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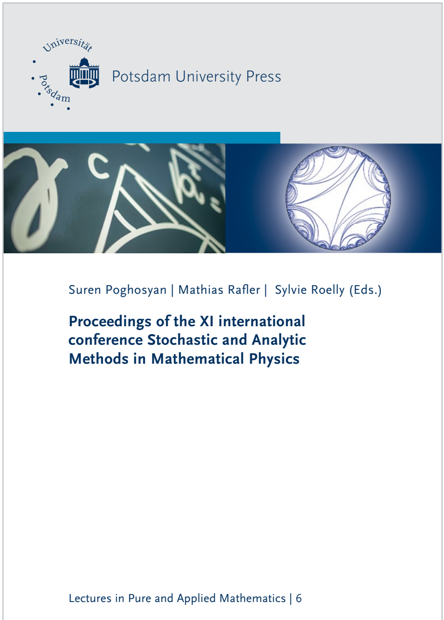

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Construction of limiting Gibbs processes and the uniqueness of Gibbs processes

*Suren Poghosyan** and *Hans Zessin*†

Abstract. *For a pair potential* Φ *in a general phase space X satisfying some natural and sufficiently general stability and regularity conditions in the sense of Poghosyan and Ueltschi we define by means of the so-called Ursell kernel a function r which is shown to be the correlation function of a unique infinitely extended process* P. *Finally, under more restrictive assumptions, we show that the Gibbs process for* Φ, *if it exists, coincides with* P. *Here we use the classical method of Kirkwood-Salsburg equations.*

1 Preliminaries

Let $(X, \mathscr{B}(X), \mathscr{B}_0(X))$ be the underlying phase space where X is a locally compact, second countable Hausdorff topological space, $\mathscr{B}(X)$ its Borel σ-field and $\mathscr{B}_0(X)$ its bounded Borel sets. Let ρ be a Radon measure on X.

Let $\mathscr{M}^{\cdot\cdot}(X)$ be the space of Radon point measures on X and $\mathfrak{X}$ be the collection of all finite point measures (finite configurations) ξ in X. $\mathfrak{X}_+$ denotes the collection of all

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non-empty ξ in $\mathfrak{X}$. For $A \in \mathscr{B}_0(X)$ let $\mathfrak{X}(A)$ be the set of finite point measures supported by A. Let $\mathscr{M}^{\cdot}_R(X)$, $R > 0$, be the space of simple point measures μ on X having the property that the minimal distance of every pair of points in any configuration μ is R, i. e. $(x, y \in \mu, x \neq y \Rightarrow d(x,y) > R)$ where d is a metric on X. Here all Dirac measures ε_x at the point $x \in X$ and the zero measure o are elements of $\mathfrak{X}$ and $\mathscr{M}^{\cdot}_R(X)$.

We call a subset $\mathfrak{X}'$ of $\mathfrak{X}$ an *environment in* X if $(\eta \in \mathfrak{X}', \xi \preceq \eta \Rightarrow \xi \in \mathfrak{X}')$. Here $\xi \preceq \eta$ if $\xi(x) \leq \eta(x)$ for all $x \in X$. Examples are $\mathfrak{X}$ and $\mathfrak{X}_R = \mathscr{M}^{\cdot}_R(X) \cap \mathfrak{X}$.

We denote by F_+ the space of $[0, +\infty]$-valued measurable functions on the corresponding space and by $\mathscr{K}$ we denote the collection of continuous functions with compact support. Define a locally finite measure Λ_ρ on $\mathfrak{X}$ by

$$\Lambda_\rho \, \varphi = \varphi(o) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_X \cdots \int_X \varphi(\varepsilon_{x_1} + \ldots + \varepsilon_{x_n}) \, \rho(\mathrm{d}x_1) \cdots \rho(\mathrm{d}x_n), \qquad \varphi \in F_+.$$

For a given configuration $\mu \in \mathscr{M}^{\cdot\cdot}$ we define the following measure on $\mathfrak{X}$:

$$\Lambda'_\mu(h) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} h(\varepsilon_{x_1} + \ldots + \varepsilon_{x_n}) \, \widetilde{\mu}^n(\mathrm{d}x_1, \ldots, \mathrm{d}x_n), \qquad h \in F_+, \text{ where}$$
$$\widetilde{\mu}^n(\mathrm{d}x_1, \ldots, \mathrm{d}x_n) = \big(\mu - \varepsilon_{x_1} - \ldots - \varepsilon_{x_{n-1}}\big)(\mathrm{d}x_n) \cdots \big(\mu - \varepsilon_{x_1}\big)(\mathrm{d}x_2) \mu(\mathrm{d}x_1).$$

$\widetilde{\mu}^n$ is called the *factorial measure* of μ of order n, and Λ'_μ the *compound factorial measure built on* μ. The term $n = 0$ of the sum is $h(o)$. Also, $\Lambda'_o(h) = h(o)$.

Below we often use the following important equation, *the Minlos' formula* [3]:

$$\int_{\mathfrak{X}} \int_{\mathfrak{X}} h(\xi, \nu - \xi) \, \Lambda'_\nu(\mathrm{d}\xi) \Lambda_\rho(\mathrm{d}\nu) = \int_{\mathfrak{X}} \int_{\mathfrak{X}} h(\xi, \nu) \, \Lambda_\rho(\mathrm{d}\xi) \Lambda_\rho(\mathrm{d}\nu), \qquad h \in F_+, \quad (5.1)$$

which is valid for all h integrable with respect to the measure on the left-hand or the right-hand side of the equation.

Let P be a point process in X that is a probability on $\mathscr{M}^{\cdot\cdot}(X)$. The *moment measure of* P *of order* k is the measure on X^k defined by

$$\nu^k_{\mathsf{P}} f = \int_{\mathscr{M}^{\cdot\cdot}(X)} \mu^{\otimes k}(f) \, \mathsf{P}(\mathrm{d}\mu), \qquad f \in \mathscr{K}(X^k),$$

whereas the *correlation measure (also called factorial moment measure) of* P *of order k* is the measure given by

$$\widetilde{\nu}_{\mathsf{P}}^{k}(f) = \int_{\mathscr{M}^{\cdot\cdot}(X)} \widetilde{\mu}^{k}(f)\,\mathsf{P}(\mathrm{d}\mu), \qquad f \in \mathscr{K}(X^{k}).$$

If $\widetilde{\nu}_{\mathsf{P}}^{k}$ has a density r_{P}^{k} with respect to some product measure $\rho^{\otimes k}$, where ρ is a Radon measure on X, then we say that r_{P}^{k} is a *correlation function of* P *of k-th order*. The process P is called *of order k* if ν_{P}^{k} is a Radon measure. P is called *of infinite order* if it is of order k for every k.

2 Ruelle's algebraic approach

We here follow Ruelle [10]. Let $\mathscr{A}$ be the set of all measurable complex functions on $\mathfrak{X}$. We define a $\star$-multiplication of two functions $h_1, h_2 \in \mathscr{A}$ by

$$h_1 \star h_2(\xi) = \int_{\mathfrak{X}} h_1(\nu) h_2(\xi - \nu)\,\Lambda'_{\xi}(\mathrm{d}\nu), \qquad \xi \in \mathfrak{X}. \tag{5.2}$$

With the $\star$-product $\mathscr{A}$ becomes a commutative algebra with the unit $\mathbb{1}(\xi) = \delta_o(\xi)$. Let $\mathscr{A}_0 = \{f \in \mathscr{A} \mid f(o) = 0\}$. We define the mapping $\Gamma : \mathscr{A}_0 \to \mathbb{1} + \mathscr{A}_0$ (*algebraic exponent*) by

$$\Gamma h = \mathbb{1} + h + \frac{1}{2!}h^{\star 2} + \ldots + \frac{1}{n!}h^{\star n} + \ldots, \qquad h \in \mathscr{A}_0. \tag{5.3}$$

Let Φ be a measurable symmetric function $\Phi : X \times X \to]-\infty, +\infty]$, a pair potential in X. $E(\xi) := \sum_{1 \le i < j \le n} \Phi(x_i, x_j)$ is the *energy* of the configuration $\xi = \varepsilon_{x_1} + \ldots + \varepsilon_{x_n}$; $B := \mathrm{e}^{-E}$ is called the Boltzmann factor. The *conditional energy* at x given the configuration ξ is given by $W_{\Phi}(x, \xi) := \int_X \Phi(x, y)\,\xi_x(\mathrm{d}y)$, where $\xi_x = \xi$ if $x \notin \xi$ and $\xi = \xi - \varepsilon_x$ otherwise.

We assume that Φ is b-*stable*, i. e. there exists a measurable function $\mathsf{b} : X \to [0, +\infty)$ such that $E(\xi) \ge -\sum_{x \in \xi} \mathsf{b}(x)$, $\xi \in \mathfrak{X}$.

We consider also $\mathscr{P}$-*stable*[1] Φ *with stability function* b in the environment $\mathfrak{X}'$. This means that there exists a measurable function $\mathsf{b} : X \to [0, +\infty)$ such that $W_{\Phi}(x, \xi_x) \ge -\mathsf{b}(x)$, $x \in \xi \in \mathfrak{X}'$, $\xi_x = \xi - \varepsilon_x$. If Φ is $\mathscr{P}$-stable with function b, then it is b-stable.

Any non-negative Φ is $\mathscr{P}$-stable in the environment $\mathfrak{X}$. Another important example is the *Penrose potential* [7] (see also [5]). Let (X, ρ) be the d-dimensional Euclidean

[1] This notion goes back to Oliver Penrose [7].

space with Lebesgue measure. Let $c,\varepsilon,R>0$ be constants. Φ is the following hard-core potential: If $|x-y|<R$ then $\Phi(x,y)=+\infty$; and if $|x-y|\geq R$ then $|\Phi|(x,y)\leq c|x-y|^{-(d+\varepsilon)}$. As Penrose has shown this potential is $\mathscr{P}$-stable in the environment $\mathfrak{X}_R$ with a constant stability which can be calculated explicitly.

The Boltzmann factor $B=\mathrm{e}^{-E}$ is an element of the algebra $(\mathscr{A},\star)$ having an inverse with respect to the $\star$ multiplication, which is denoted by $B_\star^{-1}$. Another important element U of the algebra $\mathscr{A}$ is the *Ursell function* given by

$$U(o)=0,\qquad U(\varepsilon_x)=1,\qquad U(\varepsilon_{x_1}+\ldots+\varepsilon_{x_n})=\sum_{\gamma\in\mathscr{C}_n}\prod_{(i,j)\in\gamma}\omega(x_i,x_j),\quad n\geq 2,\quad (5.4)$$

where $\mathscr{C}_n$ denotes the set of all simple, unoriented, connected graphs γ with n vertices, the product is taken over all edges (i,j) in γ and $\omega(x,y)=\mathrm{e}^{-\Phi(x,y)}-1$ is the Mayer function.

Note that $U\in\mathscr{A}_0$ and the following important relation is valid $B=\Gamma U$.

3 Ursell kernel
Representation of the correlation function

Here we follow the work of Minlos, Poghosyan [4]. Let $\mathsf{z}:X\to[0,+\infty)$ be measurable. We consider Radon measures of the form $\mathsf{z}.\rho=\rho_{\mathsf{z}}$, where ρ is Radon measure and z is a density function.

Given $A\in\mathscr{B}_0(X)$ we define the finite volume Gibbs process in A as the probability $\mathsf{Q}_{\mathsf{z},A}$ on $\mathfrak{X}(A)$ which is given by

$$\mathsf{Q}_{\mathsf{z},A}(\mathrm{d}\xi)=\frac{1}{\Xi(\mathsf{z},A)}\mathrm{e}^{-E(\xi)}\cdot\Lambda_{\mathsf{z}.\rho_A}(\mathrm{d}\xi)$$

where $\rho_A=1_A\cdot\rho$ and the normalising constant (*the partition function*) is given by

$$\Xi(\mathsf{z},A)=\int_{\mathfrak{X}(A)}\prod_{x\in\eta}\mathsf{z}(x)\mathrm{e}^{-E(\eta)}\Lambda_{\rho_A}(\mathrm{d}\eta).$$

By stability $\Xi(\mathsf{z},A)\leq\exp\left(\int_A\mathrm{e}^{\mathsf{b}(x)}\mathsf{z}(x)\,\rho(\mathrm{d}x)\right)<\infty$.

It is well known that the *correlation function of the Gibbs process* $\mathsf{Q}_{\mathsf{z},A}$ is given by

$$r_{\mathsf{z},A}(\xi) = \frac{\prod_{x\in\xi}\mathsf{z}(x)}{\Xi(\mathsf{z},A)}\int_{\mathfrak{X}(A)} \mathrm{e}^{-E(\xi+\eta)}\,\Lambda_{\mathsf{z}.\rho_A}(\mathrm{d}\eta), \qquad \xi\in\mathfrak{X}(A).$$

Proposition 5.1 The correlation function has the following remarkable representation:

$$r_{\mathsf{z},A}(\xi) = \prod_{x\in\xi}\mathsf{z}(x)\int_{\mathfrak{X}(A)} G(\xi,\eta)\,\Lambda_{\mathsf{z}.\rho_A}(\mathrm{d}\eta), \qquad \xi\in\mathfrak{X}(A) \tag{5.5}$$

where the *Ursell kernel* $G:\mathfrak{X}^2\to\mathbb{R}$ is given by $G(\xi,\eta)=(B_\star^{-1}\star D_\xi B)(\eta)$, $\xi,\eta\in\mathfrak{X}$ and $D_\xi B(\nu)=B(\xi+\nu)$, $\nu\in\mathfrak{X}$. In particular $G(\varepsilon_x,\eta)=U(\varepsilon_x+\eta)$ where U is the Ursell function.

For the proof we note that by the Minlos' formula

$$\frac{1}{\Xi(\mathsf{z},A)}\Lambda_{\mathsf{z}.\rho_A}(D_\xi B) = \frac{1}{\Xi(\mathsf{z},A)}\Lambda_{\mathsf{z}.\rho_A}(B\star B_\star^{-1}\star D_\xi B) = \Lambda_{\mathsf{z}.\rho_A}(B_\star^{-1}\star D_\xi B).$$

For a given pair potential Φ let $\overline{\Phi}=\Phi$ if Φ is finite and $\overline{\Phi}=1$ if $\Phi=+\infty$. Let a, b, c be non-negative functions on X. We will say that Φ satisfies

- ◇ c-*regularity*, if there exists a function a such that

$$\int_X |\omega|(x,y)\mathrm{e}^{(\mathsf{c}+\mathsf{a})(y)}\,\rho_{\mathsf{z}}(\mathrm{d}y) \le \mathsf{a}(x), \qquad x\in X. \tag{5.6}$$

- ◇ *Modified* b-*regularity*, if there exists a function a such that

$$\int_X |\overline{\Phi}|(x,y)\mathrm{e}^{\mathsf{b}(y)+\mathsf{a}(y)}\,\rho_{\mathsf{z}}(\mathrm{d}y) \le \mathsf{a}(x), \qquad x\in X. \tag{5.7}$$

Both assumptions (5.6) and (5.7) are introduced in [8].

Theorem 5.2 Let Φ be a b-stable pair interaction. Assume also that Φ is $2\mathsf{b}$-*regular* for a. Then the function

$$r_{\mathsf{z}}(\xi) = \prod_{x\in\xi}\mathsf{z}(x)\int_{\mathfrak{X}} G(\xi,\eta)\,\Lambda_{\rho_{\mathsf{z}}}(\mathrm{d}\eta), \qquad \xi\in\mathfrak{X} \tag{5.8}$$

is well defined and satisfies the following *Ruelle bound*

$$r_{\mathsf{z}}(\xi) \leq \prod_{x\in\xi} \mathsf{z}(x) \int_{\mathfrak{X}} \big|G\big|(\xi,\eta)\,\Lambda_{\rho_{\mathsf{z}}}(\mathrm{d}\eta) \leq \prod_{x\in\xi} \mathsf{z}(x)\mathrm{e}^{(2\mathsf{b}+\mathsf{a})(x)}, \qquad \xi \in \mathfrak{X}. \tag{5.9}$$

If Φ is $\mathscr{P}$-stable and b-regular for a then (5.9) holds with $\mathrm{e}^{(\mathsf{b}+\mathsf{a})(x)}$ instead of $\mathrm{e}^{(2\mathsf{b}+\mathsf{a})(x)}$. Moreover $r_{\mathsf{z}}(\xi) = \lim_{A\uparrow X} r_{\mathsf{z},A}(\xi)$.

The proof of this theorem is based on the so-called *forest graph estimate*. For $\xi,\eta \in \mathfrak{X}$ let $\mathscr{F}(\xi,\eta)$ be the collection of forests with the set of vertices $\xi+\eta$ and roots ξ. An unoriented simple graph is called *rooted forest* if its connected components are *rooted trees*, i. e. trees where one vertex is specified as a root.

We consider the case of b-stable Φ. The $\mathscr{P}$-stable case is entirely the same, one only needs to replace $\mathrm{e}^{2\mathsf{b}}$ by e^{b}. If Φ is modified regular, then one has to pass from ω to $\overline{\Phi}$ using the formula $\big|\omega\big|(x,y) \leq \big|\overline{\Phi}\big|(x,y)\mathrm{e}^{\Phi^-(x,y)}$.

Lemma 5.3 ([4]) For $\xi \neq o$,

$$\big|G\big|(\xi,\eta) \leq \prod_{x\in\xi+\eta} \mathrm{e}^{2\mathsf{b}(x)} \sum_{\gamma\in\mathscr{F}(\xi,\eta)} \prod_{(x,y)\in\gamma} \big|\omega\big|(x,y). \tag{5.10}$$

Denoting the right-hand side of (5.10) by $H(\xi,\eta)$ one can show that

$$H(\varepsilon_{x_1}+\ldots+\varepsilon_{x_n},\eta) = H\big(x_1,\cdot\big)\star\cdots\star H\big(x_n,\cdot\big)(\eta).$$

Then an application of the Minlos' formula and Theorem 2.1 from [8] completes the proof of Theorem 5.2.

In particular Lemma 5.3 gives the famous tree graph estimate of the Ursell function:

$$\big|U\big|(\eta) = \big|G\big|(\varepsilon_x,\eta-\varepsilon_x) \leq \prod_{x\in\eta} \mathrm{e}^{2\mathsf{b}(x)} \sum_{\gamma\in\mathscr{T}(\eta)} \prod_{(x,y)\in\gamma} \big|\omega\big|(x,y), \qquad x\in\eta. \tag{5.11}$$

Here $\mathscr{T}(\eta)$ is the set of trees with the set of vertices η.

4 Construction of limiting Gibbs processes

Theorem 5.4 Let Φ be a $\mathscr{P}$-stable pair potential in X which is b-regular for a. If $\mathrm{e}^{\mathsf{b}+\mathsf{a}}\rho$ is a Radon measure, then there exists a unique process P_{z} in X of infinite order having correlation function r_{z}, which is the limiting Gibbs process of the sequence $(\mathsf{Q}_{\mathsf{z},A_n})_n$ in the weak sense.

The proof of this theorem is based on the following lemma.

Lemma 5.5 ([11]) Let $(\mathsf{P}_n)_n$ be a sequence of point processes in X of infinite order satisfying the conditions: for each k the limits $\widetilde{\nu}^k(f) = \lim_{n\to\infty} \widetilde{\nu}^k_{\mathsf{P}_n}(f)$, $f \in \mathscr{K}(X^k)$, exist and $\sum_{\ell=1}^{\infty} \nu^\ell(A^\ell)^{-\frac{1}{2\ell}} = +\infty$ for each bounded A. Here $\nu^\ell(A^\ell) = \sum_{\mathscr{J}} \widetilde{\nu}^{|\mathscr{J}|}(A^{|\mathscr{J}|})$, where the summation is over all partitions of $\{1,\ldots,\ell\}$ into non-empty subsets. Then there exists one and only one point process P in X of infinite order such that $\mathsf{P}_n \Rightarrow \mathsf{P}$ and $\widetilde{\nu}^k_{\mathsf{P}} = \widetilde{\nu}^k$ for each k.

Lemma 5.5 combined with the Ruelle bound completes the proof of Theorem 5.4. We consider below the case where $\mathsf{z}(x) \equiv z > 0$, $x \in X$.

Proposition 5.6 Under the conditions of Theorem 5.4, $r_z(\xi) = z^{|\xi|} \int_{\mathfrak{X}} G(\xi,\eta)\Lambda_{\rho_z}(\mathrm{d}\eta)$ satisfies the Kirkwood-Salsburg (K-S) equation:

$$(\mathsf{K}\Sigma_{z\rho}) \qquad r_z(\xi) = z\mathrm{e}^{-W_\Phi(x,\xi)} \cdot \int_{\mathfrak{X}} K(x,\eta) r_z(\xi_x + \eta)\, \Lambda_\rho(\mathrm{d}\eta), \qquad x \in \xi \neq o,$$

where $K(x,\eta) := \prod_{y\in\eta} \omega(x,y)$.

The proof follows from Theorem 5.2, Minlos' formula and the fact that the Ursell kernel satisfies the equations ([PU09], [Ru69]): $G(o,\eta) = \delta_{o,\eta}$ and

$$G(\xi,\eta) = \mathrm{e}^{-W_\Phi(x,\xi_x)} \int_{\mathfrak{X}} K(x,\nu) G(\xi_x + \nu, \eta - \nu)\, \Lambda'_\eta(\mathrm{d}\nu), \qquad x \in \xi \neq o.$$

Theorem 5.7 Let Φ be a $\mathscr{P}$-stable b-regular potential for a. If $\mathrm{e}^{\mathsf{b}+\mathsf{a}}\rho$ is a Radon measure and $\sup_x \mathsf{a}(x) = C < \infty$ and if the activity satisfies $0 < z < (\mathrm{e}C)^{-1}$, the $(\mathsf{K}\Sigma_{z\rho})$ equation has a unique solution and the correlation function r_z of the process P_z is this unique solution.

Proof. We follow [10] and [3]. Let $\mathscr{E}_\delta$, $\delta > 0$, be the Banach space of all complex valued measurable functions $\varphi : \mathfrak{X}_+ \to \mathbb{C}$ such that

$$\|\varphi\|_\delta = \sup_{\xi\in\mathfrak{X}_+} \frac{|\varphi|(\xi)}{\delta^{|\xi|} \prod_{x\in\xi} \mathrm{e}^{(\mathsf{a}+\mathsf{b})(x)}} < +\infty, \tag{5.12}$$

where $|\xi| = \xi(X)$ denotes the number of particles in ξ. Since r_z satisfies the *Ruelle bound* (5.9), the correlation function r_z belongs to $\mathscr{E}_\delta$ with the norm ≤ 1 if $z \leq \delta$.

We define on $\mathscr{E}_\delta$ the linear operator K by

$$\mathsf{K}\varphi(\varepsilon_x) = z\int_{\mathfrak{X}_+} K(x,\eta)\varphi(\eta)\,\Lambda_\rho(\mathrm{d}\eta), \qquad x \in X, \tag{5.13}$$

$$\mathsf{K}\varphi(\xi) = z\mathrm{e}^{-W_\Phi(x,\xi)} \cdot \int_{\mathfrak{X}} K(x,\eta)\varphi(\eta+\xi_x)\Lambda_\rho(\mathrm{d}\eta), \qquad x \in \xi \neq o. \tag{5.14}$$

Using the operator K, we can write the K-S equation as an integral equation in the Banach space $\mathscr{E}_\delta$: $r_z = \mathsf{K}r_z + \alpha_z$, where $\alpha_z(\xi) = 0$ if $\xi(X) > 1$ and $\alpha_z(\varepsilon_x) = z$. For sufficiently small $z > 0$ the operator K is bounded. Indeed let $\varphi \in \mathscr{E}_\delta$ with $\|\varphi\| \leq 1$. Then by $\mathscr{P}$-stability and b-regularity of Φ for every $x \in \xi \in \mathfrak{X}$,

$$\begin{aligned}\big|(\mathsf{K}\varphi)\big|(\xi) &\leq z\mathrm{e}^{\mathsf{b}(x)} \int_{\mathfrak{X}} \big|\omega_x\big|(\eta)\,\delta^{|\eta|+|\xi|-1}\mathrm{e}^{(\eta+\xi_x)(\mathsf{b}+\mathsf{a})}\,\Lambda_\rho(\mathrm{d}\eta)\\ &\leq z\delta^{|\xi|-1}\mathrm{e}^{\xi(\mathsf{b}+\mathsf{a})} \cdot \exp\Big(\delta\rho\big(|\omega_x|\mathrm{e}^{\mathsf{b}+\mathsf{a}}\big)\Big) \leq \frac{z\mathrm{e}^{\delta C}}{\delta}\delta^{|\xi|}\mathrm{e}^{\xi(\mathsf{b}+\mathsf{a})}.\end{aligned}$$

Thus, if the parameters z and δ satisfy the condition $z\mathrm{e}^{\delta C}\delta^{-1} < 1$, then $\|\mathsf{K}\|_\delta < 1$ and the K-S equation has a unique solution. In particular, if we take $\delta = \frac{1}{C}$, this condition on z becomes $0 < z < (\mathrm{e}C)^{-1}$. A more detailed discussion of the choice of δ can be found in [3]. □

5 Uniqueness of Gibbs processes

In a final step we show that Gibbs processes G for Φ with activity z have correlation functions which solve the K-S equation in the same range of the parameter z. This implies that the Gibbs process G, if it exists, coincides with P_z.

We use the notion of Gibbs process introduced in [6] as a solution of an integration-by-parts formula. A point processes G is called a *Gibbs process for* (Φ,ρ), if for all $h \in F_+$

$$(\Sigma_\rho)\ \int_{\mathscr{M}^{\cdot\cdot}}\int_X h(x,\mu)\,\mu(\mathrm{d}x)\,\mathsf{G}(\mathrm{d}\mu) = \int_{\mathscr{M}^{\cdot\cdot}}\int_X h(x,\mu+\varepsilon_x)\exp\big(-W_\Phi(x,\mu)\big)\,\rho(\mathrm{d}x)\,\mathsf{G}(\mathrm{d}\mu).$$

We then write $\mathsf{G} \in \mathscr{G}(\Phi,\rho)$. This is equivalent to saying that G is a Gibbs process for (Φ,ρ) in the sense of Dobrushin, Lanford and Ruelle, cf. [6].

From now on we assume that Φ is $\mathscr{P}$-stable, modified b-regular for a and $\mathrm{e}^{\mathsf{b}+\mathsf{a}}\rho$ is a Radon measure. Then

Lemma 5.8 ([6]) Every $\mathsf{G} \in \mathscr{G}(\Phi, \rho_z)$ is of infinite order and its correlation function is given by

$$g_z(\xi) = z^{|\xi|} \int_{\mathscr{M}^{\cdot\cdot}} \exp\big(-W_\Phi(\xi, \mu)\big)\, \mathsf{G}(\mathrm{d}\mu), \qquad \xi \in \mathfrak{X}. \tag{5.15}$$

Furthermore, G is uniquely determined by its correlation functions.

Note that by modified regularity of Φ the conditional energy $W_\Phi(x, \mu)$ can be extended to the whole $\mathscr{M}^{\cdot\cdot}(X)$ and remains $\mathscr{P}$-stable with the same function b. Due to the $\mathscr{P}$-stability, the Ruelle bound takes the form $g_z(\xi) \le z^{|\xi|} \prod_{x\in\xi} \mathrm{e}^{\mathsf{b}(x)}$. Using ideas of Sabine Jansen [1], we then obtain

Proposition 5.9 Let G_z be a Gibbs process in X for (Φ, ρ_z). Then its correlation function g_z solves the $(\mathsf{K}\Sigma_{z\rho})$ equation.

In the view of the Ruelle bound we are in the situation as we had been above for the correlation function r_z.

Lemma 5.10 Let Φ be a $\mathscr{P}$-stable pair potential satisfying above mentioned conditions. Assume also $0 < z < (\mathrm{e}C)^{-1}$. Then g_z coincides with r_z which implies that G_z coincides with the limiting Gibbs process P_z.

Thus we arrive at the main result of this paper

Theorem 5.11 For all $0 < z < (\mathrm{e}C)^{-1}$ the collection $\mathscr{G}(\Phi, \rho_z)$ of Gibbs processes is either empty or the singleton $\{\mathsf{P}_z\}$.

For a large class of hard-core potentials we show in [9] that indeed for all $0 < z < (\mathrm{e}C)^{-1}$ the set $\mathscr{G}(\Phi, \rho_z)$ is not empty and therefore reduced to a unique element constructed as the limiting Gibbs process.

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