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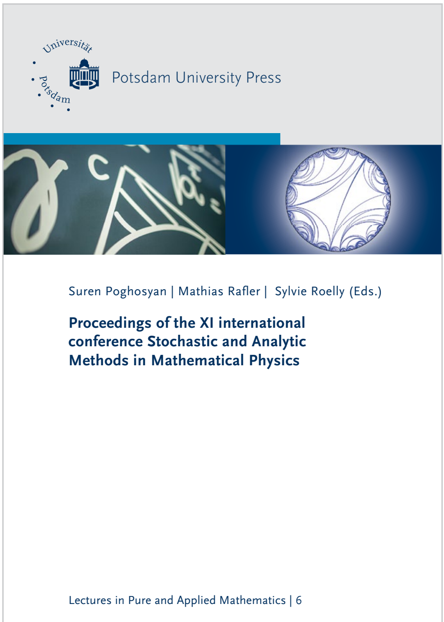

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On direct and inverse problems in the description of lattice random fields

Linda Khachatryan[*] *and Boris Nahapetian*[†]

Abstract. *Various systems of finite-dimensional distributions parameterised by boundary conditions are considered. For such systems solutions to direct and inverse problems of description of lattice random fields are given.*

1 Introduction

Let P be a random field on the integer lattice $\mathbb{Z}^d$ $(d \geq 1)$ with state space X, that is, a probability measure on the σ-algebra generated by all cylinder sets of $X^{\mathbb{Z}^d}$.

Since it is quite difficult to work directly with the probability measure P defined on the infinite product of state space X (often called *infinite-volume* measure), its study usually reduced to the analysis of a suitable system Q_P of probability distributions generated by P and defined on finite products of X. The natural requirement for such a system Q_P is that Q_P must uniquely *determine* (*restore*) the random field P, i. e., any random field P' such

[*]Institute of Mathematics, National Academy of Science, Armenia; *linda@instmath.sci.am*
[†]Institute of Mathematics, National Academy of Science, Armenia; *nahapet@instmath.sci.am*

that $\mathrm{Q}_{P'} = \mathrm{Q}_P$ must coincide with P. In this case it is necessary to note that the random field P was restored by the system Q_P, we will use the following notation: P_{Q_P}.

For a given random field P, the problem of the existence of a system Q_P for which $P_{\mathrm{Q}_P} = P$ we call the *direct problem* in the description of random fields. We will say that a system Q_P is a solution to the direct problem for a given random field P if $P_{\mathrm{Q}_P} = P$. Note that for a given random field there may exist various solutions to the direct problem.

If the direct problem is solved, the following question naturally arises: does the system Q_P possess such specific properties (consistency conditions) which allow restoring the random field P without taking into account the fact that the elements of Q_P are generated by the random field P? In the case such consistency conditions are found, it is quite possible that for any system Q endowed by these properties, there exists a random field P whose system of finite-dimensional distributions Q_P coincides with Q.

For a given system Q of finite-dimensional distributions, the problem of the existence of a random field P for which $\mathrm{Q}_P = \mathrm{Q}$ we call the *inverse problem* in the description of random fields. We will say that a random field P is a solution to the inverse problem for a given system Q if $\mathrm{Q}_P = \mathrm{Q}$. For a given system Q, any solution to the inverse problem will be denoted by P_{Q}, so that $\mathrm{Q}_{P_{\mathrm{Q}}} = \mathrm{Q}$.

By solving the direct problem, we obtain the possibility to define classes of random fields and to study their main properties; while the solution to the inverse problem makes it possible to construct models of random fields with required properties.

Historically, Kolmogorov was the first one who considered both the direct and the inverse problems of description of random processes (see [7], originally published in German in 1933). His subject of consideration was a consistent system of unconditional finite-dimensional distributions. This system, which uniquely determines a random field, is a very general one, and specific classes of random processes are defined by the corresponding restrictions on its elements. For example, by corresponding restrictions on Kolmogorov's system, classes of Gaussian processes, processes with independent increments as well as stationary processes are defined.

Over time, it became clear that in many cases it is convenient to impose restrictions not only on unconditional distributions but also on their relations, that is, on conditional distributions. This type of restrictions leads to other important classes of random processes, namely Markov processes, Gibbs random fields, martingales, etc.

It should be noted that, generally speaking, an inverse problem may have more than one solution, i. e., be incorrect. If for a given system Q the inverse problem of description of random fields is correct (has a unique solution), we will say that the system Q *specifies* (uniquely defines) a random field. Note, that Kolmogorov's system characterises a random field. However, in some cases incorrect problems lead to very interesting and useful results. For example, in the Dobrushin's theory of description of Gibbs random fields, the non-correctness of the inverse problem (non-uniqueness of its solution) is interpreted as the presence of a phase transition in the model under study (see the seminal paper [6]).

In the present work we restrict ourselves to the case of positive lattice random fields with finite state space X. We consider various systems of conditional distributions generated by a random field as well as autonomously defined consistent systems of finite-dimensional distributions parametrised by boundary conditions. For such objects solutions to the direct and the inverse problems in the description of random fields are given.

2 Preliminaries

Let $S \subset \mathbb{Z}^d$ and let $W(S) = \{V \subset S, 0 < |V| < \infty\}$ be the set of all (non-empty) finite subsets of S. For $S = \mathbb{Z}^d$ we use a simpler notation W. In some cases the braces in the notation of one-point sets $\{t\}$, $t \in \mathbb{Z}^d$, are omitted. For any function $f(\Lambda)$, $\Lambda \in W(S)$, the notation $\lim\limits_{\Lambda \uparrow S} f(\Lambda) = a$ will mean that for any increasing sequence $\{\Lambda_n\}_{n\geq 1}$ of finite sets converging to S (that is, $\Lambda_n \in W(S)$, $\Lambda_n \subset \Lambda_{n+1}$ and $\bigcup\limits_{n=1}^{\infty} \Lambda_n = S$), we have $\lim\limits_{n\to\infty} f(\Lambda_n) = a$.

Denote by $X^S = \{(x_t, t \in S)\}$, $x_t \in X$, the set of all functions (configurations) on S taking values in X. If $S = \emptyset$, we assume $X^{\emptyset} = \{\underline{o}\}$, where $\underline{o}$ is the empty configuration. For any $S, T \subset \mathbb{Z}^d$ such that $S \cap T = \emptyset$ and any $x \in X^S$, $y \in X^T$ we denote xy the concatenation of x and y defined as the configuration on $S \cup T$ equal to x on S and to y on T. If $T \subset S$, by x_T we denote the restriction of the configuration $x \in X^S$ on T.

Let P be a random field, that is, a probability measure on $X^{\mathbb{Z}^d}$. We denote by P_V the restriction of P on X^V, i. e., $P_V(A) = \big(P\big)_V(A) = P\big(\{x \in X^{\mathbb{Z}^d} : x_V \in A\}\big)$, where $A \subset X^V$, $V \in W$. A random field P is called positive if $P_V(x) > 0$ for all $x \in X^V$ and $V \in W$.

For a positive random field P, its conditional probability Q_V^z on X^V under finite condition $z \in X^S$, $S \in W(\mathbb{Z}^d \backslash V)$, is defined as $Q_V^z(x) = P_{V\cup S}(xz) \big/ P_S(z)$, $x \in X^V$, $V \in W$. In the case of infinite boundary condition $z \in X^S$, $S \subset \mathbb{Z}^d \backslash V$, $S \notin W(\mathbb{Z}^d \backslash V)$, we put

$Q_V^z(x) = \lim\limits_{\Lambda \uparrow S} Q_V^{z_\Lambda}(x)$, $x \in X^V$, $V \in W$, where the limit exists for almost all (with respect to P) configurations z.

3 Kolmogorov's system

In [7] Kolmogorov showed that any random field P is determined by its system of finite-dimensional unconditional distributions $K_P = \{P_V, V \in W\}$, and thus K_P is a solution to the direct problem.

Among the properties of the restoring system K_P, Kolmogorov singled out the following one as a consistency condition: for all $V, I \in W$, $V \cap I = \emptyset$ and $x \in X^V$

$$\sum_{y \in X^I} P_{V \cup I}(xy) = P_V(x). \tag{10.1}$$

He proved that the system $K = \{p_V, V \in W\}$ of probability distributions p_V on X^V, $V \in W$, whose elements are consistent in the sense (10.1), characterises a random field P_K, i. e., there exists a unique solution to the inverse problem for the system K.

4 Systems of probability distributions parameterised by boundary conditions

Below we examine both the direct and the inverse problems for various consistent systems of finite-dimensional distributions parameterised by boundary conditions.

Note that if a system Q defines a random field P_{Q} then the system $\mathrm{Q}_{P_{\mathrm{Q}}}$ (which coincides with Q) is a solution to the direct problem for the random field P_{Q}. Thus, further, we will mainly focus on the inverse problem.

4.1 Conditional distribution of a random field

In this section we consider the widest system of conditional probability distributions generated by a random field.

For a random field P, the system $Q_P = \{Q_V^z, z \in X^S, \emptyset \neq S \subset \mathbb{Z}^d \backslash V, V \in W\}$ of conditional probabilities Q_V^z on X^V under boundary conditions z outside V, $V \in W$, we call *conditional distribution of the random field P*.

Any random field P is restored by its conditional distribution Q_P. Thus, the conditional distribution Q_P of a random field P is a solution to the direct problem. Now let us consider the inverse problem.

We call a set $Q = \{q_V^z, z \in X^S, \emptyset \neq S \subset \mathbb{Z}^d \backslash V, V \in W\}$ of probability distributions q_V^z on X^V parametrised by boundary conditions z outside V, $V \in W$, *(general) specification* if its elements satisfy the following consistency conditions:

1. for any disjoint sets $V, I \in W$, $\emptyset \neq S \subset \mathbb{Z}^d \backslash (V \cup I)$ and $x \in X^V$, $y \in X^I$, $z \in X^S$

$$q_{V \cup I}^z(xy) = q_V^z(x) q_I^{zx}(y); \tag{10.2}$$

2. for all $V \in W$ and $\emptyset \neq S \subset \mathbb{Z}^d \backslash V$ it holds

$$q_V^z(x) = \lim_{\Lambda \uparrow S} q_V^{z_\Lambda}(x), \qquad x \in X^V, z \in X^S. \tag{10.3}$$

The consistency condition (10.2) in the case of infinite boundary conditions was considered for the first time in [1]. A specification Q will be called positive if all its elements are strictly positive.

It is not difficult to see that for any random field P, the elements of its conditional distribution Q_P satisfy the consistency conditions (10.2) and (10.3) for almost all (with respect to P) boundary conditions. However, any random field has a version of its conditional distribution Q_P being the specification.

For a given specification, there is a unique solution to the inverse problem of description of random fields.

Theorem 10.1 Let Q be a positive specification. Then there exists a unique random field P such that its conditional distribution Q_P coincides with Q.

Proof. For any $V \in W$ put

$$p_V(x) = \frac{q_V^y(x)}{q_I^x(y)} \left(\sum_{\alpha \in X^V} \frac{q_V^y(\alpha)}{q_I^\alpha(y)} \right)^{-1}, \qquad x \in X^V, \tag{10.4}$$

where $y \in X^I$, $I \in W(\mathbb{Z}^d \backslash V)$. Using (10.2) one can show that the function p_V does not depend on the choice of I and y. It is not difficult to see that the system of probability distributions $\{p_V, V \in W\}$ is consistent in Kolmogorov's sense and hence defines a unique random field P such that $K_P = \{p_V, V \in W\}$. Moreover, it can be shown that $Q_P = Q$. $\square$

4.2 One-point conditional distribution

For a given random field P, the system $Q_1(P) = \{Q_t^z, z \in X^S, \emptyset \neq S \subset \mathbb{Z}^d \backslash \{t\}, t \in \mathbb{Z}^d\}$ we call *one-point conditional distribution of the random field P*. Any random field P can be restored by its one-point conditional distribution $Q_1(P)$.

We call a system $Q_1 = \{q_t^z, z \in X^S, \emptyset \neq S \subset \mathbb{Z}^d \backslash \{t\}, t \in \mathbb{Z}^d\}$ of one-point probability distributions parametrised by boundary conditions *(general) 1-specification* if its elements satisfy the following consistency conditions:

1. for all $t, s \in \mathbb{Z}^d$, $S \subset \mathbb{Z}^d \backslash \{t, s\}$ and $x, u \in X^{\{t\}}$, $y, v \in X^{\{s\}}$, $z \in X^S$

$$q_t^{zy}(x) q_s^{zx}(v) q_t^{zv}(u) q_s^{zu}(y) = q_s^{zx}(y) q_t^{zy}(u) q_s^{zu}(v) q_t^{zv}(x); \qquad (10.5)$$

2. for all $t \in \mathbb{Z}^d$, $\emptyset \neq S \subset \mathbb{Z}^d \backslash \{t\}$ and $x \in X^{\{t\}}$, $z \in X^S$ it holds $q_t^z(x) = \lim_{\Lambda \uparrow S} q_t^{z_\Lambda}(x)$.

Theorem 10.2 Let Q_1 be a positive 1-specification. Then there exists a unique random field P such that its one-point conditional distribution $Q_1(P)$ coincides with Q_1.

Proof. Let us construct a Kolmogorov's system $K_{Q_1} = \{p_V, V \in W\}$ as follows. For $V = \{t\}$, $t \in \mathbb{Z}^d$, put

$$p_t(x) = \frac{q_t^y(x)}{q_s^x(y)} \left(\sum_{\alpha \in X^{\{t\}}} \frac{q_t^y(\alpha)}{q_s^\alpha(y)} \right)^{-1}, \qquad x \in X^{\{t\}}, \qquad (10.6)$$

where $y \in X^{\{s\}}$, $s \in \mathbb{Z}^d \backslash \{t\}$. Further, for any $V \in W$, $t \in \mathbb{Z}^d \backslash V$ and $x \in X^V$, $z \in X^{\{t\}}$ put

$$p_{t \cup V}(zx) = p_t(z) q_V^z(x),$$

$$q_V^z(x) = \prod_{j=1}^{n} \frac{q_{t_j}^{z(xu)_j}(x_{t_j})}{q_{t_j}^{z(xu)_j}(u_{t_j})} \cdot \left(\sum_{\alpha \in X^V} \prod_{j=1}^{n} \frac{q_{t_j}^{z(\alpha u)_j}(\alpha_{t_j})}{q_{t_j}^{z(\alpha u)_j}(u_{t_j})} \right)^{-1}. \qquad (10.7)$$

Here $(xu)_j = x_{t_1}\dots x_{t_{j-1}}u_{t_{j+1}}\dots u_{t_n}$ for $1 < j < n$ and $(xu)_1 = u_{t_2}\dots u_{t_n}$, $(xu)_n = x_{t_1}\dots x_{t_{n-1}}$, $n = |V|$, $V = \{t_1, t_2, \dots, t_n\}$ is some enumeration of the points of the set V, and $u \in X^V$ is an arbitrary configuration. By virtue of (10.5), p_t and q_V^z are correctly defined. It is not difficult to see that the elements of the system K_{Q_1} are consistent in Kolmogorov's sense, and hence K_{Q_1} defines a random field P such that $K_P = K_{Q_1}$. Moreover, one can verify that $Q_1(P) = Q_1$. □

4.3 Finite-conditional distribution of a random field

For a given random field P, the system $Q^{\text{fin}}(P) = \{Q_V^z, z \in X^S, S \in W(\mathbb{Z}^d \backslash V), V \in W\}$, introduced in [3], is called *finite-conditional distribution of the random field P*. The system $Q^{\text{fin}}(P)$ restores the random field P.

We call a set $Q^{\text{fin}} = \{q_V^z, z \in X^S, S \in W(\mathbb{Z}^d \backslash V), V \in W\}$ of probability distributions parametrised by finite boundary conditions *specification with finite boundary conditions* if its elements satisfy the following consistency condition: for any disjoint sets $V, I, S \in W$ and configurations $x \in X^V$, $y \in X^I$, $z \in X^S$ it holds $q_{V\cup I}^z(xy) = q_V^z(x) q_I^{zx}(y)$.

Theorem 10.3 Let Q^{fin} be a positive specification with finite boundary conditions. Then there exists a unique random field P such that $Q^{\text{fin}}(P) = Q^{\text{fin}}$.

The proof of this result is similar to the proof of Theorem 10.1.

4.4 One-point finite-conditional distribution of a random field

For a given random field P, the system $Q_1^{\text{fin}}(P) = \{Q_t^z, z \in X^S, S \in W(\mathbb{Z}^d \backslash \{t\}), t \in \mathbb{Z}^d\}$, introduced in [4], is called *one-point finite-conditional distribution of the random field P*. The system $Q_1^{\text{fin}}(P)$ restores the random field P.

We call a set $Q_1^{\text{fin}} = \{q_t^z, z \in X^S, S \in W(\mathbb{Z}^d \backslash \{t\}), t \in \mathbb{Z}^d\}$ of one-point probability distributions parametrised by finite boundary conditions *1-specification with finite boundary conditions* if its elements satisfy the following consistency condition: for all $t, s \in \mathbb{Z}^d$, $S \in W\left(\mathbb{Z}^d \backslash \{t, s\}\right)$ and $x, u \in X^{\{t\}}$, $y, v \in X^{\{s\}}$, $z \in X^S$

$$q_t^{zy}(x) q_s^{zx}(v) q_t^{zv}(u) q_s^{zu}(y) = q_s^{zx}(y) q_t^{zy}(u) q_s^{zu}(v) q_t^{zv}(x). \tag{10.8}$$

Theorem 10.4 Let Q_1^{fin} be a positive 1-specification with finite boundary conditions. Then there exists a unique random field P such that $Q_1^{\text{fin}}(P) = Q_1^{\text{fin}}$.

The proof of this result is similar to the proof of Theorem 10.2.

Note that in [4] the inverse problem for the system Q_1^{fin} was solved under the following (equivalent to (10.8)) consistency conditions:

1. for all $t,s \in \mathbb{Z}^d$, $S \in W\left(\mathbb{Z}^d \backslash \{t,s\}\right)$ and $x \in X^{\{t\}}$, $y \in X^{\{s\}}$, $z \in X^S$
$$q_t^z(x)q_s^{zx}(y) = q_s^z(y)q_t^{zy}(x);$$
2. for all $t,s \in \mathbb{Z}^d$ and $x,u \in X^{\{t\}}$, $y,v \in X^{\{s\}}$
$$q_t^y(x)q_s^x(v)q_t^v(u)q_s^u(y) = q_s^x(y)q_t^y(u)q_s^u(v)q_t^v(x).$$

4.5 Palm-type conditional distribution of a random field

For a given random field P, the system $Q^\Pi(P) = \left\{Q_V^z, z \in X^{\{t\}}, t \in \mathbb{Z}^d \backslash V, V \in W\right\}$ of conditional probabilities under condition at a point we call *Palm-type conditional distribution of the random field* P. The system $Q^\Pi(P)$ restores the random field P.

We call a set $Q^\Pi = \left\{q_V^z, z \in X^{\{t\}}, t \in \mathbb{Z}^d \backslash V, V \in W\right\}$ of probability distributions parametrised by boundary condition at a point *Palm specification* if its elements satisfy the following consistency conditions:

1. for any disjoint sets $V, I \in W$ and $z \in X^{\{t\}}$, $t \in \mathbb{Z}^d \backslash (V \cup I)$
$$\sum_{y \in X^I} q_{V \cup I}^z(xy) = q_V^z(x), \qquad x \in X^V; \tag{10.9}$$
2. for any $t,s \in \mathbb{Z}^d$, $V \in W(\mathbb{Z}^d \backslash \{t,s\})$ and $x \in X^{\{t\}}$, $y \in X^{\{s\}}$, $u \in X^V$
$$q_t^y(x)q_{s\cup V}^x(yu) = q_s^x(y)q_{t\cup V}^y(xu). \tag{10.10}$$

Theorem 10.5 Let Q^Π be a positive Palm specification. Then there exists a unique random field P such that $Q^\Pi(P) = Q^\Pi$.

Proof. For any $V \in W$ put $p_V(x) = p_t(x_t)q_{V\backslash\{t\}}^{x_t}(x_{V\backslash\{t\}})$, $x \in X^V$, $t \in V$, where p_t is defined by (10.6). By virtue of (10.10), the probability distribution p_V does not depend on the choice of the point $t \in V$. According to (10.9), the system $\{p_V, V \in W\}$ is consistent in Kolmogorov sense, and hence there exists a unique random field P such that $K_P = \{p_V, V \in W\}$. Moreover, it can be shown that $Q^{\Pi}(P) = Q^{\Pi}$. □

4.6 Dobrushin's conditional distribution of a random field

Among the conditional distributions of a random field P under infinite boundary conditions, there is a system, introduced by Dobrushin in [5], which has a special place.

For a given random field P, the system $Q^{\mathrm{D}}(P) = \{Q_V^z, z \in X^{\mathbb{Z}^d \setminus V}, V \in W\}$ of conditional probabilities on X^V under a boundary condition everywhere outside V, $V \in W$, we call *Dobrushin's conditional distribution of the random field P*.

Generally speaking, Dobrushin's system does not solve the direct problem, since there may exist various random fields with the same Dobrushin's conditional distribution (see [6]). However, there are random fields which can be restored by Dobrushin's conditional distributions (see Theorem 2 in [5]).

We call a set $Q^{\mathrm{D}} = \{q_V^z, z \in X^{\mathbb{Z}^d \setminus V}, V \in W\}$ of probability distributions parametrised by infinite boundary conditions *Dobrushin's specification* if its elements satisfy the following consistency condition: for all disjoint sets $V, I \in W$ and $x \in X^V$, $y \in X^I$, $z \in X^{\mathbb{Z}^d \setminus (V \cup I)}$

$$q_{V \cup I}^z(xy) = q_I^{zx}(y) \sum_{v \in X^I} q_{V \cup I}^z(xv).$$

For a given specification Q^{D}, Dobrushin in [5] presented a condition (quasilocality) under which there exists a random field P such that $Q^{\mathrm{D}}(P) = Q^{\mathrm{D}}$. But there may exist more than one random field whose conditional distribution $Q^{\mathrm{D}}(P)$ coincides with Q^{D}, i. e., for Dobrushin's specification the inverse problem is incorrect. However, there are conditions under which such a random field is unique (see, for example, Theorem 2 in [5]).

4.7 One-point Dobrushin-type conditional distribution of a random field

For a given random field P, the system $Q_1^{\mathrm{D}}(P) = \{Q_t^z, z \in X^{\mathbb{Z}^d \setminus \{t\}}, t \in \mathbb{Z}^d\}$ was considered in [2]. We will call it *one-point Dobrushin-type conditional distribution of the random field P*. The system $Q_1^{\mathrm{D}}(P)$, generally speaking, does not solve the direct problem for the random field P.

We call a set $Q_1^{\mathrm{D}} = \{q_t^z, z \in X^{\mathbb{Z}^d \setminus \{t\}}, t \in \mathbb{Z}^d\}$ of one-point probability distributions parametrised by infinite boundary conditions *Dobrushin-type 1-specification* if its elements satisfy the following consistency condition: for all $t, s \in \mathbb{Z}^d$ and $x, u \in X^{\{t\}}$, $y, v \in X^{\{s\}}$, $z \in X^{\mathbb{Z}^d \setminus \{t,s\}}$

$$q_t^{zy}(x) q_s^{zx}(v) q_t^{zv}(u) q_s^{zu}(y) = q_s^{zx}(y) q_t^{zy}(u) q_s^{zu}(v) q_t^{zv}(x).$$

In [2] it was shown that for a given quasilocal Dobrushin-type 1-specification Q_1^{D} there exists a random field P such that $Q_1^{\mathrm{D}}(P) = Q_1^{\mathrm{D}}$; the conditions of uniqueness of P are the same as for Q^{D}.

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Bibliography

[1] Arzumanyan, V. A., Nahapetian, B. S.: *Consistent systems of finite dimensional distributions*, Armen. J. Math. **7**(2), 146–163 (2015).

[2] Dachian, S., Nahapetian, B. S.: *Description of random fields by means of one–point conditional distributions and some applications*, Markov Process. Relat. Fields **7**, 193–214 (2001).

[3] Dachian, S., Nahapetian, B. S.: *On Gibbsiannes of random fields*, Markov Process. Relat. Fields **15**, 81–104 (2009).

[4] Dalalyan, A., Nahapetian, B. S.: *Description of random fields by means of one-point finite conditional distribution*, J. Contemp. Math. Anal. Arm. Acad. Sci. **46** (2), 113–119 (2011).

[5] Dobrushin, R. L.: *The description of a random field by means of conditional probabilities and conditions of its regularity*, Theory Probab. Appl. **13** (2), 197–224 (1968).

[6] Dobrushin, R. L.: *The problem of uniqueness of a gibbsian random field and the problem of phase transitions*, Funct. Anal. Appl. **2** (4), 302–312, (1968).

[7] Kolmogorov, A. N.: *Foundations of the theory of probability*, Oxford, England: Chelsea Publishing Co. (1950).