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# An Iterative Approach to Operators on Manifolds with Singularities

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**von**
**Jamil Abed**

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# Introduction

Partial differential equations arise in various branches of mathematics, physics, and engineering in a natural way. They describe a large variety of different situations. Elliptic equations are an important subclass with applications in almost all areas of mathematics, from harmonic analysis to geometry, as well as in numerous fields of physics. The standard example of an elliptic equation is Laplace's equation, $\Delta u = 0$, its solutions describe the behaviour of electric, gravitational, and fluid potentials, and are therefore significant in many applications, especially in electromagnetism, astronomy, and fluid dynamics. An important method to express the solutions of elliptic partial differential equations is to extend the class of the operators to the so-called pseudo-differential operators. A basic reference is the work of Kohn and Nirenberg [15] where pseudo-differential operators have been established as a calculus, see also Hörmander [12], [11], Kumano-go [19], Shubin [46].

The analysis on manifolds with geometric singularities (such as conical points, edges, or corners) is motivated by models of the applied sciences, especially of mechanics, elasticity theory, particle physics, and astronomy, as well as by pure mathematics, such as geometry and topology. More information on the general role of the singular analysis for models in mechanics may be found in [9]. The singularities can arise either from the geometry of the underlying configuration or from the operator itself. For example, the standard Laplacian in polar coordinates takes the form of a singular operator, an example of a special class of differential operators, the so-called Fuchs type operators.
The "traditional" analysis is based on adequate algebras of pseudo-differential operators that contain geometric differential operators, e.g., Laplacians, associated with corresponding singular Riemannian metrics, together with the parametrices of elliptic elements. This paper is aimed at studying pseudo-differential operators on configurations with such singularities.

Our investigations are focused on new elements of the analysis on configurations with higher singularities, especially on problems appearing on infinite cones which require the development of pseudo-differential structures from the point of view of conical exits to infinity. The new difficulty in the case of higher singularities comes from singularities on cross sections of cones that generate non-compact edges going to infinity with the new corner axis variable. To illustrate the idea, let us first consider, for example, the Laplacian on a manifold with conical singularities (say, without boundary). In this case the ellipticity does not only refer to the "standard" principal homogeneous symbol but also to the so-called conormal symbol. The latter one,

contributed by the conical point, is operator-valued and singles out the weights in Sobolev spaces, where the operator has the Fredholm property.
Another example of ellipticity with different principal symbolic components is the case of boundary value problems. The boundary, say smooth, interpreted as an edge, contributes the operator-valued boundary (or edge) symbol which is responsible for the nature of boundary conditions (for instance, of Dirichlet or Neumann type in the case of the Laplacian). In general, if the configuration has polyhedral singularities of order $k$, we have to expect a principal symbolic hierarchy of length $k+1$, with components contributed by the various strata. In order to characterise the solvability of elliptic equations, especially, the regularity of solutions in suitable scales of spaces, it is natural to embed the problem in a pseudo-differential calculus, and to construct a parametrix. For higher singularities this is a program of tremendous complexity. It is therefore advisable to organise the general elements of the calculus by means of an axiomatic framework which contains the typical features, such as the cone- or edge-degenerate behaviour of symbols but ignores the (in general) huge tail of $k-1$ iterative steps to reach the singularity level $k$.

At present the analysis of PDEs on manifolds (or, more generally, stratified spaces) with regular singularities is an important research field with many open problems and new challenges. Moreover, there are traditional aspects with a long history, motivated by applications to models in physics and other sciences. Let us give some references on crucial results and recent development of the calculus.
The "concrete" (pseudo-differential) calculus of operators on manifolds with conical or edge singularities may be found in several papers and monographs, see, for instance, [32], [36], [35], [5]. Operators on manifolds of singularity order 2 are studied in [37], [41], [20], [7]. Theories of that kind are also possible for boundary value problems with the transmission property at the (smooth part of the) boundary, see, for instance, [31], [14], [9]. This is useful in numerous applications, for instance, to models of elasticity or crack theory, see [14], [10], [8]. Elements of operator structures on manifolds with higher singularities are developed, for instance, in [40], [1]. The nature of such theories depends very much on specific assumptions on the degeneracy of the involved symbols. There are worldwide different schools studying operators on singular manifolds, partly motivated by problems of geometry, index theory, and topology, see, for instance, Melrose [21], Melrose and Piazza [22], Nistor [27], Nazaikinskij, Savin, Sternin [23], [24], [25], and many others. We do not study here operators of "multi-Fuchs" type, often associated with the notation "corner manifolds". Our operators are of a rather different behaviour with respect to the degeneracy of symbols. Nevertheless the various theories have intersections and common sources, see the paper of Kondratyev [16] or papers and monographs of other representatives of a corresponding Russian school, see, for instance, [29], [30].

## Tools and technical background

Among the tools used here to investigate elliptic partial differential equations are pseudo-differential operators. They have the important property that they form an algebra of operators, which contains the differential operators. In the classical framework they are established as a well-developed theory. To each pseudo-differential op-

erator one associates a symbolic structure; a chosen mapping in opposite direction from symbols to operators is often called quantisation. This is a general idea, which means that the quantisation process does not only make sense for elliptic operators, but also for parabolic and hyperbolic ones. Symbols are much easier objects than operators, and it may be very efficient to reduce questions on the nature of pseudo-differential operators or on the solvability of equations to the level of symbols. For example, when $X$ is a smooth compact manifold, the standard notion of ellipticity requires the invertibility of the associated homogeneous principal symbol $\sigma_\psi(\cdot)$ which is defined on the cotangent bundle $T^*X \setminus 0$.
The calculus of pseudo-differential operators is motivated by the task of expressing parametrices of elliptic partial differential equations in terms of classical pseudo-differential operators that invert the elliptic partial differential operators up to integral operators with smooth kernels. Based on such considerations, Schulze founded and developed pseudo-differential theories for degenerate elliptic operators, where the degeneracy reflects in a natural way the presence of singularities on the underlying configuration. Recall that there are several pseudo-differential scenarios, adapted to specific degenerate operators, and there are, of course, interactions between different approaches. The desired algebras and the notion of ellipticity is not only governed by the symbol coming from the interior part of the configuration, but every singularity has its own precise contribution represented by a symbol which is now operator-valued. In this new situation, ellipticity means the invertibility of all symbolic components. This entails the existence of parametrices within each algebra and also the Fredholm property in appropriate scales of weighted Sobolev spaces.

## Singular operator calculus

Before we give an overview of the main content of the work let us here recall some elements of the singular operator calculus.
Let $M$ be a manifold with a conical singularity $v \in M$, i.e., $M \setminus \{v\}$ is smooth, and $M$ is close to $v$ modelled on a cone $X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ with base $X$, where $X$ is a closed compact $C^\infty$ manifold. We then have differential operators of order $\mu \in \mathbb{N}$ on $M \setminus \{v\}$, locally near $v$ in the splitting of variables $(r, x) \in \mathbb{R}_+ \times X$ of the form

$$A := r^{-\mu} \sum_{j=0}^{\mu} a_j(r) \left( -r \frac{\partial}{\partial r} \right)^j \tag{0.0.1}$$

with coefficients $a_j \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}(X))$ (here $\mathrm{Diff}^\nu(\cdot)$ denotes the space of all differential operators of order $\nu$ on the manifold in parentheses, with smooth coefficients). Observe that when we consider a Riemannian metric on $\mathbb{R}_+ \times X := X^\wedge$ of the form $dr^2 + r^2 g_X$, where $g_X$ is a Riemannian metric on $X$, then the associated Laplace-Beltrami operator is just of the form (0.0.1) for $\mu = 2$. For such operators we have the homogeneous principal symbol $\sigma_\psi(A) \in C^\infty\big(T^*(M \setminus \{v\}) \setminus 0\big)$, and locally near $v$ in the variables $(r, x)$ with covariables $(\rho, \xi)$ the reduced symbol

$$\tilde{\sigma}_\psi(A)(r, x, \rho, \xi) := r^\mu \sigma_\psi(A)(r, x, r^{-1}\rho, \xi)$$

which is smooth up to $r = 0$. If a symbol (or an operator function) contains $r$ and $\rho$ in the combination $r\rho$ we speak of degeneracy of Fuchs type. The ellipticity condition with respect to $\sigma_\psi$ is the usual one for the homogeneous principal symbol on the main stratum of the configuration, plus an extra requirement for the reduced symbol, namely,

$$\tilde{\sigma}_\psi(A)(r, x, \rho, \xi) \neq 0 \text{ for } (\rho, \xi) \neq 0.$$

We will then shortly speak about $\sigma_\psi$-ellipticity.
It is interesting to ask the nature of an operator algebra that contains Fuchs type differential operators of the from (0.0.1) on $X^\Delta$, together with the parametrices of elliptic elements. An analogous problem is meaningful on $M$. Answers may be found in [36], including the tools of the resulting so-called cone algebra. As noted above the ellipticity close to the tip $r = 0$ is connected with a second symbolic structure, namely, the conormal symbol

$$\sigma_c(A)(w) := \sum_{j=0}^{\mu} a_j(0) w^j : H^s(X) \to H^{s-\mu}(X) \tag{0.0.2}$$

which is a family of operators, depending on $w \in \Gamma_{\frac{n+1}{2}-\gamma}$, $\Gamma_\beta := \{w \in \mathbb{C} : \operatorname{Re} w = \beta\}$, $n = \dim X$. Here $H^s(X)$ is the standard Sobolev spaces of smoothness $s \in \mathbb{R}$ on $X$. Ellipticity of $A$ with respect to a weight $\gamma \in \mathbb{R}$ means that (0.0.2) is a family of isomorphisms for all $w \in \Gamma_{\frac{n+1}{2}-\gamma}$. The bijectivity of (0.0.2) is a condition on a kind of non-linear eigenvalues of a (in general) meromorphic operator function in the complex plane.
On the infinite cone $X^\Delta$ the ellipticity refers to a further principal symbolic structure, to be observed when $r \to \infty$. The behaviour in that respect is not symmetric under the substitution $r \to r^{-1}$. The present axiomatic approach will refer to "abstract" corners represented by $r \to 0$. The considerations are based on specific insight on families of reductions of orders in given scales of spaces (in the simplest case $H^s(X)$, $s \in \mathbb{R}$, when the corner is a conical singularity). In order to motivate our general constructions we briefly recall the form of corner operators of second generation.

First, a differential operator on an open stretched wedge $\mathbb{R}_+ \times X \times \Omega \ni (r, x, y)$, $\Omega \subseteq \mathbb{R}^q$ open, is called edge-degenerate, if it has the form

$$A = r^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j\alpha}(r, y) \left(-r\frac{\partial}{\partial r}\right)^j (rD_y)^\alpha, \tag{0.0.3}$$

$a_{j\alpha} \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \operatorname{Diff}^{\mu-(j+|\alpha|)}(X))$. Observe that (0.0.3) can be written in the form $A = r^{-\mu}\operatorname{Op}_{r,y}(p)$ for an operator-valued symbol $p$ of the form $p(r, y, \rho, \eta) = \tilde{p}(r, y, r\rho, r\eta)$, $\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}))$, and

$$\operatorname{Op}_{r,y}(p)u(r, y) = \iint e^{i(r-r')\rho + i(y-y')\eta} p(r, y, \rho, \eta) u(r', y') dr' dy' đ\rho đ\eta.$$

Here $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^l_\lambda)$ denotes the space of classical parameter-dependent pseudo-differential operators on $X$ of order $\mu$, with parameter $\lambda \in \mathbb{R}^l$, that is, locally on $X$ the operators are given in terms of amplitude functions $a(x, \xi, \lambda)$, where $(\xi, \lambda)$ is treated as an $(n + l)$-dimensional covariable, and we have $L^{-\infty}(X; \mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l, L^{-\infty}(X))$ with $L^{-\infty}(X)$ being the (Fréchet) space of smoothing operators on $X$.

Analogously as in the conical case the homogeneous principal symbol $\sigma_\psi(A)$ gives rise to a reduced symbol $\tilde{\sigma}_\psi(A)$ close to $r=0$ which is again smooth up to $r=0$. Then we define $\sigma_\psi$-ellipticity in a similar manner as in the case of conical singularities. If we assume $\sigma_\psi$-ellipticity of the operator (0.0.3) then the values of the principal edge symbol

$$\sigma_\wedge(A)(y,\eta) := r^{-\mu} \sum_{j+|\alpha|\leq\mu} a_{j\alpha}(0,y)\left(-r\frac{\partial}{\partial r}\right)^j (r\eta)^\alpha, \tag{0.0.4}$$

$(y,\eta) \in T^*Y \setminus 0$, consist of Fredholm operators

$$\sigma_\wedge(A)(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge), \tag{0.0.5}$$

however, only when their subordinate conormal symbol

$$\sigma_c\sigma_\wedge(A)(y,z) = \sum_{j=0}^{\mu} a_{j0}(0,y)z^j : H^s(X) \to H^{s-\mu}(X) \tag{0.0.6}$$

is a bijective family for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$ (the spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ here are explained in Subsection 3.1.3). Clearly, a reasonable concept of ellipticity requires the bijectivity of (0.0.5), not only the Fredholm property, since parametrices should be associated with the tuple of inverse symbols. At this moment we see that, similarly as in the calculus of boundary value problem, where ellipticity is connected with additional elliptic boundary conditions, we need here elliptic edge conditions, characterised on the level of symbols by additional entries of a family of block matrices

$$\sigma_\wedge(\mathcal{A})(y,\eta) := \begin{pmatrix} \sigma_\wedge(A) & \sigma_\wedge(K) \\ \sigma_\wedge(T) & \sigma_\wedge(Q) \end{pmatrix}(y,\eta) : \begin{matrix} \mathcal{K}^{s,\gamma}(X^\wedge) \\ \oplus \\ \mathbb{C}^{j_-} \end{matrix} \longrightarrow \begin{matrix} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \\ \oplus \\ \mathbb{C}^{j_+} \end{matrix}$$

which fill up the upper left corner $\sigma_\wedge(A)$ to a family of isomorphisms by suitable finite-rank entries $\sigma_\wedge(T)$, $\sigma_\wedge(K)$ and $\sigma_\wedge(Q)$, respectively. For the ellipticity, a first essential question is what can be really said about the additional edge conditions, especially about the dimensions $j_-, j_+$. By virtue of

$$j_+ - j_- = \operatorname{ind} \sigma_\wedge(A)(y,\eta)$$

we have to answer an index question, and we even need more, namely the kernels and cokernels of $\sigma_\wedge(A)(y,\eta)$ including their dimensions.

Now let $\mathrm{Diff}^\mu_{\deg}(M)$, for a manifold $M$ with edge $Y$, denote the space of all differential operators on $M \setminus Y$ of order $\mu$ that are locally near $Y$ in the splitting of variables $(r,x,y) \in \mathbb{R}_+ \times X \times \Omega$ of the form (0.0.3). If we replace in the definition the edge covariable $\eta$ by $(\eta,\lambda) \in \mathbb{R}^{q+l}$ ($q = \dim Y$) we obtain parameter-dependent families of operators in $\mathrm{Diff}^\mu_{\deg}(M)$. Similarly as (0.0.1) an operator of the form

$$A := t^{-\mu} \sum_{j=0}^{\mu} a_j(t)\left(-t\frac{\partial}{\partial t}\right)^j$$

is called corner-degenerate if $a_j \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}_{\deg}(M))$, $j = 0,1,\ldots,\mu$. The corner conormal symbol $\sigma_c(A)(z) = \sum_{j=0}^{\mu} a_j(0)z^j$, $z \in \Gamma_{\frac{\dim M+1}{2}-\delta}$ for a corner weight $\delta \in$

$\mathbb{R}$, is just a parameter-dependent family in $\mathrm{Diff}^{\mu}_{\mathrm{deg}}(M)$ with parameter $\mathrm{Im} z$ on the indicated weight line. The program to study such operators close to the tip $t \to 0$ (see [1], [7]) is just a concrete realisation of the present theory.

## Structure of the work and main results

This thesis is divided into three chapters. Chapter 1 is devoted to necessary elements of the analysis of pseudo-differential operators. In the first section we establish basics on the calculus of pseudo-differential operators on open smooth manifolds, with symbolic structures, distributional kernels, Sobolev spaces, operator algebra aspects, and ellipticity. Here we recall the calculus in the scalar case, which can be generalised to the operator valued case. Then we pass to material on operators on manifolds with conical singularities, with tools such as Mellin transform, Mellin quantisation, weighted spaces, and cone algebra.
In the second section we investigate a class of operators defined on manifolds with conical exits to infinity. The manifold $M$ with conical exits to infinity is here defined in terms of free $\mathbb{R}_+$-actions, where the orbit space $X$ is a smooth manifold and represents the base of the cone. In the general case it is not necessary that $X$ is connected. So the manifold $M$ may have several exits to infinity, according to the finite number of connected components of $X$. First we consider pseudo-differential operators defined in the Euclidean space, as we see that the latter is an example of a manifold with exits to infinity. Then we study the effect of changing the coordinates of the open conical set, on which the operator is defined. In this respect we partly follow the approach of Schrohe [33]. Finally we define the desired operators on $M$. Here we also address some important results concerning the existence of a parametrix for elliptic operators and the elliptic regularity.

In Chapter 2 we study operators on (infinite) cylindrical manifolds $X^{\asymp} \cong \mathbb{R} \times X$, with a closed compact manifold $X$ as the base, from an alternative point of view. First we consider families of parameter-dependent operator functions on $X^{\asymp}$, with special degeneracy in the parameter and show that the push forward of the associated operators from cylinders to cones forms a class of operators in the exit calculus on infinite cones. Motivated by this, we develop a new calculus of pseudo-differential operators on cylindrical manifolds with conical exits to infinity. The operator-valued symbols studied here have two orders, one along the axial variable $r$ and the other on the inner base of the cone $X$. To start with we consider the operators on smooth functions with compact support. Then, after characterising the smoothing elements in the calculus, we prove the continuity between Schwartz spaces and extend them to continuous operators between tempered distribution spaces. It seems in a way astonishing that the smoothing operators within this calculus depend only on the "inner" order, so they exist for any order along the axial variable. Actually, because of the degeneracy of the parameters one can compensate the order along $r$. The bigger the $r$-order is, the more times one needs to differentiate in order to make estimates as in (2.2.34) possible. Finally we define a scale of Sobolev spaces on cylindrical spaces, based on $L^2$-norms, such that our new operators are continuous between them and the elliptic ones are even isomorphisms when the parameter is large enough. It turned out that this new calculus is a step for the iterative calculus which is the intention of this approach. The technique partly refers to Kumano-go's formalism on oscillatory integrals which is in its classical form very effective. In our case we are faced with a new (very substantial) difficulty, namely, the degenerate behaviour of amplitude functions near infinity. One of the main issues it to overcome this difficulty. We do this in combination with interpreting the amplitude functions as operator families globally acting on $X$, the cross section of the cylinder. This is also a very essential element to make the iterative ideas of the corner theory work.

In Chapter 3 we study operators with certain degenerate operator-valued amplitude functions, motivated by the iterative calculus of pseudo-differential operators on manifolds with higher singularities. In the first section we introduce spaces of symbols based on families of reductions of orders in given scales of (analogues of Sobolev) spaces, followed by a standard example from the parameter-dependent cone calculus. Here, in contrast to [41], [42], we develop the aspect of symbols, based on "abstract" reductions of orders which makes the approach transparent from a new point of view. A typical feature here are certain new estimates of norms in weighted spaces on the base with respect to a growing parameter. In a similar case those were also observed in Chapter 2. The second section is devoted to the specific effects of an axiomatic calculus near the tip of the corner. The full calculus involves two separate theories, one near the tip of the corner and the other at the conical exit to infinity. The corner axis is represented by a real axis $\mathbb{R} \ni r$, and the operators take values in vector-valued analogues of Sobolev spaces in $r$. The solutions are expected to have asymptotics near the tip of the corner determined by the non-bijectivity points of the Mellin symbol. In this context, we prove the existence of a parametrix and show a continuity statement between weighted spaces with asymptotics. Worth mentioning that here we only consider scales of Hilbert spaces having the compact embedding property, although one can sometimes drop this condition by imposing that the smoothing operators within the calculus should be compact operators.
The theory presented in this chapter holds as an iterative calculus where the example considered in Subsection 3.1.3 obeys it. In other words, if the manifold $X$ has conical singularities then the axiomatic calculus holds true with suitable scales of spaces and reductions of order. However, when $X$ has edges or higher singularities, it seems that this approach is not so convenient for some aspects of the iterative calculus. There is therefore another approach in preparation by my supervisor Schulze that is more adapted to the case when $X$ has higher singularities. The structures studied in the second section of Chapter 2 are motivated by the iterative calculus in that way.

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# Chapter 1

# The pseudo-differential cone calculus

## 1.1 Basics in pseudo-differential operators

### 1.1.1 Spaces of symbols

**Definition 1.1.1.** (i) *The space of symbols $S^\mu(U \times \mathbb{R}^n)$ for an open set $U \subseteq \mathbb{R}^m$ and an order $\mu \in \mathbb{R}$ is defined to be the set of all $a(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$ such that*

$$\sup_{x \in K, \xi \in \mathbb{R}^n} \langle \xi \rangle^{-\mu+|\beta|} |D_x^\alpha D_\xi^\beta a(x,\xi)| \tag{1.1.1}$$

*is finite for all multi-indices $\alpha \in \mathbb{N}^m$, $\beta \in \mathbb{N}^n$, and compact $K \subset U$. Here $D_x^\alpha := \left(-i\frac{\partial}{\partial_{x_1}}\right)^{\alpha_1} \cdot \ldots \cdot \left(-i\frac{\partial}{\partial_{x_m}}\right)^{\alpha_m}$, $i := \sqrt{-1}$ and $\langle \xi \rangle := (1+|\xi|^2)^{\frac{1}{2}}$.*

(ii) *Let $S^{(\mu)}\big(U \times (\mathbb{R}^n \setminus \{0\})\big)$ denote the space of all $f(x,\xi) \in C^\infty\big(U \times (\mathbb{R}^n \setminus \{0\})\big)$ that are positive homogeneous in $\xi$ of degree $\mu$, i.e.,*

$$f(x, \lambda\xi) = \lambda^\mu f(x,\xi)$$

*for all $\lambda \in \mathbb{R}_+$ and all $(x,\xi) \in U \times (\mathbb{R}^n \setminus \{0\})$.*

(iii) *The space $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^n)$ of so-called classical symbols is defined as the subspace of all $a(x,\xi) \in S^\mu(U \times \mathbb{R}^n)$ for which there is a sequence of homogeneous components $a_{(\mu-j)}(x,\xi) \in S^{(\mu-j)}\big(U \times (\mathbb{R}^n \setminus \{0\})\big)$, $j \in \mathbb{N}$, such that for every excision function $\chi(\xi)$ (i.e., $\chi \in C^\infty(\mathbb{R}^n)$, $\chi(\xi) = 0$ for $|\xi| < c_0$, $\chi(\xi) = 1$ for $|\xi| > c_1$ for certain $0 < c_0 < c_1$) we have*

$$\mathrm{ord}\left(a(x,\xi) - \sum_{j=0}^{N} \chi(\xi) a_{(\mu-j)}(x,\xi)\right) \to -\infty$$

*as $N \to \infty$.*

We call $\sigma_\psi^\mu(x,\xi) := a_{(\mu)}(x,\xi)$ the homogeneous principal symbol of $a$ of order $\mu$. In view of $a_{(\mu)}(x,\xi) = \lim_{\lambda\to\infty} \lambda^{-\mu} a(x,\lambda\xi)$ we can recover $\sigma_\psi^\mu$ in a unique way. Applying the same procedure to $a_1(x,\xi) := a(x,\xi) - \chi(\xi)a_{(\mu)}(x,\xi) \in S^{\mu-1}(U\times\mathbb{R}^n)$ for the order $\mu-1$ we obtain $a_{(\mu-1)}(x,\xi)$. Thus we obtain the unique homogeneous components $a_{(\mu-j)}(x,\xi)$, $j\in N$. Here we refer to the standard fact that

$$\chi S^{(\mu)}\big(U\times(\mathbb{R}^n\setminus\{0\})\big) \subset S^\mu(U\times\mathbb{R}^n)$$

for every excision function $\chi$ and any $\mu\in\mathbb{R}$. If a relation or an assertion is valid for general and classical symbols we write "(cl)" as subscript.
Note that estimates with respect to $\langle\xi\rangle$ control growth properties for $|\xi|\to\infty$. Later on we also employ some function $\xi\to[\xi]$, $[\xi] > c_0 > 0$ and $[\xi] = |\xi|$ for $|\xi|\geq c_1$ for some fixed $c_1>0$. The specific choice will be unimportant. Therefore, if we say nothing else, from now on we assume $c_1 = 1$.

**Remark 1.1.2.** *All notions and constructions concerning symbols on $U\times\mathbb{R}^n$ can be generalised to symbols given in an open conic subset $\Gamma\subset U\times\mathbb{R}^n$. For instance, $a\in S^\mu(\Gamma)$ means that $a(x,\xi)\in C^\infty(\Gamma)$, and the condition* (1.1.1) *is replaced by*

$$\sup\langle\xi\rangle^{-\mu+|\beta|}|D_x^\alpha D_\xi^\beta a(x,\xi)| < \infty$$

*for all $(x,\xi)\in\Gamma$, $|\xi|\geq 1$, $(x,\frac{\xi}{|\xi|})$ varying over a compact subset of $U\times S^{n-1}$.*

**Remark 1.1.3.** *The spaces $S^\mu(U\times\mathbb{R}^n)$, $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^n)$ are Fréchet spaces in a natural way. The expression* (1.1.1) *for $\alpha\in\mathbb{N}^m,\beta\in\mathbb{N}^n, K\subset U$ compact forms a semi-norm system for the space $S^\mu(U\times\mathbb{R}^n)$. Moreover, on $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^n)$ we have a system of linear operators*

$$\eta_j : S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^n) \longrightarrow S^{(\mu-j)}\big(U\times(\mathbb{R}^n\setminus\{0\})\big),\ j\in\mathbb{N}, \tag{1.1.2}$$

*that determine the* (*unique*)*components of $a(x,\xi)$ of homogeneities $\mu-j$, for every $j\in\mathbb{N}$, and*

$$\rho_N : S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^n) \longrightarrow S^{\mu-(N+1)}(U\times\mathbb{R}^n),\ N\in\mathbb{N} \tag{1.1.3}$$

*where $(\rho_N a)(x,\xi) := a(x,\xi) - \sum_{j=0}^N \chi(\xi)\eta_j(a)(x,\xi)$. We then endow $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^n)$ with the topology of the projective limit with respect to the mappings* (1.1.2)*, $j\in\mathbb{N}$ and* (1.1.3)*, $N\in\mathbb{N}$.*

For future references let us recall here the definitions of the projective and inductive limit topologies. Let $E$ be a vector space and $\{E_\alpha,\alpha\in I\}$ be a family of Fréchet spaces.

- Suppose that we are given, for each index $\alpha\in I$, a linear map $p_\alpha : E\to E_\alpha$. On $E$ we define the weakest locally convex topology such that all the mappings $p_\alpha$ are continuous. Equipped with this topology, $E$ is a Fréchet space, we call it the projective limit of the spaces $E_\alpha$ with respect to the mappings $p_\alpha$, denoted by $\varprojlim_{\alpha\in I} E_\alpha$. A basis of neighbourhoods of zero in this topology is defined as follows: in each $E_\alpha$, we consider a basis of neighbourhoods of zero $U_{\alpha,\beta}(\beta\in J_\alpha)$; let $V_{\alpha,\beta}$ be the preimage of $U_{\alpha,\beta}$ under $p_\alpha$; then, all the finite intersections of sets $V_{\alpha,\beta}$, when $\alpha$ and $\beta$ vary in all possible ways, form a basis of neighbourhoods of zero in the projective topology on $E$.

- Suppose now that we are given, for each index $\alpha \in I$, a linear map $q_\alpha : E_\alpha \to E$, such that $E = \bigcup_\alpha q_\alpha(E_\alpha)$. We may then define on $E$ the strongest locally convex topology such that all the mappings $q_\alpha$ are continuous. A convex subset $U$ of $E$ is a neighbourhood of zero in this topology if, for every $\alpha$, $U \cap q_\alpha(E_\alpha)$ is of the form $q_\alpha(U_\alpha)$, where $U_\alpha$ is a neighbourhood of zero in $E_\alpha$. We call $E$ with this topology the inductive limit of the spaces $E_\alpha$ with respect to the mappings $q_\alpha$ and we denote it by $\varinjlim_{\alpha \in I} E_\alpha$.

Let us now fix some convenient notation. The subspace $S^\mu_{\mathrm{cl}}(\mathbb{R}^n)$ of all $x$-independent symbols $a(\xi)$ is closed in $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^n)$, and we have

$$S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^n) = C^\infty\big(U, S^\mu_{\mathrm{cl}}(\mathbb{R}^n)\big).$$

We now recall the following well known theorem.

**Theorem 1.1.4.** *Let $a_j(x,\xi) \in S^{\mu_j}_{(\mathrm{cl})}(U \times \mathbb{R}^n)$, $j \in \mathbb{N}$, with $\mu_j \to -\infty$ for $j \to \infty$ (and $\mu_j := \mu - j$ in the classical case for some $\mu \in \mathbb{R}$). Then there exists an $a(x,\xi) \in S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^n)$, $\mu = \max\{\mu_j : j \in \mathbb{N}\}$, such that* $\mathrm{ord}\big(a(x,\xi) - \sum_{j=0}^N a_j(x,\xi)\big) \longrightarrow -\infty$ *as $N \to \infty$, and $a(x,\xi)$ is unique modulo $S^{-\infty}(U \times \mathbb{R}^n)$.*

An explicit proof may be found, for instance, in [39, Section 1.1.2]. As usual $a(x,\xi)$ is called an asymptotic sum of the symbols $a_j(x,\xi)$, $j \in \mathbb{N}$, written

$$a(x,\xi) \sim \sum_{j=0}^\infty a_j(x,\xi).$$

We can construct $a(x,\xi)$ as a convergent series

$$a(x,\xi) = \sum_{j=0}^\infty \chi\left(\frac{\xi}{c_j}\right) a_j(x,\xi)$$

in $S^\mu(U \times \mathbb{R}^n)$, with an excision function $\chi$ and constants $c_j > 0$ tending to $\infty$ sufficiently fast as $j \to \infty$, where for every $M > 0$ there exists an $N = N(M) \in \mathbb{N}$ such that $\sum_{j=N+1}^\infty \chi(\xi/c_j) a_j(x,\xi)$ converges in $S^{\mu-M}(U \times \mathbb{R}^n)$.

**Example 1.1.5.** *Given $a(x,\xi) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$, $b(x,\xi) \in S^\nu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ for open $\Omega \subseteq \mathbb{R}^n$ we have*

$$\big(\partial_\xi^\alpha a(x,\xi)\big) D_x^\alpha b(x,\xi) \in S^{\mu+\nu-|\alpha|}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$$

*for every $\alpha \in \mathbb{N}^n$. The asymptotic sum*

$$(a\#b)(x,\xi) := \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \big(\partial_\xi^\alpha a(x,\xi)\big) D_x^\alpha b(x,\xi) \tag{1.1.4}$$

*is called the Leibniz product of the symbols $a(x,\xi)$ and $b(x,\xi)$. In order to fix notation by the Leibniz product we understand a choice of an element $(a\#b)(x,\xi) \in S^{\mu+\nu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ such that $(a\#b)(x,\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\big(\partial_\xi^\alpha a(x,\xi)\big) D_x^\alpha b(x,\xi)$.*

**Remark 1.1.6.** *The Leibniz product defines an associative multiplication between symbols (always modulo symbols of order $-\infty$), i.e.,*

$$a(x,\xi)\#\big(b(x,\xi)\#c(x,\xi)\big) = \big(a(x,\xi)\#b(x,\xi)\big)\#c(x,\xi).$$

*In particular, parentheses may be omitted and for $j \in \mathbb{N}$ we may set*

$$a^{\# j} := \underbrace{a(x,\xi)\#a(x,\xi)\#\cdots\#a(x,\xi)}_{\text{j factors}}$$

### 1.1.2 Pseudo-differential operators and distributional kernels

With symbols as in Definition 1.1.1 we associate operators as follows. For $U := \Omega\times\Omega$ for an open set $\Omega \subseteq \mathbb{R}^n$ we will mainly write $(x,x') \in \Omega\times\Omega$ instead of $x$.

**Definition 1.1.7.** (i) *For $a(x,x',\xi) \in S^\mu(\Omega\times\Omega\times\mathbb{R}^n)$ we set*

$$\mathrm{Op}_x(a)u(x) := \iint e^{i(x-x')\xi}a(x,x',\xi)u(x')dx'đ\xi$$

*for $đ\xi := (2\pi)^{-n}d\xi$, $u \in C_0^\infty(\Omega)$;*

(ii) *we set*

$$L^\mu_{(\mathrm{cl})}(\Omega) := \Big\{\mathrm{Op}_x(a) : a(x,x',\xi) \in S^\mu_{(\mathrm{cl})}(\Omega\times\Omega\times\mathbb{R}^n)\Big\}.$$

*The elements of $L^\mu_{(\mathrm{cl})}(\Omega)$ are called (classical) pseudo-differential operators in $\Omega$ of order $\mu$. Sometimes we omit the subscript $x$ and simply write* $\mathrm{Op}(a)$ *if there is no ambiguity.*

Let us recall a few well known properties of pseudo-differential operators. First of all every $A \in L^\mu(\Omega)$ defines a continuous operator

$$A : C_0^\infty(\Omega) \to C^\infty(\Omega).$$

Let $K_A \in \mathcal{D}'(\Omega\times\Omega)$ denote the distributional kernel of $A$. We then have $\operatorname{sing\,supp} K_A \subseteq \operatorname{diag}(\Omega\times\Omega)$. We obtain $K_A$ in a unique way by forming

$$\langle K_A, u\otimes v\rangle = \int (Au)(x)v(x)dx$$

for $u,v \in C_0^\infty(\Omega)$. Another representation of $K_A$ is

$$K_A(x,x') = \int e^{i(x-x')\xi}a(x,x',\xi)đ\xi.$$

In particular, taking a symbol $a(\xi) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^n)$ (with constant coefficients) and defining

$$k(a)(\zeta) := \int e^{i\zeta\xi}a(\xi)đ\xi = \left(\mathcal{F}^{-1}_{\xi\to\zeta}a\right)(\zeta),$$

$\mathcal{F}^{-1}$ is the inverse Fourier transform, we have

$$K_A(x, x') = k(a)(x - x').$$

The operator $A$ is called properly supported if for arbitrary compact $K \subset \Omega$ the sets $(K \times \Omega) \cap \operatorname{supp} K_A$ and $(\Omega \times K) \cap \operatorname{supp} K_A$ are compact.

**Proposition 1.1.8.** *Every $A \in L^\mu(\Omega)$ can be written in the form $A = A_0 + C$, where $A_0 \in L^\mu(\Omega)$ is properly supported and $C \in L^{-\infty}(\Omega)$.*

*Proof.* Let us choose an arbitrary function $\omega(x, x') \in C^\infty(\Omega \times \Omega)$ that equals 1 in an open neighbourhood of $\operatorname{diag}(\Omega \times \Omega)$ and such that both $(\Omega \times M') \bigcap \operatorname{supp} \omega$ and $(M \times \Omega) \bigcap \operatorname{supp} \omega$ are compact for arbitrary compact $M, M' \subset \Omega$. Then, for $A = \operatorname{Op}(a)$ with $a(x, x', \xi) \in S^\mu(\Omega \times \Omega \times \mathbb{R}^n)$ we can set $A_0 = \operatorname{Op}(\omega a)$ which is properly supported, and $C = \operatorname{Op}\big((1-\omega)a\big)$. The kernel of $C$ is $K_C = \big(1-\omega(x, x')\big)K_A(x, x') \in C^\infty(\Omega \times \Omega)$. $\square$

**Remark 1.1.9.** *Every $A \in L^\mu(\Omega)$ can be extended (in the distributional sense) to an operator $A : \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$. If $A$ is properly supported then $A$ induces continuous operators*

$$A : C_0^\infty(\Omega) \to C_0^\infty(\Omega), \quad A : C^\infty(\Omega) \to C^\infty(\Omega),$$

*which extends to*

$$A : \mathcal{E}'(\Omega) \to \mathcal{E}'(\Omega), \quad A : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega).$$

### 1.1.3 Kernel cut-off

In this section we outline some useful material on the so-called kernel cut-off operator, first applied to symbols $a(\xi) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^n)$. Observe that for every $\mu \in \mathbb{R}$ we have

$$S^\mu_{(\mathrm{cl})}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n), \quad S^{-\infty}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n).$$

Recall that the Fourier transform $\mathcal{F}u(\xi) := \int e^{-ix\xi} u(x)dx$ defines isomorphisms

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \quad \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

We then have the following important lemma.

**Lemma 1.1.10.** *For every $a(\xi) \in S^\mu(\mathbb{R}^n)$, $\mu \in \mathbb{R}$, the distribution $k(a)(\zeta) = (\mathcal{F}^{-1}a)(\zeta) \in \mathcal{S}'(\mathbb{R}^n)$ has the following properties:*

(i) $\operatorname{sing\,supp} k(a) \subseteq \{0\}$;

(ii) *if $\psi(\zeta)$ is any cut-off function at 0 (i.e., $\psi(\zeta) \in C_0^\infty(\mathbb{R}^n), \psi(\zeta) = 1$ for $|\zeta| < c_0$, $\psi(\zeta) = 0$ for $|\zeta| > c_1$ for certain $0 < c_0 < c_1$), we have*

$$\big(1 - \psi(\zeta)\big)k(a)(\zeta) \in \mathcal{S}(\mathbb{R}^n).$$

*Proof.* (i) The Fourier transform satisfies the identities

$$\mathcal{F}^{-1}_{\xi\to\zeta}(\xi^\gamma D^\delta_\xi a)(\zeta) = (-\zeta)^\delta D^\gamma_\zeta(\mathcal{F}^{-1}_{\xi\to\zeta} a)(\zeta)$$

for arbitrary $a \in \mathcal{S}'(\mathbb{R}^n), \gamma, \delta \in \mathbb{N}^n$. In particular,

$$k\left(\xi^\gamma(-\Delta_\xi)^N a\right)(\zeta) = |\zeta|^{2N} D^\gamma_\zeta k(a)(\zeta), N \in \mathbb{N}. \tag{1.1.5}$$

From $(-\Delta_\xi)^N a \in S^{\mu-2N}(\mathbb{R}^n)$ we obtain $|\zeta|^{2N}k(a)(\zeta) = k\left((-\Delta_\xi)^N a\right)(\zeta) \in C^r(\mathbb{R}^n)$ for $\mu - 2N < -n - r$. Since $N$ is arbitrary, it follows that $k(a)(\zeta) \in C^\infty(\mathbb{R}^n \setminus \{0\})$.

(ii) We have $\xi^\gamma(-\Delta_\xi)^N a(\xi) \in S^{\mu-2N+|\gamma|}(\mathbb{R}^n)$ for every $\gamma \in \mathbb{N}^n$, $N \in \mathbb{N}$. For $N$ satisfying $\mu - 2N + |\gamma| < -n$ we get $|k\left(\xi^\gamma(-\Delta_\xi)^N a\right)(\zeta)| < c$ with a constant $c = c(\gamma, N)$, for all $\zeta \in \mathbb{R}^n$. In other words (1.1.5) gives us

$$\sup_{\zeta\in\mathbb{R}^n}\left|(1-\psi(\zeta))\,|\zeta|^{2N} D^\gamma_\zeta k(a)(\zeta)\right| < \infty$$

for all $\gamma$ and sufficiently large $N = N(\gamma)$. This implies the same estimate for all $N$. This means $\big(1-\psi(\zeta)\big)k(a)(\zeta) \in \mathcal{S}(\mathbb{R}^n)$. □

For every $\varphi \in C^\infty_0(\mathbb{R}^n_\zeta)$ we now define an operator

$$H_\mathcal{F}(\varphi) : S^\mu_{(\mathrm{cl})}(\mathbb{R}^n) \to S^\mu_{(\mathrm{cl})}(\mathbb{R}^n)$$

by setting

$$H_\mathcal{F}(\varphi)a(\xi) := \mathcal{F}_{\zeta\to\xi}\left\{\varphi(\zeta)k(a)(\zeta)\right\}. \tag{1.1.6}$$

An important example is the case $H_\mathcal{F}(\psi)$ for a cut-off function $\psi$ as in Lemma 1.1.10. In this case we have

$$H_\mathcal{F}(1-\psi)a(\xi) = a(\xi) \mod S^{-\infty}(\mathbb{R}^n).$$

**Definition 1.1.11.** *Let $S^\mu_{(\mathrm{cl})}(\boldsymbol{C}^n)$ denotes the subspace of all $h(z) \in \mathcal{A}(\mathbb{C}^n)$ such that*

$$h(\xi + i\eta) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^n_\xi)$$

*for every $\eta \in \mathbb{R}^n$, uniformly for $\eta$ varying in compact sets.*

The space $S^\mu_{(\mathrm{cl})}(\boldsymbol{C}^n)$ is Fréchet in a natural way.

**Theorem 1.1.12.** *For every $\varphi \in C^\infty_0(\mathbb{R}^n)$ the function $H_\mathcal{F}(\varphi)a(\xi)$ has an extension as a holomorphic function in $\xi + i\eta \in \mathbb{C}^n$, and we have*

$$H_\mathcal{F}(\varphi)a(\xi + i\eta) \in S^\mu_{(\mathrm{cl})}(\boldsymbol{C}^n). \tag{1.1.7}$$

*Proof.* Let us first show that $H_\mathcal{F}(\varphi)a \in S^\mu(\mathbb{R}^n)$, i.e., that $|D^\alpha_\xi H_\mathcal{F}(\varphi)a(\xi)| \le c\langle\xi\rangle^{\mu-\alpha}$ for every $\alpha \in \mathbb{N}^n$ with some constant $c = c(\alpha) > 0$, for all $\xi \in \mathbb{R}^n$. We have

$$D^\alpha_\xi H_\mathcal{F}(\varphi)a(\xi) = \int e^{-i\xi\zeta}\varphi(\zeta)(-\zeta)^\alpha k(a)(\zeta)d\zeta = \int e^{-i\xi\zeta}\varphi(\zeta)k(D^\alpha_\xi a)(\zeta)d\zeta.$$

Using $\mathcal{F}(uv) = (\mathcal{F}u) * (\mathcal{F}v)$, it follows that

$$D_\xi^\alpha H_{\mathcal{F}}(\varphi)a(\xi) = \int \hat{\varphi}(\xi - \tilde{\xi}) D^\alpha a(\tilde{\xi}) d\tilde{\xi}. \tag{1.1.8}$$

Since $\hat{\varphi}$ is a Schwartz function, we obtain

$$|D_\xi^\alpha H_{\mathcal{F}}(\varphi)a(\xi)| \leq \int |\hat{\varphi}(\xi - \tilde{\xi})||D^\alpha a(\tilde{\xi})| d\tilde{\xi} \leq c_N \int \langle \xi - \tilde{\xi}\rangle^{-N} \langle \tilde{\xi}\rangle^{\mu - |\alpha|} d\tilde{\xi}$$

for every $N \in \mathbb{N}$, with a suitable constant $c_N = c_N(\varphi, \alpha)$. Applying Peetre's inequality

$$\left(\frac{1+|x|^2}{1+|y|^2}\right)^s \leq 2^{|s|}\left(1+|x-y|^2\right)^{|s|} \text{ for every } x, y \in \mathbb{R}^n \text{ and } s \in \mathbb{R}, \tag{1.1.9}$$

we get

$$\langle \tilde{\xi}\rangle^{\mu - |\alpha|} \leq c\langle \xi\rangle^{\mu - |\alpha|} \langle \xi - \tilde{\xi}\rangle^{|\mu - |\alpha||}.$$

This gives us

$$D_\xi^\alpha H_{\mathcal{F}}(\varphi)a(\xi) \leq c_N \langle \xi\rangle^{\mu - |\alpha|} \int \langle \xi - \tilde{\xi}\rangle^{-N + |\mu - |\alpha||} d\tilde{\xi}.$$

Choosing $N$ so large that $-N + |\mu - |\alpha|| < -n$ we get the asserted estimates, namely $D_\xi^\alpha H_{\mathcal{F}}(\varphi)a(\xi) \leq d_N(\varphi, \alpha)\langle \xi\rangle^{\mu - |\alpha|}$, with constants $d_N(\varphi, \alpha)$ that can be estimated by a finite number of semi-norms with respect to $\varphi$ in the Schwartz space.
Now $H_{\mathcal{F}}(\varphi)a(\xi)$ extends to a function in $\mathcal{A}(\mathbb{C}^n)$, since it is the Fourier transform of a distribution with compact support in $\xi$. We have

$$H_{\mathcal{F}}(\varphi)a(\xi + i\eta) = \int e^{-i\zeta(\xi + i\eta)} \varphi(\zeta) k(a)(\zeta) d\zeta = H_{\mathcal{F}}(\varphi_\eta)a(\xi)$$

for $\varphi_\eta(\zeta) := e^{\zeta\eta}\varphi(\zeta) \in C_0^\infty(\mathbb{R}^n)$. To prove (1.1.7) notice first that from (1.1.8) it is easy to prove that $\varphi(\zeta) \to \left(H_{\mathcal{F}}(\varphi)a\right)(\xi)$ is a continuous operator $C_0^\infty(\mathbb{R}^n_\zeta) \to S^\mu(\mathbb{R}^n_\xi)$ for every $a(\xi) \in S^\mu(\mathbb{R}^n)$. Then the latter continuity yields the assertion. □

**Remark 1.1.13.** *For every $\varphi \in C_0^\infty(\mathbb{R}^n)$ the operator $H_{\mathcal{F}}(\varphi)$ induces a continuous map*

$$H_{\mathcal{F}}(\varphi) : S^\mu_{(\mathrm{cl})}(\mathbb{R}^n) \to S^\mu_{(\mathrm{cl})}(\boldsymbol{C}^n)$$

*for every $\mu \in \mathbb{R}$.*

### 1.1.4 Elements of the calculus

Let $A \in L^\mu(\Omega)$ be a pseudo-differential operator. Then we call an element $\sigma(x, \xi) \in S^\mu(\Omega \times \mathbb{R}^n)$ a complete symbol of $A$ if $A - \mathrm{Op}(\sigma) \in L^{-\infty}(\Omega)$.

**Theorem 1.1.14.** *To every $a(x, x', \xi) \in S^\mu(\Omega \times \Omega \times \mathbb{R}^n)$ there is a $\sigma(x, \xi) \in S^\mu(\Omega \times \mathbb{R}^n)$ with $\mathrm{Op}(a) = \mathrm{Op}(\sigma) \mod L^{-\infty}(\Omega)$ and $\sigma(x, \xi)$ admits the asymptotic sum*

$$\sigma(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} D_\xi^\alpha \partial_{x'}^\alpha a(x, x', \xi)|_{x' = x}. \tag{1.1.10}$$

*Moreover, there is a* $\tilde{\sigma}(x', \xi) \in S^\mu(\Omega \times \mathbb{R}^n)$ *with* $\operatorname{Op}(a) = \operatorname{Op}(\tilde{\sigma}) \mod L^{-\infty}(\Omega)$ *that has the asymptotic expansion*

$$\tilde{\sigma}(x', \xi) \sim \sum_\alpha \frac{1}{\alpha!}(-D_\xi)^\alpha \partial_x^\alpha a(x, x', \xi)|_{x=x'}. \tag{1.1.11}$$

*We call* $\sigma(x, \xi)$ *a left symbol of the operator and* $\tilde{\sigma}(x', \xi)$ *a right symbol.*

*Proof.* The proof basically follows from Taylor's formula:

Let $f \in C^\infty(\mathbb{R}^n)$. Then for all positive integers $N$,

$$f(\zeta + \eta) = \sum_{|\alpha|<N} \frac{(\partial^\alpha f)(\zeta)}{\alpha!}\eta^\alpha + N \sum_{|\gamma|=N} \frac{\eta^\gamma}{\gamma!} \int_0^1 (1-\theta)^{N-1}(\partial^\gamma f)(\zeta + \theta\eta)d\theta \tag{1.1.12}$$

for all $\zeta, \eta \in \mathbb{R}^n$.

Applying (1.1.12) on $a(x, x', \xi)$ with respect to the second variable we obtain

$$a(x, x', \xi) = \sum_{|\alpha|\le M} \frac{1}{\alpha!}(x'-x)^\alpha \partial_{x'}^\alpha a(x, x', \xi)|_{x'=x} + r_M(x, x', \xi) \tag{1.1.13}$$

with a remainder $r_M(x, x', \xi) \in S^\mu(\Omega \times \Omega \times \mathbb{R}^n)$. For every $N$ we can choose $M$ so large that $|x - x'|^{-2N} r_M(x, x', \xi) \in S^\mu(\Omega \times \Omega \times \mathbb{R}^n)$. This means that we can find an $a_N \in S^{\mu-2N}(\Omega \times \Omega \times \mathbb{R}^n)$ such that $\operatorname{Op}(r_M) = \operatorname{Op}(a_N)$. Applying Op on both sides of (1.1.13) we obtain

$$\operatorname{Op}(a) = \sum_{|\alpha|\le M} \frac{1}{\alpha!} \operatorname{Op}\big(D_\xi^\alpha \partial_{x'}^\alpha a(x, x', \xi)|_{x'=x}\big) + \operatorname{Op}(a_N).$$

If we form (1.1.10) by carrying out the asymptotic sum we obtain immediately $\operatorname{Op}(a) - \operatorname{Op}(\sigma) = \operatorname{Op}(\tilde{a}_N)$ for another $\tilde{a}_N \in S^{\mu-2N}(\Omega \times \Omega \times \mathbb{R}^n)$. This is true for every $N$ and hence $\operatorname{Op}(a) - \operatorname{Op}(\sigma) \in L^{-\infty}(\Omega)$. The second statement can be proved in an analogous manner by interchanging the role of $x$ and $x'$. □

In other words, every $A \in L^\mu(\Omega)$ has a complete symbol. Furthermore, the map

$$\operatorname{Op} : S^\mu(\Omega \times \mathbb{R}^n) \to L^\mu(\Omega)$$

induces an (algebraic) isomorphism

$$S^\mu(\Omega \times \mathbb{R}^n)/S^{-\infty}(\Omega \times \mathbb{R}^n) \cong L^\mu(\Omega)/L^{-\infty}(\Omega), \tag{1.1.14}$$

and $\sigma_\psi^\mu(A)$ gives us a linear mapping

$$\sigma_\psi^\mu : L_{\mathrm{cl}}^\mu(\Omega) \to S^{(\mu)}\big(\Omega \times (\mathbb{R}^n \setminus \{0\})\big)$$

that is surjective, and $\ker \sigma_\psi^\mu = L_{\mathrm{cl}}^{\mu-1}(\Omega)$.

**Theorem 1.1.15.** *Let $A \in L^\mu(\Omega)$ and $A^*$ its formal adjoint, defined by*

$$(Au, v) = (u, A^* v), \; u, v \in C_0^\infty(\Omega)$$

*with the $L^2$-scalar product $(\cdot, \cdot)$. Then $A^* \in L^\mu(\Omega)$, and if $A = \mathrm{Op}(a) \mod L^{-\infty}(\Omega)$ for some $a(x,\xi) \in S^\mu(\Omega \times \mathbb{R}^n)$, we have $A^* = \mathrm{Op}(a^*) \mod L^{-\infty}(\Omega)$ with an $a^*(x,\xi) \in S^\mu(\Omega \times \mathbb{R}^n)$, and*

$$a^*(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} D_\xi^\alpha \partial_x^\alpha \overline{a(x,\xi)}. \tag{1.1.15}$$

*Analogously, if ${}^{\mathrm{t}}A$ is the formal transposed of $A$ with respect to $\langle \cdot, \cdot \rangle$ we have ${}^{\mathrm{t}}A \in L^\mu(\Omega)$ and ${}^{\mathrm{t}}A = \mathrm{Op}(c) \mod L^{-\infty}(\Omega)$, with $c(x,\xi) \in S^\mu(\Omega \times \mathbb{R}^n)$, and*

$$c(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} D_\xi^\alpha \partial_x^\alpha a(x, -\xi). \tag{1.1.16}$$

**Theorem 1.1.16.** *Let $A \in L^\mu(\Omega), B \in L^\nu(\Omega)$ and $A$ or $B$ be properly supported. Then $AB \in L^{\mu+\nu}(\Omega)$. If $A = \mathrm{Op}(a) \mod L^{-\infty}(\Omega)$, $a(x,\xi) \in S^\mu(\Omega \times \mathbb{R}^n)$, $B = \mathrm{Op}(b)$ $\mathrm{mod}\ L^{-\infty}(\Omega)$, $b(x,\xi) \in S^\nu(\Omega \times \mathbb{R}^n)$, then $AB = \mathrm{Op}(c) \mod L^{-\infty}(\Omega)$ for a symbol $c(x,\xi) \in S^{\mu+\nu}(\Omega \times \mathbb{R}^n)$ given by the formula* (1.1.4).

The prove can be found in any basic book on pseudo-differential operators, for example, in [46] or [39].

**Remark 1.1.17.** *Let $A \in L^\mu_{\mathrm{cl}}(\Omega)$, $B \in L^\nu_{\mathrm{cl}}(\Omega)$. Then $A^* \in L^\mu_{\mathrm{cl}}(\Omega)$, $AB \in L^{\mu+\nu}_{\mathrm{cl}}(\Omega)$ (if one factor is properly supported) and*

$$\sigma_\psi^\mu(A^*) = \overline{\sigma_\psi^\mu(A)}, \qquad \sigma_\psi^{\mu+\nu}(AB) = \sigma_\psi^\mu(A)\sigma_\psi^\nu(B).$$

The Leibniz product $c(x,\xi) = a(x,\xi) \# b(x,\xi)$ of $a$ and $b$ is unique mod $S^{-\infty}(\Omega \times \mathbb{R}^n)$. Let us set $e_\xi(x) := e^{ix\xi}$ and consider a properly supported $A \in L^\mu(\Omega)$. Then

$$\sigma_A(x,\xi) := e_{-\xi}(x) A e_\xi \tag{1.1.17}$$

is a $C^\infty$ function in $(x,\xi)$. Moreover, (1.1.17) is a complete symbol of $A$ satisfying $A = \mathrm{Op}(\sigma_A)$. If the operator $A$ is also given by $A = \mathrm{Op}(a)$ for an $a(x, x', \xi) \in S^\mu(\Omega \times \Omega \times \mathbb{R}^n)$, then (1.1.17) has the asymptotic sum (1.1.10).

### 1.1.5 Continuity in Sobolev spaces

As noted above every $A \in L^\mu(\Omega)$ for open $\Omega \subseteq \mathbb{R}^n$ can be regarded as a continuous operator

$$A : C_0^\infty(\Omega) \to C^\infty(\Omega).$$

In this section we extend $A$ to continuous operators between Sobolev spaces. The Sobolev space $H^s(\mathbb{R}^n)$ of smoothness $s \in \mathbb{R}$ is defined as the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_s := \left\{ \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |(\mathcal{F}u)(\xi)|^2 d\xi \right\}^{\frac{1}{2}}. \tag{1.1.18}$$

In particular, $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$. The operator norm in $\mathcal{L}\big(H^s(\mathbb{R}^n), H^r(\mathbb{R}^n)\big)$ will be denoted by $\| \cdot \|_{s,r}$.

**Proposition 1.1.18.** *Let $a(\xi) \in S^\mu(\mathbb{R}^n)$, $\mu \in \mathbb{R}$, and $A = \mathrm{Op}(a)$. Then $A$ extends to a continuous operator $A : H^s(\mathbb{R}^n) \to H^{s-\mu}(\mathbb{R}^n)$ and $a \to A$ induces a continuous operator $S^\mu(\mathbb{R}^n) \to \mathcal{L}\big(H^s(\mathbb{R}^n), H^{s-\mu}(\mathbb{R}^n)\big)$ for every $s \in \mathbb{R}$.*

*Proof.* Let $u \in C_0^\infty(\mathbb{R}^n)$. Then we have

$$\begin{aligned}\|Au\|_{s-\mu}^2 &= \int \langle\xi\rangle^{2(s-\mu)} |(\mathcal{F}Au)(\xi)|^2 d\xi \\ &= \int \langle\xi\rangle^{2(s-\mu)} |a(\xi)\mathcal{F}(\xi)|^2 d\xi \\ &= \int \langle\xi\rangle^{2(s-\mu)} |a(\xi)|^2 |\mathcal{F}(\xi)|^2 d\xi \le c^2 \|u\|_s^2\end{aligned}$$

for $c = \sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{-\mu}|a(\xi)| < \infty$. □

**Theorem 1.1.19.** *Let $a(x,\xi) \in S^\mu(\mathbb{R}^n \times \mathbb{R}^n)$ be a symbol with $a(x,\xi) = 0$ for $x \notin K$ for some compact set $K \subset \mathbb{R}^n$. Then $\mathrm{Op}(a)$ has a continuous extension*

$$A : H^s(\mathbb{R}^n) \to H^{s-\mu}(\mathbb{R}^n)$$

*for every $s \in \mathbb{R}$ and we have $\|A\|_{s,s-\mu} \le \tilde{c} c_N$ for a constant $\tilde{c} > 0$ and*

$$c_N = \sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{-\mu} \int \big|(1-\Delta_x)^N a(x,\xi)\big|\, dx$$

*for any natural number $N > \frac{n+|s-\mu|}{2}$. Here $\Delta_x := \sum_{i=0}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in $\mathbb{R}^n$.*

*Proof.* The function $b(\zeta,\xi) := \int e^{-ix\zeta} a(x,\xi)dx$ satisfies

$$|b(\zeta,\xi)| \le c_N \langle\zeta\rangle^{-2N} \langle\xi\rangle^\mu \tag{1.1.19}$$

for every $N \in \mathbb{N}$ with some constant $c_N > 0$. In fact, we have $\zeta^\alpha b(\zeta,\xi) = \int e^{-ix\zeta} D_x^\alpha a(x,\xi)dx$ for every $\alpha \in \mathbb{N}^n$ which gives us for arbitrary $N$

$$|\langle\zeta\rangle^{2N} b(\zeta,\xi)| = \left|\int e^{-ix\zeta}(1-\Delta_x)^N a(x,\xi)dx\right| \le c_N\langle\xi\rangle^\mu. \tag{1.1.20}$$

Next we set $K(\xi,\eta) := b(\eta-\xi,\xi)\langle\xi\rangle^{-s}\langle\eta\rangle^{s-\mu}$ and observe that

$$\int |K(\xi,\eta)|d\xi, \int |K(\xi,\eta)|d\eta \le c \tag{1.1.21}$$

for all $\eta$, and $\xi$ for some constant $c > 0$. In fact, (1.1.19) gives us together with Peetre's inequality

$$|K(\xi,\eta)| \le c_N\langle\xi-\eta\rangle^{-2N} \left(\frac{\langle\eta\rangle}{\langle\xi\rangle}\right)^{s-\mu} \le c_N\langle\xi-\eta\rangle^{|s-\mu|-2N}$$

which yields (1.1.21) when we choose $2N > n + |s - \mu|$. Applying now the Fourier transform $\mathcal{F}_{x \to \eta}$ to $Au(x)$, $u \in C_0^\infty(\mathbb{R}^n)$, we obtain

$$(\widehat{Au})(\eta) = \iint e^{-ix(\eta-\xi)} a(x,\xi)\hat{u}(\xi) đ\xi dx = \int b(\eta - \xi, \xi)\hat{u}(\xi) đ\xi.$$

This yields for any $v \in H^{s-\mu}(\mathbb{R}^n)$

$$\begin{aligned}\int (\widehat{Au})(\eta)\hat{v}(\eta) d\eta &= \iint b(\eta-\xi,\xi)\hat{v}(\eta)\hat{u}(\xi) đ\xi d\eta \\ &= \iint K(\xi,\eta)\hat{v}(\eta)\langle\eta\rangle^{\mu-s}\hat{u}(\xi)\langle\xi\rangle^{s} đ\xi d\eta.\end{aligned}$$

We then obtain

$$\begin{aligned}&|\langle Au, v\rangle| = |\langle \widehat{Au}, \hat{v}\rangle| \leq \\ &(2\pi)^{-n} \left( \iint |K(\xi,\eta)| \langle\eta\rangle^{2(\mu-s)} |\hat{v}(\eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} \left( \iint |K(\xi,\eta)| \langle\xi\rangle^{2s} |\hat{u}(\xi)|^2 d\xi d\eta \right)^{\frac{1}{2}} \\ &\qquad \leq c\|v\|_{\mu-s}\|u\|_s.\end{aligned}$$

In the latter estimate we have employed (1.1.21). Hence it follows that $\sup \frac{|\langle Au,v\rangle|}{\|v\|_{\mu-s}} \leq c\|u\|_s$, where sup is taken over all $v \in H^{\mu-s}(\mathbb{R}^n)\backslash\{0\}$. This yields $\|Au\|_{s-\mu} \leq c\|u\|_s$ for some $c > 0$ (since $\sup_{v \in H^{\mu-s}(\mathbb{R}^n)\backslash\{0\}} \frac{|\langle u,v\rangle|}{\|v\|_{\mu-s}}$ defines an equivalent norm in $H^{s-\mu}(\mathbb{R}^n)$). The latter constant is of the form $c = \tilde{c}c_N$ with $\tilde{c} > 0$ and $c_N$ from (1.1.19). For $c_N$ we have from (1.1.20)

$$c_N = \sup_{\xi \in \mathbb{R}^n} \langle\xi\rangle^{-\mu} \int |(1-\Delta_x)^N a(x,\xi)| dx$$

for any $N > \frac{n+|s-\mu|}{2}$. This completes the proof. □

**Corollary 1.1.20.** *The operator $\mathcal{M}_\varphi$ of multiplication by $\varphi \in C_0^\infty(\mathbb{R}^n)$ induces continuous operators*

$$\mathcal{M}_\varphi : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n) \tag{1.1.22}$$

*for all $s \in \mathbb{R}$. Moreover, $\varphi \to \mathcal{M}_\varphi$ represents a continuous operator $C_0^\infty(\mathbb{R}^n) \to \mathcal{L}(H^s(\mathbb{R}^n))$ for every $s \in \mathbb{R}$.*

In fact $\varphi \in C_0^\infty(\mathbb{R}^n)$ can be regarded as an element in $S^0(\mathbb{R}^n \times \mathbb{R}^n)$ with compact support with respect to $x$. Then $\mathcal{M}_\varphi$ corresponds to $\mathrm{Op}(\varphi)$, and we can apply Theorem 1.1.19 which shows at the same time the continuity $C_0^\infty(\mathbb{R}^n) \to \mathcal{L}(H^s(\mathbb{R}^n))$.

**Remark 1.1.21.** *Theorem* 1.1.19 *has an obvious generalisation to symbols $a(x,\xi) \in \mathcal{S}\big(\mathbb{R}^n_x, S^\mu(\mathbb{R}^n_\xi)\big)$. Then, in particular, $\mathcal{M}_\varphi$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ induces continuous operator* (1.1.22) *and the corresponding mapping $\mathcal{S}(\mathbb{R}^n) \to \mathcal{L}(H^s(\mathbb{R}^n))$ is continuous for every $s \in \mathbb{R}$.*

For an open $\Omega \subseteq \mathbb{R}^n$ we denote by $H^s_{\mathrm{loc}}(\Omega)$ the subspace of all $u \in \mathcal{D}'(\Omega)$ such that $\varphi u \in H^s(\mathbb{R}^n)$ for every $\varphi \in C_0^\infty(\Omega)$. Moreover, $H^s_{\mathrm{comp}}(\Omega)$ denotes the subspace of all $u \in H^s(\mathbb{R}^n)$ with compact support $\operatorname{supp} u \subseteq \Omega$. $u \to \|\varphi u\|_s$, $\varphi \in C_0^\infty(\Omega)$, represents a semi-norm system on $H^s_{\mathrm{loc}}(\Omega)$ which defines a Fréchet topology in the space $H^s_{\mathrm{loc}}(\Omega)$. Moreover, $H^s_{\mathrm{comp}}(\Omega)$ may be regarded as the inductive limit of the spaces $H^s(K) := \{u \in H^s_{\mathrm{comp}}(\Omega) : \operatorname{supp} u \subseteq K\}$ for compact $K \subset \Omega$. Every $H^s(K)$ can be interpreted as a closed subspace of $H^s(\mathbb{R}^n)$. The following theorem is an easy consequence of the above Theorem 1.1.19.

**Theorem 1.1.22.** *Each $A \in L^\mu(\Omega)$ induces continuous operators*

$$A : H^s_{\mathrm{comp}}(\Omega) \to H^{s-\mu}_{\mathrm{loc}}(\Omega)$$

*for all $s \in \mathbb{R}$. If $A$ is properly supported the corresponding operators*

$$A : H^s_{\mathrm{comp}}(\Omega) \to H^{s-\mu}_{\mathrm{comp}}(\Omega), \quad A : H^s_{\mathrm{loc}}(\Omega) \to H^{s-\mu}_{\mathrm{loc}}(\Omega)$$

*are continuous for all $s \in \mathbb{R}$.*

### 1.1.6 Ellipticity

In this section we study the ellipticity of pseudo-differential operators in any open set $\Omega \subseteq \mathbb{R}^n$.

**Definition 1.1.23.** *An operator $A \in L^\mu_{(\mathrm{cl})}(\Omega)$ is called elliptic (of order $\mu$) if for any left symbol $a(x,\xi) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ of $A$ (i.e., $A = \mathrm{Op}(a) \mod L^{-\infty}(\Omega)$) there exists a $p(x,\xi) \in S^{-\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ such that*

$$1 - p(x,\xi)a(x,\xi) \in S^{-1}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n). \tag{1.1.23}$$

*In this case we also say that the symbol $a(x,\xi)$ is elliptic.*

**Remark 1.1.24.** *If $a(x,\xi) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)$ is elliptic and $\sigma^\mu_\psi(x,\xi)$ the homogeneous principal symbol, then $a(x,\xi)$ is elliptic if and only if $\sigma^\mu_\psi(x,\xi) \neq 0$ for all $x \in \Omega$, $\xi \in \mathbb{R}^n \setminus \{0\}$. In this case we find a $p(x,\xi) \in S^{-\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)$ which satisfies the relation* (1.1.23).

In fact, we can set $p(x,\xi) = \chi(\xi)\big(\sigma^\mu_\psi(x,\xi)\big)^{-1}$ for any excision function $\chi(\xi)$.

**Theorem 1.1.25.** *If $A \in L^\mu_{(\mathrm{cl})}(\Omega)$ is elliptic there exists a properly supported operator $P \in L^{-\mu}_{(\mathrm{cl})}(\Omega)$ such that*

$$I - PA, I - AP \in L^{-\infty}(\Omega).$$

*The operator $P$ is called a parametrix of $A$.*

*Proof.* For the proof it is enough to find a $p(x,\xi) \in S^{-\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ such that for any left symbol $a(x,\xi) \in S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ of the operator A we have

$$1 - p(x,\xi)\#a(x,\xi), 1 - a(x,\xi)\#p(x,\xi) \in S^{-\infty}(\Omega \times \mathbb{R}^n).$$

In fact, it then suffices to define $P$ as a properly supported representative (modulo $L^{-\infty}(\Omega)$) of $\mathrm{Op}(p)$. Let us construct $p(x,\xi)$ in such a way that $1 - p(x,\xi)\#a(x,\xi)$ is of order $-\infty$. Then, in a similar manner we can construct another $\tilde{p}(x,\xi)$ such that $1 - a(x,\xi)\#\tilde{p}(x,\xi)$ is of order $-\infty$. A simple algebraic argument then gives us $p = \tilde{p}$ mod $S^{-\infty}$. In other words we will content ourselves with the multiplication from the left. In this proof for abbreviation we omit $(x,\xi)$ in the symbols and also write $S^{\mu}_{(\mathrm{cl})}$ rather than $S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$.

By assumption there is a $p_1 \in S^{-\mu}_{(\mathrm{cl})}$ such that

$$1 - p_1 a \in S^{-1}_{(\mathrm{cl})}.$$

By virtue of $p_1 a - p_1\#a \in S^{-1}_{(\mathrm{cl})}$ we also obtain

$$c := 1 - p_1\#a \in S^{-1}_{(\mathrm{cl})}.$$

This gives us $c^{\#j} \in S^{-j}_{(\mathrm{cl})}$, cf. the notation in Remark 1.1.6, i.e., we can form the asymptotic sum

$$\sum_{j=0}^{\infty} c^{\#j} \in S^{0}_{(\mathrm{cl})},$$

cf. Theorem 1.1.4. It easily follows that

$$\left(\sum_{j=0}^{\infty} c^{\#j}\right) \#(1-c) = 1.$$

This gives us together with $p_1\#a = 1 - c$

$$\left(\sum_{j=0}^{\infty} c^{\#j}\right) \#p_1\#a = 1,$$

i.e., we may set $p := \left(\sum_{j=0}^{\infty} c^{\#j}\right) \#p_1$, which is as desired.

### 1.1.7 Mellin pseudo-differential operators

In this section we introduce basic notations and observations about the Mellin transform on $\mathbb{R}_+$, weighted Sobolev spaces and operators in those spaces.
The classical Mellin transform is defined by the formula

$$\mathcal{M}u(z) = \int_0^{\infty} r^{z-1} u(r)\,dr, \tag{1.1.24}$$

first for $u \in C_0^\infty(\mathbb{R}_+)$ and then extended to more general distribution spaces, especially, weighted Sobolev spaces, also vector-valued ones.
Let $\mathcal{A}(U)$, for some open $U \subseteq \mathbb{C}$, denote the space of all holomorphic functions in $U$ (in the Fréchet topology of uniform convergence on compact subsets); more generally, if $E$ is a Fréchet space, $\mathcal{A}(U,E)$ will denote the space of all holomorphic functions in $U$ with values in $E$, with the topology of the projective tensor product $\mathcal{A}(U,E) = \mathcal{A}(U) \otimes_\pi E$. Let us recall here that the topology of $\mathcal{A}(U) \otimes_\pi E$ is defined as the strongest locally convex topology on the space $\mathcal{A}(U) \otimes E$ for which the canonical bilinear mapping $(h,e) \to h \otimes e$, $\mathcal{A}(U) \times E \to \mathcal{A}(U) \otimes E$ is continuous. A semi-norm on the space $\mathcal{A}(U,E)$ can be defined as follows. Let $p$ and $q$ be semi-norms on $\mathcal{A}(U)$ and $E$, respectively. Then for $\theta \in \mathcal{A}(U) \otimes E$ we set

$$(p \otimes q)(\theta) := \inf \sum_j p(h_j) q(e_j),$$

where the infimum is taken over all finite sets of pairs $(h_j, e_j)$ such that $\theta = \sum_j h_j \otimes e_j$.

Let us now go back to the Mellin transform and recall that $\mathcal{M}$ defines a continuous operator

$$\mathcal{M} : C_0^\infty(\mathbb{R}_+) \to \mathcal{A}(\mathbb{C}).$$

Let $\Gamma_\beta := \{z \in \mathbb{C}, \mathrm{Re} z = \beta\}$. Then $\mathcal{M}$ composed with the restriction to $\Gamma_\beta$ gives rise to a continuous operator $C_0^\infty(\mathbb{R}_+) \to \mathcal{S}(\Gamma_\beta)$ for every $\beta$. Moreover, it is well known that the weighted Mellin transform

$$\mathcal{M}_\gamma u(z) := \mathcal{M}u(z)|_{\Gamma_{\frac{1}{2}-\gamma}},$$

$$\mathcal{M}_\gamma : C_0^\infty(\mathbb{R}_+) \to \mathcal{S}(\Gamma_{\frac{1}{2}-\gamma})$$

extends by continuity to an isomorphism

$$\mathcal{M}_\gamma : r^\gamma L^2(\mathbb{R}_+) \to L^2(\Gamma_{\frac{1}{2}-\gamma})$$

for every $\gamma \in \mathbb{R}$, and the identity

$$\|r^{-\gamma} u\|_{L^2(\mathbb{R}_+)} = (2\pi)^{-\frac{1}{2}} \|\mathcal{M}_\gamma u\|_{L^2(\Gamma_{\frac{1}{2}-\gamma})} \tag{1.1.25}$$

holds. The inverse has the form

$$\left(\mathcal{M}_\gamma^{-1} g\right)(r) = \frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{1}{2}-\gamma}} r^{-z} g(z) dz.$$

We call $\mathcal{M}_\gamma$ the weighted Mellin transform (with the weight $\gamma$).

For $u(r) \in C_0^\infty(\mathbb{R}_+)$ we set

$$(S_\gamma u)(t) = e^{-(\frac{1}{2}-\gamma)t} u(e^{-t}),$$

which induces an isomorphism $S_\gamma : C_0^\infty(\mathbb{R}_+) \to C_0^\infty(\mathbb{R})$ and we have

$$(\mathcal{M}_\gamma u)(\frac{1}{2} - \gamma + i\tau) = (\mathcal{F} S_\gamma u)(\tau).$$

These relations allow us to define Mellin pseudo-differential operators. In the simplest case we can take symbols $f(z) \in S^{\mu}_{(\mathrm{cl})}(\Gamma_{\frac{1}{2}-\gamma})$ and then set

$$\mathrm{op}^{\gamma}_{M}(f)u(r) := \mathcal{M}^{-1}_{\gamma, z\to r}\big\{f(z)(\mathcal{M}_{\gamma, r'\to z}u)(z)\big\}, \tag{1.1.26}$$

first for $u \in C_0^{\infty}(\mathbb{R}_+)$. Using the identity $\mathcal{M}\left(r^{-\gamma}u\right)(z) = \mathcal{M}u(z-\gamma)$ we can also write

$$\mathrm{op}^{\gamma}_{M}(f)u(r) = r^{\gamma}\mathrm{op}_{M}\left(T^{-\gamma}f\right)r^{-\gamma}$$

where $\left(T^{-\gamma}f\right)(z) := f(z-\gamma)$ and $\mathrm{op}_M := \mathrm{op}^0_M$. Another interpretation of (1.1.26) is

$$\mathrm{op}^{\gamma}_{M}(f)u(r) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\frac{r}{r'}\right)^{-(\frac{1}{2}-\gamma+i\rho)} f(\frac{1}{2}-\gamma+i\rho)u(r')\frac{dr'}{r'}đ\rho$$

for $đ\rho = (2\pi)^{-1}d\rho$. Writing $x = -\log r$, $x' = -\log r'$ we obtain

$$\mathrm{op}^{\gamma}_{M}(f)u(r) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(x-x')\rho}e^{(x-x')(\frac{1}{2}-\gamma)}f(\frac{1}{2}-\gamma+i\rho)u(e^{-x'})dx'đ\rho.$$

By virtue of

$$e^{(x-x')(\frac{1}{2}-\gamma)}f(\frac{1}{2}-\gamma+i\rho) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}_x \times \mathbb{R}_{x'} \times \mathbb{R}_{\rho})$$

it follows that

$$\mathrm{op}^{\gamma}_{M}(f) \in L^{\mu}_{(\mathrm{cl})}(\mathbb{R}_+)$$

with $L^{\mu}_{(\mathrm{cl})}(\mathbb{R}_+)$ being defined as the space of pseudo-differential operators on $\mathbb{R}_+$, based on the Fourier transform.

**Definition 1.1.26.** *Let $\Sigma \subseteq \mathbb{R}^n$ be an open set, and $\mu \in \mathbb{R}$. Then $S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \mathbb{C} \times \mathbb{R}^n)_{\mathrm{hol}}$ is defined to be the space of all $h(r,x,z,\xi) \in \mathcal{A}\big(\mathbb{C}, S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \mathbb{R}^n)\big)$ such that*

$$h(r,x,\beta+i\rho,\xi) \in S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \mathbb{R}^{1+n}_{\rho,\xi})$$

*for every $\beta \in \mathbb{R}$, uniformly in compact $\beta$-intervals.*

Observe that the kernel cut-off procedure of Subsection 1.1.3 can be formulated in terms of Mellin transform rather than the Fourier transform. More precisely, we have the following result.
Let $C^{\infty}_B(\mathbb{R}_+)$ denote the subspace of all $\varphi \in C^{\infty}(\mathbb{R}_+)$ such that

$$\sup_{t\in\mathbb{R}_+}|(t\partial_t)^k\varphi(t)| < \infty \text{ for all } k \in \mathbb{N}.$$

For every such $\varphi \in C^{\infty}_B(\mathbb{R}_+)$ and every $\beta \in \mathbb{R}$ we have a kernel cut-off operator

$$H_{\mathcal{M}}(\varphi) : f(r,x,\beta+i\tau,\xi) \to h(r,x,\beta+i\tau,\xi)$$

given by $H_{\mathcal{M}}(\varphi)f := \mathcal{M}\varphi(t)\mathcal{M}^{-1}f$, which defines a bilinear continuous map

$$C_B^\infty(\mathbb{R}_+) \times S^\mu_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Gamma_\beta \times \mathbb{R}^n) \to S^\mu_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Gamma_\beta \times \mathbb{R}^n)$$

for every $\mu \in \mathbb{R}$. In particular, for $\operatorname{supp}\varphi$ compact we obtain a bilinear continuous operator

$$C_0^\infty(\mathbb{R}_+) \times S^\mu_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Gamma_\beta \times \mathbb{R}^n) \to S^\mu_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Gamma_\beta \times \mathbb{R}^n).$$

**Theorem 1.1.27 (Mellin quantisation).** *Let* $\tilde{p}(r,x,\tilde{\rho},\xi) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \mathbb{R}^{1+n}_{\tilde{\rho},\xi})$ *and*

$$p(r,x,\rho,\xi) := \tilde{p}(r,x,r\rho,\xi),$$

*then there exists an* $h(r,x,z,\xi) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \mathbb{C} \times \mathbb{R}^n)_{\mathrm{hol}}$ *such that*

$$\mathrm{Op}_{r,x}(p) = \mathrm{op}_M^\beta \mathrm{Op}_x(h) \mod L^{-\infty}(\mathbb{R}_+ \times \Sigma)$$

*for every* $\beta \in \mathbb{R}$.

We now introduce weighted Sobolev spaces on $\mathbb{R}_+$ based on the Mellin transform. They will contain the smoothness $s \in \mathbb{R}$ and in addition the weight $\gamma \in \mathbb{R}$.

**Definition 1.1.28.** $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ *for* $s,\gamma \in \mathbb{R}$ *is the closure of* $C_0^\infty(\mathbb{R}_+)$ *with respect to the norm*

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+)} := \left\{ \frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{2}-\gamma}} (1+|z|^2)^s |\mathcal{M}_\gamma u(z)|^2 dz \right\}^{\frac{1}{2}}.$$

*We set*

$$\mathcal{H}^s(\mathbb{R}_+) := \mathcal{H}^{s,0}(\mathbb{R}_+).$$

In view of (1.1.25) we have

$$\mathcal{H}^{0,\gamma}(\mathbb{R}_+) = r^\gamma L^2(\mathbb{R}_+), \quad \mathcal{H}^0(\mathbb{R}_+) = L^2(\mathbb{R}_+).$$

The transformation $u(r) \to (S_\gamma u)(t)$ extends to an isomorphism

$$S_\gamma : \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \to H^s(\mathbb{R})$$

for every $s,\gamma \in \mathbb{R}$. In other words, we have

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+)} \sim \|S_\gamma u\|_{H^s(\mathbb{R})}$$

in the sense of equivalence of norms.

## 1.2 Operators on a manifold with conical exits to infinity

### 1.2.1 Manifolds with conical exits to infinity

Let $M$ be a $C^\infty$ manifold, $m = \dim M$, equipped with a free $\mathbb{R}_+$-action $z \to \delta_\lambda z$, $\lambda \in \mathbb{R}_+$, $z \in M$. Then the orbit space $X$ is a $C^\infty$ manifold, and we have a diffeomorphism

$$e : M \to \mathbb{R}_+ \times X,\ e(z) =: (r, x), \tag{1.2.1}$$

such that $e(\delta_\lambda z) = (\lambda r, x)$ for all $\lambda \in \mathbb{R}_+$, $z \in M$. We will say that $M$ is endowed with the structure of an infinite (straight) cone, if there is fixed an open covering $\mathfrak{V}$ of $M$ by neighbourhoods of the form $V = e^{-1}(\mathbb{R}_+ \times U)$, where $U$ runs over an open covering $\mathfrak{U}$ of $X$ by coordinate neighbourhoods, and charts

$$\chi : V \to \Gamma$$

for conical $\Gamma \subset \mathbb{R}^m$ (i.e., $x \in \Gamma \Leftrightarrow \lambda x \in \Gamma$ for every $\lambda \in \mathbb{R}_+$), such that

$$\chi(\delta_\lambda z) = \lambda\chi(z)$$

for every $\lambda \in \mathbb{R}_+$, $z \in M$. The manifold $X$ will also be called the base of the cone.

In this notation we mainly focus on what happens "at infinity", i.e., over $e^{-1}((R, \infty) \times X)$ for any $R > 0$.

**Definition 1.2.1.** *A $C^\infty$ manifold $M$ is said to be a manifold with conical exits (to infinity), if $M$ contains a submanifold $M_\infty$ endowed with the structure of an infinite (straight) cone such that if $X$ is the base of the cone and $e : M_\infty \to \mathbb{R}_+ \times X$ a map in the sense of* (1.2.1) *the set*

$$M_0 := M \setminus e^{-1}((R, \infty) \times X) \tag{1.2.2}$$

*is a $C^\infty$ manifold with boundary $\partial M_0 \cong X$, for a certain $R > 0$.*

On $M_{[R,\infty)} := M_\infty \setminus e^{-1}((0, R) \times X)$ for some $R \geq 1$ we can define dilations $\delta_\lambda : M_{[R,\infty)} \to M_{[R,\infty)}$ for $\lambda \geq 1$.
By definition a manifold $M$ with conical exit can be written as a union

$$M = M_0 \cup M_\infty.$$

Let us fix a corresponding partition of unity $\{\varphi_0, \varphi_\infty\}$ in such a way that $\varphi_0 \in C_0^\infty(\mathrm{int} M_0)$ and $\varphi_\infty \in C_0^\infty\big(e^{-1}((R_0, \infty) \times X)\big)$ for some $0 < R_0 < R$ with $R$ as in (1.2.2).

**Example 1.2.2.** (i) *$M = \mathbb{R}^m$ can be regarded as a manifold with conical exits when we set $M_\infty := \mathbb{R}^m \setminus \{0\}$; then $X = S^{m-1}$ and $M_0 := \{x \in \mathbb{R}^m : |x| \leq R\}$ for any $R > 0$.*

(ii) *The finite cone $X^\wedge = \mathbb{R}_+ \times X$ for a $C^\infty$ manifold $X$ has a conical exit. In this case we can set $M_\infty = X^\wedge$, and $M_0 := (0, R] \times X$ for any $R > 0$.*

Let us now endow $M$ with a Riemannian metric $g$ as follows. Choose any Riemannian metric $g_0$ on $M_0 \cup C_\varepsilon$, where $C_\varepsilon \cong [R, \varepsilon) \times X$, for some $\varepsilon > 0$, is a collar neighbourhood of $X$ in $M_\infty$, and a Riemannian metric $g_1$ on $M_\infty$, and set $g = \omega g_0 + (1-\omega) g_1$, where $\omega \in C^\infty(M_0 \cup C_\varepsilon)$ is a function with $0 \leq \omega \leq 1$ and $\omega \equiv 1$ in $M_0 \cup C_{\varepsilon/2}$. We define $g_1$ as a conical metric by setting $g_1 = dr^2 + r^2 g_X$ for any Riemannian metric $g_X$ on $X$.

In the following notation we assume, for simplicity, that the base $X$ of the cone is a compact closed $C^\infty$ manifold. Let $H^{s;g}(M)$ for $s, g \in \mathbb{R}$ denote the space of all $u \in H^s_{\mathrm{loc}}(M)$ such that

$$e_*(\varphi_\infty u)(r, x) \in H^{s;g}_{\mathrm{cone}}(X^\wedge)(:= \langle r \rangle^{-g} H^s_{\mathrm{cone}}(X^\wedge)). \tag{1.2.3}$$

Here $H^s_{\mathrm{cone}}(X^\wedge)$ is the subspace of all $f \in H^s_{\mathrm{loc}}(\mathbb{R} \times X)|_{\mathbb{R}_+ \times X}$ such that for every coordinate neighbourhood $U$ on $X$ and every diffeomorphism $\kappa : \mathbb{R}_+ \times U \to \Gamma$ to an open conical subset $\Gamma \in \mathbb{R}^{n+1}_{\tilde{x}}$ such that $\kappa(\lambda r, x) = \lambda \kappa(r, x)$ for all $\lambda \in \mathbb{R}_+$, $(r, x) \in \mathbb{R}_+ \times U$, we have

$$\big((1-\omega)\varphi u\big)\big(\kappa^{-1}(\tilde{x})\big) \in H^s(\mathbb{R}^{n+1}_{\tilde{x}})$$

for every cut-off function $\omega$ and $\varphi \in C^\infty_0(U)$. We set

$$\|u\|_{H^{s;g}(M)} = \Big\{ \|\varphi_0 u\|^2_{H^s(\mathrm{int} M_0)} + \|e_*(\varphi_\infty u)\|^2_{H^{s;g}_{\mathrm{cone}}(X^\wedge)} \Big\}^{\frac{1}{2}}, \tag{1.2.4}$$

where $H^s(\mathrm{int}\cdot)$ means the standard Sobolev space of smoothness $s$ on a compact $C^\infty$ manifold with boundary. The space $H^{s;g}(M)$ can also be equipped with a Hilbert space scalar product such that the associated norm is equivalent to (1.2.4).

By

$$\mathcal{S}(M) := \varprojlim_{k \in \mathbb{N}} H^{k;k}(M)$$

we obtain an analogue of the Schwartz space on a manifold $M$ with conical exits. Furthermore, we need an analogue of the Schwartz space $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m)$ for the case $M \times M$ that we define to be the complete projective tensor product

$$\mathcal{S}(M \times M) := \mathcal{S}(M) \hat{\otimes}_\pi \mathcal{S}(M).$$

Concerning the complete projective tensor product see (1.2.10).

**Remark 1.2.3.** *The spaces $H^{s;g}(M)$ have similar properties as those in $\mathbb{R}^m$. We have continuous embeddings $H^{s';g'}(M) \hookrightarrow H^{s;g}(M)$ for $s' \geq s$, $g' \geq g$ that are compact for $s' > s$, $g' > g$. Moreover, the scalar product $(\cdot,\cdot)_{L^2(M)}$, which we take linear in the first and anti-linear in the second argument, induces a non-degenerate sesquilinear pairing*

$$H^{s;g}(M) \times H^{-s;-g}(M) \to \mathbb{C},$$

*such that $H^{-s;-g}(M)$ can be identified with the dual of $H^{s;g}(M)$.*

### 1.2.2 Calculus in the Euclidean space

According to Example 1.2.2 (i) we interpret the Euclidean space $\mathbb{R}^m$ as a manifold with a conical exit. Pseudo-differential operators will refer to symbols in the sense of the following definition.

**Definition 1.2.4.** (i) *The space $S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ for $\mu, \nu \in \mathbb{R}$ is defined to be the set of all $a(x,\xi) \in C^\infty(\mathbb{R}^m_x \times \mathbb{R}^m_\xi)$ such that*

$$\sup_{x,\xi \in \mathbb{R}^m} \langle x \rangle^{-\nu+|\alpha|} \langle \xi \rangle^{-\mu+|\beta|} \big| D_x^\alpha D_\xi^\beta a(x,\xi) \big| \tag{1.2.5}$$

*is finite for every $\alpha, \beta \in \mathbb{N}^m$.*

(ii) *The space $S^{\mu;\nu,\nu'}(\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m)$ for $\mu, \nu, \nu' \in \mathbb{R}$ is defined to be the set of all $a(x,x',\xi) \in C^\infty(\mathbb{R}^m_x \times \mathbb{R}^m_{x'} \times \mathbb{R}^m_\xi)$ such that*

$$\sup_{x,x',\xi \in \mathbb{R}^m} \langle x \rangle^{-\nu+|\alpha|} \langle x' \rangle^{-\nu'+|\alpha'|} \langle \xi \rangle^{-\mu+|\beta|} \big| D_x^\alpha D_{x'}^{\alpha'} D_\xi^\beta a(x,x',\xi) \big| \tag{1.2.6}$$

*is finite for every $\alpha, \alpha', \beta \in \mathbb{N}^m$.*

**Remark 1.2.5.** *The expressions* (1.2.5) *for $\alpha, \beta \in \mathbb{N}^m$ form a countable semi-norm system that turns $S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ to a Fréchet space. A similar observation is true of* (1.2.6) *in connection with the space $S^{\mu;\nu,\nu'}(\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m)$.*

**Theorem 1.2.6.** *Let $a_j(x,\xi) \in S^{\mu_j;\nu_j}(\mathbb{R}^m \times \mathbb{R}^m)$, $j \in \mathbb{N}$, be an arbitrary sequence, such that $\mu_j \to -\infty$, $\nu_j \to -\infty$ as $j \to \infty$. Then there exists an $a(x,\xi) \in S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ for $\mu = \max\{\mu_j; j \in \mathbb{N}\}$, $\nu = \max\{\nu_j; j \in \mathbb{N}\}$, such that for every $N \in \mathbb{N}$ there is an $M = M(N) \in \mathbb{N}$ such that*

$$a(x,\xi) - \sum_{j=0}^{M} a_j(x,\xi) \in S^{\mu-(N+1);\nu-(N+1)}(\mathbb{R}^m \times \mathbb{R}^m).$$

*For any other $\tilde{a}(x,\xi) \in S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ with this property we have*

$$a(x,\xi) - \tilde{a}(x,\xi) \in S^{-\infty;-\infty}(\mathbb{R}^m \times \mathbb{R}^m) := \bigcap_{\mu,\nu} S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m).$$

Any choice of $a$ as in Theorem 1.2.6 will be called an asymptotic sum of the $a_j$'s, written $a \sim \sum_{j=0}^\infty a_j$. It can be proved that when $\chi(x,\xi)$ is an excision function in $(x,\xi) \in \mathbb{R}^{2m}$, then there exists a sequence of constants $c_j > 0$ such that $a(x,\xi) := \sum_{j=0}^\infty \chi\left(\frac{x,\xi}{c_j}\right) a_j(x,\xi)$ converges in $S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$.

**Remark 1.2.7.** *Given $a(x,\xi) \in S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$, $b(x,\xi) \in S^{\sigma;\tau}(\mathbb{R}^m \times \mathbb{R}^m)$, $\mu,\nu,\sigma,\tau \in \mathbb{R}$, we can form the Leibniz product $(a\#b)(x,\xi)$ as the asymptotic sum*

$$(a\#b)(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} \big(\partial_\xi^\alpha a(x,\xi)\big) D_x^\alpha b(x,\xi)$$

*in the space $S^{\mu+\sigma;\nu+\tau}(\mathbb{R}^m \times \mathbb{R}^m)$, taking into consideration that $\big(\partial_\xi^\alpha a(x,\xi)\big) D_x^\alpha b(x,\xi)$ belongs to $S^{\mu+\sigma-|\alpha|;\nu+\tau-|\alpha|}(\mathbb{R}^m \times \mathbb{R}^m)$ and Theorem* 1.2.6.

**Lemma 1.2.8.** *For $\rho > 0$ There is a function $\omega(x, x') \in S^{0;0,0}(\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m)$ (independent of the covariable $\xi$) such that*

$$\omega(x, x') = \begin{cases} 1 & |x - x'| \leq \frac{\rho}{2}\langle x \rangle, \\ 0 & |x - x'| > \rho\langle x \rangle. \end{cases}$$

*Proof.* Let $\psi \in C^\infty(\mathbb{R})$ be a function such that $\psi(t) = 1$ for $|t| \leq \frac{1}{2}$ and $\psi(t) = 0$ for $|t| > 1$; then it suffices to set $\omega(x, x') := \psi(|x - x'|/\rho\langle x \rangle)$. □

**Remark 1.2.9.** *$a(x, \xi) \in S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ implies $\omega(x, x')a(x, \xi) \in S^{\mu;\nu,0}(\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m)$ for any $\omega$ as in Lemma* 1.2.8.

Let us set

$$\mathrm{Op}(a)u(x) := \iint e^{i(x-x')\xi} a(x, x', \xi)u(x')dx'đ\xi \qquad (1.2.7)$$

for an $a(x, x', \xi) \in S^{\mu;\nu,\nu'}(\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m)$ and $u \in \mathcal{S}(\mathbb{R}^m)$. Then, using standard technique on pseudo-differential operators (especially, oscillatory integral arguments) we obtain a continuous operator

$$\mathrm{Op}(a) : \mathcal{S}(\mathbb{R}^m) \to \mathcal{S}(\mathbb{R}^m).$$

**Remark 1.2.10.** *Operators* (1.2.7) *make sense for amplitude functions that may be much more general than $S^{\mu;\nu,\nu'}(\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m)$, see, for instance, Kumano-go* [19]. *If necessary we will give more comments on possible specific properties of $a(x, x', \xi)$.*

Let us set

$$L^{\mu;\nu}(\mathbb{R}^m) := \{\mathrm{Op}(a) : a(x, \xi) \in S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)\},$$

and

$$L^{-\infty;-\infty}(\mathbb{R}^m) := \bigcap_{\mu,\nu \in \mathbb{R}} L^{\mu;\nu}(\mathbb{R}^m).$$

We now formulate a number of properties of these spaces.

**Theorem 1.2.11.** (i) *The space $L^{-\infty;-\infty}(\mathbb{R}^m)$ coincides with the space of all integral operators of the form $Cu(x) = \int c(x, x')u(x')dx'$ for $c(x, x') \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m)$.*

(ii) *We have*

$$L^{\mu;\nu}(\mathbb{R}^m) = \{\mathrm{Op}(a) + C : a(x, x', \xi) \in S^{\mu;\nu,0}(\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m), C \in L^{-\infty;-\infty}(\mathbb{R}^m)\}.$$

(iii) *The map*

$$\mathrm{Op} : S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m) \to L^{\mu;\nu}(\mathbb{R}^m) \qquad (1.2.8)$$

*is an isomorphism.*

**Remark 1.2.12.** *Using Remark* 1.2.5 *and Theorem* 1.2.11 (iii) *we obtain a Fréchet space structure in $L^{\mu;\nu}(\mathbb{R}^m)$.*

**Theorem 1.2.13.** *Every $A \in L^{\mu;\nu}(\mathbb{R}^m)$ induces continuous operators*

$$A : H^{s;g}(\mathbb{R}^m) \to H^{s-\mu;g-\nu}(\mathbb{R}^m)$$

*for all $s, g \in \mathbb{R}$. Here $H^{s;g}(\mathbb{R}^m) := \langle x \rangle^{-g} H^s(\mathbb{R}^m)$.*

Let us now consider classical symbols. First identify $S^\mu(\mathbb{R}^m)$ with the set of all $a(\xi) \in S^{\mu;0}(\mathbb{R}^m \times \mathbb{R}^m)$ which are independent of $x$. Then we have the subclass $S^\mu_{\mathrm{cl}}(\mathbb{R}^m)$ of classical symbols, its Fréchet topology is induced from the one on $S^\mu(\mathbb{R}^m)$ plus the Fréchet topology of the projective limit of the mappings

$$\eta_j : S^\mu_{\mathrm{cl}}(\mathbb{R}^m) \to S^{(\mu-j)}(\mathbb{R}^m \setminus \{0\}), j \in \mathbb{N}$$
$$\rho_N : S^\mu_{\mathrm{cl}}(\mathbb{R}^m) \to S^{\mu-(N+1)}(\mathbb{R}^m), N \in \mathbb{N}$$

as in Remark 1.1.3. This turns $S^\mu_{\mathrm{cl}}(\mathbb{R}^m)$ to a nuclear Fréchet space.
We now consider the spaces $S^\mu_{\mathrm{cl}}(\mathbb{R}^m_\xi)$ and $S^\nu_{\mathrm{cl}}(\mathbb{R}^m_x)$ with respect to the variables $\xi$ and $x$, respectively, and set

$$S^{\mu;\nu}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^m \times \mathbb{R}^m) := S^\mu_{\mathrm{cl}}(\mathbb{R}^m_\xi) \hat{\otimes}_\pi S^\nu_{\mathrm{cl}}(\mathbb{R}^m_x). \tag{1.2.9}$$

We then obtain the subspace

$$S^{\mu;\nu}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^m \times \mathbb{R}^m) \subset S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m),$$

of so-called classical symbols (in $\xi$ and $x$), and we set

$$L^{\mu;\nu}_{\mathrm{cl}}(\mathbb{R}^m) := \{\mathrm{Op}(a) : a(x,\xi) \in S^{\mu;\nu}_{\mathrm{cl}}(\mathbb{R}^m \times \mathbb{R}^m)\}.$$

For brevity, we sometimes drop the subscripts $\xi; x$ and write $S^{\mu;\nu}_{\mathrm{cl}}$, $L^{\mu;\nu}_{\mathrm{cl}}$ instead of $S^{\mu;\nu}_{\mathrm{cl}_{\xi;x}}$ and $L^{\mu;\nu}_{\mathrm{cl}_{\xi;x}}$, respectively. If a consideration is valid both for classical and general symbols or operators, we write subscripts $(\mathrm{cl})_{\xi;x}$ and (cl), respectively.
Recall that $\hat{\otimes}_\pi$ stands for the complete projective tensor product. If $E$ and $F$ are two Fréchet spaces, every element $\theta \in E \hat{\otimes}_\pi F$ is the sum of an absolutely convergent series

$$\theta = \sum_{n=0}^{\infty} \lambda_n x_n \otimes y_n, \tag{1.2.10}$$

where $(\lambda_n)$ is a sequence of complex numbers satisfying $\sum_{n=0}^\infty |\lambda_n| < \infty$, and $(x_n)$ (respectively $(y_n)$) is a sequence converging to zero in $E$ (respectively $F$). For more information about the projective tensor product and its completion see the book of Treves [47] or Köthe [17]. If $G$ is another Fréchet space and $\sigma : E \to G$ a continuous map, we also obtain a continuous operator

$$\sigma \otimes \mathrm{id} : E \hat{\otimes}_\pi F \to G \hat{\otimes}_\pi F.$$

A similar relation holds with respect to the second factor. In particular, let

$$\sigma_\psi : S^\mu_{\mathrm{cl}}(\mathbb{R}^m_\xi) \to S^{(\mu)}(\mathbb{R}^m_\xi \setminus \{0\})$$

and

$$\sigma_{\mathrm{e}} : S^{\nu}_{\mathrm{cl}}(\mathbb{R}^m_x) \to S^{(\nu)}(\mathbb{R}^m_x \setminus \{0\})$$

denote the operators that map a symbol to its homogeneous principal components of order $\mu$ and $\nu$ in the corresponding variables $\xi$ and $x$, respectively. This induces operators

$$\sigma_{\psi} : S^{\mu;\nu}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^m \times \mathbb{R}^m) \to S^{(\mu)}(\mathbb{R}^m \setminus \{0\})\hat{\otimes}_{\pi} S^{\nu}_{\mathrm{cl}}(\mathbb{R}^m) \tag{1.2.11}$$

and

$$\sigma_{\mathrm{e}} : S^{\mu;\nu}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^m \times \mathbb{R}^m) \to S^{\mu}_{\mathrm{cl}}(\mathbb{R}^m)\hat{\otimes}_{\pi} S^{(\nu)}(\mathbb{R}^m \setminus \{0\}) \tag{1.2.12}$$

(here, for simplicity, we omitted corresponding identity maps for the other factors). Now we can apply $\sigma_{\mathrm{e}}$ in (1.2.11) with respect to $x$ and $\sigma_{\psi}$ in (1.2.12) with respect to $\xi$. It is well known that the resulting maps coincide and define a map

$$\sigma_{\psi,\mathrm{e}} := \sigma_{\psi} \otimes \sigma_{\mathrm{e}} : S^{\mu;\nu}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^m \times \mathbb{R}^m) \to S^{(\mu)}(\mathbb{R}^m \setminus \{0\})\hat{\otimes}_{\pi} S^{(\nu)}(\mathbb{R}^m \setminus \{0\}).$$

We call

$$\sigma(a) := (\sigma_{\psi}(a), \sigma_{\mathrm{e}}(a), \sigma_{\psi,\mathrm{e}}(a))$$

the principal symbol of the classical symbol $a(x,\xi) \in S^{\mu;\nu}_{(\mathrm{cl})_{\xi;x}}(\mathbb{R}^m \times \mathbb{R}^m)$. For $A = \mathrm{Op}(a)$ we also write

$$\sigma(A) := \sigma(a),$$

and $\sigma_{\psi}(A) := \sigma_{\psi}(a)$, $\sigma_{\mathrm{e}}(A) := \sigma_{\mathrm{e}}(a)$, $\sigma_{\psi,\mathrm{e}}(A) := \sigma_{\psi,\mathrm{e}}(a)$.

**Remark 1.2.14.** *Let us consider arbitrary elements*

$$\begin{aligned} p_{\psi}(x,\xi) &\in S^{(\mu)}(\mathbb{R}^m_{\xi} \setminus \{0\})\hat{\otimes}_{\pi} S^{\nu}_{\mathrm{cl}}(\mathbb{R}^m_x), \\ p_{\mathrm{e}}(x,\xi) &\in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^m_{\xi})\hat{\otimes}_{\pi} S^{(\nu)}(\mathbb{R}^m_x \setminus \{0\}), \\ p_{\psi,\mathrm{e}}(x,\xi) &\in S^{(\mu)}(\mathbb{R}^m_{\xi} \setminus \{0\})\hat{\otimes}_{\pi} S^{(\nu)}(\mathbb{R}^m_x \setminus \{0\}), \end{aligned}$$

*such that*

$$\sigma_{\mathrm{e}}(p_{\psi})(x,\xi) = \sigma_{\psi}(p_{\mathrm{e}})(x,\xi) = p_{\psi,\mathrm{e}}(x,\xi). \tag{1.2.13}$$

*Then there exists an element $p(x,\xi) \in S^{\mu;\nu}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^m \times \mathbb{R}^m)$ such that*

$$\sigma_{\psi}(p) = p_{\psi}, \quad \sigma_{\mathrm{e}}(p) = p_{\mathrm{e}}, \quad \sigma_{\psi,\mathrm{e}}(p) = p_{\psi,\mathrm{e}}.$$

In fact, let $\chi$ be an excision function in $\mathbb{R}^m$. Then it suffices to set

$$p(x,\xi) = \chi(\xi)p_{\psi}(x,\xi) + \chi(x)\{p_{\mathrm{e}}(x,\xi) - \chi(\xi)p_{\psi,\mathrm{e}}(x,\xi)\}. \tag{1.2.14}$$

Moreover, if $a(x,\xi) \in S^{\mu;\nu}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^m \times \mathbb{R}^m)$ is an arbitrary symbol with

$$\sigma_{\psi}(a) = p_{\psi}, \quad \sigma_{\mathrm{e}}(a) = p_{\mathrm{e}}, \quad \sigma_{\psi,\mathrm{e}}(a) = p_{\psi,\mathrm{e}},$$

then we have

$$a(x,\xi) - p(x,\xi) \in S^{\mu-1,\nu-1}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^m \times \mathbb{R}^m).$$

**Definition 1.2.15.** *An operator $A \in L^{\mu;\nu}(\mathbb{R}^m)$ is called elliptic of order $(\mu, \nu)$ if for the symbol $a(x,\xi) \in S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ in the representation $A = \mathrm{Op}(a)$, (cf. Theorem* 1.2.11 (iii)*), there is a $p(x,\xi) \in S^{-\mu;-\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ such that*

$$1 - p(x,\xi)a(x,\xi) \in S^{-1;-1}(\mathbb{R}^m \times \mathbb{R}^m). \tag{1.2.15}$$

**Remark 1.2.16.** *An Operator $A \in L^{\mu;\nu}_{\mathrm{cl}}(\mathbb{R}^m)$ is elliptic if and only if*

$$\begin{aligned} \sigma_\psi(A)(x,\xi) &\neq 0 \text{ for all } (x,\xi) \in \mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\}), \\ \sigma_{\mathrm{e}}(A)(x,\xi) &\neq 0 \text{ for all } (x,\xi) \in (\mathbb{R}^m \setminus \{0\}) \times \mathbb{R}^m, \\ \sigma_{\psi,\mathrm{e}}(A)(x,\xi) &\neq 0 \text{ for all } (x,\xi) \in (\mathbb{R}^m \setminus \{0\}) \times (\mathbb{R}^m \setminus \{0\}). \end{aligned}$$

*These conditions are independent of each other. For instance, $A = \mathrm{Op}(a)$ for $a(x,\xi) = \langle\xi\rangle^\mu + \langle x\rangle^\nu$ satisfies the first two conditions, but the third one is violated.*

**Theorem 1.2.17.** *For an operator $A \in L^{\mu;\nu}(\mathbb{R}^m)$ the following conditions are equivalent:*

(i) *the operator*

$$A : H^{s;g}(\mathbb{R}^m) \to H^{s-\mu;g-\nu}(\mathbb{R}^m) \tag{1.2.16}$$

*is Fredholm for some $s = s_0$, $g = g_0 \in \mathbb{R}$;*

(ii) *the operator $A$ is elliptic.*

*The Fredholm property of* (1.2.16) *for $s_0, g_0 \in \mathbb{R}$ implies the same for all $s, g \in \mathbb{R}$.*

**Theorem 1.2.18.** *An elliptic operator $A \in L^{\mu;\nu}_{(\mathrm{cl})}(\mathbb{R}^m)$ has a parametrix $P \in L^{-\mu;-\nu}_{(\mathrm{cl})}(\mathbb{R}^m)$, i.e.,*

$$I - PA,\ I - AP \in L^{-\infty;-\infty}(\mathbb{R}^m).$$

*Proof.* Let us first consider the non-classical case. Let $a(x,\xi)$ be the (according to the bijection (1.2.8) unique) symbol associated with $A$ and choose $p(x,\xi)$ as in (1.2.15). Then

$$p(x,\xi)a(x,\xi) = 1 \mod S^{-1;-1}(\mathbb{R}^m \times \mathbb{R}^m)$$

implies

$$p(x,\xi)\#a(x,\xi) = 1 \mod S^{-1;-1}(\mathbb{R}^m \times \mathbb{R}^m).$$

Thus there is a $c(x,\xi) \in S^{-1;-1}(\mathbb{R}^m \times \mathbb{R}^m)$ such that

$$p\#a = 1 - c. \tag{1.2.17}$$

Then, if $c^{\#j}$ means $c\# \cdots \#c$ with $j$ factors, and using that the Leibniz product is associative, it follows that

$$\sum_{j=0}^{\infty} c^{\#j}\#(1-c) = 1 \mod S^{-\infty;-\infty}(\mathbb{R}^m \times \mathbb{R}^m);$$

(the infinite sum is interpreted as an asymptotic sum). Thus (1.2.17) gives us

$$\Big(\sum_{j=0}^{\infty} c^{\#j}\Big)\#p\#a = 1 \mod S^{-\infty;-\infty}(\mathbb{R}^m \times \mathbb{R}^m).$$

Therefore, $q := (\sum_{j=0}^{\infty} c^{\#j})\#p \in S^{-\mu;-\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ has the property $q\#a = 1 \mod S^{-\infty;-\infty}(\mathbb{R}^m \times \mathbb{R}^m)$. By virtue of $\mathrm{Op}(q)\mathrm{Op}(a) - \mathrm{Op}(q\#a) \in L^{-\infty;-\infty}(\mathbb{R}^m)$, and $\mathrm{Op}(q\#a) = 1 \mod L^{-\infty;-\infty}(\mathbb{R}^m)$ shows that $\mathrm{Op}(q)$ is a left parametrix of $A$. In a similar manner we obtain a right parametrix which shows that $\mathrm{Op}(q)$ is a two-sided parametrix, and we may set $P = \mathrm{Op}(q)$. In the classical case we can proceed in an analogous manner using that asymptotic summation preserves classical symbols. Let $a(x,\xi)$ be classical; then if we set

$$p_\psi(x,\xi) := \sigma_\psi^{-1}(a)(x,\xi),\ p_{\mathrm{e}}(x,\xi) := \sigma_{\mathrm{e}}^{-1}(a)(x,\xi),\ p_{\psi,\mathrm{e}}(x,\xi) := \sigma_{\psi,\mathrm{e}}^{-1}(a)(x,\xi),$$

it can be easily verified that

$$\sigma_\psi(p_{\mathrm{e}})(x,\xi) = \sigma_{\mathrm{e}}(p_\psi)(x,\xi) = p_{\psi,\mathrm{e}}(x,\xi),$$

and hence, according to Remark 1.2.14, there exits an element $p(x,\xi) \in S^{-\mu;-\nu}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^m \times \mathbb{R}^m)$ such that

$$\sigma_\psi(p)(x,\xi) = p_\psi(x,\xi),\ \sigma_{\mathrm{e}}(p)(x,\xi) = p_{\mathrm{e}}(x,\xi),\ \sigma_{\psi,\mathrm{e}}(p)(x,\xi) = p_{\psi,\mathrm{e}}(x,\xi).$$

Now we continue as in the first part of the proof. Since the Leibniz product of classical symbols is again classical, we have $q := (\sum_{j=0}^{\infty} c^{\#j})\#p \in S^{-\mu;-\nu}_{\mathrm{cl}_{\xi;x}}(\mathbb{R}^m \times \mathbb{R}^m)$ and satisfies $q\#a = 1 \mod S^{-\infty;-\infty}(\mathbb{R}^m \times \mathbb{R}^m)$. This shows us that $\mathrm{Op}(q) \in L^{-\mu;-\nu}_{\mathrm{cl}}(\mathbb{R}^m)$ is a parametrix of $A$. □

### 1.2.3 Invariance under push forwards

Let $\Gamma \subset \mathbb{R}^m$ be an open set of the form

$$\Gamma = \{x \in \mathbb{R}^m \setminus \{0\} : x/|x| \in U\} \tag{1.2.18}$$

for a coordinate neighbourhood $U$ on the unit sphere $S^{m-1}$. Let

$$S_0^{\mu;\nu}(\Gamma \times \mathbb{R}^m) \quad (S_0^{\mu;\nu,\nu'}(\Gamma \times \Gamma \times \mathbb{R}^m))$$

denote the subspace of all $a(x,\xi) \in S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ $(a(x,x',\xi) \in S^{\mu;\nu,\nu'}(\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m))$ such that there is a $\Gamma_0 \subset \Gamma$ of the form

$$\Gamma_0 = \{x \in \mathbb{R}^m : |x| > \varepsilon,\ x/|x| \in U_0\} \tag{1.2.19}$$

for some $\varepsilon > 0$ and $U_0$ open, $U_0 \subset U$, with

$$a(x,\xi) = 0 \text{ for } x \notin \Gamma_0 \quad (a(x,x',\xi) = 0 \text{ for } x,x' \notin \Gamma_0).$$

Moreover, let $\mathcal{S}_0(\Gamma\times\Gamma)$ denote the set of all $c(x,x') \in \mathcal{S}(\mathbb{R}^m\times\mathbb{R}^m)$ such that $c(x,x') = 0$ whenever $x \notin \Gamma_0$ or $x' \notin \Gamma_0$ for some $\Gamma_0 \subset \Gamma$ of the form (1.2.19). Define $L_0^{-\infty;-\infty}(\Gamma)$ to be the space of all $C \in L^{-\infty;-\infty}(\mathbb{R}^m)$ with kernel in $\mathcal{S}_0(\Gamma\times\Gamma)$, and

$$L_0^{\mu;\nu}(\Gamma) := \{\mathrm{Op}(a) + C : a(x,\xi) \in S_0^{\mu;\nu}(\Gamma\times\mathbb{R}^m),\ C \in L_0^{-\infty;-\infty}(\Gamma)\}.$$

In an analogous manner we define $L_{0,\mathrm{cl}}^{\mu;\nu}(\Gamma)$ as the corresponding subspace of $L_0^{\mu;\nu}(\Gamma)$ with symbols in $S_{0,\mathrm{cl}_{\xi;x}}^{\mu;\nu}(\Gamma\times\mathbb{R}^m) := S_{\mathrm{cl}_{\xi;x}}^{\mu;\nu}(\mathbb{R}^m\times\mathbb{R}^m)\cap S_0^{\mu;\nu}(\Gamma\times\mathbb{R}^m)$.

Consider a diffeomorphism

$$\varphi : \widetilde{\Gamma} \to \Gamma$$

between open conical sets of the form (1.2.18), where

$$\varphi(\lambda\tilde{x}) = \lambda\varphi(\tilde{x})$$

for all $\lambda \in \mathbb{R}_+$, $\tilde{x} \in \widetilde{\Gamma}$. Given an $\widetilde{A} \in L_0^{\mu;\nu}(\widetilde{\Gamma})$ we now form the operator push forward $A := \varphi_*\widetilde{A}$ by setting

$$A = (\varphi^*)^{-1}\widetilde{A}\varphi^*$$

with the function pull back $\varphi^*$.

**Theorem 1.2.19.** *The operator push forward $\varphi_*$ induces an isomorphism*

$$\varphi_* : L_0^{\mu;\nu}(\widetilde{\Gamma}) \to L_0^{\mu;\nu}(\Gamma) \tag{1.2.20}$$

*for every $\mu,\nu \in \mathbb{R}$.*

The proof will be given in several steps. In particular, following the lines of Schrohe [33], we derive an asymptotic expansion for the symbol $a(x,\xi)$ of $\varphi_*\widetilde{A}$. First it is evident that $\varphi_*$ restricts to an isomorphism

$$\varphi_* : L_0^{-\infty;-\infty}(\widetilde{\Gamma}) \to L_0^{-\infty;-\infty}(\Gamma). \tag{1.2.21}$$

Therefore, we may assume

$$\widetilde{A}\tilde{u}(\tilde{x}) = \iint e^{i(\tilde{x}-\tilde{x}')\tilde{\xi}}\tilde{a}(\tilde{x},\tilde{\xi})\tilde{u}(\tilde{x}')d\tilde{x}'đ\tilde{\xi},\ \tilde{u} \in \mathcal{S}(\mathbb{R}^m),$$

for an $\tilde{a}(\tilde{x},\tilde{\xi}) \in S_0^{\mu;\nu}(\widetilde{\Gamma}\times\mathbb{R}^m)$. Let $\omega(\tilde{x},\tilde{x}')$ be a function supported near the diagonal, as in Lemma 1.2.8, and write

$$\tilde{a}(\tilde{x},\tilde{\xi}) = \tilde{a}_1(\tilde{x},\tilde{x}',\tilde{\xi}) + \tilde{a}_2(\tilde{x},\tilde{x}',\tilde{\xi})$$

for $\tilde{a}_1(\tilde{x},\tilde{x}',\tilde{\xi}) := \omega(\tilde{x},\tilde{x}')\tilde{a}(\tilde{x},\tilde{\xi}), \tilde{a}_2(\tilde{x},\tilde{x}',\tilde{\xi}) := \big(1-\omega(\tilde{x},\tilde{x}')\big)\tilde{a}(\tilde{x},\tilde{\xi})$. We then have

$$\widetilde{A} = \mathrm{Op}(\tilde{a}) = \mathrm{Op}(\tilde{a}_1) + \mathrm{Op}(\tilde{a}_2). \tag{1.2.22}$$

We shall show in Lemma 1.2.25 below that $\mathrm{Op}(\tilde{a}_2) \in L_0^{-\infty;-\infty}(\widetilde{\Gamma})$. Theorefore, it is enough to consider $\widetilde{A}_1 := \mathrm{Op}(\tilde{a}_1)$. We now insert $\tilde{u}(\tilde{x}') = u(x')$ where $x' = \varphi(\tilde{x}')$, i.e., $\tilde{u} = \varphi^*u$. Let us set, for convenience, $\chi := \varphi^{-1}$. Then $A_1u(x) := (\varphi^*)^{-1}\widetilde{A}_1\varphi^*u$ takes the form

$$A_1u(x) = \iint e^{i(\chi(x)-\chi(x'))\tilde{\xi}}\tilde{a}_1\big(\chi(x),\chi(x'),\tilde{\xi}\big)u(x')|\det d\chi(x')|dx'đ\tilde{\xi}. \tag{1.2.23}$$

In order to reformulate the latter expression we employ the following lemma.

**Lemma 1.2.20.** *Let $x, x' \in \Gamma$ and $|x - x'| < \delta\langle x\rangle$ for a sufficiently small constant $\delta > 0$. Let*

$$D(x, x') := \int_0^1 d\chi\big(x' + t(x - x')\big)dt.$$

*Then we have*

(i) $\chi(x) - \chi(x') = D(x, x')(x - x')$, *moreover,* $D(\lambda x, \lambda x') = D(x, x')$ *for every* $\lambda \in \mathbb{R}_+$*;*

(ii) $D(x, x')$ *is invertible for* $|x - x'| \le \rho\langle x\rangle$ *for a constant* $0 < \rho < \delta$.

*Proof.* (i) is the mean value theorem. The property (ii) follows from the invertibility of $d\chi(x) = D(x, x)$ and from the homogeneity of the transformation $\chi$. ☐

We now return to the formula (1.2.23) and write

$$\begin{aligned} A_1 u(x): \quad &= \iint e^{i(\chi(x)-\chi(x'))\tilde{\xi}} \tilde{a}_1\big(\chi(x), \chi(x'), \tilde{\xi}\big)u(x')|\det d\chi(x')|dx' đ\tilde{\xi}, \\ &= \iint e^{i(x-x')\xi} c(x, x', \xi)u(x')dx' đ\xi \qquad (1.2.24) \end{aligned}$$

for $\xi := {}^{\mathrm{t}}D(x, x')\tilde{\xi}$ and

$$c(x, x', \xi) := \tilde{a}_1\big(\chi(x), \chi(x'), {}^{\mathrm{t}}D^{-1}(x, x')\xi\big)|\det d\chi(x')||\det D^{-1}(x, x')|. \qquad (1.2.25)$$

Let us now characterise the behaviour of $c(x, x', \xi)$. First observe that

$$c(x, x', \xi) \in S^{\mu;\nu,0}(\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m).$$

Applying Taylor's formula (1.1.12) with respect to the second variable at $x$ we have for every $N \in \mathbb{N}$

$$c(x, x', \xi) = \sum_{|\theta| \le N} \frac{(-i)^{|\theta|}}{\theta!} D^\theta_{x'} c(x, x', \xi)\Big|_{x'=x} (x - x')^\theta + r_N(x, x', \xi)$$

where

$$r_N(x, x', \xi) = (N + 1) \sum_{|\theta|=N+1} \frac{(-i)^{N+1}}{\theta!} r_\theta(x, x', \xi)(x - x')^\theta$$

with $r_\theta(x, x', \xi) = \int_0^1 (1 - t)^N D^\theta_{x'} c\big(x, x + t(x' - x), \xi\big)dt$. It follows that

$$\mathrm{Op}(c) = \mathrm{Op}\left( \sum_{|\theta| \le N} \frac{1}{\theta!} \partial^\theta_\xi D^\theta_{x'} c(x, x', \xi)\Big|_{x'=x} \right) + \mathrm{Op}(r_N) \qquad (1.2.26)$$

where $\mathrm{Op}(r_N) = \mathrm{Op}\Big( \sum_{|\theta|=N+1} \frac{1}{\theta!} \partial^\theta_\xi r_\theta(x, x', \xi)\Big)$. More precisely, setting

$$c_\theta(x, x', \xi) := \frac{1}{\theta!} \partial^\theta_\xi D^\theta_{x'} c(x, x', \xi), \qquad (1.2.27)$$

$\theta \in \mathbb{N}^m$, and

$$c_N(x, x', \xi) := \sum_{|\theta|=N+1} \frac{N+1}{\theta!} \int_0^1 \partial_\xi^\theta D_{x'}^\theta c\big(x, x + t(x'-x), \xi\big)(1-t)^N dt, \tag{1.2.28}$$

we have

$$\iint e^{i(x-x')\xi} \frac{(-i)^{|\theta|}}{\theta!} D_{x'}^\theta c(x, x', \xi)\Big|_{x'=x} (x-x')^\theta u(x') dx' đ\xi$$
$$= \iint e^{i(x-x')\xi} c_\theta(x, x, \xi) u(x') dx' đ\xi \tag{1.2.29}$$

and

$$\iint e^{i(x-x')\xi} r_N(x, x', \xi) u(x') dx' đ\xi = \iint e^{i(x-x')\xi} c_N(x, x', \xi) u(x') dx' đ\xi \tag{1.2.30}$$

for $u \in \mathcal{S}(\mathbb{R}^m)$, $N \geq \mu + m$. This follows by integration by parts and using the relation $\partial_\xi^\alpha e^{i(x-x')\xi} = i^{|\alpha|}(x-x')^\alpha e^{i(x-x')\xi}$, $\alpha \in \mathbb{N}^m$.

**Theorem 1.2.21.** *We have* $b_\theta(x, \xi) := c_\theta(x, x, \xi) \in S^{\mu-|\theta|;\nu-|\theta|}(\mathbb{R}^m \times \mathbb{R}^m)$ *for every* $\theta \in \mathbb{N}^m$. *Moreover, if we define a symbol* $b(x, \xi) \in S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ *by the asymptotic sum* $b(x, \xi) \sim \sum_{\theta \in \mathbb{N}^m} b_\theta(x, \xi)$*, we have*

$$\operatorname{Op}(c) = \operatorname{Op}(b) \bmod L^{-\infty;-\infty}(\mathbb{R}^m).$$

For the proof we establish a number of auxiliary results. First, for $\theta \in \mathbb{N}^m$ and $0 \leq t \leq 1$ we define the functions

$$\begin{aligned}
c_{\theta,t}(x, x', \xi) &:= \partial_\xi^\theta D_{x'}^\theta c\big(x, x + t(x'-x), \xi\big), \\
f_t(x, x', \xi) &:= \tilde{a}_1\Big(\chi(x), \chi\big(x + t(x'-x)\big), {}^{\mathrm{t}}D^{-1}\big(x, x + t(x'-x)\big)\xi\Big), \\
g_t(x, x') &:= \big|\det d\chi\big(x + t(x'-x)\big)\big|, \\
h_t(x, x') &:= \big|\det D^{-1}\big(x, x + t(x'-x)\big)\big|.
\end{aligned}$$

By definition we have the relations

$$c = f_1 g_1 h_1, \; c_\theta = \frac{1}{\theta!} \partial_\xi^\theta D_{x'}^\theta (f_1 g_1 h_1), \; c_{\theta,t} = \partial_\xi^\theta D_{x'}^\theta (f_t g_t h_t).$$

**Lemma 1.2.22.** (i) *We have for* $j \geq 0$

$$\langle x + t(x'-x)\rangle^{-j} \leq C\langle x\rangle^{-j}\langle x - x'\rangle^j$$

*for a constant* $C > 0$ *independent of* $t$ *for* $0 \leq t \leq 1$.

(ii) *For every* $\alpha, \alpha', \beta \in \mathbb{N}^m$ *we have*

$$\begin{aligned}
|D_x^\alpha D_{x'}^{\alpha'} g_t(x, x')| &\leq C\langle x\rangle^{-|\alpha+\alpha'|}\langle x - x'\rangle^{|\alpha+\alpha'|}, \\
|D_x^\alpha D_{x'}^{\alpha'} h_t(x, x')| &\leq C\langle x\rangle^{-|\alpha+\alpha'|}\langle x - x'\rangle^{|\alpha+\alpha'|}, \\
|D_x^\alpha D_{x'}^{\alpha'} D_\xi^\beta f_t(x, x', \xi)| &\leq C\langle \xi\rangle^{\mu-|\beta|}\langle x\rangle^{\nu-|\alpha+\alpha'|}\langle x - x'\rangle^{|\alpha+\alpha'|},
\end{aligned}$$

*where* $C > 0$ *is a constant independent of* $t$ *for* $0 \leq t \leq 1$.

*Proof.* (i) is a consequence of Peetre's inequality (1.1.9). The estimates in (ii) follow by using (i), the homogeneity of the transformation $\chi$, and the Leibniz rule, together with induction. □

**Corollary 1.2.23.** *For all multi-indices $\alpha, \alpha', \beta, \theta \in \mathbb{N}^m$ we have*

(i) $|D_x^\alpha D_{x'}^{\alpha'} D_\xi^\beta c(x,x',\xi)| \leq C\langle\xi\rangle^{\mu-|\beta|}\langle x\rangle^{\nu-|\alpha+\alpha'|}\langle x-x'\rangle^{|\alpha+\alpha'|}$;

(ii) $|D_x^\alpha D_{x'}^{\alpha'} D_\xi^\beta c_{\theta,t}(x,x',\xi)| \leq C\langle\xi\rangle^{\mu-|\beta+\theta|}\langle x\rangle^{\nu-|\theta+\alpha+\alpha'|}\langle x-x'\rangle^{|\theta+\alpha+\alpha'|}$;

(iii) $c_\theta(x,x,\xi) \in S^{\mu-|\theta|;\nu-|\theta|}(\mathbb{R}^m \times \mathbb{R}^m)$;

(iv) $|D_x^\alpha D_{x'}^{\alpha'} D_\xi^\beta c_N(x,x',\xi)| \leq C\langle\xi\rangle^{\mu-N-1-|\beta|}\langle x\rangle^{\nu-N-1-|\alpha+\alpha'|}\langle x-x'\rangle^{N+1+|\alpha+\alpha'|}$,

*for all $(x,x',\xi) \in \mathbb{R}^{3m}$, with some $C > 0$ (depending on the multi-indices).*

**Remark 1.2.24.** *There is a constant $C > 0$ such that*

$$|D_x^\alpha D_{x'}^{\alpha'} D_\xi^\beta c(x,x',\xi)| \leq C\langle\xi\rangle^{\mu-|\beta|}\langle x\rangle^{\nu}$$

*for all $(x,x',\xi) \in \mathbb{R}^{3m}$.*

In fact, by virtue of Corollary 1.2.23 (i) it is enough to observe that $\langle x-x'\rangle \leq C\langle x\rangle$ on the set $\{(x,x') \in \mathbb{R}^{2m} : c(x,x',\xi) \neq 0\}$, for some $C > 0$.

*Proof of Theorem* 1.2.21. Let us set $b_N(x,\xi) = \sum_{|\theta|\leq N} c_\theta(x,x,\xi)$ and form

$$g(x,x',\xi) := c_N(x,x',\xi) - (b - b_N)(x,\xi).$$

Then $\mathrm{Op}(c) - \mathrm{Op}(b) = \mathrm{Op}(g)$.

By virtue of (1.2.26) and (1.2.30) we have $\mathrm{Op}(c) - \mathrm{Op}(r_N) = \mathrm{Op}(b_N)$, which implies

$$\begin{aligned}\mathrm{Op}(c) - \mathrm{Op}(b) &= \mathrm{Op}(c - b_N) - \mathrm{Op}(b - b_N) = \mathrm{Op}(r_N) - \mathrm{Op}(b - b_N) \\ &= \mathrm{Op}(c_N) - \mathrm{Op}(b - b_N).\end{aligned}$$

The function $g(x,x',\xi)$ satisfies the estimates

$$|D_x^\alpha D_{x'}^{\alpha'} D_\xi^\beta g(x,x',\xi)| \leq C\langle\xi\rangle^{\mu-N-|\beta|}\langle x\rangle^{\nu-N-|\alpha+\alpha'|}\langle x-x'\rangle^{N+|\alpha+\alpha'|}. \quad (1.2.31)$$

For $c_N(x,x',\xi)$ this is contained in Corollary 1.2.23 (iv), while $b(x,\xi) - b_N(x,\xi) \in S^{\mu-N;\nu-N}(\mathbb{R}^m \times \mathbb{R}^m)$ also satisfies such estimates. To complete the proof of Theorem 1.2.21 it is enough to show that $\mathrm{Op}(g) \in L^{-\infty;-\infty}(\mathbb{R}^m)$, i.e., that this operator has an integral kernel $k(x,x') \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m)$. The kernel has the form

$$k(x,x') = \int e^{i(x-x')\xi} g(x,x',\xi) đ\xi.$$

By virtue of the estimates (1.2.31) where $N$ is arbitrary we have

$$D_x^\alpha D_{x'}^{\alpha'} k(x,x') = \int D_x^\alpha D_{x'}^{\alpha'} \{e^{i(x-x')\xi} g(x,x',\xi)\} đ\xi \tag{1.2.32}$$

with admitted differentiation under the integral sign. Next observe that for arbitrary natural $M$ and $\alpha, \alpha' \in \mathbb{N}^m$ we have

$$\sup |\langle x\rangle^M \langle x-x'\rangle^M D_x^\alpha D_{x'}^{\alpha'} k(x,x')| < \infty, \tag{1.2.33}$$

where sup is taken over all $x, x' \in \mathbb{R}^m$. In fact, inserting $\langle x-x'\rangle^{-2L}(1-\Delta_\xi)^L e^{i(x-x')\xi} = e^{i(x-x')\xi}$ in (1.2.32) for arbitrary $L$, integrating by parts and using (1.2.31) gives us immediately (1.2.33), for $N$ large enough. The estimate (1.2.33) implies $k(x,x') \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m)$. In fact, it is enough to observe that

$$1 + |x| + |x'| \le c\langle x\rangle\langle x-x'\rangle$$

for some $c > 0$.

□

**Lemma 1.2.25.** *Let* $d(x,x',\xi) := \big(1-\omega(x,x')\big)a(x,\xi)$ *for* $a(x,\xi) \in S_0^{\mu;\nu}(\Gamma \times \mathbb{R}^m)$ *and* $\omega(x,x')$ *supported near the diagonal as in Lemma* 1.2.8. *Then* $\mathrm{Op}(d) \in L^{-\infty;-\infty}(\Gamma)$.

*Proof.* We apply Taylor's formula on $d(x,x',\xi)$ with respect to $x'$

$$d(x,x',\xi) = \sum_{|\alpha|\le N} \frac{(-i)^{|\alpha|}}{\alpha!} D_{x'}^\alpha d(x,x',\xi)\Big|_{x'=x} (x-x')^\alpha + r_N(x,x',\xi) = r_N(x,x',\xi),$$

where

$$r_N(x,x',\xi) = (N+1) \sum_{|\alpha|=N+1} \frac{(-i)^{N+1}}{\alpha!} r_\alpha(x,x',\xi)(x-x')^\alpha$$

and

$$r_\alpha(x,x',\xi) = \int_0^1 (1-t)^N D_{x'}^\alpha d\big(x, x+t(x'-x), \xi\big) dt.$$

Now if we set

$$d_N(x,x',\xi) := \sum_{|\alpha|=N+1} \frac{N+1}{\alpha!} \partial_\xi^\alpha r_\alpha(x,x',\xi), N \in \mathbb{N},$$

we get that $\mathrm{Op}(d) = \mathrm{Op}(d_N)$. This implies that $\mathrm{Op}(d)$ has a kernel of the form

$$k(x,x') = \int e^{i(x-x')\xi} d_N(x,x',\xi) đ\xi.$$

For $d_N$ we have the estimate

$$|D_x^\alpha D_{x'}^{\alpha'} D_\xi^\beta d_N(x,x',\xi)| \le C\langle\xi\rangle^{\mu-N-1-|\beta|}\langle x\rangle^{\nu-N-1-|\alpha+\alpha'|}\langle x-x'\rangle^{N+1+|\alpha+\alpha'|},$$

which can be proved in a similar manner as (1.2.31). It is now easy to see, as in the proof of Theorem 1.2.21, that $k(x,x') \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m)$. □

*Proof of Theorem* 1.2.19. The formula (1.2.22) together with Lemma 1.2.25 and relation (1.2.21) reduces the assertion to the operator $\widetilde{A}_1 = \mathrm{Op}(\tilde{a}_1)$. We then have $A_1 := \varphi_* \widetilde{A}_1 = \mathrm{Op}(c)$, cf. the formula (1.2.24). The expression (1.2.25) for $c(x, x', \xi)$ shows us that

$$c(x, x', \xi) \in S_0^{\mu;\nu,0}(\Gamma \times \Gamma \times \mathbb{R}^m),$$

provided that $\omega(\tilde{x}, \tilde{x}')$ is chosen in a suitable manner. Now Theorem 1.2.21 gives us $A_1 = \mathrm{Op}(b) \bmod L^{-\infty;-\infty}(\mathbb{R}^m)$ for $b(x,\xi) \sim \sum_\theta c_\theta(x,x,\xi)$. By virtue of formula (1.2.27) the asymptotic sum can be carried out in $S_0^{\mu;\nu}(\Gamma \times \mathbb{R}^m)$. Thus $\mathrm{Op}(b) \in L_0^{\mu;\nu}(\Gamma)$. Moreover, (1.2.26) together with (1.2.28) and (1.2.30) gives us

$$\mathrm{Op}(c) = \mathrm{Op}(b) \bmod L_0^{-\infty;-\infty}(\Gamma),$$

where, as we saw, $\mathrm{Op}(b) \in L_0^{\mu;\nu}(\Gamma)$, and hence, $A_1 \in L_0^{\mu;\nu}(\Gamma)$. □

**Theorem 1.2.26.** *The operator push forward induces an isomorphism*

$$\varphi_* : L_{0,\mathrm{cl}}^{\mu;\nu}(\widetilde{\Gamma}) \to L_{0,\mathrm{cl}}^{\mu;\nu}(\Gamma)$$

*for every* $\mu, \nu \in \mathbb{R}$.

The proof will be given in Subsection 1.2.4 below, after some additional material on classical symbols and operators.

### 1.2.4 Classical symbols and operators with exit property

In this section we deepen the material on classical symbols and operators in the pseudo-differential calculus with exit property. First observe the following relations.

**Example 1.2.27.** *Let us take functions* $p_\psi, p_\mathrm{e}, p_{\psi,\mathrm{e}}$ *of the form*

$$\begin{aligned} p_\psi(x,\xi) &:= f_{(\mu)}(\xi)\ g(x), \\ p_\mathrm{e}(x,\xi) &:= f(\xi)\ g_{(\nu)}(x), \\ p_{\psi,\mathrm{e}}(x,\xi) &:= f_{(\mu)}(\xi)\ g_{(\nu)}(x) \end{aligned}$$

*for arbitrary* $f(\xi) \in S_\mathrm{cl}^\mu(\mathbb{R}_\xi^m), g(x) \in S_\mathrm{cl}^\nu(\mathbb{R}_x^m)$, *where* $f_{(\mu)}$ *and* $g_{(\nu)}$ *are the respective homogeneous principal parts of* $f$ *and* $g$. *Then the relation* (1.2.13) *is satisfied, and we can form the function* $p(x,\xi)$ *by* (1.2.14).

**Remark 1.2.28.** *An* $a(x,\xi) \in S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ *belongs to* $S_{\mathrm{cl}_{\xi;x}}^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ *if and only if there exist elements*

$$p_{\psi,j}(x,\xi) \in S^{(\mu-j)}(\mathbb{R}_\xi^m \setminus \{0\}) \hat{\otimes}_\pi S_\mathrm{cl}^{\nu-j}(\mathbb{R}_x^m), \tag{1.2.34}$$

$$p_{\mathrm{e},j}(x,\xi) \in S_\mathrm{cl}^{\mu-j}(\mathbb{R}_\xi^m) \hat{\otimes}_\pi S^{(\nu-j)}(\mathbb{R}_x^m \setminus \{0\}), \tag{1.2.35}$$

$$p_{\psi,\mathrm{e},j}(x,\xi) \in S^{(\mu-j)}(\mathbb{R}_\xi^m \setminus \{0\}) \hat{\otimes}_\pi S^{(\nu-j)}(\mathbb{R}_x^m \setminus \{0\}) \tag{1.2.36}$$

*$j \in \mathbb{N}$, such that if we form $p_j$ analogously as* (1.2.14) *in terms of* (1.2.34), (1.2.35), *and* (1.2.36), *then we have*

$$a(x,\xi) - \sum_{j=0}^{N} p_j(x,\xi) \in S^{\mu-(N+1);\nu-(N+1)}(\mathbb{R}^m \times \mathbb{R}^m).$$

*for every $N \in \mathbb{N}$. In this case we also have*

$$p_{\psi,0} = \sigma_\psi(a),\ p_{\mathrm{e},0} = \sigma_{\mathrm{e}}(a),\ p_{\psi,\mathrm{e},0} = \sigma_{\psi,\mathrm{e}}(a),$$

*and*

$$p_{\psi,j} = \sigma_\psi\left(a - \sum_{l=0}^{j-1} p_l\right),\ p_{\mathrm{e},j} = \sigma_{\mathrm{e}}\left(a - \sum_{l=0}^{j-1} p_l\right),\ p_{\psi,\mathrm{e},j} = \sigma_{\psi,\mathrm{e}}\left(a - \sum_{l=0}^{j-1} p_l\right).$$

Let $S_\xi^{[\mu]}(\mathbb{R}^m \times \mathbb{R}^m)$ be the space of all $a(x,\xi) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m)$ such that

$$a(x,\lambda\xi) = \lambda^\mu a(x,\xi) \text{ for all } \lambda \geq 1, x \in \mathbb{R}^m, |\xi| \geq c$$

for a $c = c(a) > 0$. In an analogous manner we define $S_x^{[\nu]}(\mathbb{R}^m \times \mathbb{R}^m)$ by interchanging the role of $x$ and $\xi$. Set

$$S^{\mu;[\nu]} := S^{\mu;\nu} \cap S_x^{[\nu]}, \quad S^{[\mu];\nu} := S^{\mu;\nu} \cap S_\xi^{[\mu]}.$$

Let $S_{\mathrm{cl}_\xi}^{\mu;[\nu]}(\mathbb{R}^m \times \mathbb{R}^m)$ denote the subspace of all $a(x,\xi) \in S^{\mu;[\nu]}(\mathbb{R}^m \times \mathbb{R}^m)$ such that there are elements $a_k(x,\xi) \in S_\xi^{[\mu-k]} \cap S_x^{[\nu]}, k \in \mathbb{N}$, satisfying

$$a(x,\xi) - \sum_{k=0}^{N} a_k(x,\xi) \in S^{\mu-(N+1);\nu}(\mathbb{R}^m \times \mathbb{R}^m)$$

for all $N \in \mathbb{N}$. Moreover, define $S_{\mathrm{cl}_\xi}^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ to be the subspace of all $a(x,\xi) \in S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ such that there are elements $a_k(x,\xi) \in S^{[\mu-k];\nu}(\mathbb{R}^m \times \mathbb{R}^m), k \in \mathbb{N}$, satisfying

$$a(x,\xi) - \sum_{k=0}^{N} a_k(x,\xi) \in S^{\mu-(N+1);\nu}(\mathbb{R}^m \times \mathbb{R}^m)$$

for all $N \in \mathbb{N}$. By interchanging the role of $x$ and $\xi$ we obtain analogously the spaces $S_{\mathrm{cl}_x}^{[\mu];\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ and $S_{\mathrm{cl}_x}^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$.

The following theorem gives us an equivalent definition of the spaces $S_{\mathrm{cl}_{\xi;x}}^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$, cf. (1.2.9), which we will use for proving Theorem 1.2.26. A proof of Theorem 1.2.29 can be found in [48].

**Theorem 1.2.29.** *$S_{\mathrm{cl}_{\xi;x}}^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ is the space of all $a(x,\xi) \in S^{\mu;\nu}(\mathbb{R}^m \times \mathbb{R}^m)$ for which there are sequences*

$$a_k(x,\xi) \in S_{\mathrm{cl}_x}^{[\mu-k];\nu}, k \in \mathbb{N} \quad \text{and} \quad b_l(x,\xi) \in S_{\mathrm{cl}_\xi}^{\mu;[\nu-l]}, l \in \mathbb{N} \tag{1.2.37}$$

*such that*

$$a(x,\xi) - \sum_{k=0}^{N} a_k(x,\xi) \in S^{\mu-(N+1);\nu}_{\mathrm{cl}_x}, \quad a(x,\xi) - \sum_{l=0}^{N} b_l(x,\xi) \in S^{\mu;\nu-(N+1)}_{\mathrm{cl}_\xi} \tag{1.2.38}$$

*for all* $N \in \mathbb{N}$.

*Proof of Theorem* 1.2.26. After the proof of Theorem 1.2.19 in the preceding section it is enough to verify that the symbol $b(x,\xi)$ is classical which follows when we show that the summands $b_\theta(x,\xi)$ in the asymptotic expansion of $b(x,\xi)$ are classical. Let us mainly consider $c(x,x,\xi)$; the arguments for $c_\theta(x,x,\xi), \theta \neq 0$, are similar.
Since the function $\tilde{a}(\tilde{x},\tilde{\xi})$ is classical in $\tilde{\xi}$ and $\tilde{x}$ we can find homogeneous components

$$\tilde{a}_k(\tilde{x},\tilde{\xi}) \in S^{[\mu-k];\nu}_{\mathrm{cl}_{\tilde{x}}}(\mathbb{R}^m \times \mathbb{R}^m),\ \tilde{b}_l(\tilde{x},\tilde{\xi}) \in S^{\mu;[\nu-l]}_{\mathrm{cl}_{\tilde{\xi}}}(\mathbb{R}^m \times \mathbb{R}^m), \tag{1.2.39}$$

$k, j \in \mathbb{N}$ such that

$$\tilde{a}(\tilde{x},\tilde{\xi}) - \sum_{k=0}^{N} \tilde{a}_k(\tilde{x},\tilde{\xi}) \in S^{\mu-(N+1);\nu}_{\mathrm{cl}_{\tilde{x}}}(\mathbb{R}^m \times \mathbb{R}^m)$$

$$\tilde{a}(\tilde{x},\tilde{\xi}) - \sum_{l=0}^{M} \tilde{b}_l(\tilde{x},\tilde{\xi}) \in S^{\mu;\nu-(M+1)}_{\mathrm{cl}_{\tilde{\xi}}}(\mathbb{R}^m \times \mathbb{R}^m)$$

for every $N, M \in \mathbb{N}$. Apart from the factor $\omega(\tilde{x},\tilde{x}')$, which drops out when we restrict ourselves to the diagonal, we can substitute the variables $(\tilde{x},\tilde{\xi})$ as

$$\tilde{x} = \chi(x), \quad \tilde{\xi} = {}^{\mathrm{t}}d\chi^{-1}(x)\xi,$$

in the homogeneous components (1.2.39) (using that ${}^{\mathrm{t}}D^{-1}(x,x) = {}^{\mathrm{t}}d\chi^{-1}(x)$, see Lemma 1.2.20), and obtain a resulting homogeneous components in $x$ and $\xi$ of $c(x,x,\xi) = \tilde{a}\big(\chi(x), {}^{\mathrm{t}}d\chi^{-1}(x)\xi\big)$.
For every $\tilde{a}_k$ of the first sequence in (1.2.39) there is a sequence

$$\tilde{a}_k^j \in S^{[\mu-k]}_{\tilde{\xi}} \cap S^{[\nu-j]}_{\tilde{x}}, j \in \mathbb{N},$$

such that

$$\tilde{a}_k(\tilde{x},\tilde{\xi}) - \sum_{j=0}^{L} \tilde{a}_k^j(\tilde{x},\tilde{\xi}) \in S^{\mu-k;\nu-(L+1)}(\mathbb{R}^m \times \mathbb{R}^m), L \in \mathbb{N}.$$

Setting $a_k^j(x,\xi) := \tilde{a}_k^j\big(\chi(x), {}^{\mathrm{t}}d\chi^{-1}(x)\xi\big)$, and because of the homogeneity properties of $\chi$, it is easy to see that

$$a_k^j(x,\xi) \in S^{[\mu-k]}_{\xi} \cap S^{[\nu-j]}_{x},$$

for every $k, j \in \mathbb{N}$, and that

$$a_k(x,\xi) - \sum_{j=0}^{L} a_k^j(x,\xi) \in S^{\mu-k;\nu-(L+1)}(\mathbb{R}^m \times \mathbb{R}^m), L \in \mathbb{N},$$

where $a_k(x,\xi) := \tilde{a}_k\big(\chi(x), {}^{\mathrm{t}}d\chi^{-1}(x)\xi\big)$. This means that $a_k(x,\xi) \in S^{[\mu-k];\nu}_{\mathrm{cl}_x}(\mathbb{R}^m \times \mathbb{R}^m)$, $k \in \mathbb{N}$, from which it follows that

$$c(x,x,\xi) - \sum_{k=0}^{N} a_k(x,\xi) \in S^{\mu-(N+1);\nu}_{\mathrm{cl}_x}(\mathbb{R}^m \times \mathbb{R}^m),$$

for every $N \in \mathbb{N}$. For the second sequence of (1.2.39) we can argue in a similar way. This completes the proof. □

**Remark 1.2.30.** *The invariance under $\varphi_* : \widetilde{A} \to A$ of the principal symbols is as follows:*

$$\sigma_\psi(\widetilde{A})(\tilde{x},\tilde{\xi}) = \sigma_\psi(A)(x,\xi), \quad (\tilde{x},\tilde{\xi}) \in \mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\});$$
$$\sigma_{\mathrm{e}}(\widetilde{A})(\tilde{x},\tilde{\xi}) = \sigma_{\mathrm{e}}(A)(x,\xi), \quad (\tilde{x},\tilde{\xi}) \in (\mathbb{R}^m \setminus \{0\}) \times \mathbb{R}^m;$$
$$\sigma_{\psi,\mathrm{e}}(\widetilde{A})(\tilde{x},\tilde{\xi}) = \sigma_{\psi,\mathrm{e}}(A)(x,\xi), \quad (\tilde{x},\tilde{\xi}) \in (\mathbb{R}^m \setminus \{0\}) \times (\mathbb{R}^m \setminus \{0\}).$$

### 1.2.5 Exit calculus on manifolds

In this subsection we recall some elements of the exit calculus on a manifold $M$ with conical exits to infinity.
On the manifold $M$ we fix the partition of unity $\{\varphi_0, \varphi_1, \dots, \varphi_N\}$, $N \in \mathbb{N}$, in such a way that $\varphi_0 \in C_0^\infty(\operatorname{int} M_0)$ and $\varphi_j \in C_0^\infty(V_j)$, where $V_j$ is of the form $V_j = e^{-1}\big((R_+,\infty) \times U_j\big) \cong \Gamma_j$, $j = 1,\dots,N$, for some conical sets $\Gamma_j \subseteq \mathbb{R}^m$ and $\{U_1,\dots,U_N\}$ forms an open covering of $X$ by coordinate neighbourhoods (cf. the notations at the beginning of Subsection 1.2.1). Furthermore, we choose another system $\{\psi_0,\psi_1,\dots,\psi_N\}$ of functions $\psi_0 \in C_0^\infty(\operatorname{int} M_0)$ and $\psi_j \in C^\infty(V_j)$ that are for $j \geq 1$ pull backs of elements in $C^\infty(\Gamma_j)$ that vanish near $\partial\Gamma_j$ and are homogeneous of order zero for large $|x|$, such that $\varphi_j\psi_j = \varphi_j$ for all $j = 0,\dots,N$.

**Definition 1.2.31.** $L^{\mu;\nu}_{(\mathrm{cl})}(M)$ *for $\mu,\nu \in \mathbb{R}$ is defined to be the space of all operators $A = \sum_{j=0}^N \varphi_j A_j \psi_j + C$ for arbitrary $A_0 \in L^{\mu}_{(\mathrm{cl})}(\mathrm{int} M_0)$, $A_j \in L^{\mu;\nu}_{0,(\mathrm{cl})}(V_j)$, $j = 1,\dots,N$, $C \in L^{-\infty;-\infty}(M)$.*

Let us now define complete symbols for $L^{\mu;\nu}(M)$ and principal symbols for $L^{\mu;\nu}_{\mathrm{cl}}(M)$. The manifold $M$ is written as a union $\bigcup_{j=0}^N V_j$. We choose coordinate neighbourhoods $\{O_1,\dots,O_L\}$ on $V_0 = \mathrm{int} M_0$ such that $\overline{V}_0 \subset \bigcup_{l=0}^L O_j$ and charts $\kappa_l : O_l \to \Omega_l$ for open sets $\Omega_l \subseteq \mathbb{R}^p$, $l = 1,\dots,L$, ($p$ is the dimension of the manifold $M_0$), and $\chi_j : V_j \to \Gamma_j$, $j = 1,\dots,N$. Then $\{O_1,\dots,O_L,V_1,\dots,V_N\}$ is an open covering of $M$ with the charts $\{\kappa_l\}_{l=1,\dots,L}$, and $\{\chi_j\}_{j=1,\dots,N}$. Consider the system of operators $(\kappa_l)_* A|_{O_l} \in L^\mu(\Omega_l)$, $(\chi_j)_* A|_{V_j} \in L^{\mu;\nu}(\Gamma_j)$ for all $l$ and $j$. Using the isomorphisms

$$L^\mu(\Omega_l)/L^{-\infty}(\Omega_l) \cong S^\mu(\Omega_l \times \mathbb{R}^p)/S^{-\infty}(\Omega_l \times \mathbb{R}^p),$$
$$L^{\mu;\nu}(\Gamma_j)/L^{-\infty;-\infty}(\Gamma_j) \cong S^{\mu;\nu}(\Gamma_j \times \mathbb{R}^m)/S^{-\infty;-\infty}(\Gamma_j \times \mathbb{R}^m)$$

we find corresponding local symbols $\sigma_{\Omega_l}(A)$ and $\sigma_{\Gamma_j}(A)$ that are unique modulo symbols of order $-\infty$ and $(-\infty;-\infty)$, respectively. We set

$$\boldsymbol{\sigma}(A) = \{\sigma_{\Omega_1}(A), \dots, \sigma_{\Omega_L}(A), \sigma_{\Gamma_1}(A), \dots, \sigma_{\Gamma_N}(A)\} \tag{1.2.40}$$

and call $\boldsymbol{\sigma}(A)$ a complete symbol of $A$. Furthermore, for $A \in L^{\mu;\nu}_{\mathrm{cl}}(M)$ the components of (1.2.40) are classical, and we obtain the system of homogeneous principal components

$$\sigma_\psi(A) = \{\sigma_{\psi,\Omega_1}(A), \dots, \sigma_{\psi,\Omega_L}(A), \sigma_{\psi,\Gamma_1}(A), \dots, \sigma_{\psi,\Gamma_N}(A)\}$$

that represents an invariantly defined function $\sigma_\psi(A) \in C^\infty(T^*M \setminus 0)$, homogeneous of order $\mu$. In addition, we have the principal exit symbols

$$\sigma_{\mathrm{e}}(A) = \{\sigma_{\mathrm{e},\Gamma_j}(A)\}_{j=1,\dots,N}, \quad \sigma_{\psi,\mathrm{e}}(A) = \{\sigma_{\psi,\mathrm{e},\Gamma_j}(A)\}_{j=1,\dots,N}$$

which are defined on $S^{\mu;\nu}_{\mathrm{cl}_{\xi;x}}(\Gamma_j \times \mathbb{R}^m)$ in an analogous manner as for $\mathbb{R}^m$. In the classical case we set

$$\boldsymbol{\sigma}(A) = \{\sigma_\psi(A), \sigma_{\mathrm{e}}(A), \sigma_{\psi,\mathrm{e}}(A)\}.$$

**Theorem 1.2.32.** *$A \in L^{\mu;\nu}(M)$, $B \in L^{\tilde{\mu};\tilde{\nu}}(M)$ implies $AB \in L^{\mu+\tilde{\mu};\nu+\tilde{\nu}}(M)$ and we have $\boldsymbol{\sigma}(AB) = \boldsymbol{\sigma}(A)\#\boldsymbol{\sigma}(B)$, where $\#$ means the Leibniz product of the local representatives in corresponding local coordinates. In the classical case we obtain $\boldsymbol{\sigma}(AB) = \boldsymbol{\sigma}(A)\boldsymbol{\sigma}(B)$ with component-wise multiplication.*

**Definition 1.2.33.** *An operator $A \in L^{\mu;\nu}(M)$ is called elliptic (of order $(\mu;\nu)$) if $\boldsymbol{\sigma}(A)$ is elliptic in the sense that there is a tuple of symbols*

$$\boldsymbol{p} = \{p_{\Omega_1}, \dots, p_{\Omega_L}, p_{\Gamma_1}, \dots, p_{\Gamma_N}\},$$

*$p_{\Omega_l} \in S^{-\mu}(\Omega_l \times \mathbb{R}^p)$, $p_{\Gamma_j} \in S^{-\mu;-\nu}(\Gamma_j \times \mathbb{R}^m)$, such that the components of $\boldsymbol{\sigma}(A)\#\boldsymbol{p}$ are equal to 1 modulo symbols of order $-\infty$ and $(-\infty;-\infty)$, respectively.*

**Theorem 1.2.34.** *For an operator $A \in L^{\mu;\nu}(M)$ the following conditions are equivalent:*

(i) *$A$ is elliptic (of order $(\mu,\nu)$);*

(ii)
$$A : H^{s;g}(M) \to H^{s-\mu;g-\nu}(M) \tag{1.2.41}$$
*is a Fredholm operator for any choice of $s = s_0, g = g_0 \in \mathbb{R}$.*

If $A$ is elliptic there is a parametrix $P \in L^{-\mu;-\nu}(M)$ in the sense that $1 - PA, 1 - AP \in L^{-\infty;-\infty}(M)$ holds, and (1.2.41) is a Fredholm operator for all $s, g \in \mathbb{R}$.

**Remark 1.2.35.** *Let $A \in L^{\mu;\nu}$ be elliptic. Then $Au = f \in H^{s;g}(M)$, $u \in H^{-\infty;-\infty}(M)$ implies $u \in H^{s+\mu;g+\nu}(M)$ for all $s, g \in \mathbb{R}$.*
*Moreover, $\ker A$ is a finite-dimensional subspace $V$ of $\mathcal{S}(M)$, and there is another finite-dimensional subspace $W \subset \mathcal{S}(M)$ such that $\operatorname{im} A \cap W = \{0\}$ and $\operatorname{im} A + W = H^{s;g}(M)$ when $A$ considered as an operator on $H^{s-\mu;g-\nu}(M)$. Thus $\ker A, \operatorname{coker} A$ and $\operatorname{ind} A$ are independent of the Sobolev smoothness and of weights at infinity.*

# Chapter 2

# Operators on infinite cylinders

## 2.1 The behaviour of push forwards from cylinders to cones

### 2.1.1 Characterisation of push forwards

Let $X$ be a closed compact manifold, say of dimension $n$, and $U \subset X$ a coordinate neighbourhood such that there is a diffeomorphism $\chi_1 : U \to B$ to the open unit ball $B$ in $\mathbb{R}^n$. Let $[\cdot] : \mathbb{R} \to \mathbb{R}_+$ be a positive function such that $[r] = |r|$ for $|r| \geq$const$> 0$ and set

$$\Gamma := \{(r, \tilde{x}) \in \mathbb{R}_+ \times \mathbb{R}^n : r \in \mathbb{R}_+, \tilde{x} = [r]x, x \in B\}, \tag{2.1.1}$$

and consider the diffeomorphism

$$\beta : \mathbb{R}_+ \times U \to \Gamma,\ \beta\big(r, \chi_1^{-1}(x)\big) = (r, [r]x). \tag{2.1.2}$$

Set

$$\mathcal{S}(X^\wedge) := \big\{u \in C^\infty(X^\wedge) : \big(1 - \omega(r)\big)u \in \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)\big\}$$

for some cut-off function $\omega$. This is a Fréchet space in a natural way. Moreover, define $\mathcal{S}(X^\wedge \times X^\wedge) := \mathcal{S}(X^\wedge)\hat{\otimes}_\pi \mathcal{S}(X^\wedge)$.

As a moderate generalisation of the definitions in the preceding section we define a class of pseudo-differential operators on $X^\wedge$, now with the parameter $\eta \in \mathbb{R}^q$, which will play later on the role of the edge-covariable. It can be easily verified that, for $q = 0$, the following definition is compatible with Definition 1.2.31.

**Definition 2.1.1.** *The space* $L^{\mu;\nu}(X^\wedge; \mathbb{R}^q)$ *is defined to be the set of all* $A(\eta) \in L^\mu(X^\wedge; \mathbb{R}^q)$ *with the following properties:*

(i) *For every* $\varphi, \psi \in C^\infty(X)$ *with* $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$ *we have*

$$\varphi A(\eta)\psi \in L^{-\infty;-\infty}(X^\wedge, \mathbb{R}^q) := \mathcal{S}\left(\mathbb{R}^q, L^{-\infty;-\infty}(X^\wedge)\right);$$

(ii) *for every* $\varphi, \psi \in C^\infty(X)$ *supported in the same coordinate neighbourhood of* $X$ *the push forward of* $(1-\omega)\varphi A(\eta)\psi(1-\omega')$ *under* $\beta$ *for arbitrary cut-off functions* $\omega(r), \omega'(r)$ *is equal to* $\operatorname{Op}(a)(\eta)$ *for some* $a(r, \tilde{x}, \rho, \tilde{\xi}, \eta) \in S^{\mu;\nu}(\mathbb{R}^{1+n} \times \mathbb{R}^{1+n} \times \mathbb{R}^q)$, *cf. Definition* 1.2.4 (i) (*the parameter* $\eta$ *is considered as a covariable*).

To motivate Definition 2.1.1 consider a differential operator on $X^\wedge \times \Omega \ni (r, x, y)$ of the form

$$A = r^{-\mu} \sum_{j+|\alpha|\leq\mu} a_{j\alpha}(r, y)(-r\partial_r)^j (rD_y)^\alpha$$

with coefficients $a_{j\alpha} \in C^\infty\big(\overline{\mathbb{R}}_+ \times \Omega, \operatorname{Diff}^{\mu-(j+|\alpha|)}(X)\big)$, $X$ closed compact manifold, $\Omega \subseteq \mathbb{R}^q$ open. Such an operator is called edge-degenerate. Here $\operatorname{Diff}^\nu(X)$ denotes the space of all differential operators of order $\nu \in \mathbb{R}$ on $X$ with smooth coefficients.
We are interested in the behaviour of the so-called homogeneous principal edge symbol

$$\sigma_\wedge(A)(y, \eta) := r^{-\mu} \sum_{j+|\alpha|\leq\mu} a_{j\alpha}(0, y)(-r\partial_r)^j (r\eta)^\alpha \tag{2.1.3}$$

which is a family of differential operators on $X^\wedge$. The dependence on $y \in \Omega$ will be ignored in this section. For every fixed $\eta \in \mathbb{R}^q$ the operator $\sigma_\wedge(A)(\eta)$ is of Fuchs type close to $r = 0$. To employ (2.1.3) as an edge symbol we need $\sigma_\wedge(A)(\eta)$ for $\eta \neq 0$ on the infinite stretched cone $X^\wedge$, including $r \to \infty$. The structures for $r \to 0$ and $r \to \infty$ cannot be reduced to each other by the transformation $r \to r^{-1}$. Therefore, the properties for $r \to \infty$ are discussed separately, and we just encounter an aspect of operators on a manifold with conical exits to infinity. Let us identify a coordinate neighbourhood on $X$ with the open unit ball $B \subset \mathbb{R}^n_x$; then we obtain

$$\sigma_\wedge(A)(\eta) = r^{-\mu} \sum_{j+|\gamma|+|\alpha|\leq\mu} b_{j\gamma\alpha}(x) D_x^\gamma (-r\partial_r)^j (r\eta)^\alpha, \tag{2.1.4}$$

$b_{j\gamma\alpha} \in C^\infty(B)$. Transforming $\mathbb{R}_+ \times B \ni (r, x)$ to the conical set $\Gamma := \{(r, \tilde{x}) \in \mathbb{R}^{1+n} : \tilde{x}/r = x \in B\}$, $\tilde{x} := (\tilde{x}_1, \ldots, \tilde{x}_n)$, the operator (2.1.4) in the coordinates $(r, \tilde{x}) \in \Gamma$ takes the form

$$\sigma_\wedge(A)(\eta) = r^{-\mu} \sum_{j+|\gamma|+|\alpha|\leq\mu} r^{j+|\gamma|+|\alpha|} \tilde{b}_{j\gamma\alpha}(\tilde{x}) D_r^j D_{\tilde{x}}^\gamma \eta^\alpha \tag{2.1.5}$$

for certain $\tilde{b}_{j\gamma\alpha} \in C^\infty(\Gamma)$ only depending on $\tilde{x}/r$. For the parameter-dependent homogeneous principal symbol of (2.1.4) with the covariables $(\rho, \xi)$ and parameter $\eta \in \mathbb{R}^q$ we have

$$\sigma_\psi\left(\sigma_\wedge(A)\right)(r, x, \rho, \xi, \eta) = r^{-\mu} \sum_{j+|\gamma|+|\alpha|=\mu} b_{j\gamma\alpha}(x)(-ir\rho)^j \xi^\gamma (r\eta)^\alpha$$

and for (2.1.5)

$$\sigma_\psi\left(\sigma_\wedge(A)\right)(\tilde{x},\rho,\tilde{\xi},\eta)=\sum_{j+|\gamma|+|\alpha|=\mu}\tilde{b}_{j\gamma\alpha}(\tilde{x})\rho^j\tilde{\xi}^\gamma\eta^\alpha$$

where $\tilde{b}_{j\gamma\alpha}(\tilde{x})=b_{j\gamma\alpha}(x)$ for $\tilde{x}/r=x$.

**Remark 2.1.2.** (i) *For every fixed $\eta\neq 0$ the operator* (2.1.5) *belongs to $L^{\mu;0}(\Gamma)$;*

(ii) *the ellipticity of* (2.1.4) *in the sense*

$$\tilde{\sigma}_\psi(\sigma_\wedge(A))(r,x,\tilde{\rho},\xi,\tilde{\eta}):=\sum_{j+|\gamma|+|\alpha|=\mu}b_{j\gamma\alpha}(x)(-i\tilde{\rho})^j\xi^\gamma\tilde{\eta}^\alpha\neq 0$$

*for all $(\tilde{\rho},\xi,\tilde{\eta})\in\mathbb{R}^{1+n+q}\setminus\{0\}$ entails the exit ellipticity of* (2.1.5)*, for every fixed $\eta\neq 0$, i.e.,*

$$\begin{aligned}\sigma_\psi(\sigma_\wedge(A))(\tilde{x},\rho,\tilde{\xi})&=\sum_{j+|\gamma|=\mu}\tilde{b}_{j\gamma 0}(\tilde{x})\rho^j\tilde{\xi}^\gamma\neq 0\text{ for all }(\rho,\tilde{\xi})\in\mathbb{R}^{1+n}\setminus\{0\}\\ \sigma_{\mathrm{e}}(\sigma_\wedge(A))(\tilde{x},\rho,\tilde{\xi})&=\sum_{j+|\gamma|+|\alpha|=\mu}\tilde{b}_{j\gamma\alpha}(\tilde{x})\rho^j\tilde{\xi}^\gamma\eta^\alpha\neq 0\text{ for all }(\rho,\tilde{\xi})\in\mathbb{R}^{1+n},\\ \sigma_{\psi,\mathrm{e}}(\sigma_\wedge(A))(\tilde{x},\rho,\tilde{\xi})&=\sigma_\psi\sigma_\wedge(A)(\tilde{x},\rho,\tilde{\xi})\neq 0\text{ for all }(\rho,\tilde{\xi})\in\mathbb{R}^{1+n}\setminus\{0\},\end{aligned}$$

*for all $(r,\tilde{x})\in\Gamma$.*

To study operators on $X^\wedge$ for $r\to\infty$ it is convenient to ignore specific effects for $r\to 0$ and to consider operators on $X^\asymp=\mathbb{R}\times X$ (now with two conical exits $r\to\pm\infty$). After that we can localise the results again to $r>R$ for some $R>0$ by a multiplication by $1-\sigma(r)$ for a cut-off function $\sigma(r)$.

In our case for $\sigma_\wedge(A)(\eta)$ there is an element

$$\tilde{p}(r,\tilde{\rho},\tilde{\eta})\in C^\infty\big(\mathbb{R},L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)\tag{2.1.6}$$

such that, when we set

$$p(r,\rho,\eta):=\tilde{p}(r,[r]\rho,[r]\eta),\tag{2.1.7}$$

we have

$$\big(1-\sigma(r)\big)\sigma_\wedge(A)(\eta)=r^{-\mu}\mathrm{Op}_r(p)(\eta).\tag{2.1.8}$$

In fact, $r^\mu\sigma_\wedge(A)(\eta)$ is of the form $\mathrm{Op}_r\big(\tilde{p}_1(r\rho,r\eta)\big)$ for some $\tilde{p}_1(\tilde{\rho},\tilde{\eta})\in L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$, and we obtain (2.1.8) when we first fix the function $r\to[r]$ such that $[r]=|r|$ for $|r|\geq R$ and then choose $\sigma(r)$ in such a way that $1-\sigma(r)$ vanishes for $r\leq R$ (including the negative $r$ half-axis). Then it suffices to set

$$\tilde{p}(r,\tilde{\rho},\tilde{\eta})=\big(1-\sigma(r)\big)\tilde{p}_1(\tilde{\rho},\tilde{\eta}).$$

Let $L^\mu_{(\mathrm{cl})}\big(X;\mathbb{R}\times(\mathbb{R}^q\setminus\{0\})\big)$ denote the set of (classical) parameter-dependent families of operators in $L^\mu_{(\mathrm{cl})}(X)$ with parameters $(\rho,\eta)\in\mathbb{R}\times(\mathbb{R}^q\setminus\{0\})$. The definition is analogous to the case of parameters in $\mathbb{R} \times \mathbb{R}^q$, but for completeness we formulate it in detail. If $(U_1, \dots, U_N)$ is an open covering of $X$ by coordinate neighbourhoods, $(\varphi_1, \dots, \varphi_N)$ a subordinate partition of unity, $(\psi_1, \dots, \psi_N)$ a system of functions $\psi_j \in C_0^\infty(U_j)$ such that $\psi_j \equiv 1$ on $\operatorname{supp} \varphi_j$, $j = 1, \dots, N$, and $\chi_j : U_j \to \Sigma$ charts, $\Sigma \subseteq \mathbb{R}^n$ open, then every $A(\rho, \eta) \in L^\mu_{(\mathrm{cl})}\big(X; \mathbb{R} \times (\mathbb{R}^q \setminus \{0\})\big)$ has the form

$$A(\rho,\eta) = \sum_{j=1}^{N} \varphi_j \left\{ (\chi_j^{-1})_* \mathrm{Op}_x(a_j)(\rho,\eta) \right\} \psi_j + C(\rho,\eta). \tag{2.1.9}$$

Here $a_j(x, x', \rho, \xi, \eta)$ is an element of $S^\mu\big(\Sigma \times \Sigma \times \mathbb{R}^{1+n}_{\rho,\xi} \times (\mathbb{R}^q_\eta \setminus \{0\})\big)$. The space $S^\mu\big(U \times \mathbb{R}^{1+n} \times (\mathbb{R}^q \setminus \{0\})\big)$, $U \subseteq \mathbb{R}^d$ open, consists of all

$$p(x,\rho,\xi,\eta) \in C^\infty\big(U \times \mathbb{R}^{n+1} \times (\mathbb{R}^q \setminus \{0\})\big)$$

such that

$$\sup \langle \rho, \xi, \eta \rangle^{-\mu + |\beta|} |D_x^\alpha D^\beta_{\rho,\xi,\eta} p(x,\rho,\xi,\eta)| \tag{2.1.10}$$

is finite for every $\alpha \in \mathbb{N}^d$, $\beta \in \mathbb{N}^{1+n+q}$, where sup is taken over all $x \in K$, $(\rho, \xi, \eta) \in \mathbb{R}^{1+q} \times (\mathbb{R}^q \setminus \{0\})$, $|\eta| \geq h$, for every compact $K \subset U \subset \mathbb{R}^d$ and $h > 0$. The subspace $S^\mu_{\mathrm{cl}}\big(U \times \mathbb{R}^{1+n} \times (\mathbb{R}^q \setminus \{0\}\big)$ is defined similarly as in the case of covariables $(\rho, \eta) \in \mathbb{R}^{1+q}$. The operator $C(\rho, \eta)$ in (2.1.9) is smoothing in the sense $C(\rho, \eta) \in \mathcal{S}\big(\mathbb{R} \times (\mathbb{R}^q \setminus \{0\}), L^{-\infty}(X)\big)$; here $\mathcal{S}\big(\mathbb{R} \times (\mathbb{R}^q \setminus \{0\})\big)$ is the Schwartz space over $\mathbb{R} \times (\mathbb{R}^q \setminus \{0\})$, defined to be the set of all $f(\rho, \eta) \in C^\infty\big(\mathbb{R} \times (\mathbb{R}^q \setminus \{0\})\big)$ such that $\chi(\eta) f(\rho, \eta) \in \mathcal{S}(\mathbb{R}^{1+q})$ for every excision function $\chi(\eta)$ in $\mathbb{R}^q$.
The spaces $L^\mu_{(\mathrm{cl})}\big(X; \mathbb{R} \times (\mathbb{R}^q \setminus \{0\})\big)$ are Fréchet in a natural way.

The idea of the following considerations is to construct operators in the exit calculus on $X^\asymp$ in terms of operator functions $\mathrm{Op}_r(a)(\eta)$ with symbols

$$a(r, \rho, \eta) = \tilde{a}(r, [r]\rho, [r]\eta),$$

$\tilde{a}(r, \tilde{\rho}, \tilde{\eta}) \in C^\infty\Big(\mathbb{R}, L^\mu_{(\mathrm{cl})}\big(X; \mathbb{R}_{\tilde{\rho}} \times (\mathbb{R}^q_{\tilde{\eta}} \setminus \{0\})\big)\Big)$, with a suitable dependence on $r$ at infinity. Let us study the behaviour of our operator families under push forward from coordinates $(r, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ to $(r, \tilde{x}) \in \mathbb{R}_+ \times \mathbb{R}^n$ via

$$\chi : (r, x) \to (r, \tilde{x}), \ \ \tilde{x} := [r]x. \tag{2.1.11}$$

To this end we follow the lines of Schrohe and Schulze [34]. We choose the function $r \to [r]$ in such a way that $[r] = r$ for $r \geq 1$. The main aspect concerns $r \to \infty$. In order to avoid a cutting out factor for a neighbourhood of $r = 0$ for simplicity we take symbols that are smooth up to $r = 0$ and employ (2.1.11) as a diffeomorphism

$$\chi : \overline{\mathbb{R}}_+ \times \mathbb{R}^n \to \overline{\mathbb{R}}_+ \times \mathbb{R}^n.$$

The Jacobi matrix of its inverse $\chi^{-1}(r, \tilde{x}) = (r, \tilde{x}/[r])$ has the form

$$J(r, \tilde{x}) = \begin{pmatrix} 1 & 0 \\ -(\partial_r [r])\tilde{x}/[r]^2 & [r]^{-1} I \end{pmatrix} \tag{2.1.12}$$

with $I$ being the $(n \times n)$ identity matrix. Let us define

$$F(r,\tilde{x},r',\tilde{x}') = \int_0^1 J\big(r+\vartheta(r'-r), \tilde{x}+\vartheta(\tilde{x}'-\tilde{x})\big)d\vartheta, \quad (r',\tilde{x}') \in \mathbb{R}_+ \times \mathbb{R}^n$$

which is an invertible $(n+1)\times(n+1)$-matrix. For $r, r' \geq 1$ from (2.1.12) we obtain

$$F(r,\tilde{x},r',\tilde{x}') = \begin{pmatrix} 1 & 0 \\ -N(r,\tilde{x},r',\tilde{x}') & M(r,r')I \end{pmatrix}$$

for

$$M(r,r') := \begin{cases} \frac{\log r - \log r'}{r-r'}, & r \neq r' \\ \frac{1}{r}, & r = r' \end{cases}, \quad N(r,\tilde{x},r',\tilde{x}') = \int_0^1 \frac{\tilde{x}+\vartheta(\tilde{x}'-\tilde{x})}{\big(r+\vartheta(r'-r)\big)^2} d\vartheta.$$

Let us show that

$$N(r,\tilde{x},r',\tilde{x}') = \begin{cases} \frac{1}{r'-r}\left(\frac{r'\tilde{x}-r\tilde{x}'}{rr'} - \frac{\tilde{x}-\tilde{x}'}{r-r'}(\log r - \log r')\right), & r \neq r' \\ \frac{\tilde{x}+\tilde{x}'}{2r^2}, & r = r' \end{cases} \tag{2.1.13}$$

In fact, for the case $r \neq r'$, we can write $\frac{\tilde{x}+\vartheta(\tilde{x}'-\tilde{x})}{(r+\vartheta(r'-r))^2} = \frac{a}{(r+\vartheta(r'-r))^2} + \frac{b}{r+\vartheta(r'-r)}$ with coefficients $a, b$, determined by $a + b\big(r+\vartheta(r'-r)\big) = \tilde{x}+\vartheta(\tilde{x}'-\tilde{x})$ which holds when $a + br = \tilde{x}$, $b(r'-r) = \tilde{x}'-\tilde{x}$, i.e., $a = \frac{r'\tilde{x}-r\tilde{x}'}{r'-r}$, $b = \frac{\tilde{x}'-\tilde{x}}{r'-r}$. Thus we obtain

$$\begin{aligned} \int_0^1 \frac{\tilde{x}+\vartheta(\tilde{x}'-\tilde{x})}{\big(r+\vartheta(r'-r)\big)^2} d\vartheta &= a\int_0^1 \frac{d\vartheta}{\big(r+\vartheta(r'-r)\big)^2} + b\int_0^1 \frac{d\vartheta}{r+\vartheta(r'-r)} \\ &= a\left[-\frac{1}{(r'-r)\big(r+\vartheta(r'-r)\big)}\right]_0^1 + b\left[\frac{1}{r'-r}\log\big(r+\vartheta(r'-r)\big)\right]_0^1 \\ &= a\frac{1}{r'r} + b\frac{\log r' - \log r}{r'-r}. \end{aligned}$$

Otherwise, for $r = r'$, we simply obtain $N$ by the integral $r^{-2}\int_0^1 \big(\tilde{x}+\vartheta(\tilde{x}'-\tilde{x})\big)d\vartheta$ which yields altogether (2.1.13).
Observe that $\det F$ depends only on $(r,r')$. For $r, r' \geq 1$ it follows that

$$^{\mathrm{t}}F^{-1} = \begin{pmatrix} 1 & RN \\ 0 & RI \end{pmatrix} \tag{2.1.14}$$

for $R(r,r') := M^{-1}(r,r')$; this gives us $\det {}^{\mathrm{t}}F^{-1} = R^n(r,r')$.

**Lemma 2.1.3.** *Let* $\tilde{p} \in C_{\mathrm{b}}^\infty\Big(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, S^\mu\big(\mathbb{R}^n_x \times \mathbb{R}^n_{x'} \times \mathbb{R}_{\tilde{\rho}} \times \mathbb{R}^n_{\xi} \times (\mathbb{R}^q_{\tilde{\eta}} \setminus \{0\})\big)\Big)$, *and set*

$$p(r,r',x,x',\rho,\xi,\eta) := \tilde{p}(r,r',x,x',r\rho,\xi,r\eta)$$

*for any fixed* $\eta \in \mathbb{R}^q \setminus \{0\}$. *Then we have*

$$\chi_*\left(\mathrm{Op}_{r,x}(p)\right) = \mathrm{Op}_{r,\tilde{x}}(q)$$

*for* $q(r,r',\tilde x,\tilde x',\rho,\xi,\eta)=\tilde p\left(r,r',\frac{\tilde x}{[r]},\frac{\tilde x'}{[r']},{}^{\mathrm t}F^{-1}\begin{pmatrix} r\rho\\ \xi\end{pmatrix},r\eta\right)|\det{}^{\mathrm t}F^{-1}(r,r')|[r']^{-n}$. *For* $r,r'\ge 1$ *we obtain the simpler expression*

$$\tilde p\left(r,r',\frac{\tilde x}{r},\frac{\tilde x'}{r'},r\rho+RN\xi,R\xi,r\eta\right)|R^n(r,r')||r'|^{-n}.$$

*Proof.* First recall that if $V$ is a Fréchet space with the countable semi-norm system $(\pi_\iota)_{\iota\in\mathbb N}$ then

$$C_{\mathrm b}^\infty(\overline{\mathbb R}_+\times\overline{\mathbb R}_+,V):=\Big\{f(r,r')\in C^\infty(\overline{\mathbb R}_+\times\overline{\mathbb R}_+,V):\\ \sup_{r,r'\in\overline{\mathbb R}_+\times\overline{\mathbb R}_+}\pi_\iota\big(\partial_r^k\partial_{r'}^{k'}f(r,r')\big)<\infty \text{ for every } \iota,k,k'\in\mathbb N\Big\}.$$

For $u(r,\tilde x)\in C_0^\infty(\mathbb R_+\times\mathbb R^n),(\chi^*u)(r,x)=u(r,[r]x)$ we have

$$\begin{aligned}&\chi_*\big(\mathrm{Op}_{r,x}(p)\big)u(r,\tilde x)=\mathrm{Op}(p)(\chi^*u)\big(\chi^{-1}(r,\tilde x)\big) \qquad (2.1.15)\\ &=\iiiint e^{i(r-r')+i(\frac{\tilde x}{[r]}-x')\xi}\tilde p(r,r',\frac{\tilde x}{[r]},x',r\rho,\xi,r\eta)u(r',[r']x')dr'dx'đ\rho đ\xi\\ &=\iiiint e^{i(r-r')\rho+i(\frac{\tilde x}{[r]}-\frac{\tilde x'}{[r']})\xi}\tilde p(r,r',\frac{\tilde x}{[r]},\frac{\tilde x'}{[r']},r\rho,\xi,r\eta)u(r',\tilde x')[r']^{-n}dr'd\tilde x'đ\rho đ\xi;\end{aligned}$$

here we have substituted $x'=\dfrac{\tilde x'}{[r']}$. We now observe that

$$\begin{aligned}(r-r')\rho+\left(\frac{\tilde x}{[r]}-\frac{\tilde x'}{[r']}\right)\xi&=\big(\chi^{-1}(r,\tilde x)-\chi^{-1}(r',\tilde x')\big)\begin{pmatrix}\rho\\ \xi\end{pmatrix}\\ &=\big((r,\tilde x)-(r',\tilde x')\big)\,{}^{\mathrm t}F(r,\tilde x,r',\tilde x')\begin{pmatrix}\rho\\ \xi\end{pmatrix}.\end{aligned}$$

For the right hand side of (2.1.15) we thus obtain

$$\iiiint e^{i(r-r')\rho+i(\tilde x-\tilde x')\xi}\tilde p\left(r,r',\frac{\tilde x}{[r]},\frac{\tilde x'}{[r']},{}^{\mathrm t}F^{-1}\begin{pmatrix}r\rho\\ \xi\end{pmatrix},r\eta\right)\\ u(r',\tilde x')[r']^{-n}|\det{}^{\mathrm t}F^{-1}|dr'd\tilde x đ\rho đ\xi.$$

The assertion then follows from (2.1.14). □

**Proposition 2.1.4.** *Let* $p(r,r',x,x',\rho,\xi,\eta)$ *be as in Lemma* 2.1.3 *and fix again* $\eta\in\mathbb R^q\setminus\{0\}$. *Then if* $\varphi(x),\varphi'(x)\in C_0^\infty(\mathbb R_x^n)$ *have disjoint supports and* $\omega(r),\tilde\omega(r)$ *are cut-off functions (equal to* 1 *for* $r\le 1$*), the push forward*

$$\chi_*\left(\big(1-\omega(r)\big)\varphi(x)\mathrm{Op}_{r,x}(p)\varphi'(x)\big(1-\omega'(r)\big)\right)$$

*is an integral operator with kernel in* $\mathcal S\big(\mathbb R\times\mathbb R,C^\infty(\mathbb R^n\times\mathbb R^n)\big)$.

*Proof.* Let us set $w(r,r',x,x') := \big(1-\omega(r)\big)\varphi(x)\varphi'(x')\big(1-\omega'(r')\big)$. For $u(r,x) \in C_0^\infty(\mathbb{R}^{1+n})$ and $L \in \mathbb{N}$ with $\mu - L < -n-1$ we have

$$\begin{aligned} Cu(r,x) &:= (1-\omega)\varphi \mathrm{Op}(p)\varphi'(1-\omega')u(r,x) \\ &= \iiiint e^{i(r-r')\rho + i(x-x')\xi} w(r,r',x,x')\tilde{p}(r,r',x,x',r\rho,\xi,r\eta)u(r',x')dr'dx'đ\rho đ\xi \\ &= \iiiint e^{i(r-r')\rho+i(x-x')\xi}|x-x'|^{-2L}w(r,x,r',x')\Delta_\xi^L\tilde{p}(r,r',x,x',r\rho,\xi,r\eta) \\ &\qquad\qquad u(r',x')dr'dx'đ\rho đ\xi. \end{aligned}$$

The integral exists, and we obtain $Cu(r,x) = \iint K(r,r',x,x')u(r',x')dr'dx'$ for

$$K(r,r',x,x') = r^{-1}\iint e^{i(\frac{r-r'}{r})\rho + i(x-x')\xi} w_L(r,r',x,x')(\Delta_\xi^L \tilde{p})(r,r',x,x',\rho,\xi,r\eta)đ\rho đ\xi,$$

$w_L(r,r',x,x') := |x-x'|^{-2L}w(r,x,r',x')$. Using the identity

$$b^{-2N}(r,r',x,x')\Delta_{\rho,\xi}^N e^{i\left(\frac{r-r'}{r}\right)\rho+i(x-x')\xi} = e^{i\left(\frac{r-r'}{r}\right)\rho+i(x-x')\xi}, N \in \mathbb{N} \tag{2.1.16}$$

for $b(r,r',x,x') := \left(\left|\frac{r-r'}{r}\right|^2 + |x-x'|^2\right)^{1/2}$, and applying once again integration by parts it follows that

$$\begin{aligned} Cu(r,x) = r^{-1}\iiiint & e^{i(\frac{r-r'}{r})\rho+i(x-x')\xi} w_L(r,r',x,x') \\ & b^{-2N}(r,r',x,x')\Delta_{\rho,\xi}^N\Delta_\xi^L\tilde{p}(r,r',x,x',\rho,\xi,r\eta)u(r',x')dr'dxđ\rho đ\xi. \end{aligned}$$

Thus $C$ is an integral operator with kernel

$$\begin{aligned} K(r,r',x,x') = r^{-1}\iint & e^{i(\frac{r-r'}{r})\rho+i(x-x')\xi} b^{-2N}(r,r',x,x') \\ & w_L(r,r',x,x')\Delta_{\rho,\xi}^N\Delta_\xi^L\tilde{p}(r,r',x,x',\rho,\xi,r\eta)đ\rho đ\xi. \end{aligned}$$

From the symbolic estimates $|\Delta_{\rho,\xi}^N\Delta_\xi^L\tilde{p}(r,r',x,x',\rho,\xi,r\eta)| \le c\langle\rho,\xi,r\eta\rangle^{\mu-2N-2L}$ (that are only relevant for $r \ge 1$, $r' \ge 1$ and for $(x,x') \in \operatorname{supp}\varphi \times \operatorname{supp}\varphi'$) we obtain

$$|K(r,r',x,x')| \le cr^{-1}|w_L(r,r',x,x')|b^{-2N}(r,r',x,x')\iint\langle\rho,\xi,r\eta\rangle^{\mu-2N-2L}đ\rho đ\xi.$$

For $\eta \in \mathbb{R}^q \setminus \{0\}$ fixed we employ the estimate

$$\langle\rho,\xi,r\eta\rangle^{\mu-2N-2L} = \langle\rho,\xi,r\eta\rangle^{\mu-L}\langle\rho,\xi,r\eta\rangle^{-L-2N} \le c\langle\rho,\xi\rangle^{\mu-L}\langle r\eta\rangle^{-L-2N} \tag{2.1.17}$$

$r \ge 1$, for some $c > 0$. For $b^{-2N}$ we have $b^{-2N}(r,r',x,x')\langle r\eta\rangle^{-2N} \le c|r-r'|^{-2N}$ which gives us for the kernel the estimate $|K(r,r',x,x')| \le cr^{-1}|r-r'|^{-2N}\langle r\eta\rangle^{-L}$. Choosing $N$ so large that $L = 2N$ it follows that

$$|K(r,r',x,x')| \le c\langle r\rangle^{-N}\langle r'\rangle^{-N}$$

for $r \ge 1$. The strong decrease in $x, x'$ is clear anyway, since $\varphi$ and $\varphi'$ are of compact support. For the derivatives of the kernel we can argue in an analogous manner. The push forward of $K$ under $\chi$ gives us $[r]^{-n}K(r,r',\tilde{x}/[r],\tilde{x}'/[r']) \in \mathcal{S}\big(\mathbb{R}\times\mathbb{R}, C^\infty(\mathbb{R}^n \times \mathbb{R}^n)\big)$. □

**Proposition 2.1.5.** *Let $p, \tilde{p}, \omega, \omega'$ be defined as in Lemma* 2.1.3 *and let $\varphi, \varphi' \in C_0^\infty(\mathbb{R}^n)$. Moreover, let $\psi(r,r') \in C_{\mathrm{b}}^\infty(\mathbb{R}\times\mathbb{R})$, where $\psi(r,r') = 0$ for $|r-r'| \le \varepsilon$ and $\psi(r,r') = 1$ for $|r-r'| \ge 2\varepsilon$ for some $\varepsilon > 0$. Then $\chi_*\big((1-\omega)\varphi \mathrm{Op}_{r,x}(p)\psi\varphi'(1-\omega')\big)$ is an integral operator with kernel in $\mathcal{S}\big(\mathbb{R}\times\mathbb{R}, C^\infty(\mathbb{R}^n\times\mathbb{R}^n)\big)$.*

*Proof.* Let $u(r,x) \in C_0^\infty(\mathbb{R}^{1+n})$ and define $w(r,r',x,x')$ as in the preceding proof. Then we have

$$\begin{aligned} &Cu(r,x) := (1-\omega)\varphi(\mathrm{Op}(p))\psi\varphi'(1-\omega')u(r,x) \\ &= \iiiint e^{i(r-r')\rho + i(x-x')\xi} w(r,r',x,x')\psi(r,r') \\ &\qquad\qquad \tilde{p}(r,r',x,x',r\rho,\xi,r\eta)u(r',x')dr'dx'đ\rho đ\xi. \end{aligned}$$

Substituting $r\rho = \rho'$ and then going back to $\rho$ it follows that

$$\begin{aligned} Cu(r,x) = \ & r^{-1}\iiiint e^{i(\frac{r-r'}{r})\rho + i(x-x')\xi} w(r,r',x,x')\psi(r,r') \\ &\qquad \tilde{p}(r,r',x,x',\rho,\xi,r\eta)u(r',x')dr'dx'đ\rho, đ\xi. \end{aligned} \tag{2.1.18}$$

Using the identity (2.1.16) for $N$ large enough and integrating by parts we obtain

$$\begin{aligned} Cu(r,x) = r^{-1}\iiiint & e^{i(\frac{r-r'}{r})\rho + i(x-x')\xi} b^{-2N}(r,r',x,x')w(r,r',x,x')\psi(r,r') \\ &\qquad \Delta_{\rho,\xi}^N \tilde{p}(r,r',x,x',\rho,\xi,r\eta)u(r',x')dr'dx'đ\rho đ\xi. \end{aligned}$$

Thus $C$ is an operator with kernel

$$\begin{aligned} K(r,r',x,x') = r^{-1}\iint & e^{i(\frac{r-r'}{r})\rho + (x-x')\xi} b^{-2N}(r,r',x,x') \\ &\qquad w(r,r',x,x')\psi(r,r')\Delta_{\rho,\xi}^N \tilde{p}(r,r',x,x',\rho,\xi,r\eta)đ\rho đ\xi. \end{aligned}$$

Due to the symbolic estimates $|\Delta_{\rho,\xi}^N \tilde{p}(r,r',x,x',\rho,\xi,r\eta)| \le c\langle\rho,\xi,r\eta\rangle^{\mu-2N}$ the integral with respect to $\rho,\xi$ exists. Let us choose any fixed $L \in \mathbb{N}$ such that $\mu - 2L < -n-1$ and write $\mu - 2N = \mu - 2L - 2M$ for $N = M + L$. Similarly as (2.1.17) we have an estimate $\langle\rho,\xi,r\eta\rangle^{\mu-2N} \le c\langle\rho,\xi\rangle^{\mu-2L}\langle r\eta\rangle^{-2M}$ for all $r \ge 1$, $\eta \in \mathbb{R}^q \setminus \{0\}$ fixed. This gives us an estimate for the kernel $K$ of the form

$$|K(r,r',x,x')| \le \ cr^{-1}|b^{-2L}(r,r',x,x')w(r,r',x,x')||b^{-2M}(r,r',x,x')\psi(r,r')|\langle r\eta\rangle^{-2M} \tag{2.1.19}$$

There is a constant $c > 0$ such that

$$|b^{-2M}(r,r',x,x')\psi(r,r')|\langle r\eta\rangle^{-2M} \le c\langle r\rangle^{-M}\langle r'\rangle^{-M}.$$

Taking into account that $|b^{-2L}(r,r',x,x')w(r,r',x,x')|$ is at most of polynomial growth in $r,r'$ for large $r,r'$ and that $M$ independently of $L$ can be chosen as large as we want, we see that $K(r,r',x,x')$ is strongly decreasing for $r \to \infty$, $r' \to \infty$. Moreover, the support with respect to $x,x'$ is bounded because of the involved factors $\varphi(x), \varphi'(x')$. Similar considerations are valid for the derivatives of $K$ of any order. Thus $K$ belongs to $\mathcal{S}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n)$, and the push-forward under $\chi$ is as desired. □

### 2.1.2 Estimates near the diagonal

We will here restrict the variables to the set

$$W := \left\{(r, r', \tilde{x}, \tilde{x}', \rho, \xi) : r, r' \geq 1, \left|\frac{r}{r'} - 1\right| < \frac{1}{2}, \left|\frac{\tilde{x}}{r}\right| \leq C, \left|\frac{\tilde{x}'}{r'}\right| \leq C\right\}$$

for some $C > 0$. Observe that on $W$ we have

$$[r] \sim [r'] \sim [r, \tilde{x}] \sim [r', \tilde{x}']$$

(for instance, $[r] \sim [r']$ means the existence of constants $c_1, c_2 > 0$ such that $c_1[r'] \leq [r] \leq c_2[r']$ for all $r, r'$).

**Lemma 2.1.6.** *For* $R(r, r') = M^{-1}(r, r')$ *and* $S(r, r', \tilde{x}, \tilde{x}') := R(r, r')N(r, r', \tilde{x}, \tilde{x}')$ (*cf. the notation in* (2.1.14))*we have on the set* $W$

(i) $|D_r^k D_{r'}^l R(r, r')| \leq c(r^{1-k-l})$ *for some* $c > 0$*, and* $R(r, r') \geq c_0 r$ *for some* $c_0 > 0$*;*

(ii) $|D_r^k D_{r'}^l D_{\tilde{x}}^\alpha D_{\tilde{x}'}^\beta S(r, r', \tilde{x}, \tilde{x}')| \leq c(r^{-|\alpha|-|\beta|-k-l})$ *for some* $c > 0$*.*
*The left hand side vanishes for* $|\alpha| + |\beta| > 1$*.*

*On* $W$ *the function* $S$ *satisfies the* $S^{0;0,0}$*-estimates and* $R$ *the* $S^{0;1,0}$*-estimates* (*cf. the notation in Definition* 1.2.4).

*Proof.* (i) We consider the case $r \neq r'$, since otherwise the function $R$ obviously satisfies the desired properties. Let us write $R(r, r') = r\varphi(r'/r)$ for $\varphi(t) = \frac{t-1}{\log t}$. We then have $R(r, r') \leq cr$ and $R(r, r') \geq c_0 r$. For the derivatives we have also such estimates, namely, $r\partial_r R(r, r') = r\left(\varphi(t) - t\partial_t\varphi(t)\right)|_{t=r'/r} \leq cr$, and, similarly, $r'\partial_{r'} R(r, r') \leq cr$. Since $r^k D_r^k$ can be written as a linear combination of operators $(rD_r)^j$, $1 \leq j \leq k$, we obtain $r^k D_r^k (r')^l D_{r'}^l R(r, r') \leq cr$.

(ii) We have

$$|N(r, r', \tilde{x}, \tilde{x}')| \leq \max\{|\tilde{x}|, |\tilde{x}'|\} \int_0^1 \big(r + \vartheta(r' - r)\big)^{-2} d\vartheta = \max\{|\tilde{x}|, |\tilde{x}'|\}(rr')^{-1}.$$

Since $\tilde{x}/r$ and $\tilde{x}'/r$ are both bounded and $R(r, r') \leq c\min\{r, r'\}$, it follows that $|S| \leq c$. Moreover, we have

$$D_r^k D_{r'}^l N(r, r', \tilde{x}, \tilde{x}') = c_{kl} \int_0^1 (1 - \vartheta)^k \vartheta^l \frac{\tilde{x} + \vartheta(\tilde{x}' - \tilde{x})}{\big(r + \vartheta(r' - r)\big)^{2+k+l}} d\vartheta.$$

Then, analogous estimates as above show that these terms can be estimated by

$$c\max\{|\tilde{x}|, |\tilde{x}'|\} r^{-2-k-l}.$$

Now (i) together with the Leibniz rule gives us the assertion for $\alpha = \beta = 0$. If $|\alpha| + |\beta| = 1$ the integrand is equal to $(1 - \vartheta)^{k+|\alpha|}\vartheta^{l+|\beta|}\big(r + \vartheta(r' - r)\big)^{-2-k-l}$, and we can argue as before. $\square$

**Lemma 2.1.7.** *Let $S$ be as in the preceding lemma. For every fixed $\eta \neq 0$ there are constants $c, \delta > 0$ such that*

$$\delta r[\rho, \xi] \leq [r\rho + S\xi, R\xi, r\eta] \leq cr[\rho, \xi] \tag{2.1.20}$$

*on the set $W$.*

*Proof.* In this proof we again denote by $c$ different positive constants. For the first estimate of (2.1.20) we observe that if $|\rho, \xi| \leq 1$ the term in the middle of (2.1.20) is $\geq [r\eta] \geq cr \geq cr[\rho, \xi]$. Therefore, we may assume $|\rho, \xi| > 1$. Let $\gamma = \sup|S| + 1$ and consider the cases $|\rho| \leq 2\gamma|\xi|$ and $|\rho| \geq 2\gamma|\xi|$ separately. In the first case we have $1 < (|\rho|^2 + |\xi|^2)^{\frac{1}{2}} \leq \big(((2\gamma)^2 + 1)|\xi|^2\big)^{\frac{1}{2}} = \langle 2\gamma\rangle|\xi|$. For $|\rho| \leq 2\gamma|\xi|$ this implies

$$r[\rho, \xi] = r|\rho, \xi| \leq r\big((2\gamma)^2|\xi|^2 + |\xi|^2\big)^{\frac{1}{2}} = \langle 2\gamma\rangle r|\xi|. \tag{2.1.21}$$

Moreover, we have

$$[r\rho + S\xi, R\xi, r\eta] \geq c[R\xi] \geq c|R\xi| \geq cr|\xi|$$

for a suitable $c > 0$, and (2.1.21) entails

$$[r\rho + S\xi, R\xi, r\eta] \geq cr[\rho, \xi].$$

For $|\rho| \geq 2\gamma|\xi|$ we observe that

$$|r\rho + S\xi| \geq |r\rho| - |S\xi| \geq \frac{r}{2}|\rho| + \frac{1}{2}|\rho| - \gamma|\xi| \geq \frac{r}{2}|\rho| \geq cr[\rho, \xi].$$

In the last inequality we employed that

$$[\rho, \xi] = |\rho, \xi| = (|\rho|^2 + |\xi|^2)^{\frac{1}{2}} \leq \left(|\rho|^2 + \frac{1}{(2\gamma)^2}|\rho|^2\right)^{\frac{1}{2}} = \langle(2\gamma)^{-1}\rangle|\rho|.$$

Thus we have proved the estimate from below.
For the second inequality we first observe that $R \leq cr$ and hence

$$[R\xi] \leq cr[\xi] \leq cr[\rho, \xi].$$

In a similar manner we obtain $[r\rho + S\xi] \leq c([r\rho] + [S\xi]) \leq cr[\rho, \xi]$. This yields the desired estimate from above. □

**Theorem 2.1.8.** *Let $\tilde{p}(x, x', \tilde{\rho}, \xi, \tilde{\eta}) \in S^{\mu}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{1+n} \times \mathbb{R}^q)$, $\mu \in \mathbb{R}$, $\varphi(x), \varphi'(x) \in C_0^{\infty}(\mathbb{R}^n)$, and choose cut-off functions $\omega(r), \omega'(r) = 1$ for $r \leq 1$. Fix $\eta \neq 0$, and define*

$$p(r, x, x', \rho, \xi, \eta) := \tilde{p}(x, x', r\rho, \xi, r\eta).$$

*Then for the push forward under $\chi$ we have*

$$\chi_*\left(\big(1 - \omega(r)\big)\varphi(x)\mathrm{Op}_{r,x}(p)\varphi'(x)\big(1 - \omega'(r)\big)\right) = \mathrm{Op}_{r,\tilde{x}}(q)$$

*for a symbol $q(r, r', \tilde{x}, \tilde{x}', \rho, \xi) \in S^{\mu;\mu,0}(\mathbb{R}^{1+n} \times \mathbb{R}^{1+n} \times \mathbb{R}^{1+n})$. Its symbol semi-norms can be estimated by those for $\tilde{p}$.*

*Proof.* The terms $(1-\omega),(1-\omega')$ vanish for $r \le 1$. From Lemma 2.1.3 we know that the symbol $q(r,r',\tilde{x},\tilde{x}',\rho,\xi)$ of the push forward is of the form

$$\tilde{p}(\tilde{x}/r,\tilde{x}'/r',r\rho+S\xi,R\xi,r\eta)\big((1-\omega(r)\big)\big(1-\omega'(r')\big)\varphi(\tilde{x}/r)\varphi'(\tilde{x}'/r')R^n(r,r')(r')^{-n}.$$

Since $\operatorname{supp}\varphi$ and $\operatorname{supp}\varphi'$ are bounded, $\tilde{x}/r$ and $\tilde{x}'/r'$ may assumed to be bounded. In addition we may assume $|r-r'|$ to be small, since by Proposition 2.1.5 it is admitted to multiply the symbol by a function supported near $r=r'$, modulo some rapidly decreasing remainder. In fact, since $r,r' \ge 1$ we may assume $|r/r'-1| < \frac{1}{2}$. Therefore, we only need to verify the symbolic estimates on the set $W$, for $q$ vanishes on the complement. In the following let

$$V := (\tilde{x}/r,\tilde{x}'/r',r\rho+S\xi,R\xi,r\eta);$$

then we have

$$\begin{aligned}
D_{\tilde{x}_j}\tilde{p}(V) &= \big(D_{\tilde{x}_j}\tilde{p}\big)(V)r^{-1} + \big(D_\rho\tilde{p}\big)(V)\sum_{k=1}^n \partial_{\tilde{x}_j}S_k\xi_k,\\
D_r\tilde{p}(V) &= \sum_{k=1}^n\big\{\big(D_{\tilde{x}_k}\tilde{p}\big)(V)\big(-\tilde{x}_k/r^2\big)+\big(D_\rho\tilde{p}\big)(V)\big(\rho+\partial_rS_k\xi_k\big) \\
&\quad + \big(D_{\xi_k}\tilde{p}\big)(V)\partial_rR\xi_k + \big(D_{\tilde{\eta}_k}\tilde{p}\big)(V)\eta_k\big\},\\
D_\rho\tilde{p}(V) &= \big(D_\rho\tilde{p}\big)(V)r, D_{\xi_j}\tilde{p}(V) = \big(D_{\tilde{\rho}}\tilde{p}\big)(V)S_j + \big(D_{\xi_j}\tilde{p}\big)(V)R.
\end{aligned} \tag{2.1.22}$$

Here $S_k$ and $\xi_k$ denote the components of $S$ and $\xi$, respectively. The derivatives with respect to $\tilde{x}'$ and $r'$ can be easily deduced from those.
If we concentrate on $W$ then $\partial_{\tilde{x}_j}S_k\xi_k$ satisfies the estimates for an $S^{1;-1,0}$-symbol, $\rho+\partial_rS_k\xi_k$ and $\partial_rR\xi_k$ for an $S^{1;0,0}$-symbol, while $r^{-1}$ and $\tilde{x}_k/r^2$ satisfy those for an $S^{0;-1,0}$-symbol. Also $(r,\tilde{x}) \to \varphi(\tilde{x}/r)$ and $(r',\tilde{x}') \to \varphi(\tilde{x}'/r')$ are $S^0$-symbols on $W$. According to Lemma 2.1.7 and using the relations $[r,\tilde{x}] \sim [r] \sim R \sim [r'] \sim [r',\tilde{x}']$ on $W$ we can estimate the derivatives of $\tilde{p}$ as follows:

$$\begin{aligned}
&|D_r\tilde{p}(V)|, |D_{\tilde{x}_k}\tilde{p}(V)| \le c[r,\tilde{x}]^{\mu-1}[\rho,\xi]^\mu,\\
&|D_\rho\tilde{p}(V)|, |D_{\xi_j}\tilde{p}(V)| \le c[r,\tilde{x}]^{\mu}[\rho,\xi]^{\mu-1},
\end{aligned}$$

and hence we obtain

$$|D_\rho^lD_\xi^\alpha D_{\tilde{x}}^\beta D_{\tilde{x}'}^{\beta'}D_r^kD_{r'}^{k'}q(r,r',\tilde{x},\tilde{x}',\rho,\xi)| \le c[r,\tilde{x}]^{\mu-|\beta|-k}[r',\tilde{x}']^{-|\beta'|-k'}[\rho,\xi]^{\mu-|\alpha|-l},$$

provided that the total number of derivatives is $\le 1$. The form of the derivatives in (2.1.22), together with the above observations on the functions $\partial_{\tilde{x}_j}S_k\xi_k,\dots,\tilde{x}_k/r^2$ shows that the general result follows in an analogous manner. □

### 2.1.3 Global operators

Given a $C^\infty$ manifold $X$, $n=\dim X$, with a system of charts $\kappa_\iota: U_\iota \to \mathbb{R}^n$, $\iota \in I$ we consider the cylinder $\mathbb{R}\times X$ with the charts

$$1\times\kappa_\iota : \mathbb{R}\times U_\iota \to \mathbb{R}\times\mathbb{R}^n.$$

The cylinder can also be equipped with the structure of $X^{\asymp}$, a manifold with conical exits to infinity, $r \to \pm\infty$, and we define a diffeomorphism

$$\chi : \mathbb{R} \times X \to X^{\asymp} \tag{2.1.23}$$

by the local transformations $\chi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$, $\chi(r, x) = (r, [r]x)$.

**Theorem 2.1.9.** *Let $X$ be a $C^\infty$ manifold and $\widetilde{P}(\tilde{\rho}, \tilde{\eta}) \in L^\mu(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$. Then, if $\omega(r), \omega'(r)$ are cut-off functions, and $P(r, \rho) := \widetilde{P}(r\rho, r\eta)$, for any fixed $\eta \neq 0$, we have*

$$\chi_* \big(\big(1 - \omega(r)\big)\mathrm{Op}_r(P)\big(1 - \omega'(r)\big)\big) \in L^{\mu;\mu}(X^{\asymp}).$$

*Proof.* Let $(U_\iota)_{\iota \in I}$ be an open covering of $X$ by coordinate neighbourhoods, diffeomorphic to $\mathbb{R}^n$, and let $(\varphi_\iota)_{\iota \in I}$ be a subordinate partition of unity. Then we can write $\mathrm{Op}_r(P) = \sum_{\iota,\kappa \in I} \varphi_\iota \mathrm{Op}_r(P)\varphi_\kappa$, and it suffices to show the assertion for $(1 - \omega)\varphi_\iota \mathrm{Op}_r(P)\varphi_\kappa(1 - \omega')$. However, this is an immediate consequence of Theorem 2.1.8. □

It is instructive to study the other way around, i.e., to express pseudo-differential operators in $\mathbb{R}^{1+n}_{\tilde{x}}$ with smooth symbols across $\tilde{x} = 0$ in polar coordinates with respect to $\tilde{x}$ for $\tilde{x} \neq 0$. Let $M$ be a $C^\infty$ manifold and let us fix a point $v \in M$, interpreted as a conical singularity. We define $L^\mu_{\mathrm{deg}}(M)$ as the space of all operators of the form $\sigma r^{-\mu}\mathrm{Op}_r(p)\sigma' + (1 - \sigma)A_{\mathrm{int}}(1 - \sigma'') + C$, for a symbol $p(r, \rho) = \tilde{p}(r, r\rho)$, $\tilde{p}(r, \tilde{\rho}) \in C^\infty\big(\overline{\mathbb{R}}_+, L^\mu_{\mathrm{cl}}(X; \mathbb{R}_{\tilde{\rho}})\big)$, $A_{\mathrm{int}} \in L^\mu_{\mathrm{cl}}(M \setminus \{v\})$ and $C \in L^{-\infty}(M \setminus \{v\})$. $\sigma, \sigma', \sigma''$ are cut-off functions satisfying

$$\sigma'' \prec \sigma \prec \sigma'.$$

(If $f$ and $g$ are two functions such that $g \equiv 1$ on the support of $f$ then we write this symbolically as $f \prec g$.)

**Theorem 2.1.10.** *There is a canonical embedding*

$$L^\mu_{\mathrm{cl}}(M)|_{M\setminus\{v\}} \hookrightarrow L^\mu_{\mathrm{deg}}(M)$$

*for every $\mu \in \mathbb{R}$.*

*Proof.* Set $n + 1 := \dim M$, and choose a chart $\chi : U \to \mathbb{R}^{1+n}$ for a coordinate neighbourhood $U$ of the point $v$ with $\chi(v) = 0$. Due to the pseudo-locality of pseudo-differential operators it suffices to show that $\chi_*(A|_U)|_{\mathbb{R}^{1+n}\setminus\{0\}}$ belongs to $L^\mu_{\mathrm{deg}}(\mathbb{R}^{1+n})$ for some fixed operator $A \in L^\mu_{\mathrm{cl}}(M)$.
In other words, we may assume $M = \mathbb{R}^{1+n}$, $v = 0$. Modulo a rotation in $\mathbb{R}^{1+n}$ it suffices to concentrate on a cone $\Gamma \subset \mathbb{R}^{1+n}$ written as

$$\Gamma = \{\tilde{x} = (r, rx_1, \ldots, rx_n) : r \in \mathbb{R}_+,\ x \in B\}$$

for $B := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : |x| < 1\}$. We consider the diffeomorphism

$$\beta : \mathbb{R}_+ \times B \to \Gamma,\ \beta(r, x) = (r, rx),$$

and show that

$$(\beta^{-1})_* A|_\Gamma \in L^\mu_{\mathrm{deg}}(B^\Delta),$$

$B^\Delta = (\overline{\mathbb{R}}_+ \times B)/(\{0\} \times B)$.

The reason why we take the simpler transformation $\beta$ instead of passing to the polar coordinates is that they are both diffeomorphic to each other. In fact, for $C = \Gamma \cap S^n$ there is a diffeomorphism $\kappa : \overline{\mathbb{R}}_+ \times C \to \overline{\mathbb{R}}_+ \times B$ induced by the orthogonal projection of $\mathbb{R}^{1+n} \ni (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n)$ to the hyperplane $\{\tilde{x}_0 = 1, (\tilde{x}_1, \ldots, \tilde{x}_n) \in \mathbb{R}^n\}$. The transformation $\kappa$, which is homogeneous of degree 1 in $r$, induces an isomorphism $C^\Delta \to B^\Delta$ in the category of the respective manifolds with conical singularities. This gives rise to an isomorphism $L^\mu_{\mathrm{deg}}(C^\Delta) \to L^\mu_{\mathrm{deg}}(B^\Delta)$, and we may identify $C^\Delta$ with $\Gamma \cup \{0\}$.

Modulo a remainder in $L^{-\infty}(\Gamma)$ (which can be ignored) we assume

$$(A|_\Gamma u)(\tilde{x}) = \mathrm{Op}_{\tilde{x}}(a)u(\tilde{x}) = \iint e^{i(\tilde{x}-\tilde{x}')\tilde{\xi}} a(\tilde{x}, \tilde{\xi}) u(\tilde{x}') d\tilde{x}' đ\xi$$

for an $a(\tilde{x}, \tilde{\xi}) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^{1+n} \times \mathbb{R}^{1+n})$. By definition, the points $\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n) \in \Gamma$ and $(r, x_1, \ldots, x_n) \in \mathbb{R}_+ \times B$ are related via $\beta^{-1}(\tilde{x}) = (r, \tilde{x}_1/r, \ldots, \tilde{x}_n/r) = (r, x)$, $x = (x_1, \ldots, x_n)$. Then for the associated covariables $(\rho, \xi)$ we have

$$\left((\beta^{-1})_* A|_\Gamma\right) f(r, x) = \mathrm{Op}(b) f(r, x),$$

(modulo a smoothing operator that is again negligible) where

$$b(r, x, \rho, \xi)\big|_{(r,x)=\beta^{-1}(\tilde{x})} \sim \sum_{\alpha \in \mathbb{N}^{1+n}} \frac{1}{\alpha!} \left(\partial^\alpha_{\tilde{\xi}} a\right) \left(\tilde{x}, {}^{\mathrm{t}}d\beta^{-1}(\tilde{x}) \begin{pmatrix} \rho \\ \xi \end{pmatrix}\right) \Pi_\alpha \left(\tilde{x}, \begin{pmatrix} \rho \\ \xi \end{pmatrix}\right) \tag{2.1.24}$$

for

$$\Pi_\alpha \left(\tilde{x}, \begin{pmatrix} \rho \\ \xi \end{pmatrix}\right) = D^\alpha_{\tilde{z}} e^{i\delta(\tilde{x}, \tilde{z})\binom{\rho}{\xi}}\Big|_{\tilde{z}=\tilde{x}} \tag{2.1.25}$$

and $\delta(\tilde{x}, \tilde{z}) = \beta^{-1}(\tilde{z}) - \beta^{-1}(\tilde{x}) - d\beta^{-1}(\tilde{x})(\tilde{z} - \tilde{x})$, $\tilde{x} = \tilde{x}(r, x) = (r, rx)$. Let us simplify the notations in the asymptotic sum of (2.1.24). We have $\beta^{-1}(\tilde{x}) = (r(\tilde{x}), x_1(\tilde{x}), \ldots, x_n(\tilde{x}))$ for

$$r(\tilde{x}) = \tilde{x}_0, x_j(\tilde{x}) = \tilde{x}_j/\tilde{x}_0 \text{ for } j = 1, \ldots, n.$$

Writing for the moment $r(\tilde{x}) = x_0(\tilde{x})$, we have $d\beta^{-1}(\tilde{x}) = \left(\partial_{\tilde{x}_j} x_i(\tilde{x})\right)_{\substack{i=0,\ldots,n \\ j=0,\ldots,n}}$ and hence we can write $d\beta^{-1}(\tilde{x})$ in the variables $(r, x) \in \mathbb{R}_+ \times B$ as follows

$$d\beta^{-1}(\tilde{x}) = r^{-1} \begin{pmatrix} r & 0 \\ -{}^{\mathrm{t}}x & 1 \end{pmatrix},$$

i.e., ${}^{\mathrm{t}}d\beta^{-1}(\tilde{x}) = r^{-1} \begin{pmatrix} r & -x \\ 0 & 1 \end{pmatrix}$, where 1 stands for $\mathrm{id}_{\mathbb{R}^n}$, and $\tilde{x}(r, x) = (r, rx)$. Thus the summands in the asymptotic expansion (2.1.24) have the form

$$\frac{1}{\alpha!} \left(\partial^\alpha_{\tilde{\xi}} a\right) \left(\tilde{x}(r, x), r^{-1} \begin{pmatrix} r\rho - x\xi \\ \xi \end{pmatrix}\right) \Pi_\alpha \left(\tilde{x}(r, x), \begin{pmatrix} \rho \\ \xi \end{pmatrix}\right). \tag{2.1.26}$$

In order to characterise the function $\Pi_\alpha\left((r,rx),\begin{pmatrix}\rho\\ \xi\end{pmatrix}\right)$ we first note that

$$\delta(\tilde{x},\tilde{z}) = (t-r, z-x) - r^{-1}\begin{pmatrix} r & 0\\ -{}^{\mathrm{t}}x & 1\end{pmatrix}\begin{pmatrix} t-r\\ tz-rx\end{pmatrix},$$

i,e.,

$$\delta(\tilde{x},\tilde{z})\begin{pmatrix}\rho\\ \xi\end{pmatrix} = (1-r^{-1}t)(z-x)\xi.$$

Let $\alpha = (\alpha', \alpha'')$ for $\alpha' \in \mathbb{N}$ and $\alpha'' \in \mathbb{N}^n$. Using $\partial_{\tilde{z}_j} = t^{-1}\partial_{z_j}$, $j = 1,\dots,n$, it follows that $D_{\tilde{z}}^{\alpha} = D_{\tilde{z}_0}^{\alpha'} t^{-|\alpha''|} D_z^{\alpha''}$, $z = (z_1,\dots,z_n)$. Moreover, we have $D_{\tilde{z}_0}^{\alpha'} = D_t^{\alpha'}$, $\alpha' \in \mathbb{N}$. This yields

$$\begin{aligned} D_{\tilde{z}}^{\alpha} e^{i(1-r^{-1}t)(z-x)\xi} &= D_t^{\alpha'} t^{-|\alpha''|} D_z^{\alpha''} e^{i(1-r^{-1}t)(z-x)\xi}\\ &= D_t^{\alpha'}\left\{ t^{-|\alpha''|}(1-r^{-1}t)^{|\alpha''|}\xi^{\alpha''} e^{i(1-r^{-1}t)(z-x)\xi}\right\}\\ &= D_t^{\alpha'}\left\{ t^{-|\alpha''|}(1-r^{-1}t)^{|\alpha''|}\right\}\xi^{\alpha''} e^{i(1-r^{-1}t)(z-x)\xi} \end{aligned}$$

since the differentiation of the exponent produces vanishing terms after the substitution $t = r$, $z = x$. We can write $D_t^{\alpha'}\left\{t^{-|\alpha''|}(1-r^{-1}t)^{|\alpha''|}\right\}$ as a sum of the form

$$\sum_{k\le\alpha'}\begin{pmatrix}\alpha'\\ k\end{pmatrix} D_t^{\alpha'-k} t^{-|\alpha''|} D_t^k (1-r^{-1}t)^{|\alpha''|} = \sum_{k\le\alpha'} c_{\alpha,k} t^{-|\alpha''|-\alpha'+k} r^{-k}(1-r^{-1}t)^{|\alpha''|-k}$$

for some constants $c_{\alpha,k}$. After substituting $t = r$, $z = x$ we see that the last sum vanishes unless $|\alpha''| \le \alpha'$, then it contains only one term, namely, the one corresponding to $k = |\alpha''|$. In other words, we can ignore all other terms and write

$$D_t^{\alpha'}\left\{t^{-|\alpha''|}(1-r^{-1}t)^{|\alpha''|}\right\} = c_\alpha t^{-\alpha'} r^{-|\alpha''|}$$

for some constant $c_\alpha$. We substitute in (2.1.25) and obtain

$$\begin{aligned} \Pi_\alpha\left((r,rx),\begin{pmatrix}\rho\\ \xi\end{pmatrix}\right) &= D_{\tilde{z}}^{\alpha} e^{i(1-r^{-1}t)(z-x)\xi}\Big|_{\substack{t=r\\ z=x}}\\ &= c_\alpha t^{-\alpha'} r^{-|\alpha''|}\xi^{\alpha''} e^{i(1-r^{-1}t)(z-x)\xi}\Big|_{\substack{t=r\\ z=x}}\\ &= c_\alpha r^{-|\alpha|}\xi^{\alpha''}, \end{aligned} \tag{2.1.27}$$

where for $\alpha''$ we have the inequality $|\alpha|/2 = (\alpha' + |\alpha''|)/2 \ge |\alpha''|$, since $\Pi_\alpha \equiv 0$ otherwise.

In order to show that

$$b(r,x,\rho,\xi) = r^{-\mu}\tilde{p}(r,x,r\rho,\xi) \mod S^{-\infty}(\mathbb{R}_+ \times B \times \mathbb{R}^{1+n}_{\rho,\xi})$$

for a function $\tilde{p}(r,x,\tilde{\rho},\xi) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}\times B\times\mathbb{R}^{1+n}_{\tilde{\rho},\xi})$, it suffices to know that the homogeneous components $b_{(\mu-j)}(r,x,\rho,\xi)$, $j \in \mathbb{N}$, have the form

$$b_{(\mu-j)}(r,x,\rho,\xi) = r^{-\mu}\tilde{p}_{(\mu-j)}(r,x,r\rho,\xi)$$

for certain $\tilde{p}_{(\mu-j)}(r, x, \tilde{\rho}, \xi) \in C^\infty\big(\overline{\mathbb{R}}_+ \times B, S^{(\mu-j)}(\mathbb{R}^{1+n}_{\tilde{\rho},\xi} \setminus \{0\})\big)$. In fact, the asymptotic sum $\tilde{p}(r, x, \tilde{\rho}, \xi) \sim \sum_{j=0}^{\infty} \vartheta(\tilde{\rho}, \xi)\tilde{p}_{(\mu-j)}(r, x, \tilde{\rho}, \xi)$ (for some excision function $\vartheta(\tilde{\rho}, \xi)$) can be carried out in $S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times B \times \mathbb{R}^{1+n}_{\tilde{\rho},\xi})$. Thus it remains to show that $\tilde{p}_{(\mu-j)}$ are as desired. We first consider the homogeneous components of

$$\left(\partial_\xi^\alpha a\right)\left(\tilde{x}(r, x), r^{-1}(r\rho - x\xi), r^{-1}\xi\right)$$

of homogeneity order $\mu-|\alpha|-j$, $j \in \mathbb{N}$, that are of the form $f_j(r, x, r^{-1}(r\rho-x\xi), r^{-1}\xi)$ for some $f_j(r, x, \tau, \xi) \in C^\infty\big(\overline{\mathbb{R}}_+ \times B, S^{(\mu-|\alpha|-j)}(\mathbb{R}^{1+n}_{\xi} \setminus \{0\})\big)$, $j \in \mathbb{N}$. This means, we have

$$f_j(r, x, r^{-1}(r\rho - x\xi), r^{-1}\xi) = r^{-\mu+|\alpha|+j} f_j(r, x, r\rho - x\xi, \xi).$$

However, that $f_j(r, x, \tilde{\rho} - x\xi, \xi) \in C^\infty\left(\overline{\mathbb{R}}_+ \times B \times (\mathbb{R}^{1+n} \setminus \{0\})\right)$ and $f_j\big(r, x, \lambda(\tilde{\rho} - x\xi), \lambda\xi\big) = \lambda^{\mu-|\alpha|-j} f_j(r, x, \tilde{\rho} - x\xi, \xi)$ for all $\lambda \in \mathbb{R}_+$ give us

$$f_j(r, x, \tilde{\rho} - x\xi, \xi) = |\tilde{\rho}, \xi|^{\mu-|\alpha|-j} f_{1,j}\left(r, x, \frac{\tilde{\rho} - x\xi}{|\tilde{\rho}, \xi|}, \frac{\xi}{|\tilde{\rho}, \xi|}\right)$$

where $f_{1,j}(r, x, \tilde{\rho}-x\xi, \xi) \in C^\infty\left(\overline{\mathbb{R}}_+ \times B \times S^n\right)$ with $S^n$ being the unit sphere in $\mathbb{R}^{1+n}_{\tilde{\rho},\xi}$. It follows that $f_j(r, x, \tilde{\rho} - x\xi, \xi)$ can be written as a function

$$\tilde{f}_{(\mu-|\alpha|-j)}(r, x, \tilde{\rho}, \xi) \in C^\infty\big(\overline{\mathbb{R}}_+ \times B, S^{(\mu-|\alpha|-j)}(\mathbb{R}^{1+n}_{\tilde{\rho},\xi} \setminus \{0\})\big),$$

and we obtain that the component of $\big(\partial_\xi^\alpha a\big)\big(\tilde{x}(r, x), r^{-1}(r\rho-x\xi), r^{-1}\xi\big)$ of homogeneity $\mu - |\alpha| - j$ has the form $r^{-\mu}\big(r^{|\alpha|+j}\tilde{f}_{(\mu-|\alpha|-j)}(r, x, r\rho, \xi)\big)$. From (2.1.24), (2.1.26), and (2.1.27) we finally write $b(r, x, \rho, \xi)$ as an asymptotic sum

$$b(r, x, \rho, \xi) = r^{-\mu} \sum_{\substack{\alpha\in\mathbb{N}^{1+n} \\ j\in\mathbb{N}}} \chi\left(\frac{r\rho, \xi}{c_j}\right) r^j \tilde{f}_{(\mu-|\alpha|-j)}(r, x, r\rho, \xi)\xi^{\alpha''},$$

for some excision function $\chi(\tilde{\rho}, \xi)$ and some sequence of constant $(c_j)_{j\in\mathbb{N}}$ tending to $\infty$ fast enough. $\square$

## 2.2 A new parameter-dependent calculus on infinite cylinders

### 2.2.1 Operator-valued symbols with parameter

We now consider (operator-valued) symbols depending on the covariables $\rho, \eta$ in edge-degenerate form. Since we are interested in operators on the manifold $X^{\asymp}$ modelled on a cylinder $\mathbb{R} \times X \ni (r, x)$, interpreted as a manifold with conical exits $|r| \to \infty$, $X$ smooth and closed, we ignore the edge-degeneracy at $r = 0$ and consider symbols of the form

$$a(r, \rho, \eta) = \tilde{a}(r, [r]\rho, [r]\eta) \tag{2.2.1}$$

for $\tilde{a}(r,\tilde{\rho},\tilde{\eta}) \in C^\infty\big(\mathbb{R}, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$. To be more precise, we assume

$$\tilde{a}(r,\tilde{\rho},\tilde{\eta}) \in S^\nu\big(\mathbb{R}, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big), \tag{2.2.2}$$

i.e., we take standard symbols in $r \in \mathbb{R}$ of order $\nu \in \mathbb{R}$ with values in $L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$, $\mu \in \mathbb{R}$. We study operators

$$\mathrm{Op}_r\big(\tilde{a}(r,[r]\rho,[r]\eta)\big) : C_0^\infty(\mathbb{R}\times X) \to C^\infty(\mathbb{R}\times X) \tag{2.2.3}$$

for any fixed $\eta \in \mathbb{R}^q \setminus \{0\}$ and observe the behaviour for $|r| \to \infty$. The continuity of (2.2.3) follows from $\mathrm{Op}_r\big(\tilde{a}(r,[r]\rho,[r]\eta)\big) \in L^\mu_{\mathrm{cl}}(\mathbb{R}\times X)$ for every $\eta$. Some aspects can be deduced from what we did in Subsection 2.1.1 as we proved that the push forward under the transformation (2.1.11), say, for $r > 0$, gives rise to an operator with exit behaviour. In particular, we see that (2.2.3) induces a continuous operator

$$\mathrm{Op}_r\big(\tilde{a}(r,[r]\rho,[r]\eta])\big) : \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big) \to \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big) \tag{2.2.4}$$

for every fixed $\eta \neq 0$. Also other properties of operators (2.2.3) for $|r| \to \infty$ can be deduced from the results of Section 2.1.1. However, when we replace later on the cross section $X$ by a manifold with singularities, it is instructive also to refer to the degenerate behaviour of the operator-valued amplitude functions in a more direct manner. This is just what we are doing here in the case when $X$ is smooth. In particular, we will prove the continuity (2.2.4) once again. At the same time we observe some general properties of the symbols (2.2.1) for (2.2.2). In other words, the crucial definition is as follows:

**Definition 2.2.1.** (i) *Let $E$ be a Fréchet space with the (countable) system of semi-norms $(\pi_j)_{j\in\mathbb{N}}$; then $S^\nu(\mathbb{R},E)$, $\nu \in \mathbb{R}$, is defined to be the set of all $a(r) \in C^\infty(\mathbb{R},E)$ such that*

$$\pi_j\left(D_r^k a(r)\right) \le c[r]^{\nu-k}$$

*for all $r \in \mathbb{R}, k \in \mathbb{N}$, with constants $c = c(k,j) > 0$,*

(ii) $\boldsymbol{S}^{\mu,\nu}$ *for $\mu,\nu \in \mathbb{R}$ denotes the set of all operator families*

$$a(r,\rho,\eta) = \tilde{a}(r,[r]\rho,[r]\eta)$$

*for $\tilde{a}(r,\tilde{\rho},\tilde{\eta}) \in S^\nu\big(\mathbb{R}, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$ (referring to the natural nuclear topology of the space $L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$).*

For future references we state and prove a standard property on the norm growth of parameter-dependent pseudo-differential operators.

**Theorem 2.2.2.** *Let $M$ be a closed compact $C^\infty$ manifold and $A(\lambda) \in L^\mu_{\mathrm{cl}}(M;\mathbb{R}^l)$ a parameter-dependent family of order $\mu$, and let $\nu \ge \mu$. Then there is a constant $c = c(s,\mu,\nu) > 0$ such that*

$$\|A(\lambda)\|_{\mathcal{L}(H^s(M),H^{s-\nu}(M))} \le c\langle\lambda\rangle^{\max\{\mu,\mu-\nu\}}. \tag{2.2.5}$$

*In particular, for $\mu \le 0$, $\nu = 0$ we have*

$$\|A\|_{\mathcal{L}(H^s(M),H^s(M))} \le c\langle\lambda\rangle^{\mu}. \tag{2.2.6}$$

*Moreover, for every $s', s'' \in \mathbb{R}$ and every $N \in \mathbb{N}$ there exists a $\mu(N) \in \mathbb{R}$ such that for every $\mu \leq \mu(N)$, $k := \mu(N) - \mu$, and $A(\lambda) \in L^{\mu}_{\mathrm{cl}}(M; \mathbb{R}^l)$ we have*

$$\|A\|_{\mathcal{L}(H^{s'}(M), H^{s''}(M))} \leq c\langle\lambda\rangle^{-N-k}. \tag{2.2.7}$$

*for all $\lambda \in \mathbb{R}^l$, and a constant $c = c(s', s'', \mu, N, k) > 0$.*

*Proof.* In this proof we write $\|\cdot\|_{s',s''} = \|\cdot\|_{\mathcal{L}(H^{s'}(M), H^{s''}(M))}$. The estimates (2.2.5) and (2.2.6) are standard. Concerning (2.2.7) we first observe that we have to choose $\mu$ so small that $A(\lambda) : H^{s'}(M) \to H^{s''}(M)$ is continuous. This is the case when $s'' \leq s' - \mu$, i.e., $\mu \leq s' - s''$. Let $R^{s''-s'}(\lambda) \in L^{s''-s'}_{\mathrm{cl}}(M, \mathbb{R}^l)$ be an order reducing family with the inverse $R^{s'-s''}(\lambda) \in L^{s'-s''}_{\mathrm{cl}}(M, \mathbb{R}^l)$. Then we have

$$R^{s''-s'}(\lambda) : H^{s''}(M) \to H^{s'}(M),$$

i.e., $R^{s''-s'}(\lambda)A(\lambda) : H^{s'}(M) \to H^{s'}(M)$. The estimate (2.2.6) gives us

$$\|R^{s''-s'}(\lambda)A(\lambda)\|_{s',s'} \leq c\langle\lambda\rangle^{\mu+(s''-s')}$$

for $\mu \leq s' - s''$. Moreover, (2.2.5) yields $\|R^{s'-s''}(\lambda)\|_{s',s''} \leq c\langle\lambda\rangle^{s'-s''}$. Thus

$$\begin{aligned}\|A(\lambda)\|_{s',s''} &= \|R^{s'-s''}(\lambda)R^{s''-s'}(\lambda)A(\lambda)\|_{s',s''} \\ &\leq \|R^{s'-s''}(\lambda)\|_{s',s''}\|R^{s''-s'}(\lambda)A(\lambda)\|_{s',s'} \leq c\langle\lambda\rangle^{(s'-s'')+\mu+(s''-s')} = c\langle\lambda\rangle^{\mu}.\end{aligned}$$

In other words, when we choose $\mu(N)$ in such a way that $\mu \leq s' - s''$, and $\mu(N) \leq -N$, then (2.2.7) is satisfied. In addition, if we take $\mu = \mu(N) - k$ for some $k \geq 0$ then (2.2.7) follows in general. □

**Corollary 2.2.3.** *Let $A(\lambda) \in L^{\mu}_{\mathrm{cl}}(M; \mathbb{R}^l)$, and assume that the estimate*

$$\|A(\lambda)\|_{s',s''} \leq c\langle\lambda\rangle^{-N}$$

*holds for given $s', s'' \in \mathbb{N}$ and some $N$. Then we have*

$$\|D^{\alpha}_{\lambda}A(\lambda)\|_{s',s''} \leq c\langle\lambda\rangle^{-N-|\alpha|}$$

*for every $\alpha \in \mathbb{N}^l$.*

Now we go back to Definition 2.2.1 and establish some properties of the $\boldsymbol{S}^{\mu,\nu}$ spaces that play a role in our calculus.

**Proposition 2.2.4.** (i) *$\varphi(r) \in S^{\sigma}(\mathbb{R})$, $a(r, \rho, \eta) \in \boldsymbol{S}^{\mu,\nu}$ implies $\varphi(r)a(r, \rho, \eta) \in \boldsymbol{S}^{\mu,\nu+\sigma}$.*

(ii) *For every $k, l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^q$ we have*

$$a \in \boldsymbol{S}^{\mu,\nu} \Rightarrow \partial^l_r a \in \boldsymbol{S}^{\mu,\nu-l}, \partial^k_{\rho} a \in \boldsymbol{S}^{\mu-k,\nu+k}, \partial^{\alpha}_{\eta} a \in \boldsymbol{S}^{\mu-|\alpha|,\nu+|\alpha|}.$$

(iii) *$a(r, \rho, \eta) \in \boldsymbol{S}^{\mu,\nu}$, $b(r, \rho, \eta) \in \boldsymbol{S}^{\tilde{\mu},\tilde{\nu}}$ implies $a(r, \rho, \eta)b(r, \rho, \eta) \in \boldsymbol{S}^{\mu+\tilde{\mu},\nu+\tilde{\nu}}$.*

*Proof.* (i) is evident. (ii) For simplicity we assume $q = 1$ and compute

$$\partial_r \tilde{a}(r, [r]\rho, [r]\eta) = \big((\partial_r + [r]'\rho\partial_{\tilde{\rho}} + [r]'\eta\partial_{\tilde{\eta}})\tilde{a}\big)(r, [r]\rho, [r]\eta)$$

where $[r]' := \partial_r[r]$. Since $\tilde{\rho}\tilde{a}(r, \tilde{\rho}, \tilde{\eta})$, $\tilde{\eta}\tilde{a}(r, \tilde{\rho}, \tilde{\eta}) \in S^{\nu}(\mathbb{R}, L^{\mu+1}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}))$, and $\partial_{\tilde{\rho}}\tilde{a}, \partial_{\tilde{\eta}}\tilde{a} \in S^{\nu}(\mathbb{R}, L^{\mu-1}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}))$, we obtain

$$\partial_r \tilde{a}(r, [r]\rho, [r]\eta) = \big((\partial_r + ([r]'/[r])\tilde{\rho}\partial_{\tilde{\rho}} + ([r]'/[r])\tilde{\eta}\partial_{\tilde{\eta}})\tilde{a}\big)(r, [r]\rho, [r]\eta) \in \boldsymbol{S}^{\mu,\nu-1}.$$

By induction it follows that $\partial_r^l a \in \boldsymbol{S}^{\mu,\nu-l}$ for all $l \in \mathbb{N}$. Moreover, we have

$$\partial_\rho \tilde{a}(r, [r]\rho, [r]\eta) = [r](\partial_{\tilde{\rho}}\tilde{a})(r, [r]\rho, [r]\eta)$$

which gives us $\partial_\rho a \in \boldsymbol{S}^{\mu-1,\nu+1}$, and, by iteration, $\partial_\rho^k a \in \boldsymbol{S}^{\mu-k,\nu+k}$. In a similar manner we can argue for the $\eta$-derivatives.

(iii) By definition we have

$$a(r, \rho, \eta) = \tilde{a}(r, [r]\rho, [r]\eta), \ b(r, \rho, \eta) = \tilde{b}(r, [r]\rho, [r]\eta)$$

for $\tilde{a}(r, \tilde{\rho}, \tilde{\eta}) \in S^{\nu}\big(\mathbb{R}, L^{\mu}_{\mathrm{cl}}(X, \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$, $\tilde{b}(r, \tilde{\rho}, \tilde{\eta}) \in S^{\tilde{\nu}}\big(\mathbb{R}, L^{\tilde{\mu}}_{\mathrm{cl}}(X, \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$. Then the assertion is a consequence of the relation

$$(\tilde{a}\tilde{b})(r, \tilde{\rho}, \tilde{\eta}) \in S^{\nu+\tilde{\nu}}\big(\mathbb{R}, L^{\mu+\tilde{\mu}}_{\mathrm{cl}}(X, \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big).$$

□

**Corollary 2.2.5.** *For $a(r, \rho, \eta) \in \boldsymbol{S}^{\mu,\nu}$, $b(r, \rho, \eta) \in \boldsymbol{S}^{\tilde{\mu},\tilde{\nu}}$ for every $k \in \mathbb{N}$ we have*

$$\partial_\rho^k a(r, \rho, \eta) D_r^k b(r, \rho, \eta) \in \boldsymbol{S}^{\mu+\tilde{\mu}-k,\nu+\tilde{\nu}}$$

**Remark 2.2.6.** (i) *Let $\varphi_1, \varphi_2 \in C^\infty(\mathbb{R})$ be strictly positive functions such that $\varphi_j(r) = |r|$ for $|r| \geq c_j$ for some $c_j > 0$, $j = 1, 2$. Then we have*

$$\boldsymbol{S}^{\mu,\nu} = \Big\{ a\big(r, \varphi_1(r)\rho, \varphi_2(r)\eta\big) : a(r, \tilde{\rho}, \tilde{\eta}) \in S^{\nu}\big(\mathbb{R}, L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big) \Big\};$$

(ii) *$a(r, \rho, \eta) \in \boldsymbol{S}^{\mu,\nu}$ implies $a(\lambda r, \rho, \eta) \in \boldsymbol{S}^{\mu,\nu}$ for every $\lambda \in \mathbb{R}_+$.*

*Proof.* (i) We can write

$$a\big(r, \varphi_1(r)\rho, \varphi_2(r)\eta\big) = a\big(r, \psi_1(r)[r]\rho, \psi_2(r)[r]\eta\big)$$

for $\psi_j(r) \in C^\infty(\mathbb{R})$, $\psi_j(r) = 1$ for $|r| > c$ for some $c > 0$, $j = 1, 2$. Then it suffices to verify that

$$a\big(r, \psi_1(r)\tilde{\rho}, \psi_2(r)\tilde{\eta}\big) \in S^{\nu}\big(\mathbb{R}, L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big);$$

however, this is straightforward.

(ii) It is evident that the relation $\tilde{a}(r, \tilde{\rho}, \tilde{\eta}) \in S^{\nu}\big(\mathbb{R}, L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$ implies $\tilde{a}(\lambda r, \tilde{\rho}, \tilde{\eta}) \in S^{\nu}\big(\mathbb{R}, L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$. Therefore, it suffices to show $\tilde{a}(r, [\lambda r]\rho, [\lambda r]\eta) \in \boldsymbol{S}^{\mu,\nu}$. Let us write

$$\tilde{a}(r, [\lambda r]\rho, [\lambda r]\eta) = \tilde{a}\big(r, \varphi_\lambda(r)[r]\rho, \varphi_\lambda(r)[r]\eta\big)$$

for $\varphi_\lambda(r) := [\lambda r]/[r]$. We have $\varphi_\lambda(r) = \lambda$ for $|r| > c$ for a constant $c > 0$, i.e., $\varphi_\lambda(r) - \lambda \in C_0^\infty(\mathbb{R})$. Thus there is an $r$-excision function $\chi(r)$ (i.e., $\chi \in C^\infty(\mathbb{R})$, $\chi(r) = 0$ for $|r| \le c_0$, $\chi(r) = 1$ for $|r| \ge c_1$ for certain $0 < c_0 < c_1$) such that

$$\chi(r)\tilde{a}(r, [\lambda r]\rho, [\lambda r]\eta) = \chi(r)\tilde{a}(r, [r]\lambda\rho, [r]\lambda\eta),$$

which belongs to $\boldsymbol{S}^{\mu,\nu}$. It remains to characterise $\big(1 - \chi(r)\big)\tilde{a}\big(r, \varphi_\lambda(r)[r]\rho, \varphi_\lambda(r)[r]\eta\big)$ which vanishes for $|r| \ge c_1$, and a simple calculation shows

$$\big(1 - \chi(r)\big)\tilde{a}\big(r, \varphi_\lambda(r)\tilde{\rho}, \varphi_\lambda(r)\tilde{\eta}\big) \in C_0^\infty\big(\mathbb{R}, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big),$$

which is contained in $\boldsymbol{S}^{\mu,-\infty}$. □

**Proposition 2.2.7 (Asymptotic summation).** *Let*

$$\tilde{a}_j(r, \tilde{\rho}, \tilde{\eta}) \in S^\nu\big(\mathbb{R}, L^{\mu-j}_{\mathrm{cl}}(X; \mathbb{R}^{1+q})\big), j \in \mathbb{N},$$

*be an arbitrary sequence, $\mu, \nu \in \mathbb{R}$. Then there is an $\tilde{a}(r, \tilde{\rho}, \tilde{\eta}) \in S^\nu\big(\mathbb{R}, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q})\big)$ such that*

$$a - \sum_{j=0}^{N} a_j \in S^\nu\big(\mathbb{R}, L^{\mu-(N+1)}_{\mathrm{cl}}(X; \mathbb{R}^{1+q})\big)$$

*for every $N \in \mathbb{N}$, and $a$ is unique modulo $S^\nu\big(\mathbb{R}, L^{-\infty}_{\mathrm{cl}}(X; \mathbb{R}^{1+q})\big)$.*

*Proof.* The proof is similar to the standard one on asymptotic summation of symbols. We can find an asymptotic sum as a convergent series $\tilde{a}(r, \tilde{\rho}, \tilde{\eta}) = \sum_{j=0}^\infty \chi\left((\tilde{\rho}, \tilde{\eta})/c_j\right) \tilde{a}_j(r, \tilde{\rho}, \tilde{\eta})$ for some excision function $\chi$ in $\mathbb{R}^{1+q}$, with a sequence $c_j > 0$, $c_j \to \infty$ as $j \to \infty$ so fast, that $\sum_{j=N+1}^\infty \chi\left((\tilde{\rho}, \tilde{\eta})/c_j\right) \tilde{a}(r, \tilde{\rho}, \tilde{\eta})$ converges in $S^\nu(\mathbb{R}, L^{\mu-(N+1)}_{\mathrm{cl}})$ for every $N$. □

### 2.2.2 Continuity in Schwartz spaces

The Schwartz space $\mathcal{S}(\mathbb{R}, E)$ with values in a Fréchet space $E$ can be interpreted as the projective tensor product $\mathcal{S}(\mathbb{R})\hat{\otimes}_\pi E$, using the nuclearity of $\mathcal{S}(\mathbb{R})$. In particular we have the space $\mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$. Occasionally we will write $\mathcal{S}(\mathbb{R} \times X) := \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$.

**Theorem 2.2.8.** *Let $p(r, \rho, \eta) = \tilde{p}(r, [r]\rho, [r]\eta)$, $\tilde{p}(r, \tilde{\rho}, \tilde{\eta}) \in S^\nu\big(\mathbb{R}, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$, i.e., $p(r, \rho, \eta) \in \boldsymbol{S}^{\mu,\nu}$. Then $\mathrm{Op}_r(p)(\eta)$ induces a family of continuous operators*

$$\mathrm{Op}_r(p)(\eta) : \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big) \to \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$$

*for every fixed $\eta \ne 0$.*

*Proof.* We have

$$\mathrm{Op}_r(p)(\eta)u(r) = \int e^{ir\rho} p(r, \rho, \eta)\hat{u}(\rho)đ\rho,$$

first for $u \in C_0^\infty\big(\mathbb{R}, C^\infty(X)\big)$. In the space $\mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$ we have the semi-norm system

$$\pi_{m,s}(u) = \max_{\alpha+\beta\le m} \sup_{r\in\mathbb{R}} \|[r]^\alpha \partial_r^\beta u(r)\|_{H^s(X)}$$

for $m \in \mathbb{N}$, $s \in \mathbb{Z}$, which defines the Fréchet topology of $\mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$. If necessary we indicate the variable $r$, i.e., write $\pi_{m,s;r}$ rather than $\pi_{m,s}$.
The Fourier transform $\mathcal{F}_{r\to\rho}$ induces an isomorphism

$$\mathcal{F} : \mathcal{S}\big(\mathbb{R}_r, H^s(X)\big) \to \mathcal{S}\big(\mathbb{R}_\rho, H^s(X)\big)$$

for every $s$. For every $m \in \mathbb{N}$ there exists a $C > 0$ such that

$$\pi_{m,s;\rho}(\mathcal{F}u) \le C\pi_{m+2,s;r}(u) \tag{2.2.8}$$

for all $u \in \mathcal{S}\big(\mathbb{R}, H^s(X)\big)$ (see [19, Chapter 1] for scalar functions; the case of functions with values in a Hilbert space is completely analogous). We have to show that for every $\tilde{m} \in \mathbb{N}$ and $\tilde{s} \in \mathbb{Z}$ there exist $m \in \mathbb{N}$ and $s \in \mathbb{Z}$, such that

$$\pi_{\tilde{m},\tilde{s}}\big(\mathrm{Op}(p)u(r)\big) \le c\pi_{m,s}(u) \tag{2.2.9}$$

for all $u \in \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$, for some $c = c(\tilde{m}, \tilde{s}) > 0$. According to Proposition 2.2.10 below we write the operator $\mathrm{Op}(p)(\eta)$ in the form

$$\mathrm{Op}_r(p)(\eta)\circ\langle r\rangle^{-M}\circ\langle r\rangle^{M} = \langle r\rangle^{-M}\mathrm{Op}_r(b_{MN})(\eta)\circ\langle r\rangle^{M} + \mathrm{Op}_r(d_{MN})(\eta)\circ\langle r\rangle^{M} \tag{2.2.10}$$

for a symbol $b_{MN}(r,\rho,\eta) \in \boldsymbol{S}^{\mu,\nu}, N \in \mathbb{N}$ and a remainder $d_{MN}(r,\rho,\eta)$ satisfying estimates similar to (2.2.19).
We have

$$\|\mathrm{Op}_r(p)(\eta)u(r)\|_{H^{\tilde{s}}(X)} = \left\|\int e^{ir\rho}p(r,\rho,\eta)\hat{u}(\rho)đ\rho\right\|_{H^{\tilde{s}}(X)}$$
$$\le \left\|\int e^{ir\rho}\langle r\rangle^{-M}b_{MN}(r,\rho,\eta)\big(\widehat{\langle r\rangle^M u}\big)(\rho)đ\rho\right\|_{H^{\tilde{s}}(X)} + \|\mathrm{Op}_r(d_{MN})(\eta)\langle r\rangle^M u(r)\|_{H^{\tilde{s}}(X)}. \tag{2.2.11}$$

For the first term on the right of (2.2.11) we obtain for $s := \tilde{s} + \mu$ and arbitrary $\widetilde{M} \in \mathbb{N}$

$$\left\|\int e^{ir\rho}\langle\rho\rangle^{-\widetilde{M}}\langle r\rangle^{-M}b_{MN}(r,\rho,\eta)\langle\rho\rangle^{\widetilde{M}}\big(\widehat{\langle r\rangle^M u}\big)(\rho)đ\rho\right\|_{H^{\tilde{s}}(X)}$$
$$\le \int \|\langle\rho\rangle^{-\widetilde{M}}\langle r\rangle^{-M}b_{MN}(r,\rho,\eta)\langle\rho\rangle^{\widetilde{M}}\big(\widehat{\langle r\rangle^M u}\big)(\rho)\|_{H^{\tilde{s}}(X)}đ\rho$$
$$\le c \sup_{(r,\rho)\in\mathbb{R}^2}\langle\rho\rangle^{-\widetilde{M}}\langle r\rangle^{-M}\|b_{MN}(r,\rho,\eta)\|_{\mathcal{L}(H^s(X),H^{\tilde{s}}(X))}\int\langle\rho\rangle^{\widetilde{M}}\|\big(\widehat{\langle r\rangle^M u}\big)(\rho)\|_{H^s(X)}đ\rho.$$

Moreover, we have

$$\int\langle\rho\rangle^{\widetilde{M}}\|\big(\widehat{\langle r\rangle^M u}\big)(\rho)\|_{H^s(X)}đ\rho \le \sup_{\rho\in\mathbb{R}}\langle\rho\rangle^{\widetilde{M}+2}\|\big(\widehat{\langle r\rangle^M u}\big)(\rho)\|_{H^s(X)}\int\langle\rho\rangle^{-2}đ\rho$$
$$\le c\pi_{\widetilde{M}+2,s;\rho}\big(\big(\widehat{\langle r\rangle^M u}\big)(\rho)\big) \le \pi_{\widetilde{M}+4,s;r}(\langle r\rangle^M u)) \le c\pi_{M+\widetilde{M}+4,s;r}(u)$$

Here we employed the estimate (2.2.8). Thus (2.2.11) yields

$$\begin{aligned}\pi_{0,\tilde{s}}\big(\mathrm{Op}(p)(\eta)u\big) &\\ \le c \sup_{(r,\rho)\in\mathbb{R}^2} \langle\rho\rangle^{-\widetilde{M}}\langle r\rangle^{-M}\|b_{MN}(r,\rho,\eta)\|_{\mathcal{L}(H^s(X),H^{\tilde{s}}(X))}\pi_{M+\widetilde{M}+4,s;r}(u) &\\ + \pi_{0,\tilde{s}}\big(\mathrm{Op}_r(d_{MN})(\eta)(\langle r\rangle^M u)\big). &\quad (2.2.12)\end{aligned}$$

The factor $c\sup_{(r,\rho)\in\mathbb{R}^2}\langle\rho\rangle^{-\widetilde{M}}\langle r\rangle^{-M}\|b_{MN}(r,\rho,\eta)\|_{\mathcal{L}(H^s(X),H^{\tilde{s}}(X))}$ is finite when we adequately choose $M$ and $\widetilde{M}$. In fact, since $b_{MN}(r,\rho,\eta)\in \boldsymbol{S}^{\mu,\nu}$ we see from Theorem 2.2.2 that

$$\|b_{MN}(r,\rho,\eta)\|_{\mathcal{L}(H^s(X),H^{\tilde{s}}(X))} \le c\langle r\rangle^{\nu}\langle [r]\rho,[r]\eta\rangle^{\max\{\mu,0\}}.$$

Thus

$$\langle\rho\rangle^{-\widetilde{M}}\langle r\rangle^{-M}\|b_{MN}(r,\rho,\eta)\|_{\mathcal{L}(H^s(X),H^{\tilde{s}}(X))} \le c\langle\rho\rangle^{-\widetilde{M}}\langle r\rangle^{\nu-M}\langle [r]\rho,[r]\eta\rangle^{\max\{\mu,0\}}.$$

If $\mu\le 0$ then it suffices to choose $M\ge\nu$ and $\widetilde{M}\ge 0$. Let $\mu\ge 0$ then we can write

$$\begin{aligned}\langle\rho\rangle^{-\widetilde{M}}\langle r\rangle^{-M}\|b_{MN}(r,\rho,\eta)\|_{\mathcal{L}(H^s(X),H^{\tilde{s}}(X))} &\le c\langle\rho\rangle^{-\widetilde{M}}\langle r\rangle^{\nu-M}\langle [r]\rho,[r]\eta\rangle^{\mu}\\ &= c\frac{(1+[r]^2\rho^2+[r]^2|\eta|^2)^{\mu/2}}{(1+r^2)^{(M-\nu)/2}(1+\rho^2)^{\widetilde{M}/2}}\\ &\le c\frac{(1+[r]^2\rho^2+c[r]^2)^{\mu/2}}{(1+r^2)^{(M-\nu)/2}(1+\rho^2)^{\widetilde{M}/2}}.\end{aligned}$$

This means that if we choose $M\ge\mu+\nu$, $\widetilde{M}\ge\mu$ then

$$\sup_{(r,\rho)\in\mathbb{R}^2}\langle\rho\rangle^{-\widetilde{M}}\langle r\rangle^{-M}\|b_{MN}(r,\rho,\eta)\|_{\mathcal{L}(H^s(X),H^{\tilde{s}}(X))}<\infty.$$

Next we consider the second term on the right hand side of (2.2.12). We have

$$\begin{aligned}&\|\mathrm{Op}_r(d_{MN})(\eta)\langle r\rangle^M u(r)\|_{H^{\tilde{s}}(X)}\\ &\quad= \Big\|\int e^{ir\rho}\langle\rho\rangle^{-M}d_{MN}(r,\rho,\eta)\langle\rho\rangle^{M}\big(\widehat{\langle r\rangle^M u}\big)(\rho)đ\rho\Big\|_{H^{\tilde{s}}(X)}\\ &\quad\le \int \|\langle\rho\rangle^{-M}d_{MN}(r,\rho,\eta)\langle\rho\rangle^{M}\big(\widehat{\langle r\rangle^M u}\big)(\rho)\|_{H^{\tilde{s}}(X)}đ\rho\\ &\quad\le \sup_{(r,\rho)\in\mathbb{R}^2}\|\langle\rho\rangle^{-M}d_{MN}(r,\rho,\eta)\|_{\mathcal{L}(H^s(X),H^{\tilde{s}}(X))}\int\|\langle\rho\rangle^{M}\big(\widehat{\langle r\rangle^M u}\big)(\rho)\|_{H^s(X)}đ\rho.\end{aligned}$$

From the analogue of the estimate (2.2.19) for $d_{MN}(r,\rho,\eta)$ we see that for $N$ sufficiently large it follows that the right hand side of the latter expression can be estimated by

$$\begin{aligned}&c\int\|\langle\rho\rangle^{M}\big(\widehat{\langle r\rangle^M u}\big)(\rho)\|_{H^s(X)}đ\rho\\ &\qquad\le\sup_{\rho\in\mathbb{R}}\langle\rho\rangle^{M+2}\|\big(\widehat{\langle r\rangle^M u}\big)(\rho)\|_{H^s(X)}\int\langle\rho\rangle^{-2}đ\rho\le c\pi_{2M+4,s;r}(u).\end{aligned}$$

In other words we have proved

$$\pi_{0,\tilde{s}}\big(\mathrm{Op}(p)(\eta)u\big) \le c\big(\pi_{M+\widetilde{M}+4,s}(u) + \pi_{2M+4,s}(u)\big) \le c\pi_{L,s}(u) \tag{2.2.13}$$

for $s = \tilde{s} + \mu$, $L := \max\{M + \widetilde{M} + 4, 2M + 4\}$. For $\pi_{1,\tilde{s}}$ we write

$$\partial_r \mathrm{Op}(p)(\eta)u(r) = \int e^{ir\rho}\partial_r p(r,\rho,\eta)\hat{u}(\rho)đ\rho + \int e^{ir\rho} p(r,\rho,\eta)\widehat{Du}(\rho)đ\rho,$$
$$r\mathrm{Op}(p)(\eta)u(r) = \int e^{ir\rho}\big(i\partial_\rho p(r,\rho,\eta)\big)\hat{u}(\rho)đ\rho + \int e^{ir\rho} ip(r,\rho,\eta)\partial_\rho \hat{u}(\rho)đ\rho.$$

From Proposition 2.2.4 we have

$$\partial_r p(r,\rho,\eta) \in \boldsymbol{S}^{\mu,\nu-1}, \quad i\partial_\rho p(r,\rho,\eta) \in \boldsymbol{S}^{\mu-1,\nu+1}.$$

Thus we obtain

$$\sup_{r\in\mathbb{R}} \|\partial_r \mathrm{Op}(p)(\eta)u(r)\|_{H^{\tilde{s}}(X)} \le c\pi_{L+1,s}(u),$$
$$\sup_{r\in\mathbb{R}} \|r\mathrm{Op}(p)(\eta)u(r)\|_{H^{\tilde{s}}(X)} \le c\big(\pi_{L,s-1}(u) + \pi_{L+1,s}(u)\big) \le c\pi_{L+1,s}(u),$$

i.e., $\pi_{1,\tilde{s}}\big(\mathrm{Op}(p)(\eta)u\big) \le c\pi_{L+1,s}(u)$. Analogously, one can prove (2.2.9) for arbitrary $\tilde{m} \in \mathbb{N}$, $\tilde{s} \in \mathbb{Z}$, and suitable $m, s$. □

### 2.2.3 Leibniz products and remainder estimates

Let $\tilde{a}(r,\tilde{\rho},\tilde{\eta}) \in S^\nu(\mathbb{R}, L^\mu_{\mathrm{cl}})$, $\tilde{b}(r,\tilde{\rho},\tilde{\eta}) \in S^{\tilde{\nu}}(\mathbb{R}, L^{\tilde{\mu}}_{\mathrm{cl}})$ where $L^\mu_{\mathrm{cl}} = L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$. The operator functions

$$a(r,\rho,\eta) := \tilde{a}(r,[r]\rho,[r]\eta), \quad b(r,\rho,\eta) := \tilde{b}(r,[r]\rho,[r]\eta)$$

will be interpreted as amplitude functions of a pseudo-differential calculus on $\mathbb{R}$ containing $\eta$ as a parameter (below we assume $\eta \neq 0$). We intend to apply an analogue of Kumano-go's technique [19] and form the oscillatory integral

$$a\#b(r,\rho,\eta) = \iint e^{-it\tau} a(r,\rho+\tau,\eta)b(r+t,\rho,\eta)dtđ\tau \tag{2.2.14}$$

which has the meaning of a Leibniz product, associated with the composition of operators. The rule

$$\mathrm{Op}_r(a)(\eta)\mathrm{Op}_r(b)(\eta) = \mathrm{Op}_r(a\#b)(\eta) \tag{2.2.15}$$

for $\eta \neq 0$ will be justified afterwards. Similarly as in [19], applying Taylor's formula on $a(r,\rho+\tau,\eta)$ with respect to the second variable at the point $\rho$ we get for any $N \in \mathbb{N}$

$$a(r,\rho+\tau,\eta) = \sum_{k=0}^{N} \frac{\tau^k}{k!}\partial_\rho^k a(r,\rho,\eta) + \frac{\tau^{N+1}}{N!}\int_0^1 (1-\theta)^N (\partial_\rho^{N+1}a)(r,\rho+\theta\tau,\eta)d\theta.$$

We substitute in (2.2.14) and, using the identity $e^{-it\tau}\tau^k = (-D_t)^k e^{-it\tau}, k \in \mathbb{N}$, we write

$$a\#b(r,\rho,\eta) = \sum_{k=0}^{N} \frac{1}{k!} \iint e^{-it\tau} \partial_\rho^k a(r,\rho,\eta)(D_r^k b)(r+t,\rho,\eta) dt đ\tau$$
$$+ \frac{1}{N!} \iint e^{-it\tau} \left\{ \int_0^1 (1-\theta)^N (\partial_\rho^{N+1} a)(r,\rho+\theta\tau,\eta) d\theta \right\} (D_r^{N+1} b)(r+t,\rho,\eta) dt đ\tau \tag{2.2.16}$$

Then, by means of the Fourier inversion formula we see that the first term of the right hand side of (2.2.16) is equal to $\sum_{k=0}^{N} \frac{1}{k!} \partial_\rho^k a(r,\rho,\eta) D_r^k b(r,\rho,\eta)$.
Setting $c_k(r,\rho,\eta) := \frac{1}{k!} \partial_\rho^k a(r,\rho,\eta) D_r^k b(r,\rho,\eta)$ and

$$r_N(r,\rho,\eta) :=$$
$$\frac{1}{N!} \iint e^{-it\tau} \left\{ \int_0^1 (1-\theta)^N (\partial_\rho^{N+1} a)(r,\rho+\theta\tau,\eta) d\theta \right\} (D_r^{N+1} b)(r+t,\rho,\eta) dt đ\tau \tag{2.2.17}$$

we can decompose $a\#b$ in the form

$$a\#b(r,\rho,\eta) = \sum_{k=0}^{N} c_k(r,\rho,\eta) + r_N(r,\rho,\eta). \tag{2.2.18}$$

By virtue of Corollary 2.2.5 we have $c_k(r,\rho,\eta) = \tilde{c}_k(r,[r]\rho,[r]\eta)$ for some $\tilde{c}_k(r,\tilde{\rho},\tilde{\eta}) \in S^{\nu+\tilde{\nu}}(\mathbb{R}, L_{\mathrm{cl}}^{\mu+\tilde{\mu}-k})$. Let us now characterise the remainder.

**Lemma 2.2.9.** *For every $s', s'' \in \mathbb{R}$, $k,l,m \in \mathbb{N}$, there is an $N \in \mathbb{N}$ such that*

$$\|D_r^i D_\rho^j r_N(r,\rho,\eta)\|_{s',s''} \le c\langle\rho\rangle^{-k}\langle r\rangle^{-l}\langle\eta\rangle^{-m} \tag{2.2.19}$$

*for all $(r,\rho) \in \mathbb{R}^2$, $|\eta| \ge \varepsilon > 0$, $i,j \in \mathbb{N}$, for some constant $c = c(s',s'',k,l,m,N,\varepsilon) > 0$, here $\|\cdot\|_{s',s''} = \|\cdot\|_{\mathcal{L}(H^{s'}(X),H^{s''}(X))}$.*

*Proof.* By virtue of Proposition 2.2.4 we have

$$\partial_\rho^k \tilde{a}(r,[r]\rho,[r]\eta) \in \boldsymbol{S}^{\mu-k,\nu+k}, \quad \partial_r^k \tilde{b}(r,[r]\rho,[r]\eta) \in \boldsymbol{S}^{\tilde{\mu},\tilde{\nu}-k}$$

for every $k$. Let us set

$$\tilde{a}_{N+1}(r,[r]\rho+[r]\theta\tau,[r]\eta) := (\partial_\rho^{N+1} a)(r,\rho+\theta\tau,\eta),$$
$$\tilde{b}_{N+1}(r+t,[r+t]\rho,[r+t]\eta) := (D_r^{N+1} b)(r+t,\rho,\eta).$$

By virtue of Theorem 2.2.2 for every $s_0, s'' \in \mathbb{R}$ and every $M$ there exists a $\mu(M)$ such that for every $\mu \le \mu(M)$ and $p(\tilde{\rho},\tilde{\eta}) \in L_{\mathrm{cl}}^{\mu}(X;\mathbb{R}_{\tilde{\rho},\tilde{\eta}}^{1+q})$ we have

$$\|p(\tilde{\rho},\tilde{\eta})\|_{s_0,s''} \le c\langle\tilde{\rho},\tilde{\eta}\rangle^{-M} \tag{2.2.20}$$

for all $(\tilde{\rho},\tilde{\eta}) \in \mathbb{R}^{1+q}$, $c = c(s_0, s'', \mu, M) > 0$. In addition, for $s', s_0 \in \mathbb{R}$ there exists a $B \in \mathbb{R}$ such that $\|p(\tilde{\rho},\tilde{\eta})\|_{s',s_0} \leq c\langle\tilde{\rho},\tilde{\eta}\rangle^B$ for all $(\tilde{\rho},\tilde{\eta}) \in \mathbb{R}^{1+q}$, $c = c(s', s_0, \mu) > 0$. We apply this to $\tilde{a}_{N+1}(r,\tilde{\rho},\tilde{\eta})$ and $\tilde{b}_{N+1}(r,\tilde{\rho},\tilde{\eta})$, combined with the dependence on $r \in \mathbb{R}$ as a symbol in this variable. In other words, we have the estimates

$$\|\tilde{a}_{N+1}(r,\tilde{\rho},\tilde{\eta})\|_{s_0,s''} \leq c\langle r\rangle^{\nu+(N+1)}\langle\tilde{\rho},\tilde{\eta}\rangle^{-M}, \tag{2.2.21}$$
$$\|\tilde{b}_{N+1}(r,\tilde{\rho},\tilde{\eta})\|_{s',s_0} \leq c\langle r\rangle^{\tilde{\nu}-(N+1)}\langle\tilde{\rho},\tilde{\eta}\rangle^{B}; \tag{2.2.22}$$

here we applied the above mentioned result to $\tilde{a}_{N+1}$ for the pair $(s_0, s'')$ for $N$ sufficiently large, and for $\tilde{b}_{N+1}$ the second estimate for $(s', s_0)$ with some exponent $B$. Let us take $s_0 := s' - \tilde{\mu}$; then we can set $B = \max\{\tilde{\mu}, 0\}$. The remainder (2.2.17) is regularised as an oscillatory integral in $(t,\tau)$, i.e., we may write

$$\begin{aligned} r_N(r,\rho,\eta) = \frac{1}{N!}\iint e^{-it\tau}\langle t\rangle^{-2L}(1-\partial_\tau^2)^L\langle\tau\rangle^{-2K}(1-\partial_t^2)^K \\ \left\{\int_0^1 (1-\theta)^N\tilde{a}_{N+1}(r,[r]\rho+[r]\theta\tau,[r]\eta)d\theta\right\}\tilde{b}_{N+1}(r+t,[r+t]\rho,[r+t]\eta)dt đ\tau \end{aligned} \tag{2.2.23}$$

for sufficiently large $L, K \in \mathbb{N}$. For simplicity from now on we assume $q = 1$; the considerations for the general case are completely analogous. Then we have for every $l \leq L$

$$\partial_\tau^{2l}\tilde{a}_{N+1}(r,[r]\rho+[r]\theta\tau,[r]\eta) = \left(\partial_{\tilde{\rho}}^{2l}\tilde{a}_{N+1}\right)(r,[r]\rho+[r]\theta\tau,[r]\eta)([r]\theta)^{2l},$$

and for every $k \leq K$

$$\begin{aligned} \partial_t^{2k}\tilde{b}_{N+1}(r+t,[r+t]\rho,[r+t]\eta) &= \left(\partial_t^{2k}\tilde{b}_{N+1}\right)(r+t,[r+t]\rho,[r+t]\eta) \\ &+ \left(\partial_{\tilde{\rho}}^{2k}\tilde{b}_{N+1}\right)(r+t,[r+t]\rho,[r+t]\eta)(\rho\partial_t[r+t])^{2k} \\ &+ \left(\partial_{\tilde{\eta}}^{2k}\tilde{b}_{N+1}\right)(r+t,[r+t]\rho,[r+t]\eta)(\eta\partial_t[r+t])^{2k} + R, \end{aligned}$$

where $R$ denotes a linear combination of other mixed derivatives, for example,

$$\begin{aligned} &\partial_t^{2k-1}\{(\partial_{\tilde{\rho}}\tilde{b}_{N+1})(r+t,[r+t]\rho,[r+t]\eta)(\rho\partial_t[r+t])\} \\ &\quad= \sum_{i+j=2k-1} c_{ij}\left(\partial_t^i(\partial_{\tilde{\rho}}\tilde{b}_{N+1})(r+t,[r+t]\rho,[r+t]\eta)\right)\rho\partial_t^{j+1}[r+t], \\ &\partial_t^{2k-1}\{(\partial_{\tilde{\eta}}\tilde{b}_{N+1})(r+t,[r+t]\rho,[r+t]\eta)(\eta\partial_t[r+t])\} \\ &\quad= \sum_{i+j=2k-1} c_{ij}\left(\partial_t^i(\partial_{\tilde{\eta}}\tilde{b}_{N+1})(r+t,[r+t]\rho,[r+t]\eta)\right)\eta\partial_t^{j+1}[r+t] \end{aligned}$$

for some coefficients $c_{ij}$.
From (2.2.21) we have

$$\|\partial_\tau^{2l}\tilde{a}_{N+1}(r,[r]\rho+r[\theta]\tau,[r]\eta)\|_{s_0,s''} \leq c\langle r\rangle^{\nu+(N+1)}\langle[r]\rho+[r]\theta\tau,[r]\eta\rangle^{-M-2l}([r]\theta)^{2l}, \tag{2.2.24}$$

see Corollary 2.2.3, and (2.2.22) gives us

$$\|(\partial_t^{2k}\tilde b_{N+1})(r+t,[r+t]\rho,[r+t]\eta)\|_{s',s_0}\le c\langle r+t\rangle^{\tilde\nu-(N+1)}\langle[r+t]\rho,[r+t]\eta\rangle^B \quad (2.2.25)$$

(where we take $N$ so large that $\tilde\nu-(N+1)\le 0$), and

$$\begin{aligned}&\|(\partial_{\tilde\rho}^{2k}\tilde b_{N+1})(r+t,[r+t]\rho,[r+t]\eta)(\rho\partial_t[r+t])^{2k}\|_{s',s_0}\\ &\qquad\le c\langle r+t\rangle^{\tilde\nu-(N+1)}\langle[r+t]\rho,[r+t]\eta\rangle^{B-2k}|\rho\partial_t[r+t]|^{2k},\end{aligned} \quad (2.2.26)$$

$$\begin{aligned}&\|(\partial_{\tilde\eta}^{2k}\tilde b_{N+1})(r+t,[r+t]\rho,[r+t]\eta)(\eta\partial_t[r+t])^{2k}\|_{s',s_0}\\ &\qquad\le c\langle r+t\rangle^{\tilde\nu-(N+1)}\langle[r+t]\rho,[r+t]\eta\rangle^{B-2k}|\eta\partial_t[r+t]|^{2k}.\end{aligned} \quad (2.2.27)$$

The above mentioned mixed derivatives admit similar estimates (in fact, better ones; so we concentrate on those contributed by (2.2.24), (2.2.25), (2.2.26), (2.2.27)).

We now derive an estimate for $\|r_N(r,\rho,\eta)\|_{s',s''}$. From (2.2.23) we obtain

$$\begin{aligned}\|r_N(r,\rho,\eta)\|_{s',s''}\le\iiint_0^1\|\langle t\rangle^{-2L}(1-\partial_\tau^2)^L\langle\tau\rangle^{-2K}(1-\partial_t^2)^K\\ (1-\theta)^N\tilde a_{N+1}(r,[r]\rho+[r]\theta\tau,[r]\eta)\tilde b_{N+1}(r+t,[r+t]\rho,[r+t]\eta)\|_{s',s''}d\theta dt đ\tau.\end{aligned}$$

The operator norm under the integral can be estimated by expressions of the kind

$$\begin{aligned}I:=c\langle r\rangle^{\nu+(N+1)}\langle r+t\rangle^{\tilde\nu-(N+1)}\langle t\rangle^{-2L}\langle\tau\rangle^{-2K}\langle[r]\rho+[r]\theta\tau,[r]\eta\rangle^{-M-2l}([r]\theta)^{2l}\\ \langle[r+t]\rho,[r+t]\eta\rangle^B\big\{1+\langle[r+t]\rho,[r+t]\eta\rangle^{-2k}(|\rho|^{2k}+|\eta|^{2k})|(\partial_t[r+t])^{2k}|\big\}\end{aligned}$$

$l\le L$, $k\le K$, plus terms from $R$ of a similar character. We have, using Peetre's inequality,

$$\langle r\rangle^{\nu+(N+1)}\langle r+t\rangle^{\tilde\nu-(N+1)}\le\langle r\rangle^{\nu+\tilde\nu}\langle t\rangle^{|\tilde\nu-(N+1)|}.$$

Moreover, we have $\langle[r]\rho+[r]\theta\tau,[r]\eta\rangle^{-2l}([r]\theta)^{2l}\le c\langle[r]\eta\rangle^{-2l}[r]^{2l}\le c$ for $|\eta|\ge\varepsilon>0$ (as always, $c$ denotes different constants), and

$$\begin{aligned}&\langle[r+t]\rho,[r+t]\eta\rangle^{-2k}(|\rho|^{2k}+|\eta|^{2k})|(\partial_t[r+t])^{2k}|\\ &\quad\le c\big\{\langle[r+t]\rho\rangle^{-2k}([r+t]|\rho|)^{2k}+\langle[r+t]\eta\rangle^{-2k}([r+t]|\eta|)^{2k}\big\}[r+t]^{-2k}\le c,\end{aligned}$$

using $|(\partial_t[r+t])^{2k}|\le c,[r+t]^{-2k}\le c$ for all $r,t\in\mathbb{R}$ and. This yields

$$I\le c\langle r\rangle^{\nu+\tilde\nu}\langle t\rangle^{|\tilde\nu-(N+1)|}\langle t\rangle^{-2L}\langle\tau\rangle^{-2K}\langle[r]\rho+[r]\theta\tau,[r]\eta\rangle^{-M}\langle[r+t]\rho,[r+t]\eta\rangle^B.$$

Writing $M=M'+M''$ for suitable $M',M''\ge 0$ to be fixed later on, we have

$$\begin{aligned}&\langle[r]\rho+[r]\theta\tau,[r]\eta\rangle^{-M}=\langle[r]\rho+[r]\theta\tau,[r]\eta\rangle^{-M'}\langle[r]\rho+[r]\theta\tau,[r]\eta\rangle^{-M''}\\ &\qquad\le c\langle[r]\eta\rangle^{-M'}\langle[r]\rho,[r]\eta\rangle^{-M''}\langle[r]\theta\tau\rangle^{M''}\le c\langle[r]\eta\rangle^{-M'}\langle[r]\rho\rangle^{-M''}\langle[r]\theta\tau\rangle^{M''}.\end{aligned}$$

We applied once again Peetre's inequality which gives us also

$$\langle[r+t]\rho,[r+t]\eta\rangle^B\le c\langle[r+t]\rho\rangle^B\langle[r+t]\eta\rangle^B$$

since $B \geq 0$. Thus

$$I \leq \ c\langle r\rangle^{\nu+\tilde{\nu}}\langle t\rangle^{|\tilde{\nu}-(N+1)|-2L}\langle\tau\rangle^{-2K}\langle[r]\theta\tau\rangle^{M''}\langle[r+t]\rho\rangle^{B}\langle[r]\rho\rangle^{-M''}\langle[r+t]\eta\rangle^{B}\langle[r]\eta\rangle^{-M'}.$$

Let us show that $\langle t\rangle^{-B}\langle[r+t]\rho\rangle^{B}\langle[r]\rho\rangle^{-B} \leq c$. In fact, this is evident in the regions $|r| \leq C$, $|t| \leq C$ or $|r| \geq C$, $|t| \leq C$ for some $C > 0$. For $|r| \leq C$, $|t| \geq C$ the estimate essentially follows from $1 + t^2\rho^2 \leq (1+t^2)(1+\rho^2)$. For $|r| \geq C$, $|t| \geq C$, we can suppose that $[r+t] \geq C$, and hence, $[r+t] = |r+t|$, $[r] = |r|$. Then the estimate follows from

$$\langle t\rangle^{-2}\langle[r+t]\rho\rangle^{2}\langle[r]\rho\rangle^{-2} = \frac{1+|r+t|^2|\rho|^2}{(1+|t|^2)(1+|r\rho|^2)} \leq \frac{1+|r\rho|^2+|t\rho|^2+2|rt\rho^2|}{1+|t|^2+|r\rho|^2+|rt\rho|^2}$$
$$\leq c\frac{1+|r\rho|^2+|t\rho|^2+2|rt\rho^2|}{1+|r\rho|^2+|t\rho|^2} \leq c\left(1+\frac{2|rt\rho^2|}{1+|r\rho|^2+|t\rho|^2}\right) \leq \text{const.}$$

In the last inequality we employed that

$$\frac{|rt\rho^2|}{1+|r\rho|^2+|t\rho|^2} \leq \frac{|rt|}{r^2+t^2} = \frac{|r|}{r^2+t^2}\frac{|t|}{r^2+t^2} \leq \text{const.}$$

Analogously we have $\langle t\rangle^{-B}\langle[r+t]\eta\rangle^{B}\langle[r]\eta\rangle^{-B} \leq c$. This gives us the estimate

$$I \leq c\langle r\rangle^{\nu+\tilde{\nu}}\langle t\rangle^{|\tilde{\nu}-(N+1)|-2L+2B}\langle\tau\rangle^{-2K}\langle[r]\theta\tau\rangle^{M''}\langle[r]\rho\rangle^{B-M''}\langle[r]\eta\rangle^{B-M'}.$$

Finally, using $\langle\tau\rangle^{-M''}\langle r\rangle^{-M''}\langle[r]\theta\tau\rangle^{M''} \leq c$ for all $0 \leq \theta \leq 1$ and all $r, \tau$, we obtain

$$I \leq c\langle r\rangle^{\nu+\tilde{\nu}+M''}\langle t\rangle^{|\tilde{\nu}-(N+1)|-2L+2B}\langle\tau\rangle^{-2K+M''}\langle[r]\rho\rangle^{B-M''}\langle[r]\eta\rangle^{B-M'}$$

for all $r, t \in \mathbb{R}$, $\rho, \tau \in \mathbb{R}$, $0 \leq \theta \leq 1$. Choosing $K$ and $L$ so large that

$$-2K + M'' < -1, \quad |\tilde{\nu} - (N+1)| - 2L + 2B < -1,$$

it follows that $\|r_N(r,\rho,\eta)\|_{s',s''} \leq c\langle r\rangle^{\nu+\tilde{\nu}+M''}\langle[r]\eta\rangle^{B-M'}\langle\rho\rangle^{B-M''}$ for $\eta \neq 0$. Here we used that $\langle[r]\rho\rangle^{B-M''} \leq c\langle\rho\rangle^{B-M''}$ for $B - M'' \leq 0$. Let us now show that for $B - M' \leq 0$

$$\langle[r]\eta\rangle^{B-M'} \leq c[r]^{B-M'}\langle\eta\rangle^{B-M'} \tag{2.2.28}$$

for all $|\eta| \geq \varepsilon > 0$ and some $c = c(\varepsilon) > 0$. In fact, we have

$$\frac{[r]^2\langle\eta\rangle^2}{1+|[r]\eta|^2} = \frac{[r]^2}{1+|[r]\eta|^2}\frac{\langle\eta\rangle^2}{1+|[r]\eta|^2} \sim c\frac{1}{[r]^{-2}+|\eta|^2}\frac{1}{|\eta|^{-2}+[r]^2} \leq c,$$

i.e., $(1+|[r]\eta|^2)^{-1} \leq c[r]^{-2}\langle\eta\rangle^{-2}$ which entails the estimate (2.2.28). It follows

$$\|r_N(r,\rho,\eta)\|_{s',s'} \leq c\langle r\rangle^{\nu+\tilde{\nu}+M''+B-M'}\langle\rho\rangle^{B-M''}\langle\eta\rangle^{B-M'}.$$

Now $B$ is fixed, and $M, M''$ can be chosen independently so large that

$$B - M'' \leq -k, \quad B - M' \leq -m, \quad \nu + \tilde{\nu} + M'' + B - M' \leq -l.$$

Therefore, we proved that for every $s', s'' \in \mathbb{R}$ and $k, l, m \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that

$$\|r_N(r,\rho,\eta)\|_{s',s''} \leq c\langle\rho\rangle^{-k}\langle r\rangle^{-l}\langle\eta\rangle^{-m} \tag{2.2.29}$$

for all $(r,\rho) \in \mathbb{R}^2$, $|\eta| \geq \varepsilon > 0$. In an analogous manner we can show the estimates (2.2.19) for all $i, j$. $\square$

**Proposition 2.2.10.** *For every* $a(r,\rho,\eta) \in \boldsymbol{S}^{\mu,\nu}$ *and* $\varphi(r) = [r]^{\tilde{\nu}}$ *(which belongs to* $\boldsymbol{S}^{0,\tilde{\nu}}$*) for every* $\eta \neq 0$ *we have (as operators* $\mathrm{Op}_r(\tilde{a}(r,[r]\rho,[r]\eta)) : C_0^\infty(\mathbb{R}, C^\infty(X)) \to C^\infty(\mathbb{R}, C^\infty(X))$*)*

$$\mathrm{Op}_r(a)(\eta) \circ \varphi = \varphi \circ \mathrm{Op}_r(d)(\eta) + R(\eta) \tag{2.2.30}$$

*for some* $d(r,\rho,\eta) \in \boldsymbol{S}^{\mu,\nu}$ *and a remainder* $R(\eta) = \mathrm{Op}_r(r_N)(\eta)$ *which is an operator function* $r_N(r,\rho,\eta) \in C^\infty\big(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^q_\eta, \mathcal{L}\big(H^{s'}(X), H^{s''}(X)\big)\big)$ *for every given* $s', s''$ *and sufficiently large* $N = N(s', s'') \in \mathbb{N}$*, satisfying the estimates* (2.2.19) *for all* $(r,\rho) \in \mathbb{R}^2$ *and all* $|\eta| \geq \varepsilon > 0$.

*Proof.* We apply the relation (2.2.18) to the case $b(r,\rho,\eta) = \varphi(r)$, where $N$ is so large that the remainder forms a bounded operator $H^{s'}(X) \to H^{s''}(X)$, and obtain

$$\mathrm{Op}(a) \circ \varphi = \mathrm{Op}(a \# \varphi) = \sum_{k=0}^{N} \mathrm{Op}\left(\frac{1}{k!}\partial_\rho^k a(r,\rho,\eta) D_r^k \varphi(r)\right) + R(\eta) \tag{2.2.31}$$

for $R(\eta) = \mathrm{Op}(r_N)$ and $r_N(r,\rho,\eta)$ is as in (2.2.19). For the sum on the right hand side of (2.2.31) we have

$$\sum_{k=0}^{N} \mathrm{Op}\left(\frac{1}{k!}\partial_\rho^k a(r,\rho,\eta) D_r^k \varphi(r)\right)$$
$$= \varphi(r) \circ \mathrm{Op}\left(a(r,\rho,\eta) + \partial_\rho a(r,\rho,\eta)\tilde{\nu}\frac{[r]'}{[r]} + \sum_{k=2}^{N} \frac{1}{k!}\partial_\rho^k a(r,\rho,\eta)\varphi_k(r)\right),$$

for some $\varphi_k(r) \in C_0^\infty(\mathbb{R})$, $k = 2, \dots, N$. Let us set

$$d(\eta) := a(r,\rho,\eta) + \partial_\rho a(r,\rho,\eta)\tilde{\nu}\frac{[r]'}{[r]} + \sum_{k=2}^{N} \frac{1}{k!}\partial_\rho^k a(r,\rho,\eta)\varphi_k(r).$$

Then, using Proposition 2.2.4, we easily see that $d(\eta) \in \boldsymbol{S}^{\mu,\nu}$. This completes the proof. □

Let us now return to the interpretation of (2.2.14) as a left symbol of a composition of operators. From Theorem 2.2.8 we know that

$$\mathrm{Op}_r(a)(\eta), \mathrm{Op}_r(b)(\eta) : \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big) \to \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$$

are continuous operators. Thus also $\mathrm{Op}_r(a)(\eta)\mathrm{Op}_r(b)(\eta)$ is continuous between the Schwartz spaces. This shows, in particular, that the oscillatory integral techniques of [19] also apply for our (here operator-valued) amplitude functions, and we obtain the relation (2.2.15).

Let $A(\eta) = \mathrm{Op}_r(a)(\eta)$ for

$$a(r,\rho,\eta) := \tilde{a}(r,[r]\rho,[r]\eta),\ \tilde{a}(r,\tilde{\rho},\tilde{\eta}) \in S^\nu\big(\mathbb{R}, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big).$$

Then we form the formal adjoint $A^*(\eta)$ with respect to the $L^2(\mathbb{R} \times X)$-scalar product, according to

$$\big(A(\eta)u, v\big)_{L^2(\mathbb{R}\times X)} = \big(u, A^*(\eta)v\big)_{L^2(\mathbb{R}\times X)}$$

for all $u, v \in \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$. As usual we obtain

$$A^*(\eta)v(r') = \mathrm{Op}_{r'}(a^*)(\eta)v(r')$$

for the right symbol $a^*(r', \rho, \eta) = \bar{a}(r', \rho, \eta) = \bar{\tilde{a}}(r', [r']\rho, [r']\eta)$. Similarly as before we can prove that

$$\mathrm{Op}_{r'}(a^*)(\eta) : \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big) \to \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$$

is continuous for every $\eta \neq 0$. Thus by duality it follows that

$$\mathrm{Op}_r(a)(\eta) : \mathcal{S}'\big(\mathbb{R}, \mathcal{E}'(X)\big) \to \mathcal{S}'\big(\mathbb{R}, \mathcal{E}'(X)\big) \tag{2.2.32}$$

is continuous for every $\eta \neq 0$. By $\mathcal{S}'\big(\mathbb{R}, \mathcal{E}'(X)\big)$ we mean here the space of all continuous functionals on $\mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$ with the strong topology of the bounded convergence. Another way to interpret the space $\mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$ is to say that $f \in \mathcal{E}'(X) \Leftrightarrow f \in H^s(X)$ for some real $s \in \mathbb{R}$; then $\mathcal{S}'\big(\mathbb{R}, \mathcal{E}'(X)\big)$ means the inductive limit of the spaces $\mathcal{L}\big(\mathcal{S}(\mathbb{R}), H^s(X)\big)$ over $s \in \mathbb{R}$.

**Remark 2.2.11.** *From the identifications*

$$\mathcal{E}'(X) = \bigcup_{s\in\mathbb{R}} H^s(X), \quad \mathcal{S}(\mathbb{R}) = \bigcap_{m,g\in\mathbb{R}} H^{m;g}(\mathbb{R}),$$

*for* $H^{m;g}(X) = \langle r\rangle^{-g} H^m(X)$, *we see that*

$$u \in \mathcal{S}'\big(\mathbb{R}, \mathcal{E}'(X)\big) \quad \Leftrightarrow \quad u \in \mathcal{L}\big(H^{m;g}(\mathbb{R}), H^s(X)\big)$$

*for a certain* $s \in \mathbb{R}$ *and some* $m, g \in \mathbb{R}$ *dependent on* $s$.

**Lemma 2.2.12.** *For every* $s', s'' \in \mathbb{R}$ *and* $k, l, m \in \mathbb{N}$ *there exists a real* $\mu(s', s'', k, l, m)$ *such that for every* $a(r, \rho, \eta) \in \boldsymbol{S}^{\mu,\nu}$, $\nu \in \mathbb{R}$ *we have*

$$\|a(r, \rho, \eta)\|_{s',s''} \leq c\langle\rho\rangle^{-k}\langle r\rangle^{-l}\langle\eta\rangle^{-m}$$

$(r, \rho) \in \mathbb{R}^2$, *whenever* $\mu \leq \mu(s', s'', k, l, m)$, $|\eta| \geq \varepsilon > 0$.

*Proof.* The proof is straightforward, using Theorem 2.2.2, more precisely, writing $a(r, \rho, \eta) = \tilde{a}(r, [r]\rho, [r]\eta)$, we have the estimate

$$\|\tilde{a}(r, \tilde{\rho}, \tilde{\eta})\|_{s',s''} \leq c\langle r\rangle^{\nu}\langle\tilde{\rho}, \tilde{\eta}\rangle^{-N}$$

for every fixed $N \in \mathbb{N}$ when $\mu$ is chosen sufficiently negative (depending on $N$), uniformly in $r \in \mathbb{R}$. Then, similarly as in the proof of Lemma 2.2.9, we obtain for suitable $N$ and given $k, l, m$ that $\langle [r]\rho, [r]\eta\rangle^{-N} \leq c\langle\rho\rangle^{-k}\langle r\rangle^{-l+\nu}\langle\eta\rangle^{-m}$ for $|\eta| \geq \varepsilon > 0$. □

**Corollary 2.2.13.** *Let* $a(r, \rho, \eta) \in \boldsymbol{S}^{-\infty,\nu}$ $\Big(:= \bigcap_{\mu\in\mathbb{R}} \boldsymbol{S}^{\mu,\nu}\Big)$. *Then for every* $s', s'' \in \mathbb{R}$, $k, l, m \in \mathbb{N}$ *we have*

$$\|D_r^i D_\rho^j a(r, \rho, \eta)\|_{s',s''} \leq c\langle\rho\rangle^{-k}\langle r\rangle^{-l}\langle\eta\rangle^{-m}$$

*for all* $(r, \rho) \in \mathbb{R}^2$, $|\eta| \geq \varepsilon > 0$, $i, j \in \mathbb{N}$, *for some constants* $c = c(s', s'', k, l, m, \varepsilon) > 0$.

**Proposition 2.2.14.** *The kernels $c(r,r',\eta)$ of operators $\mathrm{Op}_r(a)(\eta)$ for $a\in\boldsymbol{S}^{-\infty,\nu}$, $\nu\in\mathbb{R}$, belong to*

$$C^\infty\big(\mathbb{R}^q\setminus\{0\},\mathcal{S}\big(\mathbb{R}\times\mathbb{R},\mathcal{L}(H^{s'}(X),H^{s''}(X))\big)\big),\tag{2.2.33}$$

*for all $s',s''\in\mathbb{R}$, and are strongly decreasing in $\eta$ for $|\eta|\geq\varepsilon>0$ together with all $\eta$-derivatives, more precisely, we have*

$$\sup\|\langle\eta\rangle^\alpha D_\eta^\beta\langle r,r'\rangle^\sigma D_r^\tau D_{r'}^{\tau'}c(r,r',\eta)\|_{s',s''}<\infty\tag{2.2.34}$$

*for every $\beta\in\mathbb{N}^q$, $\alpha,\sigma,\tau,\tau'\in\mathbb{R}$ with* sup *being taken over all $|\eta|\geq\varepsilon>0$, $(r,r')\in\mathbb{R}$.*

*Proof.* If we show the result for $\nu=0$ from Proposition 2.2.4 it follows immediately for all $\nu$. Write $a(r,\rho,\eta)=\tilde{a}(r,[r]\rho,[r]\eta)$ for $\tilde{a}(r,\tilde{\rho},\tilde{\eta})\in S^0\big(\mathbb{R},L^{-\infty}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$. Then we have

$$\|D^\gamma_{\tilde{\rho},\tilde{\eta}}\tilde{a}(r,\tilde{\rho},\tilde{\eta})\|_{s',s''}\leq c\langle\tilde{\rho},\tilde{\eta}\rangle^{-N}$$

for $\gamma\in\mathbb{N}^{1+q}$ and each $N\in\mathbb{N}$. For a sufficiently large $N$ this easily gives us

$$\|D_r^iD_\rho^jD_\eta^\alpha a(r,\rho,\eta)\|_{s',s''}\leq c\langle\rho\rangle^{-k}\langle r\rangle^{-l}\langle\eta\rangle^{-m}$$

for every $k,l,m\in\mathbb{N}$, $|\eta|\geq\varepsilon>0$. Now the kernel of $\mathrm{Op}_r(a)(\eta)$ has the form

$$\int e^{i(r-r')\rho}a(r,\rho,\eta)đ\rho=\int e^{i(r-r')\rho}\big(1+|r-r'|^2\big)^{-M}(1-\Delta_\rho)^Ma(r,\rho,\eta)đ\rho\tag{2.2.35}$$

for every sufficiently large $M$. This implies

$$\begin{aligned}\Big\|\int e^{i(r-r')\rho}a(r,\rho,\eta)đ\rho\Big\|_{s',s''}&\leq\int\|\big(1+|r-r'|^2\big)^{-M}(1-\Delta_\rho)^Ma(r,\rho,\eta)\|_{s',s''}đ\rho\\&\leq c\big(1+|r-r'|^2\big)^{-M}\langle r\rangle^{-l}\langle\eta\rangle^{-m}\int\langle\rho\rangle^{-k}đ\rho\\&\leq c\big(1+|r-r'|^2\big)^{-M}\langle r\rangle^{-l}\langle\eta\rangle^{-m}<\infty\end{aligned}$$

for $k\geq 2$. In a similar manner we can treat the $(r,r')$- and $\eta$-derivatives of the kernel. ☐

**Definition 2.2.15.** (i) *Let $\boldsymbol{L}^{-\infty,-\infty}(X^\asymp;\mathbb{R}^q\setminus\{0\})$ denote the space of all operators with kernels $c(r,r',\eta)$ as in Proposition 2.2.14. Moreover, for purposes below, let $\boldsymbol{L}^{-\infty,-\infty}(X^\asymp)$ denote the space of all operators with kernels*

$$c(r,r')\in\bigcap_{s',s''\in\mathbb{R}}\mathcal{S}\big(\mathbb{R}\times\mathbb{R},\mathcal{L}\big(H^{s'}(X),H^{s''}(X)\big)\big).$$

(ii) *Let $\boldsymbol{L}^{\mu,\nu}(X^\asymp;\mathbb{R}^q\setminus\{0\})$ denote the space of all operators of the form*

$$A(\eta)=\mathrm{Op}_r(a)(\eta)+C(\eta),$$

*depending on the parameter $\eta\in\mathbb{R}^q\setminus\{0\}$, for arbitrary $a(r,\rho,\eta)\in\boldsymbol{S}^{\mu,\nu}$ and operators $C(\eta)\in\boldsymbol{L}^{-\infty,-\infty}(X^\asymp;\mathbb{R}^q\setminus\{0\})$.*

**Theorem 2.2.16.** (i) *For every* $\tilde{p}(\tilde{\rho},\tilde{\eta}) \in L^s_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$, $s \le 0$, *and* $p(r,\rho,\eta) = \tilde{p}([r]\rho,[r]\eta)$, *the operator*

$$\mathrm{Op}_r(p)(\eta) : L^2(\mathbb{R}\times X) \to L^2(\mathbb{R}\times X) \tag{2.2.36}$$

*is continuous for every* $\eta \in \mathbb{R}^q \setminus \{0\}$, *and we have*

$$\|\mathrm{Op}_r(p)(\eta)\|_{\mathcal{L}(L^2(\mathbb{R}\times X))} \le c\langle\eta\rangle^s \tag{2.2.37}$$

*for all* $|\eta| \ge \varepsilon, \varepsilon > 0$ *and a constant* $c = c(\varepsilon) > 0$.

(ii) *In the case* $s < 0$ *the operator*

$$[r]^{-s+g}\mathrm{Op}_r(p)(\eta) : L^2(\mathbb{R}\times X) \to L^2(\mathbb{R}\times X) \tag{2.2.38}$$

*is compact for every* $g < 0$ *and* $\eta \ne 0$.

*Proof.* (i) For the continuity (2.2.36) and the estimate (2.2.37) we apply a version of Calderón-Vaillancourt theorem which states that if $H$ is a Hilbert space and $a(r,\rho) \in C^\infty\big(\mathbb{R}\times\mathbb{R},\mathcal{L}(H)\big)$ is a symbol satisfying the estimate

$$\pi(a) := \sup_{\substack{k,l=0,1\\(r,\rho)\in\mathbb{R}^2}} \|D^k_r D^l_\rho a(r,\rho)\|_{\mathcal{L}(H)} < \infty \tag{2.2.39}$$

the operator

$$\mathrm{Op}_r(a) : L^2(\mathbb{R},H) \to L^2(\mathbb{R},H)$$

is continuous, where

$$\|\mathrm{Op}_r(a)\|_{\mathcal{L}(L^2(\mathbb{R},H))} \le c\pi(a)$$

for a constant $c > 0$. In the present case we have

$$a(r,\rho) = \tilde{p}([r]\rho,[r]\eta) \tag{2.2.40}$$

where $\eta \ne 0$ appears as an extra parameter. It is evident that the right hand side of (2.2.40) belongs to $C^\infty\big(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^q,\mathcal{L}(L^2(X))\big)$. From the assumption on $\tilde{p}(\tilde{\rho},\tilde{\eta})$ we have

$$\sup_{r\in\mathbb{R}} \|\tilde{p}(\tilde{\rho},\tilde{\eta})\|_{\mathcal{L}(L^2(X))} \le c\langle\tilde{\rho},\tilde{\eta}\rangle^s \tag{2.2.41}$$

for all $(\tilde{\rho},\tilde{\eta}) \in \mathbb{R}^{1+q}$ and some $c > 0$. In fact, the latter estimate corresponds to (2.2.5) for $s = \nu = 0$ and $\mu = s \le 0$. For (2.2.39) we first check the case $l = k = 0$. We have

$$\sup_{(r,\rho)\in\mathbb{R}^2} \langle[r]\rho,[r]\eta\rangle^s \le c\langle\eta\rangle^s \tag{2.2.42}$$

for all $|\eta| \ge \varepsilon > 0$ and some $c = c(\varepsilon) > 0$. Thus (2.2.41) gives us

$$\sup_{(r,\rho)\in\mathbb{R}^2} \|\tilde{p}([r]\rho,[r]\eta)\|_{\mathcal{L}(L^2(X))} \le c\langle\eta\rangle^s$$

for such a $c(\varepsilon) > 0$. Assume now for simplicity $q = 1$ (the general case is then straightforward). For the first order derivatives of $\tilde{p}([r]\rho,[r]\eta)$ in $r$ we have

$$\partial_r\tilde{p}([r]\rho,[r]\eta) = [r]'(\rho\partial_{\tilde{\rho}} + \eta\partial_{\tilde{\eta}})\tilde{p}([r]\rho,[r]\eta) \tag{2.2.43}$$

for $[r]' = \frac{d}{dr}[r]$. For the derivatives of $\tilde{p}$ with respect to $\tilde{\rho}, \tilde{\eta}$ we employ that $\partial_{\tilde{\rho}}\tilde{p}(\tilde{\rho}, \tilde{\eta})$, $\partial_{\tilde{\eta}}\tilde{p}(\tilde{\rho}, \tilde{\eta}) \in L^{s-1}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$. Thus, similarly as before we obtain

$$\|\partial^{\alpha}_{\tilde{\rho},\tilde{\eta}}\tilde{p}(\tilde{\rho}, \tilde{\eta})\|_{\mathcal{L}(L^2(X))} \leq c\langle\tilde{\rho}, \tilde{\eta}\rangle^{s-1}$$

for any $\alpha \in \mathbb{N}^2$, $|\alpha| = 1$. This gives us for the right hand side of (2.2.43)

$$\begin{aligned} \sup_{(r,\rho)\in\mathbb{R}^2} &\|[r]^{-1}[r]'([r]\rho\partial_{\tilde{\rho}} + [r]\eta\partial_{\tilde{\eta}})p([r]\rho, [r]\eta)\|_{\mathcal{L}(L^2(X))} \\ &\leq \sup[r]^{-1}|[r]\rho + [r]\eta|\langle[r]\rho, [r]\eta\rangle^{s-1} \\ &\leq c\langle\eta\rangle^s \sup[r]^{-1}|[r]\rho, [r]\eta|\langle[r]\rho, [r]\eta\rangle^{-1} \leq c\langle\eta\rangle^s. \end{aligned}$$

Here we employed (2.2.42). For the derivative of $p([r]\rho, [r]\eta)$ in $\rho$ we have

$$\begin{aligned} \sup \|\partial_\rho\tilde{p}([r]\rho, [r]\eta)\|_{\mathcal{L}(L^2(X))} &= \sup \|[r](\partial_{\tilde{\rho}}\tilde{p})([r]\rho, [r]\eta)\|_{\mathcal{L}(L^2(X))} \\ &\leq c \sup[r]\langle[r]\rho, [r]\eta\rangle^{s-1} \leq c\langle\eta\rangle^s \end{aligned}$$

for all $|\eta| \geq \varepsilon > 0$. This gives altogether the estimate (2.2.37). To obtain (2.2.36) we only need to note that $L^2\big(\mathbb{R}, L^2(X)\big) = L^2(\mathbb{R} \times X)$.

(ii) For $s < 0$ the operator

$$[r]^{-s+g}\mathrm{Op}_r(p)(\eta), \eta \neq 0 \text{ fixed},$$

can be regarded as an operator with symbol

$$a(r, \rho) \in S^{s;g}\big(\mathbb{R} \times \mathbb{R}; L^2(X), L^2(X)\big) \tag{2.2.44}$$

with values in compact operators $L^2(X) \to L^2(X)$, since $X$ is compact. The symbol class on the right of (2.2.44) refers to the trivial group action on $L^2(X)$ (cf. the notation in (3.1.8)). In order to verify (2.2.44) we have to check the estimate

$$\|\partial^k_r\partial^l_\rho[r]^{-s+g}\tilde{p}([r]\rho, [r]\eta)\|_{\mathcal{L}(L^2(X))} \leq c\langle r\rangle^{g-k}\langle\rho\rangle^{s-l} \tag{2.2.45}$$

for all $(r, \rho) \in \mathbb{R}^2$ and $k, l \in \mathbb{N}$. First, because of

$$\|[r]^{-s+g}\tilde{p}([r]\rho, [r]\eta)\|_{\mathcal{L}(L^2(X))} \leq c[r]^{-s+g}\langle[r]\rho, [r]\eta\rangle^s \leq c\langle r\rangle^g\langle\rho\rangle^s \tag{2.2.46}$$

for all $(r, \rho) \in \mathbb{R}^2$ and some constant $c > 0$, the estimate (2.2.45) is true for $k = l = 0$. Let us now check (2.2.45) for $k = 0$, $l = 1$. In this case we have $\partial_\rho[r]^{-s+g}\tilde{p}([r]\rho, [r]\eta) = [r]^{-s+g+1}(\partial_{\tilde{\rho}}\tilde{p})([r]\rho, [r]\eta)$. Since $(\partial_{\tilde{\rho}}\tilde{p})(\tilde{\rho}, \tilde{\eta}) \in L^{s-1}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$ we are in the same situation as (2.2.46) with $s - 1$ instead of $s$. Inductively we obtain (2.2.45) for arbitrary $l \in \mathbb{N}$ and $k = 0$. For $k = 1$ we have

$$\begin{aligned} \partial_r\Big([r]^{-s+g+l}(\partial^l_{\tilde{\rho}}\tilde{p})([r]\rho, [r]\eta)\Big) &= (-s + g + l)[r]'[r]^{-s+g+l-1}(\partial^l_{\tilde{\rho}}\tilde{p})([r]\rho, [r]\eta) \\ + [r]^{-s+g+l-1}[r]'([r]\rho)(\partial^{l+1}_{\tilde{\rho}}\tilde{p})([r]\rho, [r]\eta) &+ [r]^{-s+g+l-1}[r]'([r]\eta)(\partial_{\tilde{\eta}}\partial^l_{\tilde{\rho}}\tilde{p})([r]\rho, [r]\eta). \end{aligned}$$

For the first term on the right hand side we have

$$\|(-s + g + l)[r]'[r]^{-1}[r]^{-s+g+l}(\partial^l_{\tilde{\rho}}\tilde{p})([r]\rho, [r]\eta)\|_{\mathcal{L}(L^2(X))} \leq c[r]^{g-1}\langle\rho\rangle^{s-l}.$$

The other two terms admit analogous estimates since $\tilde{\rho}(\partial_{\tilde{\rho}}^{l+1}\tilde{p})(\tilde{\rho},\tilde{\eta}), \tilde{\eta}(\partial_{\tilde{\eta}}\partial_{\tilde{\rho}}^{l}\tilde{p})(\tilde{\rho},\tilde{\eta}) \in L_{\mathrm{cl}}^{s-l}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$. In other words we obtain (2.2.45) for $k=1$ and arbitrary $l$. The higher $r$-derivatives can be treated in an analogous manner, i.e., we obtain (2.2.45) in general. Consequently we have proved that

$$[r]^{-s+g}\tilde{p}([r]\rho,[r]\eta) \in S^{s;g}\big(\mathbb{R}\times\mathbb{R}; L^2(X), L^2(X)\big).$$

The values of the operator function $[r]^{-s+g}\tilde{p}([r]\rho,[r]\eta)$ are compact for $s,g<0$. Finally we use the following lemma, its proof can be found in [43, Chapter 7]:

**Lemma 2.2.17.** *Let $H, \widetilde{H}$ be two Hilbert spaces with group actions, and $a(y,\eta) \in S^{s;g}(\mathbb{R}^q\times\mathbb{R}^q; H,\widetilde{H})$, $s,g<0$; moreover, let $a(y,\eta): H\to\widetilde{H}$ be a compact operator for every $(y,\eta)\in\mathbb{R}^{2q}$. Then*

$$\mathrm{Op}(a): \langle y\rangle^{-g'}\mathcal{W}^{s'}(\mathbb{R}^q,H) \to \langle y\rangle^{g'}\mathcal{W}^{s'}(\mathbb{R}^q,\widetilde{H})$$

*is compact for every $s',g'\in\mathbb{R}$.*

By virtue of Lemma 2.2.17 the operator

$$[r]^{-s+g}\mathrm{Op}(p): \mathcal{W}^0\big(\mathbb{R},L^2(X)\big) \to \mathcal{W}^0\big(\mathbb{R},L^2(X)\big)$$

is compact. It remains to note that

$$\mathcal{W}^0\big(\mathbb{R},L^2(X)\big) = L^2\big(\mathbb{R},L^2(X)\big) = L^2(\mathbb{R}\times X).$$

□

**Remark 2.2.18.** *Using Calderón-Vaillancourt theorem one can also prove that any operator $A\in \boldsymbol{L}^{0;0}(X^{\asymp};\mathbb{R}^q\setminus\{0\})$ induces continuous operators*

$$A(\eta): L^2(\mathbb{R}\times X)\to L^2(\mathbb{R}\times X).$$

**Theorem 2.2.19.** *Let $a\in\boldsymbol{S}^{\mu,\nu}, b\in\boldsymbol{S}^{\tilde{\mu},\tilde{\nu}}$; then we have*

$$\mathrm{Op}_r(a)(\eta)\mathrm{Op}_r(b)(\eta) \in \boldsymbol{L}^{\mu+\tilde{\mu},\nu+\tilde{\nu}}(X^{\asymp};\mathbb{R}^q\setminus\{0\}).$$

*Proof.* According to (2.2.15) the composition can be expressed by $a\#b$, given by the formula (2.2.14). By virtue of Corollary 2.2.5 we have

$$\frac{1}{k!}\partial_\rho^k a(r,\rho,\eta)D_r^k b(r,\rho,\eta) \in \boldsymbol{S}^{\mu+\tilde{\mu}-k,\nu+\tilde{\nu}},$$

i.e., this symbol has the form $c_k(r,\rho,\eta)=\tilde{c}_k(r,[r]\rho,[r]\eta)$ for some

$$\tilde{c}_k(r,\tilde{\rho},\tilde{\eta}) \in S^{\nu+\tilde{\nu}}\big(\mathbb{R}, L_{\mathrm{cl}}^{\mu+\tilde{\mu}-k}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big).$$

Applying Proposition 2.2.7 we form the asymptotic sum

$$\sum_{k=0}^{\infty}\tilde{c}_k(r,\tilde{\rho},\tilde{\eta}) \sim \tilde{c}(r,\tilde{\rho},\tilde{\eta}) \in S^{\nu+\tilde{\nu}}\big(\mathbb{R}, L_{\mathrm{cl}}^{\mu+\tilde{\mu}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big).$$

Setting $c(r,\rho,\eta) = \tilde{c}(r,[r]\rho,[r]\eta)$ from (2.2.18) we obtain

$$\mathrm{Op}_r(a\#b)(\eta) = \mathrm{Op}_r(c)(\eta) + \mathrm{Op}_r\left(\sum_{k=0}^{N} c_k - c\right)(\eta) + \mathrm{Op}_r(r_N)(\eta)$$

modulo $\boldsymbol{L}^{-\infty,-\infty}(X^{\asymp};\mathbb{R}^q\setminus\{0\})$, where $\left(\sum_{k=0}^{N} c_k - c\right)(r,\rho,\eta) \in \boldsymbol{S}^{\mu+\tilde{\mu}-(N+1),\nu}$. Since this is true for every $N \in \mathbb{N}$ Lemma 2.2.12 gives us the right remainder estimate also for $\|\mathrm{Op}_r(\sum_{k=0}^{N} c_k - c)\|_{s',s''}$, and it follows altogether that the kernel of $\mathrm{Op}_r(a\#b)(\eta) - \mathrm{Op}_r(c)(\eta)$ has finite semi-norms (2.2.34) as indicated in Proposition 2.2.14 for arbitrary $\alpha,\beta \in \mathbb{N}^q$, $\sigma,\tau,\tau' \in \mathbb{R}$, $s',s'' \in \mathbb{R}$, $|\eta| \geq \varepsilon > 0$. □

**Theorem 2.2.20.** *Let $\tilde{p}(\tilde{\rho},\tilde{\eta}) \in L^s_{\mathrm{cl}}(X;\mathbb{R}^{1+q})$ be parameter-dependent elliptic of order $s \in \mathbb{R}$, and set $p(r,\rho,\eta) = \tilde{p}([r]\rho,[r]\eta)$. Then there exists a $C > 0$ such that for every $|\eta| \geq C$ the operator*

$$[r]^{-s}\mathrm{Op}_r(p)(\eta) : \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big) \to \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big) \tag{2.2.47}$$

*extends to an injective operator*

$$[r]^{-s}\mathrm{Op}_r(p)(\eta) : L^2(\mathbb{R}\times X) \to \mathcal{S}'\big(\mathbb{R}, \mathcal{E}'(X)\big). \tag{2.2.48}$$

*More precisely, considering $[r]^{-s}\mathrm{Op}_r(p)(\eta)$ as an operator*

$$[r]^{-s}\mathrm{Op}_r(p)(\eta) : L^2(\mathbb{R}\times X) \to \mathcal{L}\big(\mathcal{S}(\mathbb{R}), H^t(X)\big), \tag{2.2.49}$$

*which is continuous for some $t \in \mathbb{R}$, then it is injective.*

*Proof.* First, according to (2.2.32) there is a $t$ such that (2.2.49) is continuous for all $g, l \in \mathbb{R}$. For the injectivity we show that the operator has a left inverse. This will be approximated by $\mathrm{Op}_r(a)$ for

$$a(r,\rho,\eta) := [r]^s \tilde{p}^{(-1)}([r]\rho,[r]\eta) \tag{2.2.50}$$

where $\tilde{p}^{(-1)}(\tilde{\rho},\tilde{\eta}) \in L^{-s}_{\mathrm{cl}}(X;\mathbb{R}^{1+q})$ is a parameter-dependent parametrix of $\tilde{p}(\tilde{\rho},\tilde{\eta})$. Setting

$$b(r,\rho,\eta) := [r]^{-s}\tilde{p}([r]\rho,[r]\eta) \tag{2.2.51}$$

we can write the composition of the associated pseudo-differential operators in $r$ for every $N \in \mathbb{N}$ in the form

$$\mathrm{Op}_r(a)(\eta)\mathrm{Op}_r(b)(\eta) = \mathrm{Op}_r(a\#b)(\eta) = \mathrm{Op}_r\big(1 + c_N(r,\rho,\eta) + r_N(r,\rho,\eta)\big) \tag{2.2.52}$$

for $c_N(r,\rho,\eta) = \sum_{k=1}^{N} \frac{1}{k!}\partial_\rho^k a(r,\rho,\eta) D_r^k b(r,\rho,\eta)$ has the form

$$c_N(r,\rho,\eta) = \tilde{c}_N(r,[r]\rho,[r]\eta) \text{ for some } \tilde{c}_N(r,\tilde{\rho},\tilde{\eta}) \in S^0\big(\mathbb{R}, L^{-1}_{\mathrm{cl}}(X;\mathbb{R}^{1+q})\big).$$

Moreover, the remainder $r_N$ is as in (2.2.17). From Theorem 2.2.16 for $s = -1$ we know that

$$\|\mathrm{Op}_r(c_N)(\eta)\|_{\mathcal{L}(L^2(\mathbb{R}\times X))} \leq c\langle\eta\rangle^{-1}$$

for $|\eta| > \varepsilon$. Moreover, Lemma 2.2.9, applied to $s' = s'' = 0$ together with an operator-valued version of the Calderón-Vaillancourt theorem, gives us

$$\|\mathrm{Op}_r(r_N)(\eta)\|_{\mathcal{L}(L^2(\mathbb{R}\times X))} \leq c\langle\eta\rangle^{-1}$$

for sufficiently large $N$. Thus for every $|\eta|$ sufficiently large the operator on the right of (2.2.52) is invertible in $L^2(\mathbb{R}\times X)$, i.e., $\mathrm{Op}_r(b)(\eta)$ has a left inverse which implies the injectivity. □

**Theorem 2.2.21.** *$A \in \boldsymbol{L}^{\mu,\nu}(X^{\asymp};\mathbb{R}^q\setminus\{0\})$, $B \in \boldsymbol{L}^{\tilde\mu,\tilde\nu}(X^{\asymp};\mathbb{R}^q\setminus\{0\})$ implies $AB \in \boldsymbol{L}^{\mu+\tilde\mu,\nu+\tilde\nu}(X^{\asymp};\mathbb{R}^q\setminus\{0\})$.*

*Proof.* Let us write

$$A(\eta) = \mathrm{Op}_r(a)(\eta) + C_1(\eta),\ B(\eta) = \mathrm{Op}_r(b)(\eta) + C_2(\eta)$$

for $a \in \boldsymbol{S}^{\mu,\nu}$, $b \in \boldsymbol{S}^{\tilde\mu,\tilde\nu}$, and $C_1(\eta), C_2(\eta) \in \boldsymbol{L}^{-\infty,-\infty}(X^{\asymp};\mathbb{R}^q\setminus\{0\})$. Then we have

$$AB = \mathrm{Op}_r(a)(\eta)\mathrm{Op}_r(b)(\eta) + C_1(\eta)\mathrm{Op}_r(b)(\eta) + \mathrm{Op}_r(a)(\eta)C_2(\eta) + C_1(\eta)C_2(\eta).$$

Theorem 2.2.19 implies that $\mathrm{Op}_r(a)(\eta)\mathrm{Op}_r(b)(\eta) \in \boldsymbol{L}^{\mu+\tilde\mu,\nu+\tilde\nu}(X^{\asymp};\mathbb{R}^q\setminus\{0\})$. Moreover, the composition of smoothing families is again smoothing. It remains to show that

$$C_1(\eta)\mathrm{Op}_r(b)(\eta), \mathrm{Op}_r(a)(\eta)C_2(\eta) \in \boldsymbol{L}^{-\infty,-\infty}(X^{\asymp};\mathbb{R}^q\setminus\{0\}). \tag{2.2.53}$$

To this end we write $C_1(\eta) = \mathrm{Op}_r(c_1)(\eta)$, $C_2(\eta) = \mathrm{Op}_r(c_2)(\eta)$ for $c_1 \in \boldsymbol{S}^{-\infty,\nu_1}$, $c_2 \in \boldsymbol{S}^{-\infty,\nu_2}$ for some $\nu_1,\nu_2 \in \mathbb{R}$. Then, by virtue of (2.2.15), $C_1(\eta)\mathrm{Op}_r(b)(\eta) = \mathrm{Op}_r(c_1\#b)(\eta)$, $\mathrm{Op}_r(a)C_2(\eta) = \mathrm{Op}_r(a\#c_2)(\eta)$. Finally we observe that $c_1\#b \in \boldsymbol{S}^{-\infty,\nu_1+\tilde\nu}$ and $a\#c_2 \in \boldsymbol{S}^{-\infty,\nu+\nu_2}$ and hence they are as in Corollary 2.2.13, which completes the proof. □

**Remark 2.2.22.** *It is sometimes desirable to consider operators of the form $A(\eta^1)$ for some $A(\eta) \in \boldsymbol{L}^{\mu,\nu}(X^{\asymp};\mathbb{R}^q\setminus\{0\})$ where $\eta^1 \in \mathbb{R}^q\setminus\{0\}$ fixed. Then we can easily pass to new parameter-dependent situation by replacing $\eta^1$ by $\delta\eta^1, \delta \in \mathbb{R}$. This produces a family $A(\delta\eta^1) \in \boldsymbol{L}^{\mu,\nu}(X^{\asymp};\mathbb{R}\setminus\{0\})$. For instance, if $A$ and $B$ are two operators, in order to characterise the composition*

$$A(\eta^1)B(\eta^2)$$

*for fixed $\eta^1,\eta^2 \in \mathbb{R}^q\setminus\{0\}$ we can apply Theorem 2.2.21 to $A(\delta\eta^1), B(\delta\eta^2)$, and then set $\delta = 1$.*

**Definition 2.2.23.** *An $A \in \boldsymbol{L}^{\mu,-\mu+\nu}(X^{\asymp};\mathbb{R}^q\setminus\{0\})$ is called elliptic if it can be written in the form*

$$A(\eta) = \mathrm{Op}_r(a)(\eta) + C(\eta)$$

*for $C(\eta) \in \boldsymbol{L}^{-\infty,-\infty}(X^{\asymp};\mathbb{R}^q\setminus\{0\})$, $a(r,\rho,\eta) \in \boldsymbol{S}^{\mu,-\mu+\nu}$ for which there is a $b(r,\rho,\eta) \in \boldsymbol{S}^{-\mu,\mu-\nu}$ such that*

$$1 - a(r,\rho,\eta)b(r,\rho,\eta), 1 - b(r,\rho,\eta)a(r,\rho,\eta) \in \boldsymbol{S}^{-1,-1}.$$

**Proposition 2.2.24.** *Let* $A_1 \in \boldsymbol{L}^{\mu,-\mu+\nu}(X^{\asymp};\mathbb{R}^q \setminus \{0\})$, $A_2 \in \boldsymbol{L}^{\tilde{\mu},-\tilde{\mu}+\tilde{\nu}}(X^{\asymp};\mathbb{R}^q \setminus \{0\})$ *be elliptic. Then* $A_1A_2 \in \boldsymbol{L}^{\mu+\tilde{\mu},-(\mu+\tilde{\mu})+\nu+\tilde{\nu}}(X^{\asymp};\mathbb{R}^q \setminus \{0\})$ *is also elliptic.*

*Proof.* By definition we can write

$$A_i = \mathrm{Op}(a_i) + C_i$$

$a_i(r,\rho,\eta) = \tilde{a}_i(r,[r]\rho,[r]\eta)$, $i = 1,2$, and there are corresponding symbols $b_i(r,\rho,\eta) = \tilde{b}_i(r,[r]\rho,[r]\eta)$, $i = 1,2$. Then we have

$$A_1A_2 = \mathrm{Op}(a_1\#a_2) = \mathrm{Op}(a_3) + C_3$$

for $a_3(r,\rho,\eta) = \tilde{a}_3(r,[r]\rho,[r]\eta) \in \boldsymbol{S}^{\mu+\tilde{\mu},-(\mu+\tilde{\mu})+\nu+\tilde{\nu}}$, $C_3 \in \boldsymbol{L}^{-\infty,-\infty}$. Now it suffices to set $b_3(r,\rho,\eta) := b_2(r,\rho,\eta)b_1(r,\rho,\eta)$, and it follows that

$$1 - a_3b_3 = 1 - (a_1\#a_2)b_2b_1 = 1 - a_1a_2b_2b_1 - (a_1\#a_2 - a_1a_2)b_2b_1. \tag{2.2.54}$$

Using $a_2b_2 = 1 + c_2$, $a_1b_1 = 1 + c_1$ for $c_1, c_2 \in \boldsymbol{S}^{-1,-1}$ it follows that

$$1-a_1a_2b_2b_1 = 1-a_1(1+c_2)b_1 = 1-a_1b_1-a_1c_2b_1 = 1-(1+c_1)-a_1c_2b_1 = -c_1-a_1c_2b_1.$$

From Proposition 2.2.4 (iii) it follows that the right hand side of the latter relation belongs to $\boldsymbol{S}^{-1,-1}$. Moreover, we have

$$(a_1\#a_2 - a_1a_2)b_1b_2 \in \boldsymbol{S}^{\mu+\tilde{\mu}-1,-(\mu+\tilde{\mu})+\nu+\tilde{\nu}-1}\boldsymbol{S}^{-(\mu+\tilde{\mu}),\mu+\tilde{\mu}-(\nu+\tilde{\nu})} \subset \boldsymbol{S}^{-1,-1}.$$

Thus the right hand side of (2.2.54) belongs to $\boldsymbol{S}^{-1,-1}$. In a similar manner we can check that $1 - b_3a_3 \in \boldsymbol{S}^{-1,-1}$. □

**Theorem 2.2.25.** *Let* $A \in \boldsymbol{L}^{\mu,-\mu+\nu}(X^{\asymp};\mathbb{R}^q \setminus \{0\})$ *be elliptic. Then there exists a parametrix* $P \in \boldsymbol{L}^{-\mu,\mu-\nu}(X^{\asymp};\mathbb{R}^q \setminus \{0\})$ *in the sense that*

$$1 - AP,\ 1 - PA \in \boldsymbol{L}^{-\infty,-\infty}(X^{\asymp};\mathbb{R}^q \setminus \{0\}).$$

*Proof.* By definition we can write $A = \mathrm{Op}(a) + C$ for an $a \in \boldsymbol{S}^{\mu,-\mu+\nu}$ such that for some $p \in \boldsymbol{S}^{-\mu,\mu-\nu}$ we have $1 - ap, 1 - pa \in \boldsymbol{S}^{-1,-1}$, and $C \in \boldsymbol{L}^{-\infty,-\infty}(X^{\asymp};\mathbb{R}^q \setminus \{0\})$. Let us form $P_0 = \mathrm{Op}(p)$; then

$$AP_0 = \mathrm{Op}(a\#p) \mod \boldsymbol{L}^{-\infty,-\infty}.$$

Let us write $a\#p = ap + (a\#p - ap)$. From $a\#p = ap \mod \boldsymbol{S}^{-1,-1}$ and $ap = 1 \mod \boldsymbol{S}^{-1,-1}$ it follows that $AP_0 = 1 + D$ for some $D \in \boldsymbol{L}^{-1,-1}$. A formal Neumann series argument gives us a $K \in \boldsymbol{L}^{-1,-1}$ such that $(1 + D)(1 + K) = 1 + C$ for some $C \in \boldsymbol{L}^{-\infty,-\infty}$, and $P_0(1 + K) \in \boldsymbol{L}^{-\mu,\mu-\nu}(X^{\asymp};\mathbb{R}^q \setminus \{0\})$ is then a right parametrix of $A$. In a similar manner we conclude that there is a left parametrix, i.e., we may set $P = P_0(1 + K)$. □

## 2.3 Parameter-dependent operators on an infinite cylinder

### 2.3.1 Weighted cylindrical spaces

**Definition 2.3.1.** *Let $s, g \in \mathbb{R}$ and fix some $\tilde{p}(\tilde{\rho}, \tilde{\eta}) \in L^s_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$ as in Theorem* 2.2.20. *Then $H^{s;g}_{\mathrm{cone}}(X^{\asymp})$ is defined to be the completion of $\mathcal{S}(\mathbb{R} \times X)$ with respect to the norm*

$$\|[r]^{-s+g+\frac{n}{2}} \mathrm{Op}_r(p)(\eta^1)u\|_{L^2(\mathbb{R}\times X)}$$

*for any fixed $\eta^1 \in \mathbb{R}^q$, $|\eta^1| \geq C$ for some $C > 0$ sufficiently large, and $n = \dim X$.*

Setting $p^{s,g}(r, \rho, \eta) := [r]^{-s+g+\frac{n}{2}} \tilde{p}([r]\rho, [r]\eta)$, from Definition 2.3.1 it follows that

$$\mathrm{Op}(p^{s,g})(\eta^1) : \mathcal{S}(\mathbb{R} \times X) \to \mathcal{S}(\mathbb{R} \times X)$$

extends to a continuous operator

$$\mathrm{Op}(p^{s,g})(\eta^1) : H^{s;g}_{\mathrm{cone}}(X^{\asymp}) \to L^2(\mathbb{R} \times X). \tag{2.3.1}$$

Moreover, the operator $P(\eta) = \mathrm{Op}(p^{s,g})(\eta) \in \boldsymbol{L}^{s,-s+g+\frac{n}{2}}(X^{\asymp}; \mathbb{R}^q \setminus \{0\})$ is elliptic and hence it has a parametrix $P^{(-1)}(\eta) \in \boldsymbol{L}^{-s,s-g-\frac{n}{2}}(X^{\asymp}; \mathbb{R}^q \setminus \{0\})$. We can choose $P^{(-1)}(\eta)$ in such a way that for some $C > 0$

$$P^{(-1)}(\eta) = P^{-1}(\eta) \text{ for } |\eta| > C.$$

In fact, the relation

$$1 - P(\eta)P^{(-1)}(\eta) = C(\eta) \in \boldsymbol{L}^{-\infty,-\infty}(X^{\asymp}; \mathbb{R}^q \setminus \{0\})$$

allows us to replace $P^{(-1)}(\eta)$ by $\big(1 - \chi(\eta)\big)P^{(-1)}(\eta) + \chi(\eta)P^{(-1)}(\eta)\big(1 - C(\eta)\big)^{-1}$ for an excision function $\chi(\eta)$ such that $\chi(\eta) = 1$ for $|\eta| \geq C$ so large that $\big(1 - C(\eta)\big)^{-1}$ exists.

**Theorem 2.3.2.** *The operator* (2.3.1) *is an isomorphism for every fixed $s, g \in \mathbb{R}$ and $|\eta^1|$ sufficiently large.*

*Proof.* We show the invertibility by verifying that there is a right and a left inverse. By notation we have $p^{s,g}(r, \rho, \eta) = [r]^{-s+g+\frac{n}{2}} \tilde{p}([r]\rho, [r]\eta) \in \boldsymbol{S}^{s,-s+g+\frac{n}{2}}$. The operator family $\tilde{p}(\tilde{\rho}, \tilde{\eta}) \in L^s_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$ is invertible for large $|\tilde{\rho}, \tilde{\eta}| \geq C$ for some $C > 0$. There exists a parameter-dependent parametrix $\tilde{p}^{(-1)}(\tilde{\rho}, \tilde{\eta}) \in L^{-s}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$ such that $\tilde{p}^{(-1)}(\tilde{\rho}, \tilde{\eta}) = \tilde{p}^{-1}(\tilde{\rho}, \tilde{\eta})$ for $|\tilde{\rho}, \tilde{\eta}| \geq C$. Let us set

$$p^{-s,-g}(r, \rho, \eta) := [r]^{s-g-\frac{n}{2}} \tilde{p}^{(-1)}([r]\rho, [r]\eta) \in \boldsymbol{S}^{-s,s-g-\frac{n}{2}},$$

and $P^{s,g}(\eta) := \mathrm{Op}(p^{s,g})(\eta)$, $P^{-s,-g}(\eta) := \mathrm{Op}(p^{-s,-g})(\eta)$. Then we have

$$P^{s,g}(\eta)P^{-s,-g}(\eta) = 1 + \mathrm{Op}(c_N)(\eta) + R_N(\eta) \tag{2.3.2}$$

for some $c_N(r,\rho,\eta) \in \boldsymbol{S}^{-1,0}$ and a remainder $R_N(\eta) = \mathrm{Op}(r_N)(\eta)$ where $r_N$ is as in Lemma 2.2.9. We have $\mathrm{Op}(c_N)(\eta) \to 0$ and $R_N(\eta) \to 0$ in $\mathcal{L}(L^2(\mathbb{R}\times X))$ as $|\eta| \to \infty$; the first property is a consequence of Theorem 2.2.16 (i), the second of the estimate (2.2.19). Thus (2.3.2) shows that $P^{s,g}(\eta)$ has a right inverse for $|\eta|$ sufficiently large. Such considerations remain true when we interchange the role of $s, g$ and $-s, -g$. In other words, we also have

$$P^{-s,-g}(\eta)P^{s,g}(\eta) = 1 + \mathrm{Op}(\tilde{c}_N)(\eta) + \widetilde{R}_N(\eta)$$

where $\mathrm{Op}(\tilde{c}_N)(\eta)$ and $\widetilde{R}_N(\eta)$ are of analogous behaviour as before. This shows that $P^{s,g}(\eta)$ has a left inverse for large $|\eta|$, and we obtain altogether that (2.3.1) is an isomorphism for $\eta = \eta^1$, $|\eta^1|$ sufficiently large. □

### 2.3.2 Elements of the calculus

The results of Section 2.2.3 show the behaviour of compositions of parameter-dependent families $\mathrm{Op}(a)(\eta)$ for $a(r,\rho,\eta) \in \boldsymbol{S}^{\mu,\nu}$ and $\eta \neq 0$, first on $\mathcal{S}(\mathbb{R}\times X)$. In particular, it can be proved that, when we concentrate, for instance, on the case $s' = s'' = 0$, the inverses of operators of the form $1 + K : L^2(\mathbb{R}\times X) \to L^2(\mathbb{R}\times X)$, for $K \in \boldsymbol{L}^{-\infty,-\infty}(X^{\asymp}; \mathbb{R}^q \setminus \{0\})$, can be written in the form $1 + L$ where $L$ is again an operator of such a smoothing behaviour. Moreover, there are other (more or less standard) constructions that are immediate by the results of Section 2.2. For instance, if we look at an element $c(r,\rho,\eta) \in \boldsymbol{S}^{-1,0}$ as $c_N$ in the relation (2.3.2). By a formal Neumann series argument we find a $d(r,\rho,\eta) \in \boldsymbol{S}^{-1,0}$ such that

$$\big(1 + \mathrm{Op}(c)\big)\big(1 + \mathrm{Op}(d)\big) = 1 + \mathrm{Op}(r_M)$$

for every $M \in \mathbb{N}$ with a remainder $r_M$ which is again as in Lemma 2.2.9.

**Theorem 2.3.3.** *Let $a(r,\rho,\eta) \in \boldsymbol{S}^{\mu,-\mu+\nu}$ and $|\eta| \neq 0$. Then*

$$\mathrm{Op}(a)(\eta) : \mathcal{S}(\mathbb{R}\times X) \to \mathcal{S}(\mathbb{R}\times X)$$

*extends to a continuous operator*

$$\mathrm{Op}(a)(\eta) : H^{s;g}_{\mathrm{cone}}(X^{\asymp}) \to H^{s-\mu;g-\nu}_{\mathrm{cone}}(X^{\asymp}) \tag{2.3.3}$$

*for every $s, g \in \mathbb{R}$.*

*Proof.* Let $u \in \mathcal{S}(\mathbb{R}\times X)$, and set $\|\cdot\|_{s;g} := \|\cdot\|_{H^{s;g}_{\mathrm{cone}}(X^{\asymp})}$, in particular, $\|\cdot\|_{0;0} = \|\cdot\|_{L^2(\mathbb{R}\times X)}$. By definition we have $\|u\|_{s;g} = \|\mathrm{Op}(p^{s,g})(\eta^1)u\|_{0;0}$. Then we have

$$\begin{aligned}\|\mathrm{Op}(a)(\eta)u\|_{s-\mu;g-\nu} &= \|\mathrm{Op}(p^{s-\mu;g-\nu})(\eta^1)\mathrm{Op}(a)(\eta)u\|_{0;0}\\ &= \|\mathrm{Op}(p^{s-\mu;g-\nu})(\eta^1)\mathrm{Op}(a)(\eta)\mathrm{Op}(p^{s,g})^{-1}(\eta^1)\mathrm{Op}(p^{s,g})(\eta^1)u\|_{0;0}\end{aligned} \tag{2.3.4}$$

for $|\eta^1|$ large enough. Here the parameter $\eta \in \mathbb{R}^q \setminus \{0\}$ is also fixed. In order to apply Remark 2.2.18 we pass to the operator functions $\mathrm{Op}(p^{s-\mu,g-\nu})(\delta\eta^1)$, $\mathrm{Op}(a)(\delta\eta)$, etc.,

$\delta \in \mathbb{R} \setminus \{0\}$, and obtain families $\mathrm{Op}(a) \in \boldsymbol{L}^{\mu,-\mu+\nu}(X^{\asymp}; \mathbb{R}_\delta \setminus \{0\})$, etc., cf. Remark 2.2.22. Those can be multiplied within our calculus, and it follows that

$$\mathrm{Op}(p^{s-\mu,g-\nu})(\delta\eta^1)\mathrm{Op}(a)(\delta\eta)\mathrm{Op}(p^{s,g})^{-1}(\delta\eta^1) \in \boldsymbol{L}^{0,0}(X^{\asymp}; \mathbb{R} \setminus \{0\}).$$

By virtue of Remark 2.2.18 this is a family of continuous operators in $L^2(\mathbb{R} \times X)$, and hence, returning to $\delta = 1$, the right hand side of (2.3.4) can be estimated by

$$\|\mathrm{Op}(p^{s-\mu,g-\nu})(\eta^1)\mathrm{Op}(a)(\eta)\mathrm{Op}(p^{s,g})^{-1}(\eta^1)\|_{0;0}\|\mathrm{Op}(p^{s,g})(\eta^1)u\|_{0;0} = c\|u\|_{s;g}$$

for a constant $0 < c < \infty$. □

**Theorem 2.3.4.** *Let $A \in \boldsymbol{L}^{\mu,-\mu+\nu}(X^{\asymp}; \mathbb{R}^q \setminus \{0\})$ be elliptic. Then there is a $C > 0$ such that*

$$A(\eta) : H^{s;g}_{\mathrm{cone}}(X^{\asymp}) \to H^{s-\mu;g-\nu}_{\mathrm{cone}}(X^{\asymp}) \tag{2.3.5}$$

*is an isomorphism for every $|\eta| \geq C$ and $s, g \in \mathbb{R}$. The parametrix $B(\eta) \in \boldsymbol{L}^{-\mu,\mu-\nu}(X^{\asymp}; \mathbb{R}^q \setminus \{0\})$ can be chosen in such a way that $B(\eta) = A^{-1}(\eta)$ for $|\eta| \geq C$.*

*Proof.* Let us form the operator

$$A_0(\eta) = \mathrm{Op}(p^{s-\mu,g-\nu})(\eta)A(\eta)\mathrm{Op}(p^{s,g})^{(-1)}(\eta) \tag{2.3.6}$$

where $\mathrm{Op}(p^{s,g})^{(-1)}$ is a parametrix of $\mathrm{Op}(p^{s,g})$. Then $A_0$ is elliptic, cf. Proposition 2.2.24, and hence it has a parametrix $A_0^{(-1)}$ such that

$$1 - A_0(\eta)A_0^{(-1)}(\eta) =: C_r(\eta),\ 1 - A_0^{(-1)}(\eta)A_0(\eta) =: C_l(\eta) \in \boldsymbol{L}^{-\infty,-\infty}(X^{\asymp}; \mathbb{R}^q \setminus \{0\}).$$

The operator $1 - C_l(\eta) : L^2(\mathbb{R} \times X) \to L^2(\mathbb{R} \times X)$ is invertible for large $|\eta|$, and by replacing $A_0^{(-1)}(\eta)$ by $\big(1 - \chi(\eta)\big)A_0^{(-1)}(\eta) + \chi(\eta)A_0^{(-1)}(\eta)\big(1 - C_l(\eta)\big)^{-1}$ for a suitable excision function $\chi(\eta)$ we obtain another parametrix, again denoted by $A_0^{(-1)}(\eta)$, but with the property

$$A_0^{(-1)}(\eta) = A_0^{-1}(\eta) \text{ for } |\eta| > C$$

for a suitable $C > 0$. Now, using the relation (2.3.6) we find a parametrix $B(\eta)$ of $A(\eta)$ by setting

$$B(\eta) := \mathrm{Op}(p^{s,g})^{(-1)}(\eta)A_0^{(-1)}(\eta)\mathrm{Op}(p^{s-\mu,g-\nu})(\eta)$$

which is invertible for $|\eta| \geq C$ for $C > 0$ sufficiently large. □

**Theorem 2.3.5.** *For every $s' \geq s$, $g' \geq g$ we have a continuous embedding*

$$E : H^{s';g'}_{\mathrm{cone}}(X^{\asymp}) \hookrightarrow H^{s;g}_{\mathrm{cone}}(X^{\asymp}) \tag{2.3.7}$$

*that is compact for $s' > s$, $g' > g$.*

*Proof.* We first show that there is a continuous embedding. To this end we choose an elliptic element $B \in \boldsymbol{L}^{s',-s'+g'+\frac{n}{2}}(X^{\asymp}; \mathbb{R}^q \setminus \{0\})$ that induces isomorphisms

$$B : H^{s';g'}_{\mathrm{cone}}(X^{\asymp}) \to L^2(\mathbb{R} \times X),$$
$$B : H^{s;g}_{\mathrm{cone}}(X^{\asymp}) \to H^{s-s';g-g'-\frac{n}{2}}_{\mathrm{cone}}(X^{\asymp})$$

for every $|\eta| \geq C$ for some constant $C > 0$. Then, according to the following diagram

$$\begin{array}{ccc} H^{s';g'}_{\mathrm{cone}}(X^{\asymp}) & \xrightarrow{E} & H^{s;g}_{\mathrm{cone}}(X^{\asymp}) \\ \cong \Big\downarrow B & & \cong \Big\downarrow B \\ L^2(\mathbb{R}\times X) & \xrightarrow{E_0} & H^{s-s';g-g'-\frac{n}{2}}_{\mathrm{cone}}(X^{\asymp}), \end{array} \tag{2.3.8}$$

in order to prove that $E$ is continuous it suffices to show that $E_0$ is a continuous embedding.
We write $E_0(\eta)$ as a composition $PQ$ where $Q(\eta) := \mathrm{Op}_r\big([r]^{-(s'-s)+g'-g}q(r,\rho,\eta)\big) \in \boldsymbol{L}^{-(s-s'),s-s'-(g-g')}(X^{\asymp};\mathbb{R}^q\setminus\{0\})$ for $q(r,\rho,\eta) = \tilde{q}([r]\rho,[r]\eta)$ and some parameter dependent elliptic $\tilde{q}(\tilde{\rho},\tilde{\eta}) \in L^{s'-s}_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$ of order $s'-s$, and $P(\eta) := \big(Q(\eta)|_{H^{\infty;\infty}_{\mathrm{cone}}(X^{\asymp})}\big)^{-1}$ for $|\eta|$ sufficiently large. It is clear that $Q(\eta)$ is elliptic in the sense of Definition 2.2.23 and hence represents an isomorphism

$$Q(\eta): L^2(\mathbb{R}\times X) \to H^{s-s';g-g'-\frac{n}{2}}_{\mathrm{cone}}(X^{\asymp})$$

for every $|\eta|$ large enough. Therefore, it is sufficient to show that

$$P(\eta): H^{s-s';g-g'-\frac{n}{2}}_{\mathrm{cone}}(X^{\asymp}) \to H^{s-s';g-g'-\frac{n}{2}}_{\mathrm{cone}}(X^{\asymp}) \tag{2.3.9}$$

is continuous for $\eta$ sufficiently large. Then, since $E_0(\eta) = P(\eta)Q(\eta)$ is the identity operator on $H^{\infty;\infty}_{\mathrm{cone}}(X^{\asymp})$, it is also the identity operator on $H^{s';g'}_{\mathrm{cone}}(X^{\asymp})$, i.e., it represents an embedding operator.
Now, if we define $P_0(\eta)$ by the commutative diagram

$$\begin{array}{ccc} H^{s-s';g-g'-\frac{n}{2}}_{\mathrm{cone}}(X^{\asymp}) & \xrightarrow{P} & H^{s-s';g-g'-\frac{n}{2}}_{\mathrm{cone}}(X^{\asymp}) \\ \cong \Big\downarrow \widetilde{B} & & \cong \Big\downarrow \widetilde{B} \\ L^2(\mathbb{R}\times X) & \xrightarrow{P_0} & L^2(\mathbb{R}\times X) \end{array}$$

for some elliptic $\widetilde{B}(\eta) \in \boldsymbol{L}^{s-s',-(s-s')+g-g'}(X^{\asymp};\mathbb{R}^q\setminus\{0\})$, it is easy to find an element $\tilde{p}_0(\tilde{\rho},\tilde{\eta}) \in L^{s-s'}_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$ such that $P_0(\eta)$ is written in the form

$$P_0 = \mathrm{Op}_r\Big([r]^{-(s-s')+g-g'}p_0(r,[r]\rho,[r]\eta)\Big) \quad \mathrm{mod}\ \boldsymbol{L}^{-\infty,-\infty}(X^{\asymp};\mathbb{R}^q\setminus\{0\}),$$

$|\eta| \geq C > 0$, for $p_0(r,[r]\rho,[r]\eta) = \tilde{p}_0([r]\rho,[r]\eta)$. Finally, by virtue of Theorem 2.2.16, the operator

$$P_0(\eta): L^2(\mathbb{R}\times X) \to L^2(\mathbb{R}\times X) \tag{2.3.10}$$

is continuous for $s' \geq s$, $g' \geq g$ and for every $\eta \neq 0$. Hence $P(\eta) = (\widetilde{B}(\eta))^{-1}P_0(\eta)\widetilde{B}(\eta)$ is continuous.

For the compactness we apply Theorem 2.2.16 again and obtain that (2.3.10) is compact for $s' > s$, $g' > g$, i.e., $P(\eta) = (\widetilde{B}(\eta))^{-1}P_0(\eta)\widetilde{B}(\eta)$ is compact, since it is the composition of continuous operators with a compact one. This implies that $E_0(\eta) = P(\eta)Q(\eta)$ is compact for $s' > s$, $g' > g$, and also $E$ is compact for $s' > s$, $g' > g$, because of the commutative diagram (2.3.8). □

# Chapter 3

# Axiomatic approach with corner-degenerate symbols

## 3.1 Symbols associated with order reductions

### 3.1.1 Scales and order reducing families

Let $\mathfrak{E}$ denote the set of all families $\mathcal{E} = (E^s)_{s\in\mathbb{R}}$ of Hilbert spaces with continuous embeddings $E^{s'} \hookrightarrow E^s$, $s' \geq s$, such that $E^\infty := \bigcap_{s\in\mathbb{R}} E^s$ is dense in every $E^s, s \in \mathbb{R}$ and that there is a dual scale $\mathcal{E}^* = (E^{*s})_{s\in\mathbb{R}}$ with a non-degenerate sesquilinear pairing $(\cdot,\cdot)_0 : E^0 \times E^{*0} \to \mathbb{C}$, such that $(\cdot,\cdot)_0 : E^\infty \times E^{*\infty} \to \mathbb{C}$, extends to a non-degenerate sesquilinear pairing

$$E^s \times E^{*-s} \to \mathbb{C}$$

for every $s \in \mathbb{R}$, where $\sup_{f\in E^{*-s}\setminus\{0\}} \frac{|(u,f)_0|}{\|f\|_{E^{*-s}}}$ and $\sup_{g\in E^s\setminus\{0\}} \frac{|(g,v)_0|}{\|g\|_{E^s}}$ are equivalent norms in the spaces $E^s$ and $E^{*-s}$, respectively; moreover, if $\mathcal{E} = (E^s)_{s\in\mathbb{R}}$, $\widetilde{\mathcal{E}} = (\widetilde{E}^s)_{s\in\mathbb{R}}$ are two scales in consideration and $a \in \mathcal{L}^\mu(\mathcal{E},\widetilde{\mathcal{E}}) := \bigcap_{s\in\mathbb{R}} \mathcal{L}(E^s, \widetilde{E}^{s-\mu})$, for some $\mu \in \mathbb{R}$, then

$$\sup_{s\in[s',s'']} \|a\|_{s,s-\mu} < \infty$$

for every $s' \leq s''$; here $\|\cdot\|_{s,\tilde{s}} := \|\cdot\|_{\mathcal{L}(E^s,\widetilde{E}^{\tilde{s}})}$. Later on, in the case $s = \tilde{s} = 0$ we often write $\|\cdot\| := \|\cdot\|_{0,0}$.
A scale $\mathcal{E} \in \mathfrak{E}$ is said to have the compact embedding property, if the embeddings $E^{s'} \hookrightarrow E^s$ are compact whenever $s' > s$.

**Remark 3.1.1.** *Every* $a \in \mathcal{L}^\mu(\mathcal{E},\widetilde{\mathcal{E}})$ *has a formal adjoint* $a^* \in \mathcal{L}^\mu(\widetilde{\mathcal{E}}^*,\mathcal{E}^*)$, *obtained by* $(au,v)_0 = (u,a^*v)_0$ *for all* $u \in E^\infty, v \in \widetilde{E}^{*\infty}$.

**Remark 3.1.2.** *The space* $\mathcal{L}^\mu(\mathcal{E},\widetilde{\mathcal{E}})$ *is Fréchet in a natural way for every* $\mu \in \mathbb{R}$.

**Definition 3.1.3.** *We call a system $(b^\mu(\eta))_{\mu\in\mathbb{R}}$ of operator functions $b^\mu(\eta) \in C^\infty\big(\mathbb{R}^q, \mathcal{L}^\mu(\mathcal{E},\mathcal{E})\big)$ an order reducing family of the scale $\mathcal{E}$, if $b^\mu(\eta) : E^s \to E^{s-\mu}$ is an isomorphism for every $s,\mu \in \mathbb{R}$, $\eta \in \mathbb{R}^q$, $b^0(\eta) = \mathrm{id}_{E^s}$ for every $s \in \mathbb{R}, \eta \in \mathbb{R}^q$, and*

(i) $D_\eta^\beta b^\mu(\eta) \in C^\infty\big(\mathbb{R}^q, \mathcal{L}^{\mu-|\beta|}(\mathcal{E},\mathcal{E})\big)$ *for every $\beta \in \mathbb{N}^q$;*

(ii) *for every $s \in \mathbb{R}, \beta \in \mathbb{N}^q$ we have*

$$\max_{|\beta|\le k} \sup_{\substack{\eta\in\mathbb{R}^q \\ s\in[s',s'']}} \|b^{s-\mu+|\beta|}(\eta)\{D_\eta^\beta b^\mu(\eta)\}b^{-s}(\eta)\|_{0,0} < \infty$$

*for all $k \in \mathbb{N}$, and all reals $s' \le s''$;*

(iii) *for every $\mu,\nu \in \mathbb{R}$, $\nu \ge \mu$, we have*

$$\sup_{s\in[s',s'']} \|b^\mu(\eta)\|_{s,s-\nu} \le c\langle\eta\rangle^B$$

*for all $\eta \in \mathbb{R}^q$ and $s' \le s''$ with constants $c(\mu,\nu,s), B(\mu,\nu,s) > 0$, uniformly bounded in compact $s$-intervals and compact $\mu,\nu$-intervals for $\nu \ge \mu$; moreover, for every $\mu \le 0$ we have*

$$\|b^\mu(\eta)\|_{0,0} \le c\langle\eta\rangle^\mu$$

*for all $\eta \in \mathbb{R}^q$ with constants $c > 0$, uniformly bounded in compact $\mu$-intervals, $\mu \le 0$.*

Clearly the operators $b^\mu$ in (iii) for $\nu \ge \mu$ or $\mu \le 0$, are composed with a corresponding embedding operator.
In addition we require that the operator families $(b^\mu(\eta))^{-1}$ are equivalent to $b^{-\mu}(\eta)$, according to the following notation. Another order reducing family $(b_1^\mu(\eta))_{\mu\in\mathbb{R}}, \eta \in \mathbb{R}^q$, in the scale $\mathcal{E}$ is said to be equivalent to $(b^\mu(\eta))_{\mu\in\mathbb{R}}$, if for every $s \in \mathbb{R}$, $\beta \in \mathbb{N}^q$, there are constants $c = c(\beta, s)$ such that

$$\|b_1^{s-\mu+|\beta|}(\eta)\{D_\eta^\beta b^\mu(\eta)\}b_1^{-s}(\eta)\|_{0,0} \le c,$$

$$\|b^{s-\mu+|\beta|}(\eta)\{D_\eta^\beta b_1^\mu(\eta)\}b^{-s}(\eta)\|_{0,0} \le c,$$

for all $\eta \in \mathbb{R}^q$, uniformly in $s \in [s', s'']$ for every $s' \le s''$.

**Remark 3.1.4.** *Parameter-dependent theories of operators are common in many concrete contexts. For instance, if $\Omega$ is an (open) $C^\infty$ manifold, and $L^\mu_{\mathrm{cl}}(\Omega,\mathbb{R}^q)$ is the space of all parameter-dependent pseudo-differential operators on $\Omega$ of order $\mu \in \mathbb{R}$, with parameter $\eta \in \mathbb{R}^q$, where the local amplitude functions $a(x,\xi,\eta)$ are classical symbols in $(\xi,\eta) \in \mathbb{R}^{n+q}$, treated as covariables, $n = \dim\Omega$, while $L^{-\infty}(\Omega,\mathbb{R}^q)$ is defined as the space of all Schwartz functions in $\eta \in \mathbb{R}^q$ with values in $L^{-\infty}(\Omega)$, the space of smoothing operators on $\Omega$. Later on we will also consider specific examples with more control on the dependence on $\eta$, namely, when $\Omega = M \setminus \{v\}$ for a manifold $M$ with conical singularity $v$.*

**Example 3.1.5.** *Let $X$ be a closed compact $C^\infty$ manifold, $E^s := H^s(X)$, $s \in \mathbb{R}$, the scale of classical Sobolev spaces on $X$ and $b^\mu(\eta) \in L^\mu_{\mathrm{cl}}(X;\mathbb{R}^q_\eta)$ a parameter-dependent elliptic family that induces isomorphisms $b^\mu(\eta) : H^s(X) \to H^{s-\mu}(X)$ for all $s \in \mathbb{R}$. Then, from Theorem 2.2.2, for $\nu \geq \mu$ we have*

$$\|b^\mu(\eta)\|_{\mathcal{L}(H^s(X),H^{s-\nu}(X))} \leq c\langle\eta\rangle^{\pi(\mu,\nu)}$$

*for all $\eta \in \mathbb{R}^q$, uniformly in $s \in [s', s'']$ for arbitrary $s', s''$, as well as in compact $\mu$- and $\nu$-intervals for $\nu \geq \mu$, where*

$$\pi(\mu,\nu) := \max\{\mu, \mu - \nu\} \tag{3.1.1}$$

*with a constant $c = c(\mu,\nu,s',s'') > 0$. Observe that*

$$\sup_{\xi\in\mathbb{R}^n} \frac{\langle\xi,\eta\rangle^\mu}{\langle\xi\rangle^\nu} \leq \langle\eta\rangle^{\pi(\mu,\nu)} \tag{3.1.2}$$

*for all $\eta \in \mathbb{R}^q$.*

**Remark 3.1.6.** *Let $b^s(\tilde\tau,\tilde\eta) \in L^s_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde\tau,\tilde\eta})$ be an order reducing family as in Example 3.1.5, now with the parameter $(\tilde\tau,\tilde\eta) \in \mathbb{R}^{1+q}$ rather than $\eta$, and of order $s \in \mathbb{R}$. Then, setting $b^s(t,\tau,\eta) := b^s([t]\tau,[t]\eta)$ the expression*

$$\Big\{ \int \|[t]^{-s+\frac{n}{2}} \mathrm{Op}_t(b^s)(\eta^1)u\|^2_{L^2(X)}dt \Big\}^{\frac{1}{2}}$$

*for $\eta^1 \in \mathbb{R}^q \setminus \{0\}$, $|\eta^1|$ sufficiently large, is a norm on the space $\mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$. Here $n = \dim X$. Let $H^s_{\mathrm{cone}}(\mathbb{R}\times X)$ denote the completion of $\mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$ with respect to this norm. Observe that this space is independent of the choice of $\eta^1$, $|\eta^1|$ sufficiently large. For references below we also form weighted variants $H^{s;g}_{\mathrm{cone}}(\mathbb{R}\times X) := \langle t\rangle^{-g} H^s_{\mathrm{cone}}(\mathbb{R}\times X)$, $g \in \mathbb{R}$, and set*

$$H^{s;g}_{\mathrm{cone}}(\mathbb{R}_+\times X) := H^{s;g}_{\mathrm{cone}}(\mathbb{R}\times X)|_{\mathbb{R}_+\times X}. \tag{3.1.3}$$

*As is known, cf.* [14], *the spaces $H^{s;g}_{\mathrm{cone}}(\mathbb{R}\times X)$ are weighted Sobolev spaces in the calculus of pseudo-differential operators on $\mathbb{R}_+\times X$ with $|t| \to \infty$ being interpreted as a conical exit to infinity.*

Another feature of order reducing families, known, for instance, in the case of Example 3.1.5, is that when $U \subseteq \mathbb{R}^p$ is an open set and $m(y) \in C^\infty(U)$ a strictly positive function, $m(y) \geq c$ for $c > 0$ and for all $y \in U$, the family $b^s_1(y,\eta) := b^s(m(y)\eta)$, $s \in \mathbb{R}$, is order reducing in the sense of Definition 3.1.3 and equivalent to $b(\eta)$ for every $y \in U$, uniformly in $y \in K$ for any compact subset $K \subset U$. A natural requirement is that when $m > 0$ is a parameter, there is a constant $M = M(s',s'') > 0$ such that

$$\|b^s(\eta)b^{-s}(m\eta)\|_{0,0} \leq c\max\{m, m^{-1}\}^M \tag{3.1.4}$$

for every $s \in [s', s'']$, $m \in \mathbb{R}_+$, and $\eta \in \mathbb{R}^q$.

We now turn to another example of an order reducing family, motivated by the calculus of pseudo-differential operators on a manifold with edge (here in “abstract” form), where all the above requirements are satisfied, including (3.1.4).

**Definition 3.1.7.** (i) *If $H$ is a Hilbert space and $\kappa := \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ a group of isomorphisms $\kappa_\lambda : H \to H$, such that $\lambda \to \kappa_\lambda h$ defines a continuous function $\mathbb{R}_+ \to H$ for every $h \in H$, and $\kappa_\lambda\kappa_\rho = \kappa_{\lambda\rho}$ for $\lambda, \rho \in \mathbb{R}$, we call $\kappa$ a group action on $H$.*

(ii) *Let $\mathcal{H} = (H^s)_{s\in\mathbb{R}} \in \mathfrak{E}$ and assume that $H^0$ is endowed with a group action $\kappa = \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ that restricts (for $s > 0$) or extends (for $s < 0$) to a group action on $H^s$ for every $s \in \mathbb{R}$. In addition, we assume that $\kappa$ is a unitary group action on $H^0$. We then say that $\mathcal{H}$ is endowed with a group action.*

If $H$ and $\kappa$ are as in Definition 3.1.7 (i), it is known that there are constants $c, M > 0$, such that

$$\|\kappa_\lambda\|_{\mathcal{L}(H)} \le c \max\{\lambda, \lambda^{-1}\}^M \tag{3.1.5}$$

for all $\lambda \in \mathbb{R}_+$.
Denote by $\mathcal{W}^s(\mathbb{R}^q, H)$ the completion of $\mathcal{S}(\mathbb{R}^q, H)$ with respect to the norm

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} := \left\{ \int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle}\hat{u}(\eta)\|^2_H d\eta \right\}^{\frac{1}{2}};$$

$\hat{u}(\eta) = \mathcal{F}_{y\to\eta}u(\eta)$ is the Fourier transform in $\mathbb{R}^q$. The space $\mathcal{W}^s(\mathbb{R}^q, H)$ will be referred to as edge space on $\mathbb{R}^q$ of smoothness $s \in \mathbb{R}$ (modelled on $H$). Given a scale $\mathcal{H} = (H^s)_{s\in\mathbb{R}} \in \mathfrak{E}$ with group action we have the edge spaces

$$W^s := \mathcal{W}^s(\mathbb{R}^q, H^s), \ s \in \mathbb{R}.$$

If necessary we also write $\mathcal{W}^s(\mathbb{R}^q, H^s)_\kappa$. The spaces form again a scale $\mathcal{W} := (W^s)_{s\in\mathbb{R}} \in \mathfrak{E}$.

For purposes below we now formulate a class of operator-valued symbols

$$S^\mu(U \times \mathbb{R}^q; H, \widetilde{H})_{\kappa,\tilde{\kappa}} \tag{3.1.6}$$

for open $U \subseteq \mathbb{R}^p$ and Hilbert spaces $H$ and $\widetilde{H}$, endowed with group actions $\kappa = \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}, \tilde{\kappa} = \{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively, as follows. The space (3.1.6) is defined to be the set of all $a(y, \eta) \in C^\infty\big(U \times \mathbb{R}^q, \mathcal{L}(H, \widetilde{H})\big)$ such that

$$\sup_{(y,\eta)\in K\times\mathbb{R}^q} \langle\eta\rangle^{-\mu+|\beta|} \|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} < \infty \tag{3.1.7}$$

for every compact $K \subset U, \alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^q$.

**Remark 3.1.8.** (i) *Analogous symbols can also be defined in the case when $\widetilde{H}$ is a Fréchet space with group action, i.e., $\widetilde{H}$ is written as a projective limit of Hilbert spaces $\widetilde{H}_j, j \in \mathbb{N}$, with continuous embeddings $\widetilde{H}_j \hookrightarrow \widetilde{H}_0$, where the group action on $\widetilde{H}_0$ restricts to group actions on $\widetilde{H}_j$ for every $j$. Then $S^\mu(U \times \mathbb{R}^q; H, \widetilde{H}) := \varprojlim_{j\in\mathbb{N}} S^\mu(U \times \mathbb{R}^q; H, \widetilde{H}_j)$;*

(ii) *another generalisation can be made when one controls the variable $y$ at infinity by defining the symbol class*

$$S^{\mu;\nu}(U \times \mathbb{R}^q; H, \widetilde{H})_{\kappa,\tilde{\kappa}} \tag{3.1.8}$$

*as the set of all $a(y,\eta) \in C^\infty\big(U \times \mathbb{R}^q, \mathcal{L}(H, \widetilde{H})\big)$ such that*

$$\sup_{(y,\eta)\in K\times\mathbb{R}^q} \langle\eta\rangle^{-\mu+|\beta|}\langle y\rangle^{-\nu+|\alpha|}\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} < \infty,$$

*for all $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$.*

Consider now an operator function $p(\xi,\eta) \in C^\infty\big(\mathbb{R}^{p+q}_{\xi,\eta}, \mathcal{L}^\mu(\mathcal{H},\mathcal{H})\big)$ that represents a symbol

$$p(\xi,\eta) \in S^\mu(\mathbb{R}^{p+q}_{\xi,\eta}; H^s, H^{s-\mu})_{\kappa,\kappa}$$

for every $s \in \mathbb{R}$, such that $p(\xi,\eta) : H^s \to H^{s-\mu}$ is a family of isomorphisms for all $s \in \mathbb{R}$, and the inverses $p^{-1}(\xi,\eta)$ represent a symbol

$$p^{-1}(\xi,\eta) \in S^{-\mu}(\mathbb{R}^{p+q}_{\xi,\eta}; H^s, H^{s+\mu})_{\kappa,\kappa}$$

for every $s \in \mathbb{R}$. Then $b^\mu(\eta) := \mathrm{Op}_x(p)(\eta)$ is a family of isomorphisms

$$b^\mu(\eta) : W^s \to W^{s-\mu},\ \eta \in \mathbb{R}^q,$$

with the inverses $b^{-\mu}(\eta) := \mathrm{Op}_x(p^{-1})(\eta)$. Here $W^s = \mathcal{W}^s(\mathbb{R}^p_x, H^s)$, $s \in \mathbb{R}$.

**Proposition 3.1.9.** (i) *We have*

$$\|b^\mu(\eta)\|_{\mathcal{L}(W^0,W^0)} \le c\langle\eta\rangle^\mu \tag{3.1.9}$$

*for every $\mu \le 0$, with a constant $c(\mu) > 0$.*

(ii) *For every $s,\mu,\nu \in \mathbb{R}$, $\nu \ge \mu$, we have*

$$\|b^\mu(\eta)\|_{\mathcal{L}(W^s,W^{s-\nu})} \le c\langle\eta\rangle^{\pi(\mu,\nu)+M(s)+M(s-\mu)} \tag{3.1.10}$$

*for all $\eta \in \mathbb{R}^q$, with a constant $c(\mu,s) > 0$, and $M(s) \ge 0$ defined by*

$$\|\kappa_\lambda\|_{\mathcal{L}(H^s,H^s)} \le c\lambda^{M(s)} \text{ for all } \lambda \ge 1.$$

*Proof.* (i) Let us check the estimate (3.1.9). For the computations we denote by $j : H^{-\mu} \hookrightarrow H^0$ the embedding operator. We have for $u \in W^0$

$$\begin{aligned}
\|b^\mu(\eta)u\|^2_{W^0} &= \int \|jp(\xi,\eta)(\mathcal{F}u)(\xi)\|^2_{H^0} d\xi \\
&= \int \|\kappa^{-1}_{\langle\xi,\eta\rangle} j\kappa_{\langle\xi,\eta\rangle}\kappa^{-1}_{\langle\xi,\eta\rangle} p(\xi,\eta)\kappa_{\langle\xi,\eta\rangle}\kappa^{-1}_{\langle\xi,\eta\rangle}(\mathcal{F}u)(\xi)\|^2_{H^0} d\xi \\
&\le \int \|\kappa^{-1}_{\langle\xi,\eta\rangle} j\kappa_{\langle\xi,\eta\rangle}\|^2_{\mathcal{L}(H^{-\mu},H^0)}\|\kappa^{-1}_{\langle\xi,\eta\rangle} p(\xi,\eta)\kappa_{\langle\xi,\eta\rangle}\kappa^{-1}_{\langle\xi,\eta\rangle}(\mathcal{F}u)(\xi)\|^2_{H^{-\mu}} d\xi \\
&\le c\int \|\kappa^{-1}_{\langle\xi,\eta\rangle} p(\xi,\eta)\kappa_{\langle\xi,\eta\rangle}\|^2_{\mathcal{L}(H^0,H^{-\mu})}\|\kappa^{-1}_{\langle\xi,\eta\rangle}(\mathcal{F}u)(\xi)\|^2_{H^0} d\xi \\
&\le c\sup_{\xi\in\mathbb{R}^p}\langle\xi,\eta\rangle^{2\mu}\|u\|^2_{W^0}.
\end{aligned}$$

Thus $\|b^\mu(\eta)\|_{\mathcal{L}(W^0,W^0)} \le c\sup_{\xi\in\mathbb{R}^p}\langle\xi,\eta\rangle^\mu \le c\langle\eta\rangle^\mu$, since $\mu\le 0$.

(ii) Let $j : H^{s-\mu} \hookrightarrow H^{s-\nu}$ denote the canonical embedding. For every fixed $s\in\mathbb{R}$ we have

$$\begin{aligned}\|b^\mu(\eta)u\|^2_{W^{s-\nu}} &= \int \langle\xi\rangle^{2(s-\nu)}\|\kappa^{-1}_{\langle\xi\rangle}jp(\xi,\eta)(\mathcal{F}_{x\to\xi}u)(\xi)\|^2_{H^{s-\nu}}d\xi\\ &= \int \langle\xi\rangle^{2(s-\nu)}\|\kappa^{-1}_{\langle\xi\rangle}jp(\xi,\eta)\kappa_{\langle\xi\rangle}\langle\xi\rangle^{-s}\langle\xi\rangle^{s}\kappa^{-1}_{\langle\xi\rangle}(\mathcal{F}_{x\to\xi}u)(\xi)\|^2_{H^{s-\nu}}d\xi\\ &= \sup_{\xi\in\mathbb{R}^p}\langle\xi\rangle^{-2\nu}\|\kappa^{-1}_{\langle\xi\rangle}jp(\xi,\eta)\kappa_{\langle\xi\rangle}\|^2_{\mathcal{L}(H^s,H^{s-\nu})}\int\langle\xi\rangle^{2s}\|\kappa^{-1}_{\langle\xi\rangle}\mathcal{F}_{x\to\xi}u(\xi)\|^2_{H^s}d\xi.\end{aligned}$$

For the first factor on the right hand side we obtain

$$\begin{aligned}\|\kappa^{-1}_{\langle\xi\rangle}jp(\xi,\eta)\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(H^s,H^{s-\nu})} &\le \|\kappa^{-1}_{\langle\xi\rangle}j\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(H^{s-\mu},H^{s-\nu})}\|\kappa^{-1}_{\langle\xi\rangle}p(\xi,\eta)\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(H^s,H^{s-\mu})}\\ &\le c\|\kappa^{-1}_{\langle\xi\rangle}p(\xi,\eta)\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(H^s,H^{s-\mu})}\end{aligned}$$

with a constant $c>0$. We employed here that $\|\kappa^{-1}_{\langle\xi\rangle}j\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(H^{s-\mu},H^{s-\nu})}\le c$ for all $\xi\in\mathbb{R}^p$. Moreover,

$$\begin{aligned}&\|\kappa^{-1}_{\langle\xi\rangle}p(\xi,\eta)\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(H^s,H^{s-\mu})}\\ &\quad\le \|\kappa^{-1}_{\langle\xi\rangle}\kappa_{\langle\xi,\eta\rangle}\|_{\mathcal{L}(H^{s-\mu},H^{s-\mu})}\|\kappa^{-1}_{\langle\xi,\eta\rangle}p(\xi,\eta)\kappa_{\langle\xi,\eta\rangle}\|_{\mathcal{L}(H^s,H^{s-\mu})}\|\kappa^{-1}_{\langle\xi,\eta\rangle}\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(H^s,H^s)}\\ &\quad\le c\langle\xi,\eta\rangle^\mu\|\kappa_{\langle\xi,\eta\rangle\langle\xi\rangle^{-1}}\|_{\mathcal{L}(H^{s-\mu},H^{s-\mu})}\|\kappa_{\langle\xi,\eta\rangle^{-1}\langle\xi\rangle}\|_{\mathcal{L}(H^s,H^s)}\\ &\quad\le c\langle\xi,\eta\rangle^\mu\Big(\frac{\langle\xi,\eta\rangle}{\langle\xi\rangle}\Big)^{M(s-\mu)+M(s)}.\end{aligned}$$

As usual, $c>0$ denotes different constants (they may also depend on $s$); the numbers $M(s)$, $s\in\mathbb{R}$, are determined by the estimates

$$\|\kappa_\lambda\|_{\mathcal{L}(H^s,H^s)}\le c\lambda^{M(s)} \text{ for all } \lambda\ge 1.$$

We obtain altogether that

$$\begin{aligned}\|b^\mu(\eta)\|_{\mathcal{L}(W^s,W^{s-\nu})} &\le c\sup_{\xi\in\mathbb{R}^n}\frac{\langle\xi,\eta\rangle^\mu}{\langle\xi\rangle^\nu}\Big(\frac{\langle\xi,\eta\rangle}{\langle\xi\rangle}\Big)^{M(s-\mu)+M(s)}\\ &\le c\langle\eta\rangle^{\pi(\mu,\nu)+M(s-\mu)+M(s)},\end{aligned}$$

cf. formula (3.1.2). □

It is also not difficult to check that the operators in Proposition 3.1.9 also have the uniformity properties with respect to $s,\mu,\nu$ in compact sets, imposed in Definition 3.1.3.

### 3.1.2 Symbols based on order reductions

We now turn to operator valued symbols, referring to scales

$$\mathcal{E}=(E^s)_{s\in\mathbb{R}},\quad \widetilde{\mathcal{E}}=(\widetilde{E}^s)_{s\in\mathbb{R}}\in\mathfrak{C}.$$

For purposes below we slightly generalise the concept of order reducing families by replacing the parameter space $\mathbb{R}^q \ni \eta$ by $\mathbb{H} \ni \eta$, where

$$\mathbb{H} := \{\eta = (\eta', \eta'') \in \mathbb{R}^{q'+q''} : q = q' + q'', \eta'' \neq 0\}. \tag{3.1.11}$$

In other words for every $\mu \in \mathbb{R}$ we fix order-reducing families $b^\mu(\eta)$ and $\tilde{b}^\mu(\eta)$ in the scales $\mathcal{E}$ and $\widetilde{\mathcal{E}}$, respectively, where $\eta$ varies over $\mathbb{H}$, and the properties of Definition 3.1.3 are required for all $\eta \in \mathbb{H}$. In many cases we may admit the case $\mathbb{H} = \mathbb{R}^q$ as well.

**Definition 3.1.10.** *By $S^\mu(U \times \mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}})$ for open $U \subseteq \mathbb{R}^p, \mu \in \mathbb{R}$, we denote the set of all $a(y, \eta) \in C^\infty\big(U \times \mathbb{H}, \mathcal{L}^\mu(\mathcal{E}, \widetilde{\mathcal{E}})\big)$ such that*

$$D_y^\alpha D_\eta^\beta a(y, \eta) \in C^\infty\big(U \times \mathbb{H}, \mathcal{L}^{\mu-|\beta|}(\mathcal{E}, \widetilde{\mathcal{E}})\big), \tag{3.1.12}$$

*and for every $s \in \mathbb{R}$ we have*

$$\max_{|\alpha|+|\beta| \leq k} \sup_{\substack{y \in K, \eta \in \mathbb{H}, |\eta| \geq h \\ s \in [s', s'']}} \|\tilde{b}^{s-\mu+|\beta|}(\eta)\{D_y^\alpha D_\eta^\beta a(y, \eta)\} b^{-s}(\eta)\|_{0,0} \tag{3.1.13}$$

*is finite for all compact $K \subset U$, $k \in \mathbb{N}$, $h > 0$.*

Let $S^\mu(\mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}})$ denote the subspace of all elements of $S^\mu(U \times \mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}})$ that are independent of $y$.
Observe that when $(b^\mu(\eta))_{\mu \in \mathbb{R}}$ is an order reducing family parametrised by $\eta \in \mathbb{H}$ then we have

$$b^\mu(\eta) \in S^\mu(\mathbb{H}; \mathcal{E}, \mathcal{E}) \tag{3.1.14}$$

for every $\mu \in \mathbb{R}$.

**Remark 3.1.11.** *The space $S^\mu(U \times \mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}})$ is Fréchet with the semi-norms*

$$a \to \max_{|\alpha|+|\beta| \leq k} \sup_{\substack{(y,\eta) \in K \times \mathbb{H}, |\eta| \geq h \\ s \in [s', s'']}} \|\tilde{b}^{s-\mu+|\beta|}(\eta)\{D_y^\alpha D_\eta^\beta a(y, \eta)\} b^{-s}(\eta)\|_{0,0} \tag{3.1.15}$$

*parametrised by compact $K \subset U$, $s \in \mathbb{Z}$, $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$, $h \in \mathbb{N} \setminus \{0\}$. We then have*

$$S^\mu(U \times \mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}}) = C^\infty\big(U, S^\mu(\mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}})\big) = C^\infty(U) \hat{\otimes}_\pi S^\mu(\mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}}).$$

We will also employ other variants of such symbols, for instance, when $\Omega \subseteq \mathbb{R}^m$ is an open set,

$$S^\mu(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}}) := C^\infty\big(\overline{\mathbb{R}}_+ \times \Omega, S^\mu(\mathbb{H}; \mathcal{E}, \widetilde{\mathcal{E}})\big).$$

In order to emphasise the similarity of our considerations for $\mathbb{H}$ with the case $\mathbb{H} = \mathbb{R}^q$ we often write again $\mathbb{R}^q$ and later on tacitly use the corresponding results for $\mathbb{H}$ in general.

**Remark 3.1.12.** *Let $a(y,\eta) \in S^\mu(U \times \mathbb{R}^q)$ be a polynomial in $\eta$ of order $\mu$ and $\mathcal{E} = (E^s)_{s\in\mathbb{R}}$ any scale of Hilbert spaces and identify $D_y^\alpha D_\eta^\beta a(y,\eta)$ with $\big(D_y^\alpha D_\eta^\beta a(y,\eta)\big)\iota$ with the embedding $\iota : E^s \to E^{s-\mu+|\beta|}$. Then we have*

$$\|b^{s-\mu+|\beta|}(\eta)\{D_y^\alpha D_\eta^\beta a(y,\eta)\}b^{-s}(\eta)\|_{0,0} \le |D_y^\alpha D_\eta^\beta a(y,\eta)|\|b^{-\mu+|\beta|}(\eta)\|_{0,0} \le c\langle\eta\rangle^{\mu-|\beta|}\langle\eta\rangle^{-\mu+|\beta|} = c$$

*for all $\beta \in \mathbb{N}^q$, $|\beta| \le \mu$, $y \in K \subset U$, $K$ compact (see Definition* 3.1.3 (iii)). *Thus $a(y,\eta)$ is canonically identified with an element of $S^\mu(U \times \mathbb{R}^q; \mathcal{E}, \mathcal{E})$.*

**Proposition 3.1.13.** *We have*

$$S^{-\infty}(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}}) := \bigcap_{\mu\in\mathbb{R}} S^\mu(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}}) = C^\infty\big(U, \mathcal{S}\big(\mathbb{R}^q, \mathcal{L}^{-\infty}(\mathcal{E}, \widetilde{\mathcal{E}})\big)\big).$$

*Proof.* Let us show the assertion for $y$-independent symbols; the $y$-dependent case is then straightforward. For notational convenience we set $\widetilde{\mathcal{E}} = \mathcal{E}$; the general case is analogous. First let $a(\eta) \in S^{-\infty}(\mathbb{R}^q; \mathcal{E}, \mathcal{E})$, which means that $a(\eta) \in C^\infty\big(\mathbb{R}^q, \mathcal{L}^{-\infty}(\mathcal{E}, \mathcal{E})\big)$ and

$$\|b^{s+N}(\eta)\{D_\eta^\beta a(\eta)\}b^{-s}(\eta)\|_{0,0} < c \tag{3.1.16}$$

for all $s \in \mathbb{R}$, $N \in \mathbb{N}$, $\beta \in \mathbb{N}^q$ and show that

$$\sup_{\eta\in\mathbb{R}^q} \|\langle\eta\rangle^M D_\eta^\beta a(\eta)\|_{s,t} < \infty \tag{3.1.17}$$

for every $s,t \in \mathbb{R}$, $M \in \mathbb{N}$, $\beta \in \mathbb{N}^q$. To estimate (3.1.17) it is enough to assume $t > 0$. We have

$$\|\langle\eta\rangle^M D_\eta^\beta a(\eta)\|_{s,t} = \|b^{-kt}(\eta)b^{kt}(\eta)\langle\eta\rangle^M D_\eta^\beta a(\eta)b^{-s}(\eta)b^s(\eta)\|_{s,t} \tag{3.1.18}$$

for every $k \in \mathbb{N}$, $k \ge 1$. It is sufficient to show that the right hand side is uniformly bounded in $\eta \in \mathbb{R}^q$ for sufficiently large choice of $k$. The right hand side of (3.1.18) can be estimated by

$$\langle\eta\rangle^M \|b^{-t}(\eta)\|_{0,t}\|b^{(1-k)t}(\eta)\|_{0,0}\|b^{kt}(\eta)D_\eta^\beta a(\eta)b^{-s}(\eta)\|_{0,0}\|b^s(\eta)\|_{s,0}.$$

Using $\|b^{kt}(\eta)D_\eta^\beta a(\eta)b^{-s}(\eta)\|_{0,0} \le c$, which is true by assumption and the estimates

$$\|b^s(\eta)\|_{s,0} \le c\langle\eta\rangle^B, \quad \|b^{-t}(\eta)\|_{0,t} \le c\langle\eta\rangle^{B'},$$

with different $B, B' \in \mathbb{R}$ and $\|b^{(1-k)t}(\eta)\|_{0,0} \le c\langle\eta\rangle^{(1-k)t}$ (see Definition 3.1.3 (iii)) we obtain altogether

$$\|\langle\eta\rangle^M D_\eta^\beta a(\eta)\|_{s,t} \le c\langle\eta\rangle^{M+B+B'+(1-k)t}$$

for some $c > 0$. Choosing $k$ large enough it follows that the exponent on the right hand side is $< 0$, i.e., we obtain uniform boundedness in $\eta \in \mathbb{R}^q$.

To show the reverse direction suppose that $a(\eta)$ satisfies (3.1.17), and let $\beta \in \mathbb{N}^q$, $M, s, t \in \mathbb{R}$ be arbitrary. We have

$$\|b^t(\eta)D_\eta^\beta a(\eta)b^{-s}(\eta)\|_{0,0} \le \|b^t(\eta)\langle\eta\rangle^{-M}\|_{t,0}\|\langle\eta\rangle^{2M}D_\eta^\beta a(\eta)\|_{s,t}\|\langle\eta\rangle^{-M}b^{-s}(\eta)\|_{0,s}. \tag{3.1.19}$$

Now using (3.1.17) and the estimates

$$\|b^t(\eta)\langle\eta\rangle^{-M}\|_{t,0} \leq c\langle\eta\rangle^{A-M}, \quad \|\langle\eta\rangle^{-M}b^{-s}(\eta)\|_{0,s} \leq c\langle\eta\rangle^{A'-M},$$

with constants $A, A' \in \mathbb{R}$, we obtain

$$\|b^t(\eta)D_\eta^\beta a(\eta)b^{-s}(\eta)\|_{0,0} \leq c\langle\eta\rangle^{A+A'-2M}.$$

Choosing $M$ large enough we get uniform boundedness of (3.1.19) in $\eta \in \mathbb{R}^q$, which completes the proof. □

**Proposition 3.1.14.** *Let $a(y,\eta) \in S^\mu(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$ and $\mu \leq 0$. Then we have*

$$\|a(y,\eta)\|_{0,0} \leq c\langle\eta\rangle^\mu$$

*for all $y \in K \subset U$, $K$ compact, $\eta \in \mathbb{R}^q$, with a constant $c = c(s,K) > 0$.*

*Proof.* For simplicity we consider the $y$-independent case. It is enough to show that $\|a(\eta)u\|_{\widetilde{E}^0} \leq c\langle\eta\rangle^\mu\|u\|_{E^0}$ for all $u \in E^\infty$. Let $j : E^{-\mu} \to E^0$ denote the embedding operator. We then have

$$\|a(\eta)u\|_{\widetilde{E}^0} = \|a(\eta)b^{-\mu}(\eta)jb^\mu(\eta)u\|_{\widetilde{E}^0} \leq \|a(\eta)b^{-\mu}(\eta)\|_{\mathcal{L}(E^0,\widetilde{E}^0)}\|jb^\mu(\eta)u\|_{E^0}$$
$$\leq c\langle\eta\rangle^\mu\|u\|_{E^0}.$$

□

**Proposition 3.1.15.** *A symbol $a(y,\eta) \in S^\mu(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$, $\mu \in \mathbb{R}$, satisfies the estimates*

$$\|a(y,\eta)\|_{s,s-\nu} \leq c\langle\eta\rangle^A \tag{3.1.20}$$

*for every $\nu \geq \mu$, and every $y \in K \subset U$, $K$ compact, $\eta \in \mathbb{R}^q$, $s \in \mathbb{R}$, with constants $c = c(s,\mu,\nu) > 0$, $A = A(s,\mu,\nu,K) > 0$ that are uniformly bounded when $s,\mu,\nu$ vary over compact sets, $\nu \geq \mu$.*

*Proof.* For simplicity we consider again the $y$-independent case. Let $j : \widetilde{E}^{s-\mu} \hookrightarrow \widetilde{E}^{s-\nu}$ be the embedding operator. Then we have

$$\|a(\eta)\|_{s,s-\nu} = \|j\tilde{b}^{-s+\mu}(\eta)\tilde{b}^{s-\mu}(\eta)a(\eta)b^{-s}(\eta)b^s(\eta)\|_{s,s-\nu}$$
$$\leq \|j\tilde{b}^{-s+\mu}(\eta)\|_{0,s-\nu}\|\tilde{b}^{s-\mu}(\eta)a(\eta)b^{-s}(\eta)\|_{0,0}\|b^s(\eta)\|_{s,0}.$$

Applying (3.1.13) and Definition 3.1.3 (iii) we obtain (3.1.20) with $A = B(-s+\mu,-s+\nu,0)+B(s,s,0)$, together with the uniform boundedness of the involved constants. □
Also here it can be proved that the involved constants in Propositions 3.1.14, 3.1.15 are uniform in compact sets with respect to $s,\mu,\nu$.

**Proposition 3.1.16.** *The symbol spaces have the following properties:*

(i) $S^\mu(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}}) \subseteq S^{\mu'}(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$ *for every $\mu' \geq \mu$;*

(ii) $D_y^\alpha D_\eta^\beta S^\mu(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}}) \subseteq S^{\mu-|\beta|}(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$ *for every $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$;*

(iii) $S^{\mu}(U \times \mathbb{R}^q; \mathcal{E}_0, \widetilde{\mathcal{E}}) S^{\nu}(U \times \mathbb{R}^q; \mathcal{E}, \mathcal{E}_0) \subseteq S^{\mu+\nu}(U \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$ *for every* $\mu, \nu \in \mathbb{R}$ *(the notation on the left hand side of the latter relation means the space of all* $(y, \eta)$*-wise compositions of elements in the respective factors).*

*Proof.* For simplicity we consider symbols with constant coefficients. Let us write $\| \cdot \| := \| \cdot \|_{0,0}$.

(i) $a(\eta) \in S^{\mu}(\mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$ means (3.1.12) and (3.1.13); this implies

$$\begin{aligned} \|\tilde{b}^{s-\mu'+|\beta|}(\eta)\{D_\eta^\beta a(\eta)\}b^{-s}(\eta)\| &= \|\tilde{b}^{\mu-\mu'}(\eta)\tilde{b}^{s-\mu+|\beta|}(\eta)\{D_\eta^\beta a(\eta)\}b^{-s}(\eta)\| \\ &\le c\langle\eta\rangle^{\mu-\mu'}\|\tilde{b}^{s-\mu+|\beta|}(\eta)\{D_\eta^\beta a(\eta)\}b^{-s}(\eta)\| \le c\|\tilde{b}^{s-\mu+|\beta|}(\eta)\{D_\eta^\beta a(\eta)\}b^{-s}(\eta)\|. \end{aligned}$$

Here wee employed $\mu - \mu' \le 0$ and the property (iii) in Definition 3.1.3.

(ii) The estimates (3.1.13) can be written as

$$\|\tilde{b}^{s-(\mu-|\beta|)}(\eta)\{D_\eta^\beta a(\eta)\}b^{-s}(\eta)\| \le c,$$

which just means that $D_\eta^\beta a(\eta) \in S^{\mu-|\beta|}(\mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$.

(iii) Given $a(\eta) \in S^{\mu}(\mathbb{R}^q; \mathcal{E}_0, \widetilde{\mathcal{E}})$, $\tilde{a}(\eta) \in S^{\nu}(\mathbb{R}^q; \mathcal{E}, \mathcal{E}_0)$ we have (with obvious meaning of notation)

$$\|\tilde{b}_0^{s-\nu+|\gamma|}(\eta)\{D_\eta^\gamma \tilde{a}(\eta)\}b^{-s}(\eta)\|_{\mathcal{L}(E^0,E_0^0)}, \|\tilde{b}^{s-\mu+|\delta|}(\eta)\{D_\eta^\delta a(\eta)\}b_0^{-s}(\eta)\|_{\mathcal{L}(E_0^0,\widetilde{E}^0)} \le c$$

for all $\gamma, \delta \in \mathbb{N}^q$. If $\alpha \in \mathbb{N}^q$ is any multi-index, $D_\eta^\alpha(a\tilde{a})(\eta)$ is a linear combination of compositions $D_\eta^\delta a(\eta) D_\eta^\gamma \tilde{a}(\eta)$ with $|\gamma| + |\delta| = |\alpha|$. It follows that

$$\begin{aligned} &\|\tilde{b}^{s-(\mu+\nu)+|\alpha|}(\eta)D_\eta^\delta a(\eta)\{D_\eta^\gamma \tilde{a}(\eta)\}b^{-s}(\eta)\|_{\mathcal{L}(E^0,\widetilde{E}^0)} \\ &= \|\tilde{b}^{s-(\mu+\nu)+|\alpha|}(\eta)D_\eta^\delta a(\eta)b_0^{-s+\nu-|\gamma|}(\eta)b_0^{s-\nu+|\gamma|}(\eta)D_\eta^\gamma \tilde{a}(\eta)b^{-s}(\eta)\|_{\mathcal{L}(E^0,\widetilde{E}^0)} \\ &\le \|\tilde{b}^{t-\mu+|\alpha|-|\gamma|}(\eta)D_\eta^\delta a(\eta)b_0^{-t}(\eta)\|_{\mathcal{L}(E_0^0,\widetilde{E}^0)} \, \|b_0^{s-\nu+|\gamma|}(\eta)D_\eta^\gamma \tilde{a}(\eta)b^{-s}(\eta)\|_{\mathcal{L}(E^0,E_0^0)} \end{aligned} \tag{3.1.21}$$

for $t = s - \nu + |\gamma|$; the right hand side is bounded in $\eta$, since $|\alpha| - |\gamma| = |\delta|$. □

**Remark 3.1.17.** *Observe from* (3.1.21) *that the semi-norms of compositions of symbols can be estimated by products of semi-norms of the factors.*

### 3.1.3 An example from the parameter-dependent cone calculus

We now construct a specific family of reductions of orders between weighted spaces on a compact manifold $M$ with conical singularity $v$, locally near $v$ modelled on a cone

$$X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$$

with a smooth compact manifold $X$ as a base. The parameter $\eta$ will play the role of covariables of the calculus of operators on a manifold with edge; that is why we talk about an example from the edge calculus. The associated "abstract" cone calculus according to what we did so far in the Subsections 3.1.1 and 3.1.2 and then below in Section 3.2 will be a contribution to the calculus of corner operators of second generation.

It will be convenient to pass to the stretched manifold $\mathbb{M}$ associated with $M$ which is a compact $C^\infty$ manifold with boundary $\partial\mathbb{M} \cong X$ such that when we squeeze down $\partial\mathbb{M}$ to a single point $v$ we just recover $M$. Close to $\partial\mathbb{M}$ the manifold $\mathbb{M}$ is equal to a cylinder $[0,1) \times X \ni (t,x)$ - a collar neighbourhood of $\partial\mathbb{M}$ in $M$. A part of the considerations will be performed on the open stretched cone $X^\wedge := \mathbb{R}_+ \times X \ni (t,x)$ where we identify $(0,1) \times X$ with the interior of the collar neighbourhood (for convenience, without indicating any pull backs of functions or operators with respect to that identification).

Let $\widetilde{M} := 2\mathbb{M}$ be the double of $\mathbb{M}$ (obtained by gluing together two copies $\mathbb{M}_\pm$ of $\mathbb{M}$ along the common boundary $\partial\mathbb{M}$, where we identify $\mathbb{M}$ with $\mathbb{M}_+$); then $\widetilde{M}$ is a closed compact $C^\infty$ manifold. On the space $M$ we have a family of weighted Sobolev spaces $H^{s,\gamma}(M)$, $s,\gamma \in \mathbb{R}$, that may be defined as

$$H^{s,\gamma}(M) := \big\{\sigma u + (1-\sigma)v : u \in \mathcal{H}^{s,\gamma}(X^\wedge), v \in H^s_{\mathrm{loc}}(M \setminus \{v\})\big\},$$

where $\sigma(t)$ is a cut-off function (i.e., $\sigma \in C_0^\infty(\overline{\mathbb{R}}_+)$, $\sigma \equiv 1$ near $t = 0$), $\sigma(t) = 0$ for $t > 2/3$. Here $\mathcal{H}^{s,\gamma}(X^\wedge)$ is defined to be the completion of $C_0^\infty(X^\wedge)$ with respect to the norm

$$\left\{\frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|b^\mu_{\mathrm{base}}(\mathrm{Im}\, w)(\mathcal{M}u)(w)\|^2_{L^2(X)}dw\right\}^{\frac{1}{2}}, \tag{3.1.22}$$

$n = \dim X$, where $b^\mu_{\mathrm{base}}(\tau) \in L^\mu_{\mathrm{cl}}(X;\mathbb{R}_\tau)$ is a family of reductions of order on $X$, similarly as in Example 3.1.5 (in particular, $b^s_{\mathrm{base}}(\tau) : H^s(X) \to H^0(X) = L^2(X)$ is a family of isomorphisms). Moreover, $\mathcal{M}$ is the Mellin transform, $(\mathcal{M}u)(w) = \int_0^\infty t^{w-1}u(t)dt$, $w \in \mathbb{C}$ the complex Mellin covariable, and

$$\Gamma_\beta := \{w \in \mathbb{C} : \mathrm{Re}\, w = \beta\}$$

for any real $\beta$. From $t^\delta\mathcal{H}^{s,\gamma}(X^\wedge) = \mathcal{H}^{s,\gamma+\delta}(X^\wedge)$ for all $s,\gamma,\delta \in \mathbb{R}$ it follows the existence of a strictly positive function $\mathrm{h}^\delta \in C^\infty(M \setminus \{v\})$, such that the operator of multiplication by $\mathrm{h}^\delta$ induces an isomorphism

$$\mathrm{h}^\delta : H^{s,\gamma}(M) \to H^{s,\gamma+\delta}(M) \tag{3.1.23}$$

for every $s,\gamma,\delta \in \mathbb{R}$.

Moreover, again according to Example 3.1.5, now for the smooth compact manifold $\widetilde{M}$ we have an order reducing family $\tilde{b}(\eta)$ in the scale of Sobolev spaces $H^s(\widetilde{M})$, $s \in \mathbb{R}$. More generally, we employ parameter-dependent families $\tilde{a}(\eta) \in L^\mu_{\mathrm{cl}}(\widetilde{M};\mathbb{R}^q)$. The symbols $a(\eta)$ that we want to establish in the scale $H^{s,\gamma}(M)$ on our compact manifold $M$ with conical singularity $v$ will be essentially (i.e., modulo Schwartz functions in $\eta$ with values in globally smoothing operators on $M$) constructed in the form

$$a(\eta) := \sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma} + (1-\sigma)a_{\mathrm{int}}(\eta)(1-\tilde{\tilde{\sigma}}), \tag{3.1.24}$$

$a_{\text{int}}(\eta) := \tilde{a}(\eta)|_{\text{int}\mathbb{M}}$, with cut-off functions $\sigma(t), \tilde{\sigma}(t), \tilde{\tilde{\sigma}}(t)$ on the half axis, supported in $[0, 2/3)$, with the property

$$\tilde{\tilde{\sigma}} \prec \sigma \prec \tilde{\sigma}$$

(here $\sigma \prec \tilde{\sigma}$ means the $\tilde{\sigma}$ is equal to 1 in a neighbourhood of $\operatorname{supp}\sigma$).
The "edge" part of (3.1.24) will be defined in the variables $(t, x) \in X^\wedge$. Let us choose a parameter-dependent elliptic family of operators of order $\mu$ on $X$

$$\tilde{p}(t, \tilde{\tau}, \tilde{\eta}) \in C^\infty\big(\overline{\mathbb{R}}_+, L^\mu_{\text{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\tau},\tilde{\eta}})\big).$$

Setting

$$p(t, \tau, \eta) := \tilde{p}(t, t\tau, t\eta) \tag{3.1.25}$$

we have what is known as an edge-degenerate family of operators on $X$. We now employ the following Mellin quantisation theorem.

**Definition 3.1.18.** *Let $M^\mu_{\mathcal{O}}(X; \mathbb{R}^q)$ be the set of all $h(z, \eta) \in \mathcal{A}\big(\mathbb{C}, L^\mu_{\text{cl}}(X; \mathbb{R}^q_\eta)\big)$ such that $h(\beta + i\tau, \eta) \in L^\mu_{\text{cl}}(X; \mathbb{R}^{1+q}_{\tau,\eta})$ for every $\beta \in \mathbb{R}$, uniformly in compact $\beta$-intervals (here $\mathcal{A}(\mathbb{C}, E)$ with any Fréchet space $E$ denotes the space of all $E$-valued holomorphic functions in $\mathbb{C}$, in the Fréchet topology of uniform convergence on compact sets).*

Observe that also $M^\mu_{\mathcal{O}}(X; \mathbb{R}^q)$ is a Fréchet space in a natural way. Given an $f(t, t', z, \eta) \in C^\infty\big(\mathbb{R}_+ \times \mathbb{R}_+, L^\mu_{\text{cl}}(X; \Gamma_{\frac{1}{2}-\gamma} \times \mathbb{R}^q)\big)$ we set

$$\operatorname{op}^\gamma_M(f)(\eta)u(t) := \int_{\mathbb{R}} \int_0^\infty \Big(\frac{t}{t'}\Big)^{-(\frac{1}{2}-\gamma+i\tau)} f(t, t', \frac{1}{2} - \gamma + i\tau, \eta) u(t') \frac{dt'}{t'} đ\tau,$$

$đ\tau = (2\pi)^{-1} d\tau$, which is regarded as a parameter-dependent weighted pseudo-differential operator with symbol $f$, referring to the weight $\gamma \in \mathbb{R}$. The Mellin quantisation theorem states that there exists an element

$$\tilde{h}(t, z, \tilde{\eta}) \in C^\infty\big(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}})\big), \tag{3.1.26}$$

such that, when we set

$$h(t, z, \eta) := \tilde{h}(t, z, t\eta) \tag{3.1.27}$$

we have

$$\operatorname{op}^\gamma_M(h)(\eta) = \operatorname{Op}_t(p)(\eta) \mod L^{-\infty}(X^\wedge; \mathbb{R}^q_\eta), \tag{3.1.28}$$

for every weight $\gamma \in \mathbb{R}$. Observe that when we set

$$p_0(t, \tau, \eta) := \tilde{p}(0, t\tau, t\eta), \quad h_0(t, z, \eta) := \tilde{h}(0, z, t\eta) \tag{3.1.29}$$

we also have $\operatorname{op}^\gamma_M(h_0)(\eta) = \operatorname{Op}_t(p_0)(\eta) \mod L^{-\infty}(X^\wedge; \mathbb{R}^q_\eta)$, for all $\gamma \in \mathbb{R}$.
Let us now choose cut-off functions $\omega(t), \tilde{\omega}(t), \tilde{\tilde{\omega}}(t)$ such that $\tilde{\tilde{\omega}} \prec \omega \prec \tilde{\omega}$. Fix the notation $\omega_\eta(t) := \omega(t[\eta])$, and form the operator function

$$\begin{aligned} a_{\text{edge}}(\eta) := {} & t^{-\mu}\omega_\eta(t)\operatorname{op}^{\gamma-\frac{n}{2}}_M(h)(\eta)\tilde{\omega}_\eta(t) \\ & + t^{-\mu}\big(1 - \omega_\eta(t)\big)\operatorname{Op}_t(p)(\eta)\big(1 - \tilde{\tilde{\omega}}_\eta(t)\big) + m(\eta) + g(\eta). \end{aligned} \tag{3.1.30}$$

Here $m(\eta)$ and $g(\eta)$ are smoothing Mellin and Green symbols of the edge calculus. The definition of $m(\eta)$ is based on smoothing Mellin symbols $f(z) \in M^{-\infty}(X;\Gamma_\beta)$. Here $M^{-\infty}(X;\Gamma_\beta)$ is the subspace of all $f(z) \in L^{-\infty}(X;\Gamma_\beta)$ such that for some $\varepsilon > 0$ (depending on $f$) the function $f$ extends to an

$$l(z) \in \mathcal{A}\big(U_{\beta,\varepsilon}, L^{-\infty}(X)\big)$$

where $U_{\beta,\varepsilon} := \{z \in \mathbb{C} : |\mathrm{Re}\, z - \beta| < \varepsilon\}$ and

$$l(\delta + i\tau) \in L^{-\infty}(X;\mathbb{R}_\tau)$$

for every $\delta \in (\beta-\varepsilon, \beta+\varepsilon)$, uniformly in compact subintervals. By definition we then have $f(\beta+i\tau) = l(\beta+i\tau)$; for brevity we often denote the holomorphic extension $l$ of $f$ again by $f$. For $f \in M^{-\infty}(X;\Gamma_{\frac{n+1}{2}-\gamma})$ we set

$$m(\eta) := t^{-\mu}\omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(f)\tilde{\omega}_\eta$$

for any cut-off functions $\omega, \tilde{\omega}$.
In order to explain the structure of $g(\eta)$ in (3.1.30) we first introduce weighted spaces on the infinite stretched cone $X^\wedge = \mathbb{R}_+ \times X$, namely,

$$\mathcal{K}^{s,\gamma;g}(X^\wedge) := \omega\mathcal{H}^{s,\gamma}(X^\wedge) + (1-\omega)H^{s;g}_{\mathrm{cone}}(X^\wedge) \qquad (3.1.31)$$

for any $s, \gamma, g \in \mathbb{R}$, and a cut-off function $\omega$, see (3.1.22) which defines the norm in $\mathcal{H}^{s,\gamma}(X^\wedge)$ and the formula (3.1.3) for $H^{s;g}_{\mathrm{cone}}(X)$. Moreover, we set $\mathcal{K}^{s,\gamma}(X^\wedge) := \mathcal{K}^{s,\gamma;0}(X^\wedge)$. The operator families $g(\eta)$ are so-called Green symbols in the covariable $\eta \in \mathbb{R}^q$, defined by

$$g(\eta) \in S^\mu_{\mathrm{cl}}\big(\mathbb{R}^q_\eta; \mathcal{K}^{s,\gamma;g}(X^\wedge), \mathcal{S}^{\gamma-\mu+\varepsilon}(X^\wedge)\big), \qquad (3.1.32)$$

$$g^*(\eta) \in S^\mu_{\mathrm{cl}}\big(\mathbb{R}^q_\eta; \mathcal{K}^{s,-\gamma+\mu;g}(X^\wedge), \mathcal{S}^{-\gamma+\varepsilon}(X^\wedge)\big), \qquad (3.1.33)$$

for all $s, \gamma, g \in \mathbb{R}$, where $g^*$ denotes the $\eta$-wise formal adjoint with respect to the scalar product of $\mathcal{K}^{0,0;0}(X^\wedge) = r^{-\frac{n}{2}}L^2(\mathbb{R}_+ \times X)$ and $\varepsilon = \varepsilon(g) > 0$. Here

$$\mathcal{S}^\beta(X^\wedge) := \omega\mathcal{K}^{\infty,\beta}(X^\wedge) + (1-\omega)\mathcal{S}\big(\overline{\mathbb{R}}_+, C^\infty(X)\big)$$

for any cut-off function $\omega$. The notion of operator-valued symbols in (3.1.32), (3.1.33) refers to (3.1.6) in its generalisation to Fréchet spaces $\widetilde{H}$ with group actions (see Remark 3.1.8) that are in the present case given by

$$\kappa_\lambda : u(t,x) \to \lambda^{\frac{n+1}{2}+g}u(\lambda t, x), \ \lambda \in \mathbb{R}_+ \qquad (3.1.34)$$

$n = \dim X$, both in the spaces $\mathcal{K}^{s,\gamma;g}(X^\wedge)$ and $\mathcal{S}^\beta(X^\wedge)$.

The following theorem is crucial for proving that our new order reduction family is well defined. Therefore we will sketch the main steps of the proof, which is based on the edge calculus. Various aspects of the proof can be found in the literature, for example in Kapanadze and Schulze [13, Proposition 3.3.79], Schrohe and Schulze [34], Harutyunyan and Schulze [9]. Among the tools we have the pseudo-differential operators on $X^\wedge$ interpreted as a manifold with conical exit to infinity $r \to \infty$; the general background may be found in Schulze [39]. The calculus of such exit operators goes back to Parenti [28], Cordes [2], Shubin [45], and others.

**Theorem 3.1.19.** *We have*

$$\sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma} \in S^{\mu}\big(\mathbb{R}^q; \mathcal{K}^{s,\gamma;g}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu;g}(X^\wedge)\big) \tag{3.1.35}$$

*for every $s, g \in \mathbb{R}$, more precisely,*

$$D_\eta^\beta\{\sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma}\} \in S^{\mu-|\beta|}\big(\mathbb{R}^q; \mathcal{K}^{s,\gamma;g}(X^\wedge), \mathcal{K}^{s-\mu+|\beta|,\gamma-\mu;g}(X^\wedge)\big) \tag{3.1.36}$$

*for all $s, g \in \mathbb{R}$ and all $\beta \in \mathbb{N}^q$. (The spaces of symbols in* (3.1.35), (3.1.36) *refer to the group action* (3.1.34)*).*

*Proof.* To prove the assertions it is enough to consider the case without $m(\eta) + g(\eta)$, since the latter sum maps to $\mathcal{K}^{\infty,\gamma;g}(X^\wedge)$ anyway. The first part of the Theorem is known, see, for instance, [9] or [3]. Concerning the relation (3.1.36) we write

$$\sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma} = \sigma\{a_{\mathrm{c}}(\eta) + a_\psi(\eta)\}\tilde{\sigma} \tag{3.1.37}$$

with

$$a_{\mathrm{c}}(\eta) := t^{-\mu}\omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\tilde{\omega}_\eta, \quad a_\psi(\eta) := t^{-\mu}(1-\omega_\eta)\mathrm{Op}_t(p)(\eta)(1-\tilde{\tilde{\omega}}_\eta)$$

and it suffices to take the summands separately. In order to show (3.1.36) we consider, for instance, the derivative $\partial/\partial\eta_j =: \partial_j$ for some $1 \le j \le q$. By iterating the process we then obtain the assertion. We have

$$\partial_j \sigma\{a_{\mathrm{c}}(\eta) + a_\psi(\eta)\}\tilde{\sigma} = \sigma\{\partial_j a_{\mathrm{c}}(\eta) + \partial_j a_\psi(\eta)\}\tilde{\sigma} = b_1(\eta) + b_2(\eta) + b_3(\eta)$$

with

$$\begin{aligned} b_1(\eta) &:= \sigma t^{-\mu}\Big\{\omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\partial_j\tilde{\omega}_\eta + (1-\omega_\eta)\mathrm{Op}_t(p)(\eta)\partial_j(1-\tilde{\tilde{\omega}}_\eta)\Big\}\tilde{\sigma}, \\ b_2(\eta) &:= \sigma t^{-\mu}\Big\{\omega_\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(\partial_j h)(\eta)\tilde{\omega}_\eta + (1-\omega_\eta)\mathrm{Op}_t(\partial_j p)(\eta)(1-\tilde{\tilde{\omega}}_\eta)\Big\}\tilde{\tilde{\sigma}}, \\ b_3(\eta) &:= \sigma t^{-\mu}\Big\{(\partial_j\omega_\eta)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\tilde{\omega}_\eta + \big(\partial_j(1-\omega_\eta)\big)\mathrm{Op}_t(p)(\eta)(1-\tilde{\tilde{\omega}}_\eta)\Big\}\tilde{\sigma}. \end{aligned}$$

In $b_1(\eta)$ we can apply a pseudo-locality argument which is possible since $\partial_j\tilde{\omega}_\eta \equiv 0$ on $\operatorname{supp}\omega_\eta$ and $\partial_j(1-\tilde{\tilde{\omega}}_\eta) \equiv 0$ on $\operatorname{supp}(1-\omega_\eta)$; this yields (together with similar considerations as for the proof of (3.1.35))

$$b_1(\eta) \in S^{\mu-1}\big(\mathbb{R}^q; \mathcal{K}^{s,\gamma;g}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu;g}(X^\wedge)\big).$$

Moreover we obtain

$$b_2(\eta) \in S^{\mu-1}\big(\mathbb{R}^q; \mathcal{K}^{s,\gamma;g}(X^\wedge), \mathcal{K}^{s-\mu+1,\gamma-\mu;g}(X^\wedge)\big)$$

since $\partial_j h$ and $\partial_j p$ are of order $\mu - 1$ (again combined with arguments as for (3.1.35)). Concerning $b_3(\eta)$ we use the fact that there is a $\psi \in C_0^\infty(\mathbb{R}_+)$ such that $\psi \equiv 1$ on $\operatorname{supp}\partial_j\omega$, $\tilde{\omega} - \psi \equiv 0$ on $\operatorname{supp}\partial_j\omega$ and $(1-\tilde{\tilde{\omega}}) - \psi \equiv 0$ on $\operatorname{supp}\partial_j\omega$. Thus, when we set $\psi_\eta(t) := \psi(t[\eta])$, we obtain $b_3(\eta) := c_3(\eta) + c_4(\eta)$ with

$$\begin{aligned} c_3(\eta) &:= \sigma t^{-\mu}\Big\{(\partial_j\omega_\eta)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\psi_\eta - (\partial_j\omega_\eta)\mathrm{Op}_t(p)(\eta)\psi_\eta\Big\}\tilde{\sigma}, \\ c_4(\eta) &:= \sigma t^{-\mu}\Big\{(\partial_j\omega_\eta)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)[\tilde{\omega}_\eta - \psi_\eta] - (\partial_j\omega_\eta)\mathrm{Op}_t(p)(\eta)[(1-\tilde{\tilde{\omega}}_\eta) - \psi_\eta]\Big\}\tilde{\sigma}. \end{aligned}$$

Here, using $\partial_j \omega_\eta = (\omega')_\eta \partial_j(t[\eta])$ which yields an extra power of $t$ on the left of the operator, together with pseudo-locality argument, we obtain

$$c_4(\eta) \in S^{\mu-1}\big(\mathbb{R}^q; \mathcal{K}^{s,\gamma;g}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu;g}(X^\wedge)\big).$$

To treat $c_3(\eta)$ we employ that both $\partial_j\omega_\eta$ and $\psi_\eta$ are compactly supported on $\mathbb{R}_+$. Using the property (3.1.28), we have

$$c_3(\eta) = \sigma t^{-\mu}(\partial_j\omega_\eta)\big\{\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta) - \mathrm{Op}_t(p)(\eta)\big\}\psi_\eta\tilde{\sigma} \\ \in S^{\mu-1}\big(\mathbb{R}^q; \mathcal{K}^{s,\gamma;g}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu;g}(X^\wedge)\big).$$

□

**Definition 3.1.20.** *An operator family $c(\eta) \in \mathcal{S}\big(\mathbb{R}^q, \bigcap_{s\in\mathbb{R}} \mathcal{L}\big(H^{s,\gamma}(M), H^{\infty,\delta}(M)\big)\big)$ is called a smoothing element in the parameter-dependent cone calculus on $M$ associated with the weight data $(\gamma,\delta) \in \mathbb{R}^2$, written $c \in C_G(M,(\gamma,\delta);\mathbb{R}^q)$, if there is an $\varepsilon = \varepsilon(c) > 0$ such that*

$$c(\eta) \in \mathcal{S}\big(\mathbb{R}^q, \mathcal{L}\big(H^{s,\gamma}(M), H^{\infty,\delta+\varepsilon}(M)\big)\big),$$
$$c^*(\eta) \in \mathcal{S}\big(\mathbb{R}^q, \mathcal{L}\big(H^{s,-\delta}(M), H^{\infty,-\gamma+\varepsilon}(M)\big)\big);$$

*for all $s \in \mathbb{R}$; here $c^*$ is the $\eta$-wise formal adjoint of $c$ with respect to the $H^{0,0}(M)$-scalar product.*

The $\eta$-wise kernels of the operators $c(\eta)$ are in $C^\infty\big((M\setminus\{v\})\times(M\setminus\{v\})\big)$. However, they are of flatness $\varepsilon$ in the respective distance variables to $v$, relative to the weights $\delta$ and $\gamma$, respectively. Let us look at a simple example to illustrate the structure. We choose elements $k \in \mathcal{S}\big(\mathbb{R}^q, H^{\infty,\delta+\varepsilon}(M)\big)$, $k' \in \mathcal{S}\big(\mathbb{R}^q, H^{\infty,-\gamma+\varepsilon}(M)\big)$ and assume for convenience that both $k$ and $k'$ vanish outside a neighbourhood of $v$, for all $\eta \in \mathbb{R}^q$. Then, with respect to a local splitting of variables $(t,x)$ near $v$, we can write $k = k(t,x,\eta)$ and $k' = k'(t',x',\eta)$. Set

$$c(\eta)u(t,x) := \iint k(t,x,\eta)k'(t',x',\eta)u(t',x')t'^n dt'dx'$$

with the formal adjoint

$$c^*(\eta)v(t',x') := \iint \overline{k'(t',x',\eta)k(t,x,\eta)}v(t,x)t^n dtdx.$$

Then $c(\eta)$ is a smoothing element in the parameter-dependent cone calculus.

**Definition 3.1.21.** *By $C^\mu(M,(\gamma,\gamma-\mu);\mathbb{R}^q)$ we denote the set of all operator families*

$$a(\eta) = \sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma} + (1-\sigma)a_{\mathrm{int}}(\eta)(1-\tilde{\tilde{\sigma}}) + c(\eta) \qquad (3.1.38)$$

*where $a_{\mathrm{edge}}$ is of the form (3.1.30), $a_{\mathrm{int}} \in L^\mu_{\mathrm{cl}}(M\setminus\{v\};\mathbb{R}^q)$, while $c(\eta)$ is a parameter-dependent smoothing operator on $M$, associated with the weight data $(\gamma,\gamma-\mu)$.*

**Remark 3.1.22.** *According to* [6] *an operator family* $a(\eta) \in C^{\mu}(M, (\gamma, \gamma - \mu); \mathbb{R}^q)$ *can be equivalently written in the form*

$$a(\eta) = \sigma t^{-\mu} \mathrm{op}_M^{\gamma - \frac{n}{2}}(h)(\eta)\tilde{\sigma} + (1 - \sigma) a_{\mathrm{int}}(\eta)(1 - \tilde{\tilde{\sigma}}) + (m + g)(\eta) + c(\eta) \tag{3.1.39}$$

*where* $h$ *is as in* (3.1.27), $a_{\mathrm{int}}(\eta) \in L^{\mu}_{\mathrm{cl}}(M \setminus \{v\}; \mathbb{R}^q)$, $m + g$ *is a smoothing Mellin and Green operator and* $c(\eta) \in C_G(M, (\gamma, \gamma - \mu); \mathbb{R}^q)$.

**Theorem 3.1.23.** *Let* $M$ *be a compact manifold with a conical singularity. Then the* $\eta$*-dependent families* (3.1.24), *which define continuous operators*

$$a(\eta) : H^{s,\gamma}(M) \to H^{s-\nu,\gamma-\nu}(M) \tag{3.1.40}$$

*for all* $s \in \mathbb{R}$, $\nu \geq \mu$, *have the properties:*

$$\|a(\eta)\|_{\mathcal{L}(H^{s,\gamma}(M), H^{s-\nu,\gamma-\nu}(M))} \leq c\langle\eta\rangle^{B} \tag{3.1.41}$$

*for all* $\eta \in \mathbb{R}^q$, $s \in \mathbb{R}$, *with constants* $c = c(\mu, \nu, s) > 0$, $B = B(\mu, \nu, s)$, *and, when* $\mu \leq 0$,

$$\|a(\eta)\|_{\mathcal{L}(H^{0,0}(M), H^{0,0}(M))} \leq c\langle\eta\rangle^{\mu} \tag{3.1.42}$$

*for all* $\eta \in \mathbb{R}$, $s \in \mathbb{R}$, *with constants* $c = c(\mu, s) > 0$.

*Proof.* The result is known for the summand $(1 - \sigma) a_{\mathrm{int}}(\eta)(1 - \tilde{\tilde{\sigma}})$ as we see from Example 3.1.5. Therefore, we may concentrate on

$$p(\eta) := \sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma} : H^{s,\gamma}(M) \to H^{s-\nu,\gamma-\nu}(M).$$

To show (3.1.41) we pass to

$$\sigma a_{\mathrm{edge}}(\eta)\tilde{\sigma} : \mathcal{K}^{s,\gamma}(X^{\wedge}) \to \mathcal{K}^{s-\nu,\gamma-\nu}(X^{\wedge}).$$

Then Theorem 3.1.19 shows that we have symbolic estimates, especially

$$\|\kappa^{-1}_{\langle\eta\rangle} p(\eta) \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}))} \leq c\langle\eta\rangle^{\mu}.$$

We have

$$\|p(\eta)\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{K}^{s-\nu,\gamma-\nu}(X^{\wedge}))} \leq \|p(\eta)\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}))},$$

and

$$\begin{aligned} \|p(\eta)\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}))} &= \|\kappa_{\langle\eta\rangle} \kappa^{-1}_{\langle\eta\rangle} p(\eta) \kappa_{\langle\eta\rangle} \kappa^{-1}_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}))} \\ &\leq \|\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}), \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}))} \|\kappa^{-1}_{\langle\eta\rangle} p(\eta) \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}))} \\ &\qquad \|\kappa^{-1}_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{K}^{s,\gamma}(X^{\wedge}))} \leq c\langle\eta\rangle^{\mu + \widetilde{M} + M}. \end{aligned}$$

Here we used that $\kappa_{\langle\eta\rangle}, \kappa^{-1}_{\langle\eta\rangle}$ satisfy estimates like (3.1.5).
For (3.1.42) we employ that $\kappa_{\lambda}$ is operating as a unitary group on $\mathcal{K}^{0,0}(X^{\wedge})$. This gives us

$$\begin{aligned} \|p(\eta)\|_{\mathcal{L}(\mathcal{K}^{0,0}(X^{\wedge}), \mathcal{K}^{0,0}(X^{\wedge}))} &= \|\kappa^{-1}_{\langle\eta\rangle} p(\eta) \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{K}^{0,0}(X^{\wedge}), \mathcal{K}^{0,0}(X^{\wedge}))} \\ &\leq \|\kappa^{-1}_{\langle\eta\rangle} p(\eta) \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{K}^{0,0}(X^{\wedge}), \mathcal{K}^{-\mu,-\mu}(X^{\wedge}))} \leq c\langle\eta\rangle^{\mu}. \quad \square \end{aligned}$$

**Theorem 3.1.24.** *For every $k \in \mathbb{Z}$ there exists an $f_k(z) \in M^{-\infty}(X; \Gamma_{\frac{n+1}{2}-\gamma})$ such that for every cut-off functions $\omega, \tilde{\omega}$ the operator*

$$A := 1 + \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(f_k)\tilde{\omega} : H^{s,\gamma}(M) \to H^{s,\gamma}(M) \tag{3.1.43}$$

*is Fredholm of index $k$, for all $s \in \mathbb{R}$.*

*Proof.* We employ the result (cf. [38]) that for every $k \in \mathbb{Z}$ there exists an $f_k(z)$ such that

$$\widetilde{A} := 1 + \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(f_k)\tilde{\omega} : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge) \tag{3.1.44}$$

is Fredholm of index $k$. Recall that the proof of the latter result follows from a corresponding theorem in the case $\dim X = 0$. The Mellin symbol $f_k$ is constructed in such a way that $1 + f_k(z) \neq 0$ for all $z \in \Gamma_{\frac{1}{2}-\gamma}$ and the argument of $1 + f_k(z)|_{\Gamma_{\frac{1}{2}-\gamma}}$ varies from 1 to $2\pi k$ when $z \in \Gamma_{\frac{1}{2}-\gamma}$ goes from $\mathrm{Im}\, z = -\infty$ to $\mathrm{Im}\, z = +\infty$. The choice of $\omega, \tilde{\omega}$ is unessential; so we assume that $\omega, \tilde{\omega} \equiv 0$ for $t \geq 1 - \varepsilon$ with some $\varepsilon > 0$. Let us represent the cone $\widetilde{M} := X^\Delta$ as a union of $\big([0, 1 + \frac{\varepsilon}{2}) \times X\big)/(\{0\} \times X) =: \widetilde{M}_-$ and $(1 - \frac{\varepsilon}{2}, \infty) \times X =: \widetilde{M}_+$. Then

$$\widetilde{A}|_{\widetilde{M}_-} = 1 + \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(f_k)\tilde{\omega}, \quad \widetilde{A}|_{\widetilde{M}_+} = 1. \tag{3.1.45}$$

Moreover, without loss of generality, we represent $M$ as a union $\big([0, 1+\frac{\varepsilon}{2}) \times X\big)/(\{0\} \times X) \cup M_+$ where $M_+$ is an open $C^\infty$ manifold which intersects $\big([0, 1 + \frac{\varepsilon}{2}) \times X\big)/(\{0\} \times X) =: M_-$ in a cylinder of the form $(1 - \frac{\varepsilon}{2}, 1 + \frac{\varepsilon}{2}) \times X$. Let $B$ denote the operator on $M$, defined by

$$B_- := A|_{M_-} = 1 + \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(f_k)\tilde{\omega}, \quad B_+ := A|_{M_+} = 1 \tag{3.1.46}$$

We are then in a special situation of cutting and pasting of Fredholm operators. We can pass to manifolds with conical singularities $N$ and $\widetilde{N}$ by setting

$$N = \widetilde{M}_- \cup M_+, \quad \widetilde{N} = M_- \cup \widetilde{M}_+$$

and transferring the former operators in (3.1.45), (3.1.46) to $N$ and $\widetilde{N}$, respectively, by gluing together the $\pm$ pieces of $\widetilde{A}$ and $A$ to belong to $\widetilde{M}_\pm$ and $M_\pm$ to corresponding operators $\widetilde{B}$ on $\widetilde{N}$ and $B$ on $N$. We then have the relative index formula

$$\mathrm{ind} A - \mathrm{ind} B = \mathrm{ind} \widetilde{A} - \mathrm{ind} \widetilde{B} \tag{3.1.47}$$

(see [26]). In the present case $\widetilde{A}$ and $\widetilde{M}$ are the same as $\widetilde{B}$ and $\widetilde{N}$ where $B$ and $N$ are the same as $A$ and $M$. It follows that

$$\mathrm{ind} \widetilde{A} - \mathrm{ind} \widetilde{B} = \mathrm{ind} B - \mathrm{ind} A. \tag{3.1.48}$$

From (3.1.47), (3.1.48) it follows that $\mathrm{ind} A = \mathrm{ind} B = \mathrm{ind} \widetilde{A}$. □

Let us now give more information about the above mentioned space

$$C^\mu(M, \boldsymbol{g}; \mathbb{R}^q), \ \boldsymbol{g} = (\gamma, \gamma - \mu),$$

of parameter-dependent cone operators on $M$ of order $\mu \in \mathbb{R}$, with the weight data $\boldsymbol{g}$, cf. Definition 3.1.21. The elements $a(\eta) \in C^\mu(M, \boldsymbol{g}; \mathbb{R}^q)$ have a principal symbolic hierarchy

$$\sigma(a) := (\sigma_\psi(a), \sigma_\wedge(a)) \tag{3.1.49}$$

where $\sigma_\psi(a)$ is the parameter-dependent homogeneous principal symbol of order $\mu$, defined through $a(\eta) \in L^\mu_{\mathrm{cl}}(M \setminus \{v\}; \mathbb{R}^q)$. This determines the reduced symbol

$$\tilde{\sigma}_\psi(a)(t, x, \tau, \xi, \eta) := t^\mu \sigma_\psi(a)(t, x, t^{-1}\tau, \xi, t^{-1}\eta)$$

given close to $v$ in the splitting of variables $(t, x)$ with covariables $(\tau, \xi)$. By construction $\tilde{\sigma}_\psi(a)$ is smooth up to $t = 0$. The second component, the edge symbol $\sigma_\wedge(a)(\eta)$ is defined as

$$\sigma_\wedge(a)(\eta) := t^{-\mu}\omega_{|\eta|}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0)(\eta)\tilde{\omega}_{|\eta|} + t^{-\mu}(1-\omega_{|\eta|})\mathrm{Op}_t(p_0)(\eta)(1-\tilde{\tilde{\omega}}_{|\eta|}) + \sigma_\wedge(m+g)(\eta),$$

cf. (3.1.29), where $\sigma_\wedge(m + g)(\eta)$ is just the (twisted) homogeneous principal symbol of $m + g$ as a classical operator-valued symbol.

Every element $a(\eta)$ of $C^\mu(M, \boldsymbol{g}; \mathbb{R}^q)$ represents families of continuous operators

$$a(\eta) : H^{s,\gamma}(M) \to H^{s-\mu,\gamma-\mu}(M) \tag{3.1.50}$$

for all $s \in \mathbb{R}$.

**Definition 3.1.25.** *An element $a(\eta) \in C^\mu(M, \boldsymbol{g}; \mathbb{R}^q)$ is called elliptic, if*

(i) *$\sigma_\psi(a)$ never vanishes as a function on $T^*\big((M \setminus \{v\}) \times \mathbb{R}^q\big) \setminus 0$, and if $\tilde{\sigma}_\psi(a)$ does not vanish for all $(t, x, \tau, \xi, \eta)$, $(\tau, \xi, \eta) \neq 0$, up to $t = 0$;*

(ii) *$\sigma_\wedge(a)(\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$ is a family of isomorphisms for all $\eta \neq 0$, and any $s \in \mathbb{R}$.*

**Remark 3.1.26.** *There is an extended notion of ellipticity for $2 \times 2$ block matrix families which includes extra trace and potential families.*

**Theorem 3.1.27.** *If $a(\eta) \in C^\mu(M, \boldsymbol{g}; \mathbb{R}^q)$, $\boldsymbol{g} = (\gamma, \gamma - \mu)$, is elliptic, there exists an element $a^{(-1)}(\eta) \in C^{-\mu}(M, \boldsymbol{g}^{-1}; \mathbb{R}^q)$, $\boldsymbol{g}^{-1} := (\gamma - \mu, \gamma)$, such that*

$$1 - a^{(-1)}(\eta)a(\eta) \in C_G(M, \boldsymbol{g}_l; \mathbb{R}^q), \quad 1 - a(\eta)a^{(-1)}(\eta) \in C_G(M, \boldsymbol{g}_r; \mathbb{R}^q),$$

*where $\boldsymbol{g}_l := (\gamma, \gamma)$, $\boldsymbol{g}_r := (\gamma - \mu, \gamma - \mu)$.*

The proof employs known elements of the edge symbolic calculus (cf. [39]); so we do not recall the details here. Let us only note that the inverses of $\sigma_\psi(a), \tilde{\sigma}_\psi(a)$ and $\sigma_\wedge(a)$ can be employed to construct an operator family $b(\eta) \in C^{-\mu}(M, \boldsymbol{g}^{-1}; \mathbb{R}^q)$ such that

$$\sigma_\psi(a^{(-1)}) = \sigma_\psi(b), \quad \tilde{\sigma}_\psi(a^{(-1)}) = \tilde{\sigma}_\psi(b), \quad \sigma_\wedge(a^{(-1)}) = \sigma_\wedge(b).$$

This gives us $1 - b(\eta)a(\eta) =: c_0(\eta) \in C^{-1}(M, \boldsymbol{g}_l; \mathbb{R}^q)$, and a formal Neumann series argument allows us to improve $b(\eta)$ to a left parametrix $a^{(-1)}(\eta)$ by setting $a^{(-1)}(\eta) := \Big(\sum_{j=0}^\infty c_0^j(\eta)\Big) b(\eta)$ (using the existence of the asymptotic sum in $C^0(M, \boldsymbol{g}; \mathbb{R}^q)$). In a similar manner we can construct a right parametrix, i.e., $a^{(-1)}(\eta)$ is as desired.

**Corollary 3.1.28.** *If $a(\eta)$ is as in Theorem 3.1.27, then (3.1.50) is a family of Fredholm operators of index 0, and there is a constant $C > 0$ such that the operators (3.1.50) are isomorphisms for all $|\eta| \geq C$, $s \in \mathbb{R}$.*

**Theorem 3.1.29.** *The space $C^\mu(M, (\gamma, \gamma - \mu); \mathbb{R}^q)$ contains an element $a(\eta)$ which induces a family of isomorphisms*

$$a(\eta) : H^{s,\gamma}(M) \to H^{s-\mu,\gamma-\mu}(M) \tag{3.1.51}$$

*for all $s \in \mathbb{R}$ and all $\eta \in \mathbb{R}^q$.*

*Proof.* The strategy of the proof is to construct an operator family $a(\eta, \zeta, \lambda)$ in $C^\mu(M, (\gamma, \gamma - \mu); \mathbb{R}^{q+r+l}_{\eta,\zeta,\lambda})$, $q, r, l \in \mathbb{N} \setminus \{0\}$, which is parameter-dependent elliptic in the sense of Definition 3.1.25, then to apply Theorem 3.1.27 and finally to set $a(\eta) := a(\eta, \zeta^1, \lambda^1)$ for $\zeta^1, \lambda^1$ fixed.

We choose a function

$$p_\lambda(t, \tau, \eta, \zeta) := \tilde{p}_\lambda(t\tau, t\eta, t\zeta)$$

similarly as in (3.1.25) where $\tilde{p}_\lambda(\tilde{\tau}, \tilde{\eta}, \tilde{\zeta}) \in L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q+r+l}_{\tilde{\tau},\tilde{\eta},\tilde{\zeta},\lambda})$ is parameter-dependent elliptic with parameters $\tilde{\tau}, \tilde{\eta}, \tilde{\zeta}, \lambda$. We specify $\tilde{p}_\lambda$ in such a way that the parameter-dependent homogeneous principal symbol in $(x, \tilde{\tau}, \xi, \lambda, \tilde{\eta}, \tilde{\zeta})$ for $(\tilde{\tau}, \xi, \lambda, \tilde{\eta}, \tilde{\zeta}) \neq 0$ is equal to

$$(|\tilde{\tau}|^2 + |\xi, \lambda|^2 + |\tilde{\eta}|^2 + |\tilde{\zeta}|^2)^{\frac{\mu}{2}}.$$

We now form an element

$$\tilde{h}_\lambda(z, \tilde{\eta}, \tilde{\zeta}) \in M^\mu_{\mathcal{O}}(X; \mathbb{R}^{q+r+l}_{\tilde{\eta},\tilde{\zeta},\lambda})$$

analogously as (3.1.26) such that

$$h_\lambda(t, z, \eta, \zeta) := \tilde{h}_\lambda(z, t\eta, t\zeta)$$

satisfies

$$\mathrm{op}^\gamma_M(h_\lambda)(\eta, \zeta) = \mathrm{Op}_t(p_\lambda)(\eta, \zeta) \mod L^{-\infty}(X^\wedge; \mathbb{R}^{q+r+l}_{\eta,\zeta,\lambda}).$$

For every fixed $(\zeta, \lambda) \in \mathbb{R}^{r+l}$ this is exactly as before, but in this way we obtain corresponding $(\zeta, \lambda)$-dependent families of such objects. We set

$$\sigma b_{\mathrm{edge},\lambda}(\eta, \zeta)\tilde{\sigma} := t^{-\mu}\sigma \left\{\omega_{\eta,\zeta}\mathrm{op}^{\gamma-\frac{n}{2}}_M(h_\lambda)(\eta, \zeta)\tilde{\omega}_{\eta,\zeta} + \chi_{\eta,\zeta}\mathrm{Op}_t(p_\lambda)(\eta, \zeta)\widetilde{\chi}_{\eta,\zeta}\right\}\tilde{\sigma}$$

with

$$\chi_{\eta,\zeta}(t) := 1 - \omega_{\eta,\zeta}(t), \quad \widetilde{\chi}_{\eta,\zeta}(t) := 1 - \tilde{\tilde{\omega}}_{\eta,\zeta}(t).$$

Let us form the principal edge symbol

$$\sigma_\wedge(\sigma b_{\mathrm{edge},\lambda}\tilde{\sigma})(\eta, \zeta) = t^{-\mu}\left\{\omega_{|\eta,\zeta|}\mathrm{op}^{\gamma-\frac{n}{2}}_M(h_\lambda)(\eta, \zeta)\tilde{\omega}_{|\eta,\zeta|} + \chi_{|\eta,\zeta|}\mathrm{Op}_t(p_\lambda)(\eta, \zeta)\widetilde{\chi}_{|\eta,\zeta|}\right\}$$

for $|\eta, \zeta| \neq 0$. The latter is interpreted as a family of continuous operators

$$\sigma_\wedge(\sigma b_{\mathrm{edge},\lambda}\tilde{\sigma})(\eta, \zeta) : \mathcal{K}^{s,\gamma;g}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu;g}(X^\wedge) \tag{3.1.52}$$

which is elliptic as a family of classical pseudo-differential operators on $X^\wedge$. In addition it is "exit" elliptic on $X^\wedge$ in the sense of Remark 1.2.15 with respect to the conical exit of $X^\wedge$ to infinity. In order that (3.1.52) is Fredholm for the given weight $\gamma \in \mathbb{R}$ and all $s, g \in \mathbb{R}$ it is necessary and sufficient that the subordinate conormal symbol

$$\sigma_{\mathrm{c}}\sigma_\wedge(\sigma b_{\mathrm{edge},\lambda}\tilde{\sigma})(z) : H^s(X) \to H^{s-\mu}(X)$$

is a family of isomorphisms for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$. This is standard information from the calculus on the stretched cone $X^\wedge$. By definition the conormal symbol is just

$$\tilde{h}_\lambda(z, 0, 0) : H^s(X) \to H^{s-\mu}(X). \tag{3.1.53}$$

Since by construction $\tilde{h}_\lambda(\beta + i\tau, 0, 0)$ is parameter-dependent elliptic on $X$ with parameters $(\tau, \lambda) \in \mathbb{R}^{1+l}$, for every $\beta \in \mathbb{R}$ (uniformly in finite $\beta$-intervals) there is a $C > 0$ such that (3.1.53) becomes bijective whenever $|\tau, \lambda| > C$. In particular, choosing $\lambda$ large enough, the bijectivity follows for all $\tau \in \mathbb{R}$, i.e., for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$. Let us fix $\lambda^1$ in that way and write again

$$\begin{aligned} p(t, \tau, \eta, \zeta) &:= \tilde{p}_{\lambda^1}(t\tau, t\eta, t\zeta), \\ h(t, z, \eta, \zeta) &:= \tilde{h}_{\lambda^1}(z, t\eta, t\zeta), \\ b_{\mathrm{edge}}(\eta, \zeta) &:= b_{\mathrm{edge},\lambda^1}(\eta, \zeta). \end{aligned}$$

We are now in the same situation we started with, but we know in addition that (3.1.52) is a family of Fredholm operators of a certain index, say $-k$ for some $k \in \mathbb{Z}$. With the smoothing Mellin symbol $f_k(z, \eta, \zeta) \in M^{-\infty}(X, \Gamma_{\frac{n+1}{2}-\gamma} \times \mathbb{R}^{q+r}_{\eta,\zeta})$ as in Theorem 3.1.24 we now form the composition

$$F(\eta, \zeta) := \sigma b_{\mathrm{edge}}(\eta, \zeta)\tilde{\sigma}\big(1 + \omega_{\eta,\zeta}\mathrm{op}_M^{\gamma-\frac{n}{2}}(f_k)\tilde{\omega}_{\eta,\zeta}\big), \tag{3.1.54}$$

which is of the form

$$\sigma b_{\mathrm{edge}}(\eta, \zeta)\tilde{\sigma} + \omega_{\eta,\zeta}\mathrm{op}_M^{\gamma-\frac{n}{2}}(f)\tilde{\omega}_{\eta,\zeta} + g(\eta, \zeta) \tag{3.1.55}$$

for another smoothing Mellin symbol $f(z, \eta, \zeta)$ and a certain Green symbol $g(\eta, \zeta)$. Here, by a suitable choice of $\omega, \tilde{\omega}$, without loss of generality we assume that $\sigma \equiv 1$ and $\tilde{\sigma} \equiv 1$ on $\operatorname{supp}\omega_{\eta,\zeta} \cup \operatorname{supp}\tilde{\omega}_{\eta,\zeta}$, for all $(\eta, \zeta) \in \mathbb{R}^{q+r}$. Since (3.1.54) is a composition of parameter-dependent cone operators the associated edge symbol is equal to

$$\sigma_\wedge F(\eta, \zeta) = \sigma_\wedge(\sigma b_{\mathrm{edge}}\tilde{\sigma})(\eta, \zeta)\big(1+\omega_{|\eta,\zeta|}\mathrm{op}_M^{\gamma-\frac{n}{2}}(f_k)\tilde{\omega}_{|\eta,\zeta|}\big) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{3.1.56}$$

which is a family of Fredholm operators of index 0. By construction (3.1.56) depends only on $|\eta, \zeta|$. For $(\eta, \zeta) \in S^{q+r-1}$, the unit sphere in $\mathbb{R}^{q+r}$, we now add a Green operator $g_0$ on $X^\wedge$ such that

$$F(\eta, \zeta) + g_0(\eta, \zeta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$$

is an isomorphism; it is known that such $g_0$ (of finite rank) exists (for $N = \dim\ker F(\eta, \zeta)$ it can be written in the form $g_0 u := \sum_{j=1}^N (u, v_j)w_j$, where $(\cdot, \cdot)$ is the $\mathcal{K}^{0,0}(X^\wedge)$-scalar product and $(v_j)_{j=1,\dots,N}$, $(w_j)_{j=1,\dots,N}$ are orthonormal systems of functions in $C_0^\infty(X^\wedge)$). Setting

$$g(\eta,\zeta) := \sigma\vartheta(\eta,\zeta)|\eta,\zeta|^\mu \kappa_{|\eta,\zeta|} g_0 \kappa_{|\eta,\zeta|}^{-1}\tilde{\sigma}$$

with an excision function $\vartheta(\eta,\zeta)$ in $\mathbb{R}^{q+r}$ we obtain a Green symbol with $\sigma_\wedge(g)(\eta,\zeta) = |\eta,\zeta|^\mu \kappa_{|\eta,\zeta|} g_0 \kappa_{|\eta,\zeta|}^{-1}$ and hence

$$\sigma_\wedge(F(\eta,\zeta)+g(\eta,\zeta)) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$$

is a family of isomorphisms for all $(\eta,\zeta) \in \mathbb{R}^{q+r} \setminus \{0\}$. Setting

$$\begin{aligned} &a_{\mathrm{edge}}(\eta,\zeta) \\ &:= t^{-\mu}\Big\{\omega_{\eta,\zeta}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta,\zeta)\tilde{\omega}_{\eta,\zeta} + \chi_{\eta,\zeta}\mathrm{Op}_t(p)(\eta,\zeta)\widetilde{\chi}_{\eta,\zeta}\Big\}\Big(1+\omega_{\eta,\zeta}\mathrm{op}_M^{\gamma-\frac{n}{2}}(f_k)\tilde{\omega}_{\eta,\zeta}\Big) \\ &\qquad + |\eta,\zeta|^\mu \vartheta(\eta,\zeta)\kappa_{|\eta,\zeta|} g_0 \kappa_{|\eta,\zeta|}^{-1} \end{aligned} \tag{3.1.57}$$

we obtain an operator family

$$\sigma a_{\mathrm{edge}}(\eta,\zeta)\tilde{\sigma} = F(\eta,\zeta) + g(\eta,\zeta)$$

as announced before.

Next we choose a parameter-dependent elliptic $a_{\mathrm{int}}(\eta,\zeta) \in L^\mu_{\mathrm{cl}}(M\setminus\{v\};\mathbb{R}^{q+r}_{\eta,\zeta})$ such that its parameter-dependent homogeneous principal symbol close to $t=0$ (in the splitting of variables $(t,x)$) is equal to

$$(|\tau|^2+|\xi|^2+|\eta|^2+|\zeta|^2)^{\frac{\mu}{2}}.$$

Then we form

$$a(\eta,\zeta) := \sigma a_{\mathrm{edge}}(\eta,\zeta)\tilde{\sigma} + (1-\sigma)a_{\mathrm{int}}(\eta,\zeta)(1-\tilde{\tilde{\sigma}})$$

with $\sigma,\tilde{\sigma},\tilde{\tilde{\sigma}}$ as in (3.1.24). This is now a parameter-dependent elliptic element of the cone calculus on $M$ with parameter $(\eta,\zeta) \in \mathbb{R}^{q+r}$. It is known (see Theorem 3.1.27) that there is a constant $C > 0$ such that the operators (3.1.51) are isomorphisms for all $|\eta,\zeta| \geq C$. Now, in order to construct $a(\eta)$ such that (3.1.51) are isomorphisms for all $\eta \in \mathbb{R}^q$ we simply fix $\zeta^1$ so that $|\zeta^1| > C$ and define $a(\eta) := a(\eta,\zeta^1)$. □

Observe that the operator functions of Theorem 3.1.29 refer to scales of spaces with two parameters, namely, $s \in \mathbb{R}$, the smoothness, and $\gamma \in \mathbb{R}$, the weight. Compared with Definition 3.1.10 we have here an additional weight. There are two ways to make the different view points compatible. One is to apply weight reducing isomorphisms

$$\mathrm{h}^{-\mu} : H^{s,\gamma}(M) \to H^{s,\gamma-\mu}(M) \tag{3.1.58}$$

as in (3.1.23). Then, passing from

$$a(\eta) : H^{s,\gamma}(M) \to H^{s-\mu,\gamma-\mu}(M) \tag{3.1.59}$$

to

$$b^\mu(\eta) := \mathrm{h}^{-\gamma+\mu}a(\eta)\mathrm{h}^\gamma : H^{s,0}(M) \to H^{s-\mu,0}(M) \tag{3.1.60}$$

we obtain operator functions between spaces only referring to $s$ but with properties as required in Definition 3.1.10.

**Remark 3.1.30.** *The spaces $E^s := H^{s,0}(M)$, $s \in \mathbb{R}$, form a scale with the properties at the beginning of Section* 3.1.1.

Another way is to modify the abstract framework by admitting scales $E^{s,\gamma}$ rather than $E^s$, where in general $\gamma$ may be in $\mathbb{R}^k$ (which is motivated by the higher corner calculus). We do not study the second possibility here but we only note that the variant with $E^{s,\gamma}$-spaces is very similar to the one without $\gamma$.

Let us now look at operator functions of the form (3.1.60).

**Theorem 3.1.31.** *The operators* (3.1.60) *constitute an order reducing family in the spaces $E^s := H^{s,0}(M)$, where the properties* (i)-(iii) *of Definition* 3.1.3 *are satisfied.*

*Proof.* In this proof we concentrate on the properties of our operators for every fixed $s, \mu, \nu$ with $\nu \geq \mu$. The uniformity of the involved constants can easily be deduced; however, the simple (but lengthy) considerations will be left out.
(i) We have to show that

$$D_\eta^\beta b^\mu(\eta) = D_\eta^\beta \{\mathrm{h}^{-\gamma+\mu} a(\eta) \mathrm{h}^\gamma\} \in C^\infty\big(\mathbb{R}^q, \mathcal{L}(E^s, E^{s-\mu+|\beta|})\big)$$

for all $s \in \mathbb{R}$, $\beta \in \mathbb{N}^q$. According to (3.1.24) the operator function is a sum of two contributions. The second summand

$$(1-\sigma)\mathrm{h}^{-\gamma+\mu} a_{\mathrm{int}}(\eta) \mathrm{h}^\gamma (1-\tilde{\tilde{\sigma}})$$

is a parameter-dependent family in $L^\mu_{\mathrm{cl}}(2\mathbb{M}; \mathbb{R}^q)$ and obviously has the desired property. The first summand is of the form

$$\sigma \mathrm{h}^{-\gamma+\mu} \{a_{\mathrm{edge}}(\eta) + m(\eta) + g(\eta)\} \mathrm{h}^\gamma \tilde{\sigma}.$$

From the proof of Theorem 3.1.23 we have

$$D_\eta^\beta \sigma a_{\mathrm{edge}}(\eta) \tilde{\sigma} \in S^{\mu-|\beta|}\big(\mathbb{R}^q; \mathcal{K}^{s,\gamma;g}(X^\wedge), \mathcal{K}^{s-\mu+|\beta|,\gamma-\mu;g}(X^\wedge)\big)$$

for every $\beta \in \mathbb{N}^q$. In particular, these operator functions are smooth in $\eta$ and the derivatives improve the smoothness in the image by $|\beta|$. This gives us the desired property of $\sigma \mathrm{h}^{-\gamma+\mu} a_{\mathrm{edge}}(\eta) \mathrm{h}^\gamma \tilde{\sigma}$. The $C^\infty$ dependence of $m(\eta) + g(\eta)$ in $\eta$ is clear (those are operator-valued symbols), and they map to $\mathcal{K}^{\infty,\gamma-\mu;g}(X^\wedge)$ anyway. Therefore, the desired property of $\sigma \mathrm{h}^{-\gamma+\mu}\{m(\eta) + g(\eta)\} \mathrm{h}^\gamma \tilde{\sigma}$ is satisfied as well.
(ii) This property essentially corresponds to the fact that the product in consideration close to the conical point is a symbol in $\eta$ of order zero and that the group action in $\mathcal{K}^{0,0}(X^\wedge)$-spaces is unitary. Far from the conical point the boundedness is as in Example 3.1.5.
(iii) The proof of this property close to the conical point is of a similar structure as Proposition 3.1.9, since our operators are based on operator-valued symbols referring to spaces with group action. The contribution outside the conical point is as in Example 3.1.5. □

**Remark 3.1.32.** *For $E^s := H^{s,0}(M)$, $s \in \mathbb{R}$, $\mathcal{E} = (E^s)_{s\in\mathbb{R}}$, the operator functions $b^\mu(\eta)$ of the form* (3.1.60) *belong to $S^\mu(\mathbb{R}^q; \mathcal{E}, \mathcal{E})$* (*see the notation after Definition* 3.1.10).

## 3.2 Operators referring to a corner point

### 3.2.1 Weighted spaces

Let $\mathcal{E} = (E^s)_{s\in\mathbb{R}} \in \mathfrak{E}$ be a scale of Hilbert spaces with the compact embedding property and $(b^\mu(\rho))_{\mu\in\mathbb{R}}$, $\rho \in \mathbb{R}$, be an order reducing family (see Definition 3.1.3 with $q = 1$). We define a new scale of spaces adapted to the Mellin transform (1.1.24) and the approach of the cone calculus. In the following definition the Mellin transform refers to the variable $r \in \mathbb{R}_+$, i.e., $\mathcal{M} = \mathcal{M}_{r\to w}$.

**Definition 3.2.1.** *For every $s, \gamma \in \mathbb{R}$ we define the space $\mathcal{H}^{s,\gamma}(\mathbb{R}_+, \mathcal{E})$ to be the completion of $C_0^\infty(\mathbb{R}_+, E^\infty)$ with respect to the norm*

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+,\mathcal{E})} = \left\{\frac{1}{2\pi i}\int_{\Gamma_{\frac{d+1}{2}-\gamma}} \|b^s(\operatorname{Im} w)\mathcal{M}u(w)\|^2_{E^0} dw\right\}^{\frac{1}{2}} \tag{3.2.1}$$

*for a $d = d_{\mathcal{E}} \in \mathbb{N}$. The Mellin transform $\mathcal{M}$ in* (3.2.1) *is interpreted as the weighted Mellin transform $\mathcal{M}_{\gamma-\frac{d}{2}}$.*

The role of $d_{\mathcal{E}}$ is an extra information, given together with the scale $\mathcal{E}$. In the example $\mathcal{E} = (H^s(X))_{s\in\mathbb{R}}$ for a closed compact $C^\infty$ manifold $X$ we have $d_{\mathcal{E}} := \dim X$.

Observe that when we replace the order reducing family in (3.2.1) by an equivalent one the resulting norm is equivalent to (3.2.1).

By virtue of the identity

$$r^\beta\mathcal{H}^{s,\gamma}(\mathbb{R}_+,\mathcal{E}) = \mathcal{H}^{s,\gamma+\beta}(\mathbb{R}_+,\mathcal{E})$$

for every $s, \gamma, \beta \in \mathbb{R}$, it is often enough to refer the considerations to one particular weight, or to set

$$d_{\mathcal{E}} = 0. \tag{3.2.2}$$

For simplicity, if nothing else is said, from now on we assume (3.2.2).

**Proposition 3.2.2.** *Let $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ be a cut-off function. Then the multiplication by $\omega$ induces continuous operator*

$$\mathcal{M}_\omega : \mathcal{H}^{s,\gamma}(\mathbb{R}_+,\mathcal{E}) \to \mathcal{H}^{s,\gamma}(\mathbb{R}_+,\mathcal{E})$$

*for every $s, \gamma \in \mathbb{R}$. Moreover, $\omega \to \mathcal{M}_\omega$ induces a continuous operator*

$$C_0^\infty(\overline{\mathbb{R}}_+) \to \mathcal{L}\big(\mathcal{H}^{s,\gamma}(\mathbb{R}_+,\mathcal{E})\big).$$

Let us consider Definition 3.1.10 for the case $U = \mathbb{R}$, $q = 1$, and denote the covariable now by $\rho \in \mathbb{R}$. Set

$$S^\mu(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \mathbb{R}; \mathcal{E}, \widetilde{\mathcal{E}}) := S^\mu(\mathbb{R}\times\mathbb{R}\times\mathbb{R}; \mathcal{E}, \widetilde{\mathcal{E}})|_{\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+\times\mathbb{R}}$$

and

$$S^\mu(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Gamma_\delta; \mathcal{E}, \widetilde{\mathcal{E}}) :=$$
$$\{a(r, r', w) \in C^\infty\big(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Gamma_\delta, \mathcal{L}^\mu(\mathcal{E}, \widetilde{\mathcal{E}})\big) : a(r, r', \delta + i\rho) \in S^\mu(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \mathbb{R}_\rho; \mathcal{E}, \widetilde{\mathcal{E}})\}$$

for any $\delta \in \mathbb{R}$. The subspaces of $r'$-independent ($(r, r')$-independent) symbols are denoted by $S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{R}; \mathcal{E}, \widetilde{\mathcal{E}})$ $(S^\mu(\mathbb{R}; \mathcal{E}, \widetilde{\mathcal{E}}'))$ and $S^\mu(\overline{\mathbb{R}}_+ \times \Gamma_\delta; \mathcal{E}, \widetilde{\mathcal{E}})$ $(S^\mu(\Gamma_\delta; \mathcal{E}, \widetilde{\mathcal{E}}))$, respectively.

Given an element $f(r, r', w) \in S^\mu(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \widetilde{\mathcal{E}})$ we set

$$\mathrm{op}_M^\gamma(f)u(r) = \frac{1}{2\pi} \int \int_0^\infty (\frac{r}{r'})^{-(\frac{1}{2}-\gamma+i\rho)} f(r, r', \frac{1}{2} - \gamma + i\rho) u(r') \frac{dr'}{r'} d\rho. \tag{3.2.3}$$

Let, for instance, $f$ be independent of $r'$. Then (3.2.3) induces a continuous operator

$$\mathrm{op}_M^\gamma(f) : C_0^\infty(\mathbb{R}_+, E^s) \to C^\infty(\mathbb{R}_+, \widetilde{E}^{s-\mu}). \tag{3.2.4}$$

In fact, we have $\mathrm{op}_M^\gamma(f) = \mathcal{M}^{-1}_{\gamma, w\to r} f(r, w) \mathcal{M}_{\gamma, r'\to w}$. The weighted Mellin transform $\mathcal{M}_\gamma$ induces a continuous operator

$$\mathcal{M}_\gamma : C_0^\infty(\mathbb{R}_+, E^s) \to \mathcal{S}(\Gamma_{\frac{1}{2}-\gamma}, E^s)$$

for every $s \in \mathbb{R}$. The subsequent multiplication of $\mathcal{M}_\gamma u(w)$ by $f(r, w)$ gives rise to an element in $C^\infty\big(\mathbb{R}_+, \mathcal{S}(\Gamma_{\frac{1}{2}-\gamma}, \widetilde{E}^{s-\mu})\big)$, and then it follows easily that $\mathrm{op}_M^\gamma(f)u \in C^\infty\big(\mathbb{R}_+, \widetilde{E}^{s-\mu}\big)$. We now formulate a continuity result, first for the case of symbols with constant coefficients.

**Theorem 3.2.3.** *For every $f(w) \in S^\mu(\Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \widetilde{\mathcal{E}})$ the operator* (3.2.4) *extends to a continuous operator*

$$\mathrm{op}_M^\gamma(f) : \mathcal{H}^{s,\gamma}(\mathbb{R}_+, \mathcal{E}) \to \mathcal{H}^{s-\mu,\gamma}(\mathbb{R}_+, \widetilde{\mathcal{E}}) \tag{3.2.5}$$

*for every $s \in \mathbb{R}$. Moreover, $f \to \mathrm{op}_M^\gamma(f)$ induces a continuous operator*

$$S^\mu(\Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \widetilde{\mathcal{E}}) \to \mathcal{L}\big(\mathcal{H}^{s,\gamma}(\mathbb{R}_+, \mathcal{E}), \mathcal{H}^{s-\mu,\gamma}(\mathbb{R}_+, \widetilde{\mathcal{E}})\big) \tag{3.2.6}$$

*for every $s \in \mathbb{R}$.*

*Proof.* We have

$$\begin{aligned}
&\|\mathrm{op}_M^\gamma(f)u\|^2_{\mathcal{H}^{s-\mu,\gamma}(\mathbb{R}_+,\widetilde{\mathcal{E}})} \\
&\qquad = \int_{\mathbb{R}} \|\tilde{b}^{s-\mu}(\rho) \mathcal{M}_\gamma(\mathcal{M}_\gamma^{-1} f(\frac{1}{2} - \gamma + i\rho))(\mathcal{M}_\gamma u)(\frac{1}{2} - \gamma + i\rho)\|^2_{\widetilde{E}^0} d\rho \\
&\qquad = \int_{\mathbb{R}} \|\tilde{b}^{s-\mu}(\rho) f(\frac{1}{2} - \gamma + i\rho) b^{-s}(\rho) b^s(\rho)(\mathcal{M}_\gamma u)(\frac{1}{2} - \gamma + i\rho)\|^2_{\widetilde{E}^0} d\rho \\
&\qquad \leq \ c^2 \|u\|^2_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+,\mathcal{E})}
\end{aligned}$$

with

$$c = \sup_{\rho\in\mathbb{R}} \|\tilde{b}^{s-\mu}(\rho) f(\frac{1}{2} - \gamma + i\rho) b^{-s}(\rho)\|_{\mathcal{L}(E^0,\widetilde{E}^0)},$$

which is finite for every $s \in \mathbb{R}$ (cf. the estimates (3.1.13)). Thus we have proved the continuity of both (3.2.5) and (3.2.6). □

In order to generalise Theorem 3.2.3 to symbols with variable coefficients we impose conditions of reasonable generality that allow us to reduce the arguments to a vector-valued analogue of Kumano-go's technique.

Given a Fréchet space $V$ with a countable system of semi-norms $(\pi_\iota)_{\iota\in\mathbb{N}}$ that defines its topology, we denote by $C_B^\infty(\mathbb{R}_+ \times \mathbb{R}_+, V)$ the set of all $u(r,r') \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, V)$ such that

$$\sup_{r,r'\in\mathbb{R}_+} \pi_\iota\Big((r\partial_r)^k (r'\partial_{r'})^{k'} u(r,r')\Big) < \infty$$

for all $k, k' \in \mathbb{N}$. In a similar manner by $C_B^\infty(\mathbb{R}_+, V)$ we denote the set of such functions that are independent of $r'$.

Moreover, we set

$$S_B^\mu(\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \widetilde{\mathcal{E}}) := C_B^\infty\big(\mathbb{R}_+ \times \mathbb{R}_+, S^\mu(\Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \widetilde{\mathcal{E}})\big)$$

and, similarly, $S_B^\mu(\mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \widetilde{\mathcal{E}}) := C_B^\infty\big(\mathbb{R}_+, S^\mu(\Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \widetilde{\mathcal{E}})\big)$.

**Theorem 3.2.4.** *For every $f(r,w) \in S_B^\mu(\mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \widetilde{\mathcal{E}})$ the operator $\mathrm{op}_M^\gamma(f)$ induces a continuous mapping*

$$\mathrm{op}_M^\gamma(f) : \mathcal{H}^{s,\gamma}(\mathbb{R}_+, \mathcal{E}) \to \mathcal{H}^{s-\mu,\gamma}(\mathbb{R}_+, \widetilde{\mathcal{E}}),$$

*and $f \to \mathrm{op}_M^\gamma(f)$ a continuous operator*

$$S_B^\mu(\mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma}; \mathcal{E}, \widetilde{\mathcal{E}}) \to \mathcal{L}\big(\mathcal{H}^{s,\gamma}(\mathbb{R}_+, \mathcal{E}), \mathcal{H}^{s-\mu,\gamma}(\mathbb{R}_+, \widetilde{\mathcal{E}})\big)$$

*for every $s \in \mathbb{R}$.*

Parallel to the spaces of Definition 3.2.1 it also makes sense to consider their "cylindrical" analogues, defined as follows.

**Definition 3.2.5.** *Let $(b^s(\eta))_{s\in\mathbb{R}}$, be an order reducing family as in Definition* 3.1.3. *For every $s \in \mathbb{R}$ we define the space $H^s(\mathbb{R}^q, \mathcal{E})$ to be the completion of $C_0^\infty(\mathbb{R}^q, E^\infty)$ with respect to the norm*

$$\|u\|_{H^s(\mathbb{R}^q,\mathcal{E})} := \left\{ \int_{\mathbb{R}^q} \|b^s(\eta)\mathcal{F}u(\eta)\|_{E^0}^2 d\eta \right\}^{\frac{1}{2}}.$$

Clearly, similarly as above, with a symbol $a(y,y',\eta) \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$ (when we impose a suitable control with respect to the dependence on $y'$ for large $|y'|$) we can associate a pseudo-differential operator

$$\mathrm{Op}_y(a)u(y) = \iint e^{i(y-y')\eta} a(y,y',\eta) u(y') dy' đ\eta.$$

In particular, if $a = a(\eta)$ has constant coefficients, then we obtain a continuous operator

$$\mathrm{Op}_y(a) : H^s(\mathbb{R}^q, \mathcal{E}) \to H^{s-\mu}(\mathbb{R}^q, \widetilde{\mathcal{E}}) \tag{3.2.7}$$

for every $s \in \mathbb{R}$. In the case of variable coefficients we need some precautions on the nature of symbols. This will be postponed for the moment.

We are mainly interested in the case $q = 1$. Consider the transformation

$$(S_\gamma u)(y) := e^{-(\frac{1}{2}-\gamma)y} u(e^{-y})$$

from functions in $r \in \mathbb{R}_+$ to functions in $y \in \mathbb{R}$. We then have the identity

$$(\mathcal{M}_\gamma u)(\frac{1}{2} - \gamma + i\rho) = (\mathcal{F} S_\gamma u)(\rho)$$

with $\mathcal{F}$ being the one-dimensional Fourier transform. This gives us

$$\left\{ \frac{1}{2\pi} \int_{\mathbb{R}} \|b^s(\eta)(\mathcal{F}S_\gamma u)(\eta)\|^2_{E^0} d\eta \right\}^{\frac{1}{2}} = \|S_\gamma u\|_{H^s(\mathbb{R},\mathcal{E})} = \|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+,\mathcal{E})},$$

i.e., $S_\gamma$ induces an isomorphism

$$S_\gamma : \mathcal{H}^{s,\gamma}(\mathbb{R}_+, \mathcal{E}) \to H^s(\mathbb{R}, \mathcal{E}). \tag{3.2.8}$$

**Remark 3.2.6.** *By reformulating the expression* (3.2.3) *we obtain*

$$\mathrm{op}_M^\gamma(f)u(r) = \frac{1}{2\pi} \iint e^{(\frac{1}{2}-\gamma+i\rho)(\log r' - \log r)} f(r, r', \frac{1}{2} - \gamma + i\rho) u(r') \frac{dr'}{r'} d\rho.$$

*Substituting* $r = e^{-y}$, $r' = e^{-y'}$ *gives us*

$$\begin{aligned} \mathrm{op}_M^\gamma(f)u(r) = \frac{1}{2\pi} \iint e^{i(y-y')\rho} e^{(\frac{1}{2}-\gamma)(y-y')} f(e^{-y}, e^{-y'}, \frac{1}{2} - \gamma + i\rho) u(e^{-y'}) dy' d\rho \\ = \mathrm{Op}_y(g_\gamma)v(y) \end{aligned}$$

*with* $v(y) := u(e^{-y})$ *and* $g_\gamma(y, y', \rho) := e^{(\frac{1}{2}-\gamma)(y-y')} f(e^{-y}, e^{-y'}, \frac{1}{2} - \gamma + i\rho)$.

*In other words, if* $\chi : \mathbb{R}_+ \to \mathbb{R}$ *is defined by* $\chi(r) = -\log r =: y$, *we have* $(\chi^* v)(r) = v(-\log r)$ *or* $((\chi^{-1})^* u)(y) = u(e^{-y})$ *and*

$$\mathrm{op}_M^\gamma(f) = \chi^* \mathrm{Op}_y(g_\gamma)(\chi^{-1})^*.$$

*Thus* $\mathrm{Op}_y(g_\gamma)$ *is the operator push forward of* $\mathrm{op}_M^\gamma(f)$ *under* $\chi$.

### 3.2.2 Mellin quantisation and kernel cut-off

The axiomatic cone calculus that we develop here is a substructure of the general calculus of operators with symbols in $a(r, \rho) \in S^\mu(\mathbb{R}_+ \times \mathbb{R}; \mathcal{E}, \widetilde{\mathcal{E}})$ of the form $a(r, \rho) = \tilde{a}(r, r\rho)$, $\tilde{a}(r, \tilde{\rho}) \in S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{R}_{\tilde{\rho}}; \mathcal{E}, \widetilde{\mathcal{E}})$ (up to a weight factor and modulo smoothing operators) with a special control near $r = 0$ via Mellin quantisation. By

$L^{-\infty}(\mathbb{R}_+;\mathcal{E},\widetilde{\mathcal{E}};\mathbb{R}^q)$ we denote the space of all Schwartz functions in $\eta\in\mathbb{R}^q$ with values in operators

$$C_0^\infty(\mathbb{R}_+,E^{-\infty})\to C^\infty(\mathbb{R}_+,\widetilde{E}^\infty).$$

We then define

$$L^\mu(\mathbb{R}_+;\mathcal{E},\widetilde{\mathcal{E}};\mathbb{R}^q)=$$
$$\{\mathrm{Op}_r(a)(\eta)+C(\eta):a(r,\rho,\eta)\in S^\mu(\mathbb{R}_+\times\mathbb{R}^{1+q}_{\rho,\eta};\mathcal{E},\widetilde{\mathcal{E}}),C(\eta)\in L^{-\infty}(\mathbb{R}_+;\mathcal{E},\widetilde{\mathcal{E}};\mathbb{R}^q)\}.$$

Our next objective is to formulate a Mellin quantisation result of symbols

$$a(r,\rho,\eta)=\tilde{a}(r,r\rho,r\eta),\quad \tilde{a}(r,\tilde{\rho},\tilde{\eta})\in S^\mu(\overline{\mathbb{R}}_+\times\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}};\mathcal{E},\widetilde{\mathcal{E}}).\tag{3.2.9}$$

**Definition 3.2.7.** *By $M^\mu_{\mathcal{O}}(\mathcal{E},\widetilde{\mathcal{E}};\mathbb{R}^q_{\tilde{\eta}})$ we denote the set of all $h(z,\tilde{\eta})\in\mathcal{A}\big(\mathbb{C},S^\mu(\mathbb{R}^q_{\tilde{\eta}};\mathcal{E},\widetilde{\mathcal{E}})\big)$ such that*

$$h(\beta+i\rho,\tilde{\eta})\in S^\mu(\mathbb{R}_\rho\times\mathbb{R}^q_{\tilde{\eta}};\mathcal{E},\widetilde{\mathcal{E}})$$

*for every $\beta\in\mathbb{R}$, uniformly in compact $\beta$-intervals. For $q=0$ we simply write $M^\mu_{\mathcal{O}}(\mathcal{E},\widetilde{\mathcal{E}})$.*

The space $M^\mu_{\mathcal{O}}(\mathcal{E},\widetilde{\mathcal{E}};\mathbb{R}^q_{\tilde{\eta}})$ is Fréchet with a natural semi-norm system, namely, the one induced by $\mathcal{A}\big(\mathbb{C}\times\mathbb{R}^q,\mathcal{L}^\mu(\mathcal{E},\widetilde{\mathcal{E}})\big)$ together with

$$\sup_{|\rho|\le k}\pi_{\iota,\rho}(h_{\Gamma_\rho\times\mathbb{R}^q}),\iota,k\in\mathbb{N},$$

where $(\pi_{\iota,\rho})_{\iota\in\mathbb{N}}$ denotes a countable semi-norm system for the Fréchet topology of $S^\mu(\Gamma_\rho\times\mathbb{R}^q;\mathcal{E},\widetilde{\mathcal{E}})$.

**Theorem 3.2.8 (Mellin quantisation).** *For every symbol $a(r,\rho,\eta)$ of the form (3.2.9) there exists an $\tilde{h}(r,z,\tilde{\eta})\in C^\infty\big(\overline{\mathbb{R}}_+,M^\mu_{\mathcal{O}}(\mathcal{E},\widetilde{\mathcal{E}};\mathbb{R}^q)\big)$ such that for $h(r,z,\eta):=\tilde{h}(r,z,r\eta)$ and every $\delta\in\mathbb{R}$ we have*

$$\mathrm{op}^\delta_M(h)(\eta)=\mathrm{Op}_r(a)(\eta)$$

*modulo operators in $L^{-\infty}(\mathbb{R}_+;\mathcal{E},\widetilde{\mathcal{E}};\mathbb{R}^q)$.*

This result in the context of operator-valued symbols based on order reductions is mentioned here for completeness. It extends a corresponding result of the edge symbolic calculus, see [6, Theorem 3.2]. More information in that case is given in [18, Chapter 4]. Here we adapt some part of this approach to realise the kernel cut-off principle that allows us to recognise how many parameter-dependent meromorphic Mellin symbols exist.

**Definition 3.2.9.** *Let $S^\mu(\boldsymbol{C}\times\mathbb{R}^q;\mathcal{E},\widetilde{\mathcal{E}})$ denote the space of all operator functions $h(\zeta,\eta)\in\mathcal{A}\big(\mathbb{C},S^\mu(\mathbb{R}^q_\eta;\mathcal{E},\widetilde{\mathcal{E}})\big)$ such that*

$$h(\rho+i\delta,\eta)\in S^\mu(\mathbb{R}^{1+q}_{\rho,\eta};\mathcal{E},\widetilde{\mathcal{E}})$$

*for every $\delta\in\mathbb{R}$, uniformly in compact $\delta$-intervals.*

Clearly the space $S^\mu(\boldsymbol{C} \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$ is a generalisation of $M^\mu_{\mathcal{O}}(\mathcal{E}, \widetilde{\mathcal{E}})$, however, with an interchanged role of real and imaginary part of the complex covariable. To produce elements of $S^\mu(\boldsymbol{C} \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$ we consider a version of the kernel cut-off operator

$$H_{\mathcal{F}} : C_0^\infty(\mathbb{R}) \times S^\mu(\mathbb{R}^{1+q}; \mathcal{E}, \widetilde{\mathcal{E}}) \to S^\mu(\boldsymbol{C} \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$$

transforming an arbitrary element $a(\rho, \eta) \in S^\mu(\mathbb{R}^{1+q}; \mathcal{E}, \widetilde{\mathcal{E}})$ into $\big(H_{\mathcal{F}}(\varphi)a\big)(\zeta, \eta) \in S^\mu(\boldsymbol{C} \times \mathbb{R}^q; \mathcal{E}, \widetilde{\mathcal{E}})$ for any $\varphi \in C_0^\infty(\mathbb{R})$ (cf. Section 1.1.3). It will be useful to admit $\varphi$ to belong to the space

$$C_{\mathrm{b}}^\infty(\mathbb{R}) := \{\varphi \in C^\infty(\mathbb{R}_\theta) : \sup_{\theta \in \mathbb{R}} |D_\theta^k \varphi(\theta)| < \infty \text{ for every } k \in \mathbb{N}\}.$$

We set

$$\big(H_{\mathcal{F}}(\varphi)a\big)(\rho, \eta) := \iint e^{-i\theta\tilde{\rho}} \varphi(\theta) a(\rho - \tilde{\rho}, \eta) d\theta đ\tilde{\rho}, \tag{3.2.10}$$

interpreted as an oscillatory integral. We now prove the following result:

**Theorem 3.2.10.** *The kernel cut-off operator $H_{\mathcal{F}} : (\varphi, a) \to H_{\mathcal{F}}(\varphi)a$ defines a bilinear and continuous mapping*

$$H_{\mathcal{F}} : C_{\mathrm{b}}^\infty(\mathbb{R}) \times S^\mu(\mathbb{R}^{1+q}; \mathcal{E}, \widetilde{\mathcal{E}}) \to S^\mu(\mathbb{R}^{1+q}; \mathcal{E}, \widetilde{\mathcal{E}}), \tag{3.2.11}$$

*and $\big(H_{\mathcal{F}}(\varphi)a\big)(\rho, \eta)$ admits an asymptotic expansion*

$$\big(H_{\mathcal{F}}(\varphi)a\big)(\rho, \eta) \sim \sum_{k=0}^\infty \frac{(-1)^k}{k!} D_\theta^k \varphi(0) \partial_\rho^k a(\rho, \eta). \tag{3.2.12}$$

*Proof.* First note that the mapping

$$C_{\mathrm{b}}^\infty(\mathbb{R}) \times S^\mu(\mathbb{R}^{1+q}_{\rho,\eta}; \mathcal{E}, \widetilde{\mathcal{E}}) \to C^\infty\big(\mathbb{R}^q_{\rho,\eta}, S^\mu_{\mathrm{b}}(\mathbb{R}_\theta \times \mathbb{R}_{\tilde{\rho}}; \mathcal{E}, \widetilde{\mathcal{E}})\big)$$
$$(\varphi, a) \to \varphi(\theta) a(\rho - \tilde{\rho}, \eta),$$

for $S^\mu_{\mathrm{b}}(\mathbb{R}_\theta \times \mathbb{R}_{\tilde{\rho}}; \mathcal{E}, \widetilde{\mathcal{E}}) := C_{\mathrm{b}}^\infty\big(\mathbb{R}_\theta, S^\mu(\mathbb{R}_{\tilde{\rho}}; \mathcal{E}, \widetilde{\mathcal{E}})\big)$ is bilinear and continuous. For the proof of the continuity of (3.2.11) it suffices to verify that $\big(H_{\mathcal{F}}(\varphi)a\big)(\rho, \eta) \in S^\mu(\mathbb{R}^{1+q}; \mathcal{E}, \widetilde{\mathcal{E}})$ and then to apply the closed graph theorem. By virtue of

$$D^\beta_{\rho,\eta}\big(H_{\mathcal{F}}(\varphi)a\big)(\rho, \eta) = \big(H_{\mathcal{F}}(\varphi)(D^\beta_{\rho,\eta} a)\big)(\rho, \eta)$$

for every $\beta \in \mathbb{N}^{1+q}$ we only have to check that for every $s \in \mathbb{R}$

$$\|\tilde{b}^{s-\mu}(\rho, \eta)\big(H_{\mathcal{F}}(\varphi)a\big)(\rho, \eta) b^{-s}(\rho, \eta)\|_{\mathcal{L}(E^0, \widetilde{E}^0)} \leq c \tag{3.2.13}$$

for all $(\rho, \eta) \in \mathbb{R}^{1+q}$, with a constant $c = c(s) > 0$. We regularise the oscillatory integral (3.2.10)

$$\big(H_{\mathcal{F}}(\varphi)a\big)(\rho, \eta) = \iint e^{-i\theta\tilde{\rho}} \langle\theta\rangle^{-2}\{(1 - \partial_\theta^2)^N \varphi(\theta)\} a_N(\rho, \tilde{\rho}, \eta) d\theta đ\tilde{\rho}$$

for

$$a_N(\rho, \tilde{\rho}, \eta) := (1 - \partial_{\tilde{\rho}}^2)\{\langle\tilde{\rho}\rangle^{-2N} a(\rho - \tilde{\rho}, \eta)\}. \tag{3.2.14}$$

The function (3.2.14) is a linear combination of terms

$$(\partial_{\tilde\rho}^j\langle\tilde\rho\rangle^{-2N})(\partial_\rho^k a)(\rho-\tilde\rho,\eta) \text{ for } 0\le j,k\le 2.$$

We have

$$\begin{aligned}
&\Big\|\tilde b^{s-\mu}(\rho,\eta)\Big\{\iint e^{-i\theta\tilde\rho}\langle\theta\rangle^{-2}(1-\partial_\theta^2)^N\varphi(\theta)\\
&\qquad\qquad(\partial_{\tilde\rho}^j\langle\tilde\rho\rangle^{-2N})(\partial_\rho^k a)(\rho-\tilde\rho,\eta)d\theta đ\tilde\rho\Big\}b^{-s}(\rho,\eta)\Big\|_{\mathcal{L}(E^0,\widetilde E^0)}\\
&=\Big\|\iint \tilde b^{s-\mu}(\rho,\eta)\tilde b^{-s+\mu}(\rho-\tilde\rho,\eta)\tilde b^{s-\mu}(\rho-\tilde\rho,\eta)\big\{e^{-i\theta\tilde\rho}\langle\theta\rangle^{-2}(1-\partial_\theta^2)^N\varphi(\theta)\\
&\quad(\partial_{\tilde\rho}^j\langle\tilde\rho\rangle^{-2N})(\partial_\rho^k a)(\rho-\tilde\rho,\eta)\big\}b^{-s}(\rho-\tilde\rho,\eta)b^{s}(\rho-\tilde\rho,\eta)b^{-s}(\rho,\eta)d\theta đ\tilde\rho\Big\|_{\mathcal{L}(E^0,\widetilde E^0)}\\
&\le c\int\|\tilde b^{s-\mu}(\rho,\eta)\tilde b^{-s+\mu}(\rho-\tilde\rho,\eta)\|_{\mathcal{L}(\widetilde E^0,\widetilde E^0)}\|\tilde b^{s-\mu}(\rho-\tilde\rho,\eta)(\partial_{\tilde\rho}^j\langle\tilde\rho\rangle^{-2N})\\
&\quad(\partial_\rho^k a)(\rho-\tilde\rho,\eta)b^{-s}(\rho-\tilde\rho,\eta)\|_{\mathcal{L}(E^0,\widetilde E^0)}\|b^s(\rho-\tilde\rho,\eta)b^{-s}(\rho,\eta)\|_{\mathcal{L}(E^0,E^0)}đ\tilde\rho.
\end{aligned}\tag{3.2.15}$$

For the norms under the integral we apply Taylor's formula

$$b^s(\rho-\tilde\rho,\eta)=\sum_{m=0}^M\frac{1}{m!}(\partial_\rho^m b^s)(\rho,\eta)(-\tilde\rho)^m+\frac{\tilde\rho}{M!}^{M+1}\int_0^1(1-t)^M(\partial_\rho^{M+1}b^s)(\rho-t\tilde\rho,\eta)dt.$$

This yields

$$\begin{aligned}
\|b^s(\rho-\tilde\rho,\eta)b^{-s}(\rho,\eta)\|_{\mathcal{L}(E^0,E^0)}&\le\sum_{m=0}^M\frac{1}{m!}\langle\tilde\rho\rangle^m\|(\partial_\rho^m b^s)(\rho,\eta)b^{-s}(\rho,\eta)\|_{\mathcal{L}(E^0,E^0)}\\
&\quad+\frac{\langle\tilde\rho\rangle^{M+1}}{M!}\int_0^1(1-t)^M\|(\partial_\rho^{M+1}b^s)(\rho-t\tilde\rho,\eta)b^{-s}(\rho,\eta)\|_{\mathcal{L}(E^0,E^0)}dt.
\end{aligned}$$

By virtue of (3.1.14), Proposition 3.1.16 and Proposition 3.1.14 we obtain

$$\|(\partial_\rho^m b^s)(\rho,\eta)b^{-s}(\rho,\eta)\|_{\mathcal{L}(E^0,E^0)}\le c\langle\rho,\eta\rangle^{-m}.$$

Moreover, using Definition 3.1.3 (iii), it follows that

$$\begin{aligned}
&\|(\partial_\rho^{M+1}b^s)(\rho-t\tilde\rho,\eta)b^{-s}(\rho,\eta)\|_{0,0}\\
&\qquad=\|(\partial_\rho^{M+1}b^s)(\rho-t\tilde\rho,\eta)b^{-s}(\rho-t\tilde\rho,\eta)b^s(\rho-t\tilde\rho,\eta)b^{-s}(\rho,\eta)\|_{0,0}\\
&\qquad\le c\|(\partial_\rho^{M+1}b^s)(\rho-t\tilde\rho,\eta)b^{-s}(\rho-t\tilde\rho,\eta)\|_{0,0}\|b^s(\rho-t\tilde\rho,\eta)\|_{s,0}\|b^{-s}(\rho,\eta)\|_{0,s}\\
&\qquad\le\langle\rho-t\tilde\rho,\eta\rangle^{-(M+1)+B_1(s)}\langle\rho,\eta\rangle^{B_2(s)}
\end{aligned}$$

with certain $B_i(s)$, $i=1,2$. Here we denoted by $\|\cdot\|_{s,l}$ the operator norm in $\mathcal{L}(E^s,E^l)$, $s,l\in\mathbb{R}$. We thus obtain

$$\|b^s(\rho-\tilde\rho,\eta)b^{-s}(\rho,\eta)\|_{\mathcal{L}(E^0,E^0)}\le c\langle\tilde\rho\rangle^{M+1}\langle\rho,\eta\rangle^{B_2(s)}\sup_{|t|\le1}\langle\rho-t\tilde\rho,\eta\rangle^{-(M+1)+B_1(s)}.$$

By Peetre's inequality for $L \geq 0$ we have $\sup_{|t|\leq 1}\langle \rho - t\tilde{\rho}, \eta\rangle^{-L} \leq c\langle\tilde{\rho}\rangle^{L}\langle\rho,\eta\rangle^{-L}$. Thus, choosing $M$ so large that

$$-(M+1) + B_1(s) \leq 0, \quad -(M+1) + B_1(s) + B_2(s) \leq 0,$$

it follows that

$$\begin{aligned}\|b^s(\rho-\tilde{\rho},\eta)b^{-s}(\rho,\eta)\|_{\mathcal{L}(E^0,E^0)} &\leq c\langle\tilde{\rho}\rangle^{M+1}\langle\tilde{\rho}\rangle^{M+1-B_1(s)}\langle\rho,\eta\rangle^{-(M+1)+B_1(s)+B_2(s)} \\ &\leq c\langle\tilde{\rho}\rangle^{A(s)} \end{aligned} \tag{3.2.16}$$

for $A(s) := 2(M+1) - B_2(s)$.

In a similar manner we can show that

$$\|\tilde{b}^{s-\mu}(\rho,\eta)\tilde{b}^{-s+\mu}(\rho-\tilde{\rho},\eta)\|_{\mathcal{L}(\widetilde{E}^0,\widetilde{E}^0)} \leq c\langle\tilde{\rho}\rangle^{\widetilde{A}(s)} \tag{3.2.17}$$

for some $\widetilde{A}(s) \in \mathbb{R}$. Applying (3.2.16) and (3.2.17) in the estimate (3.2.15) it follows that

$$\|\tilde{b}^{s-\mu}(\rho,\eta)\big(H_{\mathcal{F}}(\varphi)a\big)(\rho,\eta)b^{-s}(\rho,\eta)\|_{\mathcal{L}(E^0,\widetilde{E}^0)} \leq c\sum_{0\leq j\leq 2}\int|\partial_{\tilde{\rho}}^j\langle\tilde{\rho}\rangle^{-2N}|\langle\tilde{\rho}\rangle^{A(s)+\tilde{A}(s)}đ\tilde{\rho}. \tag{3.2.18}$$

Since $N \in \mathbb{N}$ can be chosen as large as we want, it follows that the right hand side of (3.2.18) is finite for an appropriate $N$. This completes the proof of (3.2.13). The relation (3.2.12) immediately follows by applying Taylor's formula on $\varphi$ at 0. □

**Theorem 3.2.11.** *The kernel cut-off operator* $H_{\mathcal{F}} : (\varphi, a) \to H_{\mathcal{F}}(\varphi)a$ *defines a bilinear and continuous mapping*

$$H_{\mathcal{F}} : C_0^\infty(\mathbb{R}) \times S^\mu(\mathbb{R}^{1+q};\mathcal{E},\widetilde{\mathcal{E}}) \to S^\mu(\boldsymbol{C}\times\mathbb{R}^q;\mathcal{E},\widetilde{\mathcal{E}}). \tag{3.2.19}$$

*Proof.* Writing

$$\big(H_{\mathcal{F}}(\varphi)a\big)(\rho,\eta) = \int e^{-i\theta\rho}\varphi(\theta)\left\{\int e^{i\theta\rho'}a(\rho',\eta)đ\rho'\right\}d\theta$$

we see that $\big(H_{\mathcal{F}}(\varphi)a\big)(\rho,\eta)$ is the Fourier transform of a distribution

$$\varphi(\theta)\int e^{i\theta\rho'}a(\rho',\eta)đ\rho' \in \mathcal{S}'\big(\mathbb{R}_\theta, \mathcal{L}^\mu(\mathcal{E},\widetilde{\mathcal{E}})\big)$$

with compact support. This extends to a holomorphic $\mathcal{L}^\mu(\mathcal{E},\widetilde{\mathcal{E}})$-valued function in $\zeta = \rho + i\delta$, given by

$$\big(H_{\mathcal{F}}(\varphi)a\big)(\rho+i\delta,\eta) = \big(H_{\mathcal{F}}(\varphi_\delta)a\big)(\rho,\eta)$$

for $\varphi_\delta(\theta) := e^{\theta\delta}\varphi(\theta)$. From Theorem 3.2.10 we obtain $\big(H_{\mathcal{F}}(\varphi)a\big)(\rho+i\delta,\eta) \in S^\mu(\mathbb{R}^{1+q};\mathcal{E},\widetilde{\mathcal{E}})$ for every $\delta\in\mathbb{R}$. By virtue of the continuity of $\delta \to \varphi_\delta$, $\mathbb{R}\to C_0^\infty(\mathbb{R})$ and of the continuity of (3.2.11) it follows that (3.2.19) induces a continuous mapping

$$H_{\mathcal{F}} : C_0^\infty(\mathbb{R}) \times S^\mu(\mathbb{R}^{1+q};\mathcal{E},\widetilde{\mathcal{E}}) \to S^\mu(I_\delta\times\mathbb{R}^q;\mathcal{E},\widetilde{\mathcal{E}}),$$

$I_\delta := \{\zeta\in\mathbb{C} : \mathrm{Im}\zeta = \delta\}$, which is uniform in compact $\delta$-intervals. The closed graph theorem gives us also the continuity of (3.2.19) with respect to the Fréchet topology of $S^\mu(\boldsymbol{C}\times\mathbb{R}^q;\mathcal{E},\widetilde{\mathcal{E}})$. □

### 3.2.3 Meromorphic Mellin symbols and operators with asymptotics

As an ingredient of our cone algebra we now study meromorphic Mellin symbols, starting from $M^{\mu}_{\mathcal{O}}(\mathcal{E},\widetilde{\mathcal{E}})$ (see Definition 3.2.7 for $q=0$).

**Theorem 3.2.12.** *$h \in M^{\mu}_{\mathcal{O}}(\mathcal{E},\widetilde{\mathcal{E}})$ and $h|_{\Gamma_\beta} \in S^{\mu-\varepsilon}(\Gamma_\beta;\mathcal{E},\widetilde{\mathcal{E}})$ for some $\varepsilon > 0$ entails $h \in M^{\mu-\varepsilon}_{\mathcal{O}}(\mathcal{E},\widetilde{\mathcal{E}})$.*

*Proof.* The ideas of the proof are similar to the case of the cone calculus with smooth base $X$ and the scales $\big(H^s(X)\big)_{s\in\mathbb{R}}$ (see, e.g., the thesis of Seiler [44]). Let us sketch it briefly.
Without loss of generality we assume $h|_{\Gamma_0} \in S^{\mu-\varepsilon}(\Gamma_0;\mathcal{E},\widetilde{\mathcal{E}})$. We apply Taylor's formula

$$h(\beta+i\rho) = \sum_{j=0}^{N-1} \frac{(D^j_\rho h_0)(\rho)}{j!}\beta^j + \frac{\beta^N}{(N-1)!}\int_0^1 (1-\theta)^{N-1}(\partial^N_w h)(\theta\beta+i\rho)d\theta,$$

$h_0(\rho) := h(w)|_{\Gamma_0}$. The terms in the first sum on the right hand side are continuous in $\beta$ with values in $S^{\mu-j-\varepsilon}(\mathbb{R}_\rho;\mathcal{E},\widetilde{\mathcal{E}})$. Since they are holomorphic in $w \in \mathbb{C}$ with values in $\mathcal{L}^{\mu}(\mathcal{E},\widetilde{\mathcal{E}})$, Cauchy's formula gives us elements in $M^{\mu-\varepsilon}_{\mathcal{O}}(\mathcal{E},\widetilde{\mathcal{E}})$ for all $j = 0,\dots,N-1$. Choosing $N > \varepsilon$ we obtain that $(\partial^N_w h)(\theta\beta+i\rho)$ is continuous in $\theta$ and $\beta$ with values in $S^{\mu-\varepsilon}(\mathbb{R}_\rho;\mathcal{E},\widetilde{\mathcal{E}})$. At the same time it is holomorphic in $\mathbb{C}$ with values in $\mathcal{L}^{\mu}(\mathcal{E},\widetilde{\mathcal{E}})$. Cauchy's integral formula then shows that the remainder term also belongs to $M^{\mu-\varepsilon}(\mathcal{E},\widetilde{\mathcal{E}})$. □

**Proposition 3.2.13.** *Let $h(w) \in M^{\mu}_{\mathcal{O}}(\mathcal{E}_0,\widetilde{\mathcal{E}})$, $f(w) \in M^{\nu}_{\mathcal{O}}(\mathcal{E},\mathcal{E}_0)$; then for pointwise composition we have $h(w)f(w) \in M^{\mu+\nu}_{\mathcal{O}}(\mathcal{E},\widetilde{\mathcal{E}})$.*

*Proof.* The proof is obvious. □

**Definition 3.2.14.** *An element $h(w) \in M^{\mu}_{\mathcal{O}}(\mathcal{E},\widetilde{\mathcal{E}})$ is called elliptic, if for some $\beta \in \mathbb{R}$ the operators $h(\beta+i\rho) : E^s \to \widetilde{E}^{s-\mu}$ are invertible for all $s \in \mathbb{R}$, $\rho \in \mathbb{R}$ and $h^{-1}(\beta+i\rho) \in S^{-\mu}(\mathbb{R}_\rho;\widetilde{\mathcal{E}},\mathcal{E})$.*

**Theorem 3.2.15.** *Let $h \in M^{\mu}_{\mathcal{O}}(\mathcal{E},\widetilde{\mathcal{E}})$ be elliptic. Then,*

$$h(w) : E^s \to \widetilde{E}^{s-\mu} \tag{3.2.20}$$

*is a holomorphic family of Fredholm operators of index zero for $s \in \mathbb{R}$. There is a set $D \subset \mathbb{C}$, with $D \cap \{c \le \operatorname{Re} w \le c'\}$ finite for every $c \le c'$, such that the operators* (3.2.20) *are invertible for all $w \in \mathbb{C}\setminus D$.*

*Proof.* By assumption we have $g := (h|_{\Gamma_\beta})^{-1} \in S^{-\mu}(\Gamma_\beta;\widetilde{\mathcal{E}},\mathcal{E})$. Applying a version of the kernel cut-off construction, now referring to the Mellin transform rather than the Fourier transform, cf. (1.1.6), with a function $\psi \in C^\infty_0(\mathbb{R}_+)$, $\psi \equiv 1$ near 1, we obtain a continuous operator

$$H_{\mathcal{M}}(\psi) : S^{-\mu}(\Gamma_\beta;\widetilde{\mathcal{E}},\mathcal{E}) \to M^{-\mu}_{\mathcal{O}}(\widetilde{\mathcal{E}},\mathcal{E})$$

where $H_{\mathcal{M}}(\psi)g|_{\Gamma_\beta} = g \mod S^{-\infty}(\Gamma_\beta;\widetilde{\mathcal{E}},\mathcal{E})$. Setting $h^{(-1)}(w) := H_{\mathcal{M}}(\psi)g$ we obtain $h^{(-1)}(w) \in M_{\mathcal{O}}^{-\mu}(\widetilde{\mathcal{E}},\mathcal{E})$, and from Proposition 3.2.13 it follows that

$$h(w)h^{(-1)}(w) \in M_{\mathcal{O}}^{0}(\widetilde{\mathcal{E}},\widetilde{\mathcal{E}}), \quad h^{(-1)}(w)h(w) \in M_{\mathcal{O}}^{0}(\mathcal{E},\mathcal{E})$$

and

$$h(w)h^{(-1)}(w)|_{\Gamma_\beta} - 1 \in S^{-\infty}(\Gamma_\beta;\widetilde{\mathcal{E}},\widetilde{\mathcal{E}}), \quad h^{(-1)}(w)h(w)|_{\Gamma_\beta} - 1 \in S^{-\infty}(\Gamma_\beta;\mathcal{E},\mathcal{E}), \tag{3.2.21}$$

for every $\beta \in \mathbb{R}$, and hence

$$h(w)h^{(-1)}(w) = 1 + m(w), \quad h^{(-1)}(w)h(w) = 1 + l(w) \tag{3.2.22}$$

for certain $m(w) \in M_{\mathcal{O}}^{-\infty}(\widetilde{\mathcal{E}},\widetilde{\mathcal{E}})$, $l(w) \in M_{\mathcal{O}}^{-\infty}(\mathcal{E},\mathcal{E})$. For every $s \in \mathbb{R}$ and every fixed $w \in \mathbb{C}$ the operators

$$m(w) : \widetilde{E}^s \to \widetilde{E}^\infty, \ l(w) : E^s \to E^\infty$$

are continuous. Therefore, since the scales have the compact embedding property, from (3.2.22) we obtain that $h^{(-1)}(w)$ is a two-sided parametrix of $h(w)$ for every $w$, i.e., the operators (3.2.20) are Fredholm. Since $h(w) \in \mathcal{A}\big(\mathbb{C},\mathcal{L}^\mu(E^s,E^{s-\mu})\big)$ is continuous in $w \in \mathbb{C}$ we have $\operatorname{ind} h(w_1) = \operatorname{ind} h(w_2)$ for every $w_1, w_2 \in \mathbb{C}$. However, since $h$ is invertible on the line $\Gamma_\beta$ it follows that $\operatorname{ind} h(w) = 0$ for all $w \in \mathbb{C}$. Finally, from the relations (3.2.22) we see that for every $c \le c'$ there is an $L(c,c') > 0$ such that the operators (3.2.20) are invertible for all $w \in \mathbb{C}$ with $|\operatorname{Im} w| \ge L(c,c')$, $c \le \operatorname{Re} w \le c'$. Then a general result on holomorphic Fredholm families gives us that the strip $c \le \operatorname{Re} w \le c'$ contains at most finitely many points where (3.2.20) is not invertible. Those points just constitute the set $D$, it is also independent of $s$, since $\ker h(w)$ is independent of $s$ as we easily see from (3.2.22) and the smoothing remainders; then vanishing of the index shows that the invertibility holds exactly when $\ker h(w) = \{0\}$. ☐

**Theorem 3.2.16.** *The ellipticity of $h$ with respect to $\Gamma_\beta$ as in Definition* 3.2.14 *entails the ellipticity with respect to $\Gamma_\delta$ for all $\delta \in \mathbb{R}$ satisfying $\Gamma_\delta \cap D = \emptyset$. In other words Definition* 3.2.14 *is independent of the choice of $\beta$.*

Recall here that $D$ is a discrete set in the complex plane which consists of those points $w$ for which $h(w) = 0$.

*Proof.* Let us apply the kernel cut-off operator $H_{\mathcal{M}}(\psi_\varepsilon)$, where $\psi_\varepsilon \in C_0^\infty(\mathbb{R}_+)$ is of the form $\psi_\varepsilon(t) = \psi(\varepsilon t)$, $\varepsilon > 0$, for some $\psi \in C_0^\infty(\mathbb{R}_+)$. Then, setting

$$H_{\mathcal{M}}(\psi_\varepsilon)\big(h^{-1}(\beta + i\rho)\big) =: f_\varepsilon \in M_{\mathcal{O}}^{-\mu}(\widetilde{\mathcal{E}},\mathcal{E})$$

we obtain $f_\varepsilon|_{\Gamma_\beta} \in S^{-\mu}(\Gamma_\beta;\widetilde{\mathcal{E}},\mathcal{E})$ and $f_\varepsilon|_{\Gamma_\beta} \to h^{-1}(\beta + i\rho)$ as $\varepsilon \to 0$ in the topology of $S^{-\mu}(\Gamma_\beta;\widetilde{\mathcal{E}},\mathcal{E})$. This shows us that $f_{\varepsilon_1}|_{\Gamma_\beta}$ is pointwise invertible for $\varepsilon_1 > 0$ sufficiently small. Let us set $h^{(-1)}(w) = f_{\varepsilon_1}(w)$. According to Proposition 3.2.13 we have $g(w) := h^{(-1)}(w)h(w) \in M_{\mathcal{O}}^{0}(\mathcal{E},\mathcal{E})$ and by construction

$$g|_{\Gamma_\beta} = 1 + l \text{ for some } l \in S^{-\infty}(\Gamma_\beta;\mathcal{E},\mathcal{E}).$$

Then Theorem 3.2.12 yields $g = 1 \mod M_{\mathcal{O}}^{-\infty}(\mathcal{E}, \mathcal{E})$. It follows that

$$h^{(-1)}|_{\Gamma_\delta} h|_{\Gamma_\delta} = 1 + l_\delta \text{ for some } l_\delta \in S^{-\infty}(\Gamma_\delta; \mathcal{E}, \mathcal{E})$$

and hence

$$(1 + l_\delta)^{-1} h^{(-1)}|_{\Gamma_\delta} h|_{\Gamma_\delta} = 1.$$

From Proposition 3.1.13 we know that $l_\delta \in \mathcal{S}\big(\Gamma_\delta, \mathcal{L}^{-\infty}(\mathcal{E}, \mathcal{E})\big)$ and it is also clear that $(1 + l_\delta)^{-1} = 1 + m_\delta$ for some $m_\delta \in \mathcal{S}\big(\Gamma_\delta, \mathcal{L}^{-\infty}(\mathcal{E}, \mathcal{E})\big)$. Then Proposition 3.1.16 (iii) shows that $(h|_{\Gamma_\delta})^{-1} = (1 + m_\delta) h^{(-1)}|_{\Gamma_\delta} \in S^{-\mu}(\Gamma_\delta; \widetilde{\mathcal{E}}, \mathcal{E})$. □

Add a word about asymptotics!
A sequence

$$R = \{(p_j, m_j, L_j)\}_{j \in \mathbb{Z}}$$

is called a discrete asymptotic type of Mellin symbols if $p_j \in \mathbb{C}$, $m_j \in \mathbb{N}$, and $L_j \subset \mathcal{L}^{-\infty}(\mathcal{E}, \widetilde{\mathcal{E}})$ is a finite dimensional subspace of finite rank operators; moreover, $\pi_{\mathbb{C}} R := \{p_j, j \in \mathbb{Z}\}$ is assumed to intersect the strips $\{w \in \mathbb{C} : c_1 \leq \operatorname{Re} w \leq c_2\}$ in a finite set, for every $c_1 \leq c_2$. Let $M_R^{-\infty}(\mathcal{E}, \widetilde{\mathcal{E}})$ denote the space of all functions $m \in \mathcal{A}\big(\mathbb{C} \setminus \pi_{\mathbb{C}} R, \mathcal{L}^{-\infty}(\mathcal{E}, \widetilde{\mathcal{E}})\big)$ which are meromorphic with poles at the points $p_j$ of multiplicity $m_j + 1$ and Laurent coefficients at $(w - p_j)^{-(k+1)}$ belonging to $L_j$ for $0 \leq k \leq m_j$, and $\chi(w) m(w)|_{\Gamma_\delta} \in \mathcal{S}(\Gamma_\delta; \mathcal{E}, \widetilde{\mathcal{E}})$ for every $\delta \in \mathbb{R}$, uniformly in compact $\delta$-intervals, where $\chi$ is any $\pi_{\mathbb{C}} R$-excision function. Moreover, we set

$$M_R^\mu(\mathcal{E}, \widetilde{\mathcal{E}}) := M_{\mathcal{O}}^\mu(\mathcal{E}, \widetilde{\mathcal{E}}) + M_R^{-\infty}(\mathcal{E}, \widetilde{\mathcal{E}}). \tag{3.2.23}$$

**Theorem 3.2.17.** *Let $h \in M_R^\mu(\mathcal{E}_0, \widetilde{\mathcal{E}})$, $f \in M_S^\nu(\mathcal{E}, \mathcal{E}_0)$ with asymptotic types $R, S$ and orders $\mu, \nu \in \mathbb{R}$, then we have $hf \in M_P^{\mu+\nu}(\mathcal{E}, \widetilde{\mathcal{E}})$ with some resulting asymptotic type $P$.*

*Proof.* The proof of this result is analogous to the one in the "concrete" cone calculus, see [36]. □

**Proposition 3.2.18.** *For every $m \in M_R^{-\infty}(\mathcal{E}, \mathcal{E})$ there exists an $m^{(-1)} \in M_S^{-\infty}(\mathcal{E}, \mathcal{E})$ with another asymptotic type $S$ such that*

$$\big(1 + m(w)\big)\big(1 + m^{(-1)}(w)\big) = 1.$$

For the proof we employ the following Lemma.

**Lemma 3.2.19.** *Let $E$ be a Banach space, $U \subseteq \mathbb{C}$ open, $0 \in U$, and let $h \in \mathcal{A}\big(U, \mathcal{L}(E)\big)$ be an element such that $h(w) = 0$ on a closed subspace $F \subseteq E$ of finite codimension. Moreover, let $a_1, \ldots, a_N \in \mathcal{L}(E)$ be operators of finite rank, for some $N \in \mathbb{N} \setminus \{0\}$. Then there is a $\delta > 0$ such that the meromorphic $\mathcal{L}(E)$-valued function*

$$f(w) = 1 + h(w) + \sum_{j=1}^{N} a_j w^{-j}$$

*is invertible for all $w \in V_\delta := \{w \in \mathbb{C} : 0 < |w| < \delta\}$. Moreover, $f^{-1}(w) = 1 + \tilde{h}(w) + \sum_{j=1}^{\tilde{N}} \tilde{a}_j w^{-j}$ for $w \in V_\delta$ with $\tilde{h} \in \mathcal{A}\big(V_\delta \cup \{0\}, \mathcal{L}(E)\big)$ and finite rank operators $\tilde{a}_j$.*

*Proof of Proposition* 3.2.18. First observe that if $m \in \mathcal{L}^{-\infty}(\mathcal{E},\mathcal{E})$ is an operator such that

$$1+m : E^s \to E^s$$

is invertible for all $s \in \mathbb{R}$, we can define an operator $g \in \mathcal{L}^0(\mathcal{E},\mathcal{E})$ such that $(1+m)(1+g) = 1$. This gives us $1+m+g+mg = 1$, and $m, mg \in \mathcal{L}^{-\infty}(\mathcal{E},\mathcal{E})$ implies $g = -m(1+g) \in \mathcal{L}^{-\infty}(\mathcal{E},\mathcal{E})$.

Moreover, our operator function $1+m$ is holomorphic in $\mathbb{C} \setminus \pi_{\mathbb{C}}R$. Then $g = (1+m)^{-1} - 1$ is holomorphic in $\mathbb{C}\setminus D$ with values in $\mathcal{L}^{-\infty}(\mathcal{E},\mathcal{E})$, where $D \subseteq \mathbb{C}$ is a countable set such that $\{w \in \mathbb{C} : c_1 \leq \mathrm{Re} w \leq c_2\} \cap \{w \in \mathbb{C} : \mathrm{dist}(w, \pi_{\mathbb{C}}R) > \varepsilon\} \cap D$ is finite for every $c_1 \leq c_2$ and $\varepsilon > 0$. We have a representation $(1+m)^{-1} = \sum_{j=0}^{\infty}(-1)^j m^j$ as a convergent series of functions with values in $\mathcal{L}(E^s, E^s)$, $w \in \mathbb{C}$ for $c \leq \mathrm{Re}\, w \leq c'$, $|\mathrm{Im}\, w| \geq C$ for every $c \leq c'$ and $C(c,c') > 0$ sufficiently large. In a similar manner we obtain convergence of all $w$-derivatives of $\sum_{j=0}^{\infty}(-1)^j m^j$ in a set of such a structure. Thus, from $g = -m(1+g) = -m(1+m)^{-1}$ and the Schwartz property of $m$ for large $|\mathrm{Im}\, w|$, uniformly in finite strips $c \leq \mathrm{Re}\, w \leq c'$, we obtain the same property for $g$ itself. It remains to show that $g$ is meromorphic with poles at the points of $D$, that $D$ has no accumulation points at $\pi_{\mathbb{C}}R$, and that the Laurent coefficients are of the desired kind, namely, to belong to $\mathcal{L}^{-\infty}(\mathcal{E},\mathcal{E})$ and to be of finite rank. Let us verify that there are no accumulation points of the singularities of $\big(1+m(w)\big)^{-1}$. Let $w_0$ be a pole of $m$, i.e., $w_0 \in \pi_{\mathbb{C}}R$. Then we can write

$$1+m(w) = 1+m_0(w) + \sum_{k=1}^{K} b_k(w-w_0)^{-k}$$

with suitable $K \in \mathbb{N}$, $m_0$ holomorphic in a neighbourhood of $w_0$ and $\mathcal{L}^{-\infty}(\mathcal{E},\mathcal{E})$-valued, with finite rank operators $b_k$. Note that $m_0 \not\equiv -1$. Setting $n(w) := \sum_{k=1}^{K} b_k(w-w_0)^{-k}$ we have

$$1+m(w) = \big(1+m_0(w)\big)\Big(1+\big(1+m_0(w)\big)^{-1}n(w)\Big).$$

Now $m_0$ is holomorphic near $w_0$ and $1+m_0(w)$ a Fredholm family, since $m_0$ takes values in $\mathcal{L}^{-\infty}(\mathcal{E},\mathcal{E})$, therefore the singularities of $\big(1+m_0(w)\big)^{-1}$ form a countable discrete set; therefore there is a $\delta > 0$ such that $\big(1+m_0(w)\big)^{-1}$ exists for all $w$ such that $0 < |w-w_0| < \delta$. Moreover, $\big(1+m_0(w)\big)^{-1}n(w)$ can be written in the form $h(w) + \sum_{j=1}^{N} a_j(w-w_0)^{-j}$ with a suitable $h$ which is holomorphic near $w_0$ and finite rank operators $a_j$, $1 \leq j \leq N$. The operator $\big(1+m_0(w)\big)^{-1}n(w)$ vanishes on the space $F := \bigcap_{k=1}^{K} \ker b_k$ which is of finite codimension. Setting $M := \bigcap_{j=1}^{N} \ker a_j$ it follows that $h(w)u = 0$ for all $u \in M \cap F$; the latter space is also of finite codimension. Lemma 3.2.19 then shows that $1+\big(1+m_0(w)\big)^{-1}n(w)$ is invertible in $0 < |w-w_0| < \delta$ for a suitable $\delta > 0$. □

**Theorem 3.2.20.** *Let $h \in M_{\mathcal{O}}^{\mu}(\mathcal{E},\widetilde{\mathcal{E}})$ be elliptic, then there is an $f \in M_S^{-\mu}(\widetilde{\mathcal{E}},\mathcal{E})$ with asymptotic type $S$ such that $hf = 1$.*

*Proof.* Let $h^{(-1)}(w) \in M_{\mathcal{O}}^{-\mu}(\widetilde{\mathcal{E}},\mathcal{E})$ be as in the proof of Theorem 3.2.15. Then we have the relations (3.2.22). By virtue of Proposition 3.2.18 there exists a $g \in M_P^{-\infty}(\widetilde{\mathcal{E}},\widetilde{\mathcal{E}})$, for some asymptotic type $P$, such that $\big(1+m(w)\big)\big(1+g(w)\big) = 1$. This yields $h(w)f(w)=1$ for $f := h^{(-1)}(1+g)$ which belongs to $M_S^{-\mu}(\widetilde{\mathcal{E}},\mathcal{E})$, according to Theorem 3.2.17. In a similar manner we find an $\tilde{f} \in M_{\widetilde{S}}^{-\mu}(\widetilde{\mathcal{E}},\mathcal{E})$ such that $\tilde{f}(w)h(w)=1$. This implies $f=\tilde{f}$. □

**Definition 3.2.21.** *A $g \in M_R^{\mu}(\mathcal{E},\widetilde{\mathcal{E}})$ is said to be elliptic, if there is a $\beta \in \mathbb{R}$ such that $(g|_{\Gamma_\beta})^{-1} \in S^{-\mu}(\Gamma_\beta;\widetilde{\mathcal{E}},\mathcal{E})$.*

**Theorem 3.2.22.** *If $g \in M_R^{\mu}(\mathcal{E},\widetilde{\mathcal{E}})$ is elliptic, there is an asymptotic type $S$ and an element $f \in M_S^{-\mu}(\widetilde{\mathcal{E}},\mathcal{E})$ such that $gf=1$.*

*Proof.* Applying the kernel cut-off operator to $(g|_{\Gamma_\beta})^{-1}$ we find an $h^{(-1)} \in M_{\mathcal{O}}^{-\mu}(\widetilde{\mathcal{E}},\mathcal{E})$ such that $h^{(-1)}|_{\Gamma_\beta} - (g|_{\Gamma_\beta})^{-1} \in S^{-\infty}(\Gamma_\beta;\widetilde{\mathcal{E}},\mathcal{E})$. By definition we have $g = g_0 + g_1$ for certain $g_0 \in M_{\mathcal{O}}^{\mu}(\mathcal{E},\widetilde{\mathcal{E}})$, $g_1 \in M_R^{-\infty}(\mathcal{E},\widetilde{\mathcal{E}})$. Then $h^{(-1)}g_0|_{\Gamma_\beta} = 1 \mod S^{-\infty}(\Gamma_\beta;\mathcal{E},\mathcal{E})$ implies $h^{(-1)}g_0 = 1 \mod M_{\mathcal{O}}^{-\infty}(\mathcal{E},\mathcal{E})$ (see Theorem 3.2.12). If follows that $h^{(-1)}g = 1+m$ for some $m \in M_R^{-\infty}(\mathcal{E},\mathcal{E})$ with an asymptotic type $R$ (see Theorem 3.2.17). Thus Proposition 3.2.18 gives us $g^{-1} = (1+m)^{-1}h^{(-1)} \in M_S^{-\mu}(\widetilde{\mathcal{E}},\mathcal{E})$ with some asymptotic type $S$. □

Parallel to the spaces of Mellin symbols (3.2.23) we now introduce subspaces of $\mathcal{H}^{s,\gamma}(\mathbb{R}_+,\mathcal{E})$ with discrete asymptotics. To this end it is not necessary to specify certain finite-dimensional spaces $L_j \in \mathcal{L}^{-\infty}(\mathcal{E},\widetilde{\mathcal{E}})$. We consider sequences of the form

$$P := \{(p_j, m_j)\}_{0\le j\le N} \tag{3.2.24}$$

with $N \in \mathbb{N}\cup\{+\infty\}$, $m_j \in \mathbb{N}$, $0 \le j \le N$. A sequence (3.2.24) is said to be a discrete asymptotic type, associated with weight data $(\gamma,\Theta)$ (with a weight $\gamma \in \mathbb{R}$ and a weight interval $\Theta = (\vartheta, 0]$, $-\infty \le \vartheta \le 0$), if for some $d = d_{\mathcal{E}} \in \mathbb{N}$

$$\pi_{\mathbb{C}} P := \{p_j\}_{0\le j\le N} \subset \{w \in \mathbb{C} : \frac{d+1}{2} - \gamma + \vartheta < \operatorname{Re} w < \frac{d+1}{2} - \gamma\},$$

and $\pi_{\mathbb{C}}P$ is finite when $\vartheta$ is finite, and $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$ when $\vartheta = -\infty$ and $N = +\infty$. We will say that $P$ satisfies the shadow condition, if $(p,m) \in P$ implies $(p-j,m) \in P$ for all $j \in \mathbb{N}$ with $\frac{d+1}{2} - \gamma + \vartheta < \operatorname{Re}(p-j) < \frac{d+1}{2} - \gamma$.
If $\Theta$ is finite we define the (finite-dimensional) space

$$\mathcal{S}_P(\mathbb{R}_+,\mathcal{E}) := \left\{ \sum_{j=0}^{N}\sum_{k=0}^{m_j} \omega(r) c_{jk} r^{-p_j} \log^k r : \; c_{jk} \in E^{\infty} \right\}$$

with some fixed cut-off function $\omega$ on the half-axis. We then have

$$\mathcal{S}_P(\mathbb{R}_+,\mathcal{E}) \subset \mathcal{H}^{\infty,\gamma}(\mathbb{R}_+,\mathcal{E}).$$

Moreover, we set

$$\mathcal{H}^{s,\gamma}_{\Theta}(\mathbb{R}_+,\mathcal{E}) := \omega \bigcap_{\varepsilon>0} \mathcal{H}^{s,\gamma-\vartheta-\varepsilon}(\mathbb{R}_+,\mathcal{E}) + (1-\omega)\mathcal{H}^{s,\gamma}(\mathbb{R}_+,\mathcal{E}),$$

where the intersection is endowed with the Fréchet topology of the projective limit, and

$$\mathcal{H}_P^{s,\gamma}(\mathbb{R}_+,\mathcal{E}) := \mathcal{H}_\Theta^{s,\gamma}(\mathbb{R}_+,\mathcal{E}) + \mathcal{S}_P(\mathbb{R}_+,\mathcal{E})$$

as a direct sum of Fréchet spaces.
In order to formulate spaces with discrete asymptotics of type $P$ in the case $\Theta = (-\infty, 0]$ we form $P_l := \{(p,m) \in P : \operatorname{Re} p > \frac{d+1}{2} - \gamma - (l+1)\}$ for any $l \in \mathbb{N}$. From the above construction we have the spaces $\mathcal{H}_{P_l}^{s,\gamma}(\mathbb{R}_+,\mathcal{E})$ together with continuous embeddings

$$\mathcal{H}_{P_{l+1}}^{s,\gamma}(\mathbb{R}_+,\mathcal{E}) \hookrightarrow \mathcal{H}_{P_l}^{s,\gamma}(\mathbb{R}_+,\mathcal{E}),\ l \in \mathbb{N}.$$

We then define

$$\mathcal{H}_P^{s,\gamma}(\mathbb{R}_+,\mathcal{E}) := \varprojlim_{l\in\mathbb{N}} \mathcal{H}_{P_l}^{s,\gamma}(\mathbb{R}_+,\mathcal{E}) \tag{3.2.25}$$

in the corresponding Fréchet topology of the projective limit.

**Remark 3.2.23.** *That $u \in \mathcal{H}_P^{s,\gamma}(\mathbb{R}_+,\mathcal{E})$, with $P$ being associated with $(\gamma,\Theta)$, $\Theta = (-\infty,0]$, is equivalent to the existence of (unique) coefficients $c_{jk} \in E^\infty$, $0 \le k \le m_j$, such that for every $t \in \mathbb{R}_+$ there is an $N = N(t) \in \mathbb{N}$ with*

$$\omega(r)\left(u(r) - \sum_{j=0}^{N}\sum_{k=0}^{m_j} c_{jk} r^{-p_j}\log^k r\right) \in \mathcal{H}^{s,\gamma+t}(\mathbb{R}_+,\mathcal{E}).$$

Similarly as in the “concrete” cone calculus (see [36]) we have the following continuity result:

**Theorem 3.2.24.** *Let $f \in M_R^\mu(\mathcal{E},\widetilde{\mathcal{E}})$ be such that $\pi_{\mathbb{C}}R \cap \Gamma_{\frac{d+1}{2}-\gamma} = \emptyset$, and $P$ an asymptotic type associated with the weight data $\big(\gamma,(\vartheta,0]\big)$, for some $-\infty \le \vartheta < 0$. Then the operator*

$$\operatorname{op}_M^{\gamma-\frac{d}{2}}(f) : \mathcal{H}^{s,\gamma}(\mathbb{R}_+,\mathcal{E}) \to \mathcal{H}^{s-\mu,\gamma}(\mathbb{R}_+,\widetilde{\mathcal{E}}) \tag{3.2.26}$$

*restricts to a continuous operator*

$$\operatorname{op}_M^{\gamma-\frac{d}{2}}(f) : \mathcal{H}_P^{s,\gamma}(\mathbb{R}_+,\mathcal{E}) \to \mathcal{H}_Q^{s-\mu,\gamma}(\mathbb{R}_+,\widetilde{\mathcal{E}})$$

*for every $s \in \mathbb{R}$ with some resulting asymptotic type $Q$.*

The proof will be given in several steps. To this end we establish some auxiliary lemmas. The first one is due to Cauchy's integral formula.

**Lemma 3.2.25.** *Let $f(w)$ be an (operator-valued) meromorphic function with finitely many poles at points $p_j$ of multiplicities $m_j + 1$, $j = 0,\dots,N$. Then for any piecewise smooth curve $C$ clockwise surrounding the poles we have*

$$\frac{1}{2\pi i}\int_C f(w) r^{-w}\,dw = \sum_{j=0}^{N}\sum_{k=0}^{m_j} c_{jk} r^{-p_j}\log^k r$$

*for some constant (operators) $c_{jk}$.*

**Lemma 3.2.26.** *Under the requirements of Theorem* 3.2.24, *the operator* (3.2.26) *restricts to a continuous operator*

$$\mathrm{op}_M^{\gamma-\frac{d}{2}}(f) : \mathcal{S}_P(\mathbb{R}_+, \mathcal{E}) \to \mathcal{H}^{\infty,\gamma}_{\widetilde{Q}}(\mathbb{R}_+, \widetilde{\mathcal{E}}),$$

*for some asymptotic type* $\widetilde{Q}$.

*Proof.* Let $P$ be given by (3.2.24). We consider the case $\vartheta > -\infty$, the general case follows immediately. For $u(r) \in \mathcal{S}_P(\mathbb{R}_+, \mathcal{E})$ there are unique elements $c_{jk} \in E^\infty$, $j = 0, \dots, N$, $k = 0, \dots, m_j$ such that

$$u(r) = \sum_{j=0}^{N} \sum_{k=0}^{m_j} \omega(r) c_{jk} r^{-p_j} \log^k r,$$

for some cut-off function $\omega$. Then $\mathcal{M}u(w)$ is meromorphic with poles at $\pi_{\mathbb{C}}P$ with values in $E^\infty$. Moreover, if $\chi(w) \in C^\infty(\mathbb{C})$ is a $\pi_{\mathbb{C}}P$-excision function then $\chi(w)\mathcal{M}u(w)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta, E^\infty)$ for every $\beta \in \mathbb{R}$ (cf. [4, Chapter 1]). Now multiplying by $f$ we obtain an $\widetilde{E}^\infty$-valued meromorphic function with poles at $\pi_{\mathbb{C}}P \cup \pi_{\mathbb{C}}R$ and, if we set $g(w) := f(w)\mathcal{M}u(w)$,

$$\chi_1(w) g(w)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta, \widetilde{E}^\infty)$$

for any $\pi_{\mathbb{C}}P \cup \pi_{\mathbb{C}}R$-excision function $\chi_1$ and any $\beta \in \mathbb{R}$. We have

$$\begin{aligned}\mathrm{op}_M^{\gamma-\frac{d}{2}}(f)u(r) &= \big(\mathcal{M}^{-1}_{\gamma-\frac{d}{2}} g\big)(r) = \frac{1}{2\pi i} \int_{\Gamma_{\frac{d+1}{2}-\gamma}} r^{-w} g(w)\,dw \\ &= \omega \frac{1}{2\pi i} \int_{\Gamma_{\frac{d+1}{2}-\gamma}} r^{-w} g(w)\,dw + (1-\omega) \frac{1}{2\pi i} \int_{\Gamma_{\frac{d+1}{2}-\gamma}} r^{-w} g(w)\,dw. \qquad (3.2.27)\end{aligned}$$

The second term on the right hand side of (3.2.27) belongs to $(1-\omega)\mathcal{H}^{\infty,\gamma}(\mathbb{R}_+, \widetilde{\mathcal{E}})$, so let us concentrate on the first term. Let $\widetilde{Q}$ be the asymptotic type defined by $\{(q_j, d_j)\}_{0 \le j \le L}$, $L \in \mathbb{N}$, where $\{q_j\}_{0 \le j \le L}$ are the poles of $\pi_{\mathbb{C}}P \cup \pi_{\mathbb{C}}R$ that lie between the weight lines $\Gamma_{\frac{d+1}{2}-\gamma+\vartheta}$ and $\Gamma_{\frac{d+1}{2}-\gamma}$. For $\varepsilon > 0$ we write

$$\omega \frac{1}{2\pi i} \int_{\Gamma_{\frac{d+1}{2}-\gamma}} r^{-w} g(w)\,dw = \omega \frac{1}{2\pi i} \int_{\Gamma_{\frac{d+1}{2}-\gamma+\vartheta+\varepsilon}} r^{-w} g(w)\,dw + I_\varepsilon(r), \qquad (3.2.28)$$

where

$$\begin{aligned}I_\varepsilon(r) &= \omega \frac{1}{2\pi i} \int_{\Gamma_{\frac{d+1}{2}-\gamma}} r^{-w} g(w)\,dw - \omega \frac{1}{2\pi i} \int_{\Gamma_{\frac{d+1}{2}-\gamma+\vartheta+\varepsilon}} r^{-w} g(w)\,dw \\ &= \omega \frac{1}{2\pi i} \int_{C_\varepsilon} r^{-w} g(w)\,dw\end{aligned}$$

for a rectangle $C_\varepsilon$ (clockwise oriented), consisting of sufficiently large intervals $[-R, R]$ on $\Gamma_{\frac{d+1}{2}-\gamma+\vartheta+\varepsilon}$ and $\Gamma_{\frac{d+1}{2}-\gamma}$ and straight segments parallel to the real axis of sufficiently large imaginary parts. By virtue of Lemma 3.2.25 there exist an $M = M(\varepsilon) \leq L$ and elements $\tilde{c}_{jk} \in \widetilde{E}^\infty$, $j = 0, \dots, M$, $k = 0, \dots, d_j$ such that

$$\omega \frac{1}{2\pi i} \int_{C_\varepsilon} r^{-w} g(w) dw = \sum_{j=0}^{M} \sum_{k=0}^{d_j} \omega(r) \tilde{c}_{jk} r^{-q_j} \log^k r.$$

Then, from (3.2.28) we obtain for any $\varepsilon > 0$

$$\begin{aligned} &\omega \frac{1}{2\pi i} \int_{\Gamma_{\frac{d+1}{2}-\gamma}} r^{-w} g(w) dw \\ &\qquad = \omega \frac{1}{2\pi i} \int_{\Gamma_{\frac{d+1}{2}-\gamma+\vartheta+\varepsilon}} r^{-w} g(w) dw + \sum_{j=0}^{M} \sum_{k=0}^{d_j} \omega(r) \tilde{c}_{jk} r^{-q_j} \log^k r \\ &\qquad\qquad \in \omega \mathcal{H}^{\infty, \gamma-\vartheta-\varepsilon}(\mathbb{R}_+, \widetilde{\mathcal{E}}) + \mathcal{S}_{\widetilde{Q}}(\mathbb{R}_+, \widetilde{\mathcal{E}}) \end{aligned}$$

Finally we fix $\varepsilon$ so small that all the poles of $\widetilde{Q}$ lie on the right of the weight line $\Gamma_{\frac{d+1}{2}-\gamma+\vartheta+\varepsilon}$. Then we have

$$\omega \frac{1}{2\pi i} \int_{\Gamma_{\frac{d+1}{2}-\gamma}} r^{-w} g(w) dw \in \mathcal{H}^{\infty,\gamma}_{\Theta}(\mathbb{R}_+, \widetilde{\mathcal{E}}) + \mathcal{S}_{\widetilde{Q}}(\mathbb{R}_+, \widetilde{\mathcal{E}})$$

and hence, we obtain altogether that $\mathrm{op}_M^{\gamma-\frac{d}{2}}(f)u \in \mathcal{H}^{\infty,\gamma}_{\widetilde{Q}}(\mathbb{R}_+ \widetilde{\mathcal{E}})$. □

**Lemma 3.2.27.** *The operator* (3.2.26) *restricts to a continuous operator*

$$\mathrm{op}_M^{\gamma-\frac{d}{2}}(f) : \mathcal{H}^{s,\gamma}_{\Theta}(\mathbb{R}_+, \mathcal{E}) \to \mathcal{H}^{s-\mu,\gamma}_{R_\Theta}(\mathbb{R}_+, \widetilde{\mathcal{E}})$$

*for* $R_\Theta := \{(q, n) \in R : \frac{d+1}{2} - \gamma + \vartheta < \mathrm{Re}\, r < \frac{d+1}{2} - \gamma\}$.

*Proof.* Let us first show that the weighted Mellin transform induces an isomorphism

$$\mathcal{M}_{\gamma-\frac{d}{2}} : \mathcal{H}^{s,\gamma}(\mathbb{R}_+, \mathcal{E}) \to \widehat{H}^s(\Gamma_{\frac{d+1}{2}-\gamma}, \mathcal{E}), \tag{3.2.29}$$

where $\widehat{H}^s(\Gamma_\beta, \mathcal{E}) := \{h(\beta + i\rho) = \mathcal{F}_{t\to\rho} v(\rho) \text{ for some } v(t) \in H^s(\mathbb{R}_t, \mathcal{E})\}$, $\beta \in \mathbb{R}$. In fact, the Mellin and the Fourier transform relate to each other by the identity

$$\mathcal{M}_{\gamma-\frac{d}{2}} u(\frac{d+1}{2} - \gamma + i\rho) = \mathcal{F}_{t\to\rho} S_{\gamma-\frac{d}{2}} u(\rho),$$

here $S_{\gamma-\frac{d}{2}} u(t) = e^{-(\frac{d+1}{2}-\gamma)t} u(e^{-t})$, cf. Section 3.2.1. Therefore (3.2.29) is consequence of the isomorphism (3.2.8).

Now let $u \in \mathcal{H}^{s,\gamma}_{\Theta}(\mathbb{R}_+, \mathcal{E})$, then $u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+, \mathcal{E})$ and $\mathcal{M}_{\gamma-\frac{d}{2}} u \in \widehat{H}^s(\Gamma_{\frac{d+1}{2}-\gamma}, \mathcal{E})$. As a consequence of (3.2.7), the multiplication operator by $f(\frac{d+1}{2} - \gamma + i\rho)$ induces continuous operator

$$\mathcal{M}_f : \widehat{H}^s(\Gamma_{\frac{d+1}{2}-\gamma}, \mathcal{E}) \to \widehat{H}^{s-\mu}(\Gamma_{\frac{d+1}{2}-\gamma}, \widetilde{\mathcal{E}}).$$

Moreover $f\mathcal{M}_{\gamma-\frac{d}{2}}u(\frac{d+1}{2} - \gamma + i\rho)$ can be extended to an $\widetilde{E}^{s-\mu}$-valued meromorphic function in the strip $\{w \in \mathbb{C} : \frac{d+1}{2} - \gamma + \vartheta < \operatorname{Re} w < \frac{d+1}{2} - \gamma\}$ with poles at the points $q_j$ at multiplicity $d_j + 1$ and Laurent coefficients in $\widetilde{E}^\infty$ such that for any $\pi_{\mathbb{C}} R_\Theta$-excision function $\chi$ we have $\chi(w)f(w)\mathcal{M}_{\gamma-\frac{d}{2}}u|_{\Gamma_\beta} \in \widehat{H}^{s-\mu}(\Gamma_\beta, \widetilde{\mathcal{E}})$ for every $\beta \in (\frac{d+1}{2} - \gamma + \vartheta, \frac{d+1}{2} - \gamma]$, uniformly in compact $\beta$-subintervals. It can be proved, analogously as in the proof of Lemma 3.2.26, that

$$\mathrm{op}_M^{\gamma-\frac{d}{2}} u(r) = \mathcal{M}_{\gamma-\frac{d}{2}} f \mathcal{M}_{\gamma-\frac{d}{2}} u(r) \in \mathcal{H}^{s-\mu,\gamma}_{R_\Theta}(\mathbb{R}_+, \widetilde{\mathcal{E}}).$$

□

*Proof of Theorem* 3.2.24. Let $u \in \mathcal{H}^{s,\gamma}_P(\mathbb{R}_+, \mathcal{E})$. Then we can write $u = u_{\mathrm{flat}} + u_{\mathrm{sing}}$ for some $u_{\mathrm{flat}} \in \mathcal{H}^{s,\gamma}_\Theta(\mathbb{R}_+, \mathcal{E})$ and $u_{\mathrm{sing}} \in \mathcal{S}_P(\mathbb{R}_+, \mathcal{E})$. Then, by virtue of Lemma 3.2.26 and 3.2.27 we have

$$\mathrm{op}_M^{\gamma-\frac{d}{2}}(f)u_{\mathrm{flat}} \in \mathcal{H}^{s-\mu,\gamma}_{R_\Theta}(\mathbb{R}_+, \widetilde{\mathcal{E}}), \quad \mathrm{op}_M^{\gamma-\frac{d}{2}}(f)u_{\mathrm{sing}} \in \mathcal{H}^{\infty,\gamma}_{\widetilde{Q}}(\mathbb{R}_+, \widetilde{\mathcal{E}}).$$

Let $Q$ be the asymptotic type defined as the union of the poles of both $R_\Theta$ and $\widetilde{Q}$ with the corresponding multiplicity or, for the common poles, the sum of the two multiplicities. Then, to prove the assertion, it is enough to notice that both $\mathcal{H}^{s-\mu,\gamma}_{R_\Theta}(\mathbb{R}_+, \widetilde{\mathcal{E}})$ and $\mathcal{H}^{\infty,\gamma}_{\widetilde{Q}}(\mathbb{R}_+, \widetilde{\mathcal{E}})$ are continuously embedded in $\mathcal{H}^{s-\mu,\gamma}_Q(\mathbb{R}_+, \widetilde{\mathcal{E}})$. □

The case of Mellin symbols with variable coefficients is also of interest in the corner calculus. It is then adequate to assume $f(r, w) \in C^\infty(\overline{\mathbb{R}}_+, M^\mu_R(\mathcal{E}, \widetilde{\mathcal{E}}))$ and to consider operators $\omega \mathrm{op}_M^\gamma(f)\tilde{\omega}$ in combination with cut-off functions $\omega(r), \tilde{\omega}(r)$. Those induce continuous operators $\mathcal{H}^{s,\gamma}_P(\mathbb{R}_+, \mathcal{E}) \to \mathcal{H}^{s-\mu,\gamma}_Q(\mathbb{R}_+, \widetilde{\mathcal{E}})$ as well.

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