

Hadamard states for bosonic quantum field theory on globally hyperbolic spacetimes

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Summary

Quantum field theory on curved spacetimes is understood as a semiclassical approximation of some quantum theory of gravitation, which models a quantum field under the influence of a classical gravitational field, that is, a curved spacetime. The most remarkable effect predicted by this approach is the creation of particles by the spacetime itself, represented, for instance, by Hawking's evaporation of black holes or the Unruh effect. On the other hand, these aspects already suggest that certain cornerstones of Minkowski quantum field theory, more precisely a preferred vacuum state and, consequently, the concept of particles, do not have sensible counterparts within a theory on general curved spacetimes. Likewise, the implementation of covariance in the model has to be reconsidered, as curved spacetimes usually lack any non-trivial global symmetry. Whereas this latter issue has been resolved by introducing the paradigm of locally covariant quantum field theory (LCQFT), the absence of a reasonable concept for distinct vacuum and particle states on general curved spacetimes has become manifest even in the form of no-go-theorems.

Within the framework of algebraic quantum field theory, one first introduces observables, while states enter the game only afterwards by assigning expectation values to them. Even though the construction of observables is based on physically motivated concepts, there is still a vast number of possible states, and many of them are not reasonable from a physical point of view. We infer that this notion is still too general, that is, further physical constraints are required. For instance, when dealing with a free quantum field theory driven by a linear field equation, it is natural to focus on so-called quasifree states. Furthermore, a suitable renormalization procedure for products of field operators is vitally important. This particularly concerns the expectation values of the energy momentum tensor, which correspond to distributional bisolutions of the field equation on the curved spacetime. J. Hadamard's theory of hyperbolic equations provides a certain class of bisolutions with fixed singular part, which therefore allow for an appropriate renormalization scheme.

By now, this specification of the singularity structure is known as the Hadamard condition and widely accepted as the natural generalization of the spectral condition of flat quantum field theory. Moreover, due to Radzikowski's celebrated results, it is equivalent to a local condition, namely on the wave front set of the bisolution. This formulation made the powerful tools of microlocal analysis, developed by Duistermaat and Hörmander, available for the verification of the Hadamard property as well as the construction of corresponding Hadamard states, which initiated much progress in this field. However, although indispensable for the investigation in the characteristics of operators and their parametrices, microlocal analysis is not practicable for the study of their non-singular features and central results are typically stated only up to smooth objects. Consequently, Radzikowski's work almost directly led to existence results and, moreover, a concrete pattern for the construction of Hadamard bidistributions via a Hadamard series. Nevertheless, the remaining properties (bisolution, causality, positivity) are ensured only modulo C^∞ .

It is the subject of this thesis to complete this construction for linear and formally self-adjoint wave operators acting on sections in a vector bundle over a globally hyperbolic Lorentzian manifold. Based on Wightman's solution of d'Alembert's equation on Minkowski space and the construction for the advanced and retarded fundamental solution, we set up a Hadamard series for local parametrices and derive global bisolutions from them. These are of Hadamard form and we show existence of smooth bisections such that the sum also satisfies the remaining properties exactly.

Zusammenfassung

Quantenfeldtheorie auf gekrümmten Raumzeiten ist eine semiklassische Näherung einer Quantentheorie der Gravitation, im Rahmen derer ein Quantenfeld unter dem Einfluss eines klassisch modellierten Gravitationsfeldes, also einer gekrümmten Raumzeit, beschrieben wird. Eine der bemerkenswertesten Vorhersagen dieses Ansatzes ist die Erzeugung von Teilchen durch die gekrümmte Raumzeit selbst, wie zum Beispiel durch Hawkings Verdampfen schwarzer Löcher und den Unruh Effekt. Andererseits deuten diese Aspekte bereits an, dass fundamentale Grundpfeiler der Theorie auf dem Minkowskiraum, insbesondere ein ausgezeichnete Vakuumzustand und damit verbunden der Teilchenbegriff, für allgemeine gekrümmte Raumzeiten keine sinnvolle Entsprechung besitzen. Gleichmaßen benötigen wir eine alternative Implementierung von Kovarianz in die Theorie, da gekrümmte Raumzeiten im Allgemeinen keine nicht-triviale globale Symmetrie aufweisen. Letztere Problematik konnte im Rahmen lokal-kovarianter Quantenfeldtheorie gelöst werden, wohingegen die Abwesenheit entsprechender Konzepte für Vakuum und Teilchen in diesem allgemeinen Fall inzwischen sogar in Form von no-go-Aussagen manifestiert wurde.

Beim algebraischen Ansatz für eine Quantenfeldtheorie werden zunächst Observablen eingeführt und erst anschließend Zustände via Zuordnung von Erwartungswerten. Obwohl die Observablen unter physikalischen Gesichtspunkten konstruiert werden, existiert dennoch eine große Anzahl von möglichen Zuständen, von denen viele, aus physikalischen Blickwinkeln betrachtet, nicht sinnvoll sind. Dieses Konzept von Zuständen ist daher noch zu allgemein und bedarf weiterer physikalisch motivierter Einschränkungen. Beispielsweise ist es natürlich, sich im Falle freier Quantenfeldtheorien mit linearen Feldgleichungen auf quasifreie Zustände zu konzentrieren. Darüber hinaus ist die Renormierung von Erwartungswerten für Produkte von Feldern von zentraler Bedeutung. Dies betrifft insbesondere den Energie-Impuls-Tensor, dessen Erwartungswert durch distributionelle Bilösungen der Feldgleichungen gegeben ist. Tatsächlich liefert J. Hadamard Theorie hyperbolischer Differentialgleichungen Bilösungen mit festem singulären Anteil, so dass ein geeignetes Renormierungsverfahren definiert werden kann.

Die sogenannte Hadamard-Bedingung an Bidistributionen steht für die Forderung einer solchen Singularitätenstruktur und sie hat sich etabliert als natürliche Verallgemeinerung der für flache Raumzeiten formulierten Spektralbedingung. Seit Radzikowskis wegweisenden Resultaten lässt sie sich außerdem lokal ausdrücken, nämlich als eine Bedingung an die Wellenfrontenmenge der Bilösung. Diese Formulierung schlägt eine Brücke zu der von Duistermaat und Hörmander entwickelten mikrolokalen Analysis, die seitdem bei der Überprüfung der Hadamard-Bedingung sowie der Konstruktion von Hadamard Zuständen vielfach Verwendung findet und rasante Fortschritte auf diesem Gebiet ausgelöst hat. Obwohl unverzichtbar für die Analyse der Charakteristiken von Operatoren und ihrer Parametrisen sind die Methoden und Aussagen der mikrolokalen Analysis ungeeignet für die Analyse von nicht-singulären Strukturen und zentrale Aussagen sind typischerweise bis auf glatte Anteile formuliert. Beispielsweise lassen sich aus Radzikowskis Resultaten nahezu direkt Existenzaussagen und sogar ein konkretes Konstruktionsschema für Hadamard Zustände ableiten, die übrigen Eigenschaften (Bilösung, Kausalität, Positivität) können jedoch auf diesem Wege nur modulo C^∞ gezeigt werden.

Es ist das Ziel dieser Dissertation, diesen Ansatz für lineare Wellenoperatoren auf Schnitten in Vektorbündeln über global-hyperbolischen Lorentz-Mannigfaltigkeiten zu vollenden und, ausgehend von einer lokalen Hadamard Reihe, Hadamard Zustände zu konstruieren. Beruhend auf Wightmans Lösung für die d'Alembert-Gleichung auf dem Minkowski-Raum und der Herleitung der avancierten und retardierten Fundamentallösung konstruieren wir lokal Parametrisen in Form von Hadamard-Reihen und fügen sie zu globalen Bilösungen zusammen. Diese besitzen dann die Hadamard-Eigenschaft und wir zeigen anschließend, dass glatte Bischnitte existieren, die addiert werden können, so dass die verbleibenden Bedingungen erfüllt sind.

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Statement of Originality

This thesis contains no material which has been accepted for the award of any other degree or diploma at any other university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due to reference has been made in the text. I give consent to this copy of my thesis, when deposited in the University Library, being made available for loan and photocopying.

Potsdam, November 29th, 2019
Max Lewandowski

*"Wir legen ab und fahr'n nach Singapur
mit 'nem Schiff aus schäbigem Holz.
Auch wenn der Wind uns das Segel zerreißt,
wir müssen weiter, immer weiter, was soll's?"*

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Bibliography

1 Introduction

"We apologize for the inconvenience.' I think, I feel good about it."

1.1 Quantum field theory on a curved spacetime

Quantum field theory on curved spacetimes is regarded as an intermediate step on the path to a not yet available quantum description of gravitation. This framework investigates the coupling of a quantum field with classical gravitation, i.e. a curved spacetime, and is widely accepted as a reasonable paradigm outside the event horizon of black holes and with the exception of the first 10^{-43} seconds of the universe, where gravitation is considered negligibly weak on Planck length scales.

Despite its still semiclassical character, this model already predicts remarkable effects beyond the understanding of Minkowski quantum field theory, such as the creation of particles by the curved spacetime itself, a phenomenon most prominently represented by Hawking's evaporation of black holes [Haw1975] and the Unruh effect [Ful1973, Dav1975, Unr1976]. Flat Minkowski space possesses a rich group of isometries, namely the Poincaré group, providing a conserved quantity "energy" and ruling out effects like those mentioned above [Haw1970]. Curved spacetimes, however, usually lack any non-trivial symmetry and thus a reasonable notion of energy at all, thereby prompting a fundamental reconsideration of essential concepts of quantum field theory. In particular, a distinct vacuum – the state of lowest energy of the quantum field – and, as a consequence, the notion of particles turn out to be non-sensible in general curved spacetimes since only a certain class of observers, namely the inert ones, would agree on the same vacuum state. For a discussion of this issue and related effects see [Dav1984] and [Wal1994] as well as the references therein.

It follows that there is no universal interpretation of states in terms of their particle content but only as expectation values of suitable observables. We should therefore think of the observables as the more fundamental objects. These should be constructed directly from the outset and not as operators on a previously specified state space.

1.2 Algebraic and locally covariant bosonic quantum field theory

This view is best addressed by the algebraic approach to quantum field theory [HK1964, Dim1980], where first of all observables are introduced as elements of rather abstract algebras $\mathcal{A}(O) \subset \mathcal{A}(M)$ associated to spacetime regions $O \subset M$ in a local and covariant manner. The most common references for an introduction to Algebraic Quantum Field Theory (AQFT) are the classic monographs [Haa1992, Ara1999]. Contemporary overviews and developments can be found in [BDFY2015, FR2019] and the connection to quantum measurement theory has been investigated in [FV2018, Few2019].

By a spacetime M , we always mean a globally hyperbolic Lorentzian manifold (Definition 1.3.8 of [BGP2007]), which because of its causality properties qualifies for a physically reasonable model. In such a spacetime, for example, closed causal curves are forbidden, and we find global time functions

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and spacelike Cauchy hypersurfaces (Definition 1.3.4 of [BGP2007]) representing submanifolds of constant time. More precisely, globally hyperbolic Lorentzian manifolds are isometric to $(\mathbb{R} \times S, -\beta dt^2 + g_t)$ with $\beta \in C^\infty(\mathbb{R} \times S, \mathbb{R}_{>0})$ and g_t a Riemannian metric on S . This metric depends smoothly on t , and each $\{t\} \times S$ is a smooth spacelike Cauchy hypersurface of M [BS2005]. From an analytic point of view, it is also sensible to work with these manifolds since well-posed Cauchy problems are admitted for linear field equations describing frequently investigated quantum fields [Fur1999, AB2018].

The local algebras of observables are most commonly modelled as C^* -algebras (for a brief introduction see chapter 1 of [BF2009]; the approach using $*$ -algebras is given, for instance, in [KM2015]). According to this model, states are introduced only afterwards as normed and positive functionals τ , where $\tau(a)$ can be thought of as the expectation value of the observable a in the state τ . The induced GNS-representation $(\pi_\tau, \mathcal{H}_\tau, \Omega_\tau)$ provides the familiar framework of a state space \mathcal{H}_τ with the observables represented as bounded operators $\pi_\tau(a)$ and a cyclic vector Ω_τ (see section 2.3 of [BR2002] for details). Hence, the selection of a distinct vacuum is shifted to that of an algebraic state τ . The pure states then correspond to the extreme points of the convex set of all states, which are equivalently characterized by certain irreducibility properties of their GNS-representation (see sections 1.8 and 2.3 of [Ara1999]).

One crucial ingredient of AQFT is the aspect of covariance, implemented by translating symmetries of the spacetime, i.e. isometries, into algebra-automorphisms. A possible vacuum state would be considered to be invariant under these automorphisms, and indeed, an appropriately large isometry group does single out such a state. For instance, the existence of a timelike Killing field, meaning that M is stationary, would suffice (see for instance [Wal1994] or [KM2015]). However, general spacetimes have no non-trivial symmetries at all, so covariance in the AQFT-sense is a trivial demand. On the other hand, full general covariance, i.e. if the group of isometries were replaced by the diffeomorphism group, is not compatible with the local structure of the theory. Therefore, covariance has to be built in more subtly.

Whereas certain no-go-theorems demonstrate the absence of a distinct vacuum or natural state in any reasonable sense [FV2012a], the approach of locally covariant quantum field theory initiated by [BFV2003] yields a suitable local and covariant generalization of the principles of AQFT to general spacetimes. Instead of fixing a spacetime from the beginning, a whole category of spacetimes is considered with arrows given by certain isometric embeddings. The actual quantum field theory is then represented by a covariant functor to the category of C^* -algebras and injective C^* -homomorphisms. Thus, covariance is implemented via isometric embeddings that correspond to C^* -algebra-monomorphisms, rather than via isometries of a fixed spacetime and C^* -algebra-automorphisms. This emphasizes the local character of the theory yet more strongly.

For the category of spacetimes, we adopt the setting of [BG2011], where the field is also taken as a datum instead of being fixed from the outset:

Definition 1.2.1. *The category GlobHypGreen consists of the following objects and morphisms:*

- An object is a triple (M, E, P) , where
 - ▶ M is globally hyperbolic Lorentzian manifold,
 - ▶ E is a finite-dimensional, real vector bundle over M with a non-degenerate inner product,
 - ▶ $P: C^\infty(M, E) \rightarrow C^\infty(M, E)$ is a formally self-adjoint, Green hyperbolic operator.
- A morphism between two objects (M_1, E_1, P_1) and (M_2, E_2, P_2) is a pair (f, F) , where
 - ▶ f is a time-orientation preserving, isometric embedding $M_1 \rightarrow M_2$ with $f(M_1) \subset M_2$ causally compatible and open,
 - ▶ F is a fiberwise isometric vector bundle isomorphism over f such that P_1, P_2 are related via $P_1 \circ \text{res} = \text{res} \circ P_2$, where $\text{res}(\varphi) := F^{-1} \circ \varphi \circ f$ the restriction of $\varphi \in C^\infty(M_2, E_2)$ to M_1 .

Note that the usual demand of a well-posed Cauchy problem for P is replaced by the weaker assumption of Green-hyperbolicity, that is, P admits an advanced and retarded Green operator G_A, G_R . For a thorough discussion of Green-hyperbolic operators, we refer to [Bär2015]). Furthermore, the focus on real vector bundles reflects the restriction to Hermitian fields, implying that, for instance, charged fields are not considered.

This thesis will mostly deal with wave operators, which are Green-hyperbolic, as is shown in [BGP2007], and bosonic quantum field theory. Therefore, we only sketch the bosonic quantization scheme given in section 3.1 of [BG2011]. For $G := G_A - G_R$, Theorem 3.5 of [BG2011] provides the exact sequence

$$\{0\} \longrightarrow \mathcal{D}(M, E) \xrightarrow{P} \mathcal{D}(M, E) \xrightarrow{G} C_{sc}^\infty(M, E) \xrightarrow{P} C_{sc}^\infty(M, E), \quad (1.1)$$

and hence, it leads to a covariant functor into the category of symplectic vector spaces with objects (V, σ) essentially given by the solution space of the field equation

$$V := \mathcal{D}(M, E) / \ker G \cong \ker P|_{C_{sc}^\infty}, \quad \sigma([\varphi], [\psi]) := (G\varphi, \psi)_M. \quad (1.2)$$

$\mathcal{D}(M, E)$ represents test sections in E , more precisely smooth sections with compact support, and $C_{sc}^\infty(M, E)$ those with only spacelike compact support, meaning that it is contained in the causal future and past of some compact subset of M . The L^2 -product $(\cdot, \cdot)_M$ of test sections is induced by the non-degenerate inner product on E .

Every real symplectic vector space (V, σ) admits a CCR-representation, i.e. a pair (w, A) consisting of a C^* -algebra A and a map w from V into the unitary elements of A such that A is generated as a C^* -algebra by $\{w(x)\}_{x \in V}$ and the Weyl relations hold:

$$w(x)w(y) = e^{-\frac{i}{2}\sigma(x,y)}w(x+y), \quad x, y \in V. \quad (1.3)$$

This construction goes back to [Man1968] (see also section 4.2 of [BGP2007] and 5.2.2.2 of [BR2002]), and it is unique in an appropriate sense, so we refer to A as the CCR-algebra $\text{CCR}(V)$ of the symplectic space V . Thus, altogether, $(M, E, P) \mapsto \text{CCR}(\mathcal{D}(M, E) / \ker G)$ provides the desired functor.

1.3 Quasifree states

The GNS-representation $(\pi_\tau, \mathcal{H}_\tau, \Omega_\tau)$ of a state τ on the CCR-algebra of some real symplectic vector space (V, σ) provides unitary operators $\pi_\tau(w(x))$ on the induced Hilbert space \mathcal{H}_τ . There is a subclass of states τ for which the unitary one-parameter group $\{\pi_\tau(w(tx))\}_{t \in \mathbb{R}}$ is strongly continuous for all $x \in V$, and hence, field operators $\Phi_\tau(x)$ can be defined as the corresponding self-adjoint generators by Stone's theorem (Theorem VIII.8 of [RS1980]). Furthermore, they ensure the existence of a dense domain $D_\tau \subset \mathcal{H}_\tau$ such that $\text{ran } \Phi_\tau(x) \subset D_\tau \subset \text{dom } \Phi_\tau(x)$ for all x , and thus, polynomials of field operators are well-defined on D_τ . Then the Weyl relations (1.3) imply the familiar canonical commutator relations

$$[\Phi_\tau(x), \Phi_\tau(y)] = i\sigma(x, y) \cdot \text{id}_{\mathcal{H}_\tau}, \quad x, y \in V, \quad (1.4)$$

and moreover, for all $n \in \mathbb{N}$, the n -point-function of the state

$$\tau_n(x_1, \dots, x_n) := \langle \Phi_\tau(x_1) \dots \Phi_\tau(x_n) \Omega_\tau, \Omega_\tau \rangle_{\mathcal{H}_\tau}, \quad x_1, \dots, x_n \in V, \quad (1.5)$$

represents a well-defined distribution $\mathcal{D}(M, E) \times \dots \times \mathcal{D}(M, E) \rightarrow \mathbb{R}$ (see section 4.2 of [BG2011] for precise definitions and proofs).

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Particularly adapted to free quantum fields are the so-called quasifree states, which have been first introduced in [Rob1965] and treated in the framework of CCR-representations in [MV1968]. They are generated by

$$\tau(w(x)) = e^{-\frac{1}{2}\eta(x,x)}, \quad x \in V,$$

for some scalar product η on V , by which the GNS-representation of τ is determined up to unitary equivalence. An overview of quasifree states and their properties can be found, for instance, in chapter 17 of [DG2013], from where we just present the most important facts. Although not regular in the strong sense of [BG2011], a quasifree state τ still provides self-adjoint field operators and n -point-functions. In particular, the two-point-function is of the form

$$\tau_2(x, y) = \eta(x, y) + \frac{i}{2}\sigma(x, y), \quad x, y \in V, \quad (1.6)$$

so it reproduces η and hence τ . Indeed, we have $\tau_n = 0$ for n odd, and for n even, (1.5) is given by some polynomial in the elements of $\{\tau_2(x_i, x_j)\}_{i,j=1,\dots,n}$. Thinking of the n -point-functions as the propagation of the state of the field, the focus on quasifree states corresponds to the perception that this propagation is essentially given by independent one-particle-propagations, which legitimates them as the natural objects to look at when dealing with free quantum fields.

For the real symplectic vector space provided by the functor (1.2), a scalar product corresponds to a bidistribution $S: \mathcal{D}(M, E) \times \mathcal{D}(M, E) \rightarrow \mathbb{R}$ with

$$S[P\psi_1, \psi_2] = 0 = S[\psi_1, P\psi_2], \quad S[\psi_1, \psi_2] = S[\psi_2, \psi_1], \quad S[\psi, \psi] \geq 0, \quad (1.7)$$

for all $\psi, \psi_1, \psi_2 \in \mathcal{D}(M, E)$, since $\ker G = \text{ran } P|_{\mathcal{D}(M, E)}$ by exactness of (1.1). With regard to (1.2) and (1.6), a quasifree state is then determined by S and G .

1.4 Hadamard states

The fundamental reasoning of general relativity – condensed in J. A. Wheeler’s statement "Space tells matter how to move and matter tells space how to curve" – motivates the investigation in the back reaction effect on the spacetime induced by the particles the curved spacetime itself creates. This would require a semiclassical Einstein equation

$$G = 8\pi \langle T \rangle, \quad (1.8)$$

where G in this instance denotes the Einstein tensor describing the spacetime curvature and $\langle T \rangle$ the expectation value of the energy-momentum tensor with respect to the state of the particle field. Since T is quadratic in the field Φ , which is an operator-valued distribution in the quantum description, corresponding to Φ_τ in (1.4) and (1.5), this expression needs some renormalization, as products of distributions are in general ill-defined. Within Minkowski quantum field theory, normal ordering of the fields yields a satisfactory procedure, basically given by subtracting the (infinite) vacuum energy. This method or, to be more precise, the expectation value of T with respect to the ground state of the field would require a preferred choice of vacuum, which is not at our disposal for general spacetimes. Among many alternative renormalization approaches (see the classic monographs [BD1984, Ful1989] for an overview), an axiomatic ansatz like the one suggested by Wald [Wal1977, Wal1978, Wal1994] has been widely accepted as most natural and general. Wald’s axioms determine $\langle T \rangle$ in a largely satisfactory sense, and furthermore, he proposes a concrete procedure, fulfilling the axiomatic framework.

Although normally, there is no such thing as a vacuum expectation value to be subtracted, we can, more generally, focus on differences of expectation values given by the application of states τ to T within the algebraic framework. Note that the expectation value of the squared field at x corresponds to $\tau_2(x, x)$, which we consider as a limit $\lim_{y \rightarrow x} \tau_2(x, y)$ of the well-defined distributions $\tau_2(x, y)$, $x \neq y$. Thus, we can in fact adopt the idea of renormalization on Minkowski spacetime by restricting ourselves to a class of quasifree states such that the expectation values of all products $\Phi(x)\Phi(y)$ have the same "singularity structure" in the sense that subtracting them from one another provides a smooth expression. J. Hadamard's theory of second order hyperbolic equations [Had1923] indeed leads to a family of bisolutions with a fixed local singular part, i.e. for x, y "close", the difference of two Hadamard bisolutions is smooth and the limit for $y \rightarrow x$ exists (see [Wal1994] and the more recent book [Hac2016] for details). The renormalization of $\langle T \rangle$ using this approach has been carried out, for instance, in [DF2008].

Accordingly, a state τ is called a Hadamard state if τ_2 has the Hadamard singularity structure. If we assume that the Hadamard property holds in a whole neighborhood of a spacelike Cauchy hypersurface – that is that no additional singularities arise for spacelike separated pairs of spacetime points – then this is invariant under Cauchy evolution [FSW1978], meaning that it then holds in some neighborhood of every Cauchy hypersurface. This additional assumption is referred to as the global Hadamard condition [NO1985, GK1989], and any globally hyperbolic spacetime admits a large class of pure (global) Hadamard states [FNW1981, SV2001].

The first mathematically precise definition of the Hadamard singularity structure have been specified in [KW1991], in which the authors also show that for a wide class of spacetimes the Hadamard property singles out an invariant quasifree state. Moreover, in any spatially compact spacetime ("closed universes"), all Hadamard states, more specifically their GNS representations, comprise one unitary equivalence class. For general spacetimes, this suggests a certain local indistinguishability, and indeed, the restrictions of quantum field constructions given by two Hadamard states on some relatively compact spacetime region turn out to be unitarily equivalent [Ver1994]. Hence, although there is no distinct vacuum, all possible notions are equivalent in the sense that inequivalent constructions can be only distinguished by measurements over unbounded regions of spacetime. In addition, Hadamard states yield finite fluctuations for all Wick polynomials [BF2000], which makes them relevant also for the perturbative construction of interacting fields (see also [HW2015, Rej2016, Düt2019] and references therein). For instance, on ultrastatic slabs with compact Cauchy hypersurface, also the converse implication holds [FV2013], and hence, under certain circumstances, there is an alternative characterization of Hadamard states less strongly tied to ultrashort distance behaviour.

Consequently, Hadamard states are by now considered a reasonable counterpart of Minkowski finite energy states and the Hadamard condition an appropriate generalization of the energy condition for Minkowski quantum field theory. Note that the replacement of a distinct vacuum state by a whole class of states somehow reflects the essence of general relativity: Just like there is no preferred coordinate system, the concept of vacuum and particles as absolute quantities has to be re-evaluated and eventually downgraded to one choice among many.

It was Radzikowski who showed that for the massive scalar field the global Hadamard condition is equivalent to a certain requirement on the wave front set of the two-point-function [Rad1992, Rad1996a], namely

$$\text{WF}(\tau_2) = \{(p, -\xi; q, \zeta) \in (T^*M \times T^*M) \setminus \{0\} \mid (p, \xi) \sim (q, \zeta), \xi \text{ is future-directed}\}, \quad (1.9)$$

where

$$(p, \xi) \sim (q, \zeta) \iff \begin{aligned} &\exists \text{ lightlike geodesic } c: I \rightarrow M \text{ and } t, t' \in I: \\ &c(t) = p, \quad c(t') = q, \quad \dot{c}(t) = \xi^\sharp, \quad \dot{c}(t') = \zeta^\sharp. \end{aligned} \quad (1.10)$$

Here we adapted the original formulation with regard to the convention used in this thesis. Furthermore, he proved equivalence of the local and global Hadamard condition [Rad1996b], and hence, (1.9) represents a local formulation of the positive energy condition.

Sahlmann and Verch generalized Radzikowski's equivalence for sections in general vector bundles [SV2001], in particular for Hadamard states of the Dirac field in the sense of [Köh1995, Kra2000, Hol2001], which have been used, for instance, for a mathematical rigorous description of the chiral anomaly [BS2016]. In addition, [SVW2002] proposed an even more elegant characterization of the Hadamard property in terms of Hilbert space valued distributions $\varphi \mapsto \Phi_\tau[\varphi]\Omega_\tau \in \mathcal{H}_\tau$, involving the GNS-representation induced by τ . Also for non-quasifree states, one can formulate (1.9) as a constraint on the whole n -point-function, which is compatible with the special case of quasifree states [San2010]. Moreover, in analytic spacetimes, this generalized Hadamard condition can be sharpened to a condition on the analytic wave front set, thereby implying the Reeh-Schlieder-property [SVW2002]. Likewise, for non-globally hyperbolic spacetimes, there is a formulation of the Hadamard condition via restriction to globally hyperbolic subregions. Hadamard states have therefore been studied in connection with the Casimir effect and on anti-de Sitter spacetime (see [DNP2014, DFM2018] and references therein). By using the weaker concept of Sobolev wave front sets, a definition of adiabatic states on globally hyperbolic spacetimes similar to (1.9) is given in [JS2002], thus implying that Hadamard states are adiabatic. However, most importantly for the purpose of this thesis, the Hadamard condition in the form (1.9) allows us to employ the techniques of microlocal analysis provided by Duistermaat and Hörmander [DH1972]. Soon after Radzikowski's work, Junker derived pure Hadamard states for the massive scalar field on spatially compact globally hyperbolic spacetimes, using a factorization of the Klein-Gordon operator by pseudo-differential operators [Jun1996, Jun2002]. Gérard, Wrochna et al. generalized this construction to a large class of spacetimes [GOW2017] and even gauge fields [GW2015]. Furthermore, they proved the existence of (not necessarily pure) Hadamard states [GW2014] in a much more concrete manner than [FNW1981]. See [Gér2019] for a recent review of these techniques.

On the other hand, there have been further proposals for physically reasonable states like the Sorkin-Johnston-states [AAS2012], which in general lack the Hadamard property [FV2012b]. Nevertheless, a modification of their construction produces Hadamard states [BF2014]. For a contemporary synopsis concerning preferred vacuum states on general spacetimes, the nature of the Hadamard property and this construction in particular, see also [Few2018]. Apart from these rather general prescriptions, Hadamard states have been constructed explicitly for a large variety of spacetimes with special (asymptotic) symmetries, and furthermore, well-established states have been tested for the Hadamard property (see the introduction sections of [GW2014, GOW2017] and the references therein as well as section 8.4 of [FV2015] and 2.4 of [Hac2016] for an overview).

1.5 Subject of the thesis

In their seminal work [DH1972], Duistermaat and Hörmander showed the existence of distinguished two-sided parametrices classified by their singularity structure for a huge class of manifolds and operators acting on (real-valued) functions. In the case of linear wave operators P and spacetime dimension $d \geq 3$, this singles out four parametrices $\tilde{G}_A, \tilde{G}_R, \tilde{G}_F, \tilde{G}_{aF}$, which correspond to the familiar advanced, retarded, Feynman and anti-Feynman Green operator for linear wave equations on Minkowski space. With $\Delta' := \{(p, \xi; p, -\xi)\}$ the primed diagonal, they are characterized by

$$\begin{aligned} \text{WF}(\tilde{G}_A) &= \Delta' \cup \{(p, \xi) \sim (q, -\zeta), q \in J_+(p)\}, & \text{WF}(\tilde{G}_R) &= \Delta' \cup \{(p, \xi) \sim (q, -\zeta), q \in J_-(p)\}, \\ \text{WF}(\tilde{G}_F) &= \Delta' \cup \{(p, \xi) \sim (q, \zeta), t > t'\}, & \text{WF}(\tilde{G}_{aF}) &= \Delta' \cup \{(p, \xi) \sim (q, \zeta), t < t'\}. \end{aligned} \tag{1.11}$$

However, Theorem 6.5.3 of [DH1972] ensures the existence of these objects only as parametrices, i.e. Green operators up to smoothing. As it happens, on globally hyperbolic Lorentzian manifolds, there is exactly one advanced and retarded Green operator G_A, G_R for P [BGP2007], but the authors of [DH1972] point out that they do not see how to fix the C^∞ -indetermination for \tilde{G}_F and \tilde{G}_{aF} .

On the other hand, for any bidistribution \tilde{H} of Hadamard form (1.9), which is a bisolution up to C^∞ and whose antisymmetric part is given by $\frac{i}{2}(\tilde{G}_A - \tilde{G}_R)$, Radzikowski proved that $i\tilde{H} + \tilde{G}_A$ yields a Feynman parametrix in the sense of (1.11), that is $\text{WF}(i\tilde{H} + \tilde{G}_A) = \text{WF}(\tilde{G}_F)$ (Theorem 5.1 of [Rad1996a]). Furthermore, section 6.6 of [DH1972] provides the identity $\tilde{G}_F + \tilde{G}_{aF} = \tilde{G}_A + \tilde{G}_R$, so Feynman and anti-Feynman parametrices can be extracted from $\tilde{H}, \tilde{G}_A, \tilde{G}_R$ via

$$\tilde{G}_F = i\tilde{H} + \tilde{G}_A, \quad \tilde{G}_{aF} = -i\tilde{H} + \tilde{G}_R, \quad (1.12)$$

and hence, \tilde{H} has to be of the form

$$\tilde{H} = \frac{i}{2}(\tilde{G}_{aF} - \tilde{G}_F + \tilde{G}_A - \tilde{G}_R). \quad (1.13)$$

Moreover, $\frac{i}{2}(\tilde{G}_{aF} - \tilde{G}_F)$ automatically fulfills (1.7) due to Theorem 6.6.2 of [DH1972] up to smooth functions. Therefore, the characterization of Hadamard states by means of microlocal analysis almost directly provides a further existence proof of (not necessarily pure) Hadamard states by employing the existence of distinguished parametrices and their positivity properties (Theorems 6.5.3 and 6.6.2 of [DH1972]). Compared, for instance, to the deformation argument in [FNW1981], this approach is rather constructive and furthermore, it covers the cases of analytic spacetimes, which, in general, can not be treated by local deformations. However, Radzikowski remarked that it is not clear how to prove that one may choose the smooth functions such that (1.7) is exactly satisfied [Rad1992]. This issue is clearly related to the previously mentioned C^∞ -indetermination via (1.13).

This thesis solves both problems in the setting given by Definition 1.2.1. We construct bisolutions $S: \mathcal{D}(M, E) \times \mathcal{D}(M, E^*) \rightarrow \mathbb{R}$ with singularity structure equal to $\text{WF}(\tilde{G}_{aF} - \tilde{G}_F)$ such that Hadamard bisolutions as well as Feynman and anti-Feynman Green operators are determined via $S + \frac{i}{2}(G_A - G_R)$ and (1.12). The crucial properties of S are invariant under the addition of some smooth bisolution, and our construction will determine bisolutions only up to this degree of freedom. The basic idea for that is to follow the lines of [BGP2007] and deduce S from the well-known object $\frac{i}{2}(G_{aF} - G_F)$ for \square on Minkowski space via local Hadamard series. The derivation of well-defined local parametrices and the following propagation procedure to globally defined bisolutions involve several choices of objects like local domains, covers, cut-offs etc., which are canonical only in the sense that the results arising from two different choices merely differ by some smooth bisolution.

Provided that Theorem 6.6.2 of [DH1972] also holds for corresponding operators acting on sections in some Riemannian vector bundle, we then show that there is a choice of bisolutions which fulfill (1.7) and hence lead to Hadamard two-point functions. By this, we mean that for each S there is a smooth bisolution u such that $S + u$ has these properties. Conversely, given Green operators G_F, G_{aF} , we provide a criterion for the existence of such a choice for more general differential operators.

Before approaching this construction, some preparation has to be done. In the first half of chapter 2, we classify certain \mathcal{L}_+^\uparrow -invariant distributions on Minkowski space and then construct families of them containing fundamental solutions for \square later identified as the distinguished parametrices. In the second half, we prove well-posedness of the Cauchy-problem for singular sections and smooth bisections, which will be needed for the eventual globalization procedure. Chapter 3 generalizes symmetry of the Hadamard coefficients for formally self-adjoint P given in [Mor2000] to the vector-valued case.

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Afterwards, in chapter 4, we start with the actual construction by providing explicit and local expressions for the Feynman and anti-Feynman fundamental solution in the prototype case $(\square, \mathbb{R}_{\text{Mink}}^d)$. We identify them as members of the previously derived families of \mathcal{L}_+^\uparrow -invariant and homogeneous distributions just like the advanced and retarded fundamental solution are represented by Riesz distributions. As depicted in chapter 5, this allows the construction of parametrices on small domains O of any Lorentzian manifold M via Hadamard series such that the singularity structure transforms naturally. Unlike the advanced and retarded fundamental solution, the objects constructed here are far from unique. Thus, in order to ensure that they produce actual two-sided parametrices, we need symmetry of the coefficients, and therefore, with regard to chapter 3, we assume formal self-adjointness of P . This leads to local Feynman and anti-Feynman parametrices and hence Hadamard bidistributions via (1.13). For M globally hyperbolic, we moreover derive local bisolutions which have the Hadamard property. Chapter 6 then finally provides the global construction on globally hyperbolic spacetimes. Here, the well-posed Cauchy problems derived in section 2.3 are the crucial instruments for the propagation as bisolutions to $M \times M$ and moreover for the preservation of the singularity structure at each propagation step. We construct (singular) bisolutions on domains of the form $O \times M$, which cover $M \times M$. We show that there are local choices of bisolutions that fit together on the overlaps and thus constitute a globally well-defined object. Altogether, we obtain global Hadamard bisolutions and finally prove that each of them can be chosen as an actual Hadamard two-point-function.

2 Preliminaries

"But time, it's on your side now."

After having briefly fixed notations and conventions (mostly by adopting [BGP2007]), we derive a classification of \mathcal{L}_+^\uparrow -invariant distributions on Minkowski space supported on the light cone C . Solutions and fundamental solutions for \square are \mathcal{L}_+^\uparrow -invariant with singular support contained in C , and thus, coincidence of two such objects can be checked directly outside of that set. In that case, their difference is a \mathcal{L}_+^\uparrow -invariant distribution supported in C , and we are going to exhibit criteria for equality also there. Therefore, we employ the close relation between \mathcal{L}_+^\uparrow -invariant distributions on $\mathbb{R}^d \setminus \{0\}$ and distributions on \mathbb{R} , and we trace back our setting to the well-known classification of distributions supported in $\{0\}$. Afterwards, we construct fundamental solutions for \square , which, by means of the previous classification, will later be revealed as the distinguished fundamental solutions corresponding to (1.11). In the second half, we use well-posedness of the smooth and singular Cauchy problem, provided in [BGP2007, BF2009, BTW2015], in order to derive a well-posed Cauchy problem for smooth bisections, and furthermore, we show propagation of singular solutions in a suitable sense.

2.1 Notations and conventions

For any d -dimensional vector space with non-degenerate inner product $\langle\langle \cdot, \cdot \rangle\rangle$ of index 1, i.e. isometric to d -dimensional Minkowski space, we adopt the notations and conventions of [BGP2007], that is, for instance, the signature $(-, +, \dots, +)$ and the squared distance

$$\gamma(x) := -\langle\langle x, x \rangle\rangle = x_0^2 - \sum_{j=1}^{d-1} x_j^2, \quad x = (x_0, \dots, x_{d-1}) \in V. \quad (2.1)$$

The two connected components I_\pm of the set of timelike vectors $I := \{\gamma(x) > 0\}$ then determine a time-orientation. Correspondingly, we set $C_\pm := \partial I_\pm$, $J_\pm := \overline{I_\pm}$, whose non-zero elements we call "lightlike" and "causal", respectively. Leaving out " \pm " means the union of both components, i.e. $I := I_+ \cup I_-$ and similarly C and J . The zero-vector and all elements of $\{\gamma(x) < 0\}$ are referred to as "spacelike".

For M a d -dimensional time-oriented Lorentzian manifold and $p \in M$, we write $I_\pm^M(p)$, $C_\pm^M(p)$ and $J_\pm^M(p)$ for the corresponding chronological/lightlike/causal future/past of p . These sets comprise all points that can be reached from p via timelike/lightlike/causal future/past directed differentiable curves, that is, curves with tangent vectors of the respective type at each point. For subsets $A \subset M$, we define $I_\pm^M(A) := \bigcup_{p \in A} I_\pm^M(p)$ and similarly $J_\pm^M(A)$. For the definitions of different types of subsets of M like future/past compact, geodesically starshaped, convex, causally compatible, causal, Cauchy hypersurface etc., we refer to section 1.3 of [BGP2007]. Moreover, we point out that in the whole thesis a Cauchy hypersurface of M is always assumed to be spacelike.

For E some real or complex finite-dimensional vector bundle over M , the spaces of C^k -, C^∞ -, \mathcal{D} -sections in E as well as distributional sections $\mathcal{D}'(M, E, W)$ with values in some finite-dimensional space W , including their (singular) support, convergence, order etc., are defined as in section 1.1 of [BGP2007].

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For F another vector bundle over M and $P: C^\infty(M, E) \rightarrow C^\infty(M, F)$ some linear differential operator, the formally transposed operator $P^t: C^\infty(M, F^*) \rightarrow C^\infty(M, E^*)$ of P is given by

$$(P^t\varphi)[\psi] := \varphi[P\psi] = \int_M \varphi(P\psi) dV, \quad \psi \in \mathcal{D}(M, E), \varphi \in \mathcal{D}(M, F^*).$$

dV denotes the volume density induced by the Lorentzian metric. Let E be equipped with a non-degenerate inner product $\langle \cdot, \cdot \rangle$, which induces the L^2 -product of test sections

$$(\psi_1, \psi_2)_M := \int_M \langle \psi_1(p), \psi_2(p) \rangle dV(p), \quad \psi_1, \psi_2 \in \mathcal{D}(M, E), \quad (2.2)$$

and the isomorphism $\Theta: E \rightarrow E^*$. We call P formally self-adjoint if $(P\psi_1, \psi_2)_M = (\psi_1, P\psi_2)_M$ for all ψ_1, ψ_2 , that is, $P = \Theta^{-1}P^t\Theta$. The focus of this thesis lies on wave operators $P: C^\infty(M, E) \rightarrow C^\infty(M, E)$, i.e. linear differential operators of second order with, effectively, scalar principal symbol determined by the metric, namely $\xi \mapsto g(\xi^\sharp, \xi^\sharp) \cdot \text{id}_E$, $\xi \in T^*M$ (see section 1.5 of [BGP2007] for details).

L. Schwartz' kernel theorem (Theorem 5.2.1 of [Hör1990]) establishes a one-to-one-correspondence between bidistributions $K: \mathcal{D}(M, E) \times \mathcal{D}(M, E^*) \rightarrow \mathbb{R}$ and linear, sequentially continuous operators $\mathcal{K}: \mathcal{D}(M, E^*) \rightarrow \mathcal{D}(M, E^*)'$, that is, $\mathcal{K}\varphi_j \rightarrow \mathcal{K}\varphi$ if $\varphi_j \rightarrow \varphi$, given by

$$K[\psi, \varphi] = (\mathcal{K}\varphi)[\psi], \quad \psi \in \mathcal{D}(M, E), \varphi \in \mathcal{D}(M, E^*). \quad (2.3)$$

K is called the Schwartz kernel of \mathcal{K} , and it is represented by a distributional section in the bundle $E^* \boxtimes E$ over $M \times M$, whose fibers we identify via

$$(E^* \boxtimes E)_{(p,q)} = E_p^* \otimes E_q \cong \text{Hom}(E_q^*, E_p^*), \quad (p, q) \in M \times M. \quad (2.4)$$

This allows us to define the wave front set of an operator via the wave front set of its Schwartz kernel. Furthermore, we introduce the concept of parametrices for differential operators P , which yields a generalized concept of both, inverse operators and fundamental solutions for P , related by (2.3).

Definition 2.1.1. *Let $P: C^\infty(M, E) \rightarrow C^\infty(M, E)$ be a linear differential operator. A linear and sequentially continuous operator $\mathcal{Q}: \mathcal{D}(M, E^*) \rightarrow C^\infty(M, E^*)$ is called*

- *left parametrix for P^t if $\mathcal{Q}P^t|_{\mathcal{D}} - \text{id}$ is smoothing,*
- *right parametrix for P^t if $P^t\mathcal{Q} - \text{id}$ is smoothing,*
- *(two-sided) parametrix for P^t if \mathcal{Q} is left and right parametrix for P^t ,*
- *Green operator for P^t if $\mathcal{Q}P^t|_{\mathcal{D}} = P^t\mathcal{Q} = \text{id}$.*

A bidistribution $Q: \mathcal{D}(M, E) \times \mathcal{D}(M, E^*) \rightarrow \mathbb{R}$ with $p \mapsto Q(p)[\varphi] \in C^\infty(M, E^*)$ for all φ is called

- *left parametrix for P at $p \in M$ if $P_{(2)}Q(p) - \delta_p \in C^\infty(M, E)$,*
- *(two-sided) parametrix for P at $p \in M$ if Q is a left parametrix for P at p , and for all $\varphi \in \mathcal{D}(M, E^*)$, we have $P_{(1)}^t(Q(\cdot)[\varphi]) - \varphi \in C^\infty(M, E^*)$,*
- *fundamental solution for P at $p \in M$ if $P_{(2)}Q(p) = \delta_p$.*

Note that \mathcal{Q} is a (left) parametrix for P^t if its Schwartz kernel $Q[\psi, \varphi] := \mathcal{Q}\varphi[\psi]$ is a (left) parametrix for P^t at all $p \in M$. In this thesis, we will mostly refer synonymously to a parametrix as an operator \mathcal{Q} or the bidistribution Q given by its Schwartz kernel.

2.2 Fundamental solutions for d'Alembert's operator on Minkowski space

2.2.1 Lorentz-invariant distributions on the light cone

Let $\mathring{\mathbb{R}}^d := \mathbb{R}^d \setminus \{0\}$ and $J_{\pm}^c := \mathring{\mathbb{R}}^d \setminus J_{\pm}$. We find submersions $\gamma_{\pm} := \gamma|_{J_{\mp}^c} : J_{\mp}^c \rightarrow \mathbb{R}$, whose preimages

$$\gamma_{\pm}^{-1}(\kappa) = \{\gamma(x) = \kappa\} \cap J_{\mp}^c =: H_{\kappa}^{\pm}, \quad \kappa \in \mathbb{R}, \quad (2.5)$$

foliate J_{\mp}^c . In particular, we have $\gamma_{+}^{-1}|_{\mathbb{R}_{<0}} = \gamma_{-}^{-1}|_{\mathbb{R}_{<0}}$. Recall that any $\mathcal{L}_{+}^{\uparrow}$ -invariant function $\phi \in C^{\infty}(\mathring{\mathbb{R}}^d)$ is constant on those preimages, i.e. $\phi(H_{\kappa}^{\pm}) = \{c_{\kappa}^{\pm}\}$, and thus determines a pair of functions

$$\phi_{\pm} : \quad \mathbb{R} \longrightarrow \mathbb{R}, \quad \kappa \longmapsto c_{\kappa}^{\pm}, \quad (2.6)$$

which satisfy $\phi_{+}|_{\mathbb{R}_{<0}} = \phi_{-}|_{\mathbb{R}_{<0}}$ and $\gamma_{\pm}^* \phi_{\pm} = \phi|_{J_{\mp}^c}$. Conversely, any such pair ϕ_{\pm} induces a $\mathcal{L}_{+}^{\uparrow}$ -invariant function $\phi : \mathring{\mathbb{R}}^d \rightarrow \mathbb{R}$ via $\phi|_{J_{\mp}^c} := \gamma_{\mp}^* \phi_{\pm}$.

Since we are interested in the more general case of distributions, we consider a pair $T_{\pm} \in \mathcal{D}(\mathbb{R})'$ with $T_{+}|_{\mathbb{R}_{<0}} = T_{-}|_{\mathbb{R}_{<0}}$. By Theorem 10.18 of [DK2010], the pushforward of $\varphi \in \mathcal{D}(\mathbb{R}^d)$ along the submersion γ_{\pm} yields a continuous linear map $\mathcal{D}(J_{\mp}^c) \rightarrow \mathcal{D}(\mathbb{R})$ with $((\gamma_{\pm})_* \varphi)(\kappa)$ given by integration of φ over the submanifold $\gamma_{\pm}^{-1}(\kappa)$ for all $\kappa \in \mathbb{R}$. Thus, we can pull back T_{\pm} along γ_{\pm} via

$$\gamma_{\pm}^* T_{\pm}[\varphi] := T_{\pm}[(\gamma_{\pm})_* \varphi], \quad \varphi \in \mathcal{D}(J_{\mp}^c), \quad (2.7)$$

and hence, we obtain a $\mathcal{L}_{+}^{\uparrow}$ -invariant distribution $T \in \mathcal{D}(\mathring{\mathbb{R}}^d)'$ via $T|_{J_{\mp}^c} := \gamma_{\mp}^* T_{\pm}$. However, for the converse construction, a pointwise definition of T_{\pm} like in the smooth case (2.6) is not available for distributions. Alternatively, pushing forward T along γ_{\pm} by pulling back the test function does not work either since $\text{supp}(\gamma_{\pm}^* \varphi)$ is not compact unless $\varphi = 0$. On the other hand, the only compactly supported $\mathcal{L}_{+}^{\uparrow}$ -invariant distributions are derivatives of δ_0 , so $(\gamma_{\pm})_* T$ is a priori ill-defined for almost all such T . [Met1954] constructs an approximating sequence of $\mathcal{L}_{+}^{\uparrow}$ -invariant modifications of T , for which the pushforward is well-defined and independent of the modification. As a result, the relation generalizes to $\mathcal{L}_{+}^{\uparrow}$ -invariant distributions on $\mathring{\mathbb{R}}^d$ and pairs of distributions on \mathbb{R} that coincide on $\mathbb{R}_{<0}$:

Theorem 2.2.1 (Théorème 2 of [Met1954]). *For any pair $T_{\pm} \in \mathcal{D}(\mathbb{R})'$ with $T_{+}|_{\mathbb{R}_{<0}} = T_{-}|_{\mathbb{R}_{<0}}$, there is a $\mathcal{L}_{+}^{\uparrow}$ -invariant distribution $T \in \mathcal{D}(\mathring{\mathbb{R}}^d)'$ given by $T|_{J_{\mp}^c} := \gamma_{\mp}^* T_{\pm}$. Conversely, for any $\mathcal{L}_{+}^{\uparrow}$ -invariant $T \in \mathcal{D}(\mathring{\mathbb{R}}^d)'$, we find a pair $T_{\pm} \in \mathcal{D}(\mathbb{R})'$ with $T_{+}|_{\mathbb{R}_{<0}} = T_{-}|_{\mathbb{R}_{<0}}$ such that $T|_{J_{\mp}^c} = \gamma_{\mp}^* T_{\pm}$.*

Theorem 2.2.1 is the crucial result for our classification since it translates $\mathcal{L}_{+}^{\uparrow}$ -invariant distributions with support on C into distributions on the real line supported only in $\{0\}$, for which a classification is well-known (e.g. see section 3.2 of [Hör1990]). The particular construction moreover shows that $(\gamma_{\pm})_*$ maps $\mathcal{D}(\{x_0 > 0\})$ surjectively onto $\mathcal{D}(\mathbb{R})$ (see equation (4.6) on page 233 in [Met1954]). For $\kappa \geq 0$, the diffeomorphisms

$$\Phi_{\pm} : \quad I_{\pm} \longrightarrow \{x_0 > 0\}, \quad x \longmapsto (\gamma(x), \hat{x}),$$

with inverse maps $\Phi_{\pm}^{-1}(x) = (\pm \sqrt{x_0 + \|\hat{x}\|^2}, \hat{x})$, $x_0 > 0$, provide the particular expressions

$$((\gamma_{\pm})_* \varphi)(\kappa) = \int_{\mathbb{R}^{d-1}} \frac{\varphi(\pm \sqrt{\kappa + \|\hat{x}\|^2}, \hat{x})}{2\sqrt{\kappa + \|\hat{x}\|^2}} \, d\hat{x}, \quad \varphi \in \mathcal{D}(\mathbb{R}^d). \quad (2.8)$$

In particular, $\text{supp} \varphi \subset J_{\mp}^c \setminus C$ leads to $\text{supp} (\gamma_{\pm})_* \varphi \subset \mathring{\mathbb{R}}$. For any $\mathcal{L}_{+}^{\uparrow}$ -invariant $T \in \mathcal{D}(\mathring{\mathbb{R}}^d)'$ only supported in C , we therefore obtain $T[\varphi] = 0$ for all $\varphi \in \mathcal{D}(J_{\mp}^c \setminus C)$, and thus, $T_{\pm}[(\gamma_{\pm})_* \varphi] = 0$ for the

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corresponding $T_{\pm} \in \mathcal{D}(\mathbb{R})'$, that is $\text{supp } T_{\pm} \subset \{0\}$. Applying Theorem 3.2.4 of [Hör1990] then provides

$$T_{\pm} = \sum_{k=0}^{\infty} a_k^{\pm} \cdot \delta_0^{(k)} \quad \Longrightarrow \quad T|_{J_{\mp}^c} = \sum_{k=0}^{\infty} a_k^{\pm} \cdot \gamma_{\pm}^* \delta_0^{(k)}, \quad (2.9)$$

where $a_k^{\pm} \neq 0$ for only finitely many k and $\gamma_{\pm}^* \delta_0^{(k)}[\varphi] = ((\gamma_{\pm})_* \varphi)^{(k)}(0)$. Since $\{0\}$ represents an orbit of \mathcal{L}_+^{\uparrow} , the classification of all \mathcal{L}_+^{\uparrow} -invariant distributions on \mathbb{R}^d will require one further result:

Theorem 2.2.2 (Théorème 1 of [Met1954]). *Any \mathcal{L}_+^{\uparrow} -invariant $T \in \mathcal{D}(\mathbb{R}^d)'$ with $\text{supp } T \subset \{0\}$ is of the form $T = \sum_{k=0}^{\infty} b_k \cdot \square^k \delta_0$ with $b_k \neq 0$ for only finitely many k .*

Corollary 2.2.3. *Any \mathcal{L}_+^{\uparrow} -invariant measure μ_{\pm} on C_{\pm} is of the form*

$$\mu_{\pm}[\varphi] = a \int_{\mathbb{R}^{d-1}} \frac{\varphi(\pm \|\hat{x}\|, \hat{x})}{2\|\hat{x}\|} d\hat{x} + b\varphi(0), \quad \varphi \in \mathcal{D}(\mathbb{R}^d),$$

for some $a, b \in \mathbb{R}_{\geq 0}$.

Proof. Recall that δ_0 is represented by a measure on \mathbb{R}^d , whereas none of its derivatives is, and by definition of $(\gamma_{\pm})_* \varphi(\kappa)$ as the integral of φ along H_{κ}^{\pm} , the same is true for $\gamma_{\pm}^* \delta_0$. Let

$$d\Omega_{\pm}^0[\varphi] := \int_{\mathbb{R}^{d-1}} \frac{\varphi(\pm \|\hat{x}\|, \hat{x})}{2\|\hat{x}\|} d\hat{x}, \quad \varphi \in \mathcal{D}(\mathbb{R}^d), \quad (2.10)$$

which is well-defined for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, and, due to (2.8) and (2.9), coincides with $(\gamma_{\pm})_* \varphi(0)$ for $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Therefore, Theorem 2.2.2 and the fact that $\square^k \delta_0$ is a measure only for $k = 0$, we obtain $\mu_{\pm} - a d\Omega_{\pm}^0 = b\delta_0$ for some $a, b \in \mathbb{R}$, which is the claim. \square

2.2.2 Riesz distributions

We saw that (2.8) provides an explicit extension for $\gamma_{\pm}^* \delta_0$ to all of \mathbb{R}^d , so with regard to (2.9), for the desired classification, it remains to give an extension also for the pullback of the derivatives $\gamma_{\pm}^* \delta_0^{(k)}$.

According to chapter 13 of [DK2010], such extensions can be derived by regarding $\delta_0^{(k)}$ as holomorphic extensions in k of certain functions. It follows that the pullback can directly be calculated and produces a family of \mathcal{L}_+^{\uparrow} -invariant functions on \mathbb{R}^d , which are holomorphic in k . By the identity theorem of complex analysis, this identifies $\gamma_{\pm}^* \delta_0^{(k)}$ with the distributional extensions of these functions. For all $k \in \mathbb{N}$ and $x \in \mathbb{R}$, we introduce

$$\chi^k(x) := \frac{x^{k-1}}{(k-1)!} \cdot H(x)$$

where H the step function at $x = 0$, which satisfies $H' = \delta_0$. We directly obtain $\partial \chi^{k+1} = \chi^k$, so χ^k yields a fundamental solution of ∂^k and thus leads to a distributional extension χ^{-k} via

$$\chi^{-k} := \partial^{k+1} \chi^1 = \partial^{k+1} H = \delta_0^{(k)}, \quad k \in \mathbb{N}_0.$$

The analytic continuations $e^{\alpha \log x}$ and $\Gamma(\alpha + 1)$ of x^k and $k!$ for $\alpha \in \mathbb{C}$ yield a further generalization

$$\chi^{\alpha}(x) := \frac{e^{(\alpha-1) \log x}}{\Gamma(\alpha)} \cdot H(x), \quad \alpha \in \mathbb{C}. \quad (2.11)$$

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Thus, for fixed $\varphi \in \mathcal{D}(\mathbb{R})$, the map $\alpha \mapsto \chi^\alpha[\varphi]$ is holomorphic on all of \mathbb{C} , so we embedded $\delta_0^{(k)}$ into a family of distributions $\{\chi^\alpha\}_{\alpha \in \mathbb{C}}$, which are continuous for $\operatorname{Re}(\alpha) > 1$. In this spirit, on \mathbb{R}^d and for all $\operatorname{Re}(\alpha) > d$, we define

$$R_{\pm}^{\alpha}(x) = \begin{cases} 2C(\alpha, d) \cdot \gamma(x)^{\frac{\alpha-d}{2}}, & x \in J_{\pm}, \\ 0, & \text{otherwise,} \end{cases}, \quad C(\alpha, d) = \frac{2^{-\alpha} \pi^{\frac{2-d}{2}}}{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha-d}{2} + 1\right)}. \quad (2.12)$$

This yields holomorphic maps $\alpha \mapsto R_{\pm}^{\alpha}[\varphi]$ on $\{\operatorname{Re}(\alpha) > d\}$ for each $\varphi \in \mathcal{D}(\mathbb{R}^d)$, and a simple calculation shows that $\square R_{\pm}^{\alpha+2} = R_{\pm}^{\alpha}$. Hence, we obtain a distributional extension for all $\alpha \in \mathbb{C}$ via

$$R_{\pm}^{\alpha} := \square^k R_{\pm}^{\alpha+2k}, \quad \operatorname{Re}(\alpha) + 2k > d, \quad (2.13)$$

which is independent of k by the identity theorem. The distributions defined by (2.12) and (2.13) are known as the Riesz distributions, and they represent the family of \mathcal{L}_+^{\uparrow} -invariant distributions corresponding to $\{\chi^\alpha\}_{\alpha \in \mathbb{C}}$, i.e. in particular $\{\gamma_{\pm}^* \delta_0^{(k)}\}_{k \in \mathbb{N}_0}$.

Proposition 2.2.4. *For all $\alpha \in \mathbb{C}$, we have*

$$R_{\pm}^{\alpha}|_{J_{\mp}^c} = \frac{2^{1-\alpha} \pi^{\frac{2-d}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)} \cdot \gamma_{\pm}^* \chi^{\frac{\alpha-d}{2}+1}, \quad \alpha \in \mathbb{C}.$$

Moreover, $R_{\pm}^0 = \delta_0$ and R_{\pm}^2 are the unique fundamental solutions of \square with support contained in J_{\pm} .

For the proof, see section 13.2 of [DK2010]. This provides us with the desired explicit expression

$$\gamma_{\pm}^* \delta_0^{(k)} = \gamma_{\pm}^* \chi^{-k} = \frac{\Gamma\left(\frac{d}{2} - (k+1)\right)}{2^{2k+3-d} \pi^{\frac{d-2}{2}}} \cdot R_{\pm}^{d-2(k+1)} \Big|_{J_{\mp}^c}. \quad (2.14)$$

Proposition 2.2.5. *Any \mathcal{L}_+^{\uparrow} -invariant $T \in \mathcal{D}(\mathbb{R}^d)'$ with $\operatorname{supp} T \subset C$ is of the form*

$$T = \sum_{k=0}^{\infty} \left(\lambda_k^+ \cdot R_+^{d-2(k+1)} + \lambda_k^- \cdot R_-^{d-2(k+1)} + b_k \cdot \square^k \delta_0 \right) \quad (2.15)$$

with only finitely many non-vanishing coefficients.

Proof. Note that (2.9) and (2.14) imply that, away from zero, T is given by some linear combination of Riesz distributions:

$$T|_{\mathbb{R}^d} = \sum_{k=0}^{\infty} \left(\lambda_k^+ \cdot R_+^{d-2(k+1)} + \lambda_k^- \cdot R_-^{d-2(k+1)} \right) \Big|_{\mathbb{R}^d}.$$

Due to \mathcal{L}_+^{\uparrow} -invariance of the Riesz distributions, it follows that the difference is a \mathcal{L}_+^{\uparrow} -invariant distribution supported only in $\{0\}$, which, according to Theorem 2.2.2, is of the form

$$T - \sum_{k=0}^{\infty} \left(\lambda_k^+ \cdot R_+^{d-2(k+1)} + \lambda_k^- \cdot R_-^{d-2(k+1)} \right) = \sum_{k=0}^{\infty} b_k \cdot \square^k \delta_0. \quad \square$$

A function $f: \mathbb{R}^d \rightarrow \mathbb{C}$ is homogeneous of degree $\alpha \in \mathbb{C}$ if $f(tx) = t^\alpha f(x)$ for all $t \in \mathbb{R}_{>0}$, $x \in \mathbb{R}^d$. This generalizes to distributions via:

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Definition 2.2.6. A distribution $u: \mathcal{D}(\mathbb{R}^d) \rightarrow \mathbb{C}$ is homogeneous of degree $\alpha \in \mathbb{C}$ if

$$u[\varphi_t] = t^\alpha \cdot u[\varphi], \quad \varphi_t(x) := t^{-d} \cdot \varphi\left(\frac{x}{t}\right), \quad \varphi \in \mathcal{D}(\mathbb{R}^d), \quad t \in \mathbb{R}_{>0}.$$

It directly follows that δ_0 is homogeneous of order $-d$. Note that $\partial^l \varphi_t = t^{-l} (\partial^l \varphi)_t$ for all $l \in \mathbb{N}$, so if u is homogeneous of degree α , then $\partial^l u$ is homogeneous of degree $\alpha - l$. In particular, R_\pm^α is homogeneous of degree $\alpha - d$ and consequently $\square^k \delta_0 = R_\pm^{-2k}$ of degree $-d - 2k$. Hence, (2.15) provides a decomposition of \mathcal{L}_+^\uparrow -invariant distributions T with support on the light cone in terms of homogeneous distributions, which is particularly useful if T itself is homogeneous:

Corollary 2.2.7. Let $T \in \mathcal{D}(\mathbb{R}^d)'$ be \mathcal{L}_+^\uparrow -invariant, supported in C and homogeneous of degree $a > -d$ with $a \neq -2k$ for all $k \in \mathbb{N}$. Then $T = 0$. Moreover, this still holds for all $a = -2k > -d$ if, in addition, T is a solution of d'Alembert's equation and symmetric, i.e. $T(x) = T(-x)$ in the distributional sense.

Proof. The first claim follows immediately from Corollary 2.2.5, since T is supposed to coincide with a finite sum of homogeneous distributions of degree $-2k$, $k \in \mathbb{N}$. Therefore, demanding $T[\varphi_t] = t^a T[\varphi]$ for all $t > 0$ shows that all coefficients have to vanish.

However, for T homogeneous of degree $a = -2k > -d$, (2.15) leads to the form

$$T = \lambda_k^+ \cdot R_+^{d-2k} + \lambda_k^- \cdot R_-^{d-2k}.$$

Due to $R_\pm^\alpha(-x) = R_\mp^\alpha(x)$, symmetry yields $\lambda_k^+ = \lambda_k^- =: \lambda_k$, and thus,

$$0 = \square T = 2\lambda_k \underbrace{(R_+^{d-2k} + R_-^{d-2k})}_{\neq 0} \implies \lambda_k = 0. \quad \square$$

2.2.3 Symmetric fundamental solutions

Due to Corollary 2.2.5, the Riesz distributions are the only \mathcal{L}_+^\uparrow -invariant distributions supported exclusively on the light cone (note that $\square^k \delta_0 = R_\pm^{-2k}$). Moreover, somehow as a side product, we found fundamental solution R_\pm^2 for \square , which are the unique ones with support contained in J_\pm , and therefore, we will later identify them with the advanced and retarded fundamental solution for \square .

In this subsection, we derive a further family of \mathcal{L}_+^\uparrow -invariant distributions from (2.11), which, unlike the Riesz distributions, do not satisfy any support restriction and thus will correspond to the Feynman and anti-Feynman fundamental solution, eventually. To this end, we sketch the construction given in the chapters I.3 and III.2 of [GS1967]. Let always be $d \geq 3$, and for $\varphi \in \mathcal{D}(\mathbb{R}^d)$, let always denote $\psi(r) := \int_{S^{d-1}} \varphi(r\hat{x}) d\hat{x}$, which is smooth at $r = 0$ and hence lies in $\mathcal{D}(\mathbb{R}_{\geq 0})$. For all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and $k \in \mathbb{N}_0$, we define

$$r^k[\varphi] := \int_{\mathbb{R}^d} \|x\|^k \varphi(x) dx = \int_0^\infty r^{k+d-1} \psi(r) dr = \Gamma(k+d) \cdot \chi^{k+d}[\psi],$$

so by holomorphicity of (2.11), $\alpha \mapsto r^\alpha[\varphi]$ extends to a meromorphic function on all of \mathbb{C} with simple poles inherited only from the Γ -function. Note that ψ is an even function, so the residues

$$\operatorname{Res}_{\alpha=-d-k} r^\alpha[\varphi] = \chi^{-k}[\psi] \cdot \operatorname{Res}_{\alpha=-k} \Gamma(\alpha) = \frac{\psi^{(k)}(0)}{k!} \quad (2.16)$$

exist for all $k \in \mathbb{N}_0$ and vanish for odd k , i.e. we obtain simple poles at $\{-d-2k\}_{k \in \mathbb{N}_0}$. We aim at defining complex powers of γ (2.1) in the sense $(\gamma \pm i0)^\alpha$, $\alpha \in \mathbb{C}$, where the branch cut is taken along the negative real axis and $\pm i0$ refers to the respective branch.

2.2 Fundamental solutions for d'Alembert's operator on Minkowski space

Let $\mathbb{H}_\pm := \{\pm \text{Im} > 0\} \subset \mathbb{C}$ denote the open upper and lower half-plane, and $A_\pm := \mathbb{H}_\pm \times \mathbb{H}_\pm$. For $(a_0, a_1) \in A_\pm$, we define the bilinear form

$$Q(a_0, a_1) := a_0 x_0^2 + a_1 \sum_{j=1}^{d-1} x_j^2 \quad (2.17)$$

with the corresponding operator

$$D_{(a_0, a_1)} := \frac{1}{a_0} \frac{\partial^2}{\partial x_0^2} + \frac{1}{a_1} \sum_{j=1}^{d-1} \frac{\partial^2}{\partial x_j^2}. \quad (2.18)$$

If $\text{Re}(\alpha) > 0$, we calculate $D_{(a_0, a_1)} Q(a_0, a_1)^{\alpha+1} = 4(\alpha+1) \left(\alpha + \frac{d}{2}\right) Q(a_0, a_1)^\alpha$, and hence, a distributional extension of (2.17) to all of \mathbb{C} is given by

$$Q(a_0, a_1)^\alpha := \frac{D_{(a_0, a_1)}^k Q(a_0, a_1)^{\alpha+k}}{4^k \prod_{j=1}^k (\alpha+j) (\alpha+j+\frac{d-2}{2})}, \quad \text{Re}(\alpha) + k > 0. \quad (2.19)$$

Note that, due to the identity theorem, this does not depend on k , and furthermore, it actually does not yield any poles for $\text{Re}(\alpha) > -\frac{d}{2}$ since

$$\text{Res}_{\alpha=-m} Q^\alpha = \lim_{\alpha \rightarrow -m} \frac{(\alpha+m) \cdot D_{(a_0, a_1)}^{m+1} Q^{\alpha+m+1}}{4^{m+1} \prod_{j=1}^{m+1} (\alpha+j) (\alpha+j+\frac{d-2}{2})} = \frac{(-1)^{m-1} \cdot D_{(a_0, a_1)}^{m+1} Q}{4^{m+1} \Gamma(m) \cdot \prod_{j=1}^{m+1} (j-m+\frac{d-2}{2})} = 0 \quad (2.20)$$

for all natural numbers $m < \frac{d}{2}$. Thus, for fixed $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the map $(a_0, a_1, \alpha) \mapsto Q(a_0, a_1)^\alpha[\varphi]$ is holomorphic on $A_\pm \times \{\text{Re} > -\frac{d}{2}\}$, and hence, so are the maps

$$\alpha \mapsto (\gamma \pm i\varepsilon)^\alpha[\varphi] := Q(1 \pm i\varepsilon, -1 \pm i\varepsilon)^\alpha[\varphi], \quad \varphi \in \mathcal{D}(\mathbb{R}^d), \quad (2.21)$$

on $\{\text{Re} > -\frac{d}{2}\}$ for fixed $\varepsilon > 0$. Next, we investigate the residues of (2.19) at $\alpha = -\frac{d}{2}$. For fixed α with $\text{Re}(\alpha) > -\frac{d}{2}$, we obtain holomorphic maps

$$(a_0, a_1) \mapsto \text{Res}_{\alpha=-\frac{d}{2}} Q(a_0, a_1)^\alpha[\varphi], \quad (a_0, a_1) \mapsto \frac{\pi^{\frac{d}{2}} \cdot \varphi(0)}{\sqrt{a_0} a_1^{\frac{d-1}{2}} \Gamma\left(\frac{d}{2}\right)}, \quad \varphi \in \mathcal{D}(\mathbb{R}^d). \quad (2.22)$$

On the other hand, for all $\varepsilon > 0$, we have

$$Q(\pm i\varepsilon, \pm i\varepsilon)^\alpha[\varphi] = (\pm i\varepsilon)^\alpha \int_{\mathbb{R}^d} \|x\|^{2\alpha} \cdot \varphi(x) \, dx = (\pm i\varepsilon)^\alpha \cdot r^{2\alpha}[\varphi], \quad \varphi \in \mathcal{D}(\mathbb{R}^d),$$

so by (2.16), the residues of $Q(\pm i\varepsilon, \pm i\varepsilon)^\alpha$ at $\alpha = -\frac{d}{2}$ are essentially given by δ_0 :

$$\text{Res}_{\alpha=-\frac{d}{2}} Q(\pm i\varepsilon, \pm i\varepsilon)^\alpha[\varphi] = \frac{1}{2} \text{Res}_{\alpha=-d} (\pm i\varepsilon)^{\frac{\alpha}{2}} r^\alpha[\varphi] = \frac{\psi(0)}{2(\pm i\varepsilon)^{\frac{d}{2}}} = \left(\mp \frac{i\pi}{\varepsilon}\right)^{\frac{d}{2}} \cdot \frac{\varphi(0)}{\Gamma\left(\frac{d}{2}\right)}. \quad (2.23)$$

It follows that the holomorphic maps (2.22) coincide on $\{\pm i\mathbb{R}_{>0}\} \times \{\pm i\mathbb{R}_{>0}\} \subset A_\pm$, and therefore, they coincide on all of A_\pm by the identity theorem (in the version given by Theorem 3.2.6 in [AF2003]). Hence,

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the residues of (2.19) can be calculated via (2.22), and in particular for (2.21), we obtain

$$\operatorname{Res}_{\alpha=-\frac{d}{2}} (\gamma \pm i\varepsilon)^\alpha [\varphi] = \operatorname{Res}_{\alpha=-\frac{d}{2}} Q(1 \pm i\varepsilon, -1 \pm i\varepsilon)^\alpha [\varphi] = \frac{\pi^{\frac{d}{2}} \cdot \varphi(0)}{\sqrt{1 \pm i\varepsilon} (-1 \pm i\varepsilon)^{\frac{d-1}{2}} \Gamma\left(\frac{d}{2}\right)}. \quad (2.24)$$

With regard to (2.19) and (2.21), the distributions $(\gamma \pm i0)^\alpha$ are well-defined as the limits $\varepsilon \rightarrow 0$ of (2.21) in the distributional sense, so taking the limit for (2.18) and (2.19) leads to

$$(\gamma \pm i0)^\alpha = \frac{\square^k (\gamma \pm i0)^{\alpha+k}}{4^k \prod_{j=1}^k (\alpha + j) (\alpha + j + \frac{d-2}{2})}, \quad (2.25)$$

where $\operatorname{Re}(\alpha) > -\frac{d}{2}$ and $k \in \mathbb{N}_0$ chosen such that $\operatorname{Re}(\alpha) + k > 0$.

Proposition 2.2.8. *The distributions (2.25) are symmetric and \mathcal{L}_+^\uparrow -invariant, and for fixed $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the maps $\alpha \mapsto (\gamma \pm i0)^\alpha [\varphi]$ are holomorphic on $\{\operatorname{Re} > -\frac{d}{2}\}$. More precisely, for $m = 1, 2, \dots, \lfloor \frac{d-1}{2} \rfloor$, we have*

$$(\gamma \pm i0)^{-m} = \frac{(-1)^{m-1} \Gamma\left(\frac{d}{2} - m\right)}{4^m \Gamma(m) \Gamma\left(\frac{d}{2}\right)} \cdot \square^m \log(\gamma \pm i0),$$

and for $d > 2$, fundamental solutions for \square are given by

$$S_\pm := (\pm i)^{d+1} \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}}} \cdot (\gamma \pm i0)^{\frac{2-d}{2}}. \quad (2.26)$$

Proof. Clearly, (2.25) is holomorphic on $\{\operatorname{Re} > -\frac{d}{2}\} \setminus \{-\mathbb{N}\}$, and for $\alpha = -m > -\frac{d}{2}$, $m \in \mathbb{N}$, this can be checked similarly to (2.20). Since $(\gamma \pm i0)^\alpha$ is symmetric and \mathcal{L}_+^\uparrow -invariant for $\operatorname{Re}(\alpha) > 0$, so is the holomorphic extension (2.25). Moreover, holomorphicity ensures $(\gamma \pm i0)^{-m} = \lim_{\alpha \rightarrow -m} (\gamma \pm i0)^\alpha$ for all natural numbers $m < \frac{d}{2}$, and thus,

$$\begin{aligned} (\gamma \pm i0)^{-m} &= \frac{\lim_{\alpha \rightarrow -m} \frac{\square^m (\gamma \pm i0)^{\alpha+m}}{\alpha+m}}{4^m \prod_{j=1}^{m-1} (j-m) \cdot \prod_{j=1}^m (j-m + \frac{d-2}{2})} \\ &= \frac{\square^m \frac{d}{d\alpha} \Big|_{\alpha=-m} (\gamma \pm i0)^{\alpha+m}}{4^m \left(\frac{d}{2} - m\right) \dots \frac{d-2}{2} (-1)^{m-1} \prod_{j=1}^{m-1} (m-j)} = \frac{(-1)^{m-1} \Gamma\left(\frac{d}{2} - m\right)}{4^m \Gamma(m) \Gamma\left(\frac{d}{2}\right)} \cdot \square^m \log(\gamma \pm i0). \end{aligned}$$

Eventually, continuity of $\varepsilon \mapsto \operatorname{Res}_{\alpha=-\frac{d}{2}} (\gamma \pm i\varepsilon)^\alpha [\varphi]$ due to (2.24) for fixed φ as well as $\sqrt{1 \pm i\varepsilon} \rightarrow 1$ and $\sqrt{-1 \pm i\varepsilon} \rightarrow \pm i$ yield

$$\operatorname{Res}_{\alpha=-\frac{d}{2}} (\gamma \pm i0)^\alpha = \lim_{\varepsilon \rightarrow 0} \operatorname{Res}_{\alpha=-\frac{d}{2}} (\gamma \pm i\varepsilon)^\alpha = (\mp i)^{d-1} \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \cdot \delta_0,$$

so holomorphicity at $\alpha = \frac{2-d}{2}$ implies

$$\square (\gamma \pm i0)^{\frac{2-d}{2}} = \lim_{\alpha \rightarrow -\frac{d}{2}} \square (\gamma \pm i0)^{\alpha+1} = 4 \cdot \frac{2-d}{2} \underbrace{\lim_{\alpha \rightarrow -\frac{d}{2}} \left(\alpha + \frac{d}{2}\right) (\gamma \pm i0)^\alpha}_{= \operatorname{Res}_{\alpha=-\frac{d}{2}} (\gamma \pm i0)^\alpha} = (\mp i)^{d+1} \frac{4\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)} \cdot \delta_0. \quad \square$$

2.3 Cauchy problems on globally hyperbolic manifolds

A crucial feature of globally hyperbolic Lorentzian manifolds M is a well-posed Cauchy problem for wave operators (see Theorems 3.2.11 and 3.2.12 of [BGP2007] as well as Theorem 13 of [BTW2015]). In this section, we adjust those results for our purposes. Furthermore, we demonstrate, how well-posedness for differential operators P, Q on globally hyperbolic Lorentzian manifolds M, N entails well-posedness for $P \otimes Q$ on $M \times N$.

2.3.1 Smooth sections

Let M be a globally hyperbolic Lorentzian manifold with Cauchy hypersurface $\Sigma \subset M$, $\pi: E \rightarrow M$ a real or complex vector bundle over M and $P: C^\infty(M, E) \rightarrow C^\infty(M, E)$ a wave operator.

For $u_0, u_1 \in C^\infty(\Sigma, E)$ and $f \in C^\infty(M, E)$, we consider the Cauchy problem

$$\begin{cases} Pu &= f, \\ u|_\Sigma &= u_0, \\ \nabla_\nu u|_\Sigma &= u_1, \end{cases} \quad (2.27)$$

which is well-posed by the Theorems 3.2.11 and 3.2.12 of [BGP2007] for compactly supported u_0, u_1, f , and the smooth solution u satisfies

$$\text{supp } u \subset J(\text{supp } u_0 \cup \text{supp } u_1 \cup \text{supp } f). \quad (2.28)$$

Adopting the exhaustion argument in the proof of Corollary 5 in chapter 3 of [BF2009], this statement extends to general smooth data:

Theorem 2.3.1. *For all $u_0, u_1 \in C^\infty(\Sigma, E)$ and $f \in C^\infty(M, E)$, the Cauchy problem (2.27) is well-posed with smooth solution u satisfying (2.28).*

2.3.2 Smooth bisections

For F a vector bundle over some further globally hyperbolic Lorentzian manifold N , recall (2.4) for the definition of the vector bundle $E \boxtimes F$ over $M \times N$.

Theorem 2.3.2. *Let M, N be globally hyperbolic Lorentzian manifolds with a Cauchy hypersurfaces Σ, Ξ and unit normal fields μ, ν . Furthermore, let P, Q denote linear differential operators of second order acting on smooth sections in vector bundles E, F over M, N , which admit well-posed Cauchy problems and only lightlike characteristic directions. Then, for all $u_i \in C^\infty(\Sigma \times \Xi, E \boxtimes F)$, $i = 1, \dots, 4$, and $f, g \in C^\infty(M \times N, E \boxtimes F)$ with $Qf = Pg$, there is some unique section $u \in C^\infty(M \times N, E \boxtimes F)$ solving*

$$\begin{cases} Pu &= f, \\ Qu &= g, \\ u|_{\Sigma \times \Xi} &= u_1, \\ \nabla_\mu u|_{\Sigma \times \Xi} &= u_2, \\ \nabla_\nu u|_{\Sigma \times \Xi} &= u_3, \\ \nabla_\nu \nabla_\mu u|_{\Sigma \times \Xi} &= u_4. \end{cases} \quad (2.29)$$

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Proof. For all $q \in N$ and $h_0, h_1 \in C^\infty(\Sigma \times N, E \boxtimes F)$, the Cauchy problem

$$\begin{cases} Pu_q &= f(\cdot, q), \\ u_q|_\Sigma &= h_0(\cdot, q), \\ \nabla_\mu u_q|_\Sigma &= h_1(\cdot, q), \end{cases} \quad (2.30)$$

has a unique solution $u_q \in C^\infty(M, E \otimes F_q)$ by Theorem 2.3.1, and furthermore, the mapping of the data to the solution u_q is linear and continuous. Thus, it remains to determine h_0, h_1 from u_1, u_2, u_3, u_4, g and to show that then $Qu = g$ is automatically fulfilled. For all $\sigma \in \Sigma$ and $\xi \in \Xi$, we define smooth sections $h_0(\sigma, \cdot), h_1(\sigma, \cdot) \in C^\infty(N, E_\sigma \otimes F)$ and $h_0(\cdot, \xi), h_2(\cdot, \xi) \in C^\infty(M, E \otimes F_\xi)$ as solutions of

$$\begin{cases} Ph_0(\cdot, \xi) &= f(\cdot, \xi), \\ h_0(\cdot, \xi)|_\Sigma &= u_1(\cdot, \xi), \\ \nabla_\mu h_0(\cdot, \xi)|_\Sigma &= u_2(\cdot, \xi), \end{cases} \quad \begin{cases} Ph_2(\cdot, \xi) &= (\nabla_\nu f)(\cdot, \xi), \\ h_2(\cdot, \xi)|_\Sigma &= u_3(\cdot, \xi), \\ \nabla_\mu h_2(\cdot, \xi)|_\Sigma &= u_4(\cdot, \xi), \end{cases} \quad (2.31)$$

$$\begin{cases} Qh_0(\sigma, \cdot) &= g(\sigma, \cdot), \\ h_0(\sigma, \cdot)|_\Xi &= u_1(\sigma, \cdot), \\ \nabla_\nu h_0(\sigma, \cdot)|_\Xi &= u_3(\sigma, \cdot), \end{cases} \quad \begin{cases} Qh_1(\sigma, \cdot) &= (\nabla_\mu g)(\sigma, \cdot), \\ h_1(\sigma, \cdot)|_\Xi &= u_2(\sigma, \cdot), \\ \nabla_\nu h_1(\sigma, \cdot)|_\Xi &= u_4(\sigma, \cdot). \end{cases}$$

By adapting the proof of Proposition A.1 of [FNW1981], we obtain smooth sections h_0, h_1, h_2 in $E \boxtimes F$ over $(M \times \Xi) \cup (\Sigma \times N), \Sigma \times N$ and $M \times \Xi$, respectively, and, following the same lines, $u(\cdot, q) := u_q$ depends smoothly on q . Hence, we found some $u \in C^\infty(M \times N, E \boxtimes F)$ solving (2.30), which yields the initial data of a solution of (2.29):

$$\begin{aligned} u|_{\Sigma \times \Xi} &= h_0|_{\Sigma \times \Xi} = u_1, & \nabla_\mu u|_{\Sigma \times \Xi} &= h_1|_{\Sigma \times \Xi} = u_2, \\ \nabla_\nu u|_{\Sigma \times \Xi} &= \nabla_\nu h_0|_{\Sigma \times \Xi} = u_3, & \nabla_\nu \nabla_\mu u|_{\Sigma \times \Xi} &= \nabla_\nu h_1|_{\Sigma \times \Xi} = u_4. \end{aligned}$$

Note that P and Q commute, since they act on different factors of $M \times N$. Therefore, (2.30) and (2.31) imply that Qu and g satisfy the same Cauchy problem:

$$\begin{cases} PQu = QPu = Qf = Pg, \\ Qu|_{\Sigma \times N} = Qh_0 = g|_{\Sigma \times N}, \\ \nabla_\mu Qu|_{\Sigma \times N} = Q\nabla_\mu u|_{\Sigma \times N} = Qh_1 = \nabla_\mu g|_{\Sigma \times N}, \end{cases}$$

and hence $Qu = g$. By the same arguments, we have

$$\begin{cases} Qu &= g, \\ u|_{M \times \Xi} &= h_0, \\ \nabla_\nu u|_{M \times \Xi} &= h_2. \end{cases} \quad (2.32)$$

Uniqueness follows directly since trivial Cauchy data in (2.29) lead to trivial data in (2.31) and therefore in (2.30), which implies $u_q = 0$ for all q , that is $u = 0$. \square

Corollary 2.3.3. *With regard to the assumptions of Theorem 2.3.2, let $(N, \Xi, \nu) = (M, \Sigma, \mu)$, $F = E^*$ and E be equipped with a non-degenerate inner product. Let $Q = P^t$ and for all $(p, q) \in M \times M$ assume*

$$\begin{aligned} f(p, q) &= \Theta_p^{-1} g(q, p)^t \Theta_q, & u_2(\sigma_1, \sigma_2) &= \Theta_{\sigma_1}^{-1} u_3(\sigma_2, \sigma_1)^t \Theta_{\sigma_2}, \\ u_1(\sigma_1, \sigma_2) &= \Theta_{\sigma_1}^{-1} u_1(\sigma_2, \sigma_1)^t \Theta_{\sigma_2}, & u_4(\sigma_1, \sigma_2) &= \Theta_{\sigma_1}^{-1} u_4(\sigma_2, \sigma_1)^t \Theta_{\sigma_2} \end{aligned} \quad (2.33)$$

with fiberwise transposition ${}^t: \text{Hom}(E_q, E_p) \rightarrow \text{Hom}(E_p^*, E_q^*)$. Then the solution of (2.29) satisfies

$$u(p, q) = \Theta_p^{-1} u(q, p)^t \Theta_q, \quad p, q \in M. \quad (2.34)$$

Proof. Well-posedness of the Cauchy problems (2.31) and the symmetries (2.33) directly lead to

$$h_0(p, \sigma) = \Theta_p^{-1} h_0(\sigma, p)^t \Theta_\sigma \quad \text{and} \quad h_1(\sigma, p) = \Theta_\sigma^{-1} h_2(p, \sigma) \Theta_p, \quad p \in M, \sigma \in \Sigma,$$

since the corresponding Cauchy data coincide. Therefore, (2.30) and (2.32) imply (2.34). \square

We proceed with stability, i.e. continuous dependence on the Cauchy data. Recall that any manifold is paracompact, so the topology of $C^\infty(M, E)$ is generated by a countable family of seminorms. Thus, it is metrizable and we obtain a Fréchet space, for which the open mapping theorem holds.

Theorem 2.3.4. *Let $Z := (\oplus^2 C^\infty(M \times N, E \boxtimes F)) \oplus (\oplus^4 C^\infty(\Sigma \times \Xi, E \boxtimes F))$ and X the subset of elements $(f, g, u_1, u_2, u_3, u_4)$ satisfying $Qf = Pg$. Then the map*

$$X \longrightarrow C^\infty(M \times N, E \boxtimes F), \quad (f, g, u_1, u_2, u_3, u_4) \longmapsto u, \quad (2.35)$$

which sends the Cauchy data to the unique solution u of (2.29), is linear continuous.

Proof. The map

$$\begin{aligned} \Phi: \quad C^\infty(M \times N, E \boxtimes F) &\longrightarrow Z \\ u &\longmapsto (Pu, Qu, u|_{\Sigma \times \Xi}, \nabla_\mu u|_{\Sigma \times \Xi}, \nabla_\nu u|_{\Sigma \times \Xi}, \nabla_\nu \nabla_\mu u|_{\Sigma \times \Xi}) \end{aligned}$$

is linear, injective and continuous. By Theorem 2.3.2, $X \subset Z$ is a closed subspace, which is contained in $\text{ran } \Phi$, and due to continuity of differential operators, the subspace $\Phi^{-1}(X) \subset C^\infty(M \times N, E \boxtimes F)$ is also closed. Hence, we obtain a continuous and bijective map $\Phi: \Phi^{-1}(X) \rightarrow X$ between Fréchet spaces, whose inverse (2.35) is continuous by the open mapping theorem. \square

A similar argumentation as for Theorem 2.3.2 and Corollary 2.3.3 leads to

Theorem 2.3.5. *Under the assumptions of Theorem 2.3.2 but P and Q assumed to be first-order operators, the Cauchy problem*

$$\begin{cases} Pu &= f, \\ Qu &= g, \\ u|_{\Sigma \times \Xi} &= u_1, \end{cases}$$

is well-posed with smooth solution $u \in C^\infty(M \times N, E \boxtimes F)$. Moreover, u is symmetric in the sense of (2.34) if u_1, f, g are in the sense of (2.33).

2.3.3 Singular sections

We close this chapter by investigating the propagation of a family of singular solutions from a neighborhood of a Cauchy hypersurface Σ to the whole spacetime by applying the well-posed Cauchy problem for singular sections, treated in [BTW2015]. Therefore, we just have to ensure the existence of the restriction to Σ by checking Hörmander's criterion.

Theorem 2.3.6. *Let M be a globally hyperbolic Lorentzian manifold with a Cauchy hypersurface $\Sigma \subset M$, $\pi: E \rightarrow M$ a real or complex vector bundle over M and $P: C^\infty(M, E) \rightarrow C^\infty(M, E)$ a wave operator. Furthermore, let $O \subset M$ be relatively compact, and for $p \in O$, let $v(p) \in \mathcal{D}(M, E, E_p^*)'$ have spacelike compact support and only lightlike singular directions. Moreover, we assume $p \mapsto v(p)[\varphi] \in C^\infty(M, E^*)$ for fixed $\varphi \in \mathcal{D}(M, E^*)$. Then the Cauchy problem*

$$\begin{cases} Pu(p) &= 0, \\ u(p)|_\Sigma &= v(p)|_\Sigma, \\ \nabla_\nu u(p)|_\Sigma &= \nabla_\nu v(p)|_\Sigma, \end{cases}$$

has a unique solution $u(p) \in \mathcal{D}(M, E, E_p^*)'$, which has spacelike compact support and provides a smooth section $p \mapsto u(p)[\varphi]$ for each $\varphi \in \mathcal{D}(M, E^*)$.

Proof. Let $t: M \rightarrow \mathbb{R}$ be a Cauchy time function on M such that $\Sigma = t^{-1}(0)$ (Theorem 1.3.13 of [BGP2007]). Therefore, the normal directions of Σ are timelike and do not match the singular directions of v , so $v(p)|_\Sigma$ and $\nabla_\nu v(p)|_\Sigma$ are well-defined and compactly supported distributions on Σ for all p due to Hörmander's criterion ((8.2.3) of [Hör1990]).

Recall that any compactly supported distribution lies in some Sobolev space H_c^k (see e.g. (31.6) of [Tre1967]), and hence, $v(p)|_\Sigma \in H_c^k(\Sigma, E_p^* \otimes E)$ and $\nabla_\nu v(p)|_\Sigma \in H_c^{k-1}(\Sigma, E_p^* \otimes E)$ for some $k \in \mathbb{R}$. Thus, for all p , Corollary 14 of [BTW2015] provides a unique solution

$$u(p) \in C_{sc}^0(t(M), H^k(\Sigma, \cdot); E_p^* \otimes E) \cap C_{sc}^1(t(M), H^{k-1}(\Sigma, \cdot); E_p^* \otimes E),$$

where this intersection is better known as the space of finite k -energy sections (see section 1.7 of [BTW2015] for details about them). Moreover, the mapping of initial data to the solution is a linear homeomorphism, so because the restriction $v(p) \mapsto (v(p)|_\Sigma, \nabla_\nu v(p)|_\Sigma)$ is linear and continuous, so is the map of distributions T given by $v(p) \mapsto u(p)$ for all p .

For D a differential operator, let $(D_{(1)}v)(p)$ denote the distribution $\varphi \mapsto (D(v(\cdot)[\varphi]))(p)$. It follows that $(D_{(1)}v)(p)$ is linearly and continuously mapped to $(D_{(1)}u)(p)$, that is, T commutes with $D_{(1)}$ (see the proof of Proposition A.1 in [FNW1981]). In particular, the map

$$p \mapsto (D_{(1)}v)(p)[\varphi] \mapsto (D_{(1)}u)(p)[\varphi] = (D(u(\cdot)[\varphi]))(p), \quad \varphi \in \mathcal{D}(M, E^*),$$

is continuous due to smoothness of the first arrow. Since this holds for all differential operators D , we obtain smoothness of $p \mapsto u(p)[\varphi]$ for fixed φ . \square

3 Symmetry of the Hadamard coefficients

"Symmetrie ist die Schönheit der Dummen."

Let M be a Lorentzian manifold, $\pi: E \rightarrow M$ a real or complex, finite-dimensional vector bundle over M with non-degenerate inner product and $P: C^\infty(M, E) \rightarrow C^\infty(M, E)$ a formally self-adjoint wave operator. Furthermore, $\Omega \subset M$ is assumed to be a non-empty and convex domain, which therefore is time-orientable, and ∇ denotes the P -compatible connection on E , that is,

$$\nabla_{\text{grad}f}s = \frac{1}{2}(f \cdot Ps - P(f \cdot s) + \square f \cdot s), \quad s \in C^\infty(M, E), f \in C^\infty(M). \quad (3.1)$$

It follows that $P = \square^\nabla + B$ for some unique endomorphism field B and $\square^\nabla = (\text{tr} \otimes \text{id}_E) \circ \nabla^{T^*M \otimes E} \circ \nabla$ the connection-d'Alembert operator (see section 1.5 of [BGP2007]). Then the Hadamard coefficients $U_k \in C^\infty(\Omega \times \Omega, E^* \boxtimes E)$, $k \in \mathbb{N}_0$, for P are defined as the unique solutions of the transport equations

$$\nabla_{\text{grad}\Gamma_p} U_p^k - \left(\frac{1}{2} \square \Gamma_p - d + 2k \right) U_p^k = 2k \cdot P U_p^{k-1}, \quad k \in \mathbb{N}, \quad (3.2)$$

with $U_0(p, p) = \text{id}_{E_p^*}$ for all $p \in \Omega$ (Proposition 2.3.1 of [BGP2007]). Γ_p denotes the squared Lorentz distance from p , and $U_p^k := U_k(p, \cdot)$. In this chapter, we show symmetry of U_k in the sense

$$U_k(p, q) = \Theta_p U_k(q, p)^t \Theta_q^{-1}, \quad p, q \in \Omega, k \in \mathbb{N}_0. \quad (3.3)$$

Recall the identification (2.4), meaning that $U_k(p, q)$ is considered as a homomorphism $E_q^* \rightarrow E_p^*$ with fiberwise transposed operator $U_k(p, q)^t \in \text{Hom}(E_p, E_q)$.

[Mor2000] already checked the scalar case $E = M \times \mathbb{R}$ and very recently, a proof for a vector bundle setting and for arbitrary signature of M has been proposed in [Kam2019]. We restrict to Lorentz signature and adopt Moretti's approach insofar that we demonstrate (3.3) for analytic P and deduce the smooth case by analytic approximation afterwards. However, for the proof in the analytic setting, we choose an alternative approach employing symmetry properties of the advanced and retarded Green operator.

3.1 A link between even and odd dimensions

For $p, q \in \Omega$, let $\phi_{pq}(t) := \exp_p(t \exp_p^{-1}(q))$ denote the unique connecting geodesic, which provides a map

$$\phi: [0, 1] \times \Omega \times \Omega \longrightarrow \Omega, \quad (t, p, q) \longmapsto \phi_{pq}(t). \quad (3.4)$$

Let $\widehat{M} := M \times \mathbb{R}$ be equipped with the metric $\widehat{g} := g + ds^2$, $\widehat{\Omega} := \Omega \times \mathbb{R}$ and $\widehat{P} := P - \frac{\partial^2}{\partial s^2}$ on \widehat{M} . Furthermore, over \widehat{M} , we consider the same vector bundle E with fibers $E_{(p,s)} := E_p$ for all $(p, s) \in \widehat{M}$.

Lemma 3.1.1. *For $(p, s), (q, s') \in \widehat{\Omega}$ the map (3.4) and the squared Lorentzian distance are given by*

$$\phi_{(p,s)(q,s')}(t) = (\phi_{pq}(t), s + t(s' - s)), \quad \widehat{\Gamma}((p, s), (q, s')) = \Gamma(p, q) - (s' - s)^2.$$

3 Symmetry of the Hadamard coefficients

Moreover, for $\hat{f} \in C^\infty(\widehat{M})$ and $\widehat{X} = (X, X^{d+1}) \in C^\infty(\widehat{M}, T\widehat{M})$, we obtain

$$\text{grad}_{\hat{g}}\hat{f} = \text{grad}_g\hat{f} + \frac{\partial\hat{f}}{\partial s} \frac{\partial}{\partial s}, \quad \text{div}_{\hat{g}}\widehat{X} = \text{div}_gX + \frac{\partial X^{d+1}}{\partial s}, \quad \square_{\hat{g}}\hat{f} = \square_g\hat{f} - \frac{\partial^2\hat{f}}{\partial s^2}.$$

Proof. The Christoffel symbols Γ_{ij}^k of the Levi-Civita-connection on $T\widehat{M}$ vanish for i, j or $k = d + 1$, so the geodesic equation separates into the one on Ω and $\ddot{\phi}_{(p,s)(q,s')}(t) = 0$ on \mathbb{R} with boundary conditions $\phi_{(p,s)(q,s')}(0) = s$ and $\phi_{(p,s)(q,s')}(1) = s'$. Thus, $\phi_{(p,s)(q,s')}(t) = (\phi_{pq}(t), s + t(s' - s))$, and consequently,

$$\begin{aligned} \widehat{\Gamma}((p, s), (q, s')) &= \int_0^1 \widehat{g}_{\phi_{(p,s)(q,s')}(t)}(\dot{\phi}_{(p,s)(q,s')}(t), \dot{\phi}_{(p,s)(q,s')}(t)) dt \\ &= \int_0^1 g_{\phi_{pq}(t)}(\dot{\phi}_{pq}(t), \dot{\phi}_{pq}(t)) dt - \int_0^1 (s' - s)^2 dt = \Gamma(p, q) - (s' - s)^2. \end{aligned}$$

From the local form $\widehat{g} = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ follows $\det \widehat{g} = \det g$, and therefore,

$$\begin{aligned} \text{grad}_{\widehat{g}}\hat{f} &= \widehat{g}^{ij} \frac{\partial\hat{f}}{\partial x^i} \frac{\partial}{\partial x^j} = \text{grad}_g\hat{f} + \frac{\partial\hat{f}}{\partial s} \frac{\partial}{\partial s} = \left(\text{grad}_g\hat{f}, \frac{\partial\hat{f}}{\partial s} \right), \\ \text{div}_{\widehat{g}}\widehat{X} &= \frac{1}{\sqrt{\det \widehat{g}}} \frac{\partial}{\partial x^j} (\widehat{X}^j \sqrt{\det \widehat{g}}) \\ &= \frac{1}{\sqrt{\det g}} \left(\frac{\partial}{\partial x^j} (X^j \sqrt{\det g}) + \frac{\partial}{\partial s} (X^{d+1} \sqrt{\det g}) \right) = \text{div}_gX + \frac{\partial X^{d+1}}{\partial s}, \\ \square_{\widehat{g}}\hat{f} &= -\text{div}_{\widehat{g}} \text{grad}_{\widehat{g}}\hat{f} = -\text{div}_g \text{grad}_g\hat{f} - \frac{\partial}{\partial s} \frac{\partial\hat{f}}{\partial s} = \square_g\hat{f} - \frac{\partial^2\hat{f}}{\partial s^2}. \quad \square \end{aligned}$$

One can say that the Hadamard coefficients somehow measure the deviation of (M, g, E, P) from $(\mathbb{R}_{\text{Mink}}^d, \square)$, and indeed, "adding" $(\mathbb{R}, ds^2, \{0\}, \frac{\partial^2}{\partial s^2})$ does not change them:

Proposition 3.1.2. *Let \widehat{U}_k, U_k denote the Hadamard coefficients associated to $(\widehat{M}, \widehat{g}, E, \widehat{P})$ and (M, g, E, P) , respectively. Then, for all $(p, s), (q, s') \in \widehat{\Omega}$ and $k \in \mathbb{N}_0$, we have*

$$\widehat{U}_k((p, s), (q, s')) = U_k(p, q).$$

Proof. For $\widehat{\nabla}$ the \widehat{P} -compatible connection on E , Lemma 3.1.1 provides

$$\widehat{\nabla}_{\text{grad}_{\widehat{g}}\hat{f}}\hat{f} = \widehat{\nabla}_{\text{grad}_g\hat{f}}\hat{f} + \frac{\partial\hat{f}}{\partial s} \cdot \widehat{\nabla}_{\frac{\partial}{\partial s}}, \quad \square_{\widehat{g}}\widehat{\Gamma}_{(p,s)} = \square_g\Gamma_p + 2.$$

Clearly, $\widehat{U}_0(p, s; p, s) = \text{id}_{E_p^*} = U_0(p, p)$ holds, and moreover, for all $p \in \Omega$, the transport equations

$$\begin{aligned} 0 &= \widehat{\nabla}_{\text{grad}_{\widehat{g}}\widehat{\Gamma}_{(p,s)}}\widehat{U}_{(p,s)}^k - \left(\frac{1}{2}\square_{\widehat{g}}\widehat{\Gamma}_{(p,s)} - (d+1) + 2k \right) \widehat{U}_{(p,s)}^k - 2k\widehat{P}\widehat{U}_{(p,s)}^{k-1} \\ &= \nabla_{\text{grad}_g\Gamma_p}\widehat{U}_{(p,s)}^k + 2(s' - s)\widehat{\nabla}_{\frac{\partial}{\partial s}}\widehat{U}_{(p,s)}^k - \left(\frac{1}{2}(\square_g\Gamma_p + 2) - (d+1) + 2k \right) \widehat{U}_{(p,s)}^k - 2kP\widehat{U}_{(p,s)}^{k-1} + 2k\frac{\partial^2}{\partial s^2}\widehat{U}_{(p,s)}^{k-1} \\ &= \nabla_{\text{grad}_g\Gamma_p}\widehat{U}_{(p,s)}^k - \left(\frac{1}{2}\square_g\Gamma_p - d + 2k \right) \widehat{U}_{(p,s)}^k - 2kP\widehat{U}_{(p,s)}^{k-1} + 2(s' - s)\widehat{\nabla}_{\frac{\partial}{\partial s}}\widehat{U}_{(p,s)}^k + 2k\frac{\partial^2}{\partial s^2}\widehat{U}_{(p,s)}^{k-1} \end{aligned}$$

are obviously solved by U_p^k . □

3.2 Analytic approximation

In the following, let P have analytic coefficients, and we deduce analyticity of $(p, q) \mapsto U_k(p, q)$ on $\Omega \times \Omega$ for all $k \in \mathbb{N}_0$. As the coefficient of the highest order of P , also the metric g is assumed to be analytic, so the Levi-Civita connection on TM and the P -compatible connection ∇ on E are analytic as well, that is, the corresponding Christoffel symbols are. Due to basic ODE-theory, the geodesic equation ensures analyticity of $(t, \xi, p) \mapsto \exp_p(t\xi)$ on the domain of existence, and furthermore, $(p, q) \mapsto \exp_p^{-1}(q)$ is analytic by the analytic inverse function theorem (Theorem 1.4.3 of [KP1992]). This provides analyticity of $(t, p, q) \mapsto \phi_{pq}(t) = \exp_p(t \exp_p^{-1}(q))$, the Lorentzian distance $(p, q) \mapsto \Gamma(p, q) = g_p(\exp_p^{-1}(q))$ and the distortion function $(p, q) \mapsto \mu(p, q) := \left| \det \left(d \exp_p \big|_{\exp_p^{-1}(q)} \right) \right|$.

Lemma 3.2.1. *The ∇ -parallel transport along ϕ_{pq} is analytic as a map*

$$[0, 1] \times \Omega \times \Omega \longrightarrow E^* \boxtimes E, \quad (t, p, q) \longmapsto \Pi_{\phi_{pq}(t)}^p. \quad (3.5)$$

Proof. Let $p \in \Omega$ be fixed and we identify $\text{Hom}(E_{\phi_{pq}(t)}^*, E_p^*) \cong \text{Hom}(E_p, E_{\phi_{pq}(t)})$. For any $e \in E_p$, the map $s_p(t, q) := \Pi_{\phi_{pq}(t)}^p e$ defines a parallel section $\phi_{pq}(t) \mapsto s_p(t, q)$ in E along ϕ_{pq} for all $q \in \Omega$, and therefore, it satisfies $s_p(0, q) = e$ and the following system of ODE's

$$\dot{s}_p^\beta(t, q) = \underbrace{-\Gamma_{i\alpha}^\beta(\phi_{pq}(t)) \phi_{pq}^i(t)}_{=: A_p(t, q)^\beta_\alpha} s_p^\alpha(t, q). \quad (3.6)$$

The columns of the corresponding fundamental matrix $\Phi_p(t, q)$ are given by $\text{rk}(E)$ linearly independent solutions of (3.6). Thus, we have $\dot{\Phi}_p(t, q) = A_p(t, q)\Phi_p(t, q)$ and the solution of (3.6) takes the form

$$s_p(t, q) = \Phi_p(t, q)\Phi_p(0, q)^{-1}s_p(0, q),$$

From the definition of s_p , we read off $\Pi_{\phi_{pq}(t)}^p = \Phi_p(t, q)\Phi_p(0, q)^{-1}$, and hence, the map

$$[0, 1] \times \Omega \rightarrow E_p^* \otimes E, \quad (t, q) \longmapsto \Pi_{\phi_{pq}(t)}^p,$$

is analytic, since $(t, q) \mapsto A_p(t, q)$ and therefore $(t, q) \mapsto \Phi_p(t, q)$ is. Moreover, from $\Pi_p^r = (\Pi_r^p)^{-1}$ follows

$$\Phi_p(1, r)\Phi_p(0, r)^{-1} = (\Phi_r(1, p)\Phi_r(0, p)^{-1})^{-1} = \Phi_r(0, p)\Phi_r(1, p)^{-1},$$

and hence, $p \mapsto \Pi_p^r$ is analytic for each $r \in \Omega$. By Osgood's Lemma [Osg1898], a map is analytic if it is with respect to each argument, which implies analyticity of (3.5). \square

Proposition 3.2.2. *The map $(p, q) \mapsto U_k(p, q)$ is analytic on $\Omega \times \Omega$ for all $k \in \mathbb{N}_0$.*

Proof. Analyticity of the zeroth Hadamard coefficient can be directly read off from

$$(p, q) \longmapsto U_0(p, q) = \frac{\Pi_q^p}{\sqrt{\mu(p, q)}},$$

and we proceed via induction. By analyticity of P , clearly $(p, q) \mapsto P_{(2)}U_{k-1}(p, \Phi_{pq}(t))$ is analytic if $(p, q) \mapsto U_{k-1}(p, q)$ is. Similarly, $(t, p, q) \mapsto U_0(p, \phi_{pq}(t))^{-1} = \sqrt{\mu(p, \phi_{pq}(t))} \cdot \Pi_p^{\phi_{pq}(t)}$ is analytic as a composition of analytic maps (recall that μ is positive). Therefore, the integrand of

$$U_k(p, q) = -kU_0(p, q) \int_0^1 t^{k-1} \cdot U_0(p, \phi_{pq}(t))^{-1} P_{(2)}U_{k-1}(p, \phi_{pq}(t)) dt$$

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is analytic in (p, q) and uniformly continuous in t on $[0, 1]$. Hence, taking the power series expression of the integrand, the sum and the integral can be swapped, which results in a uniformly converging power series for U_k . \square

Now we tackle the general case of smooth P by analytic approximation of the coefficients. This requires the following crucial result:

Proposition 3.2.3 (Proposition 2.1 of [Mor1999]). *Let M be a real, smooth and connected manifold with non-singular metric g .*

- (a) *For any local chart (x, V) of M and any relatively compact domain O with $\bar{O} \subset V$, there is a sequence $\{g^n\}_{n \in \mathbb{N}}$ of real and analytic (with respect to x) metrics with the same signature as g , which are defined on some neighborhood of \bar{O} such that $g^n \rightarrow g$ in C^∞ , that is, all derivatives of g^n converge uniformly on \bar{O} :*

$$\forall i, j = 1, \dots, D, \quad \alpha \in \mathbb{N}_0^D : \quad \max_{v \in x(\bar{O})} \left| (D^\alpha (g^n \circ x^{-1})_{ij})(v) - (D^\alpha (g \circ x^{-1})_{ij})(v) \right| \longrightarrow 0.$$

- (b) *For any $(x, V), O, \{g^n\}_{n \in \mathbb{N}}$ as in (a) and additionally any $z \in O$, there is an $n_0 \in \mathbb{N}$ and a family $\{N_z^i\}_{i \in \mathbb{R}}$ of open neighborhoods of z such that $N_z^i \subset \bar{N}_z^j \subset O$ for any $j > i$, and $\{N_z^i\}_{i \in \mathbb{R}}$ is a local base of the topology of M . Moreover, for all $i \in \mathbb{R}$, both N_z^i and \bar{N}_z^i are common convex neighborhoods of z for all metrics $\{g^n\}_{n > n_0}$ and g .*

Proposition 3.2.4. *Let $O \subset \Omega$ be relatively compact and $\{g^n\}_{n \in \mathbb{N}}$ a sequence of real and analytic metrics defined in a neighborhood of \bar{O} with the same signature as g such that O and \bar{O} are convex with respect to all g^n , $n \in \mathbb{N}$, and g and $g^n \rightarrow g$ in C^∞ . For $\{U_k^n\}_{n \in \mathbb{N}}$ the corresponding Hadamard coefficients, we obtain $U_k^n(p, q) \rightarrow U_k(p, q)$ for all $k \in \mathbb{N}_0$ and $p, q \in O$.*

Proof. The assumption directly provides $\Gamma_{ij}^{k,n} \rightarrow \Gamma_{ij}^k$, and with regard to the geodesic equation with converging right hand side, we similarly obtain $\exp^n \rightarrow \exp$ as smooth maps $(t, \xi, p) \mapsto \exp_p(t\xi)$ on their domain of existence. Then, the inverse function theorem provides convergence of $(\exp^n)^{-1} \rightarrow \exp^{-1}$ as smooth maps on $\bar{O} \times \bar{O}$ and, as a consequence, of the Lorentzian distance $\Gamma^n \rightarrow \Gamma$ and the distortion function $\mu^n \rightarrow \mu$. Eventually, we have $\phi^n \rightarrow \phi$ for the connecting geodesic (3.4).

It remains to investigate the parallel transport. For all $p \in O$, convergence of $\Gamma_{ij}^{k,n}$ and ϕ^n leads to $A_p^n \rightarrow A_p$ for the matrices defined in (3.6), and hence,

$$\Pi_{\phi_{pq}^n}^{p,n}(t) = \Phi_p^n(t, q) \Phi_p^n(0, q)^{-1} \longrightarrow \Phi_p(t, q) \Phi_p(0, q)^{-1} = \Pi_{\phi_{pq}(t)}^p$$

as smooth maps $[0, 1] \times \bar{O} \rightarrow E_p^* \otimes E$. Thus, we can directly conclude convergence of the zeroth Hadamard coefficient

$$U_0^n(p, \cdot) = \frac{\Pi^{p,n}}{\sqrt{\mu_p^n}} \longrightarrow \frac{\Pi^p}{\sqrt{\mu_p}} = U_0(p, \cdot) \quad (3.7)$$

as smooth maps $\bar{O} \rightarrow E_p^* \otimes E$ and, in particular, $U_0^n(p, q) \rightarrow U_0(p, q)$ in $\text{Hom}(E_q^*, E_p^*)$.

We proceed inductively. Due to $\phi_{pq}^n \rightarrow \phi_{pq}$ in $C^\infty([0, 1], \bar{O})$, (3.7) implies $U_0^n(p, \cdot) \circ \phi_{pq}^n \rightarrow U_0(p, \cdot) \circ \phi_{pq}$ in $C^\infty([0, 1], E_p^* \otimes E)$ and consequently, $PU_0^n(p, \cdot) \circ \phi_{pq}^n \rightarrow PU_0(p, \cdot) \circ \phi_{pq}$. Therefore, the integrand in the expression of the first Hadamard coefficient

$$U_1^n(p, q) = -k U_0^n(p, q) \int_0^1 U_0^n(p, \phi_{pq}^n(t))^{-1} P_{(2)} U_0^n(p, \phi_{pq}^n(t)) dt$$

converges to the one in the expression of $U_1(p, q)$, and as a smooth function in t , it is integrable on the compact interval $[0, 1]$. Hence, due to majorized convergence, the integral converges as well, so we have

$$-kU_0^n(p, q) \int_0^1 U_0^n(p, \phi_{pq}^n(t))^{-1} P_{(2)} U_0^n(p, \phi_{pq}^n(t)) dt \longrightarrow -kU_0(p, q) \int_0^1 U_0(p, \phi_{pq}(t))^{-1} P_{(2)} U_0(p, \phi_{pq}(t)) dt,$$

which is $U_1(p, q)$. Recursively, this implies $U_k^n(p, q) \rightarrow U_k(p, q)$ for all $k \in \mathbb{N}$ and $p, q \in O$. \square

3.3 Proof of the symmetry

Let $V_k \in C^\infty(\Omega \times \Omega, E \boxtimes E^*)$, $k \in \mathbb{N}_0$, denote the Hadamard coefficients associated to (M, g, E^*, P^t) , which, due to formal self-adjointness of P , are closely related to U_k :

Lemma 3.3.1. *The Hadamard coefficients of P and P^t are related via*

$$V_k(p, q) = \Theta_p^{-1} U_k(p, q) \Theta_q, \quad p, q \in \Omega.$$

Proof. For all $p \in \Omega$, this clearly holds for the initial condition

$$V_0(p, p) = \text{id}_{E_p} = \Theta_p^{-1} \text{id}_{E_p^*} \Theta_p = \Theta_p^{-1} U_0(p, p) \Theta_p.$$

Formal self-adjointness of P implies $P^t = \Theta P \Theta^{-1}$, so with regard to (3.1), the P^t -compatible connection ∇^t is given by $\Theta \nabla \Theta^{-1}$. Therefore, we just have to check that $\Theta_p^{-1} U_k(p, q) \Theta_q$ satisfies the transport equations (3.2) induced by P^t and ∇^t . Setting $M_p := (\frac{1}{2} \square \Gamma(p, \cdot) - d + 2k)$, we indeed obtain

$$2k P^t \Theta_p^{-1} U_p^{k-1} \Theta = 2k \Theta_p^{-1} P U_p^{k-1} \Theta = \Theta_p^{-1} \left(\nabla_{\text{grad}_g \Gamma_p} U_p^k - M_p \cdot U_p^k \right) \Theta = \left(\nabla_{\text{grad}_g \Gamma_p}^t - M_p \right) \Theta_p^{-1} U_p^k \Theta,$$

which proves the claim. \square

For $p \in \Omega$, let $\mu_p := |\det(\text{d exp}_p) \circ \exp_p^{-1}|: \Omega \rightarrow \mathbb{R}$ and $\{R_\pm^\alpha\}_{\alpha \in \mathbb{C}}$ denote the Riesz distributions. According to section 1.4 of [BGP2007], we define

$$R_\pm^\Omega(\alpha, p)[\varphi] := R_\pm^\alpha[(\mu_p \varphi) \circ \exp_p], \quad \varphi \in \mathcal{D}(\Omega), \quad (3.8)$$

which ensures the identification $R_\pm^\Omega(\alpha, p)|_{J_\pm^\Omega(p)} = C(\alpha, d) \cdot \Gamma_p^{\frac{\alpha-d}{2}}$ for $\text{Re}(\alpha) > d$. Due to Proposition 2.4.6 of [BGP2007], they comprise Hadamard series, which yield advanced and retarded parametrices $\tilde{\mathcal{R}}_\pm(p)$ for P at each $p \in \bar{O}$ on any relatively compact domain $O \subset \Omega$. More precisely, for any integer $N > \frac{d}{2}$ and cut-off function $\sigma \in \mathcal{D}((-1, 1), [0, 1])$ with $\sigma|_{[-\frac{1}{2}, \frac{1}{2}]} = 1$, there is a sequence $\{\varepsilon_k\}_{k \geq N} \subset (0, 1]$ such that

$$\tilde{\mathcal{R}}_\pm(p) = \sum_{k=0}^{\infty} \tilde{U}_k(p, \cdot) R_\pm^\Omega(2k+2, p), \quad \tilde{U}_k := \begin{cases} U_k, & k < N, \\ \left(\sigma \circ \frac{\Gamma}{\varepsilon_k} \right) \cdot U_k, & k \geq N, \end{cases} \quad (3.9)$$

represent well-defined distributions and satisfy $(p, q) \mapsto (P \tilde{\mathcal{R}}_\pm(p) - \delta_p)(q) \in C^\infty(\Omega \times \Omega, E^* \boxtimes E)$. Furthermore, we have $p \mapsto \tilde{\mathcal{R}}_\pm(p)[\varphi] \in C^\infty(\bar{O}, E^*)$ for fixed $\varphi \in \mathcal{D}(O, E^*)$, so regarded as bidistributions and due to compactness of \bar{O} , they provide continuous operators

$$\tilde{G}_\pm: \quad \mathcal{D}(O, E^*) \rightarrow C^\infty(\bar{O}, E^*), \quad \varphi \mapsto (p \mapsto \tilde{\mathcal{R}}_\pm(p)[\varphi]). \quad (3.10)$$

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Consequently, \tilde{G}_\pm yield left parametrices for P^t with $\text{supp } \tilde{G}_\pm \varphi \subset J_\pm^{\overline{O}}(\text{supp } \varphi)$, and for G_\pm the advanced and retarded Green operator for P^t , we have:

Lemma 3.3.2. *The operators $G_\pm - \tilde{G}_\pm$ are smoothing.*

Proof. This directly follows by realizing that the last equation in the proof of Proposition 2.5.1 of [BGP2007] actually yields a smooth section, since the integral representing the C^k -term corresponds to a smoothing operator applied to a C^k -section. Due to (3.8) in [BGP2007] and (3.10), this represents the Schwartz kernel of $G_\pm - \tilde{G}_\pm$. \square

Therefore, \tilde{G}_\pm actually represent an advanced and a retarded parametrix for P^t in the sense of (1.11).

Proposition 3.3.3. *For all convex and relatively compact $O \subset \Omega$, the maps*

$$(p, q) \mapsto \sum_{k=0}^{\infty} \left((\tilde{U}_k(p, \cdot) - \tilde{V}_k(\cdot, p)^t) R_\pm^\Omega(2k+2, p) \right) (q) \quad (3.11)$$

define smooth sections in $E^* \boxtimes E$ over $O \times O$.

Proof. By Lemma 3.4.4 of [BGP2007], the advanced and retarded Green operators for P are given by G_\mp^t , so formal self-adjointness of P and uniqueness of G_\pm lead to $G_\pm = \Theta G_\mp^t \Theta^{-1}$. Hence, Lemma 3.3.2 shows that the operators $\tilde{G}_\pm - \Theta \tilde{G}_\mp^t \Theta^{-1}$ are smoothing:

$$\tilde{G}_\pm - \Theta \tilde{G}_\mp^t \Theta^{-1} = \underbrace{\tilde{G}_\pm - G_\pm}_{\text{smoothing}} - \underbrace{\Theta(\tilde{G}_\mp^t - G_\mp^t)}_{\text{smoothing}} \Theta^{-1}.$$

By recalling the relation between the Schwartz kernel of an operator and its transpose

$$\tilde{G}_\pm^t[\varphi, \psi] = \tilde{G}_\pm^t \psi[\varphi] = \tilde{G}_\pm \varphi[\psi] = \tilde{G}_\pm[\psi, \varphi], \quad \psi \in \mathcal{D}(O, E), \varphi \in \mathcal{D}(O, E^*),$$

we just have to show that the Schwartz kernel of \tilde{G}_\pm^t is given by the distribution

$$\tilde{G}_\pm^t(p) = \sum_{k=0}^{\infty} \Theta_p^{-1} \tilde{V}_k(\cdot, p)^t \Theta R_\pm^\Omega(2j+2, p).$$

Indeed, Lemma 1.4.3 of [BGP2007] and Lemma 3.3.1 imply

$$\begin{aligned} \tilde{G}_\mp^t[\varphi, \psi] &= \tilde{G}_\mp[\psi, \varphi] = \sum_{k=0}^{\infty} \int_O R_\pm^\Omega(2k+2, p) [(\tilde{U}_k(p, \cdot) \varphi)(\psi(p))] dV(p) \\ &= \sum_{k=0}^{\infty} \int_O R_\pm^\Omega(2k+2, p) [\Theta_p \psi(p) (\Theta_p^{-1} \tilde{U}_k(p, \cdot) \varphi)] dV(p) \\ &= \sum_{k=0}^{\infty} \int_O R_\pm^\Omega(2k+2, p) [\Theta_p \psi(p) (\tilde{V}_k(p, \cdot) \Theta^{-1} \varphi)] dV(p) \\ &= \sum_{k=0}^{\infty} \int_O R_\mp^\Omega(2k+2, q) [\Theta \psi(\tilde{V}_k(\cdot, q) \Theta_q^{-1} \varphi(q))] dV(p) \\ &= \sum_{k=0}^{\infty} \int_O R_\mp^\Omega(2k+2, q) [(\tilde{V}_k(\cdot, q)^t \Theta \psi)(\Theta_q^{-1} \varphi(q))] dV(q), \end{aligned}$$

which reveals the desired equality. \square

Lemma 3.3.4. *Let $k \in \mathbb{N}_0$ and assume that for all quadruples (M, g, E, P) as introduced in the beginning of the chapter with odd spacetime dimension and all lightlike related $p, q \in O$, we have*

$$U_k(p, q) = V_k(q, p)^t.$$

Then this equality holds for all $p, q \in \Omega$ and all (M, g, E, P) .

Proof. Consider the setting (M, g, E, P) with odd spacetime dimension d . Let p, q be causally related, i.e. $\Gamma(p, q) \geq 0$, and choose $a, a' \in \mathbb{R}^2$ such that $\|a - a'\|^2 = \Gamma(p, q)$. It follows that $\hat{\Gamma}((p, a), (q, a')) = 0$, so $(p, a), (q, a')$ are lightlike related in $(M \times \mathbb{R}^2, g + g_{\text{Eucl}})$, which results in

$$\hat{U}_k((p, a), (q, a')) = \hat{V}_k((p, a), (q, a'))^t$$

by assumption. Therefore, Proposition 3.1.2 provides $U_k(p, q) = V_k(q, p)^t$.

Let $\{g^n\}_{n \in \mathbb{N}}$ be an analytic approximation of g and U_k^n, V_k^n the corresponding Hadamard coefficients. Write $D_k^n(p, q) := U_k^n(p, q) - V_k^n(q, p)^t$, which depends analytically on p, q due to Proposition 3.2.2 and vanishes on $\Gamma^{-1}(\mathbb{R}_{\geq 0})$, so the identity theorem for analytic maps implies $D_k^n = 0$ on all of $O \times O$. Furthermore, by Proposition 3.2.4, we have $D_k^n(p, q) \rightarrow D_k(p, q)$ and therefore $D_k(p, q) = 0$ for all p, q , which proves the claim in the case of odd spacetime dimension.

The claim for even-dimensional settings (M, g, E, P) can be deduced from $(M \times \mathbb{R}, g + ds^2, E, P - \frac{\partial^2}{\partial s^2})$, which is odd-dimensional, and Proposition 3.1.2.

Since this works for any relatively compact and convex domain $O \subset \Omega$, by uniqueness of the Hadamard coefficients, an appropriate exhaustion of Ω by such subsets proves the claim on all of $\Omega \times \Omega$. \square

Lemma 3.3.5. *Let X be a smooth manifold, $T \in \mathcal{D}(X)'$ and $f \in C^\infty(X)$. Then*

$$f \cdot T \in C^\infty(X) \quad \implies \quad f|_{\text{sing supp } T} = 0.$$

Proof. Let $x \in \text{sing supp } T$ and assume $f(x) \neq 0$. Then there is a neighborhood N_x of x such that $f|_{N_x} \neq 0$ and therefore $\frac{1}{f} \in C^\infty(N_x)$. Thus, by smoothness of $f \cdot T$, we have $\frac{1}{f} \cdot f \cdot T = T \in C^\infty(N_x)$, which contradicts $x \in \text{sing supp } T$. \square

Theorem 3.3.6. *Let M be a Lorentzian manifold of dimension d , $\pi: E \rightarrow M$ a real or complex vector bundle over M with non-degenerate inner product, $P: C^\infty(M, E) \rightarrow C^\infty(M, E)$ a formally self-adjoint wave operator and $\Omega \subset M$ a convex domain. Then the Hadamard coefficients $U_k \in C^\infty(\Omega \times \Omega, E^* \boxtimes E)$ are symmetric in the sense*

$$U_k(p, q) = \Theta_p U_k(q, p)^t \Theta_q^{-1}, \quad p, q \in \Omega, \quad k \in \mathbb{N}_0. \quad (3.12)$$

Proof. Let d be odd. For all $k, j \in \mathbb{N}_0$ with $j \leq k$, Lemma 1.4.2 (1) of [BGP2007] provides the recursion

$$R_\pm^\Omega(2k+2, p) = \underbrace{\frac{C(2k+2, d)}{C(2j+2, d)}}_{=: K_{k,j,d}} \Gamma(p, \cdot)^{k-j} \cdot R_\pm^\Omega(2j+2, p), \quad (3.13)$$

with $K_{k,j,d} \in \mathbb{R} \setminus \{0\}$ due to (2.12), so (3.11) can be rewritten into

$$R_\pm^\Omega(2, p) \sum_{k=0}^{\infty} K_{k,0,d} (\tilde{U}_k(p, \cdot) - \tilde{V}_k(\cdot, p)^t) \Gamma(p, \cdot)^k.$$

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The proof of Lemma 2.4.2 in [BGP2007] shows that

$$(p, q) \longmapsto \sum_{k=0}^{\infty} K_{k,0,d} (\tilde{U}_k(p, q) - \tilde{V}_k(q, p)^t) \Gamma(p, q)^k \in C^\infty(O \times O, E^* \boxtimes E).$$

Furthermore, we have $\text{sing supp } R_\pm^\alpha = C_\pm$ for all $\alpha \in \mathbb{C}$, and thus, $\text{sing supp } R_\pm^\Omega(\alpha, p) = C_\pm^\Omega(p)$ by (3.8). Hence, for lightlike separated p, q , Lemma 3.3.5 implies

$$0 = \sum_{k=0}^{\infty} (\tilde{U}_j(p, q) - \tilde{V}_j(q, p)^t) \Gamma(p, q)^k = U_0(p, q) - V_0(q, p)^t$$

since $\sigma\left(\frac{\Gamma(p,q)}{\varepsilon_k}\right) = \sigma(0) = 1$, that is $\tilde{U}_0(p, q) = U_0(p, q)$ and $\tilde{V}_0(p, q) = V_0(p, q)$. It follows from Lemma 3.3.4 that for $k = 0$, (3.12) is true also for even d and on all of $\Omega \times \Omega$.

Now let d again be odd, and for some $k_0 \in \mathbb{N}$, assume (3.12) to hold for all $k = 0, \dots, k_0 - 1$, i.e. the smooth section (3.11) is given by

$$\sum_{k=k_0}^{\infty} (\tilde{U}_k(p, \cdot) - \tilde{V}_k(\cdot, p)^t) R_\pm^\Omega(2k+2, p) = R_\pm^\Omega(2k_0+2, p) \sum_{k=k_0}^{\infty} K_{k,k_0,d} (\tilde{U}_k(p, \cdot) - \tilde{V}_k(\cdot, p)^t) \Gamma(p, \cdot)^{k-k_0}.$$

Analogously, we obtain

$$0 = \sum_{k=k_0}^{\infty} K_{k,k_0,d} (\tilde{U}_k(p, q) - \tilde{V}_k(q, p)^t) \Gamma(p, q)^{k-k_0} = U_{k_0}(p, q) - V_{k_0}(q, p)^t$$

if $\Gamma(p, q) = 0$, so again applying Lemma 3.3.4 completes the proof by induction. \square

Note that the induction would have been more elaborate for even d since $K_{\frac{d}{2},j,d} = 0$, which is circumvented by using Proposition 3.1.2.

4 The Prototype

"The universe is basically an animal. It grazes on the ordinary. It creates infinite idiots just to eat them."

In this chapter, we begin with the actual construction. Recall that (1.9) is a local condition and, moreover, that the singularity structure of a bisolution is related to the corresponding differential operator essentially via its principal symbol. On these grounds, we start with the prototype setting $P = \square$ on $M = \mathbb{R}_{\text{Mink}}^d$, $d \geq 3$, since, from the viewpoint of the singularity structure of the solutions, this already incorporates the characteristic properties of the solutions for the general setting of wave operators on curved spacetimes.

From Wightman's axioms, we deduce explicitly and in a rigorous manner solutions W of d'Alembert's equation. This can be directly done by employing Fourier transformation, but with regard to the subsequent local construction on curved spacetimes, we derive a local formulation instead. We introduce the distinguished fundamental solutions and identify them with R_{\pm}^2 and S_{\pm} , leading to the decomposition $W = \frac{i}{2}(S_+ - S_- + R_-^2 - R_+^2)$, from which we will read off explicitly the Hadamard form that motivated the definition given in [KW1991]. This decomposition and the corresponding identifications constitute the cornerstone of the upcoming construction in the general case.

4.1 Wightman's solution

Following Wightman's axiomatic framework on Minkowski space, a quantum field theory is regarded as a quadruple $(\mathcal{H}, U, \Phi, D)$, consisting of a Hilbert space $(\mathcal{H}, (\cdot, \cdot))$ with dense subspace D , a strongly continuous unitary representation U of the proper orthochronous Poincaré group \mathcal{P}_+^\uparrow on \mathcal{H} and a tempered distribution Φ on \mathbb{R}^d with values in the self-adjoint operators on \mathcal{H} such that Wightman's axioms are satisfied (see section IX.8 of [RS1975]). In particular, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we assume that $\text{ran}(\Phi[\varphi]) \subset D \subset \text{dom}(\Phi[\varphi])$ and the existence of a unique U -invariant unit vector $h_0 \in \mathcal{H}$, which is cyclic with respect to $\{\Phi[\varphi]\}_\varphi$. Furthermore, we demand U -invariance of $\varphi \mapsto \Phi[\varphi]$ and D , that is,

$$U(a, \Lambda)D \subset D, \quad U(a, \Lambda)\Phi[\varphi]U(a, \Lambda)^{-1} = \Phi[T_{(a, \Lambda)}\varphi], \quad \varphi \in \mathcal{S}(\mathbb{R}^d),$$

for all $(a, \Lambda) \in \mathcal{P}_+^\uparrow$ and $T_{(a, \Lambda)}\varphi(x) := \varphi(\Lambda^{-1}(x - a))$, $x \in \mathbb{R}^d$. The one-point-function then corresponds to the distribution $W_1[\varphi] := (\Phi[\varphi]h_0, h_0)_{\mathcal{H}}$, which is constant due to translation invariance of h_0 . Replacing Φ by $\Phi'[\varphi] := \Phi[\varphi] - W_1 \int \varphi \cdot \text{id}_{\mathcal{H}}$ shows that, without loss of generality, we can restrict ourselves to the case $W_1 = 0$. However, the two-point-function is given by the bidistribution

$$W_2: \quad \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathbb{C}, \quad (\varphi_1, \varphi_2) \longmapsto (\Phi[\varphi_1]\Phi[\varphi_2]h_0, h_0)_{\mathcal{H}}, \quad (4.1)$$

which is \mathcal{P}_+^\uparrow -invariant in the sense

$$W_2[\varphi_1, \varphi_2] = W_2[T_{(a, \Lambda)}\varphi_1, T_{(a, \Lambda)}\varphi_2], \quad (a, \Lambda) \in \mathcal{P}_+^\uparrow, \quad \varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d).$$

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Hence, by translation-invariance, W_2 is completely determined by some \mathcal{L}_+^\uparrow -invariant distribution W via

$$W_2[\varphi_1, \varphi_2] = \int_{\mathbb{R}^d} \varphi_1(x) \cdot W[T_{(x, -1)}\varphi] \, dx \quad (4.2)$$

for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ (see p. 66 of [RS1975]). Since Wightman's axiomatic framework does not encode dynamics, we additionally demand W_2 to be a bisolution. This results in $\square W = 0$ and we derive a local expression of the solution roughly following the lines of chapter 5 of [Ste2000]. We introduce the Fourier transformation as follows

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-i\langle\langle \xi, x \rangle\rangle} \, dx, \quad \xi \in \mathbb{R}^d, \quad (4.3)$$

which corresponds to the traditional Fourier transformation in space and the inverse transformation in time. Therefore, all important properties remain valid and in addition, \mathcal{L}_+^\uparrow -invariance is preserved. In particular, (4.3) yields a homeomorphism $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ with inverse map

$$\check{f}(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\xi) e^{i\langle\langle \xi, x \rangle\rangle} \, d\xi, \quad x \in \mathbb{R}^d,$$

and thus admits a continuous extension $\mathcal{S}(\mathbb{R}^d)' \rightarrow \mathcal{S}(\mathbb{R}^d)'$. It is for that reason that we only considered tempered distributions Φ , a restriction, which will be insignificant, eventually, since we pursue a local formulation of W not involving any Fourier transformation.

Proposition 4.1.1. *Let W be given by (4.2) and satisfy $\square W = 0$. Then \widehat{W} is a multiple of $d\Omega_0^+$ (2.10).*

Proof. $\square W = 0$ directly leads to $\text{supp } \widehat{W} \subset C$, since $\gamma^{-1}(0) = C$ and

$$0 = \square W[\varphi] = W[\square\varphi] = -\widehat{W}[\gamma \cdot \check{\varphi}], \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Moreover, the spectral condition (Property 2 in section IX.8 of [RS1975]), which is the Wightman axiom corresponding to the Hadamard condition, and Theorem IX.32 of [RS1975] lead to the constraint $\text{supp } \widehat{W} \subset C_+$. Furthermore, we consider a Hermitian theory, that is $\Phi[\check{\varphi}]|_D = \Phi[\varphi]^*|_D$, so

$$W[\varphi * \overline{R\varphi}] = \int_{\mathbb{R}^d} \varphi(x) \cdot W[\overline{\varphi(x - \cdot)}] \, dx = W_2[\varphi, \check{\varphi}] = \|\Phi[\varphi]h_0\|_{\mathcal{H}}^2 \geq 0,$$

where $R\varphi(x) := \varphi(-x)$. Therefore, W is a distribution of positive type, and hence, \widehat{W} is a measure due to Theorem IX.10 of [RS1975]. By Corollary 2.2.3, it has to be of the form $a \, d\Omega_0^+ + b\delta_0$. Recall that $(\Phi[\varphi]h_0, h_0)_{\mathcal{H}} = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, so the claim follows from Theorem IX.34 of [RS1975]. \square

It follows that $W = a \, d\check{\Omega}_0^+$ and we proceed by reformulating this in an entirely local manner. In consideration of the eventual extraction of fundamental solutions from W , we choose $a := (2\pi)^{\frac{2-d}{2}}$, i.e.

$$W := (2\pi)^{\frac{2-d}{2}} \cdot d\check{\Omega}_0^+. \quad (4.4)$$

4.2 Local expression for Wightman's solution

For $\langle\langle \cdot, \cdot \rangle\rangle$ the bilinear extension of the Minkowski product to \mathbb{C}^d and $T_{\pm} := \{z \in \mathbb{C}^d \mid \text{Im}(z) \in I_{\pm}\}$ the complex forward and backward tube, we define

$$\Delta^{\pm} : T_{\pm} \longrightarrow \mathbb{C}, \quad z \longmapsto \pm \frac{i}{(2\pi)^{d-1}} \int_{C_{\mp}} e^{i\langle\langle z, p \rangle\rangle} d\Omega_0^{\mp}(p). \quad (4.5)$$

Note that the integral always exists since for all $z'' = (z_0'', \hat{z}'') := \text{Im}(z) \in I_{\pm}$ and $p = (p_0, \hat{p}) \in C_{\mp}$, we have

$$\langle\langle z'', p \rangle\rangle = |z_0''| \cdot |p_0| + \langle \hat{z}'', \hat{p} \rangle \geq \|\hat{z}''\| \cdot \|\hat{p}\| + \langle \hat{z}'', \hat{p} \rangle \geq 0$$

due to the Cauchy-Schwarz inequality, and equality holds if and only if $p = 0$.

Lemma 4.2.1. *The functions (4.5) are holomorphic and \mathcal{L}_+^{\uparrow} -invariant, and they fulfill*

$$\Delta^{\pm}(z) = -\Delta^{\mp}(-z), \quad z \in T_{\pm}.$$

Proof. $e^{-\langle\langle \text{Im}(z), p \rangle\rangle}$ ensures the existence of the integral and, in particular, that all complex derivatives exist:

$$\frac{\partial}{\partial z_k} \Delta^{\pm}(z) = \pm \frac{i}{(2\pi)^{d-1}} \int_{C_{\mp}} i p_k e^{i\langle\langle z, p \rangle\rangle} d\Omega_0^{\mp}(p).$$

This already demonstrates holomorphicity and the other claims follow by direct calculation:

$$\begin{aligned} \Delta^{\pm}(\Lambda z) &= \pm \frac{i}{(2\pi)^{d-1}} \int_{C_{\mp}} e^{i\langle\langle z, \Lambda^{-1} p \rangle\rangle} d\Omega_0^{\mp}(p) = \pm \frac{i}{(2\pi)^{d-1}} \int_{\Lambda^{-1}(C_{\mp})} e^{i\langle\langle z, p \rangle\rangle} d\Omega_0^{\mp}(\Lambda p) = \Delta^{\pm}(z), \\ \Delta^{\pm}(z) &= \pm \frac{i}{(2\pi)^{d-1}} \int_{C_{\mp}} e^{i\langle\langle -z, -p \rangle\rangle} d\Omega_0^{\mp}(p) = \pm \frac{i}{(2\pi)^{d-1}} \int_{C_{\pm}} e^{i\langle\langle -z, p \rangle\rangle} d\Omega_0^{\pm}(p) = -\Delta^{\mp}(-z), \end{aligned}$$

for all $\Lambda \in \mathcal{L}_+^{\uparrow}$ and $z \in T_{\pm}$. □

Corollary 4.2.2. *Let e_0 denote the future-directed unit vector in time direction. Then, for all $z \in T_{\pm}$, we find $x \in \mathbb{R}^d$ and $\varepsilon > 0$ such that*

$$\Delta^{\pm}(z) = \Delta^{\pm}(x \pm i\varepsilon e_0).$$

Proof. \mathcal{L}_+^{\uparrow} -invariance yields $\Delta^{\pm}(z) = \Delta^{\pm}(\tilde{z})$ if $\gamma(z) = \gamma(\tilde{z})$. Let $z := z' + iz'' \in T_{\pm}$, that is, $z' \in \mathbb{R}^d$ and $z'' \in I_{\pm}$. Due to the transitive action of \mathcal{L}_+^{\uparrow} on H_{κ}^{\pm} (2.5), the choice $\varepsilon := \gamma(z'') > 0$ ensures the existence of some $\Lambda \in \mathcal{L}_+^{\uparrow}$ such that $\Lambda z'' = \pm \varepsilon e_0$. Choosing $x := \Lambda z'$, we have $x \pm i\varepsilon e_0 \in T_{\pm}$ and

$$\gamma(x \pm i\varepsilon e_0) = \underbrace{\gamma(x)}_{\gamma(\Lambda z')} - \underbrace{\gamma(\pm \varepsilon e_0)}_{=\gamma(\Lambda z'')} - 2i \underbrace{\langle\langle \Lambda z', \pm \varepsilon e_0 \rangle\rangle}_{=\langle\langle \Lambda z', \Lambda z'' \rangle\rangle} = \gamma(z') - \gamma(z'') - 2i \langle\langle z', z'' \rangle\rangle = \gamma(z). \quad \square$$

Hence, for all $\varepsilon > 0$, we are left with analytic functions

$$\Delta_{\varepsilon}^{\pm} : \mathbb{R}^d \longrightarrow \mathbb{C}, \quad x \longmapsto \Delta^{\pm}(x \pm i\varepsilon e_0). \quad (4.6)$$

Proposition 4.2.3. *The limit $\varepsilon \rightarrow 0$ of (4.6) exists in $\mathcal{S}(\mathbb{R}^d)'$ and we have*

$$W = i \lim_{\varepsilon \rightarrow 0} \Delta_{\varepsilon}^{-}. \quad (4.7)$$

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Proof. For all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, dominated convergence provides

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} i\Delta_{\varepsilon}^{\pm}[\varphi] &= \mp \frac{1}{(2\pi)^{d-1}} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} \int_{C_{\mp}} \varphi(x) \cdot e^{i\langle\langle x \pm i\varepsilon e_0, p \rangle\rangle} d\Omega_0^{\mp}(p) dx \\ &= \mp (2\pi)^{\frac{2-d}{2}} \lim_{\varepsilon \rightarrow 0} \int_{C_{\mp}} \check{\varphi}(p) e^{-\varepsilon\|p\|} d\Omega_0^{\mp}(p) = \mp (2\pi)^{\frac{2-d}{2}} d\Omega_0^{\mp}[\check{\varphi}], \end{aligned}$$

and hence $i \lim_{\varepsilon \rightarrow 0} \Delta_{\varepsilon}^{-} = (2\pi)^{\frac{2-d}{2}} d\check{\Omega}_0^{+} = W$. \square

Therefore, we found a formulation of W as the distributional limit of analytic functions

$$W(x) = i \lim_{\varepsilon \rightarrow 0} \Delta_{\varepsilon}^{-}(x) = \frac{1}{(2\pi)^{d-1}} \lim_{\varepsilon \rightarrow 0} \int_{C_{+}} e^{i\langle\langle x - i\varepsilon e_0, p \rangle\rangle} d\Omega_0^{+}(p),$$

so for a local formulation, we need to evaluate the integral. Due to $\mathcal{L}_{+}^{\uparrow}$ -invariance, $\Delta^{\pm}(z)$ depend on $z \in T_{\pm}$ only via $\gamma(z)$. Furthermore,

$$z = z' + iz'' \in T_{\pm} \quad \implies \quad \gamma(z) = \gamma(z') - \gamma(z'') - 2i\langle\langle z', z'' \rangle\rangle \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}, \quad (4.8)$$

since $\gamma(z'') > 0$, and moreover, $\langle\langle z', z'' \rangle\rangle = 0$ implies $\gamma(z') \leq 0$ due to the inverse Cauchy-Schwarz-inequality on I_{\pm} (chapter 5, Proposition 30 of [O'Ne1983]). It follows that $\sqrt{\gamma(z)} \in \mathbb{C} \setminus \mathbb{R}$ and thus, the square root yields a map $\sigma: T_{\pm} \rightarrow \{\pm \text{Im} > 0\} \subset \mathbb{C}$, i.e. $\sigma(z)^2 = \gamma(z)$ and the branch chosen such that $\pm \text{Im}(\sigma(z)) > 0$ for $z \in T_{\pm}$. Hence, $\sigma(z)e_0 \in T_{\pm}$ and $\gamma(z) = \gamma(\sigma(z)e_0)$, so similarly to Corollary 4.2.2, we obtain

$$\Delta^{\pm}(z) = \Delta^{\pm}(\sigma(z)e_0), \quad z \in T_{\pm}. \quad (4.9)$$

In particular, $z = x \pm i\varepsilon e_0$ leads to the expression

$$\sigma(x \pm i\varepsilon e_0) = \text{sgn}(x_0) \sqrt{\gamma_{\varepsilon}^{\pm}(x)}, \quad \gamma_{\varepsilon}^{\pm}(x) := \gamma(x \pm i\varepsilon e_0) = \gamma(x) - \varepsilon^2 \pm 2i\varepsilon x_0. \quad (4.10)$$

Proposition 4.2.4. *The distributions $\Delta^{\pm} := \lim_{\varepsilon \rightarrow 0} \Delta_{\varepsilon}^{\pm}$ are given by*

$$\Delta^{\pm} = \pm \frac{i\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}}} \cdot \lim_{\varepsilon \rightarrow 0} \left(-\gamma_{\varepsilon}^{\pm}\right)^{\frac{2-d}{2}}. \quad (4.11)$$

Proof. (4.9) and (4.10) provide

$$\Delta_{\varepsilon}^{\pm}(x) = \Delta^{\pm} \left(\text{sgn}(x_0) \sqrt{\gamma_{\varepsilon}^{\pm}(x)} e_0 \right),$$

so the integrals (4.5) can be calculated explicitly. Since $p_0 = \pm\|\hat{p}\|$ for $p \in C_{\pm}$, (2.10) implies

$$\begin{aligned} \Delta_{\varepsilon}^{\pm}(x) &= \pm \frac{i}{(2\pi)^{d-1}} \int_{C_{\mp}} e^{-i\sigma(x \pm i\varepsilon e_0)p_0} d\Omega_0^{\mp}(p) = \pm \frac{i}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} e^{\pm i\sigma(x \pm i\varepsilon e_0)\|\hat{p}\|} \frac{d\hat{p}}{2\|\hat{p}\|} \\ &= \pm \frac{i \cdot \text{vol } S^{d-2}}{2 \cdot (2\pi)^{d-1}} \int_0^{\infty} e^{\pm i\sigma(x \pm i\varepsilon e_0)p} \cdot p^{d-3} dp = \pm \frac{i}{(4\pi)^{\frac{d-1}{2}} \Gamma\left(\frac{d-1}{2}\right)} \cdot \frac{\Gamma(d-2)}{(\mp i\sigma(x \pm i\varepsilon e_0))^{d-2}}, \end{aligned}$$

and hence, the claim follows from $\mp i\sigma(x \pm i\varepsilon e_0) = \sqrt{-\gamma_{\varepsilon}^{\pm}(x)}$ and $\Gamma\left(\frac{d-2}{2}\right) \Gamma\left(\frac{d-1}{2}\right) = \frac{\sqrt{\pi}}{2^{d-3}} \Gamma(d-2)$. \square

In particular, away from the light cone, Δ^\pm is represented by the functions

$$\Delta^\pm(x) = \pm \frac{i\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}} \cdot |\gamma(x)|^{\frac{d-2}{2}}} \begin{cases} (\mp i \cdot \text{sgn}(x_0))^{2-d}, & x \in I, \\ 1, & x \notin J, \end{cases} \quad (4.12)$$

Remark 4.2.5. For $d = 4$, note that Δ^- satisfies or rather motivates the Hadamard condition given in [KW1991]. Furthermore, instead of e_0 , we could have chosen any other future-directed timelike vector and Δ^\pm would not depend on that choice. Unlike in [KW1991], there is no logarithmic contribution in our prototype setting. However, a term $\log(-\gamma_\varepsilon^\pm)$ appears already in the slightly more general case of the Klein-Gordon operator $\square + M^2$, $M > 0$.

Recall that the Fourier transform of Δ^\pm (4.4) is essentially given by the δ -measure along C_\pm , so it is constant in all lightlike and rapidly decreasing in all other directions. Hence, we obtain

$$WF(\Delta^\pm) = \{(x, \xi) \in \mathbb{R}^d \times \mathring{\mathbb{R}}^d \mid \xi \in C_\pm, x = \lambda\xi, \lambda \in \mathbb{R}\}. \quad (4.13)$$

4.3 Advanced and retarded fundamental solution

We proceed by extracting the advanced and retarded as well as the Feynman and anti-Feynman fundamental solution for \square in the sense of (1.11) from W and demonstrate coincidence with the expressions R_\pm^2, S_\pm derived in section 2.2. The antisymmetric part of the two-point-function (4.1) represents the expectation value of the commutator of the field in two spacetime regions $\text{supp } \varphi_1, \text{supp } \varphi_2$, and hence, it is supposed to vanish if these regions are non-causally related. Therefore, the following bidistribution is occasionally referred to as the causal propagator:

$$\Delta_2^C[\varphi_1, \varphi_2] := -i \langle h_0, [\Phi[\varphi_1], \Phi[\varphi_2]] h_0 \rangle_{\mathcal{H}} = -i (W_2[\varphi_1, \varphi_2] - W_2[\varphi_2, \varphi_1]), \quad \varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^d). \quad (4.14)$$

Similar to (4.2) and due to translation invariance, Δ_2^C is determined by some distribution Δ^C via

$$\begin{aligned} \Delta_2^C[\varphi_1, \varphi_2] &= -i \int_{\mathbb{R}^d} (\varphi_1(x)W[\varphi_2(x - \cdot)] - W[\varphi_2(x + \cdot)]\varphi_1(x - \cdot)) dx \\ &= -i \int_{\mathbb{R}^d} \varphi_1(x) \left(W[\varphi_2(x - \cdot)] - \underbrace{W[\varphi_2(x + \cdot)]}_{=(RW)[\varphi_2(x - \cdot)]} \right) dx \\ &=: \int_{\mathbb{R}^d} \varphi_1(x) \Delta^C[\varphi_2(x - \cdot)] dx, \end{aligned}$$

where $R\varphi(x) := \varphi(-x)$, and thus $\Delta^C = -i(W - RW)$.

Proposition 4.3.1. We have $\Delta^C = \Delta^- + \Delta^+$ and $\text{supp } \Delta^C \subset J$. Furthermore, $\text{supp } \Delta^C \subset C$ for d even, and otherwise

$$\Delta^C(x) = \pm(-1)^{\frac{d-1}{2}} \frac{\Gamma\left(\frac{d-2}{2}\right)}{2\pi^{\frac{d}{2}} \gamma(x)^{\frac{d-2}{2}}}, \quad x \in I_\pm.$$

Proof. Lemma 4.2.1 and (4.7) imply $W = i\Delta^- = -iR\Delta^+$, and hence

$$\Delta^C = -i(i\Delta^- + i\Delta^+) = \Delta^- + \Delta^+. \quad (4.15)$$

The rest now follows directly from (4.12). \square

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Therefore, Δ^C vanishes for all non-zero spacelike vectors, which confirms the expected causality property of (4.14).

Proposition 4.3.2. *Let H_0 denote the step function with respect to the time coordinate. Then $-H_0 \cdot \Delta^C$ is a well-defined distribution and yields a fundamental solution for \square .*

Proof. Since

$$\text{sing supp } \Delta^C \cap \text{sing supp } H_0 = C \cap \{x_0 = 0\} = \{0\} \neq \emptyset,$$

we have to check the wave front sets for Hörmander's criterion (Theorem 8.2.10 of [Hör1990]). (4.13) implies

$$\text{WF}(\Delta^C) \subset \{(x, \xi) \mid \xi \in C, x = \lambda \cdot \xi, \lambda \in \mathbb{R}\},$$

and due to $H_0 = H \otimes 1_{\mathbb{R}^{d-1}}$, where H the step function at 0 on \mathbb{R} , Theorem 8.2.9 of [Hör1990] provides

$$\text{WF}(H_0) \subset \underbrace{\text{WF}(H)}_{\{0\} \times \mathbb{R} \setminus \{0\}} \times (\mathbb{R}^{d-1} \times \{0\}) = \{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \mid x = (0, \hat{x}), \xi = (\xi_0, 0)\}.$$

It follows that the singular directions of H_0 are timelike, whereas those of Δ^C are lightlike, so Hörmander's criterion is satisfied and hence, $H_0 \cdot \Delta^C$ exists as a distribution.

It remains to show $\square(H_0 \cdot \Delta^C) = -\delta_0$ and we start by proving $\square \Delta_\varepsilon^\pm = 0$ for all $\varepsilon > 0$. From (4.8) follows that $\gamma(x \pm i\varepsilon e_0) \neq 0$ for all $x \in \mathbb{R}^d$, so $x \mapsto (-\gamma_\varepsilon^\pm(x))^{\frac{2-d}{2}}$ is smooth and we can directly calculate

$$\begin{aligned} \partial_j^2 (-\gamma_\varepsilon^\pm)^{\frac{2-d}{2}} &= \partial_j \left(\frac{d-2}{2} (-\gamma_\varepsilon^\pm)^{-\frac{d}{2}} \cdot \partial_j \gamma_\varepsilon^\pm \right) \\ &= \frac{d(d-2)}{4} (-\gamma_\varepsilon^\pm)^{-\frac{d+2}{2}} \cdot (\partial_j \gamma_\varepsilon^\pm)^2 + \frac{d-2}{2} (-\gamma_\varepsilon^\pm)^{-\frac{d}{2}} \cdot \partial_j^2 \gamma_\varepsilon^\pm. \end{aligned}$$

Since $\partial_0 \gamma_\varepsilon^\pm(x) = 2(x_0 \pm i\varepsilon)$ and $\partial_j \gamma_\varepsilon^\pm(x) = 2x_j$ for $j \neq 0$, we obtain

$$\square (-\gamma_\varepsilon^\pm)^{\frac{2-d}{2}} = \frac{d(d-2)}{4} (-\gamma_\varepsilon^\pm)^{-\frac{d+2}{2}} \cdot 4\gamma_\varepsilon^\pm + \frac{d-2}{2} (-\gamma_\varepsilon^\pm)^{-\frac{d}{2}} \cdot 2d = 0,$$

and hence, $\square \Delta_\varepsilon^\pm = 0$. Therefore, $\Delta_\varepsilon^C := \Delta_\varepsilon^+ + \Delta_\varepsilon^-$ is a solution for every $\varepsilon > 0$ such that for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, integration by parts yields

$$\begin{aligned} \square(H_0 \cdot \Delta^C)[\varphi] &= \lim_{\varepsilon \rightarrow 0} H_0[\Delta_\varepsilon^C \cdot \square\varphi] \\ &= \lim_{\varepsilon \rightarrow 0} \int_0^\infty \int_{\mathbb{R}^{d-1}} \Delta_\varepsilon^C(t, \hat{x}) \cdot \square\varphi(t, \hat{x}) \, d\hat{x} \, dt \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{d-1}} \left(\frac{\partial \Delta_\varepsilon^C}{\partial t}(0, \hat{x}) \cdot \varphi(0, \hat{x}) - \Delta_\varepsilon^C(0, \hat{x}) \cdot \frac{\partial \varphi}{\partial t}(0, \hat{x}) \right) d\hat{x}. \end{aligned}$$

The second integrand vanishes, since $(-\gamma_\varepsilon^\pm(0, \hat{x}))^{\frac{2-d}{2}} = (\|\hat{x}\|^2 + \varepsilon^2)^{\frac{2-d}{2}}$ and thus $\Delta_\varepsilon^C(0, \hat{x}) = 0$ by (4.11) and (4.15). For the first term

$$\frac{\partial \Delta_\varepsilon^\pm}{\partial t}(0, \hat{x}) = \pm \frac{i\Gamma\left(\frac{d-2}{2}\right) \cdot \left(\frac{2-d}{2}\right)}{4\pi^{\frac{d}{2}} (-\gamma_\varepsilon^\pm(0, \hat{x}))^{\frac{d}{2}}} \underbrace{(-\partial_0 \gamma_\varepsilon^\pm)(0, \hat{x})}_{=\mp 2i\varepsilon} = -\frac{\varepsilon \Gamma\left(\frac{d}{2}\right)}{2\pi^{\frac{d}{2}} (\|\hat{x}\|^2 + \varepsilon^2)^{\frac{d}{2}}}$$

results in

$$\frac{\partial \Delta_\varepsilon^C}{\partial t}(0, \hat{x}) = -\frac{\varepsilon \Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}} (\|\hat{x}\|^2 + \varepsilon^2)^{\frac{d}{2}}}.$$

The substitution $\hat{x} =: \varepsilon \hat{y}$ provides

$$H_0[\Delta_\varepsilon^C \cdot \square \varphi] = -\frac{\varepsilon \Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}}} \cdot \int_{\mathbb{R}^{d-1}} \frac{\varphi(0, \hat{x})}{(\|\hat{x}\|^2 + \varepsilon^2)^{\frac{d}{2}}} \mathbf{d}\hat{x} = -\frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}}} \cdot \int_{\mathbb{R}^{d-1}} \frac{\varphi(0, \varepsilon \hat{y})}{(\|\hat{y}\|^2 + 1)^{\frac{d}{2}}} \mathbf{d}\hat{y},$$

so we can take the limit $\varepsilon \rightarrow 0$ and finally obtain

$$\begin{aligned} \square(H_0 \cdot \Delta^C)[\varphi] &= -\frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}}} \cdot \varphi(0) \cdot \underbrace{\text{vol}(S^{d-2})}_{=\frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}} \int_0^\infty \frac{r^{d-2}}{(r^2 + 1)^{\frac{d}{2}}} \mathbf{d}r = -\varphi(0). \quad \square \\ &= \underbrace{\frac{\sqrt{\pi} \Gamma\left(\frac{d-1}{2}\right)}{2\Gamma\left(\frac{d}{2}\right)}} \end{aligned}$$

This provides the late motivation for the choice of the prefactor in (4.4) and furthermore, the advanced and retarded fundamental solution for \square in the sense of (1.11) are given by

$$\Delta^A := (1 - H_0) \cdot \Delta^C, \quad \Delta^R := -H_0 \cdot \Delta^C.$$

Hence, the support properties $\text{supp } \Delta^A \subset \bar{J}_-$, $\text{supp } \Delta^R \subset \bar{J}_+$ and Proposition 2.2.4 reveal the Riesz distributions R_\pm^2 as distinguished fundamental solutions:

$$\Delta^A = R_-^2, \quad \Delta^R = R_+^2.$$

Furthermore, we directly obtain that $\frac{i}{2}(\Delta^A - \Delta^R)$ represents the antisymmetric part of W .

4.4 Feynman and anti-Feynman fundamental solution

It remains to investigate the symmetric part of W . Recalling (1.12), the Feynman and anti-Feynman fundamental solution Δ^F, Δ^{aF} can be extracted via

$$\begin{aligned} \Delta^F &= iW + \Delta^A = (1 - H_0) \cdot \Delta^+ - H_0 \cdot \Delta^-, \\ \Delta^{aF} &= -iW + \Delta^R = (1 - H_0) \cdot \Delta^- - H_0 \cdot \Delta^+, \end{aligned} \quad (4.16)$$

which leads to the identities

$$W = \frac{i}{2}(\Delta^{aF} - \Delta^F + \Delta^A - \Delta^R), \quad \Delta^F + \Delta^{aF} = \Delta^A + \Delta^R. \quad (4.17)$$

We close the discussion of the prototype by showing that (4.16) correspond to the symmetric fundamental solutions S_\pm derived in paragraph 2.2.3.

Proposition 4.4.1. *For S_\pm given by (2.25), we have $\Delta^F = S_-$ and $\Delta^{aF} = S_+$.*

Proof. For $x \notin C$, (4.12) yields

$$\Delta^F(x) = (1 - H_0(x)) \cdot \Delta^+(x) - H_0(x) \cdot \Delta^-(x) = \frac{i\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}} \cdot |\gamma(x)|^{\frac{d-2}{2}}} \cdot \begin{cases} i^{2-d}, & x \in I, \\ 1, & x \in J^c, \end{cases}$$

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since $(1 - H_0(x))(-i \operatorname{sgn}(x_0))^{2-d} = i^{2-d}(1 - H_0(x))$ and $H_0(x)(i \operatorname{sgn}(x_0))^{2-d} = i^{2-d}H_0(x)$. Similarly,

$$\Delta^{aF}(x) = (1 - H_0(x)) \cdot \Delta^-(x) - H_0(x) \cdot \Delta^+(x) = -\frac{i\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}} \cdot |\gamma(x)|^{\frac{d-2}{2}}} \cdot \begin{cases} (-i)^{2-d}, & x \in I, \\ 1, & x \in J^c, \end{cases}$$

and, on the other hand, outside of C (2.25) becomes

$$S_{\pm}(x) = \mp \frac{i\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}} \cdot |\gamma(x)|^{\frac{d-2}{2}}} \cdot \begin{cases} (\mp i)^{2-d}, & x \in I, \\ 1, & x \in J^c, \end{cases}$$

Altogether, we obtain symmetric and \mathcal{L}_+^{\uparrow} -invariant solutions $\Delta^{aF} - S_+$, $\Delta^F - S_-$ with support on the light cone. Moreover, they are homogeneous of degree $2 - d$ as can be deduced directly from (2.26) and (4.11). Consequently, these differences vanish by Corollary 2.2.7 and the claimed equality holds. \square

From (4.17), we deduce the following final result of this chapter:

Theorem 4.4.2. *For Wightman's solution W and the distinguished fundamental solutions S_{\pm}, R_{\pm}^2 for \square , the following identities hold:*

$$W = \frac{i}{2}(S_+ - S_- + R_-^2 - R_+^2), \quad S_+ + S_- = R_+^2 + R_-^2.$$

5 Local Hadamard bisolutions

“Den Raum nehmen wir doch mit unseren Organen wahr, mit dem Gesichtssinn und dem Tastsinn. Schön. Aber welches ist denn unser Zeitorgan?”

We proceed with the local construction of Hadamard bidistributions for general wave operators on curved spacetimes. Therefore, let M be a time-oriented Lorentzian manifold of dimension $d \geq 3$, $\pi: E \rightarrow M$ a real or complex vector bundle over M and $P: C^\infty(M, E) \rightarrow C^\infty(M, E)$ a wave operator. Inspired by the local construction of the advanced and retarded parametrices \tilde{G}_\pm (3.10) for P in [Gün1988] and [BGP2007], we introduce families of distributions similar to the Riesz distributions (2.12) but containing S_\pm instead. After setting up a formal Hadamard series, some well-known procedure ([Fri1975, Gün1988, BGP2007]) locally produces left parametrices $\tilde{\mathcal{L}}_\pm$ for P^t . For E equipped with a non-degenerate inner product and P formally self-adjoint, they are right parametrices as well. We show that $\frac{i}{2}(\tilde{\mathcal{L}}_+ - \tilde{\mathcal{L}}_- + \tilde{G}_+ - \tilde{G}_-)$ is of Hadamard form with antisymmetric part given by $\frac{i}{2}(\tilde{G}_+ - \tilde{G}_-)$ and hence, $\tilde{\mathcal{L}}_\pm$ represents anti-Feynman and Feynman parametrices for P^t in the sense of (1.11). Finally, assuming M to be globally hyperbolic, we derive bisolutions of Hadamard form on suitably small domains.

As usual when working with wave equations, the qualitative behaviour of the solutions depends on whether the spacetime dimension d is even or odd. Thus, for notational convenience, we introduce the even numbers

$$\kappa_d := 2 \cdot \left\lfloor \frac{d}{2} \right\rfloor = \begin{cases} d+1, & d \text{ odd} \\ d, & d \text{ even} \end{cases}.$$

5.1 Families of Riesz-like distributions on Minkowski space

Let $(\gamma \pm i0)^\alpha$ denote the distributions derived in section 2.2.3 on d -dimensional Minkowski space, which are holomorphic in α on $\{\operatorname{Re} > -\frac{d}{2}\}$ by Proposition 2.2.8. For $\operatorname{Re}(\alpha) > 0$, we introduce the distributions

$$L_\pm^\alpha := C(\alpha, d) \cdot (\gamma \pm i0)^{\frac{\alpha-d}{2}}, \quad C(\alpha, d) := \frac{2^{-\alpha} \pi^{\frac{2-d}{2}}}{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha-d}{2} + 1\right)}. \quad (5.1)$$

As a product of holomorphic functions, $\alpha \mapsto C(\alpha, d)$ is holomorphic on all of \mathbb{C} , and moreover, the following recursion holds

$$C(\alpha + 2, d) = \frac{C(\alpha, d)}{\alpha(\alpha + 2 - d)}. \quad (5.2)$$

Then (2.25) implies $\square L_\pm^{\alpha+2} = L_\pm^\alpha$ and hence holomorphic extensions to all of \mathbb{C} via $L_\pm^\alpha := \square^k L_\pm^{\alpha+2k}$ with k chosen such that $\operatorname{Re}(\alpha) + 2k > 0$. These extensions are independent of k by the identity theorem, and we note that the zeros of $\Gamma\left(\frac{\alpha}{2}\right)^{-1}$ compensate the poles of $(\gamma \pm i0)^{\frac{\alpha-d}{2}}$, that is, $\Gamma\left(\frac{\alpha}{2}\right)^{-1} (\gamma \pm i0)^{\frac{\alpha-d}{2}}$ are holomorphic in α . Despite the strong resemblance of (5.1) with the Riesz distributions (2.12), a

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fundamental discrepancy consists of the missing support restriction. Moreover, the L_{\pm}^{α} yield symmetric distributions for all α . On the other hand, with regard to Lemma 1.2.4 of [BGP2007], the relations between the elements L_{\pm}^{α} are quite similar:

Proposition 5.1.1. *For all $\alpha \in \mathbb{C}$, we have*

- (1) $\gamma \cdot L_{\pm}^{\alpha} = \alpha(\alpha - d + 2)L_{\pm}^{\alpha+2}$,
- (2) $\text{grad}\gamma \cdot L_{\pm}^{\alpha} = 2\alpha \text{grad}L_{\pm}^{\alpha+2}$,
- (3) $\square L_{\pm}^{\alpha+2} = L_{\pm}^{\alpha}$,
- (4) $L_{\pm}^{d-2n} = 0$, $n \in \mathbb{N}$,
- (5) $L_{+}^{d+2n} = L_{-}^{d+2n}$, $n \in \mathbb{N}_0$,
- (6) if $\text{Re}(\alpha) > 0$, then L_{\pm}^{α} are distributions of order at most κ_d .

Proof. (1): Follows directly from (5.2).

(2): For $\sigma_0 := 1$ and $\sigma_j := -1$ if $j = 1, \dots, d-1$, we obtain $\partial_j \gamma = 2\sigma_j x_j$ and thus, (5.2) provides

$$2\alpha(\text{grad}L_{\pm}^{\alpha+2})_j = 2\alpha\partial_j L_{\pm}^{\alpha+2} = C(\alpha+2, d) \cdot 2\sigma_j x_j \cdot \alpha(\alpha+2-d)(\gamma \pm i0)^{\frac{\alpha-d}{2}} = 2\sigma_j x_j L_{\pm}^{\alpha} = (\text{grad}\gamma)_j L_{\pm}^{\alpha}.$$

(3): Follows from (2.25) and (5.2):

$$L_{\pm}^{\alpha} = \alpha(\alpha - d + 2)C(\alpha + 2, d) \cdot \frac{\square(\gamma \pm i0)^{\frac{\alpha+2-d}{2}}}{4\left(\frac{\alpha-d}{2} + 1\right)\left(\frac{\alpha-d}{2} + \frac{d}{2}\right)} = C(\alpha + 2, d) \cdot \square(\gamma \pm i0)^{\frac{\alpha+2-d}{2}} = \square L_{\pm}^{\alpha+2}.$$

(4): Due to (3), integration by parts yields

$$L_{\pm}^{d-2n}[\varphi] = L_{\pm}^d[\square^n \varphi] = C(d, d) \int_{\mathbb{R}^d} (\square^n \varphi)(x) dx = 0, \quad \varphi \in \mathcal{D}(\mathbb{R}^d), \quad n \in \mathbb{N}.$$

(5): Follows from $(\gamma \pm i0)^n = \gamma^n$ for all $n \in \mathbb{N}_0$.

(6): Since $\text{Re}(\alpha) + \kappa_d > d$, the maps $x \mapsto L_{\pm}^{\alpha+\kappa_d}(x)$ are continuous and hence distributions of order 0. As \square increases the order by at most 2, the claim follows from (3), that is, $L_{\pm}^{\alpha} = \square^{\frac{\kappa_d}{2}} L_{\pm}^{\alpha+\kappa_d}$. \square

With regard to the equality given by Proposition 5.1.1 (5), we write $L^{d+2n} := L_{\pm}^{d+2n}$. The crucial properties of the Riesz distributions for the construction of the advanced and retarded fundamental solution are $R_{\pm}^0 = \delta_0$. Clearly, this does not hold for L_{\pm}^0 if d is even, since then $n = \frac{d}{2}$, and Proposition 5.1.1 (4) implies $L_{\pm}^0 = 0$. Anyhow, it holds in the odd-dimensional case:

Proposition 5.1.2. *Let d be odd. Then $L_{\pm}^0 = \delta_0$ and $L_{\pm}^2 = S_{\pm}$.*

Proof. The first claim follows from the second one by Propositions 2.2.8 and 5.1.1 (3). For odd d , we have $\Gamma\left(\frac{d-2}{2}\right) \cdot \Gamma\left(\frac{4-d}{2}\right) = \frac{\pi}{\sin\left(\frac{d-2}{2}\pi\right)} = (-1)^{\frac{d+1}{2}} \pi$, and thus, (2.26) provides

$$L_{\pm}^2 = \frac{(-1)^{\frac{d+1}{2}} \cdot (-1)^{\frac{d+1}{2}}}{4\pi^{\frac{d-2}{2}} \Gamma\left(\frac{4-d}{2}\right)} (\gamma \pm i0)^{\frac{2-d}{2}} = \frac{(\pm i)^{d+1} \cdot \Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}}} \cdot (\gamma \pm i0)^{\frac{2-d}{2}} = S_{\pm}. \quad \square$$

Therefore, S_{\pm} is contained in $\{L_{\pm}^{\alpha}\}_{\alpha \in \mathbb{C}}$ if d is odd. In order to find corresponding families $\{\tilde{L}_{\pm}^{\alpha}\}_{\alpha \in \mathbb{C}}$ with $\tilde{L}_{\pm}^0 = \delta_0$ in the even-dimensional case, the naive idea is multiply L_{\pm}^{α} with a function of d and α , which

is singular for $\alpha = d - 2n$ and equal to one for $\alpha = d - 2n - 1$ for all $n \in \mathbb{N}$. However, this creates poles:

$$\tilde{L}_{\pm}^{\alpha} := \frac{(\pm i)^{d-\alpha-1}}{\sin\left(\frac{d-\alpha}{2}\pi\right)} \cdot L_{\pm}^{\alpha}, \quad \alpha \in \mathbb{C} \setminus \{\dots, d-2, d, d+2, \dots\}. \quad (5.3)$$

These are well-defined distributions and by employing $\Gamma\left(\frac{d-\alpha}{2}\right)\Gamma\left(\frac{\alpha-d}{2}+1\right) = \frac{\pi}{\sin\left(\frac{\pi}{2}(d-\alpha)\right)}$, we find

$$\tilde{L}_{\pm}^{\alpha} = \tilde{C}(\alpha, d) \cdot (\gamma \pm i0)^{\frac{\alpha-d}{2}}, \quad \tilde{C}(\alpha, d) := \frac{(\pm i)^{d-\alpha-1} \Gamma\left(\frac{d-\alpha}{2}\right)}{2^{\alpha} \pi^{\frac{d}{2}} \Gamma\left(\frac{\alpha}{2}\right)}. \quad (5.4)$$

Indeed, the zeros of L_{\pm}^{α} and the poles of the prefactor in (5.3) compensate, and hence, \tilde{L}_{\pm}^{α} exist as distributions for all $\alpha = d - 2n$, $n \in \mathbb{N}$. Therefore, $\alpha \mapsto \tilde{L}_{\pm}^{\alpha}[\varphi]$ are meromorphic functions with simple poles at $\alpha = d, d+2, \dots$ for fixed φ , and by definition, many properties of L_{\pm}^{α} are directly adopted:

Proposition 5.1.3. *For all $\alpha \neq d - 2, d, d + 2, \dots$, we have*

- (1) $\gamma \cdot \tilde{L}_{\pm}^{\alpha} = \alpha(\alpha - d + 2)\tilde{L}_{\pm}^{\alpha+2}$,
- (2) $\text{grad}\gamma \cdot \tilde{L}_{\pm}^{\alpha} = 2\alpha \cdot \text{grad}\tilde{L}_{\pm}^{\alpha+2}$,
- (3) $\square\tilde{L}_{\pm}^{\alpha+2} = \tilde{L}_{\pm}^{\alpha}$,
- (4) if $\text{Re}(\alpha) > 0$, then \tilde{L}_{\pm}^{α} are distributions of order at most κ_d ,
- (5) $\tilde{L}_{\pm}^0 = \delta_0$.

Proof. (1) – (4) follow directly from Proposition 5.1.1 via (5.3). For (5), we just compare (5.3) and (2.26):

$$\tilde{L}_{\pm}^2 = \tilde{C}(2, d) \cdot (\gamma \pm i0)^{\frac{2-d}{2}} = \frac{(\pm i)^{d+1} \cdot \Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}}} \cdot (\gamma \pm i0)^{\frac{2-d}{2}} = S_{\pm}. \quad \square$$

Thus, the Feynman and anti-Feynman fundamental solution S_{\pm} for \square are contained in the families $\{\tilde{L}_{\pm}^{\alpha}\}_{\alpha}$ and we close this section by particularly investigating the integers, where \tilde{L}_{\pm}^{α} are non-singular:

Proposition 5.1.4. *For $\alpha \in \mathbb{Z} \setminus \{d, d + 2, \dots\}$, we have*

$$\tilde{L}_{\pm}^{\alpha} = \begin{cases} L_{\pm}^{\alpha}, & d - \alpha \text{ odd,} \\ \pm \frac{i}{\pi} L^d \square^n \log(\gamma \pm i0), & \alpha = d - 2n, n \in \mathbb{N}, \end{cases} \quad (5.5)$$

where the distributions $\log(\gamma \pm i0)$ are given by

$$\log(\gamma \pm i0) = \frac{1}{2d} \square(\gamma \log(\gamma \pm i0)) - \frac{d+2}{d}. \quad (5.6)$$

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Proof. For $d - \alpha$ odd, we directly obtain $(\pm i)^{d-\alpha-1} = (-1)^{\frac{d-\alpha-1}{2}} = \sin\left(\frac{d-\alpha}{2}\pi\right)$ and thus, $\tilde{L}_{\pm}^{\alpha} = L_{\pm}^{\alpha}$ follows from (5.3). Furthermore, (5.1) yields $L_{\pm}^d = C(d, d) = \frac{\pi^{\frac{2-d}{2}}}{2^d \Gamma(\frac{d}{2})}$, so (5.4) and Proposition 2.2.8 imply

$$\begin{aligned}\tilde{L}_{\pm}^{d-2n} &= \tilde{C}(d-2n, d)(\gamma \pm i0)^{-n} \\ &= \pm i(-1)^{n-1} \cdot \frac{(n-1)!}{2^{d-2n} \pi^{\frac{d}{2}} \Gamma\left(\frac{d}{2} - n\right)} \cdot \frac{(-1)^{n-1} \cdot \square^n \log(\gamma \pm i0)}{4^n (n-1)! \prod_{k=1}^n \left(k - n + \frac{d-2}{2}\right)} \\ &= \pm \frac{i}{\pi} \underbrace{\frac{\pi^{\frac{2-d}{2}}}{2^d \Gamma\left(\frac{d}{2}\right)}}_{=C(d,d)=L_{\pm}^d} \square^n \log(\gamma \pm i0).\end{aligned}$$

A direct calculation provides $\square(\gamma \log(\gamma \pm i0)) = 2d \log(\gamma \pm i0) + 2(d+2)$ and thus (5.6). \square

Proposition 5.1.5. *For all $\alpha \in \mathbb{Z}$ and $n \in \mathbb{N}$, the following expressions are \mathbb{R} -valued:*

$$i(L_+^{\alpha} - L_-^{\alpha}), \quad i(\tilde{L}_+^{d-2n} - \tilde{L}_-^{d-2n}), \quad L^{d+2(n-1)}(\log(\gamma + i0) + \log(\gamma - i0)). \quad (5.7)$$

Proof. With regard to the definition of L_{\pm}^{α} as holomorphic extensions, for the first claim, it suffices to check the case $\alpha \geq d$. If $\alpha - d$ is even, Proposition 5.1.1 (5) directly shows $i(L_+^{\alpha} - L_-^{\alpha}) = 0$. On the other hand, for odd $\alpha - d$, we calculate

$$i(L_+^{\alpha} - L_-^{\alpha}) = iC(\alpha, d)\left((\gamma + i0)^{\frac{\alpha-d}{2}} - (\gamma - i0)^{\frac{\alpha-d}{2}}\right) = \begin{cases} 0, & \gamma \geq 0 \\ (-1)^{\frac{\alpha-d+1}{2}} \cdot 2(-\gamma)^{\frac{\alpha-d}{2}}, & \gamma < 0 \end{cases}.$$

Since $L^{d+2(n-1)} = C(d+2(n-1), d) \gamma^{n-1}$ is obviously real for all $n \in \mathbb{N}$, for the third term of (5.7), we just have to check the logarithms. Indeed, due to

$$\log(\gamma \pm i0) = \begin{cases} \log \gamma, & \gamma > 0, \\ \log(-\gamma) \pm i\pi, & \gamma < 0, \end{cases}$$

it equals $C(d-2(n-1), d)\gamma^{n-1} \log|\gamma|$, which is \mathbb{R} -valued. This shows the claim also for the second term using Proposition 5.1.4:

$$i(\tilde{L}_+^{d-2n} - \tilde{L}_-^{d-2n}) = -\frac{L^d}{\pi} \square^n (\log(\gamma + i0) + \log(\gamma - i0)). \quad \square$$

Remark 5.1.6. *It is not hard to show that also for $d = 1$ and $d = 2$, (5.1) and Proposition 5.1.5 provide the respective fundamental solutions:*

$$L^2(x) = \frac{|x|}{2}, \quad \tilde{L}_{\pm}^2 = \pm \frac{i}{2\pi} \log(\gamma \pm i0).$$

Therefore, it follows that $\tilde{L}_{\pm}^0 = \delta_0$ also in these cases.

5.2 Families of Riesz-like distributions on a convex domain

Following section 1.4 of [BGP2007], we now transfer the families $\{L_{\pm}^{\alpha}\}_{\alpha \in \mathbb{C}}$, $\{\tilde{L}_{\pm}^{\alpha}\}_{\alpha \in \mathbb{C} \setminus \{d, d+2, \dots\}}$ locally to M . For $p \in M$ and $\Omega \subset M$ geodesically starshaped with respect to p , let $\Gamma_p(q) := \Gamma(p, q) = \gamma(\exp_p^{-1}(q))$ denote the squared Lorentz distance to p and μ_p the distortion function defined by (1.9) in [BGP2007]. Analogous to (3.8), we define the corresponding distributions on Ω via

$$L_{\pm}^{\Omega}(\alpha, p) := (\exp_p)_* L_{\pm}^{\alpha}, \quad \tilde{L}_{\pm}^{\Omega}(\alpha, p) := (\exp_p)_* \tilde{L}_{\pm}^{\alpha}, \quad (5.8)$$

i.e. $L_{\pm}^{\Omega}(\alpha, p)[\varphi] = L_{\pm}^{\alpha}[(\mu_p \varphi) \circ \exp_p]$ for all $\varphi \in \mathcal{D}(\Omega)$ due to Theorem 10.11 of [DK2010]. These are well-defined distributions on Ω , since $\mu_p \varphi \in \mathcal{D}(\Omega)$ and

$$\text{supp}(\exp_p^*(\mu_p \varphi)) \subset \exp_p^{-1}(\text{supp}(\mu_p \varphi)),$$

which is compact due to continuity of \exp_p , and hence, $(\exp_p)^*(\mu_p \varphi) \in \mathcal{D}(\exp_p^{-1}(\Omega))$. Consequently, for fixed $\varphi \in \mathcal{D}(\Omega)$, they provide holomorphic maps $\alpha \mapsto L_{\pm}^{\Omega}(\alpha, p)[\varphi]$ and meromorphic maps $\alpha \mapsto \tilde{L}_{\pm}^{\Omega}(\alpha, p)[\varphi]$ with simple poles at $\alpha = d, d+2, \dots$

Proposition 5.2.1. *Let $p \in M$ and $\Omega \subset M$ be geodesically starshaped with respect to p . Then, for all $\alpha \in \mathbb{C}$, we have:*

(1) For $\text{Re}(\alpha) > d$, the maps $p \mapsto L_{\pm}^{\Omega}(\alpha, p)$ are continuous on Ω and given by

$$L_{\pm}^{\Omega}(\alpha, p) = C(\alpha, d) (\Gamma_p \pm i0)^{\frac{\alpha-d}{2}}. \quad (5.9)$$

(2) $\Gamma_p \cdot L_{\pm}^{\Omega}(\alpha, p) = \alpha(\alpha - d + 2) \cdot L_{\pm}^{\Omega}(\alpha + 2, p)$,

(3) $\text{grad} \Gamma_p \cdot L_{\pm}^{\Omega}(\alpha, p) = 2\alpha \text{grad} L_{\pm}^{\Omega}(\alpha + 2, p)$,

(4) $\square L_{\pm}^{\Omega}(\alpha + 2, p) = \left(\frac{\square \Gamma_p - 2d}{2\alpha} + 1 \right) \cdot L_{\pm}^{\Omega}(\alpha, p)$, $\alpha \neq 0$.

(5) For $\text{Re}(\alpha) > 0$, (5.8) yield distributions of order at most κ_d . Moreover, there is an open neighborhood U of p and some $C > 0$ such that

$$|L_{\pm}^{\Omega}(\alpha, q)[\varphi]| \leq C \cdot \|\varphi\|_{C^{\kappa_d}(\Omega)}, \quad q \in U, \varphi \in \mathcal{D}(\Omega).$$

(6) Let $U \subset \Omega$ be an open neighborhood of p such that Ω is geodesically starshaped with respect to all $q \in U$. Furthermore, let $\text{Re}(\alpha) > 0$ and $V \in C^{\kappa_d+k}(U \times \Omega)$ such that $\text{supp} V(q, \cdot) \subset \Omega$ is compact for all $q \in U$. Then $q \mapsto L_{\pm}^{\Omega}(\alpha, q)[V(q, \cdot)] \in C^k(U)$.

(7) For all $\varphi \in C_c^k(\Omega)$, the map $\alpha \mapsto L_{\pm}^{\Omega}(\alpha, p)[\varphi]$ is holomorphic on $\{\text{Re}(\alpha) > d - 2[\frac{k}{2}]\}$.

On the domain of holomorphicity of $\tilde{L}_{\pm}^{\Omega}(\alpha, p)$, the statements (1) – (7) remain true, when we replace $L_{\pm}^{\Omega}(\alpha, p)$ and $C(\alpha, d)$ by $\tilde{L}_{\pm}^{\Omega}(\alpha, p)$ and $\tilde{C}(\alpha, d)$.

(8) For $d - \alpha$ an odd integer, we have $L_{\pm}^{\Omega}(\alpha, p) = \tilde{L}_{\pm}^{\Omega}(\alpha, p)$.

(9) $\tilde{L}_{\pm}^{\Omega}(0, p) = \delta_p$.

(10) For all $n \in \mathbb{N}$, we have $L_{\pm}^{\Omega}(d - 2n, p) = 0$ and $L_{\pm}^{\Omega}(d + 2n - 2, p) = L_{\pm}^{\Omega}(d + 2n - 2, p)$.

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Proof. (1): By definition of L_{\pm}^{Ω} and μ_p , for all $\varphi \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned} L_{\pm}^{\Omega}(\alpha, p)[\varphi] &= L_{\pm}^{\alpha}[(\mu_p \cdot \varphi) \circ \exp_p] = C(\alpha, d) \int_{\exp_p^{-1}(\Omega)} (\gamma(x) \pm i0)^{\frac{\alpha-d}{2}} \cdot (\varphi \cdot \mu_p)(\exp_p(x)) \, dx \\ &= C(\alpha, d) \int_{\Omega} (\Gamma_p(q) \pm i0)^{\frac{\alpha-d}{2}} \cdot \varphi(q) \cdot \underbrace{\mu_p(q) ((\exp_p^{-1})^* dx)(q)}_{=dV(q)} = C(\alpha, d) \int_{\Omega} (\Gamma_p(q) \pm i0)^{\frac{\alpha-d}{2}} \cdot \varphi(q) \, dV(q). \end{aligned}$$

(2), (3): Follow directly from Definition (5.8) and Proposition 5.1.1.

(4): Let $\operatorname{Re}(\alpha) > d + 2$, so we can employ (5.9) and $\langle \operatorname{grad} \Gamma_p, \operatorname{grad} \Gamma_p \rangle = -4\Gamma_p$ (Lemma 1.3.19 of [BGP2007]). Then (2) and (3) yield

$$\begin{aligned} \square L_{\pm}^{\Omega}(\alpha + 2, p) &= -\operatorname{div} \operatorname{grad} L_{\pm}^{\Omega}(\alpha + 2, p) = -\frac{1}{2\alpha} \operatorname{div} (\operatorname{grad} \Gamma_p \cdot L_{\pm}^{\Omega}(\alpha, p)) \\ &= \frac{1}{2\alpha} (\square \Gamma_p \cdot L_{\pm}^{\Omega}(\alpha, p) - \langle \operatorname{grad} L_{\pm}^{\Omega}(\alpha, p), \operatorname{grad} \Gamma_p \rangle) \\ &= \frac{1}{2\alpha} \left(\square \Gamma_p \cdot L_{\pm}^{\Omega}(\alpha, p) - \frac{L_{\pm}^{\Omega}(\alpha - 2, p)}{2(\alpha - 2)} \langle \operatorname{grad} \Gamma_p, \operatorname{grad} \Gamma_p \rangle \right) \\ &= \left(\frac{\square \Gamma_p}{2\alpha} + \frac{\alpha - d}{\alpha} \right) L_{\pm}^{\Omega}(\alpha, p) = \left(\frac{\square \Gamma_p - 2d}{2\alpha} + 1 \right) L_{\pm}^{\Omega}(\alpha, p). \end{aligned}$$

Both sides are holomorphic on $\{\operatorname{Re}(\alpha) > d + 2\}$ and can be extended uniquely and holomorphically to the punctured plane $\mathbb{C} \setminus \{0\}$, where they consequently have to coincide by the identity theorem.

(5): Since μ_p is smooth and \exp_p a diffeomorphism, the order of $L_{\pm}^{\Omega}(\alpha, p)$ coincides with the one of L_{\pm}^{α} , which is at most κ_d due to Proposition 5.1.1. Hence, for $\Omega' := \exp_p^{-1}(\Omega)$, we find some C' such that $|L_{\pm}^{\alpha}[\varphi']| \leq C' \|\varphi'\|_{C^{\kappa_d}(\Omega')}$, $\varphi' \in \mathcal{D}(\Omega')$, and therefore,

$$|L_{\pm}^{\Omega}(\alpha, p)[\varphi]| \leq C' \|(\mu_p \circ \exp_p) \cdot (\varphi \circ \exp_p)\|_{C^{\kappa_d}(\Omega')} \leq C \|\varphi\|_{C^{\kappa_d}(\Omega)}.$$

(6): For any linear isometry $A(p, q): T_p\Omega \rightarrow T_q\Omega$ and all $q \in U$, we have

$$(\mu_q \cdot V(q, \cdot)) \circ \exp_q \circ A(p, q) \in C_c^{k+d}(\exp_p^{-1}(U)),$$

and hence, $q \mapsto L_{\pm}^{\Omega}(\alpha, q)[V(q, \cdot)] \in C^k(U)$ by Lemma 1.1.6 of [BGP2007] and (5).

(7): $\alpha \mapsto L_{\pm}^{\Omega}(\alpha + 2l, x)[\square^l \varphi]$ is well-defined if $2l < k$ and $\alpha + 2l > d$, for which the maximal possible integer is $l = \lfloor \frac{k}{2} \rfloor$. Hence, $\alpha \mapsto L_{\pm}^{\Omega}(\alpha, x)[\varphi]$ is holomorphic on $\{\operatorname{Re}(\alpha) > d - 2\lfloor \frac{k}{2} \rfloor\}$.

By employing the respective properties of \tilde{L}_{\pm}^{α} , the proofs for $\tilde{L}_{\pm}^{\Omega}(\alpha, p)$ are identical.

(8): Follows directly from (5.5).

(9): Due to $\tilde{L}_{\pm}^0 = \delta_0$ and $\mu_p(p) = 1$ by Lemma 1.3.17 of [BGP2007], we directly obtain

$$\tilde{L}_{\pm}^{\Omega}(0, p)[\varphi] = \tilde{L}_{\pm}^0[(\mu_p \cdot \varphi) \circ \exp_p] = (\mu_p \cdot \varphi)(\exp_p(0)) = \underbrace{\mu_p(p)}_{=1} \varphi(p), \quad \varphi \in \mathcal{D}(\Omega).$$

(10): Follows from Proposition 5.1.1. □

Note that (8) and (9) of Proposition 5.2.1 imply $L_{\pm}^{\Omega}(0, p) = \delta_p$ for odd d , and due to (10), we again set $L^{\Omega}(d + 2n, p) := L_{\pm}^{\Omega}(d + 2n, p)$ for all $n \in \mathbb{N}_0$. Similarly to (5.8), $L_{\pm}^{\Omega}(d, p) \log(\Gamma_p \pm i0)$ is given by $\mu_p \cdot (\exp_p)_* (L_{\pm}^d \log(\gamma \pm i0))$ and thus by (5.6).

With regard to the Hadamard series and the recursions in the next section, we prove the following technical Lemma, which relates $\tilde{L}_{\pm}^{\Omega}(\alpha, p)$ and $\square(L_{\pm}^{\Omega}(d, p) \log(\Gamma_p \pm i0))$ for even d .

Lemma 5.2.2. *Let d be even and $k \in \mathbb{N}$ with $k \geq \frac{d}{2}$. Then we have*

$$\begin{aligned} & \text{grad} (L^{\Omega}(2k+2, p) \cdot \log(\Gamma_p \pm i0)) \\ &= \frac{\text{grad}\Gamma_p}{4k} \cdot L^{\Omega}(2k, p) \log(\Gamma_p \pm i0) + \frac{\text{grad}\Gamma_p}{2k(2k+2-d)} \cdot L^{\Omega}(2k, p), \\ & \square (L^{\Omega}(2k+2, p) \cdot \log(\Gamma_p \pm i0)) \\ &= \frac{\frac{\square\Gamma_p}{2} - d + 2k}{2k} \cdot L^{\Omega}(2k, p) \cdot \log(\Gamma_p \pm i0) + \frac{\frac{\square\Gamma_p}{2} - 2d + 4k + 2}{2k(k - \frac{d-2}{2})} \cdot L^{\Omega}(2k, p), \end{aligned}$$

and for $k = \frac{d-2}{2}$

$$\begin{aligned} \text{grad} (L^{\Omega}(d, p) \cdot \log(\Gamma_p \pm i0)) &= \mp \frac{i\pi \cdot \text{grad}\Gamma_p}{2(d-2)} \cdot \tilde{L}_{\pm}^{\Omega}(d-2, p) \\ \square (L^{\Omega}(d, p) \cdot \log(\Gamma_p \pm i0)) &= \mp \frac{i\pi (\square\Gamma_p - 4)}{4(d-2)} \cdot \tilde{L}_{\pm}^{\Omega}(d-2, p). \end{aligned}$$

Proof. Proposition 5.2.1 (3) provides

$$\begin{aligned} & \text{grad} (L^{\Omega}(2k+2, p) \cdot \log(\Gamma_p \pm i0)) \\ &= \log(\Gamma_p \pm i0) \cdot \text{grad}L^{\Omega}(2k+2, p) + L^{\Omega}(2k+2, p) \cdot \text{grad} \log(\Gamma_p \pm i0) \\ &= \frac{\text{grad}\Gamma_p}{4k} \cdot L^{\Omega}(2k, p) \log(\Gamma_p \pm i0) + \frac{\Gamma_p \cdot L^{\Omega}(2k, p)}{2k(2k+2-d)} \cdot \frac{\text{grad}\Gamma_p}{\Gamma_p} \\ &= \frac{\text{grad}\Gamma_p}{4k} \cdot L^{\Omega}(2k, p) \log(\Gamma_p \pm i0) + \frac{\text{grad}\Gamma_p}{2k(2k+2-d)} \cdot L^{\Omega}(2k, p), \end{aligned}$$

and hence, $\langle \text{grad}\Gamma_p, \text{grad}\Gamma_p \rangle = -4\Gamma_p$ implies

$$\begin{aligned} & \square (L^{\Omega}(2k+2, p) \cdot \log(\Gamma_p \pm i0)) \\ &= -\text{div} \left(\frac{\text{grad}\Gamma_p}{4k} \cdot L^{\Omega}(2k, p) \log(\Gamma_p \pm i0) + \frac{\text{grad}\Gamma_p}{2k(2k+2-d)} \cdot L^{\Omega}(2k, p) \right) \\ &= \frac{1}{4k} \left(\square\Gamma_p \cdot L^{\Omega}(2k, p) \log(\Gamma_p \pm i0) - \frac{L^{\Omega}(2k-2, p) \log(\Gamma_p \pm i0)}{4(k-1)} \langle \text{grad}\Gamma_p, \text{grad}\Gamma_p \rangle \right. \\ & \quad \left. - L^{\Omega}(2k, p) \cdot \frac{\langle \text{grad}\Gamma_p, \text{grad}\Gamma_p \rangle}{\Gamma_p} \right) + \frac{\square\Gamma_p \cdot L^{\Omega}(2k, p) - \frac{L^{\Omega}(2k-2, p)}{4(k-1)} \cdot \langle \text{grad}\Gamma_p, \text{grad}\Gamma_p \rangle}{2k(2k+2-d)} \\ &= \frac{1}{4k} (\square\Gamma_p + 2(2k-d)) L^{\Omega}(2k, p) \log(\Gamma_p \pm i0) + \left(\frac{1}{k} + \frac{\square\Gamma_p + 2(2k-d)}{2k(2k+2-d)} \right) L^{\Omega}(2k, p) \\ &= \frac{\frac{\square\Gamma_p}{2} - d + 2k}{2k} \cdot L^{\Omega}(2k, p) \log(\Gamma_p \pm i0) + \frac{\frac{\square\Gamma_p}{2} - 2n + 4k + 2}{2k(k - \frac{d-2}{2})} \cdot L^{\Omega}(2k, p). \end{aligned}$$

By Proposition 5.2.1 (1) $L^\Omega(d, p) = C(d, d) = \mp \frac{i\pi \tilde{C}(d-2, d)}{2(d-2)}$ is constant on Ω , so again using (3) yields

$$\text{grad} (L^\Omega(d, p) \cdot \log(\Gamma_p \pm i0)) = \mp \frac{i\pi \tilde{C}(d-2, d)}{2(d-2)} \cdot \frac{\text{grad}\Gamma_p}{\Gamma_p} = \mp \frac{i\pi \cdot \text{grad}\Gamma_p}{2(d-2)} \cdot \tilde{L}_\pm^\Omega(d-2, p),$$

$$\begin{aligned} \square (L^\Omega(d, p) \cdot \log(\Gamma_p \pm i0)) &= \pm \frac{i\pi}{2(d-2)} \text{div} \left(\text{grad}\Gamma_p \cdot \tilde{L}_\pm^\Omega(d-2, p) \right) \\ &= \mp \frac{i\pi}{2(d-2)} \left(\square\Gamma_p \cdot \tilde{L}_\pm^\Omega(d-2, p) - \frac{\langle \text{grad}\Gamma_p, \text{grad}\Gamma_p \rangle}{2(d-4)} \cdot \tilde{L}_\pm^\Omega(d-4, p) \right) \\ &= \mp \frac{i\pi}{2(d-2)} \left(\square\Gamma_p + \frac{4}{2(d-4)} \cdot (d-4)(d-4-d+2) \right) \cdot \tilde{L}_\pm^\Omega(d-2, p) \\ &= \mp \frac{i\pi (\square\Gamma_p - 4)}{2(d-2)} \cdot \tilde{L}_\pm^\Omega(d-2, p). \quad \square \end{aligned}$$

For $\Omega \subset M$ convex and $\text{Re}(\alpha) > d$, the symmetry of Γ on $\Omega \times \Omega$ implies symmetry of the continuous functions $L_\pm^\Omega(\alpha), \tilde{L}_\pm^\Omega(\alpha)$ by Proposition 5.2.1 (1). We finish this section by showing that this remains true for $\text{Re}(\alpha) \leq d$ in the sense of bidistributions:

Lemma 5.2.3. *Let $\Omega \subset M$ be convex, $\alpha \in \mathbb{C}$ and $u \in \mathcal{D}(\Omega \times \Omega)$. Then we have*

$$\int_\Omega L_\pm^\Omega(\alpha, p)[u(p, \cdot)] dV(p) = \int_\Omega L_\pm^\Omega(\alpha, q)[u(\cdot, q)] dV(q),$$

which similarly holds for $\tilde{L}_\pm^\Omega(\alpha)$.

Proof. Replacing the Riesz distributions and their antisymmetry property $R_\pm^\Omega(\alpha, p)(q) = R_\mp^\Omega(\alpha, q)(p)$ by the symmetric $L_\pm^\Omega, \tilde{L}_\pm^\Omega$, respectively, the proof coincides with the one of Lemma 1.4.3 in [BGP2007]. \square

Remark 5.2.4. *Considering $L_\pm^\Omega(\alpha)$ as bidistributions via*

$$L_\pm^\Omega(\alpha)[\varphi, \psi] = \int_\Omega L_\pm^\Omega(\alpha, p)[\varphi] \cdot \psi(p) dV(p), \quad \varphi, \psi \in \mathcal{D}(\Omega), \quad (5.10)$$

Lemma 5.2.3 provides symmetry in the sense $L_\pm^\Omega(\alpha)[\varphi, \psi] = L_\pm^\Omega(\alpha)[\psi, \varphi]$.

5.3 The Hadamard series

Let E be a real vector bundle over M and $P: C^\infty(M, E) \rightarrow C^\infty(M, E)$ a wave operator. Adopting the approach pursued in section 5.2 of [Gar1964], we start the deduction of local expressions for Feynman and anti-Feynman parametrices for P by taking the following ansatz of a formal Hadamard series:

$$\mathcal{L}_\pm(p) := \begin{cases} \sum_{k=0}^{\infty} U_p^k L_\pm^\Omega(2k+2, p) + \sum_{k=\frac{d-2}{2}}^{\infty} W_p^k L^\Omega(2k+2, p), & d \text{ odd}, \\ \sum_{k=0}^{\frac{d-4}{2}} U_p^k \tilde{L}_\pm^\Omega(2k+2, p) \pm \frac{i}{\pi} \sum_{k=\frac{d-2}{2}}^{\infty} (U_p^k \log(\Gamma_p \pm i0) + W_p^k) L^\Omega(2k+2, p), & d \text{ even}, \end{cases} \quad (5.11)$$

with coefficients $U_p^k, W_p^k \in C^\infty(\Omega, E_p^* \otimes E)$ yet to be determined. For $\varphi \in \mathcal{D}(\Omega, E^*)$, we identify $U_p^k \varphi, W_p^k \varphi$ with E_p^* -valued test functions (see section 2.1 of [BGP2007]), so $\mathcal{L}_\pm(p)$ is (formally) un-

derstood as a distribution on $\mathcal{D}(\Omega, E^*)$ with values in the complexified fiber $E_p^* \otimes_{\mathbb{R}} \mathbb{C}$. Similar to the procedure in chapter 2 of [BGP2007], we determine U_p^k, W_p^k by formally demanding $P\mathcal{L}_{\pm}(p) = \delta_p$. Since Definition 3.1 of the P -compatible connection ∇ implies a product rule for $P(f \cdot s)$, we obtain for odd d

$$\begin{aligned}
L_{\pm}^{\Omega}(0, p) &= \delta_p \stackrel{!}{=} P\mathcal{L}_{\pm}(p) = \sum_{k=0}^{\infty} P(U_p^k L_{\pm}^{\Omega}(2k+2, p)) + \sum_{k=\frac{d-2}{2}}^{\infty} P(W_p^k L^{\Omega}(2k+2, p)) \\
&= U_p^0 \square L_{\pm}^{\Omega}(2, p) - 2\nabla_{\text{grad}L_{\pm}^{\Omega}(2,p)} U_p^0 + \sum_{k=1}^{\infty} \left(P U_p^{k-1} L_{\pm}^{\Omega}(2k, p) - 2\nabla_{\text{grad}L_{\pm}^{\Omega}(2k+2,p)} U_p^k + U_p^k \square L_{\pm}^{\Omega}(2k+2, p) \right) \\
&\quad + \sum_{k=\frac{d}{2}}^{\infty} \left(P W_p^{k-1} L^{\Omega}(2k, p) - 2\nabla_{\text{grad}L^{\Omega}(2k+2,p)} W_p^k + W_p^k \square L^{\Omega}(2k+2, p) \right) \\
&= U_p^0 \square L_{\pm}^{\Omega}(2, p) - 2\nabla_{\text{grad}L_{\pm}^{\Omega}(2,p)} U_p^0 + \sum_{k=1}^{\infty} \frac{1}{2k} \left(2k P U_p^{k-1} - \nabla_{\text{grad}\Gamma_p} U_p^k + \left(\frac{1}{2} \square \Gamma_p - d + 2k \right) U_p^k \right) L_{\pm}^{\Omega}(2k, p) \\
&\quad + \sum_{k=\frac{d}{2}}^{\infty} \frac{1}{2k} \left(2k P W_p^{k-1} - \nabla_{\text{grad}\Gamma_p} W_p^k + \left(\frac{1}{2} \square \Gamma_p - d + 2k \right) W_p^k \right) L^{\Omega}(2k, p).
\end{aligned}$$

Furthermore, for even d , Lemma 5.2.2 leads to

$$\begin{aligned}
\tilde{L}_{\pm}^{\Omega}(0, p) &= \delta_p \stackrel{!}{=} P\mathcal{L}_{\pm}(p) \\
&= \sum_{k=0}^{\frac{d-4}{2}} P(U_p^k \tilde{L}_{\pm}^{\Omega}(2k+2, p)) \pm \frac{i}{\pi} \sum_{k=\frac{d-2}{2}}^{\infty} P(U_p^k L^{\Omega}(2k+2, p) \log(\Gamma_p \pm i0) + W_p^k L^{\Omega}(2k+2, p)) \\
&= U_p^0 \square \tilde{L}_{\pm}^{\Omega}(2, p) - 2\nabla_{\text{grad}\tilde{L}_{\pm}^{\Omega}(2,p)} U_p^0 \pm \frac{i}{\pi} \left(\overbrace{W_p^{\frac{d-2}{2}} \square L^{\Omega}(d, p)}^{=0} - \overbrace{2\nabla_{\text{grad}L^{\Omega}(d,p)} W_p^{\frac{d-2}{2}}}^{=0} \right) \\
&\quad + \sum_{k=1}^{\frac{d-4}{2}} \left(U_p^k \square \tilde{L}_{\pm}^{\Omega}(2k+2, p) - 2\nabla_{\text{grad}\tilde{L}_{\pm}^{\Omega}(2k+2,p)} U_p^k + \tilde{L}_{\pm}^{\Omega}(2k, p) P U_p^{k-1} \right) + \tilde{L}_{\pm}^{\Omega}(d-2, p) P U_p^{\frac{d-4}{2}} \\
&\quad \pm \frac{i}{\pi} \left\{ U_p^{\frac{d-2}{2}} \square (L^{\Omega}(d, p) \log(\Gamma_p \pm i0)) - 2\nabla_{\text{grad}(L^{\Omega}(d,p) \log(\Gamma_p \pm i0))} U_p^{\frac{d-2}{2}} \right. \\
&\quad + \sum_{k=\frac{d}{2}}^{\infty} [L^{\Omega}(2k, p) \log(\Gamma_p \pm i0) P U_p^{k-1} - 2\nabla_{\text{grad}(L^{\Omega}(2k+2,p) \log(\Gamma_p \pm i0))} U_p^k + U_p^k \square (L^{\Omega}(2k+2, p) \log(\Gamma_p \pm i0))] \\
&\quad \left. + \sum_{k=\frac{d}{2}}^{\infty} (W_p^k \square L^{\Omega}(2k+2, p) - 2\nabla_{\text{grad}L^{\Omega}(2k+2,p)} W_p^k + L^{\Omega}(2k, p) P W_p^{k-1}) \right\} \\
&= U_p^0 \square \tilde{L}_{\pm}^{\Omega}(2, p) - 2\nabla_{\text{grad}\tilde{L}_{\pm}^{\Omega}(2,p)} U_p^0 + \sum_{k=1}^{\frac{d-4}{2}} \left[\left(\frac{1}{2} \square \Gamma_p - d + 2k \right) U_p^k - \nabla_{\text{grad}\Gamma_p} U_p^k + 2k P U_p^{k-1} \right] \frac{\tilde{L}_{\pm}^{\Omega}(2k, p)}{2k} \\
&\quad + \left((d-2) P U_p^{\frac{d-4}{2}} + \left(\frac{1}{2} \square \Gamma_p - 2 \right) U_p^{\frac{d-2}{2}} - \nabla_{\text{grad}\Gamma_p} U_p^{\frac{d-2}{2}} \right) \frac{\tilde{L}_{\pm}^{\Omega}(d-2, p)}{d-2} \\
&\quad \pm \frac{i}{\pi} \sum_{k=\frac{d}{2}}^{\infty} \left[\left(\frac{1}{2} \square \Gamma_p - d + 2k \right) U_p^k - \nabla_{\text{grad}\Gamma_p} U_p^k + 2k P U_p^{k-1} \right] \frac{L^{\Omega}(2k, p) \log(\Gamma_p \pm i0)}{2k} \\
&\quad \pm \frac{i}{\pi} \sum_{k=\frac{d}{2}}^{\infty} \left[\left(\frac{1}{2} \square \Gamma_p + 2 + 4k - 2d \right) U_p^k - \nabla_{\text{grad}\Gamma_p} U_p^k \right] \frac{L^{\Omega}(2k, p)}{2k(k - \frac{d-2}{2})} \\
&\quad \pm \frac{i}{\pi} \sum_{k=\frac{d}{2}}^{\infty} \left[\left(\frac{1}{2} \square \Gamma_p + 2k - d \right) W_p^k - \nabla_{\text{grad}\Gamma_p} W_p^k + 2k P W_p^{k-1} \right] \frac{L^{\Omega}(2k, p)}{2k}.
\end{aligned}$$

5 Local Hadamard bisolutions

Imposing the initial condition $U_p^0(p) = \text{id}_{E_p^*}$, we read off the transport equations

$$2k PU_p^{k-1} = \nabla_{\text{grad}\Gamma_p} U_p^k - \left(\frac{1}{2} \square \Gamma_p - d + 2k \right) U_p^k, \quad k \in \mathbb{N}_0, \quad (5.12)$$

$$2k PW_p^{k-1} = \begin{cases} \nabla_{\text{grad}\Gamma_p} W_p^k - \left(\frac{1}{2} \square \Gamma_p - d + 2k \right) W_p^k, & k + \frac{1}{2} \in \mathbb{N}, k \geq \frac{d}{2}, \\ \nabla_{\text{grad}\Gamma_p} W_p^k - \left(\frac{1}{2} \square \Gamma_p + 2k - d \right) W_p^k + \frac{2k PU_p^{k-1}}{k - \frac{d-2}{2}} - 2U_p^k, & k \in \mathbb{N}, k \geq \frac{d}{2}. \end{cases} \quad (5.13)$$

Remark 5.3.1. Note that there is no constraint on $W_{\frac{d-2}{2}}$, which is therefore free to choose. Hence, even if (5.11) converges, the requirement $P\mathcal{L}_\pm(p) = \delta_p$ determines $\mathcal{L}_\pm(p)$ only up to smooth solutions of the form $\sum_{k=0}^{\infty} \frac{(W_p^k - \widehat{W}_p^k) \Gamma^k}{4^k k! \Gamma(k + \frac{d}{2})}$ with W_p^k, \widehat{W}_p^k arising from different choices of $W_{\frac{d-2}{2}}$.

Proposition 5.3.2. Let $O \subset \Omega$ be a non-empty domain such that Ω is geodesically starshaped with respect to all $p \in O$. For any $W_{\frac{d-2}{2}} \in C^\infty(O \times \Omega, E^* \boxtimes E)$, there are unique and smooth solutions of (5.12) and (5.13) given by

$$U_0(p, q) = \frac{\Pi_q^p}{\sqrt{\mu(p, q)}},$$

$$U_k(p, q) = -k U_0(p, q) \int_0^1 t^{k-1} U_0(p, \phi_{pq}(t))^{-1} (P_{(2)} U_{k-1})(p, \phi_{pq}(t)) dt, \quad k \geq 1, \quad (5.14)$$

$$W_k(p, q) = -k U_0(p, q) \int_0^1 t^{k-1} U_0(p, \phi_{pq}(t))^{-1} \widehat{W}_{k-1}(p, \phi_{pq}(t)) dt, \quad k \geq \frac{d}{2}. \quad (5.15)$$

$\Pi_q^p: E_p \rightarrow E_q$ denotes the ∇ -parallel transport, $\phi_{pq}: [0, 1] \rightarrow \Omega$ the unique geodesic connecting p, q (3.4) and

$$\widehat{W}_{k-1} := \begin{cases} P_{(2)} W_{k-1}, & k + \frac{1}{2} \in \mathbb{N}, k \geq \frac{d}{2}, \\ P_{(2)} \left(W_{k-1} - \frac{U_{k-1}}{k - \frac{d-2}{2}} \right) + \frac{U_k}{k}, & k \in \mathbb{N}, k \geq \frac{d}{2}. \end{cases}$$

Proof. The transport equations (5.12) and for half-integer k also (5.13) coincide with (2.3) of [BGP2007]. Therefore, U_k and for $k + \frac{1}{2} \in \mathbb{N}$ also W_k are the Hadamard coefficients given by (5.14) and (5.15) due to Proposition 2.3.1 of [BGP2007]. For integer k , we can apply the same proof for W_k with PW_p^{k-1} replaced by \widehat{W}_p^{k-1} everywhere, for which the same procedure then leads to (5.15). \square

Corollary 5.3.3. For Ω convex, E equipped with a non-degenerate inner product and P formally self-adjoint, we have symmetry of U_k for all $k \in \mathbb{N}_0$ and, in case of symmetric $W_{\frac{d-2}{2}}$, of W_k for half-integer $k \geq \frac{d}{2}$ in the sense of (3.3).

Remark 5.3.4. Note that, if d is even, the transport equation (5.13) for W_p^k is coupled to the U_p^k since the derivatives of the logarithmic term have to be somewhat compensated. In the odd-dimensional case, $W_p^{\frac{d-2}{2}} = 0$ leads to $W_p^k = 0$ for all k , whereas, as a consequence of the coupling with U_p^k , for even dimensions, we have $W_p^k \neq 0$, in general.

A more remarkable discrepancy is revealed by investigating the symmetry properties of W_k in the even dimensional case, that is, for integer k . It is tempting to conjecture symmetry like for odd d , but in fact, it does not hold [Wal1978]. This phenomenon played a prominent role during the development of Wald's axiomatic approach to a renormalized energy-momentum tensor. More precisely, the lack of symmetry

of the W_k 's prohibits the implementation of $\langle T \rangle$ with vanishing divergence and trace at the same time. This is closely related to the conformal trace anomaly, which indeed does not occur in odd-dimensional spacetimes. For details concerning this issue, we also refer to [DF2008] and [Wal1994].

5.4 Local parametrices and Hadamard bidistributions

From now on let $\Omega \subset M$ always denote a convex domain. We referred to (5.11) as formal, since in general, the series do not converge, and we just used it to extract equations for the coefficients. Nevertheless, it leads to left parametrices by some well-known procedure [Fri1975, Gün1988, BGP2007], which smoothly cuts off $\mathcal{L}_\pm(p)$ away from its singular support and leads to convergent series on relatively compact domains. Due to the derivatives arising from the cut-off, this results in left parametrices for P at p rather than fundamental solutions. To be more precise, for $N > \frac{d}{2}$, some sequence $\{\varepsilon_k\}_{k \geq N} \subset (0, 1]$ and $\sigma \in \mathcal{D}([-1, 1], [0, 1])$ with $\sigma|_{[-\frac{1}{2}, \frac{1}{2}]} \equiv 1$, we define

$$\tilde{\mathcal{L}}_\pm(p) := \begin{cases} \sum_{k=0}^{\infty} \tilde{U}_p^k L_\pm^\Omega(2k+2, p) + \sum_{k=\frac{d-2}{2}}^{\infty} \tilde{W}_p^k L^\Omega(2k+2, p), & d \text{ odd,} \\ \sum_{k=0}^{\frac{d-4}{2}} U_p^k \tilde{L}_\pm^\Omega(2k+2, p) \pm \frac{i}{\pi} \sum_{k=\frac{d-2}{2}}^{\infty} (\tilde{U}_p^k \log(\Gamma_p \pm i0) + \tilde{W}_p^k) L^\Omega(2k+2, p), & d \text{ even,} \end{cases} \quad (5.16)$$

where

$$\tilde{U}_k := \begin{cases} U_k, & k < N, \\ \left(\sigma \circ \frac{\Gamma}{\varepsilon_k}\right) \cdot U_k, & k \geq N, \end{cases} \quad \tilde{W}_k := \begin{cases} W_k, & k < N, \\ \left(\sigma \circ \frac{\Gamma}{\varepsilon_k}\right) \cdot W_k, & k \geq N. \end{cases} \quad (5.17)$$

Proposition 5.4.1. *For any relatively compact domain $O \subset \Omega$ and any smooth choice of $W_{\frac{d-2}{2}}$, there is a sequence $\{\varepsilon_k\}_{k \geq N} \subset (0, 1]$ such that (5.16) yield well-defined distributions for all $p \in \bar{O}$, and*

- (i) $\text{sing supp} \left(\tilde{\mathcal{L}}_\pm(p) \right) \subset C(p)$,
- (ii) $P \tilde{\mathcal{L}}_\pm(p) = \delta_p + K_\pm(p, \cdot)$ with $K_\pm \in C^\infty(\bar{O} \times \bar{O}, E^* \boxtimes E)$,
- (iii) $p \mapsto \tilde{\mathcal{L}}_\pm(p)[\varphi] \in C^\infty(\bar{O}, E^*)$ for all $\varphi \in \mathcal{D}(O, E^*)$,
- (iv) they are of order at most κ_d ,
- (v) there is a constant $C > 0$ such that $|\tilde{\mathcal{L}}_\pm(p)[\varphi]| \leq C \|\varphi\|_{C^{\kappa_d}(O, E^*)}$ for all $p \in \bar{O}$ and $\varphi \in \mathcal{D}(O, E^*)$.

The proofs of Lemma 2.4.1 – 2.4.4 of [BGP2007] only employ smoothness of the Hadamard coefficients and $\sigma\left(\frac{\Gamma(p, q)}{\varepsilon_k}\right) = 0$ if $|\Gamma(p, q)| \geq \varepsilon_k$, so replacing R_\pm^Ω by L_\pm^Ω proves the Proposition in the odd-dimensional case. Similarly, we obtain convergence in C^∞ of the W_k -part in even dimensions. However, for the logarithmic terms we have to adapt the corresponding estimates, which is of purely technical nature and therefore removed to the Appendix. Considering $\tilde{\mathcal{L}}_\pm, K_\pm$ as Schwartz kernels, we extract the corresponding operators

$$\tilde{\mathcal{L}}_\pm: \quad \mathcal{D}(O, E^*) \longrightarrow C^\infty(\bar{O}, E^*), \quad \varphi \longmapsto (p \mapsto \tilde{\mathcal{L}}_\pm(p)[\varphi]), \quad (5.18)$$

$$\mathcal{K}_\pm: \quad C^0(O, E^*) \longrightarrow C^\infty(\bar{O}, E^*), \quad u \longmapsto \left(p \mapsto \int_{\bar{O}} K_\pm(p, q) u(q) dV(q) \right), \quad (5.19)$$

which are bounded, since \bar{O} is compact. Let E be equipped with some non-degenerate inner product and P be formally self-adjoint. The aim of the rest of the section is to show that then, for all choices involved in Proposition 5.4.1, the corresponding operators $\tilde{\mathcal{L}}_{\pm}$ represent anti-Feynman and Feynman parametrices for P^t in the sense of (1.11).

Corollary 5.4.2. *Let $\tilde{\mathcal{L}}_{\pm}$ and $\tilde{\mathcal{L}}'_{\pm}$ be the operators (5.18) arising from two different choices of $N, W_{\frac{d-2}{2}}, O$ and $\{\varepsilon_k\}_{k \in \mathbb{N}}$. Then $\tilde{\mathcal{L}}_{\pm} - \tilde{\mathcal{L}}'_{\pm}$ is a smoothing operator on $O \cap O'$.*

Proof. The Schwartz kernels of these differences are given by the bidistributions

$$(p, g) \longmapsto (\tilde{\mathcal{L}}_{\pm}(p) - \tilde{\mathcal{L}}'_{\pm}(p))(g),$$

which are smooth due to Lemma 2.4.3 of [BGP2007] and Lemma 7.1.2, since $\text{supp}(\sigma_k - \sigma'_k) \cap \Gamma^{-1}(0) = \emptyset$ for all k . \square

Note that in terms of the operators (5.18), (5.19), Proposition 5.4.1 (iii) reads $\tilde{\mathcal{L}}_{\pm} P^t = \text{id} + \mathcal{K}_{\pm}$, and hence, $\tilde{\mathcal{L}}_{\pm}$ are left parametrices for P^t . Due to formal self-adjointness of P , they also provide right parametrices:

Proposition 5.4.3. *For P formally self-adjoint, the operators $\tilde{\mathcal{L}}_{\pm}$ define two-sided parametrices for P^t .*

Proof. We just have to show that $\tilde{\mathcal{L}}_{\pm}$ yield right parametrices. From the symmetry properties of $L_{\pm}^{\Omega}(\alpha)$ and \tilde{U}_k (Theorem 3.3.6 and Lemma 5.2.3) directly follows

$$\begin{aligned} \int_O L_{\pm}^{\Omega}(2k+2, p) [(\tilde{U}_k(p, \cdot) \varphi)(\psi(p))] dV(p) &= \int_O L_{\pm}^{\Omega}(2k+2, p) [(\Theta_p \tilde{U}_k(\cdot, p)^t \Theta^{-1} \varphi)(\psi(p))] dV(p) \\ &= \int_O L_{\pm}^{\Omega}(2k+2, p) [\Theta_p \psi(p) (\tilde{U}_k(\cdot, p)^t \Theta^{-1} \varphi)] dV(p) \\ &= \int_O L_{\pm}^{\Omega}(2k+2, p) [\tilde{U}_k(\cdot, p) \Theta_p \psi(p) (\Theta^{-1} \varphi)] dV(p) \\ &= \int_O L_{\pm}^{\Omega}(2k+2, q) [\tilde{U}_k(q, \cdot) \Theta \psi(\Theta_q^{-1} \varphi(q))] dV(q) \end{aligned}$$

for all $\varphi \in \mathcal{D}(O, E^*), \psi \in \mathcal{D}(O, E)$ and $k \in \mathbb{N}_0$. This works analogously for the logarithmic and \tilde{L}_{\pm}^{Ω} -terms in (5.16). Furthermore, the series involving the coefficients \tilde{W}_k are given by convergent power series $\sum_{j=0}^{\infty} a_j \Gamma^j$, which yield smooth sections in $E^* \boxtimes E$. Altogether, we obtain the decomposition $\tilde{\mathcal{L}}_{\pm} = \mathcal{U}_{\pm} + \mathcal{W}$ with \mathcal{U}_{\pm} representing the symmetric \tilde{U}_k -part, i.e. $\mathcal{U}_{\pm}^t = \Theta^{-1} \mathcal{U}_{\pm} \Theta$, and \mathcal{W} the smooth \tilde{W}_k -part. Hence, $\tilde{\mathcal{L}}_{\pm}$ is symmetric up to smoothing in the sense

$$\tilde{\mathcal{L}}_{\pm} = \Theta \tilde{\mathcal{L}}_{\pm}^t \Theta^{-1} + \underbrace{\mathcal{W} - \Theta \mathcal{W}^t \Theta^{-1}}_{\text{smoothing}}, \quad (5.20)$$

from which we directly deduce the claim:

$$\begin{aligned} P^t \tilde{\mathcal{L}}_{\pm} &= P^t (\Theta \tilde{\mathcal{L}}_{\pm}^t \Theta^{-1} + \mathcal{W} - \Theta \mathcal{W}^t \Theta^{-1}) \\ &= \Theta P^t \tilde{\mathcal{L}}_{\pm}^t \Theta^{-1} + P^t \mathcal{W}^t - P^t \Theta \mathcal{W}^t \Theta^{-1} \\ &= \Theta (\tilde{\mathcal{L}}_{\pm} P^t)^t \Theta^{-1} + \Theta P (\Theta^{-1} \mathcal{W} - \mathcal{W}^t \Theta^{-1}) \\ &= \text{id} + \underbrace{\Theta \mathcal{K}_{\pm}^t \Theta^{-1} + \Theta P (\Theta^{-1} \mathcal{W} - \mathcal{W}^t \Theta^{-1})}_{\text{smoothing}}. \end{aligned} \quad \square$$

Define the operator

$$\tilde{\mathcal{L}} := \frac{i}{2}(\tilde{\mathcal{L}}_+ - \tilde{\mathcal{L}}_-) \quad (5.21)$$

with Schwartz kernel given by $\tilde{\mathcal{L}} = \frac{i}{2}(\tilde{\mathcal{L}}_+ - \tilde{\mathcal{L}}_-)$.

Corollary 5.4.4. *For P formally self-adjoint, $\tilde{\mathcal{L}}$ is formally self-adjoint as well and has a real-valued and symmetric Schwartz kernel $\tilde{\mathcal{L}}$.*

Proof. Let $\varphi \in \mathcal{D}(O, E^*)$ and recall that $\tilde{\mathcal{L}}_{\pm}(p)[\varphi]$ take their values in the complexified fiber $E_p^* \otimes_{\mathbb{R}} \mathbb{C}$. Note that the W_k -series in (5.16) cancel, when taking the difference (5.21), so $\tilde{\mathcal{L}} = \frac{i}{2}(\mathcal{U}_+ - \mathcal{U}_-)$ is formally self-adjoint due to (5.20). Therefore, its Schwartz kernel is symmetric and furthermore real-valued by Proposition 5.1.5. \square

So far, we found two-sided parametrices $\tilde{G}_{\pm}, \tilde{\mathcal{L}}_{\pm}$ given by Hadamard series (3.9), (5.16), and [SV2001] actually proved equivalence of the Hadamard condition (1.9) and that the bidistribution is given by a certain Hadamard series. This latter condition together with the results of chapter 4 therefore allows us to express (1.13) in terms of $\tilde{G}_{\pm}, \tilde{\mathcal{L}}_{\pm}$ by directly comparing the corresponding Hadamard series. More precisely, we confirm that $\frac{i}{2}(\tilde{\mathcal{L}}_+ - \tilde{\mathcal{L}}_- + \tilde{G}_+ - \tilde{G}_-)$ is a Hadamard bidistribution, which, up to smooth errors, moreover is a bisolution with the right antisymmetric part. By examining a further linear combination of parametrices, analogous to (1.12), it will follow that $\tilde{\mathcal{L}}_{\pm}$ represent a Feynman and an anti-Feynman parametrix.

Proposition 5.4.5. *Let $O \subset \Omega$ be relatively compact, P formally self-adjoint and $\tilde{G}_{\pm}, \tilde{\mathcal{L}}_{\pm}$ the bidistributions given by (3.10) and (5.16). Then, for*

$$\tilde{H} := \frac{i}{2}(\tilde{\mathcal{L}}_+ - \tilde{\mathcal{L}}_- + \tilde{G}_+ - \tilde{G}_-), \quad (5.22)$$

the sections $P_{(1)}^t \tilde{H}, P_{(2)} \tilde{H}$ are smooth, the antisymmetric part of \tilde{H} is given by $\frac{i}{2}(\tilde{G}_+ - \tilde{G}_-)$ and \tilde{H} has the Hadamard singularity structure (1.9). Furthermore,

$$\tilde{\mathcal{L}}_+ + \tilde{\mathcal{L}}_- - \tilde{G}_+ - \tilde{G}_- \in C^{\infty}(O \times O, E^* \boxtimes E). \quad (5.23)$$

Proof. Since $\tilde{G}_{\pm}, \tilde{\mathcal{L}}_{\pm}$ yield two-sided parametrices for P^t , the first two properties follow immediately from Corollary 5.4.4, and we proceed with the Hadamard singularity structure.

Let $k, j \in \mathbb{N}_0$ such that $k \geq j$, and for even d , let either $j, k \leq \frac{d-2}{2}$ or $j, k > \frac{d-2}{2}$. Then, for $K_{k,j,d}$ defined as in (3.13), we have $K_{k,j,d} \neq 0$, and moreover,

$$\frac{R_{\pm}^{\Omega}(2k+2)}{R_{\pm}^{\Omega}(2j+2)} = \frac{L_{\pm}^{\Omega}(2k+2)}{L_{\pm}^{\Omega}(2j+2)} = \frac{\tilde{L}_{\pm}^{\Omega}(2k+2)}{\tilde{L}_{\pm}^{\Omega}(2j+2)} = K_{k,j,d} \cdot \Gamma^{k-j}.$$

due to (5.3) and Proposition 5.2.1. $R_{\pm}^{\Omega}(\alpha)$ denote the Riesz distributions (3.8), which, similar to $L_{\pm}^{\Omega}(\alpha), \tilde{L}_{\pm}^{\Omega}(\alpha)$ in (5.10), are considered as bidistributions. Define

$$\begin{aligned} H^{\Omega}(2) &:= \frac{i}{2}(\tilde{L}_+^{\Omega}(2) - \tilde{L}_-^{\Omega}(2) + R_-^{\Omega}(2) - R_+^{\Omega}(2)), \\ H^{\Omega}(d) &:= -\frac{L^{\Omega}(d)}{2\pi}(\log(\Gamma + i0) + \log(\Gamma - i0)) + \frac{i}{2}(R_-^{\Omega}(d) - R_+^{\Omega}(d)), \end{aligned} \quad (5.24)$$

and thus, the Hadamard series (5.22) takes the form

$$\tilde{H} = \begin{cases} H^\Omega(2) \sum_{k=0}^{\infty} K_{k,0,d} \cdot \tilde{U}_k \Gamma^k, & d \text{ odd,} \\ H^\Omega(2) \sum_{k=0}^{\frac{d-4}{2}} K_{k,0,d} \cdot \tilde{U}_k \Gamma^k + H^\Omega(d) \sum_{k=\frac{d-2}{2}}^{\infty} K_{k,\frac{d-2}{2},d} \cdot \tilde{U}_k \Gamma^{k-\frac{d-2}{2}}, & d \text{ even.} \end{cases} \quad (5.25)$$

We show that this is of Hadamard form in the sense of Definition 5.1 in [SV2001], where mostly the notations and conventions of [Gün1988] are adopted. In particular, the Hadamard coefficients $U_{(k)}$ used in [SV2001] are related with U_k via $2^k k! \cdot U_{(k)} = U_k$ (see Remark 2.3.2 of [BGP2007]). In addition, with the notation $(\alpha, k) := 2^k \cdot \frac{\Gamma(\frac{\alpha}{2} + k)}{\Gamma(\frac{\alpha}{2})}$, we find $(2j+2, k-j) \cdot (2j+4-d, k-j) = K_{k,j,d}^{-1}$ and hence,

$$K_{k,0,d} \cdot U_k = \frac{2^k \cdot k! \cdot U_{(k)}}{2^k \cdot k! \cdot (4-d, k)} = \frac{U_{(k)}}{(4-d, k)},$$

$$K_{k,\frac{d-2}{2},d} \cdot U_k = \frac{2^k \cdot k! \cdot \Gamma(\frac{d}{2}) \cdot U_{(k)}}{4^{k-\frac{d-2}{2}} \cdot k! \cdot \Gamma(k+2-\frac{d}{2})} = \frac{(2, \frac{d-2}{2})}{2^{k+d-2} \cdot (k-\frac{d-2}{2})!} \cdot U_{(k)}.$$

For all $n \in \mathbb{N}$, we choose $N \geq n + \frac{\kappa_d}{2}$ in (5.17), so the series in (5.25) truncated at $k = n + \frac{\kappa_d}{2}$ coincide with $U, V^{(n)}, T^{(n)}$ given in Appendix A.1 of [SV2001]. The remainder term is then of regularity C^n and corresponds to $H^{(n)}$ in Definition 5.1 of [SV2001].

It remains to identify the singular terms (5.24) with $G^{(1)}, G^{(2)}$ given by (5.3) in [SV2001] up to some global factor, which is -2 in the odd- and $2 \cdot (-1)^{\frac{d}{2}}$ in the even-dimensional case. Moreover, note that for the squared Lorentzian distance in the definition of $G^{(1)}, G^{(2)}$, the convention $s = -\Gamma$ is used, whereas in Appendix A.1 we have $s = \Gamma$.

Let $p, q \in O$. By definition of $\tilde{L}_\pm^\Omega(\alpha, p)$ and $R_\pm^\Omega(\alpha, p)$ as well as Theorem 4.4.2, we have

$$H^\Omega(2, p) = (\exp_p)_* W = i(\exp_p)_* \Delta^-$$

with Wightman's solution W for $(\mathbb{R}_{\text{Mink}}^d, \square)$ (4.7). Recalling Δ^- from (4.11) leads to

$$H^\Omega(2, p)(q) = \frac{\Gamma(\frac{d-2}{2})}{4\pi^{\frac{d}{2}}} \lim_{\varepsilon \rightarrow 0} (-\Gamma(p, q) + 2i\varepsilon \cdot q^0 + \varepsilon^2)^{\frac{2-d}{2}}$$

in the distributional sense with $q^0 = (\exp_p^{-1}(q))^0$. Since $\Gamma(\frac{d-2}{2}) \Gamma(1 - \frac{d-2}{2}) = (-1)^{\frac{d+1}{2}} \pi$ for odd d , this coincides with $G^{(1)}$.

Furthermore, one directly calculates $H^\Omega(d) - G^{(2)} = 0$ away from $\Gamma^{-1}(0)$. For $\phi_{pq}(t) := \exp_p(t \exp_p^{-1}(q))$ with $t \in [0, 1]$, we obtain $\Gamma(p, \phi_{pq}(t)) = t^2 \cdot \Gamma(p, q)$. Similar to (4.10), we set $\Gamma_\varepsilon^\pm(p, \cdot) := \gamma_\varepsilon^\pm \circ \exp_p^{-1}$, which yields $\Gamma_\varepsilon^\pm(p, \phi_{pq}(t)) = t^2 \cdot \Gamma_\varepsilon^\pm(p, q)$ for all $t \in (0, 1]$, and hence,

$$G^{(2)}(p, \phi_{pq}(t)) = -\frac{L^\Omega(d)}{\pi} \lim_{\varepsilon \rightarrow 0} \log(-\Gamma_\varepsilon^\pm(p, \phi_{pq}(t))) = -\frac{2L^\Omega(d)}{\pi} \cdot \log t + G^{(2)}(p, q).$$

On the other hand, since $R_\pm^\Omega(d), L^\Omega(d)$ are homogeneous distributions of degree 0, (5.24) provides

$$H^\Omega(d, p)(\phi_{pq}(t)) = -\frac{L^\Omega(d)}{2\pi} (2 \log t + 2 \log t) + H^\Omega(d, p)(q) = -\frac{2L^\Omega(d)}{\pi} \cdot \log t + H^\Omega(d, p)(q).$$

Of course, both expressions have to be understood in the distributional sense. Since Ω is diffeomorphic to $\exp_p^{-1}(\Omega) \subset T_p M$ for all $p \in \Omega$, their difference corresponds to a \mathcal{L}_+^\uparrow -invariant distributions on Minkowski space \mathbb{R}^d , which is supported on the light cone and homogeneous of degree 0. Therefore, it has to vanish everywhere by Corollary 2.2.7 and thus, Theorem 5.8 of [SV2001] ensures that \tilde{H} is of Hadamard form in the sense of (1.9). It remains to show (5.23). According to (5.24), we define

$$\begin{aligned} A^\Omega(2) &:= \frac{i}{2}(\tilde{L}_+^\Omega(2) + \tilde{L}_-^\Omega(2) - R_-^\Omega(2) - R_+^\Omega(2)), \\ A^\Omega(d) &:= -\frac{L^\Omega(d)}{2\pi}(\log(\Gamma + i0) - \log(\Gamma - i0)) - \frac{i}{2}(R_-^\Omega(d) + R_+^\Omega(d)) \end{aligned} \quad (5.26)$$

such that for (5.23), we obtain the expression (5.25) with $H^\Omega(2), H^\Omega(d)$ replaced by $A^\Omega(2), A^\Omega(d)$ and it suffices to show smoothness of the bidistributions (5.26). For $A^\Omega(2)$, this follows directly from the definitions of $\tilde{L}_\pm^\Omega(2), R_\pm^\Omega(2)$ as pullbacks of S_\pm, R_\pm^2 along a diffeomorphism and Theorem 4.4.2. On the other hand, one directly calculates that $A^\Omega(d)$ is given by the constant $-iC(d, d)$. \square

Since $\tilde{\mathcal{L}}_\pm, \tilde{G}_\pm$ are determined merely up to smooth sections, without loss of generality, we regard (5.23) as the equality

$$\tilde{\mathcal{L}}_+ + \tilde{\mathcal{L}}_- = \tilde{G}_+ + \tilde{G}_-. \quad (5.27)$$

Corollary 5.4.6. *For P formally self-adjoint, the operators $\tilde{\mathcal{L}}_\pm$ represent anti-Feynman and Feynman parametrices for P^t in the sense of (1.11) on O .*

Proof. Note that the proof of (1.12) given by Theorem 5.1 of [Rad1996a] and section 6.6 of [DH1972] exclusively employs the singularity structure of the involved parametrices as well as that the antisymmetric part of (5.22) is given by $\frac{i}{2}(\tilde{G}_+ - \tilde{G}_-)$, which is the case here, since (5.21) is symmetric. Furthermore, the singularity structure of \tilde{H} is entirely carried by the scalar distributions $L^\Omega(\alpha)$ and therefore remains unaffected when multiplying smooth vector-valued Hadamard coefficients. Hence, the statement (1.12) remains valid, that is, for the distinguished parametrices $\tilde{G}_A, \tilde{G}_R, \tilde{G}_F, \tilde{G}_{aF}$ and up to smooth bisections, we have

$$\tilde{H} = \frac{i}{2}(\tilde{G}_{aF} - \tilde{G}_F + \tilde{G}_A - \tilde{G}_R), \quad \tilde{G}_{aF} + \tilde{G}_F = \tilde{G}_A + \tilde{G}_R. \quad (5.28)$$

Since \tilde{G}_\pm represent advanced and retarded parametrices, it follows from (5.27) and (5.28) that

$$\tilde{\mathcal{L}}_+ - \tilde{\mathcal{L}}_- = -2i\tilde{H} - \tilde{G}_+ + \tilde{G}_- = \tilde{G}_{aF} - \tilde{G}_F, \quad \tilde{\mathcal{L}}_+ + \tilde{\mathcal{L}}_- = \tilde{G}_+ + \tilde{G}_- = \tilde{G}_{aF} + \tilde{G}_F,$$

which provides $\tilde{\mathcal{L}}_+ = \tilde{G}_{aF}, \tilde{\mathcal{L}}_- = \tilde{G}_F$ up to smooth bisections and thus completes the proof. \square

5.5 Local fundamental solutions and Hadamard bisolutions

In the last section of this chapter, we construct bisolutions S^O for P on certain relatively compact domains $O \subset M$ of globally hyperbolic Lorentzian manifolds with singularity structure given by $\text{WF}(\tilde{\mathcal{L}})$, that is, $S^O + \frac{i}{2}(G_+ - G_-)$ provides a local Hadamard bisolution. We start by constructing fundamental solutions $\tilde{S}_\pm^O(p)$ for P at all $p \in \bar{O}$ from (5.16), so $\tilde{S}_+^O(p) - \tilde{S}_-^O(p)$ yields a solution. For M globally hyperbolic, solving a Cauchy problem then provides a bisolution with the right singularity structure.

By Proposition 5.4.1 (ii), we have $\tilde{\mathcal{L}}_\pm P^t|_{\mathcal{D}(O, E^*)} = \text{id} + \mathcal{K}_\pm$, and hence, fundamental solutions are obtained by inverting the operators $\text{id} + \mathcal{K}_\pm$. Indeed, if $\text{vol}(\bar{O}) \cdot \|\mathcal{K}_\pm\|_{C^0(\bar{O} \times \bar{O})} < 1$, that is, for O chosen

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"small enough", (5.19) provides isomorphisms $\text{id} + \mathcal{K}_\pm: C^l(\bar{O}, E^*) \longrightarrow C^l(\bar{O}, E^*)$ for all $l \in \mathbb{N}_0$ with bounded inverses given by the Neumann series

$$(\text{id} + \mathcal{K}_\pm)^{-1} = \sum_{j=0}^{\infty} (-\mathcal{K}_\pm)^j. \quad (5.29)$$

This means that all C^l -norms of the series exist, which follows from compactness of \bar{O} and smoothness of K_\pm . The full proof coincides with the one of Lemma 2.4.8 of [BGP2007]. In the following, we restrict to such small domains:

Definition 5.5.1. *We call a relatively compact and causal subdomain O of Ω admissible if Proposition 5.4.1 provides parametrices $\tilde{\mathcal{L}}_\pm$ via (5.18) such that the smooth Schwartz kernel K_\pm of $\tilde{\mathcal{L}}_\pm P^t - \text{id}$ fulfills*

$$\text{vol}(\bar{O}) \cdot \|K_\pm\|_{C^0(\bar{O} \times \bar{O})} < 1. \quad (5.30)$$

More precisely, O is admissible, if there is a choice of $\{\varepsilon_k\}_k$ and $W_{\frac{d-2}{2}}$ such that (5.30) holds for the corresponding K_\pm . Lemma 2.4.8 of [BGP2007] shows that for O admissible, the corresponding operators $\tilde{\mathcal{L}}_\pm P^t = \text{id} + \mathcal{K}_\pm$ are isomorphisms with bounded inverses (5.29).

Proposition 5.5.2. *For any admissible O , the operators*

$$\tilde{\mathcal{S}}_\pm^O := (\text{id} + \mathcal{K}_\pm)^{-1} \tilde{\mathcal{L}}_\pm: \quad \mathcal{D}(O, E^*) \rightarrow C^\infty(\bar{O}, E^*)$$

fulfill $\tilde{\mathcal{S}}_\pm^O P^t|_{\mathcal{D}(O, E^*)} = \text{id}$, and hence, the distributions $\tilde{\mathcal{S}}_\pm^O(p)$, $p \in O$, given by

$$\tilde{\mathcal{S}}_\pm^O(p)[\varphi] = ((\text{id} + \mathcal{K}_\pm)^{-1} \tilde{\mathcal{L}}_\pm \varphi)(p), \quad \varphi \in \mathcal{D}(O, E^*), \quad (5.31)$$

yield fundamental solutions for P at p . Furthermore, $\mathcal{Q}_\pm := (\text{id} + \mathcal{K}_\pm)^{-1} - \text{id}$ are smoothing operators.

Proof. The first claim follows from $\tilde{\mathcal{L}}_\pm P^t = \text{id} + \mathcal{K}_\pm$. Moreover, Proposition 5.4.1 and Lemma 2.4.10 of [BGP2007], with $\tilde{\mathcal{R}}_\pm(\cdot)[\varphi]$ and $F_\pm^\Omega(\cdot)[\varphi]$ replaced by $\tilde{\mathcal{L}}_\pm \varphi$ and $\tilde{\mathcal{S}}_\pm^O \varphi$, show that (5.31) yield fundamental solutions. Finally, (5.29) directly yields $\mathcal{Q}_\pm = (\text{id} + \mathcal{K}_\pm)^{-1} \circ \mathcal{K}_\pm$, which is smoothing, since \mathcal{K}_\pm is, and $(\text{id} + \mathcal{K}_\pm)^{-1}$ is a continuous map $C^\infty(M, E^*) \rightarrow C^\infty(M, E^*)$. \square

From now on, let E be always equipped with some non-degenerate inner product, P formally self-adjoint and O admissible.

Proposition 5.5.3. *The operators $\tilde{\mathcal{S}}_\pm^O - \tilde{\mathcal{L}}_\pm$ are smoothing.*

Proof. Note that $\tilde{\mathcal{S}}_\pm^O - \tilde{\mathcal{L}}_\pm = \mathcal{Q}_\pm \tilde{\mathcal{L}}_\pm$. Since $\mathcal{Q}_\pm, \tilde{\mathcal{L}}_\pm$ are bounded and \mathcal{Q}_\pm has a smooth Schwartz kernel, they extend to bounded maps

$$\mathcal{Q}_\pm: \quad \mathcal{D}(O, E^*)' \rightarrow C^\infty(\bar{O}, E^*), \quad \tilde{\mathcal{L}}_\pm: \quad \mathcal{E}(O, E^*)' \rightarrow \mathcal{D}(O, E^*)'.$$

Hence, $\mathcal{Q}_\pm \tilde{\mathcal{L}}_\pm: \mathcal{E}(O, E^*)' \rightarrow C^\infty(\bar{O}, E^*)$ is bounded and therefore smoothing. \square

It follows that $\tilde{\mathcal{S}}_\pm^O$ yield anti-Feynman and Feynman parametrices for P^t on O . Moreover, their Schwartz kernels determine a real-valued bidistribution via

$$\tilde{\mathcal{S}}^O[\psi, \varphi] := \frac{i}{4} \left(\tilde{\mathcal{S}}_+^O[\psi, \varphi] - \tilde{\mathcal{S}}_-^O[\psi, \varphi] - \overline{\tilde{\mathcal{S}}_+^O[\psi, \varphi]} + \overline{\tilde{\mathcal{S}}_-^O[\psi, \varphi]} \right), \quad \psi \in \mathcal{D}(O, E), \varphi \in \mathcal{D}(O, E^*), \quad (5.32)$$

which has the right singularity structure and is a solution for P in the second argument, meaning $\text{WF}(\tilde{\mathcal{S}}^O) = \text{WF}(\tilde{\mathcal{L}})$ and $P_{(2)} \tilde{\mathcal{S}}^O = 0$.

Proposition 5.5.4. *Let M be a globally hyperbolic Lorentzian manifold, $\pi: E \rightarrow M$ a real vector bundle with non-degenerate inner product and $P: C^\infty(M, E) \rightarrow C^\infty(M, E)$ a formally self-adjoint wave operator. Furthermore, let $O \subset M$ be admissible and $\tilde{\mathcal{L}}$ denote the bidistribution given by Proposition 5.4.1 and Corollary 5.4.4. Then there is a bisolution $S^O: \mathcal{D}(O, E) \times \mathcal{D}(O, E^*) \rightarrow \mathbb{R}$ for P with $WF(S^O) = WF(\tilde{\mathcal{L}})$.*

Proof. Since O is admissible, we obtain fundamental solutions $\tilde{S}_\pm^O(p)$ at each $p \in \bar{O}$, and furthermore, (5.32) provides $p \mapsto \tilde{S}^O(p)[\varphi] \in C^\infty(\bar{O}, E^*)$ for all $\varphi \in \mathcal{D}(O, E^*)$. Moreover, as a causal subdomain of a globally hyperbolic Lorentzian manifold, O is globally hyperbolic on its own right (Lemma A.5.8 of [BGP2007]). Hence, for Σ a Cauchy hypersurface of O with unit normal field ν , there is a unique smooth solution of

$$\begin{cases} P^t(S^O(\cdot)[\varphi]) &= 0, \\ S^O(\cdot)[\varphi]|_\Sigma &= \tilde{S}^O(\cdot)[\varphi]|_\Sigma, \\ \nabla_\nu(S^O(\cdot)[\varphi])|_\Sigma &= \nabla_\nu\tilde{S}^O(\cdot)[\varphi]|_\Sigma. \end{cases}$$

By continuous dependence on the Cauchy data, $S^O(p)$ defines an E_p^* -valued distribution for all $p \in O$. Furthermore, $S^O(\cdot)[P^t\varphi] = 0$ for all φ , since it satisfies the trivial Cauchy problem.

It remains to check the wave front set, that is, smoothness of $D^O := \tilde{\mathcal{L}} - S^O$. Since S^O , \tilde{S}^O and $\tilde{\mathcal{L}}$ yield parametrices for P , the sections given by $P_{(2)}D^O$, $P_{(1)}^tD^O$ and $\tilde{\mathcal{L}} - \tilde{S}^O$ are smooth, and hence, D^O is the solution of a Cauchy problem with smooth Cauchy data, which is smooth by Theorem 2.3.2. \square

Altogether, any choice of parametrices $\tilde{\mathcal{L}}_\pm$ in the sense of Proposition 5.4.1 leads to a bisolution S^O with singularity structure given by $\frac{i}{2}(\tilde{G}_{aF} - \tilde{G}_F)$ in the sense of (1.11).

We briefly investigate the relation to the original formal fundamental solutions (5.11). To this end, for all $l \in \mathbb{N}_0$ and $p \in \Omega$, we introduce the following truncated Hadamard series

$$\mathcal{L}_\pm^{N+l}(p) := \begin{cases} \sum_{k=0}^{N+l-1} U_p^k L_\pm^\Omega(2k+2, p) + \sum_{k=\frac{d-2}{2}}^{N+l-1} W_p^k L^\Omega(2k+2, p), & d \text{ odd}, \\ \sum_{k=0}^{\frac{d-4}{2}} U_p^k \tilde{L}_\pm^\Omega(2k+2, p) \pm \frac{i}{\pi} \sum_{k=\frac{d-2}{2}}^{N+l-1} (U_p^k \log(\Gamma_p \pm i0) + W_p^k) L^\Omega(2k+2, p), & d \text{ even}. \end{cases}$$

These are $(E_p^* \otimes_{\mathbb{R}} \mathbb{C})$ -valued distributions on any domain $\Omega \subset M$ geodesically starshaped with respect to p , so in particular on every admissible subset O containing p . Then $\mathcal{L}^{N+l} := \frac{i}{2}(\mathcal{L}_+^{N+l} - \mathcal{L}_-^{N+l})$ approximates S^O in the following sense:

Proposition 5.5.5. *For all $l \in \mathbb{N}_0$, we have*

$$(p, q) \mapsto (S^O(p) - \mathcal{L}^{N+l}(p))(q) \in C^l(O \times O, E^* \boxtimes E).$$

Proof. Considering the expansion $S^O - \mathcal{L}^{N+l} = S^O - \tilde{\mathcal{L}} + \tilde{\mathcal{L}} - \mathcal{L}^{N+l}$, due to Proposition 5.5.4, we just have to check C^l -regularity of $\tilde{\mathcal{L}}_\pm - \mathcal{L}_\pm^{N+l}$. Since the arguments for all series involved in (5.16) are completely analogous, we demonstrate this only for $\tilde{\mathcal{L}}_\pm(p) = \sum_{k=0}^{\infty} \tilde{U}_p^k L_\pm^\Omega(2k+2, p)$, for which we obtain the explicit expressions

$$\tilde{\mathcal{L}}_\pm - \mathcal{L}_\pm^{N+l} = \sum_{k=N}^{N+l-1} (\sigma_k - 1) U_k(p, \cdot) L_\pm^\Omega(2k+2) + \sum_{k=N+l}^{\infty} \tilde{U}_k(p, \cdot) L_\pm^\Omega(2k+2).$$

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The infinite sum is a C^l -section due to Lemma 2.4.2 of [BGP2007] with R_{\pm}^{Ω} replaced by L_{\pm}^{Ω} and Lemma 7.1.2 for the logarithmic part of (5.16), respectively. The first part is a finite sum of smooth functions, since $\sigma_k - 1$ vanishes in a neighborhood of $\Gamma^{-1}(0)$ and hence on the singular support of $\tilde{\mathcal{L}}_{\pm}$ and \mathcal{L}_{\pm}^{N+l} . \square

Altogether, we constructed bisolutions with the Hadamard singularity structure on every $O \times O$ and we summarize:

Theorem 5.5.6. *Let M be a globally hyperbolic Lorentzian manifold, $O \subset M$ an admissible domain, $\pi: E \rightarrow M$ a real vector bundle with non-degenerate inner product and $P: C^{\infty}(M, E) \rightarrow C^{\infty}(M, E)$ a formally self-adjoint wave operator. For S^O the bisolution given by Proposition 5.5.4 and G_{\pm} the advanced and retarded Green operator on O , the bisolution*

$$H^O := S^O + \frac{i}{2}(G_+ - G_-)$$

is of Hadamard form.

Proof. By Lemma 3.3.2 and Proposition 5.5.4, the bidistributions $S^O - \tilde{\mathcal{L}}$ and $G_{\pm} - \tilde{G}_{\pm}$ are smooth, which provides smoothness of $H^O - \tilde{H}$, so Proposition 5.4.5 ensures the Hadamard property of H^O . \square

Remark 5.5.7. *Due to Proposition 2.5.1 in [BGP2007] and Proposition 5.5.5, H^O is given by a Hadamard series up to terms of arbitrarily high regularity.*

6 Global Hadamard two-point-functions

"Ich bin sicher, man kann immer positive Lösungen finden."

From now on, we adopt the setting given by Definition 1.2.1 restricted to wave operators P . In the preceding chapter, we derived bisolutions S^O on $O \times O$ for any admissible domain $O \subset M$ with $\text{WF}(S^O) = \text{WF}(\tilde{G}_{aF} - \tilde{G}_F)$ in the sense of (1.11). In this chapter, we finally tackle the construction of global bisolutions S , which locally coincide with those S^O up to smooth bisolutions and thus inherit their singularity structure. It is therefore not hard to show that each S can be chosen as a symmetric bidistribution. Recall that "choice" means the existence of a smooth bisolution u such that $S + u$ is a symmetric bisolution. Assuming E to be Riemannian and the validity of Theorem 6.6.2 of [DH1972] for sections in E , we furthermore prove the existence of a positive choice for S . It follows that S provides a scalar product (1.7), leading to the two-point-function of a quasifree state via (1.6)

$$H := S + \frac{i}{2}(G_+ - G_-), \quad (6.1)$$

which has the Hadamard singularity structure due to Theorem 5.5.6.

Multiplication with some suitable cut-off allows us to regard $S^O(p)$ as a distribution on M with spatially compact support. By employing Theorem 2.3.6, we propagate it to all of M for all $p \in O$, and we show that the results arising from two different cut-offs differ merely by some smooth bisolution. In this way, bisolutions \hat{S}^O with the right singularity structure are determined on all domains $O \times M$ for O admissible, and we find a locally finite cover of $M \times M$ by such domains. By means of Čech cohomology theory, the corresponding bisolutions \hat{S}^O can be chosen in a compatible manner, meaning that they match up on the overlaps and hence form a global bisolution S . Afterwards, we show that there are symmetric and even positive choices for S .

6.1 Global construction of symmetric bisolutions

Fix a Cauchy hypersurface $\Sigma \subset M$ and two locally finite covers $\mathcal{O} := \{O_i\}_{i \in I}$, $\mathcal{O}' := \{O'_i\}_{i \in I}$ of it by admissible subsets of M with $\overline{O}_i \subset O'_j$ if and only if $i = j$. Without loss of generality, we assume $O_i \cap \Sigma$ to be a Cauchy hypersurface of O_i . For instance, we could choose for \mathcal{O} the Cauchy developments $D(\Sigma_i)$, $i \in I$, (Definition 1.3.5 of [BGP2007]) of relatively compact and sufficiently small subdomains $\Sigma_i \subset \Sigma$, which comprise a locally finite cover of Σ .

Then $N := \bigcup_{i \in I} O_i$ yields a causal normal neighborhood of Σ in the sense of Lemma 2.2 of [KW1991]. By paracompactness of M and the Hopf-Rinow-Theorem, we find an exhaustion $\{A_m\}_{m \in \mathbb{N}}$ of I by finite subsets such that the relatively compact sets $N_m := \bigcup_{i \in A_m} O_i$ exhaust N and every compact subset of N is contained in some N_m . Besides that, causality of O implies $O \subset D(O) = D(O \cap \Sigma)$ and therefore,

$$N_m \subset \bigcup_{i \in A_m} D(O_i \cap \Sigma) \subset D\left(\bigcup_{i \in A_m} O_i \cap \Sigma\right) = D(N_m \cap \Sigma). \quad (6.2)$$

It follows that every inextendible causal curve in N_m meets $N_m \cap \Sigma$ exactly once, so $N_m \cap \Sigma$ is a Cauchy hypersurface of N_m , i.e. N_m is globally hyperbolic. In addition, for all $i \in I$, we choose the corresponding local bisolutions $S^{O'_i}, S^{O_i}$, obtained by Theorem 5.5.6, such that $S^{O'_i}|_{O_i \times O_i} = S^{O_i}$.

Proposition 6.1.1. *For each $O \in \mathcal{O}$, there is a bisolution \hat{S}^O on $O \times M$ satisfying $\hat{S}^O|_{O \times O} = S^O$.*

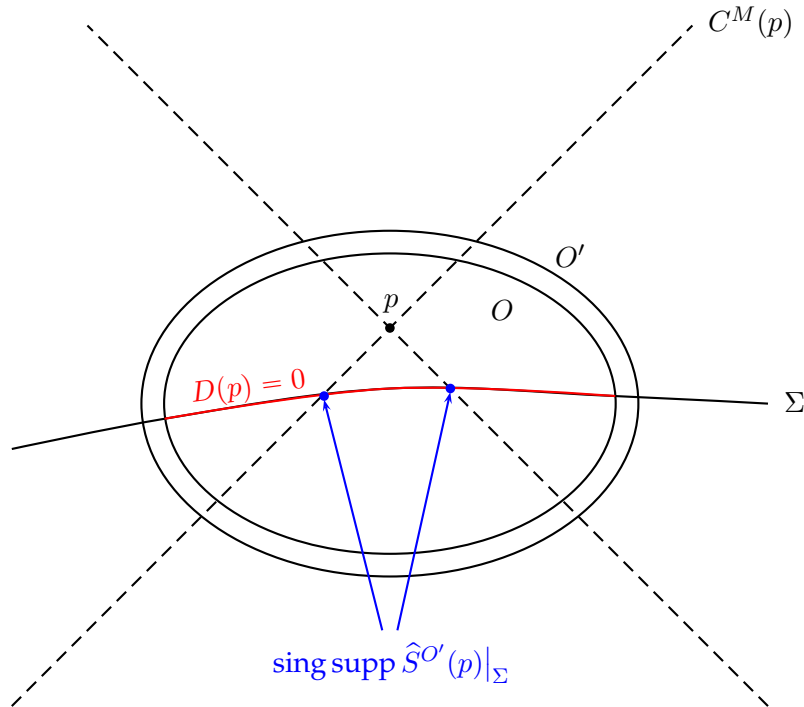
Proof. Let $O' \in \mathcal{O}'$ such that $\bar{O} \subset O'$, and $\chi \in \mathcal{D}(O')$ with $\chi|_O = 1$. Then $\chi S^{O'}(p)$ is a well-defined distribution with spacelike compact support on M for all $p \in O$, since $\chi S^{O'}(p)[\varphi] = S^{O'}(p)[\chi\varphi]$, $\varphi \in \mathcal{D}(M, E^*)$. With regard to Theorem 2.3.6, we define $\hat{S}^O(p) \in \mathcal{D}(M, E, E_p^*)'$ as the unique solution of

$$\begin{cases} P\hat{S}^O(p) &= 0, \\ \hat{S}^O(p)|_\Sigma &= \chi S^{O'}(p)|_\Sigma, \\ \nabla_\nu \hat{S}^O(p)|_\Sigma &= \nabla_\nu(\chi S^{O'}(p))|_\Sigma, \end{cases} \quad (6.3)$$

which moreover depends smoothly on p in the sense $p \mapsto \hat{S}^O(p)[\varphi] \in C^\infty(O, E^*)$ for fixed $\varphi \in \mathcal{D}(M, E^*)$. Furthermore, global hyperbolicity of O ensures $\hat{S}^O|_{O \times O} = S^O$ by Theorem 2.3.2, since the difference solves the trivial Cauchy problem on $O \times O$.

Let $T(p)[\varphi] := P^t(\hat{S}^O(\cdot)[\varphi])(p)$ and hence $T(p) \in \mathcal{D}(M, E, E_p^*)'$ for all $p \in O$. It follows that $PT(p) = 0$, and $T(p)[\varphi] = 0 = T(p)[\nabla_\nu \varphi]$ if $\text{supp } \varphi \subset O$, which leads to $T(p)|_\Sigma = \nabla_\nu T(p)|_\Sigma = 0$. Consequently, it satisfies the trivial Cauchy problem, so we have $T(p) = 0$, that is, \hat{S}^O represents a bisolution. \square

This definition of \hat{S}^O is independent of the choice of χ in an appropriate sense: Let $\tilde{\chi} \in \mathcal{D}(O')$ be another cut-off with $\tilde{\chi}|_O = 1$ and corresponding bisolution \tilde{S}^O . Then $D := \hat{S}^O - \tilde{S}^O$ is a bisolution with Cauchy data on $(O \cap \Sigma) \times \Sigma$ given by $(\chi - \tilde{\chi})S^{O'}$. Recall that $\text{sing supp } S^{O'} \subset \Gamma^{-1}(0) \cap (O' \times O')$, so causality of O yields $\text{sing supp } S^{O'}(p)|_\Sigma \subset (C^M(p) \cap O' \cap \Sigma) \subset O$ for all $p \in O$, and hence, $\text{sing supp } \chi S^{O'}|_{O \cap \Sigma \times \Sigma}$ is contained in $O \times O$. Since D satisfies the trivial Cauchy problem on $O \times O$, i.e. $D|_{O \times O} = 0$, it is a smooth bisolution by Theorem 2.3.2. Therefore, \hat{S}^O and \tilde{S}^O differ merely by some smooth bisolution.



Next, we prove the existence of a compatible choice of bisolutions $\{\widehat{S}^{O_i}\}_{i \in I}$, meaning that they coincide on the overlaps $O_i \cap O_j$, $i, j \in I$. In this way, these compatible bisolutions assemble to a well-defined object on $N \times M$. The tools for such a procedure are provided by Čech cohomology theory, for which we give a brief and purposive overview. For an introduction to this subject with details and proofs, we refer to section 5.33 of [War1983].

On $N \times M$, let \mathcal{C}^∞ denote the sheaf given by the germs of the smooth sections in $E^* \boxtimes E$ (see Example 5.2 in [War1983]). For the open cover $\mathcal{O}^M := \{O_i \times M\}_{i \in I}$ of $N \times M$, the n -simplices correspond to the non-empty $(n+1)$ -times intersections

$$O_{i_0 \dots i_n}^M := (O_{i_0} \cap \dots \cap O_{i_n}) \times M, \quad i_0, \dots, i_n \in I,$$

with $n+1$ faces $\{O_{i_0 \dots \widehat{i}_k \dots i_n}^M\}_{k=0, \dots, n}$ obtained by leaving out one O_i in the intersection, respectively. An n -cochain is a map that assigns to each non-empty $O_{i_0 \dots i_n}^M$ a section of \mathcal{C}^∞ over $O_{i_0 \dots i_n}^M$, which we identify with the elements of $C^\infty(O_{i_0 \dots i_n}^M, E^* \boxtimes E)$. The space of n -cochains is denoted by $\mathcal{C}^n(\mathcal{O}^M, \mathcal{C}^\infty)$, where $\mathcal{C}^n := \{0\}$ if $n < 0$, and the coboundary operator is defined by

$$\partial_n: \mathcal{C}^n(\mathcal{O}^M, \mathcal{C}^\infty) \longrightarrow \mathcal{C}^{n+1}(\mathcal{O}^M, \mathcal{C}^\infty), \quad (\partial_n f_n)(O_{i_0 \dots i_{n+1}}^M) := \sum_{k=0}^{n+1} (-1)^k \cdot f_n(O_{i_0 \dots \widehat{i}_k \dots i_{n+1}}^M) \Big|_{O_{i_0 \dots i_{n+1}}^M}.$$

It follows that $\partial_{n+1} \circ \partial_n = 0$ for all $n \in \mathbb{N}_0$ and we set $H^n(\mathcal{O}^M, \mathcal{C}^\infty) := \frac{\ker \partial_n}{\text{ran } \partial_{n-1}}$. These modules are trivial for all $n \in \mathbb{N}$ by some well-known construction (e.g. p. 202 in [War1983]), employing that \mathcal{C}^∞ admits a partition of unity subordinate to the locally finite cover \mathcal{O}^M :

Lemma 6.1.2. *For all $n \in \mathbb{N}$, we have*

$$H^n(\mathcal{O}^M, \mathcal{C}^\infty) = \{0\}.$$

Proof. By choice of \mathcal{O} , the cover \mathcal{O}^M is locally finite. Let $\{\chi_i\}_{i \in I}$ denote a partition of unity subordinate to \mathcal{O}^M and $f_n \in \mathcal{C}^n(\mathcal{O}^M, \mathcal{C}^\infty)$. Then, for each $i \in I$, the smooth section $\chi_i f_n(O_i^M \cap O_{i_0 \dots i_{n-1}}^M)$ is supported in $O_i^M \cap O_{i_0 \dots i_{n-1}}^M$, and thus, via extension by zero, we consider it as an element of $C^\infty(O_{i_0 \dots i_{n-1}}^M, E^* \boxtimes E)$. In this way, we obtain homomorphisms $h_n: \mathcal{C}^n(\mathcal{O}^M, \mathcal{C}^\infty) \rightarrow \mathcal{C}^{n-1}(\mathcal{O}^M, \mathcal{C}^\infty)$ via

$$h_n(f_n)(O_{i_0 \dots i_{n-1}}^M) := \sum_{i \in I} \chi_i f_n(O_i^M \cap O_{i_0 \dots i_{n-1}}^M) \in C^\infty(O_{i_0 \dots i_{n-1}}^M, E^* \boxtimes E),$$

which satisfy

$$\begin{aligned} (h_{n+1}(\partial_n f_n))(O_{i_0 \dots i_n}^M) &= \sum_{i \in I} \chi_i \cdot \partial_n f(O_i^M \cap O_{i_0 \dots i_n}^M) \\ &= \sum_{i \in I} \chi_i f_n(O_{i_0 \dots i_n}^M) + \sum_{i \in I} \sum_{k=0}^n (-1)^{k+1} \cdot \chi_i f_n(O_i^M \cap O_{i_0 \dots \widehat{i}_k \dots i_n}^M) \Big|_{O_{i_0 \dots i_n}^M} \\ &= f_n(O_{i_0 \dots i_n}^M) - (\partial_{n-1} h_n(f_n))(O_{i_0 \dots i_n}^M). \end{aligned}$$

Hence, $f_n \in \ker \partial_n$ implies $f_n = \partial_{n-1} h_n(f_n)$, that is, $f_n \in \text{ran } \partial_{n-1}$. \square

Lemma 6.1.3. *For all $i \in I$, there is a bisolution $h_i \in C^\infty(O_i \times M, E^* \boxtimes E)$ such that*

$$(\widehat{S}^{O_i} + h_i) \Big|_{O_{ij}^M} = (\widehat{S}^{O_j} + h_j) \Big|_{O_{ij}^M}, \quad i, j \in I.$$

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Proof. For $i, j \in I$, we consider the bisolution $h_{ij} := \widehat{S}^{O_i}|_{O_{ij}^M} - \widehat{S}^{O_j}|_{O_{ij}^M}$. For all $m \in \mathbb{N}$, Proposition 5.4.1 provides parametrices $\widetilde{\mathcal{L}}_{\pm}^m$ on the relative compact domains N_m such that for $\widetilde{\mathcal{L}}^m := \frac{i}{2}(\widetilde{\mathcal{L}}_+^m - \widetilde{\mathcal{L}}_-^m)$, Propositions 5.5.4 and 6.1.1 yield

$$h_{ij}|_{O_{ij} \times N_m} = \underbrace{\widehat{S}^{O_i} - \widetilde{\mathcal{L}}^m}_{\in C^\infty} - \underbrace{(\widehat{S}^{O_j} - \widetilde{\mathcal{L}}^m)}_{\in C^\infty} \in C^\infty(O_{ij} \times N_m, E^* \boxtimes E). \quad (6.4)$$

Such $\widetilde{\mathcal{L}}^m$ exist for all m and $\{N_m\}_{m \in \mathbb{N}}$ exhausts N , so we have smoothness on $O_{ij} \times N$. Furthermore, as O_{ij} is causal and N a neighborhood of a Cauchy hypersurface, h_{ij} fulfills a Cauchy problem with smooth Cauchy data and hence is smooth on all of O_{ij}^M by Theorem 2.3.2.

Therefore, recalling the identification of sections of \mathcal{C}^∞ with smooth sections in $E^* \boxtimes E$, the map $f_1: O_{ij}^M \mapsto h_{ij}$ represents a Čech-1-cochain, which moreover is a cocycle since

$$\begin{aligned} (\partial_1 f_1)(O_{ijk}^M) &= h_{jk}|_{O_{ijk}^M} - h_{ik}|_{O_{ijk}^M} + h_{ij}|_{O_{ijk}^M} \\ &= \widehat{S}^{O_j}|_{O_{ijk}^M} - \widehat{S}^{O_k}|_{O_{ijk}^M} - \widehat{S}^{O_i}|_{O_{ijk}^M} + \widehat{S}^{O_k}|_{O_{ijk}^M} + \widehat{S}^{O_i}|_{O_{ijk}^M} - \widehat{S}^{O_j}|_{O_{ijk}^M} \\ &= 0 \end{aligned}$$

for all $i, j, k \in I$. Thus, Lemma 6.1.2 ensures the existence of $f_0: O_i^M \mapsto \widetilde{h}_i \in C^\infty(O_i^M, E^* \boxtimes E)$ such that $\partial_0 f_0 = f_1$, and hence,

$$h_{ij} = f_1(O_{ij}^M) = \partial_0 f_0(O_{ij}^M) = f_0(O_j^M)|_{O_{ij}^M} - f_0(O_i^M)|_{O_{ij}^M} = \widetilde{h}_j|_{O_{ij}^M} - \widetilde{h}_i|_{O_{ij}^M}, \quad i, j \in I.$$

Recall that $O_i \cap \Sigma$ is a Cauchy hypersurface of O_i for all $i \in I$ and thus, each \widetilde{h}_i determines a bisolution $h_i \in C^\infty(O_i^M, E^* \boxtimes E)$ via Theorem 2.3.2. On the other hand, due to causality of O_{ij} , we have a well-posed Cauchy problem on O_{ij}^M , and consequently, $h_j|_{O_{ij}^M} - h_i|_{O_{ij}^M} = h_{ij}$, since their Cauchy data coincide.

This proves the claim:

$$(\widehat{S}^{O_i} + h_i)|_{O_{ij}^M} = (\widehat{S}^{O_i} + h_j - h_{ij})|_{O_{ij}^M} = (\widehat{S}^{O_j} + h_j)|_{O_{ij}^M}. \quad \square$$

For a partition of unity $\{\chi_i\}_{i \in I}$ subordinate to \mathcal{O}^M , a well-defined bisolution on $N \times M$ is given via

$$\widehat{S}^N[\psi, \varphi] := \sum_{i \in I} (\widehat{S}^{O_i} + h_i)[\chi_i \psi, \varphi], \quad \psi \in \mathcal{D}(N, E), \varphi \in \mathcal{D}(M, E^*). \quad (6.5)$$

Since \mathcal{O}^M is a locally finite cover, for each ψ , only finitely many summands are non-zero. Moreover, due to Lemma (6.1.3), this definition does not depend on the choice of the partition, and for all i , we directly read off from (6.5) that

$$\widehat{S}^N|_{O_i \times M} - \widehat{S}^{O_i} \in C^\infty(O_i \times M, E^* \boxtimes E). \quad (6.6)$$

Hence, two different constructions of such a bisolution on $N \times M$ differ only by a smooth bisolution.

Proposition 6.1.4. *There is a bisolution $S: \mathcal{D}(M, E) \times \mathcal{D}(M, E^*) \rightarrow \mathbb{R}$ such that*

$$S|_{O_i \times O_i} - S^{O_i} \in C^\infty(O_i \times O_i, E^* \boxtimes E), \quad i \in I. \quad (6.7)$$

Proof. Let \widehat{S}^N be the bisolution on $N \times M$ defined by (6.5) and recall that N is an open neighborhood of Σ . For all $\varphi \in \mathcal{D}(M, E^*)$, we define $S(\cdot)[\varphi]$ as the unique solution of

$$\begin{cases} P^t(S(\cdot)[\varphi]) &= 0, \\ S(\cdot)[\varphi]|_{\Sigma} &= \widehat{S}^N(\cdot)[\varphi]|_{\Sigma}, \\ \nabla_{\nu}(S(\cdot)[\varphi])|_{\Sigma} &= \nabla_{\nu}\widehat{S}^N(\cdot)[\varphi]|_{\Sigma}. \end{cases}$$

This yields a smooth section, which leads to a bisolution since $\widehat{S}^N(\cdot)[P^t\varphi] = 0$, and hence, $S(\cdot)[P^t\varphi]$ solves the trivial Cauchy problem. Furthermore, we have $S|_{O_i \times O_i} = \widehat{S}^N|_{O_i \times O_i}$, so (6.7) follows from (6.6) and Proposition 6.1.1. \square

Corollary 6.1.5. *There is a smooth bisolution $u \in C^\infty(M \times M, E^* \boxtimes E)$ such that*

$$(S - u)[\psi_1, \Theta\psi_2] = (S - u)[\psi_2, \Theta\psi_1], \quad \psi_1, \psi_2 \in \mathcal{D}(M, E).$$

Proof. For $(\iota S)[\psi_1, \Theta\psi_2] := S[\psi_2, \Theta\psi_1]$, let $u := \frac{1}{2}(S - \iota S)$. It follows that $S - u = \iota(S - u)$ and we show that u is smooth. For all $m \in \mathbb{N}$, let $\widetilde{\mathcal{L}}^m$ be given as in (6.4), i.e. $\widetilde{\mathcal{L}}^m = \iota\widetilde{\mathcal{L}}^m$ and $\widetilde{S}^N|_{N_m \times N_m} - \widetilde{\mathcal{L}}^m$ smooth due to Corollary 5.4.4 and Proposition 5.5.3. Therefore, u is smooth on $N_m \times N_m$ for all m :

$$2u|_{N_m \times N_m} = \widehat{S}^{N_m} - \iota\widehat{S}^{N_m} + \widetilde{\mathcal{L}}^m - \widetilde{\mathcal{L}}^m = \widehat{S}^{N_m} - \widetilde{\mathcal{L}}^m - \iota(\widehat{S}^{N_m} - \widetilde{\mathcal{L}}^m)$$

and thus on $N \times N$. Since u is a bisolution and N a neighborhood of Σ , Theorem 2.3.2 ensures smoothness on all of $M \times M$. \square

Theorem 6.1.6. *Let M be a globally hyperbolic Lorentzian manifold, $\pi: E \rightarrow M$ a real vector bundle with non-degenerate inner product over M and $P: C^\infty(M, E) \rightarrow C^\infty(M, E)$ a formally self-adjoint wave operator. Furthermore, let G_{\pm} denote the advanced and retarded Green operator for P^t and S the symmetric bisolution given by Proposition 6.1.4 and Corollary 6.1.5. Then*

$$H := S + \frac{i}{2}(G_+ - G_-) \tag{6.8}$$

is a Hadamard bisolution, and a Feynman and an anti-Feynman Green operator for P^t is determined by

$$G_F = iS + \frac{1}{2}(G_+ + G_-), \quad G_{aF} = -iS + \frac{1}{2}(G_+ + G_-). \tag{6.9}$$

Proof. For each $m \in \mathbb{N}$, let $\widetilde{\mathcal{L}}^m$ be given as in (6.4). It follows that $\text{WF}(\widetilde{\mathcal{L}}^m) = \text{WF}(\widetilde{G}_{aF} - \widetilde{G}_F)$ in the sense of (1.11) from Corollary 5.4.6, and moreover, we have

$$S|_{N_m \times N_m} - \widetilde{\mathcal{L}}^m \in C^\infty(N_m \times N_m, E^* \boxtimes E)$$

by Propositions 5.5.4 and 6.1.1 as well as (6.6). This holds for all m and hence, H is of Hadamard form in a causal normal neighborhood N of Σ due to Proposition 5.4.5. Therefore, H is globally Hadamard by Theorem 5.8 of [SV2001] or, more precisely, by (i) of the subsequent Remark.

By the same reasoning as for Corollary 5.4.6, a Feynman and an anti-Feynman parametrix for P^t are given by (1.12), that is,

$$\pm iH + G_{\pm} = \pm iS + \frac{1}{2}(G_+ + G_-).$$

These are even Green operators, since S is a bisolution and G_{\pm} are Green operators for P^t . \square

6.2 Positivity

In the previous section, we depicted the construction of symmetric bisolutions S leading to Hadamard bisolutions via (6.8). Therefore, with regard to (1.7), it only remains to show that S can be chosen such that $S[\psi, \Theta\psi] \geq 0$, $\psi \in \mathcal{D}(M, E)$, holds. The basis for the proof is Theorem 6.6.2 of [DH1972], which ensures the existence of some smooth f such that $\frac{i}{2}(\tilde{G}_{aF} - \tilde{G}_F) + f$ satisfies this positivity property. Unfortunately, it is formulated merely for the scalar setting.

Let M be a smooth manifold, $\pi: E \rightarrow M$ a real or complex vector bundle over M with non-degenerate inner product and $P: C^\infty(M, E) \rightarrow C^\infty(M, E)$ a properly supported pseudodifferential operator. For the definitions of P being of real principal type in M , pseudo-convexity of M with respect to P and the bicharacteristic relation C_P of P , we adopt Definition 3.1 of [Den1982] as well as Definition 6.3.2 and (6.5.2) of [DH1972], respectively. Assuming those properties for M and P , according to Theorem 6.5.3 of [DH1972], there are distinguished parametrices $\tilde{Q}_{C_P \setminus \Delta}, \tilde{Q}_\emptyset$ associated to the respective components of $C_P \setminus \Delta$, where Δ denotes the diagonal in $\text{Char } P \times \text{Char } P$. For P a wave operator, they correspond to Feynman and anti-Feynman parametrices, respectively.

Definition 6.2.1. *Let M be a smooth manifold, $\pi: E \rightarrow M$ a real or complex vector bundle with non-degenerate inner product and $P: C^\infty(M, E) \rightarrow C^\infty(M, E)$ a formally self-adjoint, properly supported pseudodifferential operator of real principal type in M such that M is pseudo-convex with respect to P . Then P is called of positive propagator type if there exists some $f \in C^\infty(M \times M, E^* \boxtimes E)$ such that the bidistribution $T := \frac{i}{2}(\tilde{Q}_{C_P \setminus \Delta} - \tilde{Q}_\emptyset) + f$ satisfies*

$$T[\psi, \Theta\psi] \geq 0, \quad \psi \in \mathcal{D}(M, E).$$

Note that f is not demanded to be unique and in general, a positive propagator type operator will have many such sections. For E the trivial line bundle $M \times \mathbb{R}$, every such P is of that type due to Theorem 6.6.2 of [DH1972]. On the other hand, by Proposition 5.6 of [SV2001], the Hadamard bisolutions fail to be positive if the inner product on E is not positive definite. Hence, anticipating the result of this section, wave operators acting on sections in a non-Riemannian vector bundle over a globally hyperbolic Lorentzian manifold are not of positive propagator type.

The proof of Theorem 6.6.2 of [DH1972] employs positivity of $\frac{i}{2}(\tilde{G}_{aF} - \tilde{G}_F)$ for the directional derivatives $D_n := -i\frac{\partial}{\partial x_n}$, $n = 0, \dots, d-1$ on $C^\infty(\mathbb{R}^d)$ and by applying certain operators, allowing one to keep track of the singularity structure of the corresponding parametrices, the general case is reduced to D_n . Eventually, positivity holds up to smooth functions, since there is no way to control this smooth part in terms of the singularity structure. However, in the setting of Definition 1.2.1 with E assumed to be Riemannian, we can choose the same ansatz and basically the same procedure. This strongly suggests the assumption that wave operators acting on smooth sections in some general Riemannian vector bundle over a globally hyperbolic Lorentzian manifold are of positive propagator type.

Nevertheless, even with this property and actual Green operators G_F, G_{aF} at hand, we can still only deduce the existence of some $f \in C^\infty(M \times M, E^* \boxtimes E)$ such that $(S + f)[\psi, \Theta\psi] \geq 0$ for all $\psi \in \mathcal{D}(M, E)$, where $S := \frac{i}{2}(G_{aF} - G_F)$ denotes the corresponding symmetric bisolution. It is the task of this final section to show that f can be chosen as a symmetric bisolution. As it happens, the proof actually works for a much wider class of differential operators P of positive propagator type. More precisely, they merely have to admit a Feynman and an anti-Feynman Green operator G_F, G_{aF} and a well-posed Cauchy problem. Furthermore, the characteristic set and the bicharacteristic relation have to be given by

$$\begin{aligned} \text{Char}(P) &= \{(p, \xi) \in T^*M \setminus \{0\} \mid g_p(\xi^\sharp, \xi^\sharp) = 0\}, \\ C_P &= \{(p, \xi; q, \eta) \in (T^*M \times T^*M) \setminus \{0\} \mid (p, \xi) \sim (q, \zeta)\}. \end{aligned} \tag{6.10}$$

For Σ some Cauchy hypersurface of M , the idea is to use f as initial data on $\Sigma \times \Sigma$ in order to determine a smooth bisolution u via Theorem 2.3.2. Following the lines of section 3.3 of [GW2015], positivity of the bisolution $S + u$ can be related to positivity on the level of Cauchy data, where $S + u$ and $S + f$ coincide. For the precise argument, we need some preparation.

Let $\iota: \Sigma \hookrightarrow M$ be the embedding map and $\rho := (\iota^*, \iota^* \circ \nabla_\nu)$ the corresponding pullback to the initial data on Σ , i.e.

$$\rho: C^\infty(M, E) \rightarrow C^\infty(\Sigma, E \oplus E), \quad u \mapsto (u|_\Sigma, \nabla_\nu u|_\Sigma). \quad (6.11)$$

Clearly, ρ is surjective and we have $\rho(C_{sc}^\infty(M, E)) = \mathcal{D}(M, E \oplus E)$. Furthermore, for any differential operator P with well-posed Cauchy problem, ρ yields a bijection $\ker P \rightarrow C^\infty(\Sigma, E \oplus E)$. The transposed map ρ^t is related to the pushforward along the embedding, which creates singular directions orthogonal to the embedded (spacelike) hypersurface. More precisely, according to Proposition 10.21 of [DK2010], $\iota_*\varphi$ corresponds to $\varphi\delta_\Sigma$ for any $\varphi \in C^\infty(\Sigma, E)$, and hence, ρ^t is a map

$$\rho^t: C^\infty(\Sigma, E^* \oplus E^*) \longrightarrow \mathcal{D}_{N^*\Sigma}(M, E^*)'.$$

\mathcal{D}'_Γ denotes the distributions with wave front set contained in the closed cone $\Gamma \subset T^*M \setminus \{0\}$, and we refer to section 8.2 of [Hör1990] for precise definitions and properties of these spaces. Due to Hörmander's criterion ((8.2.3) of [Hör1990]), we can pull back a distribution along ι if its wave front set does not contain the orthogonal directions mentioned above. Hence, for all closed cones $\Gamma \subset T^*M \setminus \{0\}$ with $\Gamma \cap N^*\Sigma = \emptyset$, (6.11) extends to a map

$$\rho: \mathcal{D}'_\Gamma(M, E^*)' \longrightarrow \mathcal{D}_{\iota^*\Gamma}(\Sigma, E^* \oplus E^*)', \quad u \mapsto (\chi \mapsto u[\rho^t\chi]),$$

where $\iota^*\Gamma := \{(\sigma, d\iota|_\sigma^t(\xi)) \mid (\iota(\sigma), \xi) \in \Gamma\} \subset T^*\Sigma \setminus \{0\}$ contains the projections of $\xi \in \Gamma$ onto $T^*\Sigma$. Let

$$(\chi, \zeta)_\Sigma := \int_\Sigma (\langle \chi_0, \zeta_0 \rangle + \langle \chi_1, \zeta_1 \rangle) dV_\Sigma, \quad \chi, \zeta \in \mathcal{D}(\Sigma, E \oplus E), \quad (6.12)$$

denote the inner product on $\mathcal{D}(\Sigma, E \oplus E)$ with dV_Σ the induced volume density and $\tilde{\Theta} := (\Theta, \Theta)$ the corresponding isomorphism $E \oplus E \rightarrow E^* \oplus E^*$. If P is Green-hyperbolic, we obtain the exact sequence (1.1) and thus, $\text{ran } G = \ker P|_{C_{sc}^\infty}$. This provides a further bijection $\rho G: \mathcal{D}(M, E)/\ker G \rightarrow \mathcal{D}(\Sigma, E \oplus E)$, which transfers G to a Green operator G_Σ on the space of initial data $\mathcal{D}(\Sigma, E \oplus E)$ via

$$(\rho G\psi_1, G_\Sigma \rho G\psi_2)_\Sigma := (\psi_1, G\psi_2)_M, \quad \psi_1, \psi_2 \in \mathcal{D}(M, E). \quad (6.13)$$

We finish the preparation by giving an explicit expression for G_Σ . Using the adjoints with respect to (2.2) and (6.12), as well as $G^* = -G$, (6.13) becomes $G = -G\rho^*G_\Sigma\rho G$ and hence, $-G\rho^*G_\Sigma\rho|_{\text{ran } G} = \text{id}|_{\text{ran } G}$. Then bijectivity of ρ on $\ker P = \text{ran } G$ leads to a well-defined map

$$U_\Sigma := -G\rho^*G_\Sigma: C^\infty(\Sigma, E \oplus E) \longrightarrow C^\infty(M, E),$$

which satisfies

$$\rho U_\Sigma = \text{id}, \quad U_\Sigma \rho|_{\ker P} = \text{id}|_{\ker P}, \quad P U_\Sigma = 0.$$

In other words, U_Σ maps initial data $(u_0, u_1) \in C^\infty(\Sigma, E \oplus E)$ to the solution u of the corresponding homogeneous Cauchy problem and therefore, it is frequently referred to as the Cauchy evolution operator.

6 Global Hadamard two-point-functions

On the other hand, [Dim1980] and, for the vector-valued case, [BS2018] provide the expression

$$u = G^*(\rho_1^* u_0 - \rho_0^* u_1),$$

where $\rho = (\rho_0, \rho_1)$. By uniqueness, this has to coincide with $U_\Sigma(u_0, u_1) = (\rho G)^* G_\Sigma(u_0, u_1)$, which leads to

$$\left(G_\Sigma \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \rho G \psi \right)_\Sigma = \left(U_\Sigma \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \psi \right)_M = (u, \psi)_M = \left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} \rho_1 G \psi \\ -\rho_0 G \psi \end{pmatrix} \right)_\Sigma = \left(\begin{pmatrix} -u_1 \\ u_0 \end{pmatrix}, \rho G \psi \right)_\Sigma$$

for all $\psi \in \mathcal{D}(M, E)$. By surjectivity of $\rho G: \mathcal{D}(M, E) \rightarrow \mathcal{D}(\Sigma, E \oplus E)$, this identifies G_Σ as the map

$$G_\Sigma: C^\infty(\Sigma, E \oplus E) \longrightarrow C^\infty(\Sigma, E \oplus E), \quad (u_0, u_1) \longmapsto (-u_1, u_0).$$

Theorem 6.2.2. *Let M be a globally hyperbolic Lorentzian manifold, $\pi: E \rightarrow M$ a Riemannian vector bundle and $P: C^\infty(M, E) \rightarrow C^\infty(M, E)$ a linear first- or second-order differential operator, which is of positive propagator type and admits a well-posed Cauchy problem. Assume that the characteristic set and the bicharacteristic relation of P are given by (6.10) and that $\tilde{Q}_{C_P \setminus \Delta}, \tilde{Q}_\emptyset$ can be chosen as actual Green operators $Q_{C_P \setminus \Delta}, Q_\emptyset$.*

Then there is a real-valued and symmetric bisolution S such that $S - \frac{i}{2}(Q_{C_P \setminus \Delta} - Q_\emptyset)$ is smooth and

$$S[\psi, \Theta\psi] \geq 0, \quad \psi \in \mathcal{D}(M, E).$$

Proof. The desired real-valued bisolution is given by

$$S[\psi, \varphi] := \frac{i}{4}(Q_{C_P \setminus \Delta} - Q_\emptyset)[\psi, \varphi] + \frac{i}{4}\overline{(Q_{C_P \setminus \Delta} - Q_\emptyset)[\psi, \varphi]}, \quad \psi, \varphi \in \mathcal{D}(M, E), \quad (6.14)$$

and we show the claimed properties. With regard to Corollary 6.1.5 and without loss of generality, we assume S to be symmetric, and furthermore, there is some $f \in C^\infty(M \times M, E^* \boxtimes E)$ such that

$$(S + f)[\psi, \Theta\psi] \geq 0, \quad \psi \in \mathcal{D}(M, E), \quad (6.15)$$

since P is of positive propagator type. Because $\tilde{f}[\psi_1, \Theta\psi_2] := \frac{1}{2}(f[\psi_1, \Theta\psi_2] + f[\psi_2, \Theta\psi_1])$ also satisfies (6.15), we assume symmetry of f as well, that is, $f[\psi_1, \Theta\psi_2] = f[\psi_2, \Theta\psi_1]$ for all $\psi_1, \psi_2 \in \mathcal{D}(M, E)$.

Recall that Green operators map $\mathcal{D}(M, E^*)$ to $C^\infty(M, E^*)$, so for fixed $\varphi \in C^\infty(M, E^*)$, (6.14) provides a smooth section $p \mapsto S(p)[\varphi]$ in E^* . It follows that for each $p \in M$, we obtain a well-defined E_p^* -valued distribution $S(p)$, which is a bisolution for P . Therefore, the assumptions on P imply that $\text{WF}(S(p))$ exclusively contains lightlike directions. Hence, $\text{WF}(S(p)) \cap N^*\Sigma = \emptyset$, so the restriction of $S(p)$ to Σ yields a well-defined distribution $\rho(S(p)): \mathcal{D}(\Sigma, E^* \oplus E^*) \rightarrow \mathbb{R}$ for any Cauchy hypersurface Σ . This means that, due to Theorem 8.2.13 of [Hör1990], the operator \mathcal{S} associated to (6.14) can be applied to $\rho^t \chi \in \mathcal{D}_{N^*\Sigma}(M, E^*)'$, $\chi \in \mathcal{D}(\Sigma, E^* \oplus E^*)$, and for the result, we obtain

$$\text{WF}(\mathcal{S}\rho^t \chi) \subset \{(p, \xi) \mid (p, \xi; q, 0) \in \text{WF}(S)\} \cup \{(p, \xi) \mid \exists (q, \zeta) \in \text{WF}(\rho^t \chi) : (p, \xi; q, -\zeta) \in \text{WF}(S)\}.$$

Since $\text{WF}(S) \subset C_P = \{(p, \xi) \sim (q, \zeta)\}$ and $\text{WF}(\rho^t \chi) \subset N^*\Sigma$, both contributions on the right hand side are empty. Hence, $\mathcal{S}\rho^t$ represents a map $\mathcal{D}(\Sigma, E^* \oplus E^*) \rightarrow C^\infty(M, E^*)$, so it follows that $p \mapsto \rho(S(p))[\chi] = (\mathcal{S}\rho^t \chi)(p)$ is smooth for fixed χ . With the adjoint operator $\rho^* = \Theta^{-1} \rho^t \tilde{\Theta}$, we even-

tually obtain a well-defined bidistribution $S^\Sigma: \mathcal{D}(\Sigma, E \oplus E) \times \mathcal{D}(\Sigma, E^* \oplus E^*) \rightarrow \mathbb{R}$ via

$$S^\Sigma[\lambda, \chi] := S[\rho^* \lambda, \rho^t \chi] = \int_\Sigma \tilde{\Theta} \rho \Theta^{-1} \left(S(\cdot)[\rho^t \chi] \right) (\sigma) (\lambda(\sigma)) dV_\Sigma(\sigma). \quad (6.16)$$

The bisection f determines smooth and symmetric Cauchy data on $\Sigma \times \Sigma$ and thus a smooth and symmetric bisolution u by the Theorems 2.3.2 and 2.3.5. Using the short-hand notation $S_f := S + f$ and $S_u := S + u$, this yields $S_u^\Sigma = S_f^\Sigma$ for the corresponding bidistributions (6.16), and we show that positivity is preserved under the restriction to Σ , i.e. $S_f^\Sigma[\lambda, \tilde{\Theta}\lambda] \geq 0$ for all $\lambda \in \mathcal{D}(\Sigma, E \oplus E)$. Due to Theorem 8.2.3 of [Hör1990], we find a sequence $(\psi_n)_{n \in \mathbb{N}} \subset \mathcal{D}(M, E)$ such that $\psi_n \rightarrow \rho^* \lambda$ in $\mathcal{D}_{N^*\Sigma}(M, E)'$, and consequently, $\Theta \psi_n \rightarrow \Theta \rho^* \lambda = \rho^t \tilde{\Theta} \lambda$. By continuity of S_f as a bidistribution, it follows that

$$S_f^\Sigma[\lambda, \tilde{\Theta}\lambda] = S_f[\rho^* \lambda, \rho^t \tilde{\Theta}\lambda] = \lim_{n \rightarrow \infty} \underbrace{S_f[\psi_n, \Theta \psi_n]}_{\geq 0} \geq 0. \quad (6.17)$$

The proof of Theorem 3.3.1 and Proposition 3.4.2 of [BGP2007] show that well-posedness of the Cauchy problem implies the existence of a unique advanced and retarded Green operator and hence exactness of the sequence (1.1). Thus, due to $\ker P = \text{ran } G$, S_u does not only descend to a well-defined bilinear form on $\mathcal{D}(M, E)/\ker P$, since it is a bisolution, but also to $\text{ran } G$ via

$$S'_u[G\psi_1, \Theta G\psi_2] := S_u[\psi_1, \Theta \psi_2], \quad \psi_1, \psi_2 \in \mathcal{D}(M, E).$$

By following the lines of Proposition 3.9 of [GW2015] and employing $G = -G\rho^*G_\Sigma\rho G$, this allows us to trace back the claimed positivity property to (6.17). More precisely, for all $\psi_1, \psi_2 \in \mathcal{D}(M, E)$, we have

$$\begin{aligned} S_u[\psi_1, \Theta \psi_2] &= S'_u[G\psi_1, \Theta G\psi_2] = S'_u[G\rho^*G_\Sigma\rho G\psi_1, \Theta G\rho^*G_\Sigma\rho G\psi_2] \\ &= S_u[\rho^*G_\Sigma\rho G\psi_1, \Theta \rho^*G_\Sigma\rho G\psi_2] = S_u^\Sigma[G_\Sigma\rho G\psi_1, \tilde{\Theta}G_\Sigma\rho G\psi_2] \\ &= S_f^\Sigma[G_\Sigma\rho G\psi_1, \tilde{\Theta}G_\Sigma\rho G\psi_2], \end{aligned}$$

which proves the theorem. \square

In the case of formally self-adjoint wave operators, the existence of G_F and G_{aF} is ensured by Theorem 6.1.6, so Theorem 6.2.2 leads to the final result of this thesis:

Theorem 6.2.3. *Let M be a globally hyperbolic Lorentzian manifold, $\pi: E \rightarrow M$ a Riemannian vector bundle and $P: C^\infty(M, E) \rightarrow C^\infty(M, E)$ a formally self-adjoint wave operator of positive propagator type. Then there exists a bidistribution $S: \mathcal{D}(M, E) \times \mathcal{D}(M, E^*) \rightarrow \mathbb{R}$ such that*

$$H := S + \frac{i}{2}(G_+ - G_-)$$

yields a Hadamard two-point-function, where G_\pm denotes the advanced and retarded Green operator for P^t . This means that $WF(H)$ has the Hadamard singularity structure (1.9) and satisfies

$$H[P\psi, \varphi] = 0 = H[\psi, P^t\varphi], \quad H[\psi, \varphi] - H[\Theta^{-1}\varphi, \Theta\psi] = \frac{i}{2}(G_+ - G_-)[\psi, \varphi], \quad H[\psi, \Theta\psi] \geq 0$$

for all $\psi \in \mathcal{D}(M, E)$, $\varphi \in \mathcal{D}(M, E^)$.*

Moreover, a Feynman and an anti-Feynman Green operator G_F, G_{aF} are given by (6.9).

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Note that, in general, S is far from being unique, i.e. there may be many bidistributions with the required properties. Clearly, this is related to the non-uniqueness of the many choices of smooth sections during the construction, and in most cases, it is not at all obvious, how to find these sections practically. This particularly concerns the choice of the h_i 's in Lemma 6.1.3 and the f for operators of positive propagator type.

However, the overall reasoning provides a comparatively constructive alternative to the existence proofs, which are already present in the literature ([BF2014], [FNW1981], [GOW2017]). It starts most naturally with the Hadamard condition, so the form of the bidistributions is, up to smooth terms, determined right from the start. It therefore might provide a promising starting point for a possible classification of these states up to unitary equivalence of their respective GNS-representations. This and the identification of pure states in particular would require to investigate the choices of the said smooth sections.

Furthermore, the methods used here provide an alternative procedure to the classic deformation arguments since they rely on the ability to make modifications to the metric confined to certain spacetime regions. There are situations, where this is not applicable, for instance, in the case of linearized gravity, where the background spacetime must solve the Einstein equation, or similarly for linearizations of Yang-Mills theories. They also occur if one is restricted to analytic metrics.

7 Appendix

7.1 Proof of Proposition 5.4.1 for even dimensional spacetimes

Note that in the proofs of all following Lemmas, c denotes a generic constant, i.e. its particular value can change from one line to another.

Lemma 7.1.1. *For all $l \in \mathbb{N}$ and $\beta \geq l + 1$, there is some $c_{l,\beta}$ such that for all $0 < \varepsilon \leq 1$, we have*

$$\left\| \frac{d^l}{dt^l} \left(\sigma \left(\frac{t}{\varepsilon} \right) \cdot t^\beta \cdot \log t \right) \right\|_{C^0(\mathbb{R})} \leq \varepsilon \left(\log \frac{1}{\varepsilon} + \pi + 1 \right) \cdot c_{l,\beta} \cdot \|\sigma\|_{C^l(\mathbb{R})}.$$

Proof. We start with calculating

$$\begin{aligned} & \left\| \frac{d^j}{dt^j} \left(t^\beta \cdot \log t \right) \right\|_{C^0(\mathbb{R})} \\ &= \left\| \beta(\beta-1)\dots(\beta-j+1)t^{\beta-j} \log t + t^{\beta-j} \sum_{i=1}^j \binom{j}{i} (-1)^{i-1} (i-1)! \beta(\beta-1)\dots(\beta-j+i+1) \right\|_{C^0(\mathbb{R})} \\ &\leq c_{j,\beta} \cdot |t|^{\beta-j} (|\log t| + 1) \\ &\leq c_{j,\beta} \cdot |t|^{\beta-j} (\log |t| + \pi + 1). \end{aligned}$$

Since $\sigma^{(l-j)} \left(\frac{t}{\varepsilon} \right) = 0$ for $|t| \geq \varepsilon$ and $\varepsilon^{\beta-l} \leq \varepsilon$ due to $\beta \geq l + 1$, this yields

$$\begin{aligned} \left\| \frac{d^l}{dt^l} \left(\sigma \left(\frac{t}{\varepsilon} \right) \cdot t^\beta \cdot \log t \right) \right\|_{C^0(\mathbb{R})} &\leq \sum_{j=0}^l \binom{l}{j} \cdot c_{j,\beta} \left\| \frac{\sigma^{(l-j)} \left(\frac{t}{\varepsilon} \right)}{\varepsilon^{l-j}} \cdot |t|^{\beta-j} (|\log |t|| + \pi + 1) \right\|_{C^0(\mathbb{R})} \\ &\leq \sum_{j=0}^l \binom{l}{j} \cdot c_{j,\beta} \cdot \varepsilon^{\beta-l} \left(\log \frac{1}{\varepsilon} + \pi + 1 \right) \cdot \|\sigma^{(l-j)}\|_{C^0(\mathbb{R})} \\ &\leq c_{l,\beta} \cdot \varepsilon \left(\log \frac{1}{\varepsilon} + \pi + 1 \right) \cdot \|\sigma\|_{C^l(\mathbb{R})}. \end{aligned}$$

□

Lemma 7.1.2. *For any open and relatively compact domain $O \subset \Omega$ and $l \in \mathbb{N}_0$, there is a sequence $\{\varepsilon_k\}_{k \geq N} \subset (0, 1]$ such that for all $l \geq 0$ the series*

$$(p, q) \longmapsto \sum_{k=N+l}^{\infty} \tilde{U}_k(p, q) \log(\Gamma(p, q) \pm i0) L_{\pm}^{\Omega}(2k+2, p)(q) \quad (7.1)$$

converges in $C^l(\overline{O} \times \overline{O}, E^ \boxtimes E)$. In particular, for $l = 0$, this defines a continuous section over $\overline{O} \times \overline{O}$ and a smooth section over $(\overline{O} \times \overline{O}) \setminus \Gamma^{-1}(0)$.*

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Proof. Since $k \geq N \geq \frac{d}{2}$ and d even, $\Gamma(p, q)^{k - \frac{d-2}{2}}$ is a smooth and $\Gamma(p, q)^{k - \frac{d-2}{2}} \cdot \log(\Gamma(p, q) \pm i0)$ a continuous section over $\overline{O} \times \overline{O}$, so every single summand of (7.1) is at least continuous, individually. Due to $\text{supp} \left(\sigma \circ \frac{\Gamma}{\varepsilon_k} \right) \subset \{ \Gamma(p, q) < \varepsilon_k \}$ for all $k \geq N$ and $0 \leq \sigma \leq 1$ by Lemma 7.1.1, we have

$$\left\| \tilde{U}_k \log(\Gamma \pm i0) \cdot C(2k+2, d) \Gamma^{k - \frac{d-2}{2}} \right\|_{C^0(\overline{O} \times \overline{O})} \leq c_{k,d} \|U_k\|_{C^0(\overline{O} \times \overline{O})} \cdot \varepsilon_k \left(\log \frac{1}{\varepsilon_k} + \pi + 1 \right).$$

Since $\varepsilon_k \log \frac{1}{\varepsilon_k} \rightarrow 0$ for $\varepsilon_k \rightarrow 0$, we can choose ε_k such that

$$c_{k,d} \|U_k\|_{C^0} \cdot \varepsilon_k \left(\log \frac{1}{\varepsilon_k} + \pi + 1 \right) < 2^{-k},$$

and (7.1) converges in C^0 . Now let $l > 0$ and $k \geq N + l$ such that $\Gamma(p, q)^{k - \frac{d-2}{2}} \log(\Gamma(p, q) \pm i0)$ is of C^l -regularity. Set $\rho_k(t) := \sigma \left(\frac{t}{\varepsilon_k} \right) t^{k - \frac{d-2}{2}}$, so by Lemma 7.1.1 we have

$$\|\rho_k\|_{C^l(\mathbb{R})} \leq c_{l,k,d} \cdot \varepsilon_k \|\sigma\|_{C^l(\mathbb{R})}, \quad \|\rho_k \cdot \log\|_{C^l(\mathbb{R})} \leq c_{l,k,d} \cdot \varepsilon_k \left(\log \frac{1}{\varepsilon_k} + \pi + 1 \right) \|\sigma\|_{C^l(\mathbb{R})}, \quad (7.2)$$

and Lemma 1.1.11 and 1.1.12 of [BGP2007] yield

$$\begin{aligned} \left\| \tilde{U}_k \log(\Gamma \pm i0) \cdot C(2k+2, d) \Gamma^{k - \frac{d-2}{2}} \right\|_{C^l(\overline{O} \times \overline{O})} &\leq c_{l,k,d} \|U_k\|_{C^l(\overline{O} \times \overline{O})} \cdot \left\| (\rho_k \cdot \log) \circ \Gamma \right\|_{C^l(\overline{O} \times \overline{O})} \\ &\stackrel{1.1.11}{\leq} c_{l,k,d} \|U_k\|_{C^l(\overline{O} \times \overline{O})} \cdot \left\| (\rho_k \cdot \log) \circ \Gamma \right\|_{C^l(\overline{O} \times \overline{O})} \\ &\stackrel{1.1.12}{\leq} c_{l,k,d} \|U_k\|_{C^l(\overline{O} \times \overline{O})} \cdot \max_{j=0, \dots, l} \|\Gamma\|_{C^l(\overline{O} \times \overline{O})}^j \cdot \|\rho_k \cdot \log\|_{C^l(\mathbb{R})} \\ &\stackrel{(7.2)}{\leq} c_{l,k,d} \|\sigma\|_{C^l(\mathbb{R})} \cdot \max_{j=0, \dots, l} \|\Gamma\|_{C^l(\overline{O} \times \overline{O})}^j \cdot \|U_k\|_{C^l(\overline{O} \times \overline{O})} \cdot \varepsilon_k \log \left(\frac{1}{\varepsilon_k} + \pi + 1 \right). \end{aligned}$$

Hence, for all $k \geq N - l$, we demand

$$c_{l,k,d} \|U_k\|_{C^l(\overline{O} \times \overline{O})} \cdot \varepsilon_k \log \left(\frac{1}{\varepsilon_k} + \pi + 1 \right) \leq 2^{-k}, \quad (7.3)$$

so the k . summand can be estimated by $\frac{\|\sigma\|_{C^l(\mathbb{R})}}{2^k} \cdot \max_{j=0, \dots, l} \|\Gamma\|_{C^l(\overline{O} \times \overline{O})}^j$ and (7.1) converges in $C^l(\overline{O} \times \overline{O}, E^* \boxtimes E)$. Note that for each k , we impose only finitely many conditions on ε_k , namely one for each $l \leq k - N$, which are satisfied by some positive number. Hence, for each k , there is a sufficiently small number $\varepsilon_k > 0$ such that (7.3) is fulfilled for all $l \leq k - N$.

Since all summands are smooth on $(\overline{O} \times \overline{O}) \setminus \Gamma^{-1}(0)$ and the series converges in all C^l -norms, it defines a smooth section on $(\overline{O} \times \overline{O}) \setminus \Gamma^{-1}(0)$. \square

Thus, we showed that (5.16) yield well-defined distributions with singular support on the light cone, i.e. property (i). Furthermore, (iii) follows from Proposition 5.2.1 (6). We proceed with (ii):

Lemma 7.1.3. *The sequence $(\varepsilon_k)_{k \geq N}$ can be chosen such that*

$$P_{(2)} \tilde{\mathcal{L}}_{\pm}(p) = \delta_p + K_{\pm}(p, \cdot)$$

for some $K_{\pm} \in C^{\infty}(\overline{O} \times \overline{O}, E^* \boxtimes E)$.

7.1 Proof of Proposition 5.4.1 for even dimensional spacetimes

Proof. Let $\sigma_k := \sigma \circ \frac{\Gamma}{\varepsilon_k} \in C^\infty(\Omega \times \Omega)$, so $\tilde{U}_k = \sigma_k \cdot U_k$ for all $k \geq N$ and $\text{supp } \sigma_k \subset \{\Gamma(p, q) \leq \varepsilon_k\}$. Due to Lemma 1.1.10 of [BGP2007], we can exchange P with the sum, so the transport equations (5.12) imply

$$\begin{aligned} & \sum_{k=N}^{\infty} P_{(2)} \sigma_k (U_k \log(\Gamma \pm i0) + W_k) L_{\pm}^{\Omega}(2k+2) \\ &= \sum_{k=N}^{\infty} \left(\square_{(2)} \sigma_k - 2\nabla_{\text{grad}_{(2)} \sigma_k} + \sigma_k P_{(2)} \right) (U_k \log(\Gamma \pm i0) + W_k) L_{\pm}^{\Omega}(2k+2) \\ &=: \Sigma_1 + \Sigma_2 + \sum_{k=N}^{\infty} \sigma_k P_{(2)} (U_k \log(\Gamma \pm i0) + W_k) L_{\pm}^{\Omega}(2k+2). \end{aligned}$$

Recall that the transport equations (5.12) and (5.13) are derived from the requirement that $P_{(2)}$ applied to (5.11) is a telescoping series, that is,

$$\begin{aligned} & P_{(2)} (U_k \log(\Gamma \pm i0) + W_k) L_{\pm}^{\Omega}(2k+2) \\ &= (\log(\Gamma \pm i0) \cdot P_{(2)} U_k + P_{(2)} W_k) L_{\pm}^{\Omega}(2k+2) - (\log(\Gamma \pm i0) \cdot P_{(2)} U_{k-1} + P_{(2)} W_{k-1}) L_{\pm}^{\Omega}(2k). \end{aligned}$$

Hence, the right hand side becomes

$$\Sigma_1 + \Sigma_2 - \sigma_N (\log(\Gamma \pm i0) \cdot P_{(2)} U_{N-1} + P_{(2)} W_{N-1}) L_{\pm}^{\Omega}(2N) + \Sigma_3$$

with

$$\Sigma_3 := \sum_{k=N}^{\infty} (\sigma_k - \sigma_{k+1}) (\log(\Gamma \pm i0) \cdot P_{(2)} U_k + P_{(2)} W_k) L_{\pm}^{\Omega}(2k+2).$$

Then, for $K_{\pm}(p, \cdot) := P_{(2)} \tilde{\mathcal{L}}_{\pm}(p) - \delta_p$, the transport equations (5.12) for U_k yield

$$K_{\pm} = (1 - \sigma_N) (\log(\Gamma \pm i0) \cdot P_{(2)} U_{N-1} + P_{(2)} W_{N-1}) L_{\pm}^{\Omega}(2N) + \Sigma_1 + \Sigma_2 + \Sigma_3. \quad (7.4)$$

On the right hand side, every summand individually yields a smooth section, since both $1 - \sigma_N$ and $\sigma_k - \sigma_{k+1}$ as well as all derivatives of σ_k vanish in a neighborhood of $\Gamma^{-1}(0)$, which contains the singular support of $\tilde{\mathcal{L}}_{\pm}$. Thus, K_{\pm} vanishes on $\Gamma^{-1}(0)$ to arbitrary order and it remains to show convergence in all C^l -norms, which again for the W -part is provided by the proof of Lemma 2.4.3 of [BGP2007]. Therefore, we concentrate on

$$\sum_{k=N}^{\infty} \left(\square_{(2)} \sigma_k - 2\nabla_{\text{grad}_{(2)} \sigma_k} + \sigma_k P_{(2)} \right) U_k \log(\Gamma \pm i0) L_{\pm}^{\Omega}(2k+2) =: \Sigma'_1 + \Sigma'_2 + \Sigma'_3.$$

For fixed $l \in \mathbb{N}_0$, let $k \geq 2(l+1) + N$ and $S_k := \{\frac{\varepsilon_k}{2} \leq |\Gamma(p, q)| \leq \varepsilon_k\}$. Then Lemma 1.1.12 of [BGP2007] implies for the k . summand of Σ'_2

$$\begin{aligned} & \left\| \nabla_{\text{grad}_{(2)} \sigma_k} U_k \log(\Gamma \pm i0) L_{\pm}^{\Omega}(2k+2) \right\|_{C^l(\bar{\mathcal{O}} \times \bar{\mathcal{O}})} = \left\| \nabla_{\text{grad}_{(2)} \sigma_k} U_k \log(\Gamma \pm i0) L_{\pm}^{\Omega}(2k+2) \right\|_{C^l(\bar{\mathcal{O}} \times \bar{\mathcal{O}} \cap S_k)} \\ & \leq c_{l,d} \|U_k\|_{C^{l+1}(\bar{\mathcal{O}} \times \bar{\mathcal{O}} \cap S_k)} \cdot \|\sigma_k\|_{C^{l+1}(\bar{\mathcal{O}} \times \bar{\mathcal{O}} \cap S_k)} \cdot \left\| \Gamma^{k-\frac{d-2}{2}} \log(\Gamma \pm i0) \right\|_{C^{l+1}(\bar{\mathcal{O}} \times \bar{\mathcal{O}} \cap S_k)} \\ & \leq c_{l,d} \|U_k\|_{C^{l+1}(\bar{\mathcal{O}} \times \bar{\mathcal{O}} \cap S_k)} \cdot \frac{\|\sigma\|_{C^{l+1}(\mathbb{R})}}{\varepsilon_k^{l+1}} \cdot \left\| t \mapsto t^{k-\frac{d-2}{2}} \cdot \log t \right\|_{C^{l+1}([\frac{\varepsilon_k}{2}, \varepsilon_k])} \cdot \max_{j=0, \dots, l+1} \|\Gamma\|_{C^{l+1}(\bar{\mathcal{O}} \times \bar{\mathcal{O}} \cap S_k)}^{2j} \end{aligned}$$

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$$\begin{aligned} &\leq c_{l,k,d} \|U_k\|_{C^{l+1}(\overline{O} \times \overline{O} \cap S_k)} \frac{\|\sigma\|_{C^{l+1}(\mathbb{R})}}{\varepsilon_k^{l+1}} \max_{j=0, \dots, l+1} \|\Gamma\|_{C^{l+1}(\overline{O} \times \overline{O} \cap S_k)}^{2j} \max_{t \in \left(\frac{\varepsilon_k}{2}, \varepsilon_k\right]} |t|^{k - \frac{d-2}{2} - (l+1)} (|\log t| + 1) \\ &\leq c_{l,k,d} \|U_k\|_{C^{l+1}(\overline{O} \times \overline{O})} \cdot \varepsilon_k \left(\log \frac{1}{\varepsilon_k} + \pi + 1 \right) \|\sigma\|_{C^{l+1}(\mathbb{R})} \cdot \max_{j=0, \dots, l+1} \|\Gamma\|_{C^{l+1}(\overline{O} \times \overline{O})}^{2j}, \end{aligned}$$

so we additionally demand

$$c_{l,k,d} \cdot \|U_k\|_{C^{l+1}(\overline{O} \times \overline{O})} \cdot \varepsilon_k \left(\log \frac{1}{\varepsilon_k} + \pi + 1 \right) \leq 2^{-k}.$$

Then, for all l , the C^l -norm of almost all summands (without the first $2(l+1) + N$) of Σ'_2 is bounded by $2^{-k} \cdot \|\sigma\|_{C^{l+1}(\mathbb{R})} \cdot \max_{j=0, \dots, l+1} \|\Gamma\|_{C^{l+1}(\overline{O} \times \overline{O})}^{2j}$ and thus, we have convergence in C^l for all l , i.e. Σ'_2 defines a smooth section in $E^* \boxtimes E$ over $\overline{O} \times \overline{O}$.

The treatment for Σ'_1 is completely identical, so we directly turn to the k . summand of Σ'_3 :

$$\begin{aligned} &\|(\sigma_k - \sigma_{k+1}) L_{\pm}^{\Omega}(2k+2, \cdot) \log(\Gamma \pm i0) \cdot P_{(2)} U_k\|_{C^l(\overline{O} \times \overline{O})} \\ &\leq c_{l,k,d} \left(\|\sigma_k \Gamma^{l+1} \log(\Gamma \pm i0)\|_{C^l} + \|\sigma_{k+1} \Gamma^{l+1} \log(\Gamma \pm i0)\|_{C^l} \right) \|\Gamma^{k-N-l}\|_{C^l} \cdot \|P_{(2)} U_k\|_{C^l}. \end{aligned}$$

Set $\rho_{kl}(t) := \sigma\left(\frac{t}{\varepsilon_k}\right) \cdot t^{l+1} \log t$, so we have $\sigma_k \Gamma^{l+1} \log(\Gamma \pm i0) = \rho_{kl} \circ \Gamma$. Then again Lemma 1.1.12 of [BGP2007] and Lemma 7.2 yield

$$\begin{aligned} \|\sigma_k \Gamma^{l+1} \log(\Gamma \pm i0)\|_{C^l(\overline{O} \times \overline{O})} &\leq c_l \cdot \|\rho_{kl}\|_{C^l(\mathbb{R})} \cdot \max_{j=0, \dots, l} \|\Gamma\|_{C^l(\overline{O} \times \overline{O})} \\ &\leq c_{k,l} \cdot \varepsilon_k \left(\log \frac{1}{\varepsilon_k} + \pi + 1 \right) \|\sigma\|_{C^l(\mathbb{R})} \cdot \max_{j=0, \dots, l} \|\Gamma\|_{C^l(\overline{O} \times \overline{O})}, \end{aligned}$$

so we obtain

$$\begin{aligned} \|\Sigma'_3\|_{C^l(\overline{O} \times \overline{O})} &\leq c_{l,k,d} \left(\varepsilon_k \left(\log \frac{1}{\varepsilon_k} + \pi + 1 \right) + \varepsilon_{k+1} \left(\log \frac{1}{\varepsilon_{k+1}} + \pi + 1 \right) \right) \|\sigma\|_{C^l(\mathbb{R})} \\ &\quad \cdot \max_{j=0, \dots, l} \|\Gamma\|_{C^l(\overline{O} \times \overline{O})} \cdot \|\Gamma^{k-N-l}\|_{C^l(\overline{O} \times \overline{O})} \cdot \|P_{(2)} U_k\|_{C^l(\overline{O} \times \overline{O})}. \end{aligned}$$

Hence, for all $k \geq N + l$ we demand

$$c_{l,k,d} \cdot \varepsilon_k \left(\log \frac{1}{\varepsilon_k} + \pi + 1 \right) \cdot \|P_{(2)} U_k\|_{C^l(\overline{O} \times \overline{O})} \cdot \|\Gamma^{k-N-l}\|_{C^l(\overline{O} \times \overline{O})} \leq 2^{-k-1}$$

as well as for all $k \geq N + l + 1$ that

$$c_{l,k-1,d} \cdot \varepsilon_k \left(\log \frac{1}{\varepsilon_k} + \pi + 1 \right) \cdot \|P_{(2)} U_{k-1}\|_{C^l(\overline{O} \times \overline{O})} \cdot \|\Gamma^{k-N-l-1}\|_{C^l(\overline{O} \times \overline{O})} \leq 2^{-k-2}.$$

Then the C^l -norm of almost all summands of Σ'_3 (without the first $l + N$) is bounded by $2^{-k} \cdot \|\sigma\|_{C^l(\mathbb{R})} \cdot \max_{j=0, \dots, l} \|\Gamma\|_{C^l(\overline{O} \times \overline{O})}$, so the series converges in all C^l -norms and is therefore smooth. Note that for each k we again added only finitely many conditions. \square

Finally, we show that the ε_k 's can be chosen such that for all $p \in \overline{O}$, the parametrices $\widetilde{\mathcal{L}}_{\pm}(p)$ are distributions of degree at most κ_d .

Lemma 7.1.4. *There is a sequence $(\varepsilon_k)_{k \geq N} \subset (0, 1]$, for which we find some $C > 0$ such that*

$$\left| \widetilde{\mathcal{L}}_{\pm}(p)[\varphi] \right| \leq C \cdot \|\varphi\|_{C^{\kappa_d}(\Omega)}, \quad p \in \overline{O}, \varphi \in \mathcal{D}(\Omega, E^*).$$

Furthermore, for fixed $\varphi \in \mathcal{D}(\Omega, E^*)$, the map $p \mapsto \widetilde{\mathcal{L}}_{\pm}(p)[\varphi]$ is smooth.

Proof. We show the claim only for the logarithmic part, i.e. $f := \sum_{k=\frac{d-2}{2}}^{\infty} \widetilde{U}_k \log(\Gamma \pm i0) L_{\pm}^{\Omega}(2k+2)$, since for the other two sums the proof of Lemma 2.4.4 of [BGP2007] applies identically. By Lemma 7.1.2, we have $f \in C^0(\overline{O} \times \overline{O}, E^* \boxtimes E)$ and thus,

$$|f(p)[\varphi]| \leq \|f\|_{C^0(\overline{O} \times \overline{O})} \cdot \text{vol}(\overline{O}) \cdot \|\varphi\|_{C^0(\overline{O} \times \overline{O})} \leq \|f\|_{C^{\kappa_d}(\overline{O} \times \overline{O})} \cdot \text{vol}(\overline{O}) \cdot \|\varphi\|_{C^{\kappa_d}(\overline{O} \times \overline{O})}$$

for all $p \in O$ and $\varphi \in \mathcal{D}(O, E^*)$, so the constant can be chosen via $C := \|f\|_{C^{\kappa_d}(\overline{O} \times \overline{O})} \cdot \text{vol}(\overline{O})$.

Since Proposition 5.2.1 (6) directly applies also to $\log(\Gamma_p \pm i0) L_{\pm}^{\Omega}(2k+2, p)$ and $U_k \varphi$ is smooth on $O \times O$ with $\text{supp}(U_p^k) \varphi$ compact, for every $k \geq \frac{d-2}{2}$, the map

$$p \mapsto \widetilde{U}_p^k \log(\Gamma_p \pm i0) L_{\pm}^{\Omega}(2k+2, p)[\varphi], \quad \varphi \in \mathcal{D}(O, E^*),$$

is smooth. Therefore, also $\sum_{k=\frac{d-2}{2}}^{l-1} \widetilde{U}_p^k \log(\Gamma_p \pm i0) L_{\pm}^{\Omega}(2k+2, p)[\varphi]$ is smooth for all $l > \frac{d}{2}$ and the remaining term $\sum_{k=l}^{\infty} \widetilde{U}_p^k \log(\Gamma_p \pm i0) L_{\pm}^{\Omega}(2k+2, p)[\varphi]$ is C^l by Lemma 7.1.2. This holds for all l and hence, $p \mapsto f(p)[\varphi]$ is smooth. \square

Lemma 7.1.4 shows properties (iv) and (v) of $\widetilde{\mathcal{L}}_{\pm}$, so Proposition 5.4.1 is proved also for even d .

*"I was just guessing
numbers and figures
pulling your puzzle apart."*

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