. Recall however

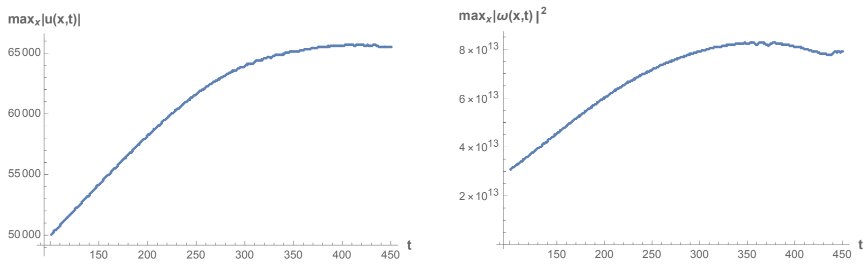

Figure 18.1: Plots of $\max_{\mathbf{x}} |\mathbf{u}(\mathbf{x},t)|$ (left) and of $\max_{\mathbf{x}} |\omega(\mathbf{x},t)|^2$ (right) as functions of time.

that the NS scaling holds: If $\mathbf{v}(\mathbf{k},t)$ is a solution of (18.5), and $\lambda > 0$ then the function $\mathbf{v}^{(\lambda)}(\mathbf{k},t) = \lambda^2 \mathbf{v}(\lambda\mathbf{k}, \lambda^{-2}t)$ is also a solution.

Simulations were performed on a mesh in **k**-space which is part of a regular lattice centered at the origin with step $\delta = 1$, and maximal configuration $[-254,254] \times [-254,254] \times [-3000,3000]$. The velocity field at a given time is described by about $5 \cdot 10^9$ real numbers, close to the maximal capacity of modern supercomputers. A comparison of the accuracy of our program to that of finite-difference methods is under way.

As for the complex blow-up, the interesting phenomena take place in a very short time, and in what follows time is measured in units of $\tau = 1.5625 \times 10^{-8}$. The enstrophy $S(t)$ grows almost threefold, from $S(0) \approx 2 \times 10^8$ to $S(T_M) \approx 6 \times 10^8$, with $T_M \approx 710\tau$, after which it decreases. The maximal values of the velocity $|\mathbf{u}(\mathbf{x},t)|$ and of the vorticity $|\omega(\mathbf{x},t)|$ also grow, as shown in Figure 18.1, reaching a maximum at $t \approx 410\tau$ and $t \approx 350\tau$, respectively.

Figure 18.2 reports the behaviour in time of the marginal distributions of the square of the vorticity along the symmetry axis in the physical **x**-space and in the Fourier **k**-space:

$$S_3(k_3,t) = \int_{\mathbb{R}\times\mathbb{R}} |\mathbf{k}|^2 |\mathbf{v}(\mathbf{k},t)|^2 \, dk_1 \, dk_2, \qquad \tilde{S}_3(x_3,t) = \int_{\mathbb{R}\times\mathbb{R}} |\omega(\mathbf{x},t)|^2 \, dx_1 \, dx_2.$$

The behaviour of S_3 shows that, as time grows, the high $|\mathbf{k}|$ modes are enhanced and a modulated periodic pattern sets in for large k_3 with distance of the peaks close to $a = 30$. In the physical space observe that at the time $t = 400\tau$ the vorticity concentrates in sharp peaks near the planes $x_3 = \pm\bar{x}_3$ with $\bar{x}_3 \approx \frac{\pi}{a}$, corresponding to the modulated periodicity. In fact a 3-D plot would show that the high velocity and vorticity values are concentrated

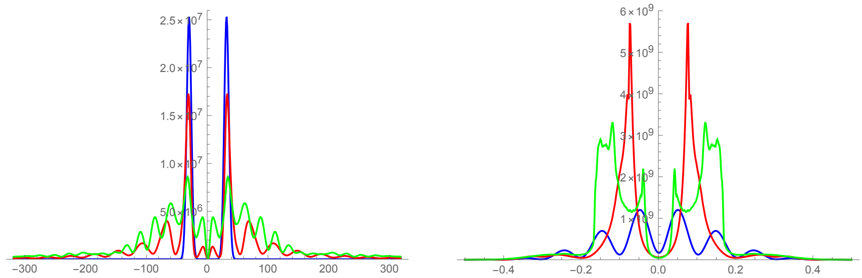

Figure 18.2: Plots of the marginal distributions $S_3(k_3,t)$ (left) and $\tilde{S}_3(x_3,t)$ (right) at the times $t = 0$ (blue), $t = 400\tau$ (red), and $t = 711\tau$ (green).

in two “doughnuts” around the x_3-axis, bisected by the planes $x_3 \approx \pm\bar{x}_3$, while elsewhere the fluid stays more or less quiet.

4 Concluding remarks

The real solution described in the previous paragraph is strongly reminiscent of tornadoes and similar phenomena, with a sharp increase and concentration of the velocity and the vorticity in an annular region around the symmetry axis. Similar solutions could be a good model of such physical phenomena, and they are likely to apply also to compressible fluids, perhaps in conditions of quasi-incompressibility.

Concerning the possibility of a blow-up, observe that our initial data are obtained by antisymmetrising the data (18.10) with $\Phi = 0$, and our solution is axially symmetric around the third axis in the physical **x**-space (and also in Fourier **k**-space), with no swirl (i. e., there is no rotation around the x_3-axis). This is a consequence of the choice of the fixed point $\mathbf{H}^{(0)}$, and also by taking a small $\Phi \neq 0$ we would stay close to axial symmetry with no swirl, which implies global regularity, according to a recent paper by Lei and Zang [5], in which the criticality of the axial symmetric case is also proved for the first time.

The research should be extended to real solutions related to fixed points $\mathbf{H} \neq \mathbf{H}^{(0)}$, which are not axial symmetric. The theoretical analysis requires the extension of the results of Li and Sinai to such fixed points and a deeper study of the behaviour of the series (18.6) in the real case. We also plan to obtain indications from computer simulations.

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