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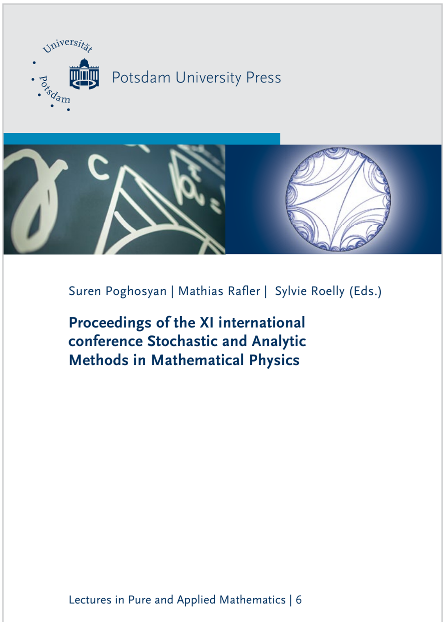

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Trotter product formula on Hilbert and Banach spaces for operator-norm convergence

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Abstract. *We review results on the operator-norm convergence of the Trotter product formula on Hilbert and Banach spaces. We concentrate here on the problem of convergence rates. Some results concerning evolution semigroups are also presented.*

1 Introduction

The product formula for matrices A and B

$$e^{-\tau C} = \lim_{n\to\infty} \left(e^{-\tau A/n} e^{-\tau B/n}\right)^n, \quad \tau \geq 0, \tag{2.1}$$

was established by S. Lie (1875). Here $C := A + B$. The proof of formula (2.1) can be easily extended to bounded operators $\mathscr{L}(\mathfrak{H})$ and $\mathscr{L}(\mathfrak{X})$ on Hilbert ($\mathfrak{H}$) and Banach ($\mathfrak{X}$) spaces. Moreover, a straightforward computation shows that the operator norm convergence rate in (2.1) is $O(1/n)$:

$$\sup_{\tau\in[0,T]} \left\|e^{-\tau A/n} e^{-\tau B/n} - e^{-\tau C/n}\right\|_{\mathscr{L}(\cdot)} = O(1/n). \tag{2.2}$$

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H. Trotter [17] has extended this result to unbounded operators A and B on Banach spaces, but now in the (weaker) *strong* operator topology: $\text{s-lim}_{n\to\infty} \mathscr{A}_n = \mathscr{A} \Leftrightarrow \lim_{n\to\infty} \|(\mathscr{A}_n - \mathscr{A})x\| = 0$ for any $x \in \mathfrak{X}$. He proved that if A and B are generators of contraction semigroups on a separable Banach space such that the algebraic sum $A+B$ is a densely defined closable operator and the closure $C = \overline{A+B}$ is a generator of a contraction semigroup, then

$$e^{-\tau C} = \underset{n\to\infty}{\text{s-lim}} \left(e^{-\tau A/n} e^{-\tau B/n}\right)^n, \tag{2.3}$$

uniformly in $\tau \in [0,T]$ for any $T > 0$. It was a long-time belief that the Trotter formula is valid only in the strong operator topology. But in the *nineties* it was discovered that under certain quite standard assumptions the strong convergence of the product formula (2.3) can be improved to the *operator-norm* convergence: $\lim_{n\to\infty} \|\mathscr{A}_n - \mathscr{A}\|_{\mathscr{L}(\mathfrak{H})} = 0 \Leftrightarrow \lim_{n\to\infty} \sup_{\{u\in\mathfrak{H}:\|u\|=1\}} \|(\mathscr{A}_n - \mathscr{A})u\| = 0$, on a Hilbert space $\mathfrak{H}$.

For the Trotter product formula in the *trace-class ideal* of $\mathscr{L}(\mathfrak{H})$ we refer to [18].

2 Trotter product formula on Hilbert spaces

2.1 Self-adjoint case. Considering the Trotter product formula on a separable Hilbert space $\mathfrak{H}$, T. Kato has shown that for non-negative self-adjoint operators A and B the Trotter formula (2.3) holds in the *strong* operator topology if $\text{dom}(\sqrt{A}) \cap \text{dom}(\sqrt{B})$ is dense in the Hilbert space and $C = A \dot{+} B$ is the form-sum of operators A and B. Naturally the problem arises whether Kato's result can be extended to the *operator-norm* convergence. A first attempt in this direction was undertaken by Dzh. Rogava [16]. He claimed that if A and B are non-negative self-adjoint operators such that $\text{dom}(A) \subseteq \text{dom}(B)$ and the operator-sum: $C = A + B$, is *self-adjoint*, then

$$\left\|(e^{-\tau A/n} e^{-\tau B/n})^n - e^{-\tau C}\right\|_{\mathscr{L}(\mathfrak{H})} = O(\ln(n)/\sqrt{n}), \qquad n \to \infty, \tag{2.4}$$

holds. In [12] it was shown that if one substitutes in above conditions the self-adjointness of the operator-sum by the A-smallness of B with a relative bound less then *one*, then (2.4) is true with the rate of convergence improved to

$$\left\|(e^{-\tau A/n} e^{-\tau B/n})^n - e^{-\tau C}\right\|_{\mathscr{L}(\mathfrak{H})} = O(\ln(n)/n), \qquad n \to \infty. \tag{2.5}$$

The problem in its original formulation was finally solved in [7]. There it was shown that the best possible in this general setup rate (2.2) holds if the operator sum: $C = A+B$, is already a self-adjoint operator. Rogava's result, as well as many other results (including [12]), when the operator sum of generators is self-adjoint, are corollary of [7]. A new direction comes due to results for the fractional-power conditions. In [14], with elucidation in [6], it was proven that assuming: $\mathrm{dom}(C^\alpha) \subseteq \mathrm{dom}(A^\alpha) \cap \mathrm{dom}(B^\alpha)$, $\alpha \in (1/2,1)$, $C = A \dot{+} B$ and $\mathrm{dom}(A^{1/2}) \subseteq \mathrm{dom}(B^{1/2})$ one obtains that

$$\sup_{\tau\in[0,T]} \left\|(e^{-\tau A/n}e^{-\tau B/n})^n - e^{-\tau C}\right\|_{\mathscr{L}(\mathfrak{H})} = O(n^{-(2\alpha-1)}).$$

Notice that formally $\alpha = 1$ yields the rate obtained in [7]. We remark also that the results of [6, 14] do *not* cover the case $\alpha = 1/2$. Although, it turns out that in this case the Trotter product formula converges on the operator norm:

$$\sup_{\tau\in[0,T]} \left\|(e^{-\tau A/n}e^{-\tau B/n})^n - e^{-\tau C}\right\|_{\mathscr{L}(\mathfrak{H})} = o(1),$$

if $\sqrt{B}$ is *relatively compact* with respect to $\sqrt{A}$, i.e. $\sqrt{B}(I+A)^{-1/2}$ is compact, see [13].

2.2 Non-self-adjoint case. Another direction was related with extension of the Trotter, and the Trotter-Kato, product formulae to the case of accretive [1, 2] and non-self-adjoint *sectorial* generators [4, 5]. Let A be a non-negative self-adjoint operator and let B be a maximal *accretive* ($\mathrm{Re}(Bf,f) \geq 0$ for $f \in \mathrm{dom}(B)$) operator, such that

$$\mathrm{dom}(A) \subseteq \mathrm{dom}(B) \quad \text{and} \quad \mathrm{dom}(A) \subseteq \mathrm{dom}(B^*).$$

If B is A-small with a relative bound less than one, then estimate (2.5) holds for generator C which is a well-defined maximal accretive operator-sum: $C = A+B$, see [1].

In [2] this result was generalised as follows. Let A be a non-negative self-adjoint operator and let B be a maximal accretive operator such that $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$ and B is A-small with relative bound less than one. If the condition

$$\mathrm{dom}\big((C^*)^\alpha\big) \subseteq \mathrm{dom}(A^\alpha) \cap \mathrm{dom}\big((B^*)^\alpha\big), \qquad C = A+B,$$

is satisfied for some $\alpha \in (0,1]$, then the norm-convergent Trotter product formula:

$$\sup_{\tau\in[0,T]} \left\|(e^{-\tau A/n}e^{-\tau B/n})^n - e^{-\tau C}\right\|_{\mathscr{L}(\mathfrak{H})} = O(\ln(n)/n^{\alpha}) \; ,$$

holds as $n \to \infty$. In fact, more results are known about the operator-norm Trotter product formula convergence for non-self-adjoint semigroups with sectorial generators, but *without* the rate estimates, see [3]. A new approach to analysis of the non-self-adjoint case was developed in [5]. Since it is based on holomorphic properties of semigroups, one can apply it even in Banach spaces. Therefore we postpone its presentation to Section 3.

3 Trotter product formula on Banach spaces

3.1 Holomorphic case. There are only few generalisations of the results of Section 2 to Banach spaces. The main obstacle for that is the fact that the concept of self-adjointness is missing. One of solutions is to relax the self-adjointness replacing the non-negative self-adjoint generator A by a generator of the holomorphic semigroup. The following result was proved in [5].

Theorem 2.1 ([5, Theorem 3.6 and Corollary 3.7]) Let A be a generator of a holomorphic contraction semigroup on the separable Banach space $\mathfrak{X}$ and let B be generator of a contraction semigroup on $\mathfrak{X}$.

i) If for some $\alpha \in (0,1)$ the condition $\mathrm{dom}(A^{\alpha}) \subseteq \mathrm{dom}(B)$, holds and $\mathrm{dom}(A^*) \subseteq \mathrm{dom}(B^*)$ is satisfied, then the operator sum $C = A + B$ is generator of a contraction semigroup and for any $T > 0$:

$$\sup_{\tau\in[0,T]} \left\|(e^{-\tau B/n}e^{-\tau A/n})^n - e^{-\tau C}\right\|_{\mathscr{L}(\mathfrak{X})} = O\big(\ln(n)/n^{1-\alpha}\big). \tag{2.6}$$

ii) If for some $\alpha \in (0,1)$ the condition $\mathrm{dom}((A^{\alpha})^*) \subseteq \mathrm{dom}(B^*)$ is satisfied and $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$ is valid, then $C = A + B$ is generator of a contraction semigroup and

$$\sup_{\tau\in[0,T]} \left\|(e^{-\tau A/n}e^{-\tau B/n})^n - e^{-\tau C}\right\|_{\mathscr{L}(\mathfrak{X})} = O\big(\ln(n)/n^{1-\alpha}\big), \tag{2.7}$$

for any $T > 0$.

Theorem 2.2 ([5, Theorem 3.6 and Corollary 3.7]) Let A be generator of a holomorphic contraction semigroup on $\mathfrak{X}$ and let B be generator of a contraction semigroup on $\mathfrak{X}$. If B is in addition a *bounded* operator, then for any $T > 0$:

$$\sup_{\tau\in[0,T]} \left\|(e^{-\tau B/n}e^{-\tau A/n})^n - e^{-\tau C}\right\|_{\mathscr{L}(\mathfrak{X})} = O\big((\ln(n))^2/n\big),$$
$$\sup_{\tau\in[0,T]} \left\|(e^{-\tau A/n}e^{-\tau B/n})^n - e^{-\tau C}\right\|_{\mathscr{L}(\mathfrak{X})} = O\big((\ln(n))^2/n\big).$$

Theorem 2.2 becomes *false* if the condition that A is generator of a holomorphic semigroup is *dropped*.

3.2 Non-holomorphic: evolution semigroup. Let A and operators $\{B(t)\}_{t\in[0,T]}$ be generators of holomorphic semigroups on a separable Banach space $\mathfrak{X}$. Consider non-autonomous Cauchy problem for $u_0 := u(0)$:

$$\partial_t u(t) = -\big(A + B(t)\big)u(t), \qquad t \in [0,T], \tag{2.8}$$

Assumptions:

(A1) Operator $A \geq I$ is generator of a holomorphic contraction semigroup in $\mathfrak{X}$.

(A2) Let $\{B(t)\}_{t\in[0,T]}$ be a family of closed operators such that for a.e. $t \in [0,T]$ and some $\alpha \in (0,1)$ the condition $\mathrm{dom}(A^\alpha) \subset \mathrm{dom}(B(t))$ is satisfied such that

$$C_\alpha := \operatorname*{ess\,sup}_{t\in[0,T]} \left\|B(t)A^{-\alpha}\right\|_{\mathscr{L}(\mathfrak{X})} < \infty\ .$$

(A3) Let $\{B(t)\}_{t\in[0,T]}$ be a family of generators of contraction semigroups in $\mathfrak{X}$ such that the function $[0,T] \ni t \mapsto \big(B(t) + \xi I\big)^{-1}x \in \mathfrak{X}$ is strongly measurable for any $x \in \mathfrak{X}$ and any $\xi > b$ for some $b > 0$.

(A4) We assume that $\mathrm{dom}(A^*) \subset \mathrm{dom}(B(t)^*)$ and

$$C_1^* := \operatorname*{ess\,sup}_{t\in[0,T]} \left\|B(t)^*(A^*)^{-1}\right\|_{\mathscr{L}(\mathfrak{X}^*)} < \infty,$$

where A^* and $B(t)^*$ denote operators which are adjoint of A and $B(t)$, respectively.

(A5) There exists $\beta \in (\alpha,1)$ and a constant $L_\beta > 0$ such that for a.e. $t,s \in [0,T]$ one has the estimate:

$$\left\|A^{-1}\big(B(t) - B(s)\big)A^{-\alpha}\right\|_{\mathscr{L}(\mathfrak{X})} \leq L_\beta |t-s|^\beta .$$

(A6) There exists a constant $L_1 > 0$ such that for a.e. $t,s \in [0,T]$ one has the estimate:

$$\big\|A^{-\alpha}\big(B(t)-B(s)\big)A^{-\alpha}\big\|_{\mathscr{L}(\mathfrak{X})} \leq L_1|t-s|.$$

The evolution equation (2.8) is associated with family $\{C(t)\}_{t\in[0,T]}$, $C(t) = A + B(t)$.

We consider the Banach space $L^p([0,T],\mathfrak{X})$ for $p \in [1,\infty)$ and introduce in this space the *multiplication* operators $\mathscr{A}$ and $\mathscr{B}$ generated by A and $\{B(t)\}_{t\in[0,T]}$, see [8, 15]. Similarly, one can introduce the multiplication operator $\mathscr{C}$ induced by the family $\{C(t)\}_{t\in[0,T]}$ which is also a generator of a holomorphic semigroup. Notice that $\mathscr{C} = \mathscr{A} + \mathscr{B}$ and $\mathrm{dom}(\mathscr{C}) = \mathrm{dom}(\mathscr{A})$. Let D_0 the generator of the right-shift *nilpotent* semigroup on $L^p([0,T],\mathfrak{X})$, i.e. $\big(e^{-\tau D_0}f\big)(t) = \chi_{[0,T]}(t-\tau)f(t-\tau), \quad f \in L^p([0,T],\mathfrak{X})$.

Next, we consider the operator

$$\begin{gathered} \widetilde{\mathscr{K}}f = D_0 f + \mathscr{A}f + \mathscr{B}f, \\ f \in \mathrm{dom}(\widetilde{\mathscr{K}}) = \mathrm{dom}(D_0)\cap\mathrm{dom}(\mathscr{A})\cap\mathrm{dom}(\mathscr{B}). \end{gathered} \tag{2.9}$$

Assuming (A1)–(A3) it was shown in [11] that the operator $\widetilde{\mathscr{K}}$ is closable and its closure $\mathscr{K}$ is generator of the *evolution semigroup* $\{e^{-\tau\mathscr{K}}\}_{\tau\geq 0}$ [8, 15], which is also nilpotent and consequently a *non-holomorphic* semigroup. Further we set $\widetilde{\mathscr{K}_0}f = D_0 f + \mathscr{A}f$ for $f \in \mathrm{dom}(\widetilde{\mathscr{K}_0}) = \mathrm{dom}(D_0)\cap\mathrm{dom}(\mathscr{A})$.

In contrast to the Hilbert space the operator $\widetilde{\mathscr{K}_0}$ is not necessary generator of a semigroup. However, the operator $\widetilde{\mathscr{K}_0}$ is closable and its closure $\mathscr{K}_0$ is a generator. Note that $\mathscr{K}$ coincides with the algebraic sum: $\mathscr{K} = \mathscr{K}_0 + \mathscr{B}$.

Theorem 2.3 ([11, Theorem 7.8]) Let the Assumptions (A1)–(A4) be satisfied for some $\alpha \in (0,1)$. If (A5) holds, then one gets for $n \to \infty$ the asymptotic:

$$\sup_{\tau\in[0,T]}\big\|\big(e^{-\tau B/n}e^{-\tau\mathscr{K}_0/n}\big)^n - e^{-\tau\mathscr{K}}\big\|_{\mathscr{L}(L^p([0,T],\mathfrak{X}))} = O\big(1/n^{\beta-\alpha}\big). \tag{2.10}$$

Assuming instead of Assumption (A5) the Assumption (A6) one finds

Theorem 2.4 ([9, Theorem 5.4]) Let the Assumptions (A1)–(A4) be satisfied for some $\alpha \in (1/2,1)$. If (A6) is valid, then for $n \to \infty$ one gets the asymptotic:

$$\sup_{\tau\in[0,T]}\big\|\big(e^{-\tau B/n}e^{-\tau\mathscr{K}_0/n}\big)^n - e^{-\tau\mathscr{K}}\big\|_{\mathscr{L}(L^p([0,T],\mathfrak{X}))} = O\big(1/n^{1-\alpha}\big). \tag{2.11}$$

3.3 Convergence rate for propagators. To construct approximations of solution operators (*propagators*) for the Cauchy problem (2.8), we apply to the problem (2.8) the *evolution semigroup* approach developed in [8, 15, 11]. The idea is to transform the *non-autonomous* Cauchy problem (2.8) into an *autonomous* problem generated by evolution semigroup $\{e^{-\tau\mathscr{K}}\}_{\tau\geq 0}$.

Definition 2.5 ([8, 15]) Linear operator $\mathscr{K}$ in $L^p([0,T],\mathfrak{X})$, $p\in[1,\infty)$, is called *evolution generator* if for multiplication operator $M(\phi)$:

(i) $\mathrm{dom}(\mathscr{K})\subset C([0,T],\mathfrak{X})$ and $M(\phi)\mathrm{dom}(\mathscr{K})\subset\mathrm{dom}(\mathscr{K})$ for $\phi\in W^{1,\infty}([0,T])$;

(ii) $\mathscr{K}M(\phi)f-M(\phi)\mathscr{K}f=M(\partial_t\phi)f$ for $f\in\mathrm{dom}(\mathscr{K})$ and $\phi\in W^{1,\infty}([0,T])$;

(iii) the domain $\mathrm{dom}(\mathscr{K})$ has a dense cross-section, i.e. for each $t\in(0,T]$ the set

$$[\mathrm{dom}(\mathscr{K})]_t:=\{x\in\mathfrak{X}:\exists f\in\mathrm{dom}(\mathscr{K})\text{ such that }x\in f(t)\},$$

is dense in $\mathfrak{X}$. Here for any $\phi\in L^\infty([0,T])$ we denote by $M(\phi)$ a bounded multiplication operator on $L^p([0,T],\mathfrak{X})$ defined as $\big(M(\phi)f\big)(t)=\phi(t)f(t)$, $f\in L^p([0,T],\mathfrak{X})$.

One can check that the operator $\mathscr{K}$ defined as the closure of $\widetilde{\mathscr{K}}$ (2.9) is an evolution generator, cf. [11, Theorem 1.2]. Evolution generators are related to *propagators*, which are defined as follows.

Definition 2.6 Let $\{U(t,s)\}_{(t,s)\in\Delta}$, $\Delta=\{(t,s)\in(0,T]\times(0,T]:s\leq t\leq T\}$, be a strongly continuous family of bounded operators on $\mathfrak{X}$. If the conditions

$$U(t,t)=I \qquad \text{for } t\in(0,T], \tag{2.12}$$

$$U(t,r)U(r,s)=U(t,s) \qquad \text{for } t,r,s\in(0,T] \text{ with } s\leq r\leq t, \tag{2.13}$$

$$\|U\|_\Delta:=\sup_{(t,s)\in\Delta}\|U(t,s)\|_{\mathscr{L}(\mathfrak{X})}<\infty \tag{2.14}$$

are satisfied. If $u(t)=U(t,0)u_0$, $t\geq 0$, for $u_0\in\mathrm{dom}(A)$, is solution of the Cauchy problem (2.8), then $\{U(t,s)\}_{(t,s)\in\Delta}$ is called *solution operator*, or *propagator*.

It is known [8, Theorem 4.12] that there is an *one-to-one* correspondence between the set of all evolution generators on $L^p([0,T],\mathfrak{X})$ and the set of all propagators in the sense of Definition 2.6. It is established by equation

$$\big(e^{-\tau\mathscr{K}}f\big)(t)=U(t,t-\tau)\chi_{[0,T]}(t-\tau)f(t-\tau), \qquad f\in L^p([0,T],\mathfrak{X}). \tag{2.15}$$

Let $\mathscr{K}_0$ be generator of evolution semigroup $\{\mathscr{U}_0(\tau)\}_{\tau\geq 0}$ and let $\mathscr{B}$ be multiplication operator induced by a measurable family $\{B(t)\}_{t\in[0,T]}$ of generators of contraction semigroups. Note that in this case the multiplication operator $\mathscr{B}$ is a generator of a contraction semigroup $\big(e^{-\tau\mathscr{B}}f\big)(t) = e^{-\tau B(t)}f(t)$, on the Banach space $L^p([0,T],\mathfrak{X})$. Since $\{\mathscr{U}_0(\tau)\}_{\tau\geq 0}$ is the evolution semigroup, then by (2.15) there exists propagator $\{U_0(t,s)\}_{(t,s)\in\Delta}$ such that the representation: $\big(\mathscr{U}_0(\tau)f\big)(t) = U_0(t,t-\tau)\chi_{[0,T]}(t-\tau)f(t-\tau)$, $f\in L^p([0,T],\mathfrak{X})$, is valid for a. e. $t\in[0,T]$ and $\tau\geq 0$. Then we define

$$Q_j(t,s;n) := U_0\left(s+j\tfrac{(t-s)}{n}, s+(j-1)\tfrac{(t-s)}{n}\right) e^{-\frac{(t-s)}{n}B\left(s+(j-1)\frac{(t-s)}{n}\right)}$$

where $j\in\{1,2,\ldots,n\}$, $n\in\mathbb{N}$, $(t,s)\in\Delta$, and we set for *approximants* $\{V_n(t,s)\}_{n\geq 1}$:

$$V_n(t,s) := \prod_{j=1}^{n\leftarrow} Q_j(t,s;n), \qquad n\in\mathbb{N},\ (t,s)\in\Delta,$$

where the product is increasingly ordered in j from the right to the left. Then by (2.15) a straightforward computation shows that the representation

$$\left(\left(e^{-\tau\mathscr{K}_0/n}e^{-\tau\mathscr{B}/n}\right)^n f\right)(t) = V_n(t,t-\tau)\chi_{[0,T]}(t-\tau)f(t-\tau)\ ,$$

$f\in L^p([0,T],\mathfrak{X})$, holds for each $\tau\geq 0$ and a.e. $t\in[0,T]$.

Similarly we can introduce

$$G_j(t,s;n) = e^{-\frac{t-s}{n}B\left(s+j\frac{t-s}{n}\right)}U_0\big(s+j\tfrac{t-s}{n}, s+(j-1)\tfrac{t-s}{n}\big)$$

where $j\in\{1,2,\ldots,n\}$, $n\in\mathbb{N}$, $(t,s)\in\Delta$. Now let the *approximants* be defined by

$$U_n(t,s) := \prod_{j=1}^{n\leftarrow} G_j(t,s;n), \qquad n\in\mathbb{N},\ (t,s)\in\Delta,$$

where the product is again increasingly ordered in j from the right to the left. Note that

$$\left(\left(e^{-\tau\mathscr{B}/n}e^{-\tau\mathscr{K}_0/n}\right)^n f\right)(t) = U_n(t,t-\tau)\chi_{[0,T]}(t-\tau)f(t-\tau)\ ,$$

$f\in L^p([0,T],\mathfrak{X})$, holds for each $\tau\geq 0$ and a.e. $t\in[0,T]$.

Proposition 2.7 ([10, Proposition 2.1]) Let $\mathscr{K}$ and $\mathscr{K}_0$ be generators of evolution semigroups on the Banach space $L^p([0,T],\mathfrak{X})$ for some $p \in [1,\infty)$. Further, let $\{B(t))\}_{t\in[0,T]}$ be a strongly measurable family of generators of contraction on $\mathfrak{X}$. Then for $n \in \mathbb{N}$,

$$\sup_{\tau\in[0,T]} \left\| e^{-\tau\mathscr{K}} - \left(e^{-\tau\mathscr{K}_0/n} e^{-\tau\mathscr{B}/n}\right)^n \right\|_{\mathscr{L}(L^p([0,T],\mathfrak{X}))} = \operatorname*{ess\,sup}_{(t,s)\in\Delta} \|U(t,s) - V_n(t,s)\|_{\mathscr{L}(\mathfrak{X})},$$
$$\sup_{\tau\in[0,T]} \left\| e^{-\tau\mathscr{K}} - \left(e^{-\tau\mathscr{B}/n} e^{-\tau\mathscr{K}_0/n}\right)^n \right\|_{\mathscr{L}(L^p([0,T],\mathfrak{X}))} = \operatorname*{ess\,sup}_{(t,s)\in\Delta} \|U(t,s) - U_n(t,s)\|_{\mathscr{L}(\mathfrak{X})}.$$

From Theorem 2.3 and Proposition 2.7 one obtains the following assertion.

Theorem 2.8 ([11, Theorem 1.4]) Let the Assumptions (A1)–(A4) be satisfied. If (A5) holds, then for $n \to \infty$ one gets the rate:

$$\operatorname*{ess\,sup}_{(t,s)\in\Delta} \|U_n(t,s) - U(t,s)\|_{\mathscr{L}(\mathfrak{X})} = O\big(1/n^{\beta-\alpha}\big). \tag{2.16}$$

On the other hand, from Theorem 2.4 and Proposition 2.7 we get

Theorem 2.9 ([9, Theorem 5.6]) Let the Assumptions (A1)–(A4) be satisfied for some $\alpha \in (1/2, 1)$. If (A6) is valid, then for $n \to \infty$ one obtains a better rate:

$$\operatorname*{ess\,sup}_{(t,s)\in\Delta} \|U_n(t,s) - U(t,s)\|_{\mathscr{L}(\mathfrak{X})} = O\big(1/n^{1-\alpha}\big).$$

4 Example of sharpness

We study bounded perturbations of the evolution generator D_0. To this aim we consider $\mathfrak{X} = \mathbb{C}$ and we denote by $L^2([0,1])$ the Hilbert space $L^2([0,1],\mathbb{C})$.

For $t \in [0,1]$, let $q : t \mapsto q(t) \in L^\infty([0,1])$. Then q induces on the Banach space $L^2([0,1])$ a bounded multiplication operator Q defined as

$$\big(Qf\big)(t) := q(t)f(t), \qquad f \in L^2([0,1]).$$

For simplicity we assume that $q \geq 0$. Then Q generates on $L^2([0,1])$ a contraction semigroup $\{e^{-\tau Q}\}_{\tau\geq 0}$. Since generator Q is bounded, the closed operator $\mathscr{K} := D_0 + Q$, with domain $\mathrm{dom}(\mathscr{K}) = \mathrm{dom}(D_0)$, is generator of a semigroup on $L^2([0,1])$. By [17] we get

$$\operatorname*{s-lim}_{n\to\infty} \left(e^{-\tau D_0/n} e^{-\tau Q/n}\right)^n = e^{-\tau(D_0+Q)}.$$

One can easily check that $\mathscr{K}$ is an evolution generator. A straightforward computation shows that

$$\left(e^{-\tau(D_0+Q)}f\right)(t) = e^{-\int_{t-\tau}^{t} q(y)dy}\chi_{[0,1]}(t-\tau)f(t-\tau).$$

This yields that the propagator corresponding to $\mathscr{K}$ is given by

$$U(t,s) = e^{-\int_s^t q(y)dy}, \qquad (t,s)\in\Delta.$$

Now a simple computation shows that

$$\left(\left(e^{-\tau D_0/n}e^{-\tau Q/n}\right)^n f\right)(t) =: V_n(t,t-\tau)\chi_{[0,T]}(t-\tau)f(t-\tau).$$

Then by straightforward calculations we find that

$$V_n(t,s) = e^{-\frac{t-s}{n}\sum_{k=0}^{n-1} q(s+k\frac{t-s}{n})}, \qquad (t,s)\in\Delta.$$

Theorem 2.10 ([10, Proposition 3.1]) Let $q\in L^\infty([0,1])$ be non-negative. Then

$$\sup_{\tau\in[0,1]}\left\|e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n}e^{-\tau Q/n}\right)^n\right\|_{\mathscr{L}(L^2([0,1]))} \leq O\left(\operatorname*{ess\,sup}_{(t,s)\in\Delta}\left|\int_s^t q(y)dy - \frac{t-s}{n}\sum_{k=0}^{n-1} q(s+k\tfrac{t-s}{n})\right|\right),$$

as $n\to\infty$.

Note that by Theorem 2.10 the operator-norm convergence rate of the Trotter product formula for the pair $\{D_0,Q\}$ coincides with the convergence rate of the integral Darboux-Riemann sum approximation of the Lebesgue integral.

Theorem 2.11 ([10, Theorem 3.2]) If the function: $q\in C^{0,\beta}([0,1])$, $\beta\in(0,1]$, is non-negative, then for $n\to\infty$ one gets

$$\sup_{\tau\in[0,1]}\left\|e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n}e^{-\tau Q/n}\right)^n\right\|_{\mathscr{L}(L^2([0,1]))} = O(1/n^\beta).$$

Theorem 2.12 ([10, Theorem 3.3]) If $q\in C([0,1])$ is continuous and non-negative, then for $n\to\infty$

$$\left\|e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n}e^{-\tau Q/n}\right)^n\right\|_{\mathscr{L}(L^2([0,1]))} = o(1)\,. \tag{2.17}$$

It follows that the convergence to zero in (2.17) may be *arbitrarily* slow.

Theorem 2.13 ([10, Theorem 3.4]) Let $\delta_n > 0$ be a sequence with $\delta_n \to 0$ as $n \to \infty$. Then there exists a continuous function $q : [0,1] \to \mathbb{R}$ such that

$$\sup_{\tau\in[0,1]} \left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n}e^{-\tau Q/n}\right)^n \right\|_{\mathscr{L}(L^2([0,1]))} = \omega(\delta_n) \,, \tag{2.18}$$

as $n \to \infty$. Here ω is the Landau symbol: $\omega(\delta_n) \Leftrightarrow \limsup_{n\to\infty} |\omega(\delta_n)/\delta_n| = \infty$.

If q is only *measurable*, it can happen that the Trotter product formula for that pair $\{D_0, Q\}$ does *not* converge in the *operator-norm* topology:

Theorem 2.14 ([10, Theorem 3.5]) There is a non-negative measurable function $q \in L^\infty([0,1])$, such that

$$\liminf_{n\to\infty} \sup_{\tau\in[0,1]} \left\| e^{-\tau(D_0+Q)} - \left(e^{-\tau D_0/n}e^{-\tau Q/n}\right)^n \right\|_{\mathscr{L}(L^2([0,1]))} > 0. \tag{2.19}$$

Theorem 2.14 does not exclude the convergence in the *strong* operator topology.

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