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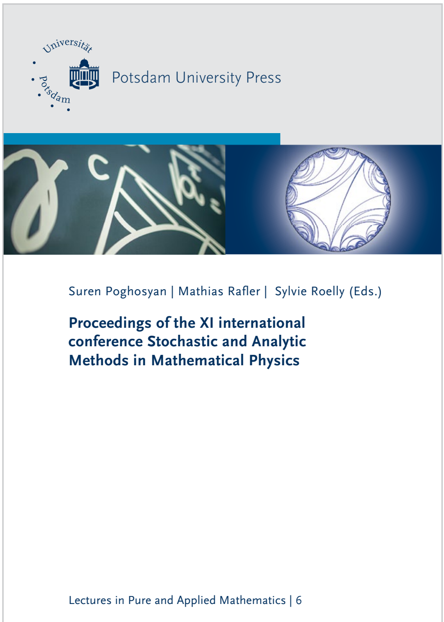

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On an approximation of 2-D stochastic Navier-Stokes equations

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Abstract. *We describe a full-discrete explicit approximation scheme for the process solution of the two-dimensional incompressible stochastic Navier-Stokes equations driven by additive noise with periodic boundary conditions. We focus on the properties which play a role in the proof of the strong convergence towards the mild solution of the equation.*

1 Introduction

Often (stochastic) evolution equations, such as stochastic Navier-Stokes equations, are mathematical models for dynamics and phenomena in physics. Therefore, the simulation of the solutions with implementable approximation schemes has become of great interest. The approximations should converge in some sense and possibly reflect the behaviour of the solutions. A *strongly convergent* scheme (i. e., in mean square) "respects", for instance, the mean of the process.

In general terms, the explicit and the linear-implicit Euler schemes do not converge strongly to the solutions of many stochastic evolution equations (see e. g. [7, Theorem 2.1]) and convergent implicit schemes have higher computational costs (for more details see, e. g. [8]). Therefore recently, different versions of the Euler method have

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been proven to converge strongly for (specific cases of) evolution equations. The techniques are different according to the specificity of the coefficients and the dimension: truncation of the drift, taming, etc.

In the case of two-dimensional stochastic Navier-Stokes equations driven by additive or multiplicative noise, several existence and uniqueness results and (strongly) convergent approximation schemes are available. We refer to [4] and references therein for existence and uniqueness results and an overview on numerical approximations such as the strong convergent ones in [5] (in the additive noise case). Other relevant strongly convergent schemes are the fully implicit and also the semi-implicit Euler schemes introduced in [3] and the splitting scheme of [1]. This list is far from extensive. We refer for instance to the introduction of the recent article [6] for a state-of-the-art summary.

In this article we describe an *explicit full-discrete non-linearity-truncated accelerated exponential Euler-type scheme* (DTAEE scheme, see Equation (8.5) below) which has been proven in [11] to converge strongly to the *mild solutions*[1] of the two-dimensional incompressible stochastic Navier-Stokes equations on the torus driven by some trace class noise in Equation (8.3). We focus on the description of the approximation scheme stressing the properties leading to the already mentioned strong convergence result. The contribution of this document is therefore a deeper insight on the properties of the DTAEE scheme.

This paper is organised as follows: We first introduce the stochastic Navier-Stokes equations under consideration (see Section 2). In Section 3 we focus on the numerical approximation scheme DTAEE. Finally in Section 3.2 we comment on the strong convergence of the approximation towards the solution.

2 The framework: 2-D stochastic Navier-Stokes equations

2.1 The 2-D stochastic Navier-Stokes equations with periodic boundary conditions on the torus and trace class noise

Let $T \in (0,\infty)$, let $\lambda_{(0,1)^2}$ denote the Lebesgue measure on $(0,1)^2$, and let $H_0 \subset H_1 \subset H \subset L^2(\lambda_{(0,1)^2};\mathbb{R}^2)$ be appropriate Hilbert subvector spaces of the Hilbert space $L^2(\lambda_{(0,1)^2};\mathbb{R}^2)$

[1] Weaker notion of solution with respect to the classical strong and weak one: any strong/weak solution is also a mild solution.

to be precised in Section 2.2. In particular, H is the separable Hilbert space having an orthonormal basis consisting of divergence-free functions with periodic boundary. Let P be the projection on H of elements of $L^2(\lambda_{(0,1)^2};\mathbb{R}^2)$, and let W be an Id_H-cylindrical Wiener process.

Let $\varepsilon_0, \varepsilon \in (0,\infty)$ and $\xi \in H_0$. It is known that the process $X\colon [0,T]\times\Omega\to H_1$ satisfying for all $t\in[0,T]$ that $\mathbb{P}$-a.s.

$$X_t = \mathrm{e}^{tA}\xi + \int_0^t \mathrm{e}^{(t-s)A}\big(F(X_s)+\varepsilon_0 X_s\big)\,ds + \int_0^t \mathrm{e}^{(t-s)A}(-A)^{-1/2-\varepsilon}\,dW_s \tag{8.1}$$

is a *mild solution* to the following stochastic partial differential equation

$$\begin{cases} dX_t(x) = \big(\Delta X_t(x) + F\big(X_t\big)(x)\big)dt + (-A)^{-1/2-\varepsilon}\,dW_t(x), \quad x\in(0,1)^2, t\in[0,T], \\ \quad X_0 = \xi \in H_0, \end{cases} \tag{8.2}$$

with *incompressibility* (i. e. divergence-free) condition $\operatorname{div} X_t = 0$ and where $A = \Delta - \varepsilon_0$, Δ is the Laplacian with periodic boundary conditions, and $F(X_s) = c_1 X_s + c_2 P(-\nabla X_s \cdot X_s)$ with $c_1, c_2 \in \mathbb{R}$.

This is a two-dimensional stochastic Navier-Stokes equations on the torus $(0,1)^2$ with periodic boundary conditions driven by some trace class noise. Indeed $(-A)^{-1/2-\varepsilon}$, $\varepsilon \in (0,\infty)$ is a Hilbert-Schmidt operator, so we are actually considering as noise a Wiener process on the Hilbert space H with covariance matrix $(-A)^{-1-2\varepsilon}$. Note that we could change the noise and/or the operator A up to a multiplicative constant, or consider a more regular noise. For simplicity, from now on, we take $c_1 = -\varepsilon_0$, $c_2 = 1$. Hence the mild solution (8.1) rewrites

$$X_t = \mathrm{e}^{tA}\xi + \int_0^t \mathrm{e}^{(t-s)A}P(-\nabla X_s\cdot X_s)\,ds + \int_0^t \mathrm{e}^{(t-s)A}(-A)^{-1/2-\varepsilon}\,dW_s. \tag{8.3}$$

To conclude, note that the mild solution expresses the process as a stochastic evolution equation. The right-hand side of equation is the sum of a Bochner integral, resulting from the convolution of the semigroup and the non-linearity, with a stochastic integral which is the noise part, also called the *stochastic convolution* process.

2.2 The state space

Let us now construct the Hilbert space of square-integrable divergence-free functions with periodic boundary conditions $H \subset L^2(\lambda_{(0,1)^2};\mathbb{R}^2)$. (Recall that $\lambda_{(0,1)^2}$ denotes the Lebesgue measure on $(0,1)^2$).

For all $k \in \mathbb{Z}$ let $\varphi_k \in C\big((0,1),\mathbb{R}\big)$ be the function

$$\varphi_k(x) := \mathbb{1}_{\{0\}}(k) + \mathbb{1}_{\mathbb{N}}(k)\sqrt{2}\cos(2k\pi x) + \mathbb{1}_{\mathbb{N}}(-k)\sqrt{2}\sin(-2k\pi x), \qquad x \in (0,1).$$

Let the following elements of $L^2(\lambda_{(0,1)^2};\mathbb{R}^2)$:

$$e_{0,0,0} \equiv (1,0), \qquad e_{0,0,1} \equiv (0,1), \quad \text{and} \quad e_{k,l,0}\colon (x,y) \mapsto \left(\tfrac{l\varphi_k(x)\varphi_l(y)}{\sqrt{k^2+l^2}}, \tfrac{k\varphi_{-k}(x)\varphi_{-l}(y)}{\sqrt{k^2+l^2}} \right)$$

for all $k,l \in \mathbb{Z}^2 \setminus \{(0,0)\}$.

Let $H \subseteq U$ be the closed subvector space of $L^2(\lambda_{(0,1)^2};\mathbb{R}^2)$ with orthonormal basis $\mathbb{H} = \{e_{0,0,1}\} \cup \{e_{i,j,0}\colon i,j \in \mathbb{Z}\}$.

In addition consider the eigenvalues of the perturbed Laplace operator $\varepsilon_0 - \Delta$:

$$\lambda_{e_{0,0,1}} = \lambda_{e_{0,0,0}} = \varepsilon_0, \quad \lambda_{e_{k,l,0}} = \varepsilon_0 + 4\pi^2(k^2+l^2), \qquad k,l \in \mathbb{Z}.$$

Note that the operator $\varepsilon_0 - \Delta$ is a diagonal operator on the basis $\mathbb{H}$ with point spectrum $\{\lambda_h\colon h \in \mathbb{H}\}$: for all v in the domain of $(\varepsilon_0 - \Delta)$

$$(\varepsilon_0 - \Delta)v = \sum_{h\in\mathbb{H}} \lambda_h \langle v,h\rangle_H.$$

Let ρ_0,ρ be positive real numbers satisfying $1/2 < \rho_0 < \rho < 1/2 + \varepsilon$, $\gamma \in (\rho,\infty)$, $\kappa \geq 0$ and let H_0, H_1, H_ρ be respectively the domains of the following fractional powers of the operator $(\kappa - \Delta)$: $(\kappa-\Delta)^\gamma$, $(\kappa-\Delta)^\rho$ and $(\kappa-\Delta)^{\rho_0}$. Therefore $\|h\|^2_{H_1} = \sum_{v\in\mathbb{H}}(\kappa - \varepsilon_0 + \lambda_v)^{2\rho}\langle h,v\rangle^2$ for every $h \in H_1$. For simplicity we take here $\kappa = \varepsilon_0$, hence

$$\|h\|^2_{H_1} = \sum_{v\in\mathbb{H}} \lambda_v^{2\rho}\langle h,v\rangle^2$$

for every $h \in H_1$.

3 The explicit full-discrete non-linearity-truncated accelerated exponential Euler-type scheme

In this section we will need the following notation: For every $n \in \mathbb{N}$, let $\mathbf{H}_n$ be the finite dimensional subspace of H spanned by

$$\mathbb{H}_n := \{e_{0,0,1}\} \cup \{e_{k,l,0} \colon k,l \in \mathbb{Z} \text{ and } k^2 + l^2 < n^2\} \subseteq \mathbb{H}$$

and $P_n \colon H \to H$ the projection on $\mathbf{H}_n$

$$P_n(u) := \sum_{h \in \mathbb{H}_n} \langle h, u \rangle_H \, h, \quad u \in H.$$

3.1 Step-by-step construction

The DTAEE scheme approximating in the strong sense the mild solution (8.3) can be constructed as follows in several steps. First, one considers a *spectral Galerkin approximation* (see [2]) combined with truncation of the non-linearity, obtaining the approximation scheme (8.4) below. Then one discretises the time, obtaining (8.5) below, and finally one notices that the quantities can be computed explicitly with the known square integrable functions belonging to the orthonormal basis $\mathbb{H}$.

Let $(h_m)_{m\in\mathbb{N}}$ be a sequence of positive real numbers converging to 0 and let P_n be projections on increasing finite dimensional spaces $\mathbf{H}_n \subseteq H_1$ specified in Section 2.2. Let $\mathscr{O}^n, \mathscr{X}^n \colon [0,T] \times \Omega \to \mathbf{H}_n$ be the stochastic processes satisfying for all $n \in \mathbb{N}$, $t \in [0,T]$:

$$\begin{aligned} \mathscr{O}_t^n &= \int_0^t P_n \mathrm{e}^{(t-s)A} (-A)^{-1/2-\varepsilon} \, dW_s + P_n e^{tA} \xi \\ \mathscr{X}_t^n &= \mathscr{O}_t^n + \int_0^t P_n \mathrm{e}^{(t-s)A} \, \mathbb{1}_{\left\{ \| \mathscr{X}^n_{\lfloor s \rfloor_{h_n}} \|_{H_1} + \| \mathscr{O}^n_{\lfloor s \rfloor_{h_n}} \|_{H_1} \le h_n^{-\chi} \right\}} (-\nabla \mathscr{X}^n_{\lfloor s \rfloor_{h_n}} \cdot \mathscr{X}^n_{\lfloor s \rfloor_{h_n}}) \, ds \end{aligned} \tag{8.4}$$

$\mathbb{P}$-a.s., where $\chi \in \left(0, \min\{\frac{1-\rho_0}{5}, \frac{\rho-\rho_0}{3}\}\right)$, $\lfloor t \rfloor_{h_n} := \max\left((-\infty, t] \cap \{0, h_n, -h_n, 2h_n, -2h_n, \ldots\}\right)$ denotes the so-called round-ground function.

The latter scheme is not full-discrete, but the fact that we know precisely how the operator acts on elements of (the orthonormal basis $\mathbb{H}$ of) H yields its fully explicit space-time discrete version. We call it DTAEE scheme and it is derived by taking for all $n \in \mathbb{N}$ the sequence $\left(\mathscr{X}^n_{(k+1)h_n}\right)_{k\in(-1,T/h_n-1)\cap\mathbb{N}}$ and making explicit the projections P_n.

Let us first consider the time discretisation: For all $n = 1,2,\ldots$, $k \in (-1, T/h_n - 1)\cap\mathbb{N}$ let $\mathbf{X}^n_0 := \mathbf{O}^n_0 := P_n\xi = \sum_{v\in\mathbb{H}_n}\langle v,\xi\rangle v$ and

$$\begin{aligned}\mathbf{O}^n_{(k+1)h_n} &= \mathrm{e}^{h_nA}\mathbf{O}^n_{kh_n} + \int_{kh_n}^{(k+1)h_n} P_n\mathrm{e}^{((k+1)h_n-s)A}(-A)^{-1/2-\varepsilon}\,dW_s,\\ \mathbf{X}^n_{(k+1)h_n} &= \mathrm{e}^{h_nA}\mathbf{X}^n_{kh_n} + \mathbf{O}^n_{(k+1)h_n} - \mathrm{e}^{h_nA}\mathbf{O}^n_{kh_n}\\ &\quad + \int_{kh_n}^{(k+1)h_n} P_n\mathrm{e}^{((k+1)h_n-s)A}\,\mathbb{1}_{\left\{\|\mathbf{X}^n_{kh_n}\|_{H_1}+\|\mathbf{O}^n_{kh_n}\|_{H_1}\le h_n^{-\chi}\right\}}(-\nabla\mathbf{X}^n_{kh_n}\cdot\mathbf{X}^n_{kh_n})\,ds.\end{aligned} \tag{8.5}$$

We can explicate further the approximation scheme in (8.5) for two-dimensional stochastic Navier-Stokes equations. Indeed, one of the main features of the scheme (8.4) is that it does not discretise the semigroup and yet a discretisation of the noise part is allowed in the following sense. Let us consider $(\beta^v)_{v\in\mathbb{H}}$ a sequence of independent standard Brownian motions such that the cylindrical Wiener process can be seen as $W = \sum_{v\in\mathbb{H}}\beta^v v$. Let us rewrite the approximation of the noise in (8.5) as

$$\mathbf{O}^n_{(k+1)h_n} = \sum_{v\in\mathbb{H}_n}\left(\mathrm{e}^{-h_n\lambda_v}\langle\mathbf{O}^n_{kh_n},v\rangle + \int_{kh_n}^{(k+1)h_n}\mathrm{e}^{-((k+1)h_n-s)\lambda_v}(\lambda_v)^{-1/2-\varepsilon}\,d\beta^v_s\right)v. \tag{8.6}$$

For every $v\in\mathbb{H}$ it holds that $\int_{kh_n}^{(k+1)h_n}\mathrm{e}^{-((k+1)h_n-s)\lambda_v}(\lambda_v)^{-1/2-\varepsilon}\,d\beta^v_s$ is independent of $\mathbf{O}^n_{kh_n}$ and is distributed as a Gaussian random variable with mean 0 and variance $\frac{1-\mathrm{e}^{-2h_n\lambda_v}}{2\lambda_v^{2(1+\varepsilon)}}$.

The other term can be explicated as follows: $\mathbf{X}^n_{(k+1)h_n} = \sum_{v\in\mathbb{H}_n}\langle\mathbf{X}^n_{(k+1)h_n},v\rangle v$ with

$$\begin{aligned}\langle\mathbf{X}^n_{(k+1)h_n},v\rangle &= \mathrm{e}^{-h_n\lambda_v}\langle\mathbf{X}^n_{kh_n},v\rangle + \langle\mathbf{O}^n_{(k+1)h_n},v\rangle - \mathrm{e}^{-h_n\lambda_v}\langle\mathbf{O}^n_{kh_n},v\rangle + \frac{1-\mathrm{e}^{-h_n\lambda_v}}{\lambda_v}\times\\ &\quad\times\mathbb{1}_{\left\{\sqrt{\sum_{w\in\mathbb{H}_n}\lambda_w^{2\rho}\langle\mathbf{X}^n_{kh_n},w\rangle^2}+\sqrt{\sum_{w\in\mathbb{H}_n}\lambda_w^{2\rho}\langle\mathbf{O}^n_{kh_n},w\rangle^2}\le h_n^{-\chi}\right\}}\langle-\nabla\mathbf{X}^n_{kh_n}\cdot\mathbf{X}^n_{kh_n},v\rangle.\end{aligned} \tag{8.7}$$

Note that this is a recursive formulation of the approximation scheme.

Finally note that the scheme in (8.4) and the derived discretised one are adaptations to the specific stochastic Navier-Stokes equation (8.2) of a type of approximation method which have been proven to converge strongly to a large class of infinite-dimensional stochastic evolution equations with non-globally monotone non-linearity (see e. g. Theorem 3.5 in [10] and references therein). The specific result in the case of (8.2) is discussed in Section 3.2.

3.2 Properties of the approximation scheme

In this section we analyse the properties of the 2-D stochastic Navier-Stokes equations we consider and the DTAEE approximation scheme which yield the following strong convergence result:

Theorem 8.1 (cf. [11]) Let X the mild solution in (8.3), and $\mathscr{X}^n$, $n \in \mathbb{N}$ as in (8.4). Then for all $p \geq 1$

$$\limsup_{n\to\infty} \sup_{s\in[0,T]} \mathsf{E}\big[\|X_s - \mathscr{X}_s^n\|_H^p\big] = 0.$$

This is strong convergence uniform in time. From the statement of the result it is clear that the same convergence holds with the (DTAEE) more explicit version of the scheme (8.5): take $\mathbf{X}_t^n$ to be the process with continuous sample paths obtained, e. g., by interpolation from $\mathbf{X}_{h_n k}^n$ (given in recursive formulation in Equations (8.6)–(8.7)). Indeed, the approximation processes have continuous sample paths which coincide a.s. on a dense countable subset of $[0,T]$.

One of the difficulties in proving the strong convergence follows from the fact that the non-linearity F, although $F \in C(H_\rho, H)$, is not globally Lipschitz. Indeed, it is only Lipschitz on bounded sets: there exists a non-negative real number $\theta \in [0,\infty)$ such that for all $v, w \in H_\rho$ it holds that

$$\|F(v) - F(w)\|_H \leq \theta\big(1 + \|v\|_{H_\rho} + \|w\|_{H_\rho}\big)\|v - w\|_{H_\rho} < \infty$$

(see [11] for the proof). However note that, roughly speaking, the approximation scheme controls the Lipschitz constant by truncating the non-linearity. In other words, the truncation prevents strong divergence (see (8.8) below).

The lack of global Lipschitzianity for the non-linear functional F has been compensated by the fact that the non-linearity satisfies the following coercivity-type condition.

For all $\delta > 0$, $v, w \in H_\rho$ it holds that

$$|\langle v, F(v+w)\rangle_H| \leq \left(\frac{3}{2}\varepsilon_0 + \frac{1}{2\delta}\left[\sup_{x\in(0,1)^2}|\underline{w}(x)|_2^2\right]\right)\|v\|_H^2 + 2\delta\|(\varepsilon_0 - \Delta)^{1/2}v\|_H^2 + \left(\frac{\varepsilon_0}{2}\left[\sup_{x\in(0,1)^2}|\underline{w}(x)|_2^2\right] + \frac{1}{2\delta}\left[\sup_{x\in(0,1)^2}|\underline{w}(x)|_2^4\right]\right),$$

where $\underline{v}, \underline{w}$ are the continuous functions belonging to the equivalence class $v, w \in H_\rho$ which exist by Sobolev's embedding. (See [11] for the proof).

The coercivity-type condition, combined with the Lipschitzianity on bounded sets above and some Gronwall-type argument, yields a-priori estimates for the approximation scheme involving a transformation of the noise part $\mathscr{O}^n$, say $\mathbb{O}^n$ (see e. g. [10, Corollary 2.6]).

To prove the strong convergence based on the mentioned a-priori bounds for $\|\mathscr{X}_t^n\|_H$, one needs to prove suitable exponential integrability properties of the process $\mathbb{O}^n$, related to the uniform norms involved in the coercivity-type condition. More precisely, given $p > 4$, there exists $\eta \in [0,\infty)$ such that $\mathbb{O}_t^n = \int_0^t P_n \mathrm{e}^{(t-s)(A-\eta)}(-A)^{-1/2-\varepsilon}\, dW_s + P_n \mathrm{e}^{t(A-\eta)}\xi$ and

$$\sup_{m\in\mathbb{N}} \mathsf{E}\Bigg[\int_0^T \exp\left(\int_s^T p\Big(\sup_{x\in(0,1)^2}|\mathbb{O}^m_{\lfloor u\rfloor_{h_m}}(x)|^2\Big)\,du\right)\cdot\max\Bigg\{1, \sup_{x\in(0,1)^2}\big|\mathbb{O}^m_{\lfloor s\rfloor_{h_m}}(x)\big|^{2p}, \|\mathbb{O}_s^m\|_H^p, \int_0^T \|\mathscr{O}_u^m + P_m\left(e^{u(A-\eta)} - e^{uA}\right)\xi\|_{H_\rho}^{6p}\,du\Bigg\}\,ds\Bigg] < \infty.$$

The proof of this statement (we refer to [11]) is quite technical but kind of natural since the noise is a Gaussian process. Indeed one of the main tools is Fernique's theorem.

The a-priori bounds are then used to prove that the approximation scheme does not explode:

$$\limsup_{n\to\infty}\sup_{t\in[0,T]} \mathsf{E}\big[\|\mathscr{X}_t^n\|_H^p\big] < \infty, \qquad p \geq 1. \tag{8.8}$$

Once obtained these uniform moments bounds for the approximation, it suffices to prove the convergence in probability uniformly in time of $\mathscr{X}^n$ towards the mild solution X in (8.3) to obtain the desired strong convergence uniform in time (see [9, Proposition 4.5]).

The coercivity-type condition satisfied by the drift is not required in the proof of the convergence in probability uniformly in time (based e. g. on [9, Proposition 3.3]). Instead, local Lipschitzianity and the following convergence for the noise are relevant. It can be easily shown that $\sup_{t\in[0,T]} \mathsf{E}\big[\|O_t - \mathscr{O}_t^n\|_{H_\rho}^p\big]$ converges to 0 for all $p \geq 1$ with an explicit polynomial rate, where the process O denotes the sum of the initial condition and the stochastic convolution in (8.3). This is not surprising either because the approximation of the noise term is essentially Galerkin approximation.

We examined here the properties of the approximation scheme (8.4) relevant for the strong convergence uniform in time towards the mild solution (8.3) of two-dimensional Navier-Stokes equations (8.2). To conclude, we would like to mention that the rate of convergence for this approximation scheme has not been proven yet. Recent results based on some stochastic non-linear integration-by-parts formulas seem promising, but have not yet been exploited in the case of 2-D Navier-Stokes equations.

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