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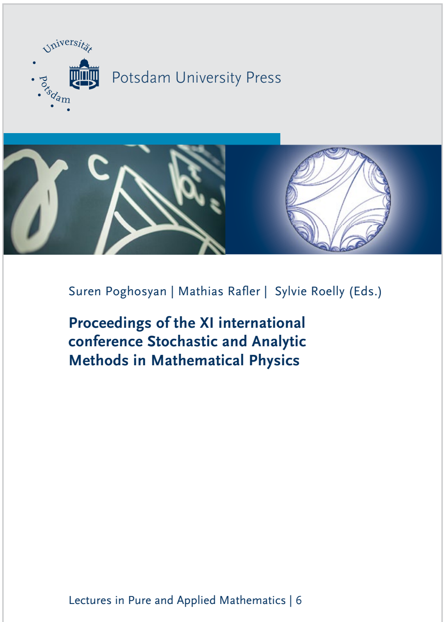

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6

Non-local convolution type parabolic equations with fractional and regular time derivative

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Abstract. *This note deals with the fundamental solutions of parabolic equations for convolution type non-local operators. Our goal is to compare the large time asymptotics of these fundamental solutions with that of the classical Gaussian heat kernel. A similar problem is considered for evolution equations with a fractional time derivative.*

1 Introduction

Parabolic equations with non-local elliptic operators play an important role in the study of population dynamics models. The presence of a non-local operator on the right-hand side of the equation reflects the fact that the interaction in these models has a non-local character. One of these models is the so-called contact model in $\mathbb{R}^d$, see e. g. [5, 6]. It is a continuous time birth and death Markov process in a continuum defined on the space of infinite (but locally finite) configurations $\gamma \in \Gamma$ lying in the space $\mathbb{R}^d$: $\gamma \subset \mathbb{R}^d$. The process is characterised by the birth and death rates. Each point $x \in \gamma$ of a configuration γ might create an offspring y independently of other points of the configuration. The

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offspring location is distributed in the space with the density $a(x-y)$ (so-called dispersal kernel), and we assume $\int_{\mathbb{R}^d} a(z)dz = 1$. In addition, any point of the configuration has an independent exponentially distributed random life time determined by the mortality rate $m(x) > 0$, and in the general case the mortality rate is a spatially inhomogeneous function $m(x) \geq 0$. The formal generator of the dynamics of this process takes the form

$$LF(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} a(x-y)\big(F(\gamma \cup y) - F(\gamma)\big)dy + \sum_{x\in\gamma} m(x)\big(F(\gamma\backslash x) - F(\gamma)\big).$$

The case of homogeneous mortality $m(x) \equiv \kappa$ has been studied in detail in the paper [5]. In the most interesting case $\kappa = 1$ (the critical regime) a family of stationary distributions exist.

One of the remarkable properties of the contact model is the fact that the first correlation function $\rho(x)$ (the so-called density of configurations) satisfies an evolution equation which is decoupled and can be considered separately. It should be noted that evolutions of the higher order correlation functions have more complicated hierarchical structure.

The dynamics of the first correlation function is described by the following Cauchy problem:

$$\frac{\partial \rho}{\partial t} = A\rho, \quad \rho = \rho(t,x), \quad x \in \mathbb{R}^d,\ t \geq 0, \qquad \rho(0,x) = \rho_0(x) \geq 0, \qquad \text{where} \tag{6.1}$$

$$A\rho(x) = -m(x)\rho(x) + \int_{\mathbb{R}^d} a(x-y)\rho(y)dy. \tag{6.2}$$

If $m(x) \equiv 1$, then the operator A takes the form

$$A\rho(x) = -\rho(x) + \int_{\mathbb{R}^d} a(x-y)\rho(y)dy = \int_{\mathbb{R}^d} a(x-y)\big(\rho(y) - \rho(x)\big)dy. \tag{6.3}$$

Thus we obtain parabolic equation (6.1) with a convolution operator on the right-hand side.

Notice that correlation functions in the contact model, as well as in other models of the population dynamics, need not vanish at infinity. Thus to study the behaviour of correlation functions we have to consider the Cauchy problem for evolution equations (6.1)–(6.3) in the classes of functions that satisfy suitable growth conditions at infinity. Then the information about the point-wise asymptotics or two-sided bounds of the corresponding fundamental solution becomes very important not only in the region where the central limit theorem applies but also in other space-time regions.

In this note we compare the large time behaviour of the fundamental solutions of problem (6.1)–(6.3) with that of the classical Gaussian heat kernel.

We also make a similar comparison of the Gaussian heat kernel and the fundamental solution of evolution equations with a fractional time derivative.

An essential part of the estimates used in this note is borrowed from our previous works [4] and [7]. However, the lower bounds for the studied fundamental solutions in the region of super-large deviations are new. For these bounds we provide a detailed proof.

2 Convolution type operators

We consider a zero order convolution type operator A in $L^2(\mathbb{R}^d)$, $d \geq 1$, defined by

$$Af(x) = \int_{\mathbb{R}^d} a(x-y)\big(f(y)-f(x)\big)dy,$$

where the convolution kernel a is a non-negative integrable function. If $\int_{\mathbb{R}^d} a(z)\,dz = 1$, then A is the generator of a continuous time Markov jump process with the jump distribution $a(z)$: $Af = a * f - f$. We assume that the convolution kernel $a(\cdot)$ has the following properties:

- ◇ Boundedness

$$a(x) \geq 0, \qquad a(x) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d). \tag{6.4}$$

- ◇ Symmetry

$$a(x) = a(-x) \quad \text{for all } x \in \mathbb{R}^d. \tag{6.5}$$

- ◇ Normalisation and second moments

$$\int_{\mathbb{R}^d} a(x)dx = 1, \qquad \int_{\mathbb{R}^d} |x|^2 a(x)dx < \infty. \tag{6.6}$$

- ◇ (Super)exponential decay

$$0 \leq a(x) \leq Ce^{-b|x|^p}, \quad \text{with } p \geq 1, b > 0, C > 0, \tag{6.7}$$

we also consider the case of compactly supported $a(x)$.

3 Non-local parabolic problem

We study the large time behaviour of the fundamental solution of the following parabolic problem

$$\begin{aligned} &\partial_t u(x,t) = Au(x,t) = a * u - u, \qquad (x,t) \in \mathbb{R}^d \times (0,+\infty), \\ &u(x,0) = \delta(x). \end{aligned} \tag{6.8}$$

Remark 6.1 Let $\xi^0(t)$ be a continuous time jump Markov process with jump intensity equal to 1 and with jump distribution $a(\cdot)$, and assume that $\xi^0(0) = 0$. Then $u(\cdot, t)$ is the law of $\xi^0(t)$.

Since A is a bounded operator in $L^2(\mathbb{R}^d)$ we have

$$e^{tA} = e^{-t}\, e^{ta\star} = e^{-t} \sum_{k=0}^{\infty} t^k \frac{a^{\star k}}{k!} = e^{-t}\mathbb{1} + e^{-t} \sum_{k=1}^{\infty} t^k \frac{a^{\star k}}{k!}$$

and

$$u(x,t) = e^{tA}\delta(x) = e^{-t}\delta(x) + e^{-t} \sum_{k=1}^{\infty} \frac{t^k}{k!} a^{\star k}(x). \tag{6.9}$$

Observe that $u(x,t)$ consists of a singular part at zero $e^{-t}\delta(x)$ and a regular part $v(x,t) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$:

$$v(x,t) = e^{-t} \sum_{k=1}^{\infty} \frac{t^k}{k!} a^{\star k}(x). \tag{6.10}$$

We focus on obtaining point-wise upper and lower bounds for the regular part $v(x,t)$ as $t \to \infty$.

Let us briefly recall some of the existing results on heat kernels. The heat kernel of the classical heat equation in $\mathbb{R}^d$

$$\partial_t g - \Delta g = 0, \qquad g|_{t=0} = \delta_x,$$

is given by the Gauss-Weierstrass function

$$g_t(x,y) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{6.11}$$

For the heat kernel of a more general parabolic equation $\partial_t g - Lg = 0$ with a uniformly elliptic second-order divergence form operator L the well-known Aronson estimates hold, see [1],

$$g_t(x,y) \asymp \frac{C}{t^{d/2}} \exp\left(-\frac{|x-y|^2}{ct}\right).$$

One of the simplest non-local heat equation is

$$\partial_t g + (-\Delta)^{\alpha/2} g = 0, \qquad \text{where } 0 < \alpha < 2.$$

Its heat kernel satisfies the following estimates, see e. g. [2],

$$g_t(x,y) \asymp \frac{C}{t^{d/\alpha}} \left(1 + \frac{|x-y|}{t^{1/\alpha}}\right)^{-(d+\alpha)} \tag{6.12}$$

Remark 6.2 Note that $(-\Delta)^{\alpha/2}$ is an integro-differential operator of the form

$$(-\Delta)^{\alpha/2} f(x) = c_{d,\alpha}\, \text{p.v.} \int_{\mathbb{R}^d} \frac{f(x)-f(y)}{|x-y|^{d+\alpha}} dy. \tag{6.13}$$

The heavy tail of the heat kernel in estimate (6.12) is a consequence of slow decay of the integral kernel in (6.13).

4 Asymptotics of $v(x,t)$ as $t \to \infty$

In this section we present results from the paper [4], where the large time behaviour of the fundamental solution to the problem (6.8) has been obtained. This asymptotic behaviour depends crucially on the relation between $|x|$ and t. We consider separately four different regions in the (x,t) space. Namely,

I. $|x| \le r t^{1/2}\big(1+o(1)\big)$ (standard deviations region)

II. $|x| = r t^{1+\delta/2}\big(1+o(1)\big)$, $\delta \in (0,1)$ (moderate deviations region)

III. $|x| = r t\big(1+o(1)\big)$ ($\delta = 1$) (large deviations region)

IV. $|x| = r t^{1+\delta/2}\big(1+o(1)\big)$, $\delta > 1$ ("extra-large" deviations region)

4.1 Normal and moderate deviations region

We begin with the case when x belongs to regions I. or II.

Theorem 6.3 (see [4]) Assume that $a(\cdot)$ satisfies conditions (6.4)–(6.7). Then for the function $v(x,t)$ defined by (6.10) the following asymptotic relation holds as $t \to \infty$:

1) if $|x| \le r t^{1/2}$ for some $r > 0$, then

$$v(x,t) = (2\pi t)^{-d/2}\big(\det(\sigma)\big)^{-1/2}\, e^{-\frac{(\sigma^{-1}x,x)}{2t}}\big(1+o(1)\big) \tag{6.14}$$

with $\sigma^{ij} = \displaystyle\int_{\mathbb{R}^d} x_i x_j\, a(x)\, dx$.

2) if $x = r t^{1+\delta/2}\big(1+o(1)\big)$ with $0 < \delta < 1$ and $r \in R^d \backslash \{0\}$, then the following asymptotic relation holds as $t \to \infty$:

$$v(x,t) = e^{-\frac{(\sigma^{-1}x,x)}{2t}(1+o(1))} = e^{-\frac{1}{2}(\sigma^{-1}r,r)t^{\delta}(1+o(1))}. \tag{6.15}$$

It should be noted that the Gaussian form of the asymptotics (6.14) in the region of standard deviations is the immediate consequence of the local limit theorem for processes with independent increments. Formula (6.14) can also be derived from the asymptotic representation of the corresponding Fourier transform. In the moderate deviations region the asymptotics (6.15) of the fundamental solution still coincide with that in the standard deviations region, but only in the logarithmic order. For the pre-exponential factor we can only state the sub-exponential rate of decay.

4.2 Large deviations region

In order to formulate the result in the region $|x| \sim t$ we should first introduce a number of auxiliary quantities. Let X be a random vector in $\mathbb{R}^d$ with distribution $a(\cdot)$. If condition (6.7) is fulfilled with some $p \geq 1$ then X has finite exponential moment $\Lambda(\gamma) = \mathbb{E}e^{\gamma \cdot X}$ at least for small enough $\gamma \in \mathbb{R}^d$.

We define the cumulant generating function $L(\gamma) = \ln \Lambda(\gamma)$, and introduce $I(r)$, $r \in \mathbb{R}^d$, as its Legendre transform: $I(r) = \sup_\gamma \big(\gamma \cdot r - L(\gamma)\big)$, $r, \gamma \in R^d$. Denote by ξ_r a positive solution of the equation

$$\ln \xi = I(\xi r) - \xi r \cdot \nabla I(\xi r), \qquad \xi \in R.$$

Lemma 6.4 Let $a(x)$ satisfy conditions (6.4)–(6.7). Then for any $r \in R^d \backslash \{0\}$ the above equation has a unique solution ξ_r, and $0 < \xi_r < 1$.

Let us define the rate function

$$\Phi(r) = 1 - \frac{1}{\xi_r}\big(1 + \ln \xi_r - I(\xi_r r)\big). \tag{6.16}$$

We introduce now some additional technical conditions on the kernel.

$(\mathbf{A}_1)$ $p = 1$ and for any $b_1 > b$ and any $\theta \in S^{d-1}$ we have $\mathbb{E}e^{b_1(X,\theta)} = \infty$, where b is the same constant as in (6.7).

$(\mathbf{A}_1^s)$ $p = 1$ and $\mathbb{E}|X|e^{b(X,\theta)} = \infty$ for any $\theta \in S^{d-1}$.

$(\mathbf{A}_p)$ $p > 1$ and for any $\theta \in S^{d-1}$

$$L(\gamma) = \ln \mathbb{E}e^{\gamma(X,\theta)} = C(b,p)|\gamma|^{p/(p-1)}\big(1 + o(1)\big), \qquad \text{as } |\gamma| \to \infty,$$

where $C(b,p) = \frac{p-1}{p}(bp)^{-1/(p-1)}$ is a constant appearing in the logarithmic asymptotics of the Laplace transform of $e^{-b|x|^p}$.

Remark 6.5 Condition $(\mathbf{A}_p)$, $p \geq 1$, can be treated as a sort of soft lower bound for $a(x)$. In particular, it holds if $a(x)$ satisfies the following two-sided estimate

$$C_0 e^{-b|x|^p} \leq a(x) \leq C_1 e^{-b|x|^p}, \qquad p \geq 1. \tag{6.17}$$

Also, Condition $(\mathbf{A}_p)$ implies that $a(x)$ could not have a bounded support.

Theorem 6.6 (see [4]) Let conditions (6.4)–(6.7) be fulfilled, and assume additionally that in the case $p = 1$ condition $(\mathbf{A_1^s})$ holds. Then for $x = rt\big(1+o(1)\big)$ with $r \in R^d \setminus \{0\}$ we have

$$v(x,t) = e^{-\Phi(r)t(1+o(1))} \qquad \text{as } t \to \infty, \tag{6.18}$$

where $\Phi(r)$ is defined by (6.16).

The rate function $\Phi(r)$ possesses the following important properties: Φ is a convex function, $\Phi(0) = 0$, $\Phi(r) > 0$ for $r \neq 0$, and

$$\Phi(r) = \frac{1}{2}\sigma^{-1} r \cdot r \big(1+o(1)\big), \qquad \text{as } r \to 0; \tag{6.19}$$

$$\Phi(r) \to \infty, \qquad \text{as } r \to \infty. \tag{6.20}$$

If $a(x)$ has a finite support, then

$$\Phi(r) \geq c|r| \ln|r|, \qquad \text{as } |r| \to \infty.$$

Furthermore, if $p = 1$ and condition $(\mathbf{A_1})$ holds, then

$$\Phi(r) = b|r| \big(1+o(1)\big), \qquad \text{as } |r| \to \infty; \tag{6.21}$$

if $p > 1$ and condition $(\mathbf{A_p})$ holds, then

$$\Phi(r) = \frac{p}{p-1}\big(b(p-1)\big)^{1/p} |r| \left(\ln|r|\right)^{\frac{p-1}{p}} \big(1+o(1)\big), \qquad \text{as } |r| \to \infty. \tag{6.22}$$

Remark 6.7 It should be also emphasised that in the case $p = 1$ conditions $(\mathbf{A_1})$, $(\mathbf{A_1^s})$ are required for proving the main result on the asymptotics of the heat kernel, while in the case $p > 1$ condition $(\mathbf{A}_p)$ is only used for determining the asymptotic behaviour of the function $\Phi(r)$ for large r.

4.3 Extra-large deviations region

In the region $|x| \gg t$ only an upper bound for $v(x,t)$ was obtained in [4].

Theorem 6.8 (see [4]) Assume that $a(x)$ satisfies conditions (6.4)–(6.7). Then for $|x| = rt^{1+\delta/2}\big(1+o(1)\big)$ with $\delta > 1$ and $r \neq 0$ the following asymptotic upper bound holds:

$$v(x,t) \le e^{-c_p t^{\delta+1/2}(\ln t)^{p-1/p}(1+o(1))} \qquad \text{as } t \to \infty, \tag{6.23}$$

where the constant $c_p = c_p(b,r)$ depends on b, r and p. If $a(\cdot)$ has a finite support, then for $|x| = rt^{\delta+1/2}\big(1+o(1)\big)$ with $\delta > 1$ we have

$$v(x,t) \le e^{-\tilde{c}t^{\delta+1/2}\ln t(1+o(1))} \qquad \text{as } t \to \infty,$$

where $\tilde{c}$ depends on r, δ and the support of $a(\cdot)$.

Here, for $a(x)$ satisfying two-sided estimate (6.17), we prove that a similar lower bound holds.

Theorem 6.9 Let Conditions (6.4)–(6.6) be fulfilled, and assume that bound (6.17) holds. Then in the region of "extra-large" deviations with $|x| \gg t$ the following two-sided bound holds for all sufficiently large t:

$$\exp\left\{-\tilde{C}_1|x|\left(\ln\left|\frac{x}{t}\right|\right)^{\frac{p-1}{p}}\right\} \le v(x,t) \le \exp\left\{-\tilde{C}_2|x|\left(\ln\left|\frac{x}{t}\right|\right)^{\frac{p-1}{p}}\right\}. \tag{6.24}$$

Proof. The upper bound in (6.24) is a direct consequence of estimate (6.23). To obtain the lower bound in (6.24) we consider an auxiliary operator

$$\big(\tilde{A}u\big)(x) = \int_{\mathbb{R}^d} C_0\tilde{a}_p(x-y)u(y)dy - C_1\int_{\mathbb{R}^d}\tilde{a}_p(y)dy\cdot u(x),$$

where $\tilde{a}_p(x) = e^{-b|x|^p}$ and C_0, C_1 are the same constants as in (6.17). Let us represent $\tilde{u}(x,t) = e^{t\tilde{A}}\delta(x)$ in the same way as (6.9):

$$\tilde{u}(x,t) = e^{-C_1\alpha_p t}\delta(x) + e^{-C_1\alpha_p t}\sum_{k=1}^{\infty}\frac{(C_0t)^k}{k!}\tilde{a}_p^{\star k}(x),$$

where $\alpha_p = \int_{\mathbb{R}^d}\tilde{a}_p(y)dy$. Thus the regular part of $\tilde{u}(x,t)$ equals to

$$\tilde{v}(x,t) = e^{-C_1\alpha_p t}\sum_{k=1}^{\infty}\frac{(C_0t)^k}{k!}\tilde{a}_p^{\star k}(x), \tag{6.25}$$

and we conclude using (6.17) that $v(x,t) \ge \tilde{v}(x,t)$ for all $x \in \mathbb{R}^d$. Therefore, it suffices to obtain the lower bound in (6.24) for the function $\tilde{v}(x,t)$. To this end we first estimate $\tilde{a}_p^{\star k}(x)$ for $k = \left(\ln\frac{|x|}{t}\right)^{-1/p}|x|$. Divide the one-dimensional segment $[0,x]$ into k equal parts and denote by $z_j,\ j=1,\dots,k$, the centers of corresponding subsegments $[r_{j-1},r_j]$, $r_j=\frac{x}{k}j$. Notice that

$$|r_j-r_{j-1}|=\frac{|x|}{k}=\Big(\ln\frac{|x|}{t}\Big)^{1/p}\to\infty \qquad \text{as } t\to\infty.$$

Let $B_1(z)\subset\mathbb{R}^d$ be a ball of the unit volume with the center at z. If $x_j\in B_1(z_j),\ x_{j-1}\in B_1(z_{j-1})$, then

$$|x_j-x_{j-1}|\le|r_j-r_{j-1}|+2=\frac{|x|}{k}\big(1+o(1)\big).$$

Consequently

$$\tilde a_p^{\star k}(x)\ge\int\limits_{B_1(z_1)}\cdots\int\limits_{B_k(z_k)}e^{-|x_1-x_2|^p-\ldots-|x_k-x|^p}dx_1\cdots dx_k\ge e^{-c_1\left(\frac{|x|}{k}\right)^p k} \tag{6.26}$$

with some constant $c_1>0$. Finally keeping in representation (6.25) for $\tilde v(x,t)$ only one term with $k=\big(\ln\frac{|x|}{t}\big)^{-1/p}|x|$ and considering estimate (6.26) we obtain the desired low bound in (6.24). □

5 Time fractional equations

In this section we present results of our work [7] where the asymptotic behaviour of solution $w_\alpha(x,t)$ of the following fractional time parabolic problem has been studied

$$\partial_t^\alpha w_\alpha=a*w_\alpha-w_\alpha, \qquad w_\alpha\big|_{t=0}=\delta_0.$$

Here ∂_t^α is the fractional derivative (the Caputo derivative) of the order α, $0<\alpha<1$, and $a(x)$ is the same convolution kernel as that in Section 2.

As follows from [3] the solution $w_\alpha(x,t)$ admits the following representation in terms of the fundamental solution $u(x,t)$ of a non-local heat equation:

$$w_\alpha(x,t)=\int_0^\infty u(x,r)d_r\mathbb{P}(S_r\ge t)=\int_0^\infty u(x,r)G_t^\alpha(r)dr,$$

here $S=\{S_r,r\ge0\}$ is the α-stable subordinator with the Laplace transform $\mathbb{E}e^{-\lambda S_r}=e^{-r\lambda^\alpha}$, and $G_t^\alpha(r)=d_r\Pr\{V_t^\alpha\le r\}$ is the density of the inverse α-stable subordinator V_t^α. Using the representation for the Laplace transform of V_t^α: $\mathbb{E}e^{-\lambda V_t^\alpha}=E_\alpha(-\lambda t^\alpha)$, E_α is the Mittag-Leffler function, and representation (6.9) for the non-local heat kernel

$$u(x,t)=e^{-t}\delta_0(x)+v(x,t) \qquad \text{with } v(x,t)=\sum_{k=1}^\infty\frac{a^{*k}(x)}{k!}t^k e^{-t}$$

we obtain $w_\alpha(x,t) = E_\alpha(-t^\alpha)\delta_0(x) + p_\alpha(x,t)$, where the regular part of $w_\alpha(x,t)$ equals

$$p_\alpha(x,t) = \sum_{k=1}^{\infty} \frac{a^{*k}(x)}{k!}\, t^{\alpha k} E_\alpha^{(k)}(-t^\alpha)$$

It turned out that in contrast with the equations studied in Section 4.1 in the case of equations with fractional time derivatives we should divide (x,t) space in 6 regions in order to describe the large time behaviour of the corresponding fundamental solutions. These regions are

- $|x|$ is bounded;
- (Subnormal deviations) $1 \ll |x| \ll t^{\frac{\alpha}{2}}$ or equivalently, $|x(t)| \to \infty$ and there exists an increasing function $r(t)$, $r(0) = 0$, $\lim\limits_{t\to\infty} r(t) = +\infty$ such that $r(t) \le |x| \le \big(r(t)+1\big)^{-1} t^{\alpha/2}$ for all sufficiently large t;
- (Normal deviations) $x = vt^{\alpha/2}\big(1+o(1)\big)$ with an arbitrary $v \in \mathbb{R}^d \setminus \{0\}$;
- (Moderate deviations) $x = vt^\beta\big(1+o(1)\big)$ with $\frac{\alpha}{2} < \beta < 1$ and $v \in \mathbb{R}^d \setminus \{0\}$;
- (Large deviations) $x = vt\big(1+o(1)\big)$ with $v \in \mathbb{R}^d \setminus \{0\}$;
- (Extra large deviations) $|x| \gg t$, i. e. $\lim\limits_{t\to\infty} \frac{|x(t)|}{t} = \infty$.

The main result from [7] is the following point-wise asymptotic formula for $p_\alpha(x,t)$ as $t \to \infty$.

- If $|x|$ is bounded, then

$$\begin{aligned} \mathsf{c}_- t^{-\alpha/2} &\le p_\alpha(x,t) \le \mathsf{c}_+ t^{-\alpha/2} && \text{if } d = 1,\\ \mathsf{c}_- t^{-\alpha}\log t &\le p_\alpha(x,t) \le \mathsf{c}_+ t^{-\alpha}\log t && \text{if } d = 2,\\ \mathsf{c}_- t^{-\alpha} &\le p_\alpha(x,t) \le \mathsf{c}_+ t^{-\alpha} && \text{if } d \ge 3. \end{aligned}$$

- If $1 \ll |x| \ll t^{\alpha/2}$, then

$$\begin{aligned} \mathsf{c}_- t^{-\alpha/2} &\le p_\alpha(x,t) \le \mathsf{c}_+ t^{-\alpha/2} && \text{if } d = 1,\\ \mathsf{c}_- t^{-\alpha}\log\frac{t^\alpha}{|x|^2} &\le p_\alpha(x,t) \le \mathsf{c}_+ t^{-\alpha}\log\frac{t^\alpha}{|x|^2} && \text{if } d = 2,\\ \mathsf{c}_- t^{-\alpha}|x|^{2-d} &\le p_\alpha(x,t) \le \mathsf{c}_+ t^{-\alpha}|x|^{2-d} && \text{if } d \ge 3. \end{aligned}$$

- $\diamond$ If $x = vt^{\alpha/2}\big(1+o(1)\big)$ with $v \in \mathbb{R}^d \setminus \{0\}$, then

$$p_\alpha(x,t) = t^{-\frac{d\alpha}{2}} \int_0^\infty W_\alpha(s)\Psi(v,s)\,ds\,\big(1+o(1)\big), \qquad \text{where}$$

$$\Psi(v,s) = \frac{1}{|\det\sigma|^{1/2}(2\pi s)^{d/2}} \exp\left(-\frac{(\sigma^{-1}v,v)}{s}\right)$$

and $W_\alpha(s)$ is the Wright function that is expressed via the density $G_t^\alpha(r)$ of the inverse subordinator.

- $\diamond$ If $x = vt^\beta\big(1+o(1)\big)$ with $\alpha/2 < \beta < 1$ and $v \in \mathbb{R}^d \setminus \{0\}$, then

$$p_\alpha(x,t) = \exp\left\{-K_v t^{2\beta-\alpha/2-\alpha}\big(1+o(1)\big)\right\}$$

with a constant K_v depending on α and v.

- $\diamond$ If $x = vt\big(1+o(1)\big)$ with $v \in \mathbb{R}^d \setminus \{0\}$, then

$$p_\alpha(x,t) = \exp\big\{-\mathtt{F}(v)t\big(1+o(1)\big)\big\}\,.$$

- $\diamond$ If $|x| \gg t$, then, combining the approach developed in [7] with the statement of Theorem 6.9, we obtain

$$\exp\left\{-\mathtt{c}_-\,|x|\,\left(\log\left|\frac{x}{t}\right|\right)^{p-1/p}\right\} \le p_\alpha(x,t) \le \exp\left\{-\mathtt{c}_+\,|x|\,\left(\log\left|\frac{x}{t}\right|\right)^{p-1/p}\right\}.$$

6 Conclusions

1. Comparing classical heat kernel (6.11) and the regular part of the fundamental solution of the non-local parabolic problem (6.8) we observe that crucial modifications of the Gaussian form of the asymptotics occurs in the region of large deviations, when $x = rt$. It is there, at the distances of order t, that the non-local character of the operator A starts to play an important role. As seen from (6.19), the fundamental solution is still close to the Gaussian function for small r, but it differs essentially from the corresponding Gaussian function for sufficiently large r, see (6.21), (6.22). In the "extra-large" deviations region this difference is further enhanced. It follows from estimate (6.24) that the non-local fundamental solution $v(x,t)$ has more heavy tail at infinity than the classical heat kernel (6.11).
2. Comparing $p_\alpha(x,t)$ and $v(x,t)$ we notice that

- ◇ in the regions of normal and moderate deviations the asymptotics of $p_\alpha(x,t)$ strongly depends on α, and in the region of subnormal deviations it additionally depends on the dimension;
- ◇ in the region of large deviations $|x| \sim t$ the form of the asymptotics of $p_\alpha(x,t)$ is similar to that of $v(x,t)$, however the rate functions are different;
- ◇ in the region of extra large deviations the asymptotic upper bounds for $p_\alpha(x,t)$ and for $v(x,t)$ are the same.

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