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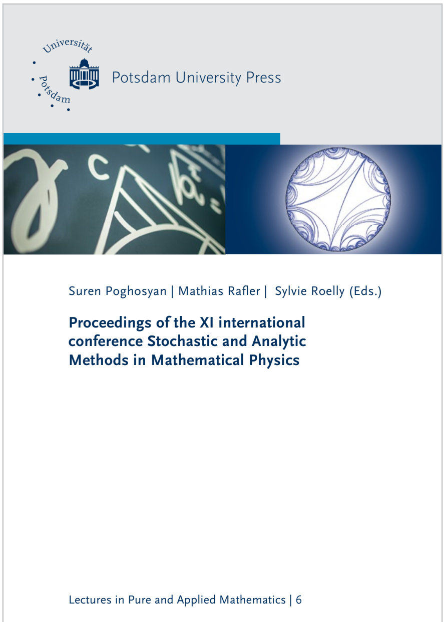

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When bounded chaos becomes unbounded

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Abstract. *We consider infinite particle systems with deterministic Newtonian dynamics. Assuming that initial conditions are uniformly bounded, we find examples and general conditions when coordinates and/or velocities remain bounded and when they can grow infinitely in time.*

1 Introduction

Here we present our first results in the field which could be called non-equilibrium deterministic mechanics of infinite systems. We hope that this field can provide a lot of models describing some qualitative phenomena in physical and biological systems. Obviously, the main interesting interaction for such models is Coulomb interaction. However, it remains difficult and as of yet unknown. That is why we use quadratic interaction, which is natural when each particle spends all its time in some potential well. Assuming that initial conditions are uniformly bounded, we find examples and general conditions when the coordinates and velocities remain bounded and when they can grow infinitely in time. Our conclusion is: when the initial deviations from the equilibrium strongly fluctuate,

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then they grow infinitely in time, and when they are sufficiently smooth, they stay uniformly bounded forever. However, we could not get necessary and sufficient conditions for this.

We consider here a countable system of point particles with unit masses on $\mathbb{R}$ with coordinates $\{x_k\}_{k\in\mathbb{Z}}$ and velocities $\{v_k\}_{k\in\mathbb{Z}}$. We define a formal energy (hamiltonian) by the following formula:

$$H=\sum_{k\in\mathbb{Z}}\frac{v_k^2}{2}+\frac{\omega_0^2}{2}\sum_{k\in\mathbb{Z}}\big(x_k(t)-ka\big)^2+\frac{\omega_1^2}{2}\sum_{k\in\mathbb{Z}}\big(x_k(t)-x_{k-1}(t)-a\big)^2,$$

with parameters $a>0$, $\omega_1>0$, $\omega_0\geqslant 0$. Particle dynamics is defined by the infinite system of ODE:

$$\ddot{x}_k(t)=-\frac{\partial H}{\partial x_k}=-\omega_0^2\big(x_k(t)-ka\big)+\omega_1^2\big(x_{k+1}(t)-x_k(t)-a\big)$$
$$-\,\omega_1^2\big(x_k(t)-x_{k-1}(t)-a\big),\qquad k\in\mathbb{Z}\tag{9.1}$$

with initial conditions $x_k(0)$, $v_k(0)$. The equilibrium state (minimum of the energy) is

$$x_k=ka,\quad v_k=0,\qquad k\in\mathbb{Z}.$$

This means that if the initial condition is the equilibrium state, then the system will not evolve, i.e. $x_k(t)=ka$, $v_k(t)=0$ for all $t\geqslant 0$. Let us introduce the deviation variables:

$$q_k(t)=x_k-ka,\qquad p_k(t)=\dot{q}_k(t)=v_k(t).$$

It is easy to see that $q_k(t)$ satisfies the following system of ODE:

$$\ddot{q}_k=-\omega_0^2q_k+\omega_1^2(q_{k+1}-q_k)-\omega_1^2(q_k-q_{k-1}),\quad k\in\mathbb{Z}.\tag{9.2}$$

The system of coupled harmonic oscillators (9.2) and its generalisations is a classical object in mathematical physics. The existence of a solution and its ergodic properties were studied in [12]. There has been an extensive research on convergence to equilibrium for an infinite harmonic chain coupled with a heat bath ([1, 7, 15, 2]). The property of uniform boundedness of particle coordinates (by time t and index k) is crucial in some applications. For instance, uniform boundedness in finite harmonic chains allows to derive Euler equations and Chaplygin gas without any stochastics (see in [13]). Uniform boundedness of a one-side non-symmetrical harmonic chain plays an important role in some traffic flow models [14]. We should also cite some physical papers [11, 8, 9]. The most closely related works to our results are [5, 6], where the author studied weighted l_2 norms of infinite harmonic chains, whereas our main interest is a max-norm.

Remark. Proofs of all forthcoming theorems will appear in the second issue of the new journal “Structure of Mathematical Physics”, 2020, No. 2.

2 l_2 initial conditions

Introduce the following function spaces on $\mathbb{Z}$:

$$l_\infty := l_\infty(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{R} : \sup_{k\in\mathbb{Z}} |f(k)| < \infty\}, \qquad |f|_\infty = \sup_{k\in\mathbb{Z}} |f(k)|,$$
$$l_2 := l_2(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{R} : \sum_{k\in\mathbb{Z}} |f(k)|^2 < \infty\}, \qquad |f|_2 = \sqrt{\sum_{k\in\mathbb{Z}} |f(k)|^2}.$$

If $q(0)$, $p(0) \in l_2(\mathbb{Z})$, then there exists unique solution $q(t)$, $p(t)$ of (9.2) which belongs to $l_2(\mathbb{Z})$, i.e. $q(t)$, $p(t) \in l_2(\mathbb{Z})$ for all $t \geqslant 0$. This assertion is well known (see [12, 3, 4]), and easily follows from the boundedness of the operator W on $l_2(\mathbb{Z})$:

$$(Wq)_k = -\omega_0^2 q_k + \omega_1^2(q_{k+1} - q_k) - \omega_1^2(q_k - q_{k-1}).$$

The first question of our interest is uniform boundedness (in k and time $t \geqslant 0$) of $|q_k(t)|$. Define the max-norm of $q_k(t)$, $M(t) := \sup_k |q_k(t)|$. We shall say that the system has the property of uniform boundedness, if $\sup_{t\geqslant 0} M(t) < \infty$.

Theorem 9.1 The following assertions hold:

1) If $\omega_0 > 0$, then $\sup_{t\geqslant 0} M(t) < \infty$.

2) If $\omega_0 = 0$,

then we have several results:

a) For all $t \geqslant 0$ the following inequality holds:

$$M(t) \leqslant \frac{2}{\sqrt{\omega_1}} \|p(0)\|_2 \sqrt{t} + \|q(0)\|_2 \tag{9.3}$$

b) Suppose that $\sum_{k\neq 0} |p_k(0)| \ln |k| < \infty$. Then there is constant $c > 0$ such that for all $t \geqslant 0$:

$$M(t) \leqslant \frac{\sqrt{2}}{\omega_1 \pi} |P| \ln(t) + \|q(0)\|_2 + c, \qquad P = \sum_k p_k(0)$$

c) For all $\delta > 1/2$, there exists at least one initial condition $q(0) = 0,\ p(0) \in l_2(\mathbb{Z})$ such that

$$\lim_{t\to\infty} \frac{q_0(t)}{\sqrt{t}} \ln^\delta t = \Gamma(\delta) > 0,$$

where Γ is the gamma function.

From case 9.1 a) we see that if $\omega_0 = 0$ and the initial velocities of the particles are all zero, then $|q_k(t)|$ are uniformly bounded. The assertions 9.1 c) is an attempt to answer the question on the accuracy in the basic inequality (9.3) from 9.1 a) with respect to the rate of growth in t.

Next we will formulate theorems concerning asymptotic behavior of $q_k(t)$ in several cases. Define Fourier transform of the sequence $u = \{u_k\} \in l_2(\mathbb{Z})$, $\widehat{u}(\lambda) = \sum_k u_k e^{ik\lambda}$, $\lambda \in \mathbb{R}$. Note that $\widehat{u}(\,\cdot\,) \in L_2([0,2\pi])$, i.e. $\int_0^{2\pi} |\widehat{u}(\lambda)|^2 \, d\lambda = 2\pi \sum_k |u_k|^2 < \infty$.

Further on we will use the Fourier transform of the initial conditions $Q(\lambda) = \widehat{q(0)}(\lambda)$, $P(\lambda) = \widehat{p(0)}(\lambda)$.

For complex valued functions f and g on $\mathbb{R}$ and a constant $c \in \mathbb{C}$ we will write $f(x) \asymp c + \frac{g(x)}{\sqrt{x}}$, if $f(x) = c + \frac{g(x)}{\sqrt{x}} + \bar{\bar{o}}\left(\frac{1}{\sqrt{x}}\right)$ as $x \to \infty$.

Theorem 9.2 **($\omega_0 > 0$)** Suppose that $\omega_0 > 0$ and Q, P are of the class $C^n(\mathbb{R})$ for some $n \geqslant 2$. Then

1) For any fixed $t \geqslant 0$ we have $q_k(t) = O(k^{-n})$.

2) For any fixed $k \in \mathbb{Z}$ and $t \to \infty$ we have the following asymptotic formula:

$$q_k(t) \asymp \frac{1}{\sqrt{t}} \Big(C_1 \cos\big(\omega_1(t)\big) + S_1 \sin\big(\omega_1(t)\big) + (-1)^k C_2 \cos\big(\omega_2(t)\big) + (-1)^k S_2 \sin\big(\omega_2(t)\big) \Big),$$

where

$$C_1 = \frac{1}{\omega_1} \sqrt{\frac{\omega_0}{2\pi}} Q(0), \qquad S_1 = \frac{1}{\omega_1 \omega_0} \sqrt{\frac{\omega_0}{2\pi}} P(0)$$

$$C_2 = \frac{1}{\omega_1} \sqrt{\frac{\omega_0'}{2\pi}} Q(\pi), \qquad S_2 = \frac{1}{\omega_1 \omega_0'} \sqrt{\frac{\omega_0'}{2\pi}} P(\pi),$$

$$\omega_1(t) = t\omega_0 + \frac{\pi}{4}, \qquad \omega_2(t) = t\omega_0' - \frac{\pi}{4}, \qquad \omega_0' = \sqrt{\omega_0^2 + 4\omega_1^2}.$$

3) Let $t = \beta |k|$, $\beta > 0$ and $k \to \infty$. Put $\gamma(\beta) = \beta^2 \omega_1^2 - 1 - \beta\omega_0$.

a) If $\gamma(\beta) > 0$, then

$$q_k(t) \asymp \frac{1}{\sqrt{|k|}}\left(\mathscr{F}_k^+[Q] - i\mathscr{F}_k^-\left[\frac{P(\lambda)}{\omega(\lambda)}\right]\right),$$

where for a complex valued function $g(\lambda)$ defined on the real line we introduce the following functionals:

$$\mathscr{F}_k^\pm[g] = c_+\left(g(\mu_+)e^{i\omega_+(k)} \pm g(-\mu_+)e^{-i\omega_+(k)}\right) + c_-\left(g(\mu_-)e^{i\omega_-(k)} \pm g(-\mu_-)e^{-i\omega_-(k)}\right)$$

$$\omega_\pm(k) = k\big(\mu_\pm + \beta\omega(\mu_\pm)\big) \pm \frac{\pi}{4}\operatorname{sign}(k), \qquad c_\pm = \frac{1}{2}\sqrt{\frac{\beta\omega(\mu_\pm)}{2\pi\Delta}},$$

$$\mu_\pm = -\arccos\frac{1}{\beta^2\omega_1^2}(1\pm\Delta),$$

$$\Delta = \sqrt{(\beta^2\omega_1^2-1)^2-\beta^2\omega_0^2}, \qquad \omega(\lambda) = \sqrt{\omega_0^2+2\omega_1^2(1-\cos\lambda)}.$$

b) If $\gamma(\beta) = 0$ and $n \geqslant 3$, then $q_k(t) = O(k^{-3})$.

c) If $\gamma(\beta) < 0$ then $q_k(t) = O(k^{-n})$ for n defined above.

Recall that a sufficient condition on $z \in l_2(\mathbb{Z})$ for $\widehat{z} \in C^n(\mathbb{R})$ is $\sum_k |k|^n|z_k| < \infty$. Thus if the following series converge for some $n \geqslant 2$:

$$\sum_k |k|^n|q_k(0)| < \infty \qquad \text{and} \qquad \sum_k |k|^n|p_k(0)| < \infty,$$

then Theorem 9.2 holds.

Theorem 9.3 ($\omega_0 = 0$) Suppose that $\omega_0 = 0$ and $Q, P \in C^n(\mathbb{R})$, $n \geqslant 6$ then

1) For any fixed $t \geqslant 0$ we have $q_k(t) = O(k^{-n})$.

2) For any fixed $k \in \mathbb{Z}$ and $t \to \infty$ one has:

$$q_k(t) \asymp \frac{P(0)}{2\omega_1} + \frac{(-1)^k}{\sqrt{t}}\Big(C\cos\big(2\omega_1 t - \tfrac{\pi}{4}\big) + S\sin\big(2\omega_1 t - \tfrac{\pi}{4}\big)\Big),$$

where

$$C = \frac{1}{\sqrt{\pi\omega_1}}Q(\pi), \qquad S = \frac{1}{2\omega_1\sqrt{\pi\omega_1}}P(\pi).$$

3 l_∞ initial conditions

Our next concern will be the uniform boundedness in k and t of the solution. Denote $p(t) = \dot{q}(t)$. Further on we always assume that $q(0) \in l_\infty$, $p(0) = 0$, that is $\sup_k |q_k(0)| < \infty$, $p_k(0) = 0$. The following result follows from Theorem 9.1.

Proposition 9.4 Let $q(0) \in l_2(\mathbb{Z})$, $p(0) = 0$, then $|q(t)|_\infty \leqslant |q(0)|_2$.

Thus the solution will be uniformly bounded. The situation drastically changes if we consider l_∞ initial conditions. Namely, the following theorem holds.

Theorem 9.5

1) Let $q(0) \in l_\infty(\mathbb{Z})$, $p(0) = 0$, then for any $t \geqslant 0$:
$$|q(t)|_\infty \leqslant \left(\sqrt{2\gamma\omega_1 t} + 2\right) |q(0)|_\infty,$$
where $\gamma > 0$ is the root of the equation $\frac{1}{\gamma} e^{\frac{1}{\gamma}} = \frac{1}{e}$.

2) For any $k \in \mathbb{Z}$ there exists a constant $c > 0$, initial conditions $q(0) \in l_\infty(\mathbb{Z})$, $p(0) = 0$ and increasing sequence of time moments $t_1 < t_2 < \ldots$, $t_n \to \infty$ as $n \to \infty$ such that
$$q_k(t_{2n}) \geqslant c\sqrt{t_{2n}}, \qquad q_k(t_{2n+1}) \leqslant -c\sqrt{t_{2n+1}}, \qquad n = 1, 2, \ldots.$$

Corollary 9.6 For any $k \in \mathbb{Z}$ there is initial condition $q(0) \in l_\infty(\mathbb{Z})$, $p(0) = 0$ such that
$$\limsup_{t\to\infty} \frac{q_k(t)}{\sqrt{t}} = c_1 > 0, \qquad \liminf_{t\to\infty} \frac{q_k(t)}{\sqrt{t}} = c_2 < 0.$$
for some constants c_1, c_2 depending on k.

Define the following operator on l_∞,
$$(Vq)_k = -\omega_1^2 (\Delta q)_k = -\omega_1^2 (q_{k+1} - 2q_k + q_{k-1}).$$
It is clear that $|V|_\infty \leqslant 4\omega_1^2$. Thus the following operator $C(t)$ is also bounded in l_∞:
$$C(t) \doteqdot \cos\left(t\sqrt{V}\right) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k} V^k}{(2k)!}. \tag{9.4}$$

Theorem 9.7 There exist constants $a, b > 0$ such that for all $t \geqslant 0$ the following inequalities hold:
$$a\sqrt{t} + 1 \leqslant \left|\cos\left(t\sqrt{V}\right)\right|_\infty \leqslant b\sqrt{t} + 1$$

We will need some definitions. For sequence $q \in l_\infty$ define new sequence:

$$q^\Delta = -\Delta q, \quad q_k^\Delta = 2q_k - q_{k+1} - q_{k-1}, \qquad k \in \mathbb{Z}.$$

Let $l^\Delta \subset l_\infty(\mathbb{Z})$ be the set of sequences $q \in l_\infty(\mathbb{Z})$, for which the following conditions hold:

1) $q^\Delta \in l_2(\mathbb{Z})$. Then the Fourier transform of q^Δ, $Q^\Delta(\lambda) = \sum_k e^{ik\lambda} q_k^\Delta$ belongs to $L_2([0, 2\pi])$.

2) For some real number $A \in \mathbb{R}$ the function
$$\phi(\lambda) = \frac{1}{\sin\frac{\lambda}{2}} \left(\frac{Q^\Delta(\lambda)}{\sin\frac{\lambda}{2}} - iA \right) \tag{9.5}$$
belongs to $L_1[0, \pi]$, where $i^2 = -1$, that is $\int_0^\pi |\phi(\lambda)| d\lambda < \infty$.

Then l^Δ becomes a linear vector space over $\mathbb{R}$.

Theorem 9.8 Assume that $q(0) \in l^\Delta$, $p(0) = 0$, then the solution $\{q_k(t)\}$ is uniformly bounded that is $\sup_{t \geqslant 0} \sup_{k \in \mathbb{Z}} |q_k(t)| < \infty$.

Theorem 9.9 Assume that $q(0) \in l^\Delta$, $p(0) = 0$, then there exists $\nu \in \mathbb{R}$ such that for any $k \in \mathbb{Z}$ the following equality holds: $\lim_{t \to \infty} q_k(t) = \nu$.

Relation of the number ν with the limit $q_k(0)$ at infinity is given in Theorem 9.11 below.

We now give examples of sequences $q \in l^\Delta$.

1) Sign sequence. Put
$$q_k = \operatorname{sign}(k) = \begin{cases} 1, & k > 0, \\ 0, & k = 0, \\ -1, & k < 0 \end{cases}.$$
It is clear that $q_k^\Delta = 0$ for $|k| > 1$. Then $q_1^\Delta = 1$, $q_{-1}^\Delta = -1$, $q_0^\Delta = 0$. That is why $Q^\Delta(\lambda) = \left(e^{i\lambda} - e^{-i\lambda}\right) = 2i\sin(\lambda)$. Put $A = 4$ in (9.5). Then
$$\phi(\lambda) = \frac{1}{\sin\frac{\lambda}{2}} \left(\frac{2\sin(\lambda)}{\sin\frac{\lambda}{2}} - 4 \right) = \frac{4}{\sin\frac{\lambda}{2}} \left(\cos\frac{\lambda}{2} - 1 \right).$$
It is clear that $\phi(\lambda) \in L_1[0, \pi]$. Thus, $\operatorname{sign}(k) \in l^\Delta$. See Figure 9.1 for the solution with intial condition $q_k(0) = \operatorname{sign}(k)$, $p(0) = 0$. and $\omega = 1/2$: Both particles, (with

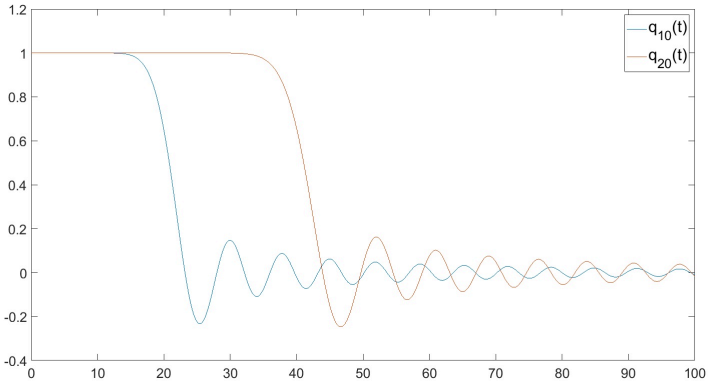

Figure 9.1: Solution with initial condition $q_k(0) = \mathrm{sign}(k)$, $p(0) = 0$

numbers 10 and 20), until $t \ll 2n$, oscillate around point 1 with exponentially small amplitude. However, such fluctuations are not visible on the picture

Then they quickly fall into a regime of relaxation oscillations around the equilibrium point. In such a case the solution is given by the formula:

$$q_n(t) = J_0(t) + 2\sum_{k=1}^{n-1} J_{2k}(t) + J_{2n}(t) = 1 + J_{2n}(t) - 2\sum_{k=2n}^{\infty} J_{2k}(t), \qquad n \geqslant 1, \quad (9.6)$$

where

$$J_n(t) = \frac{1}{\pi}\int_0^{\pi} \cos(nx - t\sin x)dx, \qquad t \geqslant 0$$

is the Bessel function of first kind. In Equality (9.6) we used the known formula ([10]):

$$2\sum_{k=1}^{\infty} J_{2k}(t) + J_0(t) = 1.$$

2) Now consider as example the in some sense opposite to the Sign sequence:

$$q_k = \begin{cases} 1, & k \neq 0 \\ b, & k = 0 \end{cases},$$

for some $b \in \mathbb{R}$. Then

$$Q^\Delta(\lambda) = e^{i\lambda}(2-b-1) + 2b - 2 + e^{-i\lambda}(2-b-1)$$
$$= 2(b-1)(1-\cos\lambda) = 4(b-1)\sin^2\frac{\lambda}{2}.$$

Put $A = 0$ in (9.5). Then $\phi(\lambda) = 4(b-1)$. Again we see that $\phi(\lambda) \in L_1[0,\pi]$ and then $q_k \in l^\Delta$.

3) Consider the sequence $q_k = (-1)^k$. Then $(\Delta q)_k = (-1)^k(-1-1-2) = -4q_k$. And thus, $q \notin l^\Delta$. Nevertheless one can prove the uniform boundedness of solution with such initial conditions. It is known that $q(t) = \cos\left(t\sqrt{V}\right)q(0)$ and $V = -\omega_1^2\Delta$. Thus

$$q(t) = \sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}V^k}{(2k)!}q = \sum_{k=0}^{\infty}(-1)^k\frac{(4\omega_1^2)^k t^{2k}}{(2k)!}q = \cos(2\omega_1 t)q.$$

Uniform boundedness of $q(t)$ follows.

Theorem 9.10 Assume that

$$\sum_{k\neq 0}|q_k^\Delta|\cdot|k|\ln|k| < \infty, \tag{9.7}$$

then $q \in l^\Delta$.

As an example, consider the sequence

$$q_k = \frac{\sin\left(\ln\ln|k|\right)}{\ln^2\left(|k|\right)} \quad \text{if } |k| > 1$$

and $q_k = 0$ for $|k| \leqslant 1$. It is not difficult to see that

$$q_k^\Delta = O\left(\frac{1}{k^2\ln^3|k|}\right).$$

Thus the conditions of (9.7) hold, and then $q \in l^\Delta$.

Theorem 9.11 Assume that $q \in l^\Delta$, then there exist finite limits

$$\lim_{k\to+\infty} q_k = L_+, \qquad \lim_{k\to-\infty} q_k = L_-$$

and the following equalities hold as well:

$$L_+ - L_- = \frac{A}{2}, \qquad \frac{L_+ + L_-}{2} = \nu,$$

where number A is defined in (9.5), and ν was introduced in Theorem 9.9.

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