On a Method for Solution of the Ordinary Differential Equations Connected with Huygens' Equations

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The problem of describing the linear second order hyperbolic equations, satisfying the Huygens' principle, is known as the Hadamards' problem [1], [2], [3]. Ch. Huygens [4] formulated this principle as a geometrical method for constructing the wave front, and it was used for explaining the main properties of light propagation. Its mathematical formulation within the theory of Cauchy problem for hyperbolic partial differential equations has been given by well-known French mathematician J. Hadamard in his 1923 Yale lectures [1]. It has received a good deal of attention and nowadays is a classical problem in mathematical physics, but is has turned out to be very difficult and, at present it is still far from its complete solution.

Let us consider the Cauchy problem for hyperbolic second order differential equations

$$\sum_{i,j=1}^{n} g^{ij}(x)u_{x^i x^j} + \sum_{i=1}^{n} b^i(x)u_{x^i} + c(x)u = 0, \tag{1}$$

$$u|_S = f, \quad \frac{\partial u}{\partial \nu}|_S = g. \tag{2}$$

Here S is a space-like manifold of the dimension $n-1$ and $\frac{\partial u}{\partial \nu}$ is a derivative with respect to the normal ν to S.

DEFINITION 1. *Equation (1) satisfies the Huygens' principle when the value of the solution of the Cauchy problem (1), (2) for any point $P = (x_0^1, \ldots, x_0^n)$ is determined by the Cauchy data given in the intersection of the initial manifold S and the characteristic conoid with vertex at P.*

It should be noted that the Huygens' principle is invariant relative to the following, so-called elemtary transformations:

(a) non-degenerate coordinant transformation;
(b) transformation of the unknown function $u \to \lambda u, \ \lambda(x) \neq 0$;
(c) multiplication of the equation by the function $\lambda(x) \neq 0$.

Two equations are called equivalent when one of them is derived from the other by means of elementary transformations. In our further considerations, these elementary transformations are necessarily taken into account.

Papers [5] and [6] contain descriptions of all the Huygens' equations of the type

$$L_n^{(k)}[u] \equiv u_{tt} - u_{xx} - \sum_{i=1}^{n-2} a^i(x-t)u_{y^i y^i} + a^k(x-t)b(y^k)u = 0, \tag{3}$$

$k = 1, \ldots, n-2$, where $n \geq 4$ is an even number, and $a^i > 0, \ i = 1, \ldots, n-2$.

We briefly describe these results assuming $k = 1$ for definiteness.

We denote

$$l_\mu = \frac{\partial}{\partial y^1} - \frac{\mu'(y^1)}{\mu(y^1)}, \quad l_\mu^* = -\frac{\partial}{\partial y^1} - \frac{\mu'(y^1)}{\mu(y^1)},$$

where $\mu(y^1)$ is an arbitrary solution of the equation[1]

$$\mu'' - b(y^1)\mu = 0. \tag{4}$$

[1]A.H. Hovhannisyan acknoweldges DAAD for financial support during his stay in Germany.

Then the $L_n^{(1)}$ may be written in the form

$$L_n^{(1)} = P + a^1(x-t)l_\mu^* l_\mu,$$

where

$$P = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \sum_{i=2}^{n-2} a^i(x-t)\frac{\partial^2}{\partial (y^i)^2}.$$

We define the operator

$$\tilde{L}_n^{(1)} = P + a^1(x-t)l_\mu l_\mu^* \tag{5}$$

and call it $l_\mu-$transform of the operator $L_n^{(1)}$. It is clear that the equality

$$l_\mu L_n^{(1)} = \tilde{L}_n^{(1)} l_\mu \tag{6}$$

takes place.

THEOREM 1. *If $L_n^{(1)}$ satisfies the Huygens' principle, then $\tilde{L}_{n+2}^{(1)}$ also satisfies the Huygens' principle.*
We denote the operator $L_n^{(1)}$ with $b(y^1) \equiv 0$ as L_n^0. From [2] it is known that the equation

$$L_n^0[u] = 0$$

satisfies the Huygens' princple for any even $n \geq 4$. Making use of the operator L_{04}, theorem 1 allows to construct Huygens' operators for $n \geq 6$ variables.

There arises a natural question: whether all the Huygens' operators of the type (3) may be obtained through the described construction basing on L_4^0. A positive answer to this question is given in [6].

THEOREM 2. *Let the operator $L_n^{(1)}$ satisfy the Huygens' princple. Then it may be derived from the operator L_{0n} through $l_\mu-$transforms applied no more than $\frac{n-4}{2}$ times.*
The aim of the present paper is to show how helpful the described algorithm for constructing the Huygens' equations is and to investigate the equation (4) in detals. Taking account of the equality (6) and equation (4), it follows from theorem 2 that there are operators

$$l_{\mu k} = \frac{\partial}{\partial y^1} - \frac{\mu_{\prime}^k}{\mu_k}, \quad k = 0, 1, \ldots, m-1, \; m \leq \frac{n-4}{2}$$

such that

$$l_{\mu_{m-1}} \cdots l_{\mu_0} L_{0n} = L_n^{(1)} l_{\mu_{m-1}} \cdots l_{\mu_0}, \tag{7}$$

where

$$\mu_0 = \alpha y^1 + \beta, \quad (\alpha, \beta = const),$$
$$(l_{\mu_k} l_{\mu_k}^*)\mu_{k+1} = 0, \quad k = 0, \ldots, m-2. \tag{8}$$

If m is the minimum number of transformations which transform the operator L_{0n} into $L_n^{(1)}$, then the functions μ_k $(k = 0, \ldots, m-1)$ is uniquely determined with the accuracy up to a constant factor [6].

The following reccurent relation is easily obtained from (8):

$$\mu_{k+1} = \frac{a_k}{\mu_k} + \frac{b_k}{\mu_k}\int \mu_k^2(y^1)dy^1.$$

However, it is not always convenient to use this formula because of the necessity to find the indefinite integral of μ_k^2. Therefor, further we study properties of the function $b(y^1)$ which allow as to determine conditions, easily checked and necessary for equation (3) to acquire Huygens' properties, and which may indicate us some other method to find functions μ_k.

First, we give the known results [7].

THEOREM 3. *We assume that the operator has Huygens' properties for even values* $n \geq N$, $m = \frac{N-4}{2}$ *and* $\{l_{\mu k}\}_{k=0}^{m-1}$ *is a unique sequence of operators determined by the formula (8), so that the equality (7) takes place. Then*

$$(a) \qquad b(y^1) = -2\frac{d^2}{d(y^1)^2}\left[ln \prod_{k=0}^{m-1} \mu_k(y^1)\right],$$

$$(b) \qquad \prod_{k=0}^{l} \mu_k(y^1) = P_l(y^1)$$

is a polynomial of degree $\frac{1}{2}(l+1)(l+2)$, $0 \leq l \geq m-1$.

Point (a) is proved by direct computations and point (b) through the following important result.

THEOREM 4. *If the sequance of functions* $\{\mu_k(y^1)\}_{k=0}^{m-1}$ *satisfies the conditions of (8), then each of* $\mu_k(y^1)$ *is a rational function.*

Using these results, we shall prove some properties of the function $b(y^1)$.

Let the polynomial $P_{m-1}(y^1)$ has the form

$$P_{m-1}(y^1) = (y^1 - a_1)^{k_1} \cdots (y^1 - a_s)^{k_s}, \quad (k_1 + \cdots + k_s = \frac{1}{2}m(m+1)).$$

Then from point (a) of theorem 3 we obtain:

$$b(y^1) = 2\frac{\sum\limits_{q=1}^{s} \prod\limits_{i \neq q} (y^1 - a_i)^2 k_q}{\prod\limits_{i=1}^{s} (y^1 - a_i)^2}.$$

Hence, singularities of the function $b(y^1)$ are poles of the second order and (4) is a Funchian equation.

We separate one of the poles, e.g. a_j, $1 \leq j \leq s$, and write the function $b(y^1)$ in the form

$$b(y^1) = \frac{b_j(y^1)}{(y^1 - a_j)^2,}$$

where

$$b_j(y^1) = 2k_j + 2\frac{\sum\limits_{q=1}^{s} \prod\limits_{i \neq q} (y^1 - a_i)^2 k_q}{\prod\limits_{i \neq j}^{s} (y^1 - a_i)^2}. \qquad (9)$$

It immediately follows from this formula that $b'(a_j) = 0$. Let $\tilde{L}_n^{(1)}$ be the $l_{\mu_m}-$transform of the operator $L_n^{(1)}$. Then μ_m is the solution of equation (4). According to theorem 4, μ_m is a rational fuction. We study its behavior in the neighborhood singular point a_j $1 \leq j \leq s$. With this purpose we write equation (4) as

$$\mu_m'' - \frac{b_j(y^1)}{(y^1 - a_j)^2} = 0.$$

Solution μ_m is found in the form of

$$\mu_m = (y^1 - a_j)^\rho \left[1 + \sum_{k=1}^{\infty} \mu_m^k (y^1 - a_j)^k \right]$$

where ρ is the root of the defining equation

$$\rho^2 - \rho - b_j(a_j) = 0. \tag{10}$$

Hence

$$\rho_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + b_j(a_j)}.$$

As μ_s is a rational function, then ρ_1 and ρ_2 must be integers

$$\frac{1}{2} \pm \frac{\sqrt{1 + 4b_j(a_j)}}{2} = q_j + 1, \quad q_j = 0, \pm 1, \dots.$$

Hence

$$b_j(a_j) = q_j(q_j + 1). \tag{11}$$

As the negative values q_j do not suggest new values for $b_j(a_j)$, it may be assumed that $q_j = 0, 1, \dots$.

We obtain from formula (9) that

$$b_j(a_j) = 2k_j.$$

Therefore, multiplicity of the root a_j of the polynomial P_{m-1} may assume the following values:

$$k_j = \frac{q_j(q_j + 1)}{2}, \quad j = 1, 2, \dots, s.$$

As the order of the polynomial P_{m-1} is $\frac{1}{2}m(m+1)$ and $m \leq \frac{n-4}{2}$ we obtain

$$\sum_{j=1}^{s} q_j(q_j + 1) \leq \frac{n-4}{2} \cdot \frac{n-2}{2}.$$

If $s = 1$, then

$$q \leq \frac{n-4}{2}$$

which is in agreement with the known result [2].

Roots of equation(10) are

$$\rho_1 = q_j + 1, \quad \rho_2 = -q_j.$$

Therefore, signularity of the rational function μ_m in the point a_j may have only the form $(y^1 - a_j)^{-q_j}$. Since μ_m is the solution of a linear equation, its singularities may appear only in the points where singularities of the equation coefficients are. Therefore the function μ_m may be found as

$$\mu_m = \frac{T_r(y^1)}{(y^1 - a_1)^{q_1} \cdots (y^1 - a_s)^{q_s}}, \tag{12}$$

where $T_r(y^1)$ is a polynomial of the order r. Acording to theorem 3 the polynomial

$$P_m = \prod_{k=0}^{m} \mu_k = P_{m-1}\mu_m$$

has the order $\frac{1}{2}(m-1)(m+2)$, and the order of P_{m-1} is $\frac{1}{2}m(m+1)$, accordingly.

Therefore,

$$r = m + 1 + \sum_{k=1}^{s} q_k.$$

Writing $T_r(y^1)$ with indefinite coefficients and substituting μ_m into the equation, it is possible to determine these coefficients, and hence, to find μ_m.

The facts, proved above, may be presented as the following result.

THEOREM 5. *Let equation (3) satisfy the Huygens' principle and let* $b(y^1) \neq 0$. *Then*

(a) singularities if the function $b(y^1)$ $a_1, \dots, a_s$ *are poles of the second order, being* $s \leq \frac{1}{2}m(m+1) \leq \frac{(n-2)(n-4)}{8}$.

(b) if the function $b(y^1)$ *is presented in the neighbourhood of a singular point* a_j $1 \leq j \leq s$ *as*

$$b(y^1) = \frac{b_j(y^1)}{(y^1 - a_j)^2},$$

then $b_j(a_j)$ *may assume only one of the following values*

$$b_j(a_j) = q_j(q_j+1), \quad q_j = 0, 1, \dots,$$

where

$$\sum_{j=1}^{s} q_j(q_j+1) \leq \frac{(n-4)(n-2)}{4}$$

and $b_j'(a_j) = 0$.

(c) solution of equation (4) has the form of (12).

Let us consider the Huygens' equation [5], [6]

$$L_8[u] \equiv u_{tt} - u_{xx} - \sum a^i(x-t)u_{y^iy^i} + a^1(x-t)\frac{6y^1[(y^1)^3+2]}{[(y^1)^3-1]^2}u = 0.$$

The function

$$b(y^1) = \frac{6y^1[(y^1)^3+2]}{[(y^1)^3-1]^2}$$

has the singularities in the points $a_1 = 1$, $a_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $a_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. For the $b_j(y^1)$, $i = 1, 2, 3$ we have

$$b_1(1) = 1 \cdot 2, \quad q_1 = 1,$$

$$b_1\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 \cdot 2, \quad q_2 = 1,$$

$$b_1\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 \cdot 2, \quad q_2 = 1.$$

Operator L_8 is obtained from the operator L_8^0

$$L_8^0 = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \sum a^i(x-t)\frac{\partial^2}{\partial y^{i2}} \tag{13}$$

with the operators l_{μ_0}, l_μ, where

$$\mu_0 = y^1$$

and μ_1 is a solution of the equation

$$\mu'' - \frac{2}{y^{i2}}\mu = 0$$

and has the form

$$\mu_1 = \frac{(y^1)^3 - 1}{y^1}.$$

Then

$$P_0 = \mu_0 = y^1,$$

$$P_1 = \mu_0\mu_1 = (y^1)^3 - 1.$$

The solution of equation

$$\mu'' - \frac{6y^1[(y^1)^3 + 2]}{[(y^1)^3 - 1]^2}\mu = 0 \tag{14}$$

according to the (12) can be find in the form

$$\mu_2 = \frac{T_2(y^1)}{(y^1)^3 - 1} = \frac{\sum_{i=1}^{6} \alpha_i (y^1)^i}{(y^1)^3 - 1}.$$

Substituding this function in (14) we obtain

$$30\alpha_6 - 36\alpha_6 + 6\alpha_6 = 0,$$

$$20\alpha_5 - 30\alpha_5 + 6\alpha_5 = 0,$$

$$12\alpha_4 - 24\alpha_4 + 6\alpha_4 = 0,$$

$$6\alpha_3 - 30\alpha_6 - 18\alpha_3 + 6\alpha_3 = 0,$$

$$2\alpha_2 - 20\alpha_5 - 12\alpha_2 + 6\alpha_2 = 0,$$

$$-12\alpha_4 - 6\alpha_1 + 6\alpha_1 = 0,$$

$$-6\alpha_3 + \alpha_0 = 0,$$

$$-2\alpha_2 = 0.$$

From here we have

$$\alpha_5 = \alpha_4 = \alpha_2 = 0,$$

$$\alpha_0 = \alpha_3 = -5\alpha_6$$

and α_1, α_6 are any numbers. If we take $\alpha_6 = 0$, $\alpha_1 = 1$, then

$$\mu_2 = \frac{y^1}{(y^1)^3 - 1}.$$

If $\alpha_6 = -1$, $\alpha_1 = 0$

$$\mu_2 = \frac{5 + 5(y^1)^3 - (y^1)^6}{(y^1)^3 - 1}.$$

Therefore, the common solution of the equation (14) has the form

$$\mu_2 = C_1 \frac{y^1}{(y^1)^3 - 1} + C_2 \frac{5 + 5(y^1)^3 - (y^1)^6}{(y^1)^3 - 1}. \tag{15}$$

For $C_1 = 1$, $C_2 = 0$ we have

$$P_2 = y^1, \quad b(y^1) = \frac{2}{(y^1)^2}$$

and we obtain the known Huygens' equation

$$u_{tt} - u_{xx} - \sum_{i=1}^{8} a^i(x-t)u_{y^iy^i} + a^1(x-t)\frac{2}{(y^1)^2}u = 0.$$

If $C_1 = 0$, $C_2 = 1$, then

$$P_2 = 5 + 5(y^1)^3 - (y^1)^6$$

$$u_{tt} - u_{xx} - \sum_{i=1}^{8} a^i(x-t)u_{y^iy^i} + a^1(x-t)\frac{2[6(y^1)^{10} + 225(y^1)^4 - 150y^1]}{[(y^1)^6 - 5(y^1)^3 - 5]^2}u = 0.$$

In the case $C_1 \neq 0$, $C_2 \neq 0$, it is follow that

$$P_2 = y^1 + \gamma[5 + 5(y^1)^3 - (y^1)^6]$$

and the following Huygens' equation

$$u_{tt} - u_{xx} - \sum_{i=1}^{8} a^i(x-t)u_{y^iy^i} + a^1(x-t)\frac{2 + 36\gamma(y^1)^5 + 6\gamma^2 y^1[2(y^1)^9 + 75(y^1)^3 - 50]}{\{y^1 + \gamma[5 + 5(y^1)^3 - (y^1)^6]\}^2}u = 0. \tag{16}$$

The solution of the equation (14) can be find in the form

$$\mu_2 = \frac{(y^1 - 1)^2 T_3(t^1)}{(y^1)^3 + y^1 + 1}$$

too, where

$$T_3(y^1) = \sum_{i=0}^{3} \alpha_i (y^1)^i.$$

Substituding this function in the equation () gives

$$\mu_2 = \frac{(y^1 - 1)^2[(y^1)^3 + 3(y^1)^2 + 6^1 + 5]}{(y^1)^2 + y^1 + 1}. \tag{17}$$

Using this solution it is possible to find the common solution of the equation (14). The function (17) can be find from (15) if we take $C_1 = 9$, $C_2 = -1$ $\left(\gamma = -\frac{1}{9}\right)$. In this case the equation (16) can be written in the form

$$u_{tt} - u_{xx} - \sum_{i=1}^{8} a^i(x-t)u_{y^iy^i} + a^1(x-t)\left[\frac{6}{(y^1 - 1)^2} + \frac{6\left[(y^1)^4 + 4(y^1)^3 - 6(y^1)^2 + 2y^1 + 2\right]}{\left[(y^1)^3 + 3(y^1)^2 + 6(y^1)^1 + 5\right]^2}\right]u = 0.$$

To constuct the new Huygens' equation, we have to know the factorization of the denominator of the function $b(y^1)$. However using the fact, that $P_3 = P_2\mu_3$ is the polynomial of degree 10, it is possible to assume, that the equation

$$\mu_3'' - \frac{2 + 36\gamma(y^1)^5 + 6\gamma^2 y^1[2(y^1)^9 + 75(y^1)^3 - 50]}{\{y^1 + \gamma[5 + 5(y^1)^3 - (y^1)^6]\}^2}\mu_3 = 0 \tag{18}$$

has the solution of the form

$$\mu_3 = \frac{T_{10}(y^1)}{y^1 + \gamma[5 + 5(y^1)^3 - (y^1)^6]}.$$

Easy to see that substitution of the function

$$\mu = \frac{T_r(y^1)}{Q_s(y^1)},$$

in the equation

$$\mu'' - \frac{R}{Q_s^2} = 0$$

lead to expresion

$$Q_s^2 T_r'' - 2Q_s Q_s' T_r' + [2(Q_s')^2 - Q_s Q_s'' - R]T_r = 0. \tag{19}$$

For μ_3 we have

$$T_r = T_{10} = \sum_{i=0}^{10} \alpha_i (y^1)^i,$$

$$Q_s = Q_6 = y^1 + \gamma[5 + 5(y^1)^3 - (y^1)^6],$$

$$R = 2 + 36\gamma(y^1)^5 + 6\gamma^2 y^1[2(y^1)^9 + 75(y^1)^3 - 50].$$

Substituding these functions in (19) we can find the common solution of the equation (18)

$$\mu_3 = C_1\mu_3^{(1)} + C_2\mu_3^{(2)},$$

where

$$\mu_3^{(1)} = \frac{3\gamma^2(y^1)^{10} - 45\gamma^2(y^1)^7 - 21\gamma(y^1)^5 - 105\gamma(y^1)^2 - 525\gamma^2 y^1 - 7}{y^1 + \gamma[5 + 5(y^1)^3 - (y^1)^6]}$$

and

$$\mu_3^{(2)} = \frac{(y^1)^3 - 1}{y^1 + \gamma[5 + 5(y^1)^3 - (y^1)^6]}.$$

Now let us consider the following Huygens' equation

$$L_8[u] \equiv u_{tt} - u_{xx} - \sum_{i=1}^{6} a^i(x-t)u_{y^iy^i} + \frac{6a^1(x-t)}{(y^1)^2}u = 0 \tag{20}$$

Operator L_8 can be obtained from L_8^0 (13) by the transformations with operators l_{μ_0} and l_{μ_0}, $\mu_0 = y^1$, $\mu_1 = (y^1)^2$. To construct the next Huygens' equation we have to solve

$$\mu_2'' - \frac{6}{(y^1)^2}\mu_2 = 0.$$

Then

$$\mu_2 = \frac{(y^1)^5 + \gamma}{(y^1)^2}$$

and

$$P_2 = \mu_0\mu_1\mu_2 = y^1[(y^1)^5 + \gamma].$$

The coresponding Huygens' equation has the form

$$u_{tt} - u_{xx} - \sum_{i=1}^{6} a^i(x-t)u_{y^iy^i} + a^1(x-t)\frac{12(y^1)^{10} - 36\gamma(y^1)^5 + 2\gamma^2}{(y^1)^2[(y^1)^5 + \gamma]^2}u = 0. \tag{21}$$

Note that in spite of the fact that in the equation (20) $q = 2$, nevertheless in (21) $q_i = 1$, $i = 1, \dots, 6$. Therefor the solution of the equation

$$\mu'' - \frac{12(y^1)^10 - 36\gamma(y^1)^5 + 2\gamma^2}{(y^1)^2[(y^1)^5 + \gamma]^2}\mu_3 = 0 \tag{22}$$

can be find in the form

$$\mu_3 = \frac{T_{10}}{y^1[(y^1)^5 + \gamma]}.$$

Substuding of this function in (22) gives

$$\mu_3^{(1)} = \frac{3(y^1)^{10} + 21\gamma(y^1)^5 - 7\gamma^2}{y^1[(y^1)^5 + \gamma]}$$

and the Huygens' equation

$$u_{tt} - u_{xx} - \sum_{i=1}^{10} a^i(x-t)u_{y^iy^i} + a^1(x-t)\frac{180(y^1)^{18} - 1260\gamma(y^1)^{13} + 8190\gamma^2(y^1)^8 - 5880\gamma^3(y^1)^3}{[3(y^1)^{10} + 21\gamma(y^1)^5 - 7\gamma^2]^2}u = 0.$$

The second linear independent solution is

$$\mu_3^{(2)} = \frac{(y^1)^2}{[(y^1)^5 + \gamma]}$$

and the corresponding Huygens' equation

$$u_{tt} - u_{xx} - \sum_{i=1}^{10} a^i(x-t)u_{y^iy^i} + a^1(x-t)\frac{6}{(y^1)^2}u = 0.$$

All these examples of the odinary differential equations of the form (4) conectes with Huygens' principle can be usefull for the investigating of the following problem.

Let us write the equation (4) with the coefficient (a) in theorem 3 in the form

$$\mu'' + \sum_{i=1}^{s} \frac{2k_i}{(y^1 - a_i)^2}\mu = 0, \tag{23}$$

where a_i $(i = 1, \dots, s)$ are the root of the polynomial $P_{m-1}(y^1)$ in formula (b) and k_i are there multiplicities. As it was shown before for an equations, conected with Huygens' equations $2k_i = q_i\ (q_i + 1)$ with some positive integers q_i. The equations of the form (23) are known as Fuchs equations [9]. The arrising problem is to discribe the a_i and k_i, $i = 1, \dots, s$ such that the equation (23) will be conected with Huygens' principle, in particular all the solutions of (23) are rational functions. Below we give a necessary and suficient condition on a_i and k_i, $i = 1, \dots, n$ for which the solutions of the equation (23) do not consist the logarithmic singularities.

In the neighborhood of a singular point a_r $(r = 1, \dots, s)$ an equation (23) can be written in the form

$$(y^1 - a_r)^2\mu'' - \Big[q_r(q_r + 1) + \sum_{\substack{k=1\\k\neq r}}^{s} q_k(q_k + 1)\frac{(y^1 - a_r)^2}{(y^1 - a_k)^2}\Big]\mu = 0.$$

If we write

$$\left(\frac{y^1 - a_r}{y^1 - a_k}\right)^2 = \sum_{n=2}^{\infty}(n-1)\frac{(y^1 - a_r)^n}{(a_k - a_r)^n}$$

and look for the solution in the form

$$\mu(y^1) = (y^1 - a_r)^q \sum_{m=0}^{\infty} \alpha_{mr}(y^1 - a_r)^m$$

we obtain

$$\sum_{m=0}^{\infty} \alpha_{mr}(m+q)(m+q-1)(y^1-a_r)^m - \Big[q_r(q_r+1) + \sum_{\substack{k=1\\k\neq r}}^{s} q_k(q_k+1)\sum_{n=2}^{\infty}\frac{(n-1)}{(a_k-a_r)^n}(y^1-a_r)^n\Big] +$$

$$+\sum_{m=0}^{\infty} \alpha(y^1-a_r)^m = 0. \tag{24}$$

For $m = 0$ we have

$$q_1 = -q_r, \quad q_r = q_r + 1$$

and α_{0r} an arbitrary number.

Let us consider $q = -q_r$, when the solution has the singularity. In this case the equation (24) has the form

$$\sum_{m=0}^{\infty} \alpha_{mr}(m-q_r)(m-q_r-1)(y^1-a_r)^m - \Big[q_r(q_r+1) + \sum_{\substack{k=1\\k\neq r}}^{s} q_k(q_k+1)\sum_{n=2}^{\infty}\frac{(n-1)(y^1-a_r)^n}{(a_k-a_r)^n}\Big]\sum_{m=0}^{\infty}\alpha_{mr}(y^1-a_r)^m = 0.$$

For $m = 1$ we obtain $\alpha_{1r} = 0$ and for $m = p \geq 2$

$$\alpha_{pr}[p(p-2q_r-1)] = \sum_{\substack{k=1\\k\neq r}}^{s} q_k(q_k+1)\sum_{l=0}^{p-2}\frac{(p-l-1)}{(a_k-a_r)^{p-l}}, \quad r = 1, \ldots, s. \tag{25}$$

From (25) we obtain

THEOREM 6. *The solutions of the equation (23) have not a logarithmic singularities if and only if*

$$\sum_{\substack{k=1\\k\neq r}}^{s} q_k(q_k+1)\sum_{l=0}^{2q_r+1}\alpha_{lr}\frac{(2q_r-l)}{(a_k-a_r)^{2q_r+1-l}} = 0. \tag{26}$$

If $s = 1$ the equation (23) is none other tahn Euler equation and has the common solution

$$\mu = C_1(y^1-a_1)^{q_1+1} + C_2(y^1-a_1)^{-q_1} \quad (q_1 > 0 \text{ is integer})$$

which does not contain the logarithmic singularity for each value of a_1.

For $s = 2$ it follows from (25) and (26), that the common solution of the equation (23) has the logarithmic singularities for any values of a_1 and a_2.

If $s > 2$, as is shown in for example (14), there are some distribution of the roots a_i, for which the common solution of (23) has no the logarithmic singularities.

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