Boundary Value Problems on Manifolds with Singularities

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Die selbständige und eigenhändige Anfertigung versichere ich an Eides statt.

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Contents

Introduction

The present dissertation is devoted to new developments in the analysis of operators on manifolds with singularities, especially differential and pseudo-differential operators of elliptic type when the geometry of the underlying space may be characterized by stratifications. On one hand our approach is based on the symbolic structure of operators, according to the background of micro-local analysis, on the other hand we focus on exciting new developments of the analysis on underlying spaces, roughly characterized by stratifications. The latter notion covers singularities of conical, edge, or corner type, and the context is altogether motivated by applications of partial differential equations in physics and technical sciences.

Let us first give some references on the involved fields in general. Both applications as well as the purely analytic content and also geometric/topological interpretations have a long history. Below we come back to more references, but let us already mention names of researchers, namely, Grisvard, Sanchez-Palencia, Dauge, Hörmander, Egorov, Treves, Boutet de Monvel, Višik, Eskin, Sternin, Agranovich, Atiyah, Singer, Bott, Melrose, Nistor, and many others. Our research is embedded into the achievements of different specific schools which are connected with the mentioned authors.

A formal element of approaching our concrete results is the program of operator algebras with symbolic structures which can be described in the simplest form by the task to express parametrices of elliptic elements within the operator algebra in consideration. Such a program is related to the process of solving an equation of the form

$$\mathcal{A}u = f$$

where $\mathcal{A}$ is just an elliptic operator in the algebra, u is the solution under some prescribed right-hand side f. In the simplest classical case $\mathcal{A}$ may be the Laplacian on a closed smooth manifold M, and the algebra the space of classical pseudo-differential operator of any real order. Then, a well-known result of pseudo-differential analysis is that there is an operator $\mathcal{P}$ of opposite order, namely, -2, such that

$$\mathcal{P}\mathcal{A}u = \mathcal{P}f \tag{0.0.1}$$

transforms the right-hand side to something known, while $\mathcal{P}$ satisfies a relation of the form $\mathcal{P}\mathcal{A} = \mathcal{I} - \mathcal{G}$ for the identity operator $\mathcal{I}$ and a remainder $\mathcal{G}$ which is in our context a Green operator, with a smooth kernel. Thus (0.0.1) gives us $u = \mathcal{G}u + \mathcal{P}f$, but the particularly regularizing property of $\mathcal{G}$ maps any distribution u into some smooth function. This gives us altogether a characterization of solutions u in terms of the structure of the parametrix $\mathcal{P}$ which incorporates all essential properties of solutions.

The program of this thesis follows such an idea, both for boundary value problems (BVPs) in the smooth case which is outlined in Chapter 2, edge problems in Chapter 3, and BVPs on a manifold with edge and boundary in Chapter 4. The other Chapters are continuing the strategy.

Chapter 1 is devoted to basic notions on the analysis of pseudo-differential operators (PDOs) first in local form in $\mathbb{R}^n$, then on a smooth manifold. We present material in a concise way and we also refer to standard text books, e.g., on Fourier and Mellin transform, distributions, Sobolev spaces and other necessary tools. From Section 1.3 on we pass to introducing parameter-dependent variants of PDOs which will become essential for different levels of quantization. This gives rise to operator-valued symbols and edge Sobolev spaces which are necessary in several generalizations, both in "abstract" functional-analytic set-up with involved, Hilbert spaces with group action and specific realizations. Chapter 1 ends by studying examples and concrete realizations occurring in the framework of boundary symbols associated with BVPs.

Chapter 2 gives an introduction to Boutet de Monvel's calculus. According to the general task to single out specific algebras of pseudo-differential operators on an open manifold, containing "all" differential operators together with parametrices of elliptic elements, the corresponding step in the "hierarchy" of more sophisticated realizations of this idea is the case of a manifold with smooth boundary. In the above-mentioned process the optimal outcome of the discussion is to achieve a "minimal" algebra satisfying the indicated criteria. Since in the present context we still start with relatively simple objects, namely, differential operators with smooth coefficients up to the boundary, and since those have the so-called transmission property at the boundary, the parametrices will also have such a property, though pseudo-differential, and hence, up to some specific observations, going back to Boutet de Monvel [7], the corresponding operator algebra is dominated by the transmission property see also the works of Grubb [19], Rempel, Schulze [48], Eskin [14]. We outline details from this calculus which give a first impression of operators having a principal symbolic hierarchy, consisting of the usual interior principal symbol, coming from the open interior of the underlying manifold X with boundary ∂X, and in addition of the principal boundary symbol, associated with ∂X. Both symbolic components determine operators up to lower order terms, compact when X itself is compact. In any case the respective pairs of symbols are responsible for ellipticity and parametrices, belonging to (Leibniz) inverted symbolic tuples, and give rise to the Fredholm property when X is compact. There are several natural extensions of this concept, in particular, parameter-dependent versions, which are the basis of higher cone and edge quantizations, when our manifold with boundary is not smooth, but has conical or edge singularities. In order to make this process more transparent we introduce in

Chapter 3 an operator calculus where we ignore the boundary and study what we need for unifying Boutet de Monvel's calculus with the so-called edge calculus. In other words the program of this part of the thesis is the edge-calculus on a manifold with edge "without boundary". The edge-calculus is a specific pseudo-differential calculus on a manifold with edge (or conical singularities) which satisfies the program to contain interesting classes of differential operators together with the parametrices of elliptic elements. "Interesting" means here that the differential operators are locally generated by vector fields of the form

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, r\frac{\partial}{\partial r}, r\frac{\partial}{\partial y_1}, \dots, r\frac{\partial}{\partial y_q} \tag{0.0.2}$$

expressed in stretched variables $(r, x, y) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q$ belonging to the open interior of the corresponding model wedge $\mathbb{R}_+ \times X \times \mathbb{R}^q$ for a closed smooth X with local coordinates $x \in \mathbb{R}^n$, and where $r^{-\mu}$ with μ being the order, appears as a multiplicative factor. Examples of such operators for $\mu = 2$ are Laplace Beltrami operators for Riemannian metric of the form

$$dr^2 + r^2 g_X + dy^2$$

for any Riemannian metric in $\mathbb{R}^n$ (or globally on a smooth manifold X with local coordinates x). Because of (0.0.2) the operators are characterized close to $r = 0$, and hence, ellipticity degenerates locally near the edge which requires a completely new machinery for establishing the corresponding operator algebra. There are world-wide various schools of active research, especially around the work of Melrose, cf. [43], [42], based on the idea of blowing up singularities. In the present exposition we follow the achievements of work of Schulze [54], [53], see also [62]-[65] jointly with Schrohe, and by other authors of the research team in Potsdam, in particular, Hirschmann [21], Dorschfeldt [13], Gil, Seiler [18], [67], [68]. The approach is relatively complex it refers to the tradition of BVPs of Boutet de Monvel or of other important schools of PDE such as of Višik, Eskin [71], [72], or Agmon, Douglis and Nirenberg [1]. Nevertheless, the so-called edge algebra equipped with scalar interior and operator-valued edge symbols is not easy to communicate because of the large variety of notions which are necessary to reflect the behaviour of degenerate operators and the adequate weighted distribution spaces and subspaces with asymptotics as $r \to 0$. Observe that in the case of a one-dimensional model cone, one the essential aspects is that the edge algebra also contains the algebra of usual BVPs, i.e., it admits arbitrary non-degenerate symbols, without the transmission property, smooth up to the respective boundary, though a manifold with edge in this case is nothing else than a manifold with smooth boundary. The edge operators in this variant form a calculus which contains Boutet de Monvel's BVPs as a very specific subclass.

Chapter 4 contains the new results of this thesis, namely, a comprehensive calculus of boundary value problems with the transmission property on a manifold N with edge Y and boundary ∂N, where the transmission property refers to the smooth part of ∂N which is itself together with Y a manifold with edge Y and without boundary. Note that there is an earlier singular case of boundary value problems [62], [63], where the manifold with boundary has conical singularities. This is a rather specific case which is also involved in the present thesis, however in more convenient quantizations, compared with the "traditional one". Here we carry out the program outlined at the beginning. We achieve the full calculus of boundary value problems on N, including boundary, trace and potential conditions on $\partial N \setminus Y$ by using a new Mellin-edge quantization, similarly as in [18] where such an approach has been demonstrated in the closed case. There are also edge conditions along Y, again of trace and potential type, where here the trace operators appear in integral form, not as restrictions to Y after possible differentiations. A new feature in this edge-"variant" of Boutet de Monvel's calculus is that instead of 2×2-block matrix we have 3×3-block matrix operators

$$\mathfrak{A} = (A_{ij})_{i,j=1,2,3}. \tag{0.0.3}$$

Clearly some entries may also vanish. A particularly interesting point is like in the edge algebra in the boundary less case that in contrast to Green contributions in upper left corners A_{11}, motivated by operator compositions or in parametrices, in the edge case we also have so-called smoothing Mellin plus Green operators, both in A_{11} and A_{22}. Various trace and potential parts are represented by A_{21}, A_{31}, and A_{12}, A_{13} and A_{32} and A_{23}, respectively, while A_{22} is an edge operator on $\partial N \cup Y$ inducing smoothing Mellin plus Green operators, A_{33} are classical block-matrix valued pseudo-differential operators on Y. In other words the complete calculus of operators (0.0.3) altogether forms a rich structure, but all entries are necessary in general, and the properties of solutions to elliptic equations

$$\mathfrak{A}\mathfrak{u} = \mathfrak{f}$$

with $\mathfrak{u}$ and $\mathfrak{f}$ being vectors of distributions with 3 components are incorporated in respective parametrices, constructed in Section 4.8. In addition we formulate properties such as characterization of remainders of solutions under natural conditions on the meromorphic structure of inverted holomorphic Mellin symbols. The recently published article [29] contains the crucial aspects of our new Mellin-edge approach which is much easier accessible than the one, called "traditional", see also more references in [29].

Chapter 1

Pseudo-Differential Operators

Pseudo-differential operators generalize the family of differential operators in a natural way. This part of the exposition is mainly devoted to notation and results. Proofs may be found in textbooks on pseudo-diffrerntial operators, e.g., [69] though we indicate the arguments in some exceptional cases.

1.1 Basics on pseudo-differential operators

In this section we introduce notation and results on scalar amplitude functions in a corresponding pseudo-differential calculus. We frequently use the Fourier transform

$$\hat{u}(\xi) := Fu(\xi) := \int_{\mathbb{R}^n} \mathrm{e}^{-ix\xi} u(x)\, dx$$

with its inverse

$$(F^{-1}g)(x) = \int \mathrm{e}^{ix\xi} g(\xi)\, đ\xi,$$

$đ\xi = (2\pi)^{-n} d\xi$. Let

$$A = \sum_{|\alpha| \le \mu} a_\alpha(x)\, D_x^\alpha$$

be a differential operator in a domain $\Omega \subseteq \mathbb{R}^n$ with coefficients $a_\alpha(x) \in C^\infty(\Omega)$, regarded as an operator $A : C^\infty(\Omega) \to C^\infty(\Omega)$. Then A can be expressed by the Fourier transform F as

$$A = F^{-1}\, a(x, \xi)\, F \quad \text{with} \quad a(x, \xi) = \sum_{|\alpha| \le \mu} a_\alpha(x)\, \xi^\alpha,$$

using the elementary identity

$$D_x^\alpha = F^{-1} \xi^\alpha F. \tag{1.1.1}$$

Here

$$D_x^\alpha := \left(\frac{1}{i} \frac{\partial}{\partial x_1} \right)^{\alpha_1} \dots \left(\frac{1}{i} \frac{\partial}{\partial x_n} \right)^{\alpha_n}$$

for $i = \sqrt{-1}$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$; here $\mathbb{N} = \{0, 1, 2, \dots\}$. Later on we also use differentiations without factors i^{-1} and write

$$\partial_x^\alpha := \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n} \right)^{\alpha_n}.$$

Thus

$$Au(x) = \int e^{ix\xi} a(x,\xi) \left\{ \int e^{-ix'\xi} u(x')\, dx' \right\} đ\xi. \tag{1.1.2}$$

This gives us a relation between A and its so-called complete symbol $a(x,\xi)$. The operator A is determined by $a(x,\xi)$. Conversely, by knowing $A : C^\infty(\Omega) \to C^\infty(\Omega)$, the complete symbol $a(x,\xi)$ can be recovered as

$$a(x,\xi) = e^{-ix\xi} A e^{ix'\xi},$$

where A acts with respect to x'.

Now the idea behind pseudo-differential operators is just to allow more general symbols than polynomials in ξ in the defining relation (1.1.2). For the calculus it will be more convenient to write (1.1.2) formally as a double integral. It will also be natural to permit so-called amplitude functions $a(x,x',\xi)$ that are C^∞-dependent on $(x,x') \in \Omega \times \Omega$. We begin by studying the appropriate classes of amplitude functions; for simplicity we also refer to these as symbols.

Definition 1.1.1. (i) *The space $S^\mu(\Omega \times \mathbb{R}^n)$ of symbols $a(x,\xi)$ of order $\mu \in \mathbb{R}$ on an open set $\Omega \subseteq \mathbb{R}^m$ is defined as the set of all $a(x,\xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ such that*

$$\sup_{(x,\xi)\in K\times\mathbb{R}^n} |D_x^\alpha D_\xi^\beta a(x,\xi)|\, \langle\xi\rangle^{-\mu+|\beta|} \tag{1.1.3}$$

is finite for every $K \Subset \Omega$, and arbitrary multi-indices $\alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^n$, here $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$. Equivalently we define $S^\mu(\Omega \times \mathbb{R}^n)$ as the set of all $a(x,\xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ satisfying the symbolic estimate

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le c\,\langle\xi\rangle^{\mu-|\beta|} \tag{1.1.4}$$

for all $(x,\xi) \in K \times \mathbb{R}^n$, $\alpha \in \mathbb{N}^m$, $\beta \in \mathbb{N}^n$, as mentioned before, for constants $c = c(\alpha,\beta,K) > 0$.

(ii) *Let $S^{(\mu)}(\Omega \times (\mathbb{R}^n \setminus \{0\}))$ denote the space of all elements $a_{(\mu)}(x,\xi) \in C^\infty(\Omega \times (\mathbb{R}^n \setminus \{0\}))$ such that*

$$a_{(\mu)}(x,\delta\xi) = \delta^\mu a_{(\mu)}(x,\xi) \tag{1.1.5}$$

for all $\delta \in \mathbb{R}_+$, $(x,\xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$.

Observe that there is an isomorphism

$$\begin{aligned} S^{(\mu)}(\Omega \times (\mathbb{R}^n \setminus \{0\})) &\longrightarrow C^\infty(\Omega \times S^{n-1}), \\ a_{(\mu)}(x,\xi) &\longrightarrow a_{(\mu)}(x, \frac{\xi}{|\xi|}). \end{aligned} \tag{1.1.6}$$

Here S^{n-1} is the unit sphere in $\mathbb{R}^n$. Then $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)$, the space of classical symbols of order μ, is defined as the subspace of all $a(x,\xi) \in S^\mu(\Omega \times \mathbb{R}^n)$, such that there are elements $a_{(\mu-j)}(x,\xi) \in S^{(\mu-j)}(\Omega \times (\mathbb{R}^n \setminus \{0\})), j \in \mathbb{N}$, with the property

$$r_{N+1} := a(x,\xi) - \chi(\xi) \sum_{j=0}^{N} a_{(\mu-j)}(x,\xi) \in S^{\mu-(N+1)}(\Omega \times \mathbb{R}^n) \tag{1.1.7}$$

for every $N \in \mathbb{N}$ and any excision function $\chi(\xi)$ (i.e., $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ such that $\chi(\xi) = 0$ for $|\xi| < c_0$, $\chi(\xi) = 1$ for $|\xi| > c_1$ for certain $0 < c_0 < c_1$). Note that

$$a_{(\mu)}(x,\xi) = \lim_{\delta \to \infty} \delta^{-\mu} a(x, \delta\xi) \tag{1.1.8}$$

which is an immediate consequence of relation (1.1.7). The homogeneous components $a_{(\mu-j)}(x,\xi)$ can also be recovered, by an iterative process.

Remark 1.1.2. (i) *The space $S^\mu(\Omega\times\mathbb{R}^n)$ is a Fréchet space with the system of semi-norms* (1.1.3)*, $\alpha \in \mathbb{N}^m$, $\beta \in \mathbb{N}^n$, $K \Subset \Omega$.*

(ii) *The (so-called) homogeneous components $a_{(\mu-j)}(x,\xi)$, $j \in \mathbb{N}$, of classical symbols $a(x,\xi) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)$ are uniquely determined by $a(x,\xi)$.*

(iii) *The space $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)$ is Fréchet in the projective limit topology with respect to the system of operators*

$$S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^n) \hookrightarrow S^\mu(\Omega \times \mathbb{R}^n)$$

(the embedding),

$$S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^n) \longrightarrow S^{(\mu-j)}(\Omega \times (\mathbb{R}^n \setminus \{0\})),\ j \in \mathbb{N}, \tag{1.1.9}$$

determined by $a \longrightarrow a_{(\mu-j)}$, (producing the unique homogeneous components), and

$$S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^n) \longrightarrow S^{\mu-(N+1)}(\Omega \times \mathbb{R}^n),\ N \in \mathbb{N},$$

determind by $a \longrightarrow r_{N+1}$, cf. (1.1.7)*. If a consideration is valid both in the general and classical case we write as subscript* "(cl)" *and we use a notation $S^\mu_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$. The space $S^\mu_{(\mathrm{cl})}(\mathbb{R}^n)$ of all elements $a(\xi)$ with constant coefficient (i.e., independent of x) is closed in $S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$. Then we have*

$$S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n) = C^\infty(\Omega, S^\mu_{(\mathrm{cl})}(\mathbb{R}^n)). \tag{1.1.10}$$

Let $\mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$ be the subspace of all u such that

$$\sup_{x\in\mathbb{R}^n} |x^\alpha D_x^\beta u(x)| \quad \text{for all} \quad \alpha, \beta \in \mathbb{N}^n$$

are finite. $\mathcal{S}(\mathbb{R}^n)$ is called the Schwartz space of rapidly decreasing functions. Setting $S^{-\infty}(\Omega\times\mathbb{R}^n) := \bigcap_{\mu\in\mathbb{R}} S^\mu(\Omega \times \mathbb{R}^n)$, we have

$$S^{-\infty}(\Omega \times \mathbb{R}^n) = C^\infty(\Omega, \mathcal{S}(\mathbb{R}^n)). \tag{1.1.11}$$

Theorem 1.1.3. *Let $a_j(x,\xi) \in S^{\mu_j}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$, $j \in \mathbb{N}$, be an arbitrary sequence, where $\mu_j \to -\infty$ as $j \to \infty$, and $\mu_j := \mu - j$ in the classical case. Then there is a symbol $a(x,\xi) \in S^\mu_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$ with $\mu = \max_{j\in\mathbb{N}}\{\mu_j\}$ such that for every M there is an $N(M)$ such that for all $N \geq N(M)$*

$$a(x,\xi) - \sum_{j=0}^{N} a_j(x,\xi) \in S^{\mu-M}(\Omega \times \mathbb{R}^n). \tag{1.1.12}$$

The element $a(x,\xi) \in S^\mu_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$ is uniquely determined by this property mod $S^{-\infty}(\Omega\times\mathbb{R}^n)$.

We call any such $a(x,\xi)$ asymptotic sum of the $a_j(x,\xi)$, $j \in \mathbb{N}$, and write

$$a(x,\xi) \sim \sum_{j=0}^{\infty} a_j(x,\xi).$$

Remark 1.1.4. *We can construct an asymptotic sum in the sense of Theorem* 1.1.3 *as a convergent sum in* $S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$

$$a(x,\xi) = \sum_{j=0}^{\infty} \chi\left(\frac{\xi}{c_j}\right) a_j(x,\xi),$$

for some excision function χ *and constant* $c_j > 0$, $c_j \to \infty$ *sufficiently fast as* $j \to \infty$.

Definition 1.1.5. *Let* $\Omega \subseteq \mathbb{R}^n$ *be open and* $\mu \in \mathbb{R}$. *An operator of the form*

$$\operatorname{Op}(a)u(x) = \iint e^{i(x-x')\xi} a(x,x',\xi)u(x')\,dx'\,đ\xi, \tag{1.1.13}$$

for $đ\xi := (2\pi)^{-n} d\xi$, *with symbol* (*or amplitude function*) $a(x,x',\xi) \in S^{\mu}(\Omega \times \Omega \times \mathbb{R}^n)$ *is called a pseudo-differential operator on* Ω *of order* μ. *Instead of* $\operatorname{Op}(a)$ *we often write* $\operatorname{Op}_x(a)$ *in order to indicate the special variable in the respective oscillatory integral. We set*

$$L^{\mu}_{(\mathrm{cl})}(\Omega) = \{\operatorname{Op}(a) : \quad a(x,x',\xi) \in S^{\mu}_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^n)\}, \tag{1.1.14}$$

here, similarly as before, we write $L^{\mu}_{(\mathrm{cl})}(\Omega)$ *when a consideration on pseudo-differential operators is valid both in the general and classical case. The elements of* (1.1.14) *are called classical pseudo-differential operators. Let* $L^{-\infty}(\Omega) := \bigcap_{\mu \in \mathbb{R}} L^{\mu}(\Omega)$, *consisting of smoothing operators, i.e., any such operator* C *can be expressed by a kernel* $c(x,x') \in C^{\infty}(\Omega \times \Omega)$ *such that,*

$$Cu(x) = \int c(x,x')u(x')\,dx'.$$

Occasionally, if $A \in L^{\mu}(\Omega)$ is written as $A = \operatorname{Op}(a)$ for $a(x,\xi) \in S^{\mu}(\Omega_x \times \mathbb{R}^n)$ ($a(x',\xi) \in S^{\mu}(\Omega_{x'} \times \mathbb{R}^n)$) we call a a left symbol, also denoted by

$$a_{\mathrm{L}}(x,\xi) \quad \text{and} \quad a_{\mathrm{R}}(x',\xi), \tag{1.1.15}$$

respectively. For $A = \operatorname{Op}(a)$, $a(x,x',\xi) \in S^{\mu}(\Omega \times \Omega \times \mathbb{R}^n)$, we also call $a(x,x',\xi)$ a double symbol. Instead of (1.1.13) for $A = \operatorname{Op}(a_{\mathrm{L}})$ we may also write

$$Au(x) = \int e^{ix\xi} a_{\mathrm{L}}(x,\xi)\hat{u}(\xi)\,đ\xi. \tag{1.1.16}$$

Moreover, for $A = \operatorname{Op}(a_{\mathrm{R}})$ we have

$$Au(x) = \int e^{ix\xi}\left\{\int e^{-ix'\xi} a_{\mathrm{R}}(x',\xi)u(x')dx'\right\} đ\xi. \tag{1.1.17}$$

Let us formulate some mapping properties of pseudo-differential operators. First it is an elementary fact that an $A \in L^{\mu}(\Omega)$ induces a continuous operator

$$A : C_0^{\infty}(\Omega) \longrightarrow C^{\infty}(\Omega).$$

Thus A has a distributional kernel $K_A(x,x') \in \mathcal{D}'(\Omega\times\Omega)$ which is uniquely determined on test functions of the form $\varphi(x)\varphi'(x')$ for $\varphi,\varphi' \in C_0^\infty(\Omega)$ by

$$\langle K_A(x,x'), \varphi(x)\varphi'(x')\rangle = \int_\Omega (A\varphi)(x)\varphi'(x')\,dx. \tag{1.1.18}$$

We call A is properly supported if $\operatorname{supp} K_A$ has proper support, i.e., for arbitrary $M \Subset \Omega$, $M' \Subset \Omega$, the intersections $\operatorname{supp} K_A \cap (M\times\Omega)$, $\operatorname{supp} K_A \cap (\Omega\times M')$ are compact in $\Omega\times\Omega$. By virtue of

$$\operatorname{sing\,supp} K_A \subseteq \operatorname{diag}(\Omega\times\Omega), \tag{1.1.19}$$

for $\operatorname{diag} := \{(x,x) : x\in\Omega\}$. Every $A \in L^\mu(\Omega)$ can be written as a sum

$$A = A_0 + C, \tag{1.1.20}$$

where $A_0 \in L^\mu(\Omega)$ is properly supported and $C \in L^{-\infty}(\Omega)$. In fact, if $\omega(x,x') \in C^\infty(\Omega\times\Omega)$ an arbitrary function such that $\omega \equiv 1$ in an open neighbourhood of $\operatorname{diag}(\Omega\times\Omega)$ and such that both $(\Omega\times M)\bigcap \operatorname{supp}\omega$ and $(\Omega\times M')\bigcap\operatorname{supp}\omega$ are compact for arbitrary $M, M' \Subset \Omega$. Then, for $A = \operatorname{Op}(a)$ with $a(x,x',\xi) \in S^\mu(\Omega\times\Omega\times\mathbb{R}^n)$, we can set $A_0 = \operatorname{Op}(\omega a)$ which is properly supported, and $C = \operatorname{Op}((1-\omega)a)$. It follows that

$$L^\mu_{(\mathrm{cl})}(\Omega) = \{\operatorname{Op}(a) + C : \quad a(x,\xi) \in S^\mu_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n), C \in L^{-\infty}(\Omega)\},$$

is an equivalent to Definition 1.1.5. Note that when $A \in L^\mu(\Omega)$ is properly supported the operator induces continuous maps

$$A : C_0^\infty(\Omega) \to C_0^\infty(\Omega), \quad C^\infty(\Omega) \to C^\infty(\Omega).$$

Remark 1.1.6. *Let $A \in L^\mu_{(\mathrm{cl})}(\Omega)$ be properly supported. Then there is a unique $a(x,\xi) \in S^\mu_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$ such that*

$$A = \operatorname{Op}(a).$$

In fact, the Fourier inversion formula

$$u(x) = \int e^{ix\xi}\,\hat{u}(\xi)\,đ\xi$$

can be composed from the left with A and it follows that

$$\begin{aligned} Au(x) &= \int Ae^{ix\xi}\,\hat{u}(\xi)\,đ\xi \\ &= \int e^{ix\xi}(e^{-ix\xi}Ae^{ix\xi})\hat{u}(\xi)\,đ\xi = \operatorname{Op}(a)u \end{aligned} \tag{1.1.21}$$

for $a(x,\xi) := e^{-ix\xi}(Ae^{ix\xi})$. Here for convenience, we do not explicitly show that $a(x,\xi) \in S^\mu_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$. Details may be found in Šhubin's book [69]. Because of (1.1.20) we can write

$$L^\mu_{(\mathrm{cl})}(\Omega) = L^\mu_{(\mathrm{cl})}(\Omega)_N + L^{-\infty}(\Omega) \tag{1.1.22}$$

where subscript N indicates the subspace of all properly operators A such that K_A is supported by a fixed compact set $N \subset \Omega\times\Omega$. Then (1.1.21) induces an isomorphism of

$$L^\mu_{(\mathrm{cl})}(\Omega)_N \longrightarrow S^\mu_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)_N, \tag{1.1.23}$$

where the symbol space on the right-hand side is Fréchet, and hence also $L^{\mu}_{(\mathrm{cl})}(\Omega)_N$. Since

$$L^{-\infty}(\Omega) \cong C^{\infty}(\Omega \times \Omega)$$

is Fréchet as well also (1.1.22) is Fréchet.

Theorem 1.1.7. (i) *Let $A \in L^{\mu}_{(\mathrm{cl})}(\Omega)$, $B \in L^{\nu}_{(\mathrm{cl})}(\Omega)$, and assume that A or B is properly supported. Writing $A = \mathrm{Op}(a)$, $B = \mathrm{Op}(b), \mathrm{mod}\, L^{-\infty}(\Omega)$ for a corresponding left symbols a, b, then for the composition we have $AB \in L^{\mu+\nu}_{(\mathrm{cl})}(\Omega)$, and*

$$AB = \mathrm{Op}(c) \,\mathrm{mod}\, L^{-\infty}(\Omega),$$

for a symbol $c(x,\xi) \in S^{\mu+\nu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$, where

$$c(x,\xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} a(x,\xi)\, D_x^{\alpha} b(x,\xi). \tag{1.1.24}$$

(ii) *Let $A \in L^{\mu}_{(\mathrm{cl})}(\Omega)$ and A^* be its formal adjoint of A defined by*

$$(u, A^* v) = (Au, v) \tag{1.1.25}$$

for all $u, v \in C_0^{\infty}(\Omega)$, with $(.\,,.)$ being the sesquilinear $L^2(\Omega)$-scalar product. Then we have $A^* \in L^{\mu}_{(\mathrm{cl})}(\Omega)$. Writing $A = \mathrm{Op}(a) \,\mathrm{mod}\, L^{-\infty}(\Omega)$ for a left symbol a, then we have $A^* = \mathrm{Op}(a^*)$, where

$$a^*(x,\xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} D_x^{\alpha} \bar{a}(x,\xi). \tag{1.1.26}$$

Remark 1.1.8. *The operation* (1.1.24)*, i.e., the correspondence*

$$(a, b) \to a\#b := c$$

between symbols a, b is called the Leibniz product between a and b. It is well-defined $\mathrm{mod}\, S^{-\infty}(\Omega \times \mathbb{R}^n)$.

Observe that the Leibniz multiplication is associative, i.e.,

$$(a_1 \# a_2) \# a_3 = a_1 \# (a_2 \# a_3)$$

$\mathrm{mod}\, S^{-\infty}(\Omega \times \mathbb{R}^n)$. Let us briefly recall the definition of Sobolev spaces, first in $\mathbb{R}^n$. The space $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, is defined as the completion of $C_0^{\infty}(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{H^s(\mathbb{R}^n)} := \left\{ \int \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 \,đ\xi \right\}^{\frac{1}{2}}, \tag{1.1.27}$$

with $\hat{u}(\xi) := (Fu)(\xi)$ being the Fourier transform. Equivalently we can define $H^s(\mathbb{R}^n)$ as the set of those $u \in \mathcal{S}'(\mathbb{R}^n)$ and that $Fu(\xi)$ locally integrable in $\mathbb{R}^n$ and the norm (1.1.27) is finite. We usually identify $H^0(\mathbb{R}^n)$ with $L^2(\mathbb{R}^n)$. The scalar product

$$(u, v) = \int u(x) \bar{v}(x)\, dx$$

of $L^2(\mathbb{R}^n)$ gives rise to a pairing

$$(.\,,.) : C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n) \longrightarrow \mathbb{C} \tag{1.1.28}$$

which extends to a non-degenerate sesquilinear pairing

$$H^s(\mathbb{R}^n) \times H^{-s}(\mathbb{R}^n) \longrightarrow \mathbb{C} \tag{1.1.29}$$

for every $s \in \mathbb{R}$. Note that the space $H^s(\mathbb{R}^n)$ for $s \in \mathbb{N}$ can be characterized as the set of all $u \in L^2(\mathbb{R}^n)$ such that

$$D_x^\alpha u \in L^2(\mathbb{R}^n)$$

for all $|\alpha| \leq s$. Moreover, on an open set $\Omega \subseteq \mathbb{R}^n$ we define $H^s_{\mathrm{comp}}(\Omega)$ to be the subset of all $u \in H^s(\mathbb{R}^n)$ such that $\operatorname{supp} u$ is compact in Ω. Let $H^s_{\mathrm{loc}}(\Omega)$ be the subset of all $u \in \mathcal{D}'(\Omega)$ such that $\varphi u \in H^s_{\mathrm{comp}}(\Omega)$ for every $\varphi \in C_0^\infty(\Omega)$. We often write

$$H^{-\infty}_{\mathrm{loc}}(\Omega) := \bigcup_{s\in\mathbb{R}} H^s_{\mathrm{loc}}(\Omega), \quad H^{\infty}_{\mathrm{loc}}(\Omega) := \bigcap_{s\in\mathbb{R}} H^s_{\mathrm{loc}}(\Omega).$$

Observe that

$$H^{\infty}_{\mathrm{loc}}(\Omega) = C^\infty(\Omega), \quad H^{-\infty}_{\mathrm{loc}}(\Omega) = \mathcal{D}'(\Omega).$$

Theorem 1.1.9. *Every $A \in L^\mu(\Omega)$ induces continuous operators*

$$A : H^s_{\mathrm{comp}}(\Omega) \longrightarrow H^{s-\mu}_{\mathrm{loc}}(\Omega), \tag{1.1.30}$$

for all $s \in \mathbb{R}$. Moreover, if A is properly supported then A induces continuous operators

$$A : H^s_{\mathrm{comp}}(\Omega) \longrightarrow H^{s-\mu}_{\mathrm{comp}}(\Omega), \quad H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^{s-\mu}_{\mathrm{loc}}(\Omega), \tag{1.1.31}$$

for all $s \in \mathbb{R}$.

More generally we can say that any A induces an operator

$$A : \mathcal{E}'(\Omega) \longrightarrow \mathcal{D}'(\Omega)$$

and in the properly supported case

$$A : \mathcal{E}'(\Omega) \longrightarrow \mathcal{E}'(\Omega), \quad \mathcal{D}'(\Omega) \longrightarrow \mathcal{D}'(\Omega). \tag{1.1.32}$$

Remark 1.1.10. *For $\Omega = \mathbb{R}^n$ we have different specific continuity results in Sobolev spaces in $\mathbb{R}^n$. In the simplest case, if $A = \mathrm{Op}(a)$ has a symbol $a(\xi) \in S^\mu(\mathbb{R}^n)$ with constant coefficients, then we have continuity*

$$A : H^s(\mathbb{R}^n) \longrightarrow H^{s-\mu}(\mathbb{R}^n) \tag{1.1.33}$$

for all $s \in \mathbb{R}$. Another result in this direction is that (1.1.33) *is continuous when $a(x, \xi)$ is independent of x for large $|x|$.*

The definition of pseudo-differential operators is based on the chosen coordinates $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$ and the Fourier transform with covariables $\xi = (\xi_1, \dots, \xi_n)$. However there is invariance under substituting a diffeomorphism

$$\chi : \Omega \longrightarrow \tilde{\Omega}$$

from Ω to another open set $\tilde{\Omega}$. First note that the function pull backs

$$\chi^* : C_0^\infty(\tilde{\Omega}) \longrightarrow C_0^\infty(\Omega)$$

or

$$\chi^* : C^\infty(\tilde{\Omega}) \longrightarrow C^\infty(\Omega)$$

define isomorphisms between the respective spaces. This allows us to form

$$\tilde{A} := \chi_* A := (\chi^{-1})^* A \, \chi^* : C_0^\infty(\tilde{\Omega}) \longrightarrow C^\infty(\tilde{\Omega}),$$

called the operator push forward under χ.

Theorem 1.1.11. *The operator push forward under χ induces isomorphisms*

$$\chi_* : L^\mu_{(\mathrm{cl})}(\Omega) \longrightarrow L^\mu_{(\mathrm{cl})}(\tilde{\Omega}),$$

for every $\mu \in \mathbb{R}$. If A is written as $A = \mathrm{Op}(a) + C$ for a symbol $a(x, \xi) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ and some $C \in L^{-\infty}(\Omega)$, then we have

$$\chi_* A = \mathrm{Op}(\tilde{a}) + \tilde{C}$$

for some $\tilde{a}(\tilde{x}, \tilde{\xi}) \in S^\mu_{(\mathrm{cl})}(\tilde{\Omega} \times \mathbb{R}^n)$ and $\tilde{C} \in L^{-\infty}(\tilde{\Omega})$, and there is an asymptotic expansion

$$\tilde{a}(\tilde{x}, \tilde{\xi})|_{\tilde{x}=\chi(x)} \sim \sum_\alpha \frac{1}{\alpha!} (\partial_\xi^\alpha a)(x, {}^{\mathrm{t}}(d\chi(x))\tilde{\xi}) \Phi_\alpha(x, \tilde{\xi}), \tag{1.1.34}$$

where $d\chi(x)$ is the Jacobian matrix of χ at x and $\Phi_\alpha(x, \tilde{\xi})$ is a polynomial in $\tilde{\xi}$ of degree $\leq$ *$|\alpha|/2$ of the form*

$$\Phi_\alpha(x, \tilde{\xi}) = D_z^\alpha \mathrm{e}^{i\delta(x,z)\tilde{\xi}}|_{x=z} \tag{1.1.35}$$

for

$$\delta(x, z) := \chi(z) - \chi(x) - d\chi(x)(z - x). \tag{1.1.36}$$

Remark 1.1.12. *For*

$$\tilde{x} = \chi(x), \quad \tilde{\xi} = {}^{\mathrm{t}} d\chi(x)^{-1} \xi$$

we have the following relation between the symbols $a(x, \xi)$ and $\tilde{a}(\tilde{x}, \tilde{\xi})$ in Theorem 1.4, *namely,*

$$\tilde{a}(\chi(x), {}^{\mathrm{t}} d\chi(x)^{-1} \xi) = a(x, \xi) \tag{1.1.37}$$

mod $S^{\mu-1}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$. *In the classical case for the principal homogeneous components $\tilde{a}_{(\mu)}(\tilde{x}, \tilde{\xi})$ and $a_{(\mu)}(x, \xi)$ we have*

$$\tilde{a}_{(\mu)}(\chi(x), {}^{\mathrm{t}} d\chi(x)^{-1} \xi) = a_{(\mu)}(x, \xi). \tag{1.1.38}$$

Relation (1.1.37) can be interpreted as a symbol push forward

$$S^\mu_{(\mathrm{cl})}(\Omega_x \times \mathbb{R}^n_\xi) \longrightarrow S^\mu_{(\mathrm{cl})}(\tilde{\Omega}_{\tilde{x}} \times \mathbb{R}^n_{\tilde{\xi}}). \tag{1.1.39}$$

In fact, (1.1.37) is a consequence of relation (1.1.34) and $\Phi_0(x, \tilde{\xi}) = 1$. Then (1.1.38) for classical symbols follows from (1.1.37).

Remark 1.1.13. *The correspondence* (1.1.34) $a \to \tilde{a}$ *will also be referred to as the symbol push forward under the diffeomorphism* χ*, written* $\tilde{a} = \chi_* a$ *which uniquely determines* $\tilde{a} \bmod S^{-\infty}(\tilde{\Omega} \times \mathbb{R}^n)$*. Then, in particular, we have*

$$\chi_*(a \# b) = \chi_* a \, \# \, \chi_* b.$$

This is a consequence of relation

$$\chi_*(AB) = \chi_* A \, \chi_* B$$

and Theorem 1.1.7.

Definition 1.1.14. *A pseudo-differential operator* $A \in L^\mu(\Omega)$ *represented as*

$$A = \mathrm{Op}(a) + C \tag{1.1.40}$$

for a left symbol $a(x,\xi) \in S^\mu(\Omega \times \mathbb{R}^n)$ *and some* $C \in L^{-\infty}(\Omega)$ *is called elliptic if there is a* $p(x,\xi) \in S^{-\mu}(\Omega \times \mathbb{R}^n)$ *such that*

$$a(x,\xi)p(x,\xi) = 1 \bmod S^{-1}(\Omega \times \mathbb{R}^n). \tag{1.1.41}$$

Clearly this definition is not affected by the choice of the decomposition (1.1.40). In the case $A \in L^\mu_{\mathrm{cl}}(\Omega)$ the homogeneous principal part $a_{(\mu)}(x,\xi)$ remains unchanged when we change (1.1.40) and A is elliptic if and only if

$$a_{(\mu)}(x,\xi) \neq 0 \quad \text{for all} \quad (x,\xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\}).$$

Definition 1.1.15. *Let* $A \in L^\mu_{(\mathrm{cl})}(\Omega)$*,* $P \in L^{-\mu}_{(\mathrm{cl})}(\Omega)$*, and assume that* A *or* P *is properly supported. Then* P *is called a parametrix of* A *if*

$$PA = 1 - C_{\mathrm{L}}, \quad AP = 1 - C_{\mathrm{R}}, \tag{1.1.42}$$

for certain C_{L}*,* $C_{\mathrm{R}} \in L^{-\infty}(\Omega)$*. Here* 1 *indicates the identity operator.*

Theorem 1.1.16. *An elliptic* $A \in L^\mu_{(\mathrm{cl})}(\Omega)$ *has a properly supported parametrix* $P \in L^{-\mu}_{(\mathrm{cl})}(\Omega)$*.*

Proof. By definition we have a $p(x,\xi) \in S^{-\mu}(\Omega \times \mathbb{R}^n)$ such that (1.1.41) holds. At the same time we have

$$p(x,\xi)\, a(x,\xi) - 1 \in S^{-1}(\Omega \times \mathbb{R}^n).$$

This implies

$$p(x,\xi) \# a(x,\xi) - 1 \in S^{-1}(\Omega \times \mathbb{R}^n),$$

using relation (1.1.24). Here and in the following we interpret the Leibniz product in terms of any fixed choice of the corresponding asymptotic sum, i.e., we ignore remainders of order $-\infty$; those do not affect the conclusions. Thus, writing

$$c(x,\xi) := p(x,\xi) \# a(x,\xi) - 1$$

we have

$$p(x,\xi) \# a(x,\xi) = 1 - c(x,\xi). \tag{1.1.43}$$

This allows us to Leibniz invert the right-hand side, we find a $d \in S^{-1}(\Omega \times \mathbb{R}^n)$ such that

$$(1-d)\#(1-c) = 1 \operatorname{mod} S^{-\infty}(\Omega \times \mathbb{R}^n).$$

In fact, it suffices to form the asymptotic sum

$$d := \sum_{j=1}^{\infty} (-1)^j c^{\#j}$$

where $c^{\#j} := c\#c\#\ldots\#c$ (j *times*). Multiplying (1.1.43) by $1-d$ we obtain

$$((1-d)\#p)\#a = 1.$$

The method which is used here is also referred to as a formal Neumann series argument. For convenience we drop remainders of order $-\infty$. Thus, since

$$(1-d)\#p := p_{\mathrm{L}} \in S^{-\mu}(\Omega \times \mathbb{R}^n)$$

we constructed a p_{L} such that $p_{\mathrm{L}}\#a = 1$. Set $P_{\mathrm{L}} := \operatorname{Op}(p_{\mathrm{L}})$. Let us now assume that A is properly supported. Then $a(x,\xi)$ may be chosen as in Remark 1.1.6. Thus we can form the composition

$$P_{\mathrm{L}}A = \operatorname{Op}(p_{\mathrm{L}}\#a) = 1 \operatorname{mod} L^{-\infty}(\Omega), \tag{1.1.44}$$

cf. Theorem 1.1.7. By interchanging factors we can also construct $p_{\mathrm{R}} \in S^{-\mu}(\Omega \times \mathbb{R}^n)$ and $P_{\mathrm{R}} = \operatorname{Op}(p_{\mathrm{R}})$ with similar properties. A standard algebraic argument then gives us

$$p_{\mathrm{L}} = p_{\mathrm{R}} \operatorname{mod} S^{-\infty}(\Omega \times \mathbb{R}^n)$$

and we may set $P = P_{\mathrm{L}}$.

$\square$

Corollary 1.1.17. *Let $A \in L^{\mu}_{(\mathrm{cl})}(\Omega)$ be elliptic and consider the equation*

$$Au = f \tag{1.1.45}$$

for $u \in H^{-\infty}_{\mathrm{comp}}(\Omega)$, $f \in H^{s-\mu}_{\mathrm{loc}}(\Omega)$, $s \in \mathbb{R}$ fixed. Then it follows that $u \in H^{s}_{\mathrm{comp}}(\Omega)$.

In fact, Theorem 1.1.16 gives us a properly supported parametrix of A. Then we can multiply (1.1.45) from the left by P and obtain

$$PAu = Pf \in H^{s}_{\mathrm{loc}}(\Omega),$$

using (1.1.31). Thus, from (1.4.11) we obtain

$$PAu = u - C_{\mathrm{L}}u \in H^{s}_{\mathrm{loc}}(\Omega),$$

and hence $u = C_{\mathrm{L}}u + Pf \in H^{s}_{\mathrm{comp}}(\Omega)$ because of $C_{\mathrm{L}}u \in H^{\infty}_{\mathrm{loc}}(\Omega)$.
The effect that solutions u to elliptic equations (1.1.45) are of a Sobolev regularity shifted by the order μ of A is also-called elliptic regularity.

1.2 Pseudo-differential operators on C^∞ manifolds

It will be essential also to formulate operators and distribution spaces on a smooth manifold M of dimension n. For convenience we first assume that M is closed and compact, with a fixed Riemannian metric. Let $L^{-\infty}(M)$ be the space of smoothing operators C on M, defined by

$$Cu(x) = \int_M c(x,x')u(x')\,dx' \tag{1.2.1}$$

for a kernel $c(x,x') \in C^\infty(M \times M)$, with dx' being determined by the Riemannian metric. Moreover, choose an open covering $(U_1,\dots,U_N)$ of M by coordinate neighbourhoods U_j, fix a subordinate partition of unity $(\varphi_1,\dots,\varphi_N)$. Moreover, choose functions $\varphi'_j \in C_0^\infty(U_j)$ such that $\varphi_j \prec \varphi'_j$ for all j (where $f \prec g$ for functions f,g means that $g \equiv 1$ on $\operatorname{supp} f$). Let $\chi_j : U_j \longrightarrow \Omega_j$, be a system of charts. For $B \in L^\mu_{(\mathrm{cl})}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ open, and a chart

$$\chi : U \longrightarrow \Omega$$

on M we define

$$\chi_*^{-1}B = A,$$

the operator push forward of B under the deffeomorphism

$$\chi^{-1} : \Omega \longrightarrow U,$$

by setting

$$\chi_*^{-1}B := \chi^* B(\chi^{-1})^*.$$

We then define

$$L^\mu(U) := \{A := (\chi^{-1})_* B \,:\, B \in L^\mu(\Omega)\}. \tag{1.2.2}$$

This gives us a space of continuous operators

$$A : C_0^\infty(U) \longrightarrow C^\infty(U).$$

It is obvious that the push forward yields an isomorphism

$$\chi_*^{-1} : L^\mu(\Omega) \longrightarrow L^\mu(U).$$

Then $L^\mu_{(\mathrm{cl})}(M)$, $\mu \in \mathbb{R}$, is defined as the set of all operators

$$A : C_0^\infty(M) \longrightarrow C^\infty(M)$$

of the form

$$A = \sum_{j=1}^N \varphi_j A_j \varphi'_j + C \tag{1.2.3}$$

for arbitrary $A_j \in L^\mu_{(\mathrm{cl})}(U_j)$ and $C \in L^{-\infty}(M)$. Note that we have

$$A_j = \sum_{j=1}^N \varphi_j A|_{U_j} \varphi'_j \mod L^{-\infty}(M). \tag{1.2.4}$$

Let us set

$$L^\mu_{(\mathrm{cl})}(M) = \left\{ A := \sum_{j=1}^N \varphi_j A_j \varphi'_j + C : A_j \in L^\mu_{(\mathrm{cl})}(U_j), C \in L^{-\infty}(M) \right\} \tag{1.2.5}$$

Remark 1.2.1. *Note that latter notation admits to choose $A_j \in L^{\mu}_{(\mathrm{cl})}(U_j)$ in a quite arbitrary manner. However, we may replace A_j by $\tilde{A}_j \in L^{\mu}_{(\mathrm{cl})}(U_j)$ such that the space $L^{\mu}_{(\mathrm{cl})}(M)$ remains unchanged but*

$$(\chi_{jk})_* \tilde{A}_j = (\chi_{kj})_* \tilde{A}_k \bmod L^{-\infty}(\Omega_j \cap \Omega_k) \tag{1.2.6}$$

where $(\chi_{jk}) : U_j \cap U_k \longrightarrow \Omega_j \cap \Omega_k$, $(\chi_{kj}) : U_k \cap U_j \longrightarrow \Omega_k \cap \Omega_j$, for any $j, k = 1, \ldots, N$. Thus local complete symbols of A over $\chi_j U_j = \Omega_j$ modulo symbols of order $-\infty$ for all j, which means that we may assume without loss of generality that the operators A_j from the very beginning have such compatibility properties. Then, denoting by $a_j(x, \xi)$ corresponding local left symbols of A_j over Ω_j, (i.e.,$A_j := \mathrm{Op}(a_j)$), the above mentioned symbol push forwards induce transformations

$$a_j|_{\Omega_j \cap \Omega_k} \longrightarrow a_k|_{\Omega_k \cap \Omega_j}$$

for all j, k.

Note that from (1.1.38) it follws that the homogeneous principal symbols of $A_j|_{U_j}$ behave like invariantly defined functions on $T^*M \setminus 0$ (the cotangent bundle minus zero section). More precisely, computing the principal symbols of

$$(\chi_j)_* A|_{U_j} \quad \text{and} \quad (\chi_k)_* A|_{U_k}$$

in the corresponding local coordinates of $\Omega_j = \chi_j U_j$ and $\Omega_k = \chi_k U_k$, respectively, then the rule (1.1.38) tells us that the principal symbols transform as functions on $T^*M \setminus 0$ under the transition diffeomorphisms

$$\kappa_{kj} : \Omega_{jk} \longrightarrow \Omega_{kj}$$

for

$$\Omega_{jk} = \chi_j(U_j \cap U_k), \quad \Omega_{kj} = \chi_k(U_k \cap U_j).$$

In other words writing

$$\chi_{jk} := \chi_j|_{U_j \cap U_k} : U_j \cap U_k \longrightarrow \Omega_{jk}, \quad \chi_{kj} := \chi_k|_{U_j \cap U_k} : U_j \cap U_k \longrightarrow \Omega_{kj}$$

it follows that

$$\kappa_{kj} = \chi_{kj} \circ \chi_{jk}^{-1} : \Omega_{jk} \longrightarrow \Omega_{kj}$$

like

$$(x, \xi) \longrightarrow (\kappa_{kj}(x), {}^{\mathrm{t}}d\kappa_{kj}^{-1}(x)\xi).$$

Which just corresponds to transition diffeomorphism

$$\Omega_{jk} \times \mathbb{R}^n \longrightarrow \Omega_{kj} \times \mathbb{R}^n$$

of the cotangent bundle. Then for $A \in L^{\mu}_{\mathrm{cl}}(M)$ the local homogeneous principal symbols $a_{(\mu)}(x, \xi)$ of $A|_{U_j}$ as functions of $(x, \xi) \in \Omega_j \times (\mathbb{R}^n \setminus \{0\})$ are invariantly defined as functions on $T^*M \setminus 0$, i.e., we have

$$a_{(\mu)}(x, \xi) = a_{(\mu)}(\tilde{x}, \tilde{\xi}) \tag{1.2.7}$$

as soon as

$$\begin{aligned} \tilde{x} &= \kappa_{kj}(x), \\ \tilde{\xi} &= {}^{\mathrm{t}}d\kappa_{kj}{}^{-1}(x)\xi. \end{aligned}$$

Definition 1.2.2. *For $s \in \mathbb{R}^n$ we define the Sobolev space $H^s(M)$ by*

$$H^s(M) = \Big\{ \sum_{j=1}^{N} \varphi_j u_j \ : u_j \circ \chi_j^{-1} \in H^s_{\mathrm{loc}}(\Omega_j),\ j = 1, \ldots, N \Big\}.$$

We can easily define Hilbert space scalar product in $H^s(M)$ where we identify $H^0(M)$ with $L^2(M)$, with the standard scalar product

$$(u, v)_{L^2(M)} = \int_M u(x)\bar{v}(x)\, dx. \tag{1.2.8}$$

Then (1.2.8) induces a non-degenerate sesquilinear pairing

$$(.\,,.) : H^s(M) \times H^{-s}(M) \longrightarrow \mathbb{C} \tag{1.2.9}$$

for every $s \in \mathbb{R}$. For any A which is continuous as an operator

$$A : H^s(M) \longrightarrow H^{s-\mu}(M)$$

for all $s \in \mathbb{R}$ we find an

$$A^* : H^s(M) \longrightarrow H^{s-\mu}(M),$$

called the formal adjoint of A which is uniquely determined by the relation

$$(Au, v) = (u, A^* v)$$

for all $u, v \in C^\infty(M)$.

Theorem 1.2.3. *An $A \in L^\mu(M)$ induces continuous operators*

$$A : H^s(M) \longrightarrow H^{s-\mu}(M)$$

for all $s \in \mathbb{R}$.

Theorem 1.2.4. (i) *$A \in L^\mu_{(\mathrm{cl})}(M)$, $B \in L^\nu_{(\mathrm{cl})}(M)$ implies $AB \in L^{\mu+\nu}_{(\mathrm{cl})}(M)$.*

(ii) *$A \in L^\mu_{(\mathrm{cl})}(M)$ implies $A^* \in L^\mu_{(\mathrm{cl})}(M)$.*

Remark 1.2.5. *By virtue of the rule* (1.2.2) *of symbol push forward it follows that ellipticity of an operator is an invariant property, i.e., globally defined on the manifold M.*

In other words the following definition makes sense.

Definition 1.2.6. *An $A \in L^\mu_{(\mathrm{cl})}(M)$ is called elliptic if $(\chi_j)_* A|_{U_j} \in L^\mu_{(\mathrm{cl})}(\Omega_j)$ for $\Omega_j := \chi_j U_j$ is elliptic in the sense of Definition* 1.1.14*, for all j.*

Theorem 1.2.7. *An elliptic operator $A \in L^\mu_{(\mathrm{cl})}(M)$ has a parametrix $P \in L^{-\mu}_{(\mathrm{cl})}(M)$, i.e.,*

$$PA = 1 - C_{\mathrm{L}}, \quad AP = 1 - C_{\mathrm{R}} \tag{1.2.10}$$

for some C_{L}, $C_{\mathrm{R}} \in L^{-\infty}(M)$.

Proof. Assuming that the local operators A_j of A in formula (1.2.5) satisfy the compatibility condition of Remark 1.2.1 then it suffices to construct parametrices P_j of $A|_{U_j} \in L^{-\mu}_{(\mathrm{cl})}(U_j)$ by applying Theorem 1.1.16 in corresponding local coordinates and then to set

$$P := \sum_{j=1}^{N} \varphi_j P_j \varphi'.$$

□

Definition 1.2.8. *An operator $A : H \longrightarrow \tilde{H}$ between Hilbert spaces $H, \tilde{H}$ is called a Fredholm operator, if*

$$\ker A := \{u \in H : Au = 0\} \tag{1.2.11}$$

and $\mathrm{coker} A = \tilde{H}/\mathrm{im} A$ *are both of finite dimension. Then*

$$\mathrm{ind}\, A := \dim \ker A - \dim \mathrm{coker}\, A \tag{1.2.12}$$

is called the index of A.

Theorem 1.2.9. *For an operator $A \in L^{\mu}(M)$, the following conditions are equivalent:*

(i) *A is elliptic,*

(ii) *A induces a Fredholm operator*

$$A : H^s(M) \longrightarrow H^{s-\mu}(M) \tag{1.2.13}$$

for some $s = s_0 \in \mathbb{R}$.

The Fredholm property (1.2.13) *of A for an $s = s_0$ entails the Fredholm property for all $s \in \mathbb{R}$.*

Proof. Let us show (i) ⇒ (ii) i.e., the ellipticity of A gives rise to the Fredholm property of (1.2.13). From Theorem 1.2.7 we find a parametrix P of A. Thus the first relation of (1.2.10) shows that ker$A$ is of finite dimension, since $1 - C_{\mathrm{L}}$ is Fredholm in $H^{s_0}(M)$, using that C_{L} as a smoothing operator is compact. In other words, $(1 - C_{\mathrm{L}})u = 0$ entails $PAu = 0$. Using $\ker PA \supseteq \ker A$, and hence $\dim \ker A < \infty$. From the second relation of (1.2.10) we obtain for the formal adjoint

$$(AP)^* = P^* A^* = 1 - C_{\mathrm{R}}^*.$$

Then, similarly as before $\dim \ker A^* < \infty$. Together with $\dim \ker A^* = \dim \mathrm{coker} A$ we conclude that A is a Fredholm operator. This holds for $s = s_0$. However, ker$A$ as well as kerA^* are finite dimensional subspaces of $H^{\infty}(M) = C^{\infty}(M)$ and hence independent of s. Concerning (ii) ⇒ (i) we refer to standard textbook, cf. [48, Page 197 Theorem 7].

□

Theorem 1.2.10. (i) *Let $A \in L^{\mu}_{(\mathrm{cl})}(M)$ be an operator such that* (1.2.13) *is an isomorphism for some $s = s_0 \in \mathbb{R}$. Then A is an isomorphism for all $s \in \mathbb{R}$, and we have $A^{-1} \in L^{-\mu}_{(\mathrm{cl})}(M)$.*

(ii) *For every $\mu \in \mathbb{R}$ there exists an elliptic operator $A \in L^{\mu}_{\mathrm{cl}}(M)$ such that*

$$A : H^s(M) \longrightarrow H^{s-\mu}(M) \tag{1.2.14}$$

is an isomorphism for every $s \in \mathbb{R}$.

1.3 Parameter-dependent pseudo-differential operators

In applications below we need a generalization of $L^\mu_{(\mathrm{cl})}(M)$ to the case of pseudo-differential operators with parameters $\lambda \in \mathbb{R}^l, l \in \mathbb{N}$. The corresponding class $L^\mu_{(\mathrm{cl})}(M;\mathbb{R}^l)$ is defined as follows. First we form

$$L^\mu_{(\mathrm{cl})}(\Omega;\mathbb{R}^l) := \{\mathrm{Op}(a)(\lambda) : a(x,x',\xi,\lambda) \in S^\mu_{(\mathrm{cl})}(\Omega\times\Omega\times\mathbb{R}^{n+l}_{\xi,\lambda})\} \tag{1.3.1}$$

where analougosly as (1.1.13)

$$\mathrm{Op}(a)(\lambda)u(x) = \iint e^{i(x-x')\xi} a(x,x',\xi,\lambda)u(x')\,dx'đ\xi. \tag{1.3.2}$$

For the respective definition on a manifold M we also need parameter-dependent smoothing operators, namely,

$$C(\lambda) \in L^{-\infty}(M;\mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l, L^{-\infty}(M))$$

with $L^{-\infty}(M)$ being identified with $C^\infty(M\times M)$, cf. (1.2.1). In the following definition we employ similar notation as in (1.2.3).

Definition 1.3.1. *We define*

$$L^\mu_{(\mathrm{cl})}(M;\mathbb{R}^l) := \left\{ \sum_{j=1}^N \varphi_j A_j(\lambda)\varphi'_j + C(\lambda) : A_j(\lambda) \in L^\mu_{(\mathrm{cl})}(U_j;\mathbb{R}^l), \right. \\ \left. C(\lambda) \in L^{-\infty}(M;\mathbb{R}^l) \right\} \tag{1.3.3}$$

where

$$L^\mu_{(\mathrm{cl})}(U_j;\mathbb{R}^l) := \{A(\lambda) = (\chi^{-1})_* B(\lambda) : B(\lambda) \in L^\mu_{(\mathrm{cl})}(\Omega;\mathbb{R}^l)\} \tag{1.3.4}$$

Similarly as in pseudo-differential operators without parameter every $A(\lambda) \in L^\mu_{(\mathrm{cl})}(\Omega;\mathbb{R}^l)$ has left and right symbols

$$a_{\mathrm{L}}(x,\xi,\lambda) \quad \text{and} \quad a_{\mathrm{R}}(x',\xi,\lambda), \tag{1.3.5}$$

respectively, cf. (1.1.15). Most of the constructions have a parameter-dependent analogue. We tacitly employ this generalization when we do not recall explicit definitions. In particular, we have parameter-dependent ellipticity of an operator $A(\lambda) \in L^\mu(\Omega;\mathbb{R}^l)$ which means that for $a(x,\xi,\lambda) \in S^\mu(\Omega\times\mathbb{R}^{n+l})$ with $A(\lambda) = \mathrm{Op}(a)(\lambda)$ there is a $p(x,\xi,\lambda) \in S^{-\mu}(\Omega\times\mathbb{R}^{n+l})$ such that

$$a(x,\xi,\lambda)p(x,\xi,\lambda) = 1 \bmod S^{-1}(\Omega\times\mathbb{R}^{n+l}) \tag{1.3.6}$$

For $A(\lambda) \in L^\mu_{\mathrm{cl}}(\Omega;\mathbb{R}^l)$ we have the parameter-dependent homogeneous principal symbol

$$a_{(\mu)} \in S^{(\mu)}(\Omega\times(\mathbb{R}^{n+l}\setminus\{0\})), \tag{1.3.7}$$

and parameter-dependent ellipticity in this case means

$$a_{(\mu)}(x,\xi,\lambda) \neq 0 \quad \text{for all } a(x,\xi,\lambda) \in \Omega\times(\mathbb{R}^{n+l}\setminus\{0\}). \tag{1.3.8}$$

There are then corresponding generalizations of Definition 1.1.15 and Theorem 1.1.16. It will be necessary also to have such constructions on a smooth manifold M. For convenience we consider the case of a closed manifold M. Because of the transformation behaviour of symbols under coordinate diffeomorphisms which is valid in analogous form also in the parameter-dependent case the ellipticity with parameters of an $A(\lambda) \in L^{\mu}_{(\mathrm{cl})}(M; \mathbb{R}^l)$ can be defined for local symbols over any coordinate neighbourhood U on M. To be more precise, looking at an element in (1.3.3), it we can push forward $A(\lambda)|_U$ under a chart

$$\chi : U \longrightarrow \Omega,$$

i.e., form $\chi_* A(\lambda) \in L^{\mu}_{(\mathrm{cl})}(\Omega; \mathbb{R}^l)$, which can be written as $\mathrm{Op}_x(a)(\lambda)$ for an $a(x, \xi, \lambda) \in S^{\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n+l})$. Then, in order to define parameter-dependent ellipticity of $A(\lambda)$ it suffices to apply it locally to a. This make sense because of the coordinate invariance of the condition.

Theorem 1.3.2. *Let M be a closed smooth manifold and $A(\lambda) \in L^{\mu}_{(\mathrm{cl})}(M; \mathbb{R}^l)$ parameter-dependent elliptic. Then*

(i) *$A(\lambda)$ has a parameter-dependent parametrix $P(\lambda) \in L^{-\mu}_{(\mathrm{cl})}(M; \mathbb{R}^l)$ which means*

$$P(\lambda)A(\lambda) - 1,\ A(\lambda)P(\lambda) - 1 \in L^{-\infty}(M; \mathbb{R}^l); \tag{1.3.9}$$

(ii) *For every $s \in \mathbb{R}$ the operators $A(\lambda)$ form a family of Fredholm operators*

$$A(\lambda) : H^s(M) \longrightarrow H^{s-\mu}(M), \tag{1.3.10}$$

and there is a constant $C > 0$ such that (1.3.10) *are isomorphisms whenever $|\lambda| \geq C$.*

Corollary 1.3.3. *For every μ there exists an elliptic operator $R^{\mu} \in L^{\mu}_{\mathrm{cl}}(M)$ which induces isomorphisms*

$$R^{\mu} : H^s(M) \longrightarrow H^{s-\mu}(M) \tag{1.3.11}$$

for all $s \in \mathbb{R}$. Any such R^{μ} will also be called an order reducing operator or a reduction of orders on M.

In fact, we can construct a parameter-dependent elliptic operator $R^{\mu}(\lambda) \in L^{\mu}_{\mathrm{cl}}(M; \mathbb{R}^l_{\lambda})$ for $l \in \mathbb{N} \setminus \{0\}$ by using parameter-dependent elliptic local symbols

$$a(x, \xi, \lambda) := (1 + |\xi|^2 + |\lambda|^2)^{\mu/2}$$

and form an associated operator via (1.3.3). Then using the above-mentioned notation (1.3.8) we have

$$a_{(\mu)}(x, \xi, \lambda) = (|\xi|^2 + |\lambda|^2)^{\mu/2},$$

cf. (1.3.8). Applying now Theorem 1.3.2 (ii) to $R^{\mu}(\lambda)$ the operator $R^{\mu} := R^{\mu}(\lambda^1)$ for λ^1 of sufficiently large absolute value induces isomorphisms (1.3.11) for all s. Note that $L^0_{\mathrm{cl}}(M) = R^{\mu} L^{\mu}_{\mathrm{cl}}(M)$; clearly the latter notation means $\{R^{\mu} A : A \in L^{\mu}_{\mathrm{cl}}(M)\}$.

1.4 Operators with operator-valued symbols

Let us first consider symbols with values in a Fréchet space E with semi-norm system $(\pi_j)_{j\in\mathbb{N}}$ we define the space

$$S^\mu(\Omega\times\mathbb{R}^q, E) \tag{1.4.1}$$

for $\mu\in\mathbb{R}, \Omega\subseteq\mathbb{R}^q$ open, as the set of all $a(y,\eta)\in C^\infty(\Omega\times\mathbb{R}^q, E)$ such that

$$\pi_j(D_y^\alpha D_\eta^\beta a(y,\eta)) \leq c\langle\eta\rangle^{\mu-|\beta|} \tag{1.4.2}$$

for all $(y,\eta)\in K\times\mathbb{R}^q, K\Subset\mathbb{R}^p$, all $\alpha\in\mathbb{N}^p, \beta\in\mathbb{N}^q$, and for all $j\in\mathbb{N}$, for constants $c=c(\alpha,\beta,j,K)>0$. In addition spaces of classical E-valued symbols

$$S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^q, E) \tag{1.4.3}$$

are defined by asymptotic expansions

$$a(y,\eta)\sim\sum_{j=0}^\infty a_{(\mu-j)}(y,\eta) \tag{1.4.4}$$

where $a_{(\nu)}\in C^\infty(\Omega\times(\mathbb{R}^q\setminus\{0\}), E), \nu\in\mathbb{R}$, is asked to satisfyies the homogeneity condition

$$a_{(\nu)}(y,\delta\eta)=\delta^\nu a_{(\nu)}(y,\eta), \delta\in\mathbb{R}_+.$$

We now pass to pseudo-differential operators with operator-valued symbols. Those will be necessary in applications in singular analysis below.

A (separable) Hilbert space H is said to be endowed with a group action

$$\kappa=\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$$

if

$$\kappa_\delta : H\longrightarrow H$$

are isomorphisms, $\kappa_\delta\,\kappa_\nu=\kappa_{\delta\nu}$ for all $\delta,\nu\in\mathbb{R}_+$, and if $\delta\longrightarrow\kappa_\delta h$ represents an element of $C(\mathbb{R}_+, H)$ for every $h\in H$.

Proposition 1.4.1. *If κ is a group action in H, then we have*

$$\|\kappa_\delta\|_{\mathcal{L}(H)}\leq c\max\{\delta,\delta^{-1}\}^M$$

for suitable constant $c>0$, $M>0$.

Example 1.4.2. (i) *For $H:=L^2(\mathbb{R}^n)$ the operators*

$$\kappa_\delta u(x):=\delta^{n/2}u(\delta x),\ \delta\in\mathbb{R}_+, \tag{1.4.5}$$

are unitary for all δ.

(ii) *For $H:=H^s(\mathbb{R}^n)$, $s\in\mathbb{R}$ the operators* (1.4.5) *represents a group action. More generally we can endow $H:=\langle x\rangle^{-g}H^s(\mathbb{R}^n)$ with the group action* (1.4.5).

(iii) *For every H the identity map is an admitted group action. This is often chosen when H is of finite dimension.*

Note that the constant $M = M(\kappa)$ and $c = c(\kappa)$ in Proposition 1.4.1 may depend on the space H, e.g., on s and g in Example 1.4.2 (ii).

It will be necessary also to consider the case of Fréchet spaces E with group action κ. Here we assume that

$$E = \varprojlim_{j\in\mathbb{N}} E^j \tag{1.4.6}$$

is a projective limit of Hilbert spaces E^j continuously embedded in E^0 for all j, where E^0 is endowed with the group action κ and $\kappa|_{E^j}$ is a group action on E^j for every j.

Example 1.4.3. *The Schwartz space*

$$\mathcal{S}(\mathbb{R}^n) = \varprojlim_{k\in\mathbb{N}} \langle x\rangle^{-k} H^k(\mathbb{R}^n),$$

is Fréchet with group action (1.4.5)*, using a generalization of Example* 1.4.2 (ii) *for $g = s = k$.*

In the following consideration, we frequently employ functions on an open set $U \subseteq \mathbb{R}^n$ taking values in a Fréchet space E, endowed with a countable system $(\rho_j)_{j\in\mathbb{N}}$ of semi-norms. In particular, we have the space $C^\infty(U, E)$ defined as the set of all functions

$$u : U \longrightarrow E,$$

such that

$$\xi_{j,K}(u) := \sup_{x\in K} \rho_j(D_x^\alpha)u(x) < \infty \tag{1.4.7}$$

for all $j \in \mathbb{N}, K \Subset U$. Then $C^\infty(U, E)$ is Fréchet with the semi-norm system (1.4.7). Another example is the Schwartz space $\mathcal{S}(\mathbb{R}^n, E)$ defined as the set of all $u : \mathbb{R}^n \longrightarrow E$

$$\sigma_{j,\alpha,\beta}(u) := \sup_{x\in K} \rho_j\big(x^\alpha D_x^\beta u(x)\big) < \infty \tag{1.4.8}$$

for every $j \in \mathbb{N}, \alpha, \beta, K \Subset \mathbb{R}^n \in \mathbb{N}^m$. Then $\mathcal{S}(\mathbb{R}^n, E)$ is Fréchet in the semi-norm system (1.4.8), $j \in \mathbb{N}, \alpha, \beta \in \mathbb{N}^n$. There are other equivalent systems of semi-norms which define $\mathcal{S}(\mathbb{R}^n, E)$. For instance, for $E = \mathbb{C}$ we can write

$$\mathcal{S}(\mathbb{R}^n) := \bigcap_{k\in\mathbb{N}} \langle x\rangle^{-k} H^k(\mathbb{R}^n)$$

and with the semi-norms

$$\sigma_k(u) := \|\langle x\rangle^k u\|_{H^k(\mathbb{R}^n)}, \quad k \in \mathbb{N}.$$

Definition 1.4.4. *Let H and $\tilde{H}$ be Hilbert spaces with group actions κ and $\tilde{\kappa}$, respectively.*

(i) *The space $S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H})$ of symbols $a(y, \eta)$ of order $\mu \in \mathbb{R}$ on an open set $\Omega \subseteq \mathbb{R}^p$ is the set of all $a(y, \eta) \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(H, \tilde{H}))$ such that*

$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D_y^\alpha D_\eta^\beta a(y, \eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\tilde{H})} \le c\,\langle\eta\rangle^{\mu-|\beta|} \tag{1.4.9}$$

for all $(y, \eta) \in K \times \mathbb{R}^q$, $K \Subset \Omega$, $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$, for constants $c = c(\alpha, \beta, K) > 0$. The elements of $S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H})$ are called operator-valued symbols, referring to twisted symbolic estimates (1.4.9).

(ii) *Let* $S^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \tilde{H})$, $\mu \in \mathbb{R}$, *denote the space of all* $a_{(\mu)}(y,\eta) \in C^\infty(\Omega \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(H, \tilde{H}))$ *such that*

$$a_{(\mu)}(y, \delta\eta) = \delta^\mu \tilde{\kappa}_\delta \, a_{(\mu)}(y,\eta) \, \kappa_\delta^{-1} \tag{1.4.10}$$

for all $\delta \in \mathbb{R}_+$, $(y,\eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$. *Relation* (1.4.10) *is also referred to as twisted homogeneity of order* μ.

(iii) *The space* $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \tilde{H})$ *of classical symbols of order* μ *is the set of all* $a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H})$ *such that there are elements* $a_{(\mu-j)}(y,\eta) \in S^{(\mu-j)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \tilde{H})$, $j \in \mathbb{N}$, *such that*

$$r_{N+1} := a(y,\eta) - \chi(\eta) \sum_{j=0}^{N} a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(\Omega \times \mathbb{R}^q; H, \tilde{H}) \tag{1.4.11}$$

for every $N \in \mathbb{N}$ *and any excision function* $\chi(\eta)$.

Remark 1.4.5. *Note that for any excision function* $\chi(\eta)$ *and* $a_{(\mu)}(y,\eta) \in S^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \tilde{H})$ *we have* $\chi(\eta) a_{(\mu)}(y,\eta) \in S^{(\mu)}(\Omega \times \mathbb{R}^q; H, \tilde{H})$.

In fact we have

$$a_{(\mu)}(y, \frac{\eta}{\langle\eta\rangle}) = \langle\eta\rangle^{-\mu} \tilde{\kappa}^{-1}_{\langle\eta\rangle} a_{(\mu)}(y,\eta) \kappa_{\langle\eta\rangle}$$

i.e.,

$$\| \tilde{\kappa}^{-1}_{\langle\eta\rangle} \big(\chi(\eta) a_{(\mu)}(y,\eta)\big) \kappa_{\langle\eta\rangle} \|_{\mathcal{L}(H,\tilde{H})} \leq c \langle\eta\rangle^\mu,$$

i.e., $\chi(\eta) a_{(\mu)}(y,\eta)$ satisfies the first symbolic estimate of (1.4.9). For the derivatives we can argue in a similar manner, using that

$$D^\alpha_y D^\beta_\eta : S^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \tilde{H}) \longrightarrow S^{(\mu-|\beta|)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \tilde{H}).$$

Remark 1.4.6. *Let* $\eta \to [\eta]$ *be any strictly positive function in* $C^\infty(\mathbb{R}^q)$ *such that* $[\eta] = |\eta|$ *for* $|\eta| >$ constant *for a* constant > 0; *then we obtain the same space* $S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H})$ *as in Definition* 1.4.4 (i) *when we replace* $\langle\eta\rangle$ *by* $[\eta]$ *in relation* (1.4.9).

Remark 1.4.7. *For any* $a(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \tilde{H})$ *we can recover the homogeneous components* $a_{(\mu-j)}(y,\eta), j \in \mathbb{N}$, *in a unique way. For* $a_{(\mu)}(y,\eta)$ *we obtain*

$$a_{(\mu)}(y,\eta) = \lim_{\delta\to\infty} \delta^{-\mu} \tilde{\kappa}_\delta^{-1} a(y,\eta) \kappa_\delta^{-\mu}. \tag{1.4.12}$$

Then $a_{(\mu-1)}(y,\eta)$ *can be recovered by applying an analogous conclusion to*

$$a(y,\eta) \to \chi(\eta) a_{(\mu)}(y,\eta)$$

which is of order $\mu - 1$, *etc.*

Let us observe some useful properties of the spaces $S^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \tilde{H})$. First we have a linear operator

$$S^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \tilde{H}) \longrightarrow C^\infty(\Omega \times S^{q-1}, \mathcal{L}(H, \tilde{H})) \tag{1.4.13}$$

defined by

$$a_{(\mu)}(y,\eta) \longrightarrow a_{(\mu)}(y,\frac{\eta}{|\eta|}).$$

Here S^{q-1} is the unit sphere in $\mathbb{R}^q$, and $C^\infty(\Omega \times S^{q-1}, \mathcal{L}(H,\tilde{H}))$ the corresponding Fréchet space of smooth functions

$$\Omega \times S^{q-1} \longrightarrow \mathcal{L}(H,\tilde{H}).$$

The operator (1.4.13) is an isomorphism. Its inverse has the form

$$a_{(\mu)}(y,\frac{\eta}{|\eta|}) \longrightarrow \tilde{\kappa}_{|\eta|}|\eta|^\mu a_{(\mu)}(y,\frac{\eta}{|\eta|})\kappa_{|\eta|}^{-1} = a_{(\mu)}(y,\eta) \tag{1.4.14}$$

because of

$$a_{(\mu)}(y,\frac{\eta}{|\eta|}) = |\eta|^{-\mu}\tilde{\kappa}_{|\eta|^{-1}}a_{(\mu)}(y,\eta)\kappa_{|\eta|^{-1}}^{-1}.$$

The right-hand side of (1.4.14) is refered to as extension of $a_{(\mu)}(y,\frac{\eta}{|\eta|})$ by homogeneity from the unit cosphere S^{q-1} to $\mathbb{R}^q \setminus \{0\}$.

In particular, because of the bijection (1.4.13) the space $S^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}; H, \tilde{H})$ is induced by $C^\infty(\Omega \times S^{q-1}, \mathcal{L}(H,\tilde{H}))$.

Remark 1.4.8. *The space of functions*

$$f_{(\mu)}(\eta) \in S^{(\mu)}(\mathbb{R}^q \setminus \{0\}; H, \tilde{H})$$

(i.e., constant with respect to $y \in \Omega$) is a closed subspace of $S^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \tilde{H})$, and we have

$$S^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \tilde{H}) = C^\infty(\Omega, S^{(\mu)}(\mathbb{R}^q \setminus \{0\}; H, \tilde{H})).$$

Remark 1.4.9. *The operators $D_y^\alpha D_\eta^\beta, \alpha, \beta \in \mathbb{N}^q$, induce continuous operators*

$$D_y^\alpha D_\eta^\beta : S^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \tilde{H}) \longrightarrow S^{(\mu-|\beta|)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \tilde{H})$$

for all $\mu \in \mathbb{R}$.

Remark 1.4.10. (i) *The space $S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H})$ is a Fréchet space with the semi-norm system*

$$a \longrightarrow \sup_{\substack{y \in K \\ \eta \in \mathbb{R}^q}} \langle\eta\rangle^{-\mu+|\beta|} \|\tilde{\kappa}_{\langle\eta\rangle}^{-1}\{D_y^\alpha\, D_\eta^\beta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\tilde{H})}$$

$\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$, $K \Subset \Omega$.

(ii) *The (so-called) homogeneous components $a_{(\mu-j)}(y,\eta)$, $j \in \mathbb{N}$, of classical symbols $a(y,\eta) \in S_{\mathrm{cl}}^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H})$ are uniquely determined by $a(y,\eta)$. Thus*

$$c_j : S_{\mathrm{cl}}^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H}) \longrightarrow S^{(\mu-j)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \tilde{H}), \tag{1.4.15}$$

determined by $c_j := a_{(\mu-j)}$, $j \in \mathbb{N}$ and r_{N+1} in formula (1.4.11) allow us to endow $S_{\mathrm{cl}}^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H})$ with the Fréchet topology with respect to the maps $c_j, j \in \mathbb{N}, r_{N+1}, N \in \mathbb{N}$ and the canonical embedding

$$S_{\mathrm{cl}}^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H}) \hookrightarrow S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H}).$$

(iii) *The space* $S^\mu_{\mathrm{cl}}(\mathbb{R}^q;\, H, \tilde{H})$*, of all* y*-independent elements is closed in* $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \tilde{H})$*. Setting* $S^{-\infty}(\Omega \times \mathbb{R}^q; H, \tilde{H}) = \bigcap_{\mu\in\mathbb{R}} S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H})$ *we have* $S^{-\infty}(\Omega \times \mathbb{R}^q; H, \tilde{H}) = C^\infty(\Omega, \mathcal{S}(\mathbb{R}^q, \mathcal{L}(H, \tilde{H}))$*.*

Similarly as for scalar symbols, if a consideration is valid both in the general and classical case we write "(cl)" as subscript. Let $S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; H, \tilde{H})$ denote the subspace of all $a(y, \eta) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \tilde{H})$ that are independent of y. It can be easily verified that $S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; H, \tilde{H})$ are closed subspace of $S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \tilde{H})$ and we have

$$S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \tilde{H}) = C^\infty(\Omega, S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; H, \tilde{H})). \tag{1.4.16}$$

Because of nuclearity of $C^\infty(\Omega)$ we have

$$S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \tilde{H}) = C^\infty(\Omega)\hat{\otimes}_\pi S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; H, \tilde{H}) \tag{1.4.17}$$

where $\hat{\otimes}_\pi$ denotes the projective tensor product between the respective Fréchet spaces, cf. Theorem 1.7.1.

Remark 1.4.11. *Let* $a(y, \eta) \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(H, \tilde{H}))$ *such that*

$$a(y, \delta\eta) = \delta^\mu\, \tilde{\kappa}_\delta\, a(y, \eta)\, \kappa_\delta^{-1} \tag{1.4.18}$$

for all $|\eta| \geq c$ *for some* $c > 0$ *and* $\delta \geq 1$*, then we have* $a(y, \eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \tilde{H})$*.*

In fact, by virtue of (1.4.18) we have

$$a(y, \eta) = \chi a_{(\mu)}(y, \eta) + \varphi(y, \eta) \tag{1.4.19}$$

for

$$a_{(\mu)}(y, \eta) = \lim_{\delta\to\infty} \delta^{-\mu}\tilde{\kappa}_\delta^{-1} a(y, \delta\eta)\kappa_\delta \in S^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \tilde{H})$$

and some $\varphi(y, \eta) \in C^\infty(\Omega \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(H, \tilde{H}))$ that vanishes for $|\eta| \geq$ constant for a constant > 0. Thus $\varphi(y, \eta) \in S^{-\infty}(\Omega \times \mathbb{R}^q; H, \tilde{H})$, and hence (1.4.19) shows that $a(y, \eta)$ is classical of order μ. In addition we see that in this case $a_{(\mu-j)}(y, \eta) = 0$ for all $j \in \mathbb{N}, j > 0$.

Remark 1.4.12. (i) *Spaces* $S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q)$ *of scalar symbols can be identified with spaces* $S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; \mathbb{C}, \mathbb{C})$ *where* $\mathbb{C}$ *is equipped with the trivial group action* κ^1*, determined by* $\kappa^1_\delta = \mathrm{id}_\mathbb{C}$ *for every* $\delta \in \mathbb{R}_+$*.*

(ii) *The* (y, η)*-wise composition of operator functions gives rise to bilinear maps*

$$S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H_0, \tilde{H}) \cdot S^\nu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, H_0) \longrightarrow S^{\mu+\nu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \tilde{H}) \tag{1.4.20}$$

for every $\mu, \nu \in \mathbb{R}$*.*

(iii) *The* (y, η)*-wise multiplication gives us bilinear maps*

$$\begin{aligned} S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q) \cdot S^\nu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \tilde{H}) &\longrightarrow S^{\mu+\nu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \tilde{H}), \\ S^\nu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \tilde{H}) \cdot S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q) &\longrightarrow S^{\mu+\nu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \tilde{H}). \end{aligned} \tag{1.4.21}$$

Example 1.4.13. *Let $p(x,\xi) \in S^\mu(\mathbb{R}^n \times \mathbb{R}^n)$, $\mu \in \mathbb{R}$, and assume p is independent of x for $|x| \geq C$ for some $C > 0$; then the associated pseudo-differential operator*

$$\operatorname{Op}_x(p) : H^s(\mathbb{R}^n) \longrightarrow H^{s-\mu}(\mathbb{R}^n) \tag{1.4.22}$$

for every $s \in \mathbb{R}$. Consider the involved Sobolev spaces with the group action of Example 1.4.2 (ii). *This allow us to form the family of operators*

$$b(\delta) := \tilde{\kappa}_\delta \operatorname{Op}_x(p)\, \kappa_\delta^{-1} \in C^\infty(\mathbb{R}_+, \mathcal{L}(H^s(\mathbb{R}^n), H^{s-\mu}(\mathbb{R}^n))). \tag{1.4.23}$$

Because of

$$\begin{aligned} b(\lambda\delta) &= \tilde{\kappa}_{\lambda\delta} \operatorname{Op}_x(p)\, \kappa_{\lambda\delta}^{-1} \\ &= \tilde{\kappa}_\lambda \big(\tilde{\kappa}_\delta \operatorname{Op}_x(p)\, \kappa_\delta^{-1}\big)\, \kappa_\lambda^{-1} \\ &= \tilde{\kappa}_\lambda\, b(\delta)\kappa_\lambda^{-1} \end{aligned} \tag{1.4.24}$$

we have

$$b(|\eta|) \in S^{(0)}(\mathbb{R}^q \setminus \{0\}; H^s(\mathbb{R}^n), H^{s-\mu}(\mathbb{R}^n)),$$

for $\delta = |\eta|$, cf. Definition 1.4.4 (ii). *Thus, if $\eta \longrightarrow [\eta]$ is defined as in Remark* 1.4.6*, then we have*

$$a(\eta) := [\eta]^\mu\, b([\eta]) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^q_\eta; H^s(\mathbb{R}^n), H^{s-\mu}(\mathbb{R}^n)),$$

cf. Remark 1.4.11.

Other examples of operator-valued symbols are as follows. Consider the case

$$H := H^s(\mathbb{R}), \quad \tilde{H} = \mathbb{C}$$

where on $H^s(\mathbb{R})$ we impose the group action of Example (1.4.2) (ii) for $n = 1$ and on $\mathbb{C}$ the trivial group action, i.e., $\tilde{\kappa}_\delta = \mathrm{id}$ for all $\delta \in \mathbb{R}_+$. Then, as is well-known, the operator of restriction

$$\mathrm{r}' : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}, \quad \mathrm{r}'u := u(0)$$

extends by continuity to a continuous map

$$\mathrm{r}' : H^s(\mathbb{R}) \longrightarrow \mathbb{C} \tag{1.4.25}$$

for every $s \in \mathbb{R}$, $s > 1/2$.

$$\begin{aligned} u(x) &= \int_{\mathbb{R}} e^{ix\xi} \hat{u}(\xi)\, đ\xi, \\ u(0) &= \int_{\mathbb{R}} \hat{u}(\xi)\, đ\xi = \int_{\mathbb{R}} \langle\xi\rangle^{-s}\langle\xi\rangle^s \hat{u}(\xi)\, đ\xi, \\ |u(0)| &= |\int_{\mathbb{R}} \langle\xi\rangle^{-s}\langle\xi\rangle^s \hat{u}(\xi)\, đ\xi| \leq \Big(\int_{\mathbb{R}} \langle\xi\rangle^{-2s}\, đ\xi\Big)^{1/2} \Big(\int_{\mathbb{R}} \langle\xi\rangle^{2s} |\hat{u}(\xi)|^2\, đ\xi\Big)^{1/2}, \\ |u(0)| &\leq c\|u\|_{H^s(\mathbb{R})}, \end{aligned}$$

for $s > \frac{1}{2}$. This yields (1.4.25). Then we have

$$\mathrm{r}' \in S^{1/2}_{\mathrm{cl}}(\mathbb{R}^q_\eta; H^s(\mathbb{R}), \mathbb{C}). \tag{1.4.26}$$

In fact, for $u(x) \in H^s(\mathbb{R})$ and $(\kappa_\delta u)(x) = \delta^{1/2} u(\delta x), \delta \in \mathbb{R}_+$ and by Remark 1.4.11 we obtain (1.4.26).

In the following definition we employ the Schwartz space $\mathcal{S}(\mathbb{R}^q, H)$ with values in Hilbert space, defined as the set of all $u \in C^\infty(\mathbb{R}^q, H)$ such that

$$\sup_{y \in \mathbb{R}^q} \|y^\alpha \, D_y^\beta u(y)\|_H < \infty$$

for all α, $\beta \in \mathbb{N}^q$. Further examples come frome differentiations

$$\partial^k := \left(\frac{d}{dt}\right)^k : H^s(\mathbb{R}) \longrightarrow H^{s-k}(\mathbb{R}).$$

In this case we have $\partial^k = \delta^k \kappa_\delta \partial^k \kappa_\delta^{-1}$, for every $\delta \in \mathbb{R}_+$.

$$\partial^k \in S^k_{\mathrm{cl}}(\mathbb{R}^q; H^s(\mathbb{R}), H^{s-k}(\mathbb{R})).$$

More generally, the differentiation in $\mathbb{R}^n$

$$D_x^\alpha : H^s(\mathbb{R}^n) \longrightarrow H^{s-|\alpha|}(\mathbb{R}^n)$$

is a symbol in $S^{|\alpha|}_{\mathrm{cl}}(\mathbb{R}^q; H^s(\mathbb{R}^n), H^{s-|\alpha|}(\mathbb{R}^n))$.

1.5 Edge Sobolev spaces

Definition 1.5.1. (i) *Let H be a Hilbert space with a group action κ. Then $\mathcal{W}^s(\mathbb{R}^q, H)$, $s \in \mathbb{R}$, is defined as the completion of $\mathcal{S}(\mathbb{R}^q, H)$ with respect to the norm*

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q, H)} = \left\{ \int \langle\eta\rangle^{2s} \, \|\kappa^{-1}_{\langle\eta\rangle}(Fu)(\eta)\|^2_H \, đ\eta \right\}^{1/2} \tag{1.5.1}$$

for $đ\eta := (2\pi)^{-q} d\eta$.

(ii) *If E is a Fréchet space with group action, cf.* (1.4.6), *by* (i) *we have the spaces $\mathcal{W}^s(\mathbb{R}^q, E^j)$, and we define*

$$\mathcal{W}^s(\mathbb{R}^q, E) = \varprojlim_{j \in \mathbb{N}} \mathcal{W}^s(\mathbb{R}^q, E^j).$$

It can be proved, cf. [21], that the space $\mathcal{W}^s(\mathbb{R}^q, H)$ can be equivalently defined as the set of all $u \in \mathcal{S}(\mathbb{R}^q, H)$ such that (1.5.1) is finite.

Remark 1.5.2. *Observe that the spaces $\mathcal{W}^s(\mathbb{R}^q, H)$ depend on κ. If necessary we use notation*

$$\mathcal{W}^s(\mathbb{R}^q, H)_\kappa.$$

For $\kappa_\delta = \mathrm{id}_H$ for all δ we have

$$\mathcal{W}^s(\mathbb{R}^q, H)_{\mathrm{id}} = H^s(\mathbb{R}^q, H).$$

In particular, if $H = \mathbb{C}$ endowed with the trivial group action we have

$$\mathcal{W}^s(\mathbb{R}^q, \mathbb{C}) = H^s(\mathbb{R}^q). \tag{1.5.2}$$

Remark 1.5.3. *There are many ways to introduce in* $\mathcal{W}^s(\mathbb{R}^q, H)_\kappa$ *equivalent norms. For instance, in* (1.5.1) *we can replace* $\langle\eta\rangle$ *by* $[\eta]$ *for any strictly positive smooth function in* $\mathbb{R}^q$ *such that* $[\eta] = |\eta|$ *for all* $\eta \geq \text{const}$, *for some* $\text{const} > 0$. *Then* (1.5.1)*turns to an equivalent norm.*

Proposition 1.5.4. *For every* p, $q \in \mathbb{N}$ *and* $s \in \mathbb{R}$ *we have*

$$\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^p)) = H^s(\mathbb{R}^{p+q}). \tag{1.5.3}$$

according to Example 1.3.23 *in* [53], *where* $H^s(\mathbb{R}^p)$ *is endowed with the group action*

$$\kappa_\delta : u(x) \longrightarrow \delta^{p/2} u(\delta x), \delta \in \mathbb{R}_+.$$

A similar result of this type concerns the case $H^s(\mathbb{R}_+)$ *with group action*

$$\kappa_\delta : u(t) \longrightarrow \delta^{1/2} u(\delta t), \delta \in \mathbb{R}_+.$$

Then

$$\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}_+)) = H^s(\mathbb{R}^q \times \mathbb{R}_+) = H^s(\mathbb{R}^{q+1}_+). \tag{1.5.4}$$

Here $H^s(\mathbb{R}^{1+q}_+) := \{u(t,y) \in H^s(\mathbb{R}^{1+q}) : u(t,y)|_{t>0}\}$.

For an open set $\Omega \subseteq \mathbb{R}^q$ we define $\mathcal{W}^s_{\mathrm{comp}}(\Omega, H)$ to be the space of all $u \in \mathcal{D}'(\mathbb{R}^q, H)$ such that $\operatorname{supp} u$ is a compact subset of Ω (here we tacitly identify distributions in Ω supported by such a compact set with distributions in $\mathbb{R}^q$ supported by that set). Moreover, $\mathcal{W}^s_{\mathrm{loc}}(\Omega, H)$ is defined as the set of all $u \in \mathcal{D}'(\Omega, H)$ such that $\varphi u \in \mathcal{W}^s_{\mathrm{comp}}(\Omega, H)$ for every $\varphi \in C_0^\infty(\Omega)$, cf. Proposition 1.5.5 below. Similar notation is used for edge spaces referring to a Fréchet space E with group action.

The following result has been proved by Hirschmann [21]. By $\mathcal{M}_\varphi$ we denote the operator of multiplication by a function φ. Instead of $\mathcal{M}_\varphi u$ we also write φu.

Proposition 1.5.5. *For any Hilbert space* H *with group action* κ *and* $\varphi \in \mathcal{S}(\mathbb{R}^q_y)$ *the operator*

$$\mathcal{M}_\varphi : \mathcal{W}^s(\mathbb{R}^q, H) \longrightarrow \mathcal{W}^s(\mathbb{R}^q, H)$$

is continuous for every $s \in \mathbb{R}$.

Proof. The Schwartz space $\mathcal{S}(\mathbb{R}^q, H)$ is dense in $\mathcal{W}^s(\mathbb{R}^q, H)$; cf. Definition 1.5.1. Thus it suffices to show

$$\|\mathcal{M}_\varphi u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} \leq c_\varphi \|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} \tag{1.5.5}$$

for all $u \in \mathcal{S}(\mathbb{R}^q, H)$, for some constant $c_\varphi \geq 0$. We have (up to equivalence of norms)

$$\|\varphi u\|^2_{\mathcal{W}^s(\mathbb{R}^q,H)} = \int [\eta]^{2s} \|\kappa^{-1}_{[\eta]} F(\varphi u)(\eta)\|^2_H \, d\eta = \|m(\eta)\|^2_{L^2(\mathbb{R}^q)} \tag{1.5.6}$$

for

$$m(\eta) := \|[\eta]^s \kappa^{-1}_{[\eta]} \int (F\varphi)(\eta - \xi) Fu(\xi) \, đ\xi\|_H,$$

using that $F(\varphi u)(\eta) = \int (F\varphi)(\eta - \xi) Fu(\xi)\, đ\xi$. Peetreś inequality, cf. Lemma 1.7.2 in Section 1.7, we have

$$[\eta]^s \leq c^{|s|}[\eta - \xi]^{|s|}[\xi]^s \tag{1.5.7}$$

for every $\xi, \eta \in \mathbb{R}^q, s \in \mathbb{R}$ for a constant $c > 0$. Applying (1.5.7) for $s = 1$ we obtain

$$K(\delta) := \begin{cases} \delta & \delta \geq 1 \\ \delta^{-1} & 0 \leq \delta \leq 1 \end{cases}$$

the estimate

$$K([\xi]/[\eta]) \leq c[\eta - \xi] \tag{1.5.8}$$

for some $c > 0$, for all $\xi, \eta \in \mathbb{R}^q$, from (1.5.7) it follows that

$$m(\eta) \leq c_s \|\kappa_{[\eta]}^{-1} \int [\eta - \xi]^{|s|} \hat{\varphi}(\eta - \xi)[\xi]^s \hat{u}(\xi)\, đ\xi\|_H \tag{1.5.9}$$

for a constant $c_s > 0$. Thus the right-hand side of (1.5.9) is equal to

$$c_s \| \int \kappa_{[\eta]/[\xi]}^{-1} [\eta - \xi]^{|s|-k}[\eta - \xi]^k \hat{\varphi}(\eta - \xi)[\xi]^s \kappa_{[\xi]}^{-1} \hat{u}(\xi)\|_H\, đ\xi. \tag{1.5.10}$$

From proposition 1.4.1 and (1.5.8) it follows that

$$\|\kappa_{[\eta]/[\xi]}^{-1}\|_{\mathcal{L}(H)} \leq cK([\xi]/[\eta])^M \leq c[\eta - \xi]^M \tag{1.5.11}$$

for some $c > 0$. Thus (1.5.10) can be estimated by

$$\begin{aligned} &c_s \int \|\kappa_{[\eta]/[\xi]}^{-1}\|_{\mathcal{L}(H)} [\eta - \xi]^{|s|-k}[\eta - \xi]^k |\hat{\varphi}(\eta - \xi)| [\xi]^s \|\kappa_{[\xi]}^{-1} \hat{u}(\xi)\|_H\, đ\xi \\ &\leq cc_s \int [\eta - \xi]^{|s|-k+M}[\eta - \xi]^k |\hat{\varphi}(\eta - \xi)| [\xi]^s \|\kappa_{[\xi]}^{-1} \hat{u}(\xi)\|_H\, đ\xi \\ &\leq cc_s \int [\eta - \xi]^{|s|-k+M}[\xi]^s \|\kappa_{[\xi]}^{-1} \hat{u}(\xi)\|_H\, đ\xi \end{aligned}$$

for $C_k(\varphi) := \sup_{\xi \in \mathbb{R}^q} [\xi]^k |\hat{\varphi}(\xi)|$. Let us set

$$g(\eta) = [\eta]^{|s|-k+M},\ h(\eta) = [\eta]^s \|\kappa_{[\eta]}^{-1} \hat{u}(\eta)\|_H,$$

and choose $k \in \mathbb{N}$ so large such that $|s| - k + M < -q$, and hence $g \in L^1(\mathbb{R}^q)$. Moreover, $\|h\|_{L^2(\mathbb{R}^q)} = \|u\|_{\mathcal{W}^s(\mathbb{R}^q, H})$ we have

$$m(\eta) \leq cc_s C_k(\varphi)(g * h)(\eta)$$

for $\eta \in \mathbb{R}^q$. From Youngs' inequality, cf. ... below, we obtain, using (1.5.6),

$$\begin{aligned} \|\varphi u\|_{\mathcal{W}^s(\mathbb{R}^q, H)} &\leq cc_s C_k(\varphi) \|g * h\|_{L^2(\mathbb{R}^q)} \leq cc_s C_k(\varphi) \|g\|_{L^1(\mathbb{R}^q)} \|h\|_{L^2(\mathbb{R}^q)} \\ &= C_s C_k(\varphi) \|u\|_{\mathcal{W}^s(\mathbb{R}^q, H)} \end{aligned}$$

$C_s := cc_s \|g\|_{L^1(\mathbb{R}^q)}$, which just corresponds to (1.5.5). □

Similarly as (1.1.13) we form pseudo-differential operators $\mathrm{Op}_y(a) = \mathrm{Op}(a)$ in the variable $y \in \Omega$, where $a(y, y', \eta)$ is a symbol in $S^{\mu}_{\mathrm{cl}}(\Omega \times \Omega \times \mathbb{R}^q; H, \tilde{H})$, cf. Definition 1.4.4 for $\Omega \times \Omega$ instead of Ω, now for open $\Omega \subseteq \mathbb{R}^q$.

Proposition 1.5.6. *For every*

$$a(\eta) \in S^{\mu}(\mathbb{R}^q; H, \tilde{H})$$

the associated pseudo-differential operator $\mathrm{Op}_y(a)$ *induces a continuous operator*

$$\mathrm{Op}_y(a) : \mathcal{W}^s(\mathbb{R}^q, H) \longrightarrow \mathcal{W}^{s-\mu}(\mathbb{R}^q, \tilde{H}) \tag{1.5.12}$$

for every $s \in \mathbb{R}$. *Moreover, the map* $a \longrightarrow \mathrm{Op}_y(a)$ *generates a continuous map*

$$S^{\mu}(\mathbb{R}^q; H, \tilde{H}) \longrightarrow \mathcal{L}(\mathcal{W}^s(\mathbb{R}^q, H), \mathcal{W}^{s-\mu}(\mathbb{R}^q, \tilde{H})) \tag{1.5.13}$$

for every $s \in \mathbb{R}$.

Proof. We have for $u \in \mathcal{W}^s(\mathbb{R}^q, H)$, using that $\mathrm{Op}(a) = F^{-1}a(\eta)F$

$$\begin{aligned}
\|\mathrm{Op}(a)u\|^2_{\mathcal{W}^{s-\mu}(\mathbb{R}^q,\tilde{H})} &= \int \langle\eta\rangle^{2(s-\mu)} \|\tilde{\kappa}^{-1}_{\langle\eta\rangle} F \,\mathrm{Op}(a)u(\eta)\|^2_{\tilde{H}} \,đ\eta \\
&= \int \langle\eta\rangle^{2(s-\mu)} \|\tilde{\kappa}^{-1}_{\langle\eta\rangle} a(\eta)Fu(\eta)\|^2_{\tilde{H}} \,đ\eta \\
&= \int \langle\eta\rangle^{2(s-\mu)} \|\tilde{\kappa}^{-1}_{\langle\eta\rangle} a(\eta)\kappa_{\langle\eta\rangle} \kappa^{-1}_{\langle\eta\rangle}Fu(\eta)\|^2_{\tilde{H}} \,đ\eta \\
&\leq \int \langle\eta\rangle^{2(s-\mu)} \|\tilde{\kappa}^{-1}_{\langle\eta\rangle} a(\eta)\kappa_{\langle\eta\rangle}\|^2_{\mathcal{L}(H,\tilde{H})} \|\kappa^{-1}_{\langle\eta\rangle}Fu(\eta)\|^2_{H} \,đ\eta \\
&\leq \sup_{\eta\in\mathbb{R}^q} \langle\eta\rangle^{-2\mu} \|\tilde{\kappa}^{-1}_{\langle\eta\rangle} a(\eta)\kappa_{\langle\eta\rangle}\|^2_{\mathcal{L}(H,\tilde{H})} \int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle}(Fu)(\eta)\|^2_{H} \,đ\eta \leq c^2 \|u\|^2_{\mathcal{W}^s(\mathbb{R}^q,H)}.
\end{aligned}$$

This shows the claimed continuity (4.1.5). Moreover,

$$||\mathrm{Op}(a)||_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q,H),\mathcal{W}^{s-\mu}(\mathbb{R}^q,\tilde{H}))} \leq \sup_{\eta\in\mathbb{R}^q} \langle\eta\rangle^{-\mu} ||\tilde{\kappa}^{-1}_{\langle\eta\rangle} a(\eta)\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\tilde{H})},$$

shows the continuity of (1.5.13), because $a(\eta) \in S^{\mu}(\mathbb{R}^q; H, \tilde{H}) \longrightarrow 0$ entails $c(a) \longrightarrow 0$ and hence

$$||\mathrm{Op}(a)||_{\mathcal{L}(W^s(\mathbb{R}^q,H),W^{s-\mu}(\mathbb{R}^q,\tilde{H}))} \longrightarrow 0.$$

□

Note that, as a corollary of relation (1.4.26),

$$\mathrm{Op}_y(\mathrm{r}') : \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R})) \longrightarrow H^{s-1/2}(\mathbb{R}^q, \mathbb{C})$$

is continuous for $s > \frac{1}{2}$. However, from (1.5.2) and (1.5.3), we get the continuity of the restriction

$$\mathrm{Op}_y(\mathrm{r}') : H^s(\mathbb{R}^{q+1}) \longrightarrow H^{s-1/2}(\mathbb{R}^q), \tag{1.5.14}$$

for $s > 1/2$, where $\mathrm{Op}_y(\mathrm{r}')$ has the meaning of restriction

$$u(y_1, \ldots, y_q, y_{q+1}) \longrightarrow u(y_1, \ldots, y_q, 0).$$

Another continuity result concerns symbols $a(y, \eta)$ that are not necessarily constant with respect to y. There are many different situations of that kind, where we either control the behavior of symbols for $|y| \longrightarrow \infty$ or the nature of distribution spaces for $|y| \longrightarrow \infty$.

The spaces of symbols in Definition 1.4.4 can be specified for an open set $\Omega \subseteq \mathbb{R}^q$ in variables y or $\Omega \times \Omega$ for open $\Omega \subseteq \mathbb{R}^q$, in variables (y, y'). Symbols $a(y, y', \eta)$ are called double symbols, and for those we form associated operators

$$\operatorname{Op}(a)u(y) = \iint \mathrm{e}^{i(y-y')\eta} a(y, y', \eta) u(y')\, dy' đ\eta, \tag{1.5.15}$$

first for $u \in C_0^\infty(\Omega, H)$, and later on for more general H-valued distributions, such as $\mathcal{W}^s_{\mathrm{loc}}(\Omega, H)$. Clearly symbols $a(y, \eta)$ and $a(y', \eta)$ are admitted as special cases, and analogously as in the scalar case, in this connection we call $a(y, \eta) =: a_{\mathrm{L}}(y, \eta)$ a left symbol, $a(y', \eta) := a_{\mathrm{R}}(y, \eta)$ a right symbol.

Theorem 1.5.7. *Assume that $a(y, \eta) \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q; H, \tilde{H})$ vanishes for large $|y|$, i.e.,*

$$a(y, \eta) = b(y, \eta) + a_1(\eta)$$

for $b(y, \eta) \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q; H, \tilde{H})$ such that $b(y, \eta) \equiv 0$ for $|y| > R$ for some $R > 0$. Then $\operatorname{Op}(a)$ induces a continuous operator

$$\operatorname{Op}(a) : \mathcal{W}^s(\mathbb{R}^q, H) \longrightarrow \mathcal{W}^{s-\mu}(\mathbb{R}^q, \tilde{H}) \tag{1.5.16}$$

for every $s \in \mathbb{R}$.

Proof. By virtue of Proposition 1.5.6 it suffices to consider $\operatorname{Op}(b)$. By assumption we have

$$b(y, \eta) \in C_0^\infty(B_R, S^\mu(\mathbb{R}^q; H, \tilde{H}))$$

for $B_R := \{y \in \mathbb{R}^q : |y| \leq R\}$ and $C_0^\infty(B_R)$ is the Fréchet space of all $\varphi \in C^\infty(\mathbb{R}^q)$ such that $\operatorname{supp} \varphi \subseteq B_R$. Because of

$$C_0^\infty(B_R, S^\mu(\mathbb{R}^q; H, \tilde{H})) = C_0^\infty(B_R) \hat{\otimes}_\pi S^\mu(\mathbb{R}^q; H, \tilde{H}),$$

applying Theorem 1.7.1in Section 1.7, we can write

$$b(y, \eta) = \sum_{j=0}^\infty \lambda_j \varphi_j(y) b_j(\eta)$$

for $\lambda_j \in \mathbb{C}, \sum_{j=0}^\infty |\lambda_j| < \infty, \varphi_j \in C_0^\infty(B_R), b_j(\eta) \in S^\mu(\mathbb{R}^q; H, \tilde{H})$, tending to zero in the respective spaces, as $j \to \infty$. The functions $\varphi_j(y)$ are interpreted as operators of multiplication $\mathcal{M}_{\varphi_j}$. We now apply Proposition 1.5.5 which tells us that

$$\mathcal{M}_{\varphi_j} : u \longrightarrow \varphi_j u$$

induces a continuous operator

$$\mathcal{M}_{\varphi_j} : \mathcal{W}^{s-\mu}(\mathbb{R}^q, \tilde{H}) \longrightarrow \mathcal{W}^{s-\mu}(\mathbb{R}^q, \tilde{H})$$

where

$$||\mathcal{M}_{\varphi_j}||_{\mathcal{L}(\mathcal{W}^{s-\mu}(\mathbb{R}^q,\tilde{H}))} \longrightarrow 0 \tag{1.5.17}$$

as $j \to \infty$. Then, writing

$$\mathrm{Op}(b) = \sum_{j=0}^{\infty} \lambda_j \mathcal{M}_{\varphi_j} \mathrm{Op}(b_j)$$

we obtain

$$\begin{aligned} &\|\mathrm{Op}(b)\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q,H),\mathcal{W}^{s-\mu}(\mathbb{R}^q,\tilde{H}))} \\ &\le \sum_{j=0}^{\infty} |\lambda_j| \|\mathrm{Op}(b_j)\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q,H),\mathcal{W}^{s-\mu}(\mathbb{R}^q,\tilde{H}))} \|\mathcal{M}_{\varphi_j}\|_{\mathcal{L}(\mathcal{W}^{s-\mu}(\mathbb{R}^q,\tilde{H}))}. \end{aligned} \tag{1.5.18}$$

From $b_j \to 0$ and Proposition 1.5.6 we obtain

$$\|\mathrm{Op}(b_j)\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q,H),\mathcal{W}^{s-\mu}(\mathbb{R}^q,\tilde{H}))} \longrightarrow 0 \tag{1.5.19}$$

as $j \to \infty$. Moreover, (1.5.17) and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$ show that the right-hand side of (1.5.18) converges. This yields the continuity of (1.5.16).

□

Conclusions using projective tensor product structure of involved spaces as in the previous proof are often referred to as tensor product argument.

Example 1.5.8. *Let* $E = C^\infty(\Omega), F = S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; H, \tilde{H})$, *then relations* (1.4.17) *mean*

$$E\hat{\otimes}_\pi F = S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \tilde{H})$$

and hence we can apply Theorem 1.7.1.

There are other continuity results for operators with operator-valued symbols. For purposes below we discuss a few cases.

Theorem 1.5.9. *For* $a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H})$, $\Omega \subseteq \mathbb{R}^q$ *open, the operator*

$$\mathrm{Op}_y(a) : C_0^\infty(\Omega, H) \longrightarrow C^\infty(\Omega, \tilde{H}) \tag{1.5.20}$$

extends to a continuous operator

$$\mathrm{Op}_y(a) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, H) \longrightarrow \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{H}) \tag{1.5.21}$$

for every $s \in \mathbb{R}$.

Note that when we consider a double symbol

$$a(y, y', \eta) \in S^\mu(\Omega \times \Omega \times \mathbb{R}^q; H, \tilde{H})$$

rather than $a(y,\eta)$ as in Theorem 1.5.9 we also have continuous operators $\mathrm{Op}(a)$ in the sense of (1.5.20) and (1.5.21).

By $L^{-\infty}(\Omega; H, \tilde{H})$ we denote the space of operators C acting by kernels $c(y,y') \in C^\infty(\Omega \times \Omega, \mathcal{L}(H, \tilde{H}))$, namely

$$Cu(y) = \int c(y,y')u(y')\,dy',$$

$u \in C_0^\infty(\Omega, H)$.

Definition 1.5.10. *We set*

$$L^{\mu}_{(\mathrm{cl})}(\Omega; H, \tilde{H}) := \{\mathrm{Op}(a) + C : a(y,\eta) \in S^{\mu}_{(\mathrm{cl})}(\Omega; H, \tilde{H}), C \in L^{-\infty}(\Omega; H, \tilde{H})\}. \tag{1.5.22}$$

A vector-valued analogue of the Schwartz kernel theorem then gives us an operator-valued distributional kernel for $A := \mathrm{Op}(a)$, namely,

$$K_A(y, y') = \int e^{i(y-y')\eta} a(y, y', \eta)\, đ\eta \tag{1.5.23}$$

which belongs to $\mathcal{D}'(\Omega \times \Omega, \mathcal{L}(H, \tilde{H}))$.

Proposition 1.5.11. *We have*

$$\mathrm{sing\,supp}\, K_A(y, y') \subseteq \mathrm{diag}\,(\Omega \times \Omega). \tag{1.5.24}$$

Proof. Using the identity

$$\Delta_\eta^N e^{i(y-y')\eta} = -|y - y'|^{2N} e^{i(y-y')\eta}.$$

We may write

$$\mathrm{Op}(a)u(y) = \iint -|y - y'|^{-2N} \Delta_\eta^N e^{i(y-y')\eta} a(y, y', \eta) u(y')\, dy' đ\eta,$$

$u \in C_0^\infty(\Omega, H)$. Integrating by part in η gives us

$$\mathrm{Op}(a)u(y) = \iint e^{i(y-y')\eta} (-|y - y'|^{-2N} \Delta_\eta^N a(y, y'\eta)) u(y')\, dy' đ\eta.$$

Writing

$$\mathrm{Op}(a)u(y) = \mathrm{Op}((\omega + (1 - \omega))a)u(y)$$

for an $\omega(y, y') \in C^\infty(\Omega \times \Omega)$ such that $\omega \equiv 1$ in a neighbourhood of $\mathrm{diag}\,(\Omega \times \Omega)$ we obtain

$$\mathrm{Op}(a)u(y) = \mathrm{Op}(\omega a)(y, \eta) + \iint e^{i(y-y')\eta} (-|y - y'|^{-2N} (1 - \omega(y, y')) \Delta_\eta^N a(y, y', \eta) u(y')\, dy' đ\eta. \tag{1.5.25}$$

Since $1 - \omega$ vanishes in a neighbourhood of $\mathrm{diag}(\Omega \times \Omega)$ we have

$$-|y - y'|^{-2N} (1 - \omega(y, y')) \in C^\infty(\Omega \times \Omega).$$

Moreover, as a consequence of Definition 1.4.4 (i) we have

$$\Delta_\eta^N a(y, y', \eta) \in S^{\mu - 2N}(\Omega \times \Omega \times \mathbb{R}^q; H, \tilde{H})$$

for every $N \in \mathbb{N}$. Now for every $k \in \mathbb{N}$ there exists an $N = N(k)$ such that the kernel of $\mathrm{Op}((1 - \omega)a)$ belongs to $C^k(\Omega \times \Omega, \mathcal{L}(H, \tilde{H}))$. Since N in (1.5.25) is arbitrary, the kernel of $\mathrm{Op}((1 - \omega)a)$ belongs to $C^\infty(\Omega \times \Omega, \mathcal{L}(H, \tilde{H}))$. Since for every neighbourhood of $\mathrm{diag}\,(\Omega \times \Omega)$ we can choose $\omega(y, y')$ in such a way that ω vanishes outside this neighbourhood, we obtain relation (1.5.24). $\square$

Remark 1.5.12. *Similarly as* (1.1.20) *an operator* $A \in L^{\mu}_{(\mathrm{cl})}(\Omega; H, \tilde{H})$ *can be written in the form*

$$A = A_0 + C \tag{1.5.26}$$

where $A_0 \in L^{\mu}_{(\mathrm{cl})}(\Omega; H, \tilde{H})$ *is properly supported in the variables* (y, y') *and* $C \in L^{-\infty}(\Omega; H, \tilde{H})$.

Similarly as Remark 1.1.6 we have a recovering process for symbols.

Remark 1.5.13. *Let* $A \in L^{\mu}_{(\mathrm{cl})}(\Omega; H, \tilde{H})$ *be properly supported. Then there is a unique* $a(y, \eta) \in S^{\mu}_{(\mathrm{cl})}(\Omega; H, \tilde{H})$ *such that*

$$A = \mathrm{Op}_y(a).$$

In fact, for any $u \in C_0^{\infty}(\Omega, H)$, using the Fourier inversion formula

$$u(y) = \int \mathrm{e}^{iy\eta} \hat{u}(\eta)\, đ\eta \tag{1.5.27}$$

we obtain

$$Au(y) = \int A\mathrm{e}^{iy\eta} \hat{u}(\eta)\, đ\eta = \mathrm{Op}_y(a)u \tag{1.5.28}$$

and we may set $a(y, \eta) = \mathrm{e}^{-iy\eta} A \mathrm{e}^{iy\eta}$ belonging to $C^{\infty}(\Omega \times \mathbb{R}^q, \mathcal{L}(H, \tilde{H}))$. An additional consideration (which is left to the reader) shows that in fact $a(y, \eta)$ is a symbol as asserted see also Shubin's book [69].

1.6 Applications of operator-valued symbols

For applications below we consider some concrete examples of operator-valued symbols. Let us form trace symbols γ^j defined by

$$\gamma^j(\zeta)u := [\zeta]^{-j-1/2} \frac{\partial^j u}{\partial t^j}\Big|_{t=0} \tag{1.6.1}$$

acting on functions $u \in H^s(\mathbb{R})$, for the meaning of notation $\zeta \to [\zeta]$ see Example 1.4.13. Then, for large $|\zeta|$ we obtain

$$\begin{aligned} \gamma^j(\delta\zeta)u(t) &= \delta^{j+1/2} \gamma^j(\zeta) \kappa_{\delta}^{-1} u(t) \\ &= \delta^{j+1/2} [\zeta]^{-j-1/2} \frac{\partial^j}{\partial t^j} \delta^{-1/2} u(\delta^{-1}t)|_{t=0} = \gamma^j(\zeta) \end{aligned} \tag{1.6.2}$$

for every $\delta \in \mathbb{R}_+$. Therefore, according to Remark 1.4.11, we have $\gamma^j \in S^0_{\mathrm{cl}}(\mathbb{R}^d_{\zeta}; H^s(\mathbb{R}), \mathbb{C})$ for any $s - j > \frac{1}{2}$. If is also useful to form potential symbols

$$k_j(\zeta) : c \longrightarrow \omega([\zeta]t) t^j [\zeta]^{j+1/2} c$$

mapping $c \in \mathbb{C}$ to $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$. Here ω is any cut-off function. Then we have

$$\gamma^j(\zeta) k_j(\zeta) = \mathrm{id}_{\mathbb{C}}$$

for every $\zeta \in \mathbb{R}^d$ and

$$k_j(\zeta)\gamma^j(\zeta) = \omega(t).$$

In addition we have

$$k_j(\delta\zeta)c = \delta^{-j-1/2} \kappa_{\delta} k_j(\zeta) c = \delta^{-j-1/2} \delta^{1/2} \omega([\zeta]\delta t) \delta^j t^j [\zeta]^{j+1/2} c = \kappa_{\delta} k_j(\zeta) \tag{1.6.3}$$

for all $\delta \in \mathbb{R}_+$ and large $|\zeta|, \kappa_{\delta} u(t) := \delta^{1/2} u(\delta t)$. Thus, because of Remark 1.4.11 we have $k_j(\zeta) \in S^0_{\mathrm{cl}}(\mathbb{R}^d_{\zeta}; H^s(\mathbb{R}), \mathbb{C})$.

1.7 Appendix

In general, if E_0, E_1 are Fréchet spaces embedded in a Hausdorff topological vector space H we have the non-direct sum

$$E_0 + E_1 := \{e_0 + e_1 : e_0 \in E_0, e_1 \in E_1\}. \tag{1.7.1}$$

Then, setting

$$\Delta := \{(e, -e) : e \in E_0 \cap E_1\}$$

we have an isomorphism

$$E_0 + E_1 \cong E_0 \oplus E_1/\Delta.$$

The quotient space is defined as space of equivalence classes of pairs (e_0, e_1) where

$$(e_0, e_1) \sim (f_0, f_1) \longleftrightarrow (e_0 - f_0, e_1 - f_1) \in \Delta.$$

Δ is closed in the Fréchet space $E_0 \oplus E_1$; then $E_0 \oplus E_1/\Delta$ is again a Fréchet space, called the non-direct sum of the involved Fréchet spaces. In particular, if E_0, E_1 are Hilbert spaces, then $E_0 \oplus E_1$ is also a Hilbert space, and Δ is a closed subspace. Then $E_0 \oplus E_1/\Delta$ can be identified with $\Delta^\perp$, the orthogonal complement of Δ, i.e., $E_0 + E_1$ is also a Hilbert space. In addition if a Fréchet space F is a module over an algebra A, we denote by $[a]F$ the completion of $\{af : f \in F\}$ in F.

Theorem 1.7.1. [51] *Let E, F be Fréchet spaces. Then every $g \in E\hat{\otimes}_\pi F$ in the respective projective tensor product (which is again as Fréchet space) can be written as convergent sum in $E\hat{\otimes}_\pi F$ of the form*

$$g = \sum_{j=0}^{\infty} \lambda_j e_j \otimes f_j \tag{1.7.2}$$

for $\lambda_j \in \mathbb{C}, \sum_{j=0}^{\infty} |\lambda_j| < \infty$ and $e_j \in E, f_j \in F$ tends to zero as $j \to \infty$.

The convergence of (1.7.2) refers to the system of projective tensor projects $p\tilde{\otimes}_\pi q$ where p and q sum over the semi-norm systems of the space E and F, respectively. In this sense we have

$$p\hat{\otimes}_\pi q\,(g) \leq \sum_{j=0}^{\infty} \lambda_j p(e_j) q(f_j) \tag{1.7.3}$$

which is convergent.

Lemma 1.7.2 (Peetreś inequality)**.** *We have*

$$(1 + |\xi - \eta|)^s \leq (1 + |\eta|)^s (1 + |\xi|)^{|s|} \tag{1.7.4}$$

for all $\xi, \eta \in \mathbb{R}^n$ and every $s \in \mathbb{R}$.

Chapter 2

Outline of Boutet de Monvel's Calculus

2.1 Boundary value problems for differential operators

Consider a smooth manifold X with boundary ∂X, endowed with a Riemannian metric there is the product metric of $\partial X \times [0,1)$ in a collar neighbourhood of the boundary for some Riemannian metric on ∂X. We also consider $2X$, the double of X obtained by gluing together two copies X_+ and X_- of X along the common boundary to a smooth manifolds. $2X$ locally close to the boundary from $\partial X \times [0,1)$, we have a splitting of variables $x = (x', x_n) \in \Omega \times [0,1)$ for an open set $\Omega \subseteq \mathbb{R}^{n-1}$ for $n = \dim X$. In order to understand operators on X near the boundary it is convenient first to look at the half-space $\Omega \times \overline{\mathbb{R}}_+$. A basic issue will be to understand on how a differential operator

$$A = \sum_{|\alpha| \leq \mu} a_\alpha(x)\, D_x^\alpha$$

of order μ with coefficients $a_\alpha \in C^\infty(\mathbb{R}^n)$ acquires an operator-valued symbol from the boundary. From Section 1.1 we have the usual symbol

$$a(x, \xi) = \sum_{|\alpha| \leq \mu} a_\alpha(x)\, \xi^\alpha,$$

and its homogeneous principal symbol

$$\sigma_\psi(A)\,(x, \xi) = \sum_{|\alpha| = \mu} a_\alpha(x)\, \xi^\alpha,$$

$\xi \neq 0$. Recall that $\sigma_\psi(A)(x, \delta\xi) = \delta^\mu \sigma_\psi(A)(x, \xi)$, $\delta \in \mathbb{R}_+$ (homogeneity property). Moreover, we have the homogeneous principal boundary symbol, defined as

$$\sigma_\partial(A)(x', \xi') := \sum_{|\alpha| = \mu} a_\alpha(x', 0)\, (\xi', D_{x_n})^\alpha \tag{2.1.1}$$

or equivalently,

$$\sigma_\partial(A)\,(x', \xi') = \sigma_\psi(A)\,(x', 0, \xi', D_{x_n}) \tag{2.1.2}$$

for $D_{x_n} = \frac{1}{i}\frac{\partial}{\partial x_n}$, $\xi' \neq 0$. The boundary symbol is regarded as a family of differential operators on the half-axis, say, between Sobolev spaces $H^s(\mathbb{R}_+) = H^s(\mathbb{R})|_{\mathbb{R}_+}$, $s \in \mathbb{R}$, namely,

$$\sigma_\partial(A)(x', \xi') : H^s(\mathbb{R}_+) \longrightarrow H^{s-\mu}(\mathbb{R}_+), \tag{2.1.3}$$

$\xi' \neq 0$. Homogeneity in the case of boundary symbols is connected with the action of a one-parameter-dependent group $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ of isomorphisms

$$\kappa_\delta : H^s(\mathbb{R}_+) \longrightarrow H^s(\mathbb{R}_+), (\kappa_\delta\, u)(x_n) = \delta^{1/2} u(\delta x_n). \tag{2.1.4}$$

Then

$$\sigma_\partial(A)(x', \delta\xi') = \delta^\mu \kappa_\delta \sigma_\partial(A)(x', \xi') \kappa_\delta^{-1}, \tag{2.1.5}$$

for all $\delta \in \mathbb{R}_+$.

Example 2.1.1. *For $A = \Delta = \sum_{j=1}^n \frac{\partial^2}{\partial {x_j}^2}$, i.e., the Laplacian in $\mathbb{R}^n$, we have*

$$\sigma_\psi(\Delta)(\xi) = -|\xi|^2,$$

and for $x = (x', x_n), \xi = (\xi', \xi_n)$,

$$\sigma_\partial(\Delta)(\xi') = -|\xi'|^2 + \frac{\partial^2}{\partial x_n^2}. \tag{2.1.6}$$

In this case (2.1.5) *turns to*

$$\sigma_\partial(\Delta)(\delta\xi') = \delta^2 \kappa_\delta \sigma_\partial(\Delta)(\xi') \kappa_\delta^{-1},\ \delta \in \mathbb{R}_+.$$

First observe that A in $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, can be written in iterated form

$$A = \mathrm{Op}_{x'}(\mathrm{Op}_{x_n}(a)), \tag{2.1.7}$$

using that $a(x, \xi) = a(x', x_n, \xi', \xi_n)$ for fixed (x', ξ') is a symbol in (x_n, ξ_n).

Proposition 2.1.2. *Let $a(x, \xi) \in S^\mu(\Omega \times \mathbb{R}^n_\xi)$ for an open set $\Omega \subseteq \mathbb{R}^n$, and form the (x', ξ')-dependent operator function*

$$p(x', \xi') := \mathrm{Op}_{x_n}(a)(x', \xi') : H^s(\mathbb{R}) \longrightarrow H^{s-\mu}(\mathbb{R})$$

between Sobolev spaces on the x_n-axis, assuming that $a(x', x_n, \xi', \xi_n)$ is independent of x_n for large $|x_n|$. Then $p(x', \xi')$ is an operator-valued symbol, in the sense of the following generalization of symbolic estimates (1.1.4)*, namely,*

$$\|\kappa^{-1}_{\langle\xi'\rangle}\{D^\alpha_{x'} D^\beta_{\xi'} p(x', \xi')\}\kappa_{\langle\xi'\rangle}\|_{\mathcal{L}(H^s(\mathbb{R}), H^{s-\mu}(\mathbb{R}))} \leq c\, \langle\xi'\rangle^{\mu-|\beta|}$$

for all $(x', \xi') \in K \times \mathbb{R}^{n-1}$, $K \Subset \Omega', \Omega' \subseteq \mathbb{R}^{n-1}$, and arbitrary multi-indices α, $\beta \in \mathbb{N}^{n-1}$ for constant $c = c(\alpha, \beta, K) > 0$. The group action $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ here refers to $H^s(\mathbb{R})$ rather than $H^s(\mathbb{R}_+)$, cf. (2.1.4).

Remark 2.1.3. *The boundary symbol* (2.1.6) *is a surjective family of Fredholm operators*

$$\sigma_\partial(\Delta)(\xi') : H^s(\mathbb{R}_+) \longrightarrow H^{s-2}(\mathbb{R}_+) \tag{2.1.8}$$

for every $s > \frac{3}{2}, \xi' \neq 0$ *and*

$$\ker \sigma_\partial(\Delta)(\xi') = \big\{ c\mathrm{e}^{-|\xi'|x_n} : c \in \mathbb{C} \big\}. \tag{2.1.9}$$

Thus, denoting by r' *the operator of restriction to* $x_n = 0$ *we have an isomorphism*

$$\mathrm{r}' : \ker \sigma_\partial(\Delta)(\xi') \longrightarrow \mathbb{C}.$$

In fact, the surjectivity of (2.1.8) follow from the existence of a right inverse of $\sigma_\partial(\Delta)(\xi')$ which is of the form

$$\mathrm{r}^+\mathrm{Op}_{x_n}(-|\xi'|^2 - \xi_n^2)^{-1}\mathrm{e}^+ \tag{2.1.10}$$

where

$$\mathrm{e}^+ : H^{s-2}(\mathbb{R}_+) \longrightarrow \mathcal{D}'(\mathbb{R})$$

is the operator of extension by 0 from $\mathbb{R}_+$ to $\mathbb{R}$, and r^+ is the operator of restriction of distribution from $\mathbb{R}$ to $\mathbb{R}_+$. Then interpreting (2.1.8) as operator $\mathrm{r}^+\big(- |\xi'|^2 + \frac{\partial^2}{\partial x_n^2}\big)\mathrm{e}^+$ we obtain

$$\begin{aligned} &\mathrm{r}^+\mathrm{Op}_{x_n}\big(- |\xi'|^2 - \xi_n^2\big)\mathrm{e}^+ \, \mathrm{r}^+\mathrm{Op}_{x_n}\big(- |\xi'|^2 - \xi_n^2\big)^{-1}\mathrm{e}^+ \\ &= \mathrm{r}^+\mathrm{Op}_{x_n}\big(- |\xi'|^2 - \xi_n^2\big)\mathrm{Op}_{x_n}\big(- |\xi'|^2 - \xi_n^2\big)^{-1}\mathrm{e}^+ = \mathrm{id}_{H^{s-2}(\mathbb{R}_+)}. \end{aligned}$$

Let us now consider the ξ'-dependent operators

$$\sigma_\partial(\mathcal{A}_0)(\xi') := \begin{pmatrix} \sigma_\partial(\Delta)(\xi') \\ \mathrm{r}' \end{pmatrix} : H^s(\mathbb{R}_+) \longrightarrow \begin{matrix} H^{s-2}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{matrix}, \tag{2.1.11}$$

$\xi' \neq 0$, where

$$\mathcal{A}_0 = \begin{pmatrix} \Delta \\ T_0 \end{pmatrix} \tag{2.1.12}$$

indicates the operator representing the Dirichlet problem. In order to fix notation, the Dirichlet problem for the Laplacian in a bounded domain $G \subset \mathbb{R}^n$ with smooth boundary ∂G means to solve

$$\Delta u = f, \tag{2.1.13}$$

$$T_0 u = g \tag{2.1.14}$$

where T_0 means the restriction of a function to ∂G. We identify (2.1.13) with a column matrix (2.1.12) of operators between spaces of smooth functions, or, alternatively, functions in Sobolev spaces for $s > 3/2$,

$$\mathcal{A}_0 : C^\infty(\overline{G}) \longrightarrow \begin{matrix} C^\infty(\overline{G}) \\ \oplus \\ C^\infty(\partial G) \end{matrix}, \quad H^s(G) \longrightarrow \begin{matrix} H^{s-2}(G) \\ \oplus \\ H^{s-1/2}(\partial G) \end{matrix}. \tag{2.1.15}$$

Here, $H^s(G) := H^s(\mathbb{R}^n)|_G$. Operators (2.1.15) (either realized in spaces of smooth functions or in Sobolev spaces) represent specific boundary value problems, both called the Dirichlet problem for the Laplacian. If is called elliptic, since both components of the principal symbolic hierarchy

$$\sigma(\mathcal{A}_0) = \big(\sigma_\psi(\mathcal{A}_0), \sigma_\partial(\mathcal{A}_0)\big) \tag{2.1.16}$$

for $\sigma_\psi(\mathcal{A}_0) := \sigma_\psi(\Delta)$, are bijective, cf. Lemma 2.1.4, and the subsequent material.
The operator function (2.1.11) is called the boundary symbol of $\mathcal{A}_0$ in corresponding coordinates, where $\mathrm{r}' := \sigma_\partial(T_0)$.
(2.1.11) is a family of isomorphisms, $\xi' \neq 0$, according to the following general observation.

Lemma 2.1.4. *Let*

$$\begin{pmatrix} A \\ T \end{pmatrix} : H \longrightarrow \begin{matrix} \tilde{H} \\ \oplus \\ L \end{matrix} \tag{2.1.17}$$

be a linear operator for Hilbert spaces $H, \tilde{H}$ and L. Then (2.1.17) *is an isomorphism if and only if*

$$A : H \longrightarrow \tilde{H}$$

is surjective and T induces an isomorphism

$$T|_{\ker A} : \ker A \longrightarrow L. \tag{2.1.18}$$

According to the general terminology the isomorphism (2.1.11) *means that the Dirichlet condition T_0 in connection with the Laplacian Δ is an elliptic boundary condition.*

Another example is T_1, the Neumann condition, defined as $T_1 u := \frac{\partial}{\partial \nu} u|_{\partial G}$ where $\frac{\partial}{\partial \nu}$ is the derivative of u in normal direction to the boundary ∂G. Instead of (2.1.15) we have operators

$$\mathcal{A}_1 = \begin{pmatrix} \Delta \\ T_1 \end{pmatrix} \tag{2.1.19}$$

$$\mathcal{A}_1 : C^\infty(\overline{G}) \longrightarrow \begin{matrix} C^\infty(\overline{G}) \\ \oplus \\ C^\infty(\partial G) \end{matrix}, \quad H^s(G) \longrightarrow \begin{matrix} H^{s-2}(G) \\ \oplus \\ H^{s-3/2}(\partial G) \end{matrix}. \tag{2.1.20}$$

Then

$$\sigma_\partial(\mathcal{A}_1)(\xi') := \begin{pmatrix} \sigma_\partial(\Delta)(\xi') \\ \mathrm{r}' \frac{\partial}{\partial x_n} \end{pmatrix} : H^s(\mathbb{R}_+) \longrightarrow \begin{matrix} H^{s-2}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{matrix}, \tag{2.1.21}$$

which is also a family of isomorphisms, since $\mathrm{r}' \frac{\partial}{\partial x_n} := \sigma_\partial(T_1)$ maps $\ker \sigma_\partial(\Delta)(\xi')$ isomorphically to $\mathbb{C}$, cf. Lemma 2.1.4. We will see later on that the operators (2.1.15) induce isomorphisms between the respective spaces. In addition we will produce the pseudo-differential structure of inverse operators. In the present case we write

$$\mathcal{P}_0 := \mathcal{A}_0^{-1}, \quad \mathcal{P}_0 = (P_0 \quad K_0).$$

In potential theory P_0 is called Green's function and K_0 the double layer potential. We will have

$$P_0 = E + G_0$$

for a fundamental solution of the Laplacian and G_0 will be called later on a Green operator.

$$\mathcal{A}_0\mathcal{P}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{P}_0\mathcal{A}_0 = P_0\Delta + K_0T_0 = 1. \tag{2.1.22}$$

In particular, we have

$$\begin{aligned} \Delta P_0 &= 1, \quad \Delta K_0 = 0, \\ T_0P_0 &= 0, \quad T_0K_0 = 1. \end{aligned}$$

Thus

$$\mathcal{A}_1\mathcal{P}_0 = \begin{pmatrix} 1 & 0 \\ T_1P_0 & T_1K_0 \end{pmatrix}. \tag{2.1.23}$$

2.2 Green, trace and potential symbols

We now study specific operators-valued symbols in the sense of Definition 1.4.4 .

Definition 2.2.1. *An element $k(x',\xi') \in S^{\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1}; \mathbb{C}, \mathcal{S}(\overline{\mathbb{R}}_+))$, $\Omega \subseteq \mathbb{R}^{n-1}$, is called a potential symbol. Here $\mathbb{C}$ is equipped with the trivial group action $\kappa^1_\delta := \mathrm{id}_{\mathbb{C}}$ and $\mathcal{S}(\overline{\mathbb{R}}_+))$ with $\kappa_\delta u(x_n) = \delta^{1/2}u(\delta x_n)$ $\delta \in \mathbb{R}_+$, with $\mathcal{S}(\overline{\mathbb{R}}_+))$ being represented as*

$$\mathcal{S}(\overline{\mathbb{R}}_+) = \varprojlim_{N\in\mathbb{N}} H^{N,N}(\mathbb{R}_+) \text{ for } H^{N,N}(\mathbb{R}_+) = \langle x_n\rangle^{-N}H^N(\mathbb{R}_+), \tag{2.2.1}$$

where $H^N(\mathbb{R}_+) = H^N(\mathbb{R})|_{\mathbb{R}_+}$, cf. Example 1.4.3.

Consider a function $g(x', x_n[\xi'])$ where $g(x',x_n) \in C^\infty(\Omega, \mathcal{S}(\overline{\mathbb{R}}_+))$. Then

$$k(x',\xi') : c \longrightarrow g(x', x_n[\xi'])c \tag{2.2.2}$$

represents a potential symbol $k(x',\xi')$ of order $-\frac{1}{2}$. In fact, we have the identity

$$k(x', \delta\xi')c = \delta^{-\frac{1}{2}}\kappa_\delta k(x',\xi')(\kappa^1_\delta)^{-1}c$$

for $|\xi'| \geq C$, $\delta \geq 1$, i.e.,

$$k(x',\delta\xi')c = \delta^{\frac{1}{2}}\delta - \frac{1}{2}g(x', x_n[\delta\xi'])c = \delta^{-\frac{1}{2}}\kappa_\delta k(x',\xi')c. \tag{2.2.3}$$

Thus

$$k(x',\xi') \in S^{-\frac{1}{2}}_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1}; \mathbb{C}, \mathcal{S}(\overline{\mathbb{R}}_+)),$$

cf. Remark 1.4.11.

Lemma 2.2.2. *The relation*

$$g(x',x_n) \longrightarrow k(x',\xi')$$

defines a continuous operator

$$C^\infty(\Omega, \mathcal{S}(\overline{\mathbb{R}}_+)) \longrightarrow S^{-1/2}_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1}; \mathbb{C}, \mathcal{S}(\overline{\mathbb{R}}_+)). \tag{2.2.4}$$

Lemma 2.2.3. *The* (x', ξ')*-dependent family of mappings* (2.2.6), *i.e.,*

$$k(x', \xi') : \mathbb{C} \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+)$$

determines an operator-valued symbol

$$k(x', \xi') \in S_{\mathrm{cl}}^{-1/2}(\Omega \times \mathbb{R}^{n-1}; \mathbb{C}, \mathcal{S}(\overline{\mathbb{R}}_+)) \hat{\otimes}_\pi S_{\mathrm{cl}}^{\mu}(\mathbb{R}^{n-1}) = S_{\mathrm{cl}}^{\mu-1/2}(\Omega \times \mathbb{R}^{n-1}; \mathbb{C}, \mathcal{S}(\overline{\mathbb{R}}_+))$$

.

Proof. By definition we have relation

$$g(x', x_n, \xi') \in C^\infty(\Omega, \mathcal{S}(\overline{\mathbb{R}}_+)) \hat{\otimes}_\pi S_{\mathrm{cl}}^{\mu}(\mathbb{R}^{n-1}).$$

Applying Theorem 1.7.1 to $E := C^\infty(\Omega, \mathcal{S}(\overline{\mathbb{R}}_+))$, $F := S_{\mathrm{cl}}^{\mu}(\mathbb{R}^{n-1})$, we obtain a convergent sum

$$g(x', x_n, \xi') = \sum_{j=0}^{\infty} \lambda_j g_j(x', x_n) p_j(\xi')$$

for $\lambda_j \in \mathbb{C}$, $\sum_{j=0}^{\infty} |\lambda_j| < \infty$ and elements $g_j(x', x_n) \in C^\infty(\Omega, \mathcal{S}(\overline{\mathbb{R}}_+))$, $p_j(\xi') \in S_{\mathrm{cl}}^{\mu}(\mathbb{R}^{n-1})$, tending to zero in the respective spaces, as $j \to \infty$. From Lemma 2.2.2 from $g_j(x', x_n)$ we obtain potential symbols

$$k_j(x', \xi') \in S_{\mathrm{cl}}^{-1/2}(\Omega \times \mathbb{R}^{n-1}; \mathbb{C}, \mathcal{S}(\overline{\mathbb{R}}_+)),$$

tending to zero in this symbol space as $j \longrightarrow \infty$. From Remark 1.4.12 (iii) together with evident continuity product in the involved factors it follows that

$$k(x', \xi') = \sum_{j=0}^{\infty} \lambda_j k_j(x', \xi') p_j(\xi')$$

converges in the space $S_{\mathrm{cl}}^{-1/2}(\Omega \times \mathbb{R}^{n-1}; \mathbb{C}, \mathcal{S}(\overline{\mathbb{R}}_+)) \hat{\otimes}_\pi S_{\mathrm{cl}}^{\mu}(\mathbb{R}^{n-1})$. □

In other words, for every

$$g(x', x_n, \xi') \in C^\infty(\Omega, \mathcal{S}(\overline{\mathbb{R}}_+)) \hat{\otimes}_\pi S_{\mathrm{cl}}^{\mu}(\mathbb{R}^{n-1}) = \mathcal{S}(\overline{\mathbb{R}}_+, S_{\mathrm{cl}}^{\mu}(\Omega \times \mathbb{R}^{n-1})) \tag{2.2.5}$$

by

$$k(x', \xi') : c \longrightarrow g(x', x_n[\xi'], \xi')c, \tag{2.2.6}$$

we obtain a potential symbol $k(x', \xi')$ of order $\mu - \frac{1}{2}$. Then we obtain a continuous operator, called a potential operator,

$$\mathrm{Op}_{x'}(k) : \mathcal{W}_{\mathrm{comp}}^{s}(\Omega, \mathbb{C}) \longrightarrow \mathcal{W}_{\mathrm{loc}}^{s-\mu+\frac{1}{2}}(\Omega, \mathcal{S}(\overline{\mathbb{R}}_+)), \tag{2.2.7}$$

for every $s \in \mathbb{R}$, cf. Theorem 1.5.9.
The counterpart of potential operators in Boutet de Monvel's calculus of boundary value problems are trace operators, belonging to trace symbols. Such a symbol is obtained by an operator function $g(x', x_n[\xi'])$ for a

$$g(x', x_n) \in C^\infty(\Omega, \mathcal{S}(\overline{\mathbb{R}}_+)),$$

where the associated (operator-valued) trace symbol acts on $H^s(\mathbb{R}_+)$ for $s > -\frac{1}{2}$ by

$$b(x', \xi')u := \int_0^\infty g(x', x_n[\xi'])u(x_n)\, dx_n. \tag{2.2.8}$$

Note that for $u \in H^s(\mathbb{R}_+)$, $-\frac{1}{2} < s < \frac{1}{2}$, the extension of u by 0 to the negative half-axis, denoted by e^+u, gives us $\mathrm{e}^+u \in H^s(\mathbb{R})$. More generally, similarly as in the discussion on potential symbols we admit functions (2.2.5) and the associated trace symbol has the form

$$b(x', \xi')u := \int_0^\infty g(x', x_n[\xi'], \xi')u(x_n)\, dx_n. \tag{2.2.9}$$

Lemma 2.2.4. *The* (x', ξ')*-dependent family of mappings* (2.2.9)*, i.e.,*

$$\mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \mathbb{C}$$

determines an operator-valued symbol $b(x', \xi') \in S_{\mathrm{cl}}^{\mu-1/2}(\Omega \times \mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+), \mathbb{C})$ *and*

$$b(x', \xi') \in S_{\mathrm{cl}}^{\mu-1/2}(\Omega \times \mathbb{R}^{n-1}; H^s(\mathbb{R}_+), \mathbb{C})$$

for every $s \in \mathbb{R}$, $s > -1/2$.

Proof. We first consider the special case (2.2.8) and show that we get a symbol in $S_{\mathrm{cl}}^{-\frac{1}{2}}(\Omega \times \mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+), \mathbb{C})$. Similarly as in the proof of Lemma 2.2.3 it suffices to check the property

$$b(x', \xi') \in C^\infty(\Omega \times \mathbb{R}^{n-1}, \mathcal{L}(\mathcal{S}(\overline{\mathbb{R}}_+), \mathbb{C}))$$

which is evident. The homogeneity

$$b(x', \delta\xi') = \delta^{-1/2} b(x', \xi')\kappa_\delta^{-1}$$

for all $\delta \geq 1$ and $|\xi'| > C$, cf. Remark 1.4.11, is obtained as follows :

$$\begin{aligned} b(x', \delta\xi')u &= \int_0^\infty g(x', x_n[\delta\xi'])u(x_n)\, dx_n \\ &= \int_0^\infty g(x', x_n[\delta\xi'])u(x_n)\, dx_n \\ &= \int_0^\infty g(x', \tilde{x_n}[\xi'])u(\delta^{-1}\tilde{x_n})\delta^{-1}\, d\tilde{x_n} \\ &= \int_0^\infty g(x', \tilde{x}_n[\xi'])(\kappa_\delta^{-1}u)(\tilde{x}_n)\delta^{-1/2}\, d\tilde{x}_n = \delta^{-1/2} b(x', \xi')\kappa_\delta^{-1}u. \end{aligned}$$

□

In Boutet de Monvel's boundary symbolic calculus there are also singular trace operators, namely,

$$\gamma^k : u(x_n) \longrightarrow \mathrm{r}'\left(\frac{\partial^k}{\partial {x_n}^k}\right) \tag{2.2.10}$$

for some $k \in \mathbb{N}$.

Lemma 2.2.5. *We have*

$$\gamma^k \in S_{\mathrm{cl}}^{k+1/2}(\mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+), \mathbb{C})$$

and also

$$\gamma^k \in S_{\mathrm{cl}}^{k+1/2}(\mathbb{R}^{n-1}; H^s(\mathbb{R}_+), \mathbb{C})$$

for all $s \in \mathbb{R}, s - k > \frac{1}{2}$.

Proof. Applying Remark 1.4.11 it suffices to show that

$$\gamma^k \in C^\infty(\mathbb{R}^{n-1}_{\xi'}, \mathcal{L}(\mathcal{S}(\overline{\mathbb{R}}_+), \mathbb{C})) \quad \text{or} \quad \gamma^k \in C^\infty(\mathbb{R}^{n-1}_{\xi'}, \mathcal{L}(H^s(\mathbb{R}^{n-1}_{\xi'}), \mathbb{C})),$$

$s > k + \frac{1}{2}$. This is clear since γ^k does not depend on ξ'. Moreover, we have twisted homogeneity

$$\gamma^k u = \delta^{k+\frac{1}{2}} \kappa_\delta^1 \gamma^k \kappa_\delta^{-1} u \tag{2.2.11}$$

for $\delta \in \mathbb{R}_+, \xi' \in \mathbb{R}^{n-1}$. Recall that $\kappa_\delta^1 = \mathrm{id}_{\mathbb{C}}$ for all δ. Thus

$$\gamma^k \kappa_\delta^{-1} u = \gamma^k \delta^{-1/2} u(\delta^{-1} x_n) = \delta^{-1/2} \delta^{-k} u(0),$$

i.e., we obtain relation (2.2.11), as desired. □

The trace symbols of the form (2.2.9) or (2.2.10) are special examples of trace symbols in general. The x'-dependence is not so essential for the structure; so we now omit x'. In order to generate such symbols in unified form we consider families of maps

$$b(\xi') : u(x_n) \longrightarrow \int_0^\infty g(x_n[\xi'], \xi') D_{x_n}^k u(x_n)\, dx_n.$$

Integration by parts gives us

$$\begin{aligned} b(\xi')u &= \int_0^\infty D_{x_n}(g(x_n[\xi'], \xi') D_{x_n}^{k-1} u(x_n))\, dx_n - \int_0^\infty g_1(x_n[\xi'], \xi') D_{x_n}^{k-1} u(x_n)\, dx_n \\ &= g(x_n[\xi'], \xi') D_{x_n}^{k-1} u(x_n)\Big|_0^\infty - \int_0^\infty g_1(x_n[\xi'], \xi') D_{x_n}^{k-1} u(x_n)\, dx_n \\ &= -g_0(0, \xi')\gamma^{k-1} u - \int_0^\infty g_1(x_n[\xi'], \xi') D_{x_n}^{k-1} u(x_n)\, dx_n \end{aligned} \tag{2.2.12}$$

for

$$\begin{aligned} g_0(0, \xi') &:= g(0, \xi') \in S_{\mathrm{cl}}^\mu(\mathbb{R}^{n-1}_{\xi'}), \\ g_1(x_n[\xi'], \xi') &:= [\xi'](D_{x_n} g_0)(x_n[\xi'], \xi') \end{aligned}$$

for $g_0 := g_0(x_n[\xi'], \xi')$. By iteration of this computation it follows that

$$b(\xi')u = \sum_{j=1}^{k-1} (-1)^j g_{j-1}(0, \xi')\gamma^{k-j} u + \int_0^\infty (-1)^k g_k(x_n[\xi'], \xi') u(x_n)\, dx_n \tag{2.2.13}$$

for

$$g_j(0, \xi') = D_{x_n} g_{j-1}(x_n[\xi'], \xi')\Big|_{x_n=0} \in S_{\mathrm{cl}}^{\mu+j}(\mathbb{R}^{n-1}_{\xi'}).$$

Here $g_{j-1}(0, \xi')\gamma^{k-j}$ belong to $S_{\mathrm{cl}}^{\mu+k+1/2}(\mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+), \mathbb{C})$ for all j. Let us consider other specific operator-valued symbols occurring in boundary value problems, namely, Green symbols.

Definition 2.2.6. (i) *A Green symbol $g(x',\xi')$ of order $\nu \in \mathbb{R}$ and of type 0 is a symbol*

$$g(x',\xi') \in S^{\nu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1}; L^2(\mathbb{R}_+), L^2(\mathbb{R}_+)) \tag{2.2.14}$$

referring to $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$, $(\kappa_\delta u)(x_n) = \delta^{1/2}u(\delta x_n)$, acting in $L^2(\mathbb{R}_+)$, such that

$$g(x',\xi') \in S^{\nu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1}; L^2(\mathbb{R}_+), \mathcal{S}(\overline{\mathbb{R}}_+))$$

and

$$g^*(x',\xi') \in S^{\nu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1}; L^2(\mathbb{R}_+), \mathcal{S}(\overline{\mathbb{R}}_+))$$

with indicating the (x',ξ')-wise adjoint in $L^2(\mathbb{R}_+)$.

(ii) *A Green symbol $g(x',\xi')$ of order ν and type $e \in \mathbb{N}$ is defined as a symbol of the form*

$$g(x',\xi') := \sum_{j=0}^{e} g_j(x',\xi')\frac{d^j}{dx_n^j} \tag{2.2.15}$$

for Green symbols $g_j(x',\xi')$ of order $\nu - j$ and of type 0, in the sense of (i).

Let $R^{\nu,e}_{\mathrm{G}}(\Omega \times \mathbb{R}^{n-1})$ denote the space of all Green symbols of order $\nu \in \mathbb{R}$ and type $e \in \mathbb{N}$. Observe that we have natural inclusions

$$R^{\nu,e}_{\mathrm{G}}(\Omega \times \mathbb{R}^{n-1}) \subset S^{\nu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1}; H^s(\mathbb{R}_+), \mathcal{S}(\overline{\mathbb{R}}_+)) \tag{2.2.16}$$

for all $s > e - \frac{1}{2}$.

Let us investigate the internal structure of such symbols. First we have the following Theorem.

Theorem 2.2.7. *For an operator $g \in \mathcal{L}(L^2(\mathbb{R}_+))$ the following properties are equivalent:*

(i) *g induces continuous operators*

$$g, g^* : L^2(\mathbb{R}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+).$$

(ii) *There exists a $c(x_n, x_n') \in \mathcal{S}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+) = \mathcal{S}(\mathbb{R}\times\mathbb{R})|_{\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+} = \mathcal{S}(\overline{\mathbb{R}}_+)\hat{\otimes}_\pi\mathcal{S}(\overline{\mathbb{R}}_+)$ such that*

$$gu(x_n) = \int_0^\infty c(x_n, x_n')u(x_n')\,dx_n'.$$

Proof. (ii) $\Rightarrow$ (i) is evident. Thus it remaines to show (i) $\Rightarrow$ (ii). The continuity of

$$g : L^2(\mathbb{R}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+) \tag{2.2.17}$$

implies that

$$gu(x_n) = \int_0^\infty c(x_n, x_n')u(x_n')\,dx_n' \tag{2.2.18}$$

for some $c(x_n, x_n') \in \mathcal{S}(\overline{\mathbb{R}}_+)\hat{\otimes}_\pi L^2(\mathbb{R}_+)$. Analogously

$$g^* : L^2(\mathbb{R}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+) \tag{2.2.19}$$

where

$$g^*v(x_n') = \int_0^\infty c(x_n, x_n')v(x_n)\, dx_n \tag{2.2.20}$$

implies $c(x_n, x_n') \in L^2(\mathbb{R}_+)\hat{\otimes}_\pi \mathcal{S}(\overline{\mathbb{R}}_+)$. Thus

$$c(x_n, x_n') \in \big(\mathcal{S}(\overline{\mathbb{R}}_+)\hat{\otimes}_\pi L^2(\mathbb{R}_+)\big) \cap \big(L^2(\mathbb{R}_+)\hat{\otimes}_\pi \mathcal{S}(\overline{\mathbb{R}}_+)\big).$$

It follows that g is Hilbert-Schmidt operator, i.e., $c(x_n, x_n') \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$. Moreover, (2.2.18), (2.2.20) implyes

$$\begin{aligned} \langle x_n\rangle^l g : L^2(\mathbb{R}_+) &\longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+), \\ \langle x_n'\rangle^{l'} g^* : L^2(\mathbb{R}_+) &\longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+), \end{aligned} \tag{2.2.21}$$

for every $l, l' \in \mathbb{N}$. Thus, $\langle x_n\rangle^l c(x_n, x_n'), \langle x_n'\rangle^{l'} c(x_n, x_n') \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$. For every $N \in \mathbb{N}$ there are $l, l' \in \mathbb{N}$ such that

$$(1 + |x_n|^2 + |x_n'|^2)^{N/2} = \langle x_n, x_n'\rangle^N \leq c\big(\langle x_n\rangle^l + \langle x_n'\rangle^{l'}\big) \tag{2.2.22}$$

for some constant $c > 0$, we can replace the left-hand side by

$$(1 + |x_n|^2 + |x_n'|^2)^{N/2}.$$

In addition, it suffices to estimate the expression by $(1 + |x_n|^2 + |x_n'|^2)^N$. Choosing l, l' large enough for N rather than $N/2$ we btain the desired estimate. Since l, l' are to be chosen large, we can again replace l, l' by $2l, 2l'$. Thus (2.2.22) will follow from

$$\big(1 + |x_n|^2 + 1 + |x_n'|^2\big)^N \leq c\big((1 + |x_n|^2)^L + (1 + |x_n'|^2)^L\big)$$

for sufficiently large L. Applying the binomial formula for $L := N$ we have

$$\frac{\big(1 + |x_n|^2 + 1 + |x_n'|^2\big)^N}{(1 + |x_n|^2)^N + (1 + |x_n'|^2)^N} \leq c$$

for all $x_n, x_n' \in \overline{\mathbb{R}}_+$. It follows that $\langle x_n, x_n'\rangle^N c(x_n, x_n') \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$ for every $N \in \mathbb{N}$.

Applying once again (2.2.17), (2.2.19) we obtain that the operators with kernels

$$\partial_{x_n}^k c(x_n, x_n'), \partial_{x_n'}^{k'} c(x_n, x_n')$$

also induce continuous operators

$$L^2(\mathbb{R}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+),$$

for arbitrary $k, k' \in \mathbb{N}$. Combined with the conclusion before we also have

$$\partial_{x_n}^k \langle x_n, x_n'\rangle^N c(x_n, x_n'), \partial_{x_n'}^{k'} \langle x_n, x_n'\rangle^N c(x_n, x_n') \in L^2(\mathbb{R}_+ \times \mathbb{R}_+) \tag{2.2.23}$$

for all $k, k', N \in \mathbb{N}$. It remains to verify also

$$\partial_{x_n}^m \partial_{x_n'}^{m'} \langle x_n, x_n'\rangle^N c(x_n, x_n') \in L^2(\mathbb{R}_+ \times \mathbb{R}_+),$$

for arbitrary $m, m' \in \mathbb{N}$. We first consider the case $m = m' = 1$. Here we employ the identity

$$\big(\partial_{x_n} + \partial_{x'_n}\big)^2 = \partial^2_{x_n} + \partial^2_{x'_n} + 2\partial_{x_n}\partial_{x'_n}.$$

From (2.2.23) for $k = k' = 1$ we conclude

$$\big(\partial_{x_n} + \partial_{x'_n}\big)\langle x_n, x'_n\rangle^N c(x_n, x'_n) \in L^2(\mathbb{R}_+ \times \mathbb{R}_+).$$

However, since the mapping property of the operator with the latter kernel has again the mapping properties (2.2.18), (2.2.20), we may repeat the conclusion and arrive at the kernel

$$\big(\partial_{x_n} + \partial_{x'_n}\big)^2\langle x_n, x'_n\rangle^N c(x_n, x'_n) \in L^2(\mathbb{R}_+ \times \mathbb{R}_+). \tag{2.2.24}$$

with the above-mentioned mapping property. Then subtracting from the kernel in (2.2.24) the kernels (2.2.23) for $k = 2$ gives us the kernel

$$2\partial_{x_n}\partial_{x'_n}\langle x_n, x'_n\rangle^N c(x_n, x'_n) \in L^2(\mathbb{R}_+ \times \mathbb{R}_+).$$

Iterating this process and combinis it with (2.2.23) yields

$$\partial^m_{x_n}\partial^{m'}_{x'_n}\langle x_n, x'_n\rangle^N c(x_n, x'_n) \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$$

for arbitrary $m, m', N \in \mathbb{N}$, which is just the claimed property $c(x_n, x'_n) \in \mathcal{S}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+)$.

$\square$

Theorem 2.2.8. *A $g(x', \xi')$ is Green symbol of order ν and type zero if and only if there is a function*

$$f(x', x_n, \xi', x'_n) \in C^\infty(\Omega, \mathcal{S}(\overline{\mathbb{R}}_+))\hat{\otimes}_\pi\mathcal{S}(\overline{\mathbb{R}}_+)\hat{\otimes}_\pi S^{\nu+1}_{\mathrm{cl}}(\mathbb{R}^{n-1})$$

such that

$$g(x', \xi')u(x_n) = \int_0^\infty f(x', \xi', x_n[\xi'], x'_n[\xi'])u(x'_n)\, dx'_n. \tag{2.2.25}$$

Proof. Let us first check that $g(x', \xi')$ given by (2.2.25) determines a symbol (2.2.14). We have to check the symbolic estimates, namely,

$$\|\kappa^{-1}_{\langle\xi'\rangle}\big\{D^\alpha_x D^\beta_{\xi'} g(x', \xi')\big\}\kappa_{\langle\xi'\rangle}\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \le c\langle\xi'\rangle^{\nu-|\beta|} \tag{2.2.26}$$

for $(x', \xi') \in K \times \mathbb{R}^{n-1}, K \Subset \Omega$, and all $\alpha, \beta \in \mathbb{N}^{n-1}$, for some constants $c = c(\alpha, \beta, K) > 0$. Let us first look at the case $\alpha = \beta = 0$. Then we have

$$\kappa^{-1}_{\langle\xi'\rangle}g(x', \xi')\kappa_{\langle\xi'\rangle}u(x_n) = \int_0^\infty \langle\xi'\rangle^{-1/2} f(x', \xi', \langle\xi'\rangle^{-1}x_n[\xi'], x'_n[\xi'])\langle\xi'\rangle^{1/2}u(\langle\xi'\rangle x'_n)\, dx'_n. \tag{2.2.27}$$

Substituting $\tilde{x}_n := \langle\xi'\rangle x'_n$ it follows that $d\tilde{x}_n = \langle\xi'\rangle dx'_n$ and (2.2.27) takes the form

$$\int_0^\infty f(x', \xi', \frac{[\xi']}{\langle\xi'\rangle}x_n, \frac{[\xi']}{\langle\xi'\rangle}\tilde{x}_n)u(\tilde{x}_n)\langle\xi'\rangle^{-1}\, d\tilde{x}_n.$$

Thus

$$\begin{aligned}
&\left\| \int_0^\infty f(x',\xi',\frac{[\xi']}{\langle\xi'\rangle}x_n,\frac{[\xi']}{\langle\xi'\rangle}x_n')u(x_n')\langle\xi'\rangle^{-1}\,dx_n' \right\|^2_{L^2(\mathbb{R}_+)} \\
&= \int_0^\infty \left|\left[\int_0^\infty f(x',\xi',\frac{[\xi']}{\langle\xi'\rangle}x_n,\frac{[\xi']}{\langle\xi'\rangle}x_n')u(x_n')\langle\xi'\rangle^{-1}\,dx_n'\right]\right|^2 dx_n \\
&\le \int_0^\infty \left(\int_0^\infty \left|\langle x_n\rangle^{-N}\langle x_n\rangle^{N}\langle x_n'\rangle^{-N}\langle x_n'\rangle^{N} f(x',\xi',\frac{[\xi']}{\langle\xi'\rangle}x_n,\frac{[\xi']}{\langle\xi'\rangle}x_n')u(x_n')\langle\xi'\rangle^{-1}\right| dx_n'\right)^2 dx_n \\
&\le \sup_{\substack{x'\in K\\ x_n,x_n'\in\mathbb{R}_+}} |\langle x_n\rangle^{N}\langle x_n'\rangle^{N} f(x',\xi',\frac{[\xi']}{\langle\xi'\rangle}x_n,\frac{[\xi']}{\langle\xi'\rangle}x_n')|^2 \int_0^\infty \left(\int_0^\infty |\langle x_n\rangle^{-N}\langle x_n'\rangle^{-N}u(x_n')\langle\xi'\rangle^{-1}|dx_n'\right)^2 dx_n \\
&\le c(\langle\xi'\rangle^{\nu+1})^2 \int \langle x_n\rangle^{-2N}\langle\xi'\rangle^{-2}\left(\int |\langle x_n'\rangle^{-N}u(x_n')|\,dx_n'\right)^2 dx_n \\
&\le c(\langle\xi'\rangle^{\nu})^2 \int_0^\infty \langle x_n\rangle^{-2N}\,dx_n \left(\int \langle x_n'\rangle^{-N}|u(x_n')|\,dx_n'\right)^2 \\
&\le c\langle\xi'\rangle^{2\nu}\left(\|\langle x_n'\rangle^{-N}\|_{L^2(\mathbb{R}_+)}\|u(x_n')\|_{L^2(\mathbb{R}_+)}\right)^2 \\
&\le c\langle\xi'\rangle^{2\nu}\|u(x_n')\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned} \tag{2.2.28}$$

In other words we proved the estimate (2.2.26) for $\alpha=\beta=0$. The corresponding estimates for arbitrary α,β are straightforward. More generally, if $p(\cdot)$ is a semi-norm on the space $\mathcal{S}(\overline{\mathbb{R}}_+)$, i.e.,

$$p(v) := \sup_{x_n\in\overline{\mathbb{R}}_+} \langle x_n\rangle^k |\partial^l_{x_n} v(x_n)|,\ k,l\in\mathbb{N}, \tag{2.2.29}$$

we have to show that

$$p(\kappa^{-1}_{\langle\xi'\rangle}\{D^\alpha_{x'}D^\beta_{\xi'}g(x',\xi')\}\kappa_{\langle\xi'\rangle}u) \le c\langle\xi'\rangle^{\nu-|\beta|}\|u\|_{L^2(\mathbb{R}_+)}.$$

Consider once again first $\alpha=\beta=0, k\in\mathbb{N}$, and $l=1$. Then we look at

$$\sup_{x_n\in\overline{\mathbb{R}}_+} \langle x_n\rangle^k \left| \int_0^\infty \frac{\partial}{\partial x_n}\langle\xi'\rangle^{-1/2} f(x',\xi',\frac{[\xi']}{\langle\xi'\rangle}x_n,[\xi']x_n')\langle\xi'\rangle^{1/2}u(\langle\xi'\rangle x_n')\,dx_n'\right|.$$

We have for the above expression

$$\begin{aligned}
&\sup\langle x_n\rangle^k\left|\int_0^\infty \langle\xi'\rangle^{-1/2}\frac{[\xi']}{\langle\xi'\rangle}(\partial_{x_n}f)(x',\xi',\frac{[\xi']}{\langle\xi'\rangle}x_n,[\xi']x_n')\langle\xi'\rangle^{1/2}u(\langle\xi'\rangle x_n')\,dx_n'\right| \\
&= \sup\langle x_n\rangle^k\left|\int_0^\infty \frac{[\xi']}{\langle\xi'\rangle}(\partial_{x_n}f)(x',\xi',\frac{[\xi']}{\langle\xi'\rangle}x_n,\frac{[\xi']}{\langle\xi'\rangle}\tilde{x}_n,\xi')u(\tilde{x}_n)\langle\xi'\rangle^{-1}\,d\tilde{x}_n\right| \\
&= \sup\langle x_n\rangle^k\left|\int_0^\infty \frac{[\xi']}{\langle\xi'\rangle}(\partial_{x_n}f)(x',\xi',\frac{[\xi']}{\langle\xi'\rangle}x_n,\frac{[\xi']}{\langle\xi'\rangle}\tilde{x}_n,\xi')\langle\tilde{x}_n\rangle^{N}\langle\tilde{x}_n\rangle^{-N}u(\tilde{x}_n)\langle\xi'\rangle^{-1}\,d\tilde{x}_n\right| \\
&\le \sup_{x_n,\tilde{x}_n\in\overline{\mathbb{R}}_+}(\langle x_n\rangle^k\frac{[\xi']}{\langle\xi'\rangle}(\partial_{x_n}f)(x',\xi',\frac{[\xi']}{\langle\xi'\rangle}x_n,\frac{[\xi']}{\langle\xi'\rangle}\tilde{x}_n)\langle\tilde{x}_n\rangle^{N}\langle\xi'\rangle^{-1})\int_0^\infty |\langle\tilde{x}_n\rangle^{-N}u(\tilde{x}_n)|\,d\tilde{x}_n \\
&\le c\langle\xi'\rangle^{\nu}\left(\int |\langle\tilde{x}_n\rangle^{-N}|^2\,d\tilde{x}_n\right)^{1/2}\left(\int |u(\tilde{x}_n)|^2\,d\tilde{x}_n\right)^{1/2} \\
&\le c\langle\xi'\rangle^{\nu}\|u\|_{L^2(\mathbb{R}_+)}.
\end{aligned}$$

The corresponding estimates for (2.2.29) for arbitrary $k, l \in \mathbb{N}$ are easy as well and also those for arbitrary α, β. Conversely, let $g(x', \xi')$ be a Green symbol of order ν and type 0, we can generate a function f as follows.

From Definition 2.2.6 the operator-valued symbol $g(x', \xi')$ has a sequence of twisted homogeneous components

$$g_{(\nu-j)}(x', \xi') \in S^{(\nu-j)}(\Omega \times (\mathbb{R}^{n-1} \setminus \{0\}); L^2(\mathbb{R}_+), \mathcal{S}(\overline{\mathbb{R}}_+)).$$

By virtue of Theorem 2.2.7, applied for every $(x', \xi') \in \Omega \times (\mathbb{R}^{n-1} \setminus \{0\})$ we find an

$$f_{(\nu-j)}(x', \xi'; x_n, x_n') \in C^\infty(\Omega \times (\mathbb{R}^{n-1} \setminus \{0\}), \mathcal{S}(\overline{\mathbb{R}}_+)\hat{\otimes}_\pi \mathcal{S}(\overline{\mathbb{R}}_+))$$

such that

$$g_{(\nu-j)}(x', \xi')u(x_n) = \int f_{(\nu-j)}(x', \xi'; x_n, x_n')u(x_n')\, dx_n'.$$

Using twisted homogeneity of order $\nu - j$, i.e.,

$$g_{(\nu-j)}(x', \delta\xi') = \kappa_\delta \delta^{\nu-j} g_{(\nu-j)}(x', \xi')\kappa_\delta^{-1}, \delta \in \mathbb{R}_+,$$

it follows that

$$\begin{aligned}\int f_{(\nu-j)}(x', \delta\xi'; x_n, x_n')u(x_n')\, dx_n' &= \kappa_\delta \delta^{\nu-j} \int f_{(\nu-j)}(x', \xi'; x_n, x_n')u(x_n')(\kappa_\delta^{-1}u)(x_n')\, dx_n' \\ &= \delta^{\nu-j} \int \delta^{1/2} f_{(\nu-j)}(x', \xi'; \delta x_n, x_n')\delta^{-1/2}u(\delta^{-1}x_n')\, dx_n' \\ &= \delta^{\nu-j} \int f_{(\nu-j)}(x', \xi'; \delta x_n, \delta x_n')\delta u(x_n')\, dx_n'.\end{aligned} \tag{2.2.30}$$

Analogously,

$$g_{(\nu-j)}(x', \xi') = \delta^{-\nu+j}\kappa_\delta^{-1} g_{(\nu-j)}(x', \delta\xi')\kappa_\delta$$

gives us

$$\begin{aligned}g_{(\nu-j)}(x', \xi')u(x_n) &= \delta^{-\nu+j} \int \kappa_\delta^{-1} f_{(\nu-j)}(x', \delta\xi'; x_n, x_n')(\kappa_\delta u)(x_n')\, dx_n' \\ &= \delta^{-\nu+j} \int f_{(\nu-j)}(x', \delta\xi'; \delta^{-1}x_n, \delta^{-1}x_n')\delta^{-1}u(x_n')\, dx_n'.\end{aligned} \tag{2.2.31}$$

The latter relation holds for every $\delta \in \mathbb{R}_+$. Inserting $\delta = [\xi']^{-1}$ yields the kernel function of $g_{(\nu-j)}(x', \xi')$ of the form $[\xi']^{\nu-j+1} f(x', \frac{\xi'}{[\xi']}; E)$ where $E = \mathcal{S}(\overline{\mathbb{R}}_{+,\tilde{x}_n} \times \overline{\mathbb{R}}_{+,\tilde{x}_n'})$ for $\tilde{x}_n := [\xi']x_n, \tilde{x}_n' := [\xi']x_n'$. Thus for any excision function $\chi(\xi')$ we have

$$\chi(\xi')[\xi']^{\nu-j+1} f_{(\nu-j)}(x', \frac{\xi'}{[\xi']}) \in S_{\mathrm{cl}}^{\nu-j+1}(\Omega \times \mathbb{R}^{n-1}, E)$$

we can form the asymptotic sum

$$f_1(x', \xi') \sim \sum_{j=0}^{\infty} \chi(\xi')[\xi']^{\nu-j+1} f_{(\nu-j)}(x', \xi') \in S_{\mathrm{cl}}^{\nu+1}(\Omega \times \mathbb{R}^{n-1}, E)$$

$f_1(x',\xi')$ is uniquely determined modulo $S^{-\infty}(\Omega\times\mathbb{R}^{n-1},E)$, i.e.,

$$g((x',\xi')u(x_n) = \int f_1(x',\xi';[\xi']x_n,[\xi']x_n')u(x_n')\,dx_n' + g_{-\infty}(x',\xi')u(x_n'),$$

$$g_{-\infty}(x',\xi')u(x_n) = \int f_{-\infty}(x',\xi';[\xi']x_n,[\xi']x_n')u(x_n')\,dx_n',$$

for an $f_{-\infty}(x',\xi';\tilde{x}_n,\tilde{x}_n') \in S^{-\infty}(\Omega\times\mathbb{R}^{n-1},\mathcal{S}(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+))$. Then we may set

$$f(x',\xi';[\xi']x_n,[\xi']x_n') := f_1(x',\xi';[\xi']x_n,[\xi']x_n') + f_{-\infty}(x',\xi';[\xi']x_n,[\xi']x_n').$$

By construction we have

$$f(x',\xi';\tilde{x}_n,\tilde{x}_n') \in \mathcal{S}(\Omega\times\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+,S_{\mathrm{cl}}^{\nu+1}(\mathbb{R}^{n-1})) \tag{2.2.32}$$

for $\mathcal{S}(\Omega\times\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+) := C^\infty(\Omega,\mathcal{S}(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+))$. The map

$$\sigma^{\nu+1-j} : S_{\mathrm{cl}}^{\nu+1}(\mathbb{R}^{n-1}) \longrightarrow S^{(\nu+1-j)}(\mathbb{R}^{n-1}\setminus\{0\}) \tag{2.2.33}$$

induces a map

$$\mathcal{S}(\Omega\times\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+,S_{\mathrm{cl}}^{\nu+1}(\mathbb{R}^{n-1})) \longrightarrow \mathcal{S}(\Omega\times\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+,S^{(\nu+1-j)}(\mathbb{R}^{n-1}\setminus\{0\})). \tag{2.2.34}$$

Then the twisted homogeneous component of order $\nu+1-j$ of the operator-valued symbol $f(x',\xi';x_n[\xi'],x_n'[\xi'])$ is equal to $\tilde{\sigma}^{\nu+1-j}(f)(x',\xi',x_n|\xi'|,x_n'|\xi'|)$. Moreover, observe that when we choose different functions $\xi'\to[\xi'],\xi'\to[\tilde{\xi}']$ which equal to $|\xi'|$ for large $|\xi'|$, then

$$f(x',\xi';x_n[\xi'],x_n'[\xi']) = f(x',\xi';x_n[\tilde{\xi}'],x_n'[\tilde{\xi}'])\bmod R_{\mathrm{G}}^{-\infty}(\Omega\times\mathbb{R}^{n-1}).$$

□

Remark 2.2.9. *If $g(x',\xi')\in R_{\mathrm{G}}^{\nu,e}(\Omega\times\mathbb{R}^{n-1})$ we have*

$$x_n^l g(x',\xi') \in R_{\mathrm{G}}^{\nu-l,e}(\Omega\times\mathbb{R}^{n-1}) \tag{2.2.35}$$

for every $l\in\mathbb{N}$. More generally, for any $\varphi(x_n)\in C^\infty(\overline{\mathbb{R}}_+)$ such that the operator of multiplication by φ preserves the space $\mathcal{S}(\overline{\mathbb{R}}_+)$ we have

$$\varphi(x_n)g(x',\xi') \in R_{\mathrm{G}}^{\nu,e}(\Omega\times\mathbb{R}^{n-1}). \tag{2.2.36}$$

In particular, if φ vanishes of order l at $x_n=0$ instead of (2.2.36) *we have*

$$\varphi(x_n)g(x',\xi') \in R_{\mathrm{G}}^{\nu-l,e}(\Omega\times\mathbb{R}^{n-1}).$$

Definition 2.2.10. (i) *A potential symbol $k(x',\xi')$ of order $\nu\in\mathbb{R}$ is a symbol*

$$k(x',\xi') \in S_{\mathrm{cl}}^{\nu}(\Omega\times\mathbb{R}^{n-1};\mathbb{C},\mathcal{S}(\overline{\mathbb{R}}_+)).$$

(ii) *A trace symbol $b(x',\xi')$ of order $\nu\in\mathbb{R}$ and type 0 is a symbol*

$$S_{\mathrm{cl}}^{\nu}(\Omega\times\mathbb{R}^{n-1};L^2(\mathbb{R}_+),\mathbb{C})$$

such that the (x',ξ')-wise adjoint $b^*(x',\xi')$ is a potential symbol in the sense of (i).

(iii) *A trace symbol* $b(x', \xi')$ *of order* ν *and type* $e \in \mathbb{N}$ *is defined as a symbol*

$$b(x', \xi') := \sum_{j=0}^{e} b_j(x', \xi') \frac{d^j}{dx_n^j} \tag{2.2.37}$$

for trace symbols $b_j(x', \xi')$ *of order* $\nu - j$ *and type* 0.

Proposition 2.2.11. *Every Green symbol* $g(x', \xi') \in R_{\mathrm{G}}^{\nu,e}(\Omega \times \mathbb{R}^{n-1})$ *cf. Definition* 2.2.6, *can be written in the form*

$$g(x', \xi') = \sum_{j=0}^{e-1} k_j(x', \xi') \circ \gamma^j + g_0(x', \xi') \tag{2.2.38}$$

for $\gamma^j u := (\frac{d^j}{dx_n^j} u)(0)$, *potential symbols* $k_j(x', \xi')$ *of order* $\nu - j - \frac{1}{2}$ *and some* $g_0(x', \xi') \in R_{\mathrm{G}}^{\nu,0}(\Omega \times \mathbb{R}^{n-1})$. *In this representation the symbols* k_j *and* g_0 *are unique. Conversely, every* $g(x', \xi')$ *of the form* (2.2.38) *belong to* $R_{\mathrm{G}}^{\nu,e}(\Omega \times \mathbb{R}^{n-1})$.

Proof. Every $g(x', \xi') \in R_{\mathrm{G}}^{\nu,e}(\Omega \times \mathbb{R}^{n-1})$ can be written as (2.2.15) for elements $g_j(x', \xi') \in R_{\mathrm{G}}^{\nu-j,0}(\Omega \times \mathbb{R}^{n-1})$. It suffices to rephrase every summand for $j > 0$ in the asserted form. We have

$$\begin{aligned}
&\int_0^\infty f_j(x', \xi', x_n[\xi'], x_n'[\xi']) \frac{d^j}{dx'^j} u(x')\, dx_n' \\
&= \int_0^\infty \frac{d}{dx_n'} \Big(f_j(x', \xi', x_n[\xi'], x_n'[\xi']) \frac{d^{j-1}}{dx_n'^{\,j-1}} u(x_n') \Big)\, dx_n' \\
&- \int_0^\infty [\xi'] (\frac{d}{dx_n'} f_j)(x', \xi', x_n[\xi'], x_n'[\xi']) \frac{d^{j-1}}{dx_n'^{\,j-1}} u(x_n')\, dx_n' \\
&= f_j(x', \xi', x_n[\xi'], x_n'[\xi']) \frac{d^{j-1}}{dx_n'^{\,j-1}} u(x_n') \Big|_0^\infty - \\
&\int_0^\infty [\xi'] (\frac{d}{dx_n'} f_j)(x', \xi', x_n[\xi'], x_n'[\xi']) \frac{d^{j-1}}{dx_n'^{\,j-1}} u(x_n')\, dx_n' \\
&= -f_j(x', \xi', x_n[\xi'], 0) \gamma^{j-1} u - \\
&\int_0^\infty [\xi'] (\frac{d}{dx_n'} f_j)(x', \xi', x_n[\xi'], x_n'[\xi']) \frac{d^{j-1}}{dx_n'^{\,j-1}} u(x_n')\, dx_n'.
\end{aligned} \tag{2.2.39}$$

By iterating the computation concerning the second summand on the right-hand side we obtain after finitely many steps the required form, namely, (2.2.38). The arguments work in both directions which gives us the assertion in converse direction. □

Proposition 2.2.12. *Every trace symbol* $b(x', \xi')$ *of order* ν *and type* e, *cf. Definition* 2.2.10 (iii) *can be written in the form*

$$b(x', \xi') u(x_n) = \sum_{j=0}^{e-1} c_j(x', \xi') \gamma^j + x_{n,0}(x', \xi') \tag{2.2.40}$$

for classical symbols $c_j(x', \xi') \in S_{\mathrm{cl}}^{\nu-j-1/2}(\Omega \times \mathbb{R}^{n-1})$, *and a trace symbol* $x_{n,0}(x', \xi')$ *of order* ν *and type* 0. *In this representation the symbols* c_j *and* $x_{n,0}$ *are unique. Conversely every expression* (2.2.40) *determines a trace symbol of order* ν *and type* e. *The proof is similar to the one of Proposition* 2.2.11.

Definition 2.2.13. *By* $R^{\nu;e}_{\mathrm{G}}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2)$*, we denote the set of all operator families*

$$\boldsymbol{g}(x',\xi') := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}(x',\xi') \tag{2.2.41}$$

for arbitrary $g_{11}(x',\xi') \in R^{\nu;e}_{\mathrm{G}}(\Omega \times \mathbb{R}^{n-1})$*, cf. Definition* 2.2.6*, and column vectors*

$$g_{21}(x',\xi') = {}^{\mathrm{t}}\big(g_{21,1}(x',\xi'), \dots, g_{21,j_2}(x',\xi')\big)$$

for trace symbols $g_{21,i}(x',\xi')$ *of order* ν *and type* e*,* $i = 1, \dots, j_2$*, cf. Definition* 2.2.10 (iii)*, row vectors*

$$g_{12}(x',\xi') = \big(g_{12,1}(x',\xi'), \dots, g_{12,j_1}(x',\xi')\big)$$

of potential symbols $g_{12,i}(x',\xi')$ *of order* ν*,* $i = 1, \dots, j_1$*, cf. Definition* 2.2.10 (i)*, and a* $j_2 \times j_1$ *matrix* $g_{22}(x',\xi')$ *of elements in* $S^{\nu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1})$*, cf. Section* 1.1*.*

Remark 2.2.14. *From Definition* 2.2.13 *it follows that*

$$R^{\nu;e}_{\mathrm{G}}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2) \subset S^{\nu}_{\mathrm{cl}}\Big(\Omega \times \mathbb{R}^{n-1}; \begin{array}{c} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_1} \end{array}, \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_2} \end{array}\Big) \tag{2.2.42}$$

for any $s > \nu - 1/2$*, cf. Definition* 1.4.4 (iii)*, where* $\begin{array}{c} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_1} \end{array}$ *is equipped with the group action* $\mathrm{diag}\,(\kappa, \mathrm{id}_{\mathbb{C}^j})$ *for* $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$*,* $(\kappa_\delta u)(x_n) = \delta^{1/2} u(\delta x_n), \delta \in \mathbb{R}_+$*, cf. formula* (1.4.5)*. Similarly,* $\begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_2} \end{array}$ *is equipped with such a group action, see also Example* 1.4.2*.*

Recall that for a classical operator-valued symbol such as (2.2.42) $\ni \boldsymbol{g}(x',\xi')$ we have a sequence of twisted homogeneous components $\boldsymbol{g}_{(\mu-j)}(x',\xi'), j \in \mathbb{N}$, cf. notation in Remark 1.4.11. Those are uniquely determined by $\boldsymbol{g}$. In particular, there is the principal part $\boldsymbol{g}_{(\mu)}(x',\xi')$ which can be computed as a limit like (1.4.12), i.e.,

$$\boldsymbol{g}_{(\mu)}(x',\delta\xi') = \delta^{\nu} \begin{pmatrix} \kappa_\delta & 0 \\ 0 & \mathrm{id}_{\mathbb{C}^{j_2}} \end{pmatrix} \boldsymbol{g}_{(\mu)}(x',\xi') \begin{pmatrix} \kappa_\delta^{-1} & 0 \\ 0 & \mathrm{id}_{\mathbb{C}^{j_1}} \end{pmatrix} \tag{2.2.43}$$

for all $\delta \in \mathbb{R}_+, \xi' \neq 0$.

2.3 The boundary symbolic calculus

In this section we develop the boundary symbolic calculus, first in the one-dimensional case. Let us ignore for the moment (x',ξ'), variables x' and covariables ξ' on the boundary.

Let us define the space $\mathcal{B}^{e}_{\mathrm{G}}(\overline{\mathbb{R}}_+; j_1, j_2)$ of operators

$$\boldsymbol{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} : \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_1} \end{array} \longrightarrow \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_2} \end{array} \tag{2.3.1}$$

where g_{11} is a Green operator of type e on $\overline{\mathbb{R}}_+$, while g_{21} is a trace operator of type e, g_{12} a potential operator, g_{22} a $j_2 \times j_1$ matrix of complex numbers.

Definition 2.3.1. *Let $\mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+)$ denote the space of upper left corners in* (2.3.1).

In the preceding section we studied the case with dependence on (x', ξ'). If we ignore this for a while we have simply operators (2.3.1)*. Recall that, say for $j_1 = j_2 = 1$,*

$$g_{11}u = \sum_{j=0}^{e} \int_0^\infty f_j(x_n, x_n') \frac{d^j}{dx_n'^{\,j}} u(x_n')\, dx_n' \tag{2.3.2}$$

for some $f_j(x_n, x_n') \in \mathcal{S}(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+)$,

$$g_{21}u = \sum_{j=0}^{e} \int_0^\infty b_j(x_n') \frac{d^j}{dx_n'^{\,j}} u(x_n')\, dx_n' \tag{2.3.3}$$

for $b_j(x_n') \in \mathcal{S}(\overline{\mathbb{R}}_+)$,

$$g_{12}c := ck(x_n) \tag{2.3.4}$$

and $k(x_n) \in \mathcal{S}(\overline{\mathbb{R}}_+), c \in \mathbb{C}$. The operator g_{22} in this case is simply the multiplication by a complex number.

Remark 2.3.2. *An operator $g_{11} \in \mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+)$, cf. formula* (2.3.2) *induces continuous operators*

$$g_{11} : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+) \tag{2.3.5}$$

and

$$g_{11} : H^s(\mathbb{R}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+) \tag{2.3.6}$$

for every $s > e - 1/2$. Interpreting (2.3.6) *as an operator*

$$g_{11} : H^s(\mathbb{R}_+) \longrightarrow H^s(\mathbb{R}_+), \quad s > e - 1/2, \tag{2.3.7}$$

by composing (2.3.6) *with the embedding*

$$\mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow H^s(\mathbb{R}_+)$$

then (2.3.7) *is compact.*

In fact, the embedding can be seen as a chain of embedding, namely,

$$\mathcal{S}(\overline{\mathbb{R}}_+) \hookrightarrow \langle x_n \rangle^{-k} H^{s'}(\mathbb{R}_+)$$

for any $k > 0, s' > s$, and

$$\langle x_n \rangle^{-k} H^{s'}(\mathbb{R}_+) \hookrightarrow H^s(\mathbb{R}_+).$$

We now study 2×2 block-matrices of the form

$$\boldsymbol{a} = \begin{pmatrix} \mathrm{Op}^+(a) & 0 \\ 0 & 0 \end{pmatrix} + \boldsymbol{g} \tag{2.3.8}$$

where $\boldsymbol{g}$ is of the form (2.3.1) and

$$\mathrm{Op}^+(a) := \mathrm{r}^+\mathrm{Op}(a)\mathrm{e}^+ : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+), \tag{2.3.9}$$

$$a(\xi_n) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}_{\xi_n}), \mu \in \mathbb{Z}, \tag{2.3.10}$$

where

$$\mathrm{Op}(a)u(x_n) = \iint \mathrm{e}^{i(x_n - x'_n)\xi_n} a(\xi_n) u(x'_n)\, dx_n đ\xi_n. \tag{2.3.11}$$

Let $\mathcal{B}^{\mu,e}(\overline{\mathbb{R}}_+)$ denote the space of upper left corners in (2.3.8), i.e., the space of operators $\mathrm{Op}^+(a)+g_{11}$ for a as in (2.3.10) and $g_{11} \in \mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+)$. In boundary value problems we assume that $a(\xi_n)$ has the transmission property which means (in one of the various equivalent definitions) that in the asymptotic expansion

$$a(\xi_n) \sim \sum_{j=0}^{\infty} a_j^{\pm}(i\xi_n)^{\mu-j}, \text{ for } \xi_n \to \pm\infty$$

we have

$$a_j^+ = a_j^- \text{ for all } j \in \mathbb{N}. \tag{2.3.12}$$

Let $S^{\mu}_{\mathrm{tr}}(\mathbb{R})$ for $\mu \in \mathbb{Z}$ denote the space of all $a(\xi_n) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R})$ satisfying condition (2.3.12).

Remark 2.3.3. *The space $S^{\mu}_{\mathrm{tr}}(\mathbb{R})$ is closed in the Fréchet topology induced by $S^{\mu}_{\mathrm{cl}}(\mathbb{R})$.*

Note that a general symbol (2.3.10) for $\mu = 0$ determines a map

$$a : \mathbb{R} \longrightarrow \mathbb{C}$$

such that

$$|a(\xi_n)| \leq \mathrm{const}$$

for all $\xi_n \in \mathbb{R}$. Condition (2.3.12) for $\mu = 0$ has the consequence that the curve $\{a(\xi_n) \in \mathbb{C} : \xi_n \in \mathbb{R}\}$ is smooth, including the endpoints $a_0^- = a_0^+$.

Remark 2.3.4. *Every $a(\xi_n) \in S^{\mu}_{\mathrm{tr}}(\mathbb{R}), \mu \in \mathbb{N}$, has a decomposition*

$$a(\xi_n) = a_0(\xi_n) + p(\xi_n) \tag{2.3.13}$$

where $a_0(\xi_n) \in S^0_{\mathrm{tr}}(\mathbb{R})$ and $p(\xi_n)$ is a polynomial in ξ_n of order μ.

In the following definition we set

$$\mathbb{C}_{\mp} := \{\zeta \in \mathbb{C} : \mathrm{Im}\,\zeta \lessgtr 0\}.$$

Definition 2.3.5. (i) *An $a(\xi_n) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}), \mu \in \mathbb{R}$, is called a plus-symbol if it has an analytic extension to a function $a_+(\zeta)$ in the complex variable $\zeta = \xi_n + i\vartheta$ for $\mathrm{Im}\,\zeta < 0$, where*

$$a_+(\zeta) \in C^{\infty}(\overline{\mathbb{C}}_-) \cap \mathcal{A}(\mathbb{C}_-),$$

in particular, $a(\xi_n) = a_+(\zeta)$, and if

$$|a_+(\zeta)| \leq c(1 + |\zeta|^2)^{\mu/2} \tag{2.3.14}$$

for all $\zeta \in \overline{\mathbb{C}}_-$.

(ii) *An* $a(\xi_n) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}), \mu \in \mathbb{R}$, *is called a minus-symbol if it extends to an*

$$a_-(\zeta) \in C^\infty(\overline{\mathbb{C}}_+) \cap \mathcal{A}(\mathbb{C}_+),$$

and (2.3.14) *holds for all* $\zeta \in \overline{\mathbb{C}}_+$.

Example 2.3.6. *For any fixed* $\delta \in \mathbb{R}_+$, *the symbols*

$$(\delta + i\xi_n)^k, k \in \mathbb{Z}, \tag{2.3.15}$$

are plus, and

$$(\delta - i\xi_n)^k, k \in \mathbb{Z} \tag{2.3.16}$$

minus symblos. Thus symbols (2.3.15) *and* (2.3.16) *have the transmission property.*

In fact, for $k \in \mathbb{N}$, we have polynomials; those are plus and minus-symbols at the same time. For $-k \in \mathbb{N}$ we see the plus/minus property from the following analytic extensions to the respective complex half-planes namely, in the plus/minus case as

$$(\delta + i(\xi_n + i\vartheta))^k = (\delta + i\xi_n - \vartheta)^k$$

where $\vartheta < 0$, then $\operatorname{Im}\zeta < 0$ and $\vartheta > 0$ then $\operatorname{Im}\zeta > 0$.

For references below we now formulate the Palay-Wiener theorem. Concerning an explicit proof, cf. [14].

Let $H^s_0(\overline{\mathbb{R}}_\pm)$ denote the subspace of all $u(x_n) \in H^s(\mathbb{R})$ such that $\operatorname{supp} u \subseteq \overline{\mathbb{R}}_\pm$.

Theorem 2.3.7. *Let* $u(x_n) \in H^s_0(\overline{\mathbb{R}}_+), s \in \mathbb{R}$. *Then the Fourier transform* $\hat{u}(\xi_n) = \int_0^\infty \mathrm{e}^{-ix_n\xi_n} u(x_n)\, dx_n$ *has an extension to a function*

$$h_+(\zeta) = \int \mathrm{e}^{-ix_n\xi_n}(\mathrm{e}^{x_n\vartheta} u(x_n))\, dx_n \tag{2.3.17}$$

$\in C(\operatorname{Im} \leq 0) \cap \mathcal{A}(\operatorname{Im}\zeta < 0)$, $\zeta = \xi_n + i\vartheta$, *such that*

$$\int (1 + |\xi_n| + |\vartheta|)^{2s} |h_+(\xi_n + i\vartheta)|^2\, d\xi_n \leq C \tag{2.3.18}$$

for all $\vartheta \leq 0$, *for some* $C > 0$, *independent of* ϑ.

Conversely, if $h_+(\xi_n + i\vartheta)$ is a locally integrable function in ξ_n for $-\infty < \vartheta < 0$, satisfying (2.3.18) for all $s \in \mathbb{R}$, for some $C > 0$ independent of ϑ and belonging to $\mathcal{A}(\operatorname{Im}\zeta < 0)$, then there is a $u(x_n) \in H^s_0(\overline{\mathbb{R}}_+)$, such that (2.3.17) holds.

An analogous theorem holds for functions in $H^s(\overline{\mathbb{R}}_-)$.

Theorem 2.3.8. (i) *Let* $a(\xi_n) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}), \mu \in \mathbb{R}$, *be a plus-symbol. Then* $\operatorname{Op}(a)$ *induces a continuous operator*

$$\operatorname{Op}(a) : H^s_0(\overline{\mathbb{R}}_+) \longrightarrow H^{s-\mu}_0(\overline{\mathbb{R}}_+) \tag{2.3.19}$$

for every $s \in \mathbb{R}$. *Analogously, if* $a(\xi_n) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R})$ *is a minus-symbol, then*

$$\operatorname{Op}(a) : H^s_0(\overline{\mathbb{R}}_-) \longrightarrow H^{s-\mu}_0(\overline{\mathbb{R}}_-) \tag{2.3.20}$$

is continuous for every $s \in \mathbb{R}$.

(ii) *Let* $a(\xi_n) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}), \mu \in \mathbb{R}$ *be a minus-symbol, and let*

$$\mathrm{e}^+_s : H^s(\mathbb{R}_+) \longrightarrow H^s(\mathbb{R}) \tag{2.3.21}$$

be a continuous extension operator for fixed $s \in \mathbb{R}$ *(i.e.,* $\mathrm{r}^+\mathrm{e}^+_s = \mathrm{id}_{H^s(\mathbb{R}_+)}$*) with* $H^s(\mathbb{R}_+)$ *being identified with* $H^s(\mathbb{R})/H^s_0(\overline{\mathbb{R}}_+)$*. Then*

$$\mathrm{r}^+\mathrm{Op}(a)\mathrm{e}^+_s : H^s(\mathbb{R}_+) \longrightarrow H^{s-\mu}(\mathbb{R}_+) \tag{2.3.22}$$

is continuous and independent of the choice of e^+_s*. Moreover, we have*

$$\mathrm{r}^+\mathrm{Op}(a)\mathrm{e}^+_s = \mathrm{r}^+\mathrm{Op}(a)\mathrm{e}^+$$

for every $s \in \mathbb{R}, s > -1/2$*.*

Proof. (i) $\mathrm{Op}(a)$ for any symbol $a(\xi_n)$ is defined as $F^{-1}a(\xi_n)F$. If $a(\xi_n)$ is a plus-symbol then the operator of multiplication by a induces a continuous map

$$FH^s_0(\overline{\mathbb{R}}_+) \longrightarrow FH^{s-\mu}_0(\overline{\mathbb{R}}_+),$$

which is consequence of Theorem 2.3.7, the Palay-Wiener theorem. Thus $\mathrm{Op}(a)$ itself defines a continuous operator (2.3.19). In fact, Theorem 2.3.7 tells us that $\hat{u} \in FH^s_0(\overline{\mathbb{R}}_+)$ is characterized by (2.3.18) in $\varphi \leq 0$. Moreover, since $a(\xi_n)$ is a plus-symbol, we have the estimate (2.3.14), i.e.,

$$|a(\xi_n + i\vartheta)| \leq c(1 + |\xi_n| + |\vartheta|)^{\mu}$$

for a constant $c > 0$, independent of $\vartheta \leq 0$. Thus

$$a(\xi_n)\hat{u}(\xi_n) \in \hat{H}^{s-\mu}(\mathbb{R}_{\xi_n})$$

extends to a holomorphic function $a(\xi_n + i\vartheta)\hat{u}(\xi_n + i\vartheta)$ in $\vartheta < 0$, which satisfies the relation

$$\begin{aligned}
&\int (1 + |\xi_n| + |\vartheta|)^{2(s-\mu)} |a(\xi_n + i\vartheta)\hat{u}(\xi_n + i\vartheta)|^2 \, d\xi_n \leq 0 \\
&\leq \int (1 + |\xi_n| + |\vartheta|)^{2(s-\mu)} |a(\xi_n + i\vartheta)|^2 |\hat{u}(\xi_n + i\vartheta)|^2 \, d\xi_n \\
&\leq c \int (1 + |\xi_n| + |\vartheta|)^{2s} |\hat{u}(\xi_n + i\vartheta)|^2 \, d\xi_n.
\end{aligned}$$

For the case of a minus-symbol $a(\xi_n)$ (2.3.20) by analogous conclusions.

(ii) If e^+_s is a continuous extension operator (2.3.21) then (2.3.22) is obviously continuous, since both

$$\mathrm{Op}(a) : H^s(\mathbb{R}) \longrightarrow H^{s-\mu}(\mathbb{R})$$

and

$$\mathrm{r}^+ : H^{s-\mu}(\mathbb{R}) \longrightarrow H^{s-\mu}(\mathbb{R}_+)$$

are continuous. In other words

$$\mathrm{r}^+\mathrm{Op}(a)\mathrm{e}^+_s u \in H^{s-\mu}(\mathbb{R}_+) \tag{2.3.23}$$

for every $s \in \mathbb{R}$. If we now replace e_s^+ for $s > -1/2$ by e^+, the extension operator by zero, then we know that

$$(\mathrm{e}_s^+ - \mathrm{e}^+)u \in H_0^{\tilde{s}}(\overline{\mathbb{R}}_-) \tag{2.3.24}$$

for some $\tilde{s} \in \mathbb{R}$, where $\tilde{s} = 0$ for $s \geq 0$ and $\tilde{s} = s$ for $-1/2 < s \leq 0$. Thus we have (2.3.24) for $\tilde{s} := \min\{0, s\}$. It follows from (2.3.20) that

$$\mathrm{Op}(a)(\mathrm{e}_s^+ - \mathrm{e}^+)u \in H_0^{\tilde{s}-\mu}(\overline{\mathbb{R}}_-).$$

However, $\mathrm{r}^+ H_0^{\tilde{s}-\mu}(\overline{\mathbb{R}}_-) = \{0\}$. It follows altogether

$$\mathrm{r}^+\mathrm{Op}(a)(\mathrm{e}_s^+ - \mathrm{e}^+)u = 0,$$

i.e.,

$$\mathrm{r}^+\mathrm{Op}(a)\mathrm{e}_s^+ u = \mathrm{r}^+\mathrm{Op}(a)\mathrm{e}^+ u.$$

□

Proposition 2.3.9. *For every $a(\xi_n) \in S_{\mathrm{tr}}^\mu(\mathbb{R})$ and every $N \in \mathbb{N}$, there exists plus and minus-symbols $p_N(\xi_n)$ and $m_N(\xi_n)$ in $S_{\mathrm{tr}}^\mu(\mathbb{R})$, respectively, such that*

$$a(\xi_n) - p_N(\xi_n) \in S_{\mathrm{cl}}^{\mu-(N+1)}(\mathbb{R}),\ a(\xi_n) - m_N(\xi_n) \in S_{\mathrm{cl}}^{\mu-(N+1)}(\mathbb{R}). \tag{2.3.25}$$

Proof. Without loss of generality we may assume $\mu = 0$, since for $\mu \in \mathbb{N}$ we can omit the corresponding polynomial part which is a plus- and minus-symbl at the same time. From the definition of the transmission property we know that there are coefficients $a_j \in \mathbb{C}, j \in \mathbb{N}$, such that

$$a(\xi_n) - \chi(\xi_n)\sum_{j=0}^{N} a_j (i\xi_n)^{-j} \in S_{\mathrm{cl}}^{-(N+1)}(\mathbb{R}). \tag{2.3.26}$$

In order to show the existence $m_N(\tau)$ we observe that

$$\begin{aligned}(i\xi_n)^{-1} &= -(1-i\xi_n)^{-1} + (i\xi_n)^{-1}(1-i\xi_n)^{-1} = \dots \\ &= -\sum_{k=1}^{N}(1-i\xi_n)^{-k} + (i\xi_n)^{-1}(1-i\xi_n)^{-N}.\end{aligned} \tag{2.3.27}$$

This gives us

$$\chi(\xi_n)(i\xi_n)^{-1} = -\sum_{k=1}^{N}(1-i\xi_n)^{-k} \bmod S_{\mathrm{cl}}^{-(N+1)}(\mathbb{R}).$$

More generally,

$$\chi(\xi_n)(i\xi_n)^{-j} = (-\sum_{k=1}^{N}(1-i\xi_n)^{-k})^{-j} \bmod S_{\mathrm{cl}}^{-(N+1)}(\mathbb{R}). \tag{2.3.28}$$

Thus, if we replace $\chi(\xi_n)(i\xi_n)^{-1}$ on the left-hand side of (2.3.26) by (2.3.28) we obtain a minus-symbol

$$m_N(\xi_n) := \sum_{j=0}^{N} a_j(-\sum_{k=1}^{N}(1-i\xi_n)^{-k})^{-j} \tag{2.3.29}$$

with the desired property, cf. also Example 2.3.6. The construction of $P_N(\xi_n)$ is analogous. □

Proposition 2.3.10. *Every* $a(\xi_n) \in S^{\mu}_{\mathrm{tr}}(\mathbb{R})$ *induces continuous operators*

$$\mathrm{r}^+\mathrm{Op}(a)\mathrm{e}^+ : H^s(\mathbb{R}_+) \longrightarrow H^{s-\mu}(\mathbb{R}_+) \tag{2.3.30}$$

for every $s \in \mathbb{R}, s > -1/2$.

Proof. From Theorem 2.3.8 (ii) we know that (2.3.30) is continuous for $s > -1/2$ when $a(\xi_n)$ is a minus-symbol. For $a(\xi_n) \in S^{\mu}_{\mathrm{tr}}(\mathbb{R})$ we employ the second decomposition of (2.3.25), namely,

$$a(\xi_n) = m_N(\xi_n) + c_N(\xi_n)$$

for a minus-symbol $m_N(\xi_n)$ and a remainder $c_N(\xi_n) \in S^{\mu-(N+1)}_{\mathrm{cl}}(\mathbb{R})$. This will be used for N sufficiently large. Then

$$\mathrm{Op}(a) = \mathrm{Op}(m_N) + \mathrm{Op}(c_N)$$

gives rise to

$$\mathrm{r}^+\mathrm{Op}(a)\mathrm{e}^+ = \mathrm{r}^+\mathrm{Op}(m_N)\mathrm{e}^+ + \mathrm{r}^+\mathrm{Op}(c_N)\mathrm{e}^+.$$

Form the minus property of m_N we already know the claimed continuity property of $\mathrm{r}^+\mathrm{Op}(m_N)\mathrm{e}^+$. Moreover, $c_N(\xi_n) \in S^{\mu-(N+1)}_{\mathrm{cl}}(\mathbb{R})$ shows the continuity of

$$\mathrm{Op}(c_N) : H^s(\mathbb{R}) \longrightarrow H^{s-\mu+N+1}(\mathbb{R}) \tag{2.3.31}$$

for any $s \in \mathbb{R}$. In the case $s \geq 0$ it follows that

$$\mathrm{e}^+ : H^s(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}) = H^0(\mathbb{R}) \tag{2.3.32}$$

is continuous, for $-1/2 < s \leq 0$ we now

$$\mathrm{e}^+ : H^s(\mathbb{R}_+) \longrightarrow H^s(\mathbb{R}) \tag{2.3.33}$$

see [14]. From (2.3.31), (2.3.32) it follows that

$$\mathrm{Op}(c_N)\mathrm{e}^+ : H^s(\mathbb{R}_+) \longrightarrow H^{s-\mu+N+1}(\mathbb{R})$$

for $-1/2 < s \leq 0$, and thus

$$\mathrm{r}^+\mathrm{Op}(c_N)\mathrm{e}^+ : H^s(\mathbb{R}_+) \longrightarrow H^{s-\mu+N+1}(\mathbb{R}_+) \hookrightarrow H^{s-\mu}(\mathbb{R}_+).$$

□

Let us set

$$l^{\nu}_{\pm}(\xi_n) := (\delta \pm i\xi_n)^{\nu} \tag{2.3.34}$$

for some $\nu \in \mathbb{R}$ and fixed $\delta \in \mathbb{R}_+$. The following proposition is related to Theorem 2.3.8.

Proposition 2.3.11. *The operator* $\mathrm{Op}^+(l^{\nu}_-), \nu \in \mathbb{R}$, *induces a continuous operator*

$$\mathrm{Op}^+(l^{\nu}_-) : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+) \tag{2.3.35}$$

which is invertible, and we have

$$(\mathrm{Op}^+(l^{\nu}_-))^{-1} = \mathrm{Op}^+(l^{-\nu}_-). \tag{2.3.36}$$

Moreover, for any choice of an extension (2.3.21) *the operator*

$$\mathrm{r}^+\mathrm{Op}(l_-^\nu)\mathrm{e}_s^+ : H^s(\mathbb{R}_+) \longrightarrow H^{s-\nu}(\mathbb{R}_+) \tag{2.3.37}$$

is an isomorphism for every $s \in \mathbb{R}$, and

$$(\mathrm{r}^+\mathrm{Op}(l_-^\nu)\mathrm{e}_s^+)^{-1} = \mathrm{r}^+\mathrm{Op}(l_-^{-\nu})\mathrm{e}_{s-\nu}^+ \tag{2.3.38}$$

again for any choice of $\mathrm{e}_{s-\nu}^+$. The operators (2.3.37), (2.3.38) *are independent of the involved extension operators, and they may be replaced by e^+, whenever $s > -1/2, s-\nu > -1/2$.*

Proof. By virtue of Seeleyś extension theorem [66] there is a continuous extension operator

$$E : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \mathcal{S}(\mathbb{R}).$$

For $u \in \mathcal{S}(\overline{\mathbb{R}}_+)$ we have

$$\mathrm{Op}^+(l_-^\nu)u = \mathrm{r}^+\mathrm{Op}(l_-^\nu)Eu,$$

independently of the choice of the operator E. This holds for similar reasons as in the proof of Theorem 2.3.8 (i). In order to express the inverse (2.3.36) we can employ the operator e^+ as well. Then we obtain,

$$\begin{aligned}&\left(\mathrm{r}^+\mathrm{Op}(l_-^{-\nu})\mathrm{e}^+\right)\left(\mathrm{r}^+\mathrm{Op}(l_-^\nu)\mathrm{e}^+\right) = \\ &\mathrm{r}^+\mathrm{Op}(l_-^{-\nu}l_-^\nu)\mathrm{e}^+ - \mathrm{r}^+\mathrm{Op}(l_-^{-\nu})(1-\mathrm{e}^+\mathrm{r}^+)\mathrm{Op}(l_-^\nu)\mathrm{e}^+ = 1\end{aligned}$$

using the minus property of the involved symbols. In a similar manner we show that $\mathrm{r}^+\mathrm{Op}(l_-^{-\nu})\mathrm{e}^+$ is a right inverse, and hence (2.3.36) is verified. The second part of Proposition 2.3.11 concerning Sobolev spaces is simple as well if we use arguments from the proof of Theorem 2.3.8.

□

Proposition 2.3.12. *Let $F = F_{x_n \to \xi_n}$ be the Fourier transform on $\mathbb{R}$, i.e.,*

$$(Fu)(\xi_n) = \int \mathrm{e}^{-ix_n\xi_n} u(x_n)\,dx_n,$$

applied to $u \in L^2(\mathbb{R})$. Then the following conditions are equivalent:

(i)
$$a(\xi_n) \in F(\mathrm{e}^+\mathcal{S}(\overline{\mathbb{R}}_+))$$
for $\mathcal{S}(\overline{\mathbb{R}}_+) := \{u|_{\mathbb{R}_+} : u \in \mathcal{S}(\mathbb{R})\}$,

(ii) *$a(\xi_n)$ is a plus-symbol of order -1*
$$a(\xi_n) \in S_{\mathrm{cl}}^{-1}(\mathbb{R}_{\xi_n}),$$
and there is an extension $a_+(\zeta)$ of $a(\xi_n)$ into the complex ζ half-plane $\overline{\mathbb{C}}_-$, where $\xi_n = \operatorname{Re}\zeta$, such that
$$a_+(\zeta) \in C^\infty(\operatorname{Im}\zeta \le 0) \cap \mathcal{A}(\operatorname{Im}\zeta < 0)$$
and there is an asymptotic expansion
$$a_+(\zeta) \sim \sum_{k \le -1} a_k \zeta^k \quad \text{for } |\zeta| \to \infty,\ \operatorname{Im}\zeta \le 0. \tag{2.3.39}$$
Moreover, all derivatives $(\partial_{\xi_n}^l a)(\xi_n), l \in \mathbb{N}$, extend to $\partial_\zeta^l a_+(\zeta)$ with analogous properties i.e., as functions in $C^\infty(\operatorname{Im}\zeta \le 0) \cap \mathcal{A}(\operatorname{Im}\zeta < 0)$ and asymptotic expansions, obtained by applying ∂_ζ^l to (2.3.39).

An analogous characterization holds for the space $F(\mathrm{e}^{-}\mathcal{S}(\overline{\mathbb{R}}_{+})))$ *where* $\overline{\mathbb{C}}_{-}$ *is replaced by* $\overline{\mathbb{C}}_{+}$.

Proof. (i) $\Rightarrow$ (ii). From the Paley-Wiener Theorem, cf. [14], we know that since $u(x_n) \in \mathrm{e}^{+}\mathcal{S}(\overline{\mathbb{R}}_{+}) \subset L^2(\mathbb{R})$, vanishes for $x_n < 0, a(\xi_n) = Fu(\xi_n)$ extends in the Fourier covariable ξ_n to an element in $\mathcal{A}(\operatorname{Im}\zeta < 0) \cap C^{\infty}(\operatorname{Im}\zeta \le 0)$ where

$$\int |a(\xi_n + i\vartheta)|^2 \, d\xi_n \le C$$

for all $\vartheta \le 0$ for some constent $C > 0$ independent of ϑ. In the present case we can easly verify that $a(\zeta) \in C^{\infty}(\operatorname{Im}\zeta \le 0)$, we have

$$a(\xi_n) = \int_0^{\infty} \mathrm{e}^{-ix_n\xi_n} u(x_n) \, dx_n,$$

and integration by parts for $\xi_n \neq 0$

$$\begin{aligned} a(\xi_n) &= -\frac{1}{i\xi_n}\mathrm{e}^{-ix_n\xi_n} u(x_n)\big|_0^{\infty} + \frac{1}{i\xi_n}\int_0^{\infty} \mathrm{e}^{ix_n\xi_n}\partial_{x_n} u(x_n)\, dx_n = \cdots = \\ &= -\frac{1}{i\xi_n}u(0) + \frac{1}{(i\xi_n)^2}\partial_{x_n}u(0) + \cdots + \\ &\frac{1}{(i\xi_n)^{k+1}}\partial_{x_n}^{k}u(0) + \frac{1}{(i\xi_n)^{k+1}}\int_0^{\infty} \mathrm{e}^{-ix_n\xi_n}\partial_{x_n}^{k+1}u(x_n)\, dx_n. \end{aligned} \tag{2.3.40}$$

We thus obtain an asymptotic expansion

$$a(\xi_n) \sim \sum_{j=0}^{\infty} \partial_{x_n}^{j} u(0)(i\xi_n)^{-(j+1)}. \tag{2.3.41}$$

For the derivatives similar conclusions yield

$$a'(\xi_n) = -ix_n \int_0^{\infty} \mathrm{e}^{-ix_n\xi_n} u(x_n)\, dx_n \sim -i\sum_{k=0}^{\infty} \partial_{x_n}^{k}(x_n u)(0)(i\xi_n)^{-(k+1)}, \tag{2.3.42}$$

using

$$\partial_{x_n}^{l}(x_n u)(0) = l\partial_{x_n}^{l-1}(u)(0)$$

it follows that $a'(\xi_n) := \sum_{k=1}^{\infty} k(\partial_{x_n}^{k-1}u)(0)(i\xi_n)^{-(k+1)}$. Thus the expansion for $a'(\xi_n)$ follows by differentiating (2.3.41) summandwise. An analogus computation gives us asymptotic expansions for arbitrary derivatives of $a(\xi_n)$ by summandwise differentiating (2.3.41). Since the involved the integrals also converge in $\operatorname{Im}\zeta < 0$ rather than in $\xi_n = \operatorname{Re}\zeta$ and can be differentiated with respect to the complex parameter ζ, the claimed asymptotic expansion also holds with respect to ζ for $\operatorname{Im}\zeta < 0$. Summing up we proved that $a(\xi_n)$ is a plus-symbol of order -1, and (i) $\Rightarrow$ (ii) is complete.

For (ii) $\Rightarrow$ (i) we argue as follows. Applying the inverse Fourier transform to $a(\xi_n)$ gives us

$$u(x_n) := \int \mathrm{e}^{ix_n\xi_n} a(\xi_n)\, đ\xi_n \in L^2(\mathbb{R}).$$

From the Paley-Wiener Theorem 2.3.7 and the properties of $a(\xi_n)$ listed in (ii) we conclude that $u(x_n)$ vanishes for almost all $x_n \in \mathbb{R}_-$. From the properties in (ii) we know that $\xi_n^k D_{\xi_n}^l a(\xi_n) \in C^\infty(\mathbb{R})$ is equal to the summ of a polynomial in ξ_n and a function $h_{kl}(\xi_n)$ with the properties in (ii), for all $k, l \in \mathbb{N}$. Thus $F^{-1}\xi_n^k D_{\xi_n}^l a(\xi_n)(x_n)$ is equal to a derivative of the Dirac distribution at $x_n = 0$ plus a function in $L^2(\mathbb{R})$. It follows that $D_{x_n}^k x_n^l u(x_n)|_{x_n>0}$ is in $L^2(\mathbb{R}_+)$. This shows that $u \in \mathcal{S}(\overline{\mathbb{R}}_+)$.
The second assertion of Proposition 2.3.12 concerning $\mathrm{e}^- \mathcal{S}(\overline{\mathbb{R}}_-)$ can be proved in an analogous manner. $\square$

Definition 2.3.13. *Let* $\mathcal{B}^{\mu,e}(\overline{\mathbb{R}}_+; j_1, j_2)$ *be the space of all* 2×2 *operator block-matrices* (2.3.8) *where* $\mathrm{Op}^+(a)$ *is given by a symbol* (2.3.9) *with the transmission property and* $\boldsymbol{g}$ *is of the form* (2.3.1).

The operators $\boldsymbol{a} \in \mathcal{B}^{\mu,e}(\overline{\mathbb{R}}; j_1, j_2)$ induce continuous maps

$$\boldsymbol{a}: \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_1} \end{array} \longrightarrow \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_2} \end{array}, \tag{2.3.43}$$

or

$$\boldsymbol{a}: \begin{array}{c} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_1} \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_2} \end{array} \tag{2.3.44}$$

for $s > e - 1/2$.

Proposition 2.3.14. *For every* $a(\xi_n) \in S_{\mathrm{tr}}^\mu(\mathbb{R})$, $\mu \in \mathbb{Z}$, *the truncated operator* $\mathrm{Op}^+(a)$ *induces a continuous map*

$$\mathrm{Op}^+(a) : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+). \tag{2.3.45}$$

Proof. Without loss of generality we assume $\mu \leq -1$. In fact, in general we can write

$$a(\xi_n) = a_0(\xi_n) + p(\xi_n)$$

for a polynomial $p(\xi_n)$ of order μ (when $\mu \in \mathbb{N}$) and a symbol $a_0(\xi_n) \in S_{\mathrm{tr}}^{-1}(\mathbb{R})$. Then

$$\mathrm{Op}^+(a) = \mathrm{Op}^+(a_0) + \mathrm{Op}^+(p),$$

and $\mathrm{Op}^+(p)u$ for $u \in \mathcal{S}(\overline{\mathbb{R}}_+)$ is equal to $p(D_{x_n})u\big|_{\mathbb{R}_+} + \mathrm{r}^+ v$ where $v \in \mathcal{E}'(\mathbb{R})$ is supported by the origin $\{0\}$. Since r^+ restrict to the open half-axis $\mathbb{R}_+$ we have $\mathrm{r}^+ v = 0$. Moreover, we have

$$\mathrm{Op}^+(a_0) = \mathrm{r}^+ \mathrm{Op}(a_0)\mathrm{e}^+ = \mathrm{r}^+ F^{-1} a_0(\xi_n) F \mathrm{e}^+,$$

and

$$S_{\mathrm{tr}}^{-1}(\mathbb{R}) = F(\mathrm{e}^+ \mathcal{S}(\overline{\mathbb{R}}_+) + \mathrm{e}^- \mathcal{S}(\overline{\mathbb{R}}_-)). \tag{2.3.46}$$

For $u \in \mathcal{S}(\overline{\mathbb{R}}_+)$ the Fourier transform of $\mathrm{e}^+ u$ belongs to $S_{\mathrm{tr}}^{-1}(\mathbb{R})$. And $u \longrightarrow F\mathrm{e}^+ u$ defines a continuous operator

$$\mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow S_{\mathrm{tr}}^{-1}(\mathbb{R}) \tag{2.3.47}$$

which is the proof of Proposition 2.3.12. Moreover, the space $S^{-1}_{\mathrm{tr}}(\mathbb{R})$ is an algebra which follows from condition (2.3.12), and it is also easy to verify that the operator $\mathcal{M}_{a_0}$ of multiplication by a_0 induces a continuous operator

$$\mathcal{M}_{a_0} : S^{-1}_{\mathrm{tr}}(\mathbb{R}) \longrightarrow S^{-1}_{\mathrm{tr}}(\mathbb{R}) \tag{2.3.48}$$

Moreover, (2.3.46) defines an isomorphism

$$F : \mathrm{e}^{+}\mathcal{S}(\overline{\mathbb{R}}_{+}) + \mathrm{e}^{-}\mathcal{S}(\overline{\mathbb{R}}_{-}) \longrightarrow S^{-1}_{\mathrm{tr}}(\mathbb{R}),$$

between the respective Fréchet spaces, in particular,

$$F^{-1} : S^{-1}_{\mathrm{tr}}(\mathbb{R}) \longrightarrow \mathrm{e}^{+}\mathcal{S}(\overline{\mathbb{R}}_{+}) + \mathrm{e}^{-}\mathcal{S}(\overline{\mathbb{R}}_{-}) \tag{2.3.49}$$

is continuous, and finally

$$\mathrm{r}^{+} : \mathrm{e}^{+}\mathcal{S}(\overline{\mathbb{R}}_{+}) + \mathrm{e}^{-}\mathcal{S}(\overline{\mathbb{R}}_{-}) \longrightarrow \mathrm{e}^{+}\mathcal{S}(\overline{\mathbb{R}}_{+}) \tag{2.3.50}$$

is continuous. Thus

$$\mathrm{Op}^{+}(a_0) : \mathcal{S}(\overline{\mathbb{R}}_{+}) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_{+})$$

is the composition of continuous operator (2.3.47), (2.3.48), (2.3.49), (2.3.50), and hence (2.3.45) itself is continuous. $\square$

Proposition 2.3.15. *Let $c(x_n, x_n') \in \mathcal{S}(\overline{\mathbb{R}}_{+})\hat{\otimes}_{\pi}\mathcal{S}(\overline{\mathbb{R}}_{+})$, and let $\mu \in \mathbb{R}, s \in \mathbb{N}, s > j - 1/2$ for some $j \in \mathbb{N}$. Then*

$$g : u \longrightarrow \mathrm{Op}^{+}(l_{-}^{s-\mu}) \int_0^{\infty} c(x_n, x_n')\partial^{j}_{x_n'} \mathrm{Op}^{+}(l_{-}^{-s})u(x_n')\, dx_n' \tag{2.3.51}$$

$u \in L^2(\mathbb{R}_{+})$, define an operator $g \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_{+})$.

Proof. From the continuity (2.3.35) we have

$$\mathrm{Op}^{+}(l_{-}^{s-\mu})c(x_n, x_n') = e(x_n, x_n')$$

for an $e(x_n, x_n') \in \mathcal{S}(\overline{\mathbb{R}}_{+})\hat{\otimes}_{\pi}\mathcal{S}(\overline{\mathbb{R}}_{+})$. Moreover, Proposition 2.3.10 gives us the continuity

$$\mathrm{Op}^{+}(l_{-}^{-s}) : L^2(\mathbb{R}_{+}) \longrightarrow H^{s}(\mathbb{R}_{+}),$$

and then

$$\partial^{j}_{x_n'} \mathrm{Op}^{+}(l_{-}^{-s}) : L^2(\mathbb{R}_{+}) \longrightarrow H^{s-j}(\mathbb{R}_{+}).$$

is continuous as well. Since $s - j > -1/2$ the integral

$$\int e(x_n, x_n')\partial^{j}_{x_n} \mathrm{Op}^{+}(l_{-}^{-s})u(x_n')\, dx_n'$$

yields an element of $\mathcal{S}(\overline{\mathbb{R}}_{+})$; this shows the continuity $g : L^2(\mathbb{R}_{+}) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_{+})$. Moreover, for the adjoint g^* we have

$$g^* v(x_n') = \mathrm{Op}^{+}(\overline{b}) \int_0^{\infty} \overline{e}(x_n, x_n')v(x_n)\, dx_n$$

for the symbol $b(\xi_n) = (i\xi_n)^j l_-^{-s}(\xi_n)$ with $\overline{b}(\xi_n)$ being the complex conjugate. Then, using the continuity

$$\mathrm{Op}^+(\overline{b}) : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+),$$

cf. Proposition 2.3.14, it follows that

$$g^* : L^2(\mathbb{R}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+)$$

is continuous. By virtue of Theorem 2.2.7 we proved $g \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$.

□

Theorem 2.3.16. (i) *For every $a(\xi_n) \in S^\mu_{\mathrm{tr}}(\mathbb{R}), b(\xi_n) \in S^\nu_{\mathrm{tr}}(\mathbb{R}), \mu, \nu \in \mathbb{Z}$, we have*

$$\mathrm{Op}^+(a)\mathrm{Op}^+(b) = \mathrm{Op}^+(ab) + g$$

for some $g \in \mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+)$, for $e = \max\{\nu, 0\}$.

(ii) *We have*

$$a(\xi_n) \in S^\mu_{\mathrm{tr}}(\mathbb{R}), g \in \mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+) \Rightarrow \mathrm{Op}^+(a)g \in \mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+), \tag{2.3.52}$$

$$k \in \mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+), b(\xi_n) \in S^\nu_{\mathrm{tr}}(\mathbb{R}) \Rightarrow k\mathrm{Op}^+(b) \in \mathcal{B}^h_{\mathrm{G}}(\overline{\mathbb{R}}_+), \tag{2.3.53}$$

for $h = \max\{\nu + e, 0\}$ and

$$k \in \mathcal{B}^d_{\mathrm{G}}(\overline{\mathbb{R}}_+), g \in \mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+) \Rightarrow kg \in \mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+). \tag{2.3.54}$$

For the proof we employ the following result.

Lemma 2.3.17. *Let $\varepsilon := \mathbb{R}_\pm \longrightarrow \mathbb{R}_\mp$ be defined as $\varepsilon(x_n) := -x_n$, and let*

$$\varepsilon^* := L^2(\mathbb{R}_\mp) \longrightarrow L^2(\mathbb{R}_\pm)$$

be the corresponding function pull back. Then for any $a(\xi_n) \in S^0_{\mathrm{tr}}(\mathbb{R})$ the operators

$$\mathrm{r}^+\mathrm{Op}(a)\mathrm{e}^-\varepsilon^*, \quad \epsilon^*\mathrm{r}^-\mathrm{Op}(a)\mathrm{e}^+$$

induce continuous maps

$$L^2(\mathbb{R}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+).$$

In addition

$$a \longrightarrow \mathrm{r}^+\mathrm{Op}(a)\mathrm{e}^-\varepsilon^*, \quad a \longrightarrow \varepsilon^*\mathrm{r}^-\mathrm{Op}(a)\mathrm{e}^+$$

are continuous as operators

$$S^0_{\mathrm{tr}}(\mathbb{R}) \longrightarrow \mathcal{L}(L^2(\mathbb{R}_+), \mathcal{S}(\overline{\mathbb{R}}_+)).$$

Proof of Theorem 2.3.16.. We write

$$a(\xi_n) = a_0(\xi_n) + p(\xi_n),\ b(\xi_n) = b_0(\xi_n) + q(\xi_n)$$

for $a_0, b_0 \in S^0_{\mathrm{tr}}(\mathbb{R})$ and polynomials p and q of order μ and ν, respectively. We have

$$\begin{aligned}\mathrm{Op}^+(a)\mathrm{Op}^+(b) =& \mathrm{Op}^+(a_0)\mathrm{Op}^+(b_0) + \mathrm{Op}^+(a_0)\mathrm{Op}^+(q)\\ &+ \mathrm{Op}^+(p)\mathrm{Op}^+(b_0) + \mathrm{Op}^+(p)\mathrm{Op}^+(q).\end{aligned} \tag{2.3.55}$$

The first summand on the right of (2.3.55) can be written in the form

$$\mathrm{Op}^+(a_0)\mathrm{Op}^+(b_0) = \mathrm{Op}^+(a_0 b_0) + \mathrm{r}^+\mathrm{Op}(a_0)(1-\mathrm{e}^+\mathrm{r}^+)\mathrm{Op}(b_0)\mathrm{e}^+.$$

We show that $g_0 := \mathrm{r}^+\mathrm{Op}(a_0)(1-\mathrm{e}^+\mathrm{r}^+)\mathrm{Op}(b_0)\mathrm{e}^+$ belongs to $\mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$, i.e., induces continuous operators

$$g_0, g_0^* : L^2(\mathbb{R}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+).$$

Using the isomorphism $\varepsilon^* := L^2(\mathbb{R}_\mp) \longrightarrow L^2(\mathbb{R}_\pm)$ coming from $\varepsilon := \mathbb{R}_\pm \longrightarrow \mathbb{R}_\mp$, $\varepsilon(x_n) = -x_n$, we obtain

$$g_0 = \mathrm{r}^+\mathrm{Op}(a_0)\mathrm{e}^-\varepsilon^*\varepsilon^*\mathrm{r}^-\mathrm{Op}(b_0)\mathrm{e}^+,\ g_0^* = \mathrm{r}^+\mathrm{Op}(\overline{b}_0)\mathrm{e}^-\varepsilon^*\varepsilon^*\mathrm{r}^-\mathrm{Op}(\overline{a}_0)\mathrm{e}^+.$$

Moreover, if $q(\xi_n)$ is a polynomial of order ν we have $\mathrm{Op}^+(q)\mathrm{Op}^+(a_0) = \mathrm{Op}^+(qa_0)$, and

$$\mathrm{Op}^+(a_0)\mathrm{Op}^+(q) = \mathrm{Op}^+(a_0 q) + h \text{ for some } h \in \mathcal{B}^\nu_{\mathrm{G}}(\overline{\mathbb{R}}_+). \tag{2.3.56}$$

For the proof of (2.3.56) we assume $q(\xi_n) := \xi_n$. The general case easily follows by iterating the arguments, while $\nu = 0$ is trivial where $h = 0$. In the computation we may assume $u \in C_0^\infty(\overline{\mathbb{R}}_+)$; then

$$\begin{aligned}\mathrm{Op}^+(a_0)\mathrm{Op}^+(q)u(x_n) &= \mathrm{r}^+\int \mathrm{e}^{ix_n\xi_n}a_0(\xi_n)\Big\{\int_0^\infty \mathrm{e}^{-ix_n'\xi_n}i^{-1}\partial_{x_n'}u(x_n')\,dx_n'\Big\}đ\xi_n\\ &= \mathrm{r}^+\int \mathrm{e}^{ix_n\xi_n}\Big\{\mathrm{e}^{-ix_n'\xi_n}u(x_n')|_0^\infty + \int_0^\infty \mathrm{e}^{-ix_n'\xi_n}\xi_n u(x_n')\,dx_n'\Big\}đ\xi_n\\ &= \mathrm{Op}^+(a_0\xi_n)u(x_n) + r^+\int \mathrm{e}^{ix_n\xi_n}(-i\xi_n)a_0(\xi_n)\,đ\xi_n(\gamma_0 u).\end{aligned}$$

Observe that the second term on the right of the latter relation represents an element of $\mathcal{B}^1_{\mathrm{G}}(\overline{\mathbb{R}}_+)$. Finally, we have $\mathrm{Op}^+(p)\mathrm{Op}^+(q) = \mathrm{Op}^+(pq)$ which is evident. Let $a(\xi_n) \in S^\mu_{\mathrm{tr}}(\mathbb{R})$ and $g \in \mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+)$. Then

$$gu(x_n) = \sum_{j=0}^{e}\int_0^\infty c_j(x_n, x_n')\partial^j_{x_n'}u(x_n')\,dx_n' \tag{2.3.57}$$

for kernels $c_j(x_n, x_n') \in \mathcal{S}(\overline{\mathbb{R}}_+)\hat{\otimes}_\pi\mathcal{S}(\overline{\mathbb{R}}_+)$. Moreover, we have a continuous operator (2.3.45), and it follows that $\mathrm{Op}^+(a)c_j(x_n, x_n') \in \mathcal{S}(\overline{\mathbb{R}}_+)\hat{\otimes}_\pi\mathcal{S}(\overline{\mathbb{R}}_+)$. This implies $\mathrm{Op}^+(a)g \in \mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+)$. Next let k and b be as in (2.3.53), and assume for simplicity

$$kv(x_n) = \int_0^\infty h(x_n, x_n')\partial^e_{x_n'}v(x_n')\,dx_n', \tag{2.3.58}$$

where $h(x_n, x_n') \in \mathcal{S}(\overline{\mathbb{R}}_+)\hat{\otimes}_\pi\mathcal{S}(\overline{\mathbb{R}}_+)$. Then

$$k\mathrm{Op}^+(b)v(x_n) = \int_0^\infty h(x_n, x_n')\mathrm{Op}^+((i\xi_n)^e b)v(x_n')\,dx_n'.$$

If $e + \nu \le 0$ we have

$$\mathrm{Op}^+((i\xi_n)^e b) : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+),$$

and it follows that $k\mathrm{Op}^+(b) \in \mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+)$. In the case $e + \nu \geq 0$ we can write $(i\xi_n)^e b(\xi_n) = f_0(\xi_n) + r(\xi_n)$ for some $f_0(\xi_n) \in S^{-1}_{\mathrm{tr}}(\mathbb{R})$ and a polynomial $r(\xi_n)$ of order $e+\nu$. We then obtain

$$k\mathrm{Op}^+(b)v(x_n) = \int_0^\infty h(x_n, x_n')\mathrm{Op}^+(f_0)v(x_n')\,dx_n' + k\mathrm{Op}^+(r)v.$$

The first summand in the latter expression defines an element of $\mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$ and the second one an element of $\mathcal{B}^{e+\nu}_{\mathrm{G}}(\overline{\mathbb{R}}_+)$.

Finally, for (2.3.54) we form

$$gu(x_n) := \sum_{j=0}^{e} \int_0^\infty c_j(x_n, x_n')\partial^j_{x_n'} u(x_n')\,dx_n',$$

and assume again (2.3.58). Then

$$(kg)u(x_n) = \int_0^\infty h(x_n, x_n'') \sum_{j=0}^{e} \int_0^\infty \partial^d_{x_n''} c_j(x_n'', x_n')\partial^j_{x_n'} u(x_n')\,dx_n' dx_n'' \tag{2.3.59}$$

gives us the claimed structure of the composition. □

Theorem 2.3.18. *Let* $\boldsymbol{a} \in \mathcal{B}^{\mu,d}(\overline{\mathbb{R}}_+; j_0, j_2)$, $\boldsymbol{b} \in \mathcal{B}^{\nu,e}(\overline{\mathbb{R}}_+; j_1, j_0)$, *then*

$$\boldsymbol{ab} \in \mathcal{B}^{\mu+\nu,h}(\overline{\mathbb{R}}_+; j_1, j_2),$$

for $h := \max\{d+\nu, e\}$.

Proof. For simplicity we assume $j_0 = j_1 = j_2 = 1$. The composition of upper left corners, namely, $\boldsymbol{a}_{11}, \boldsymbol{b}_{11}$ for

$$\boldsymbol{a}_{11} := \mathrm{Op}^+(a) + g_{11}, \quad \boldsymbol{b}_{11} := \mathrm{Op}^+(b) + h_{11},$$

for

$$g_{11} \in \mathcal{B}^d_{\mathrm{G}}(\overline{\mathbb{R}}_+), \quad h_{11} \in \mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+)$$

and $a(\xi_n) \in S^\mu_{\mathrm{tr}}(\mathbb{R})$, $b \in S^\nu_{\mathrm{tr}}(\mathbb{R})$ has been characterized by Theorem 2.3.16. The remaining entries in $\boldsymbol{ab}$ for

$$\boldsymbol{a} = \begin{pmatrix} \boldsymbol{a}_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} \boldsymbol{b}_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \tag{2.3.60}$$

are

$$g_{12}h_{21}, \quad \boldsymbol{a}_{11}h_{12} + g_{12}h_{22}, \tag{2.3.61}$$

$$g_{21}\boldsymbol{b}_{11} + g_{22}h_{21}, \quad g_{21}h_{12} + g_{22}h_{22}. \tag{2.3.62}$$

The first operator in (2.3.61) is the composition of a potential operator and a trace operator of type e. Such a composition is then Green of type e. Moreover, by assumption h_{12} and g_{12} are potential symbol; then both $\boldsymbol{a}_{11}h_{12}$ and $g_{12}h_{22}$ are potential symbols, because of (2.3.45), and $h_{22} \in \mathbb{C}$. Moreover, g_{21} is a trace symbol of type d, cf. formula (2.3.57), then for the first operator in (2.3.62) we have

$$g_{21}\boldsymbol{b}_{11} = g_{21}\mathrm{Op}^+(b) + g_{21}h_{11} \tag{2.3.63}$$

where $g_{21}h_{11}$ is a trace operator of type d and $g_{22}h_{21}$ is a trace operator of type e. In order to characterize (2.3.63) it remains to verify that $g_{21}\mathrm{Op}^+(b)$ is a trace perator of type $d+\nu$. Without loss of generality we assume $\nu \geq 0$. Then we write

$$b(\xi_n) = q(\xi_n) + b_0(\xi_n)$$

for a polynomial $q(\xi_n)$ of degree ν and some $b_0(\xi_n) \in S^0_{\mathrm{tr}}(\mathbb{R})$. It follows that

$$\mathrm{Op}^+(b) = \mathrm{Op}^+(q) + \mathrm{Op}^+(b_0).$$

g_{21} is a trace operator of type d, i.e., of the form

$$g_{21}u(x_n) = \sum_{j=0}^{d} \int_0^\infty c_j(x_n, x_n')\partial^j_{x_n'} u(x_n')\, dx_n',$$

cf. formula (2.3.57). Then

$$\begin{aligned} g_{21}\mathrm{Op}^+(q)u &= \sum_{j=0}^{d} \int_0^\infty c_j(x_n, x_n')\partial^j_{x_n'} \sum_{l=0}^{\nu} \partial^l_j u(x_n')\, dx_n' \\ &= \sum_{j=0}^{d}\sum_{l=0}^{\nu} \int_0^\infty c_j(x_n, x_n')\partial^{j+l}_{x_n'} u(x_n')\, dx_n', \end{aligned} \tag{2.3.64}$$

i.e., $g_{21}\mathrm{Op}^+(q)$ is a trace operator of order $d+\nu$. Moreover, applying $g_{21}\mathrm{Op}^+(b_0)$ to $u \in \mathcal{S}(\overline{\mathbb{R}}_+)$, gives us

$$\begin{aligned} g_{21}\mathrm{Op}^+(b_0)u &= \sum_{j=0}^{d} \int_0^\infty c_j(x_n, x_n')\mathrm{Op}^+\big((i\xi_n)^j b_0(\xi_n)\big) u(x_n')\, dx_n' \\ &= \sum_{j=0}^{d} \int_0^\infty c_j(x_n, x_n')\mathrm{Op}^+(b_0)\partial^j_{x_n'} u(x_n')\, dx_n'. \end{aligned} \tag{2.3.65}$$

Here we employed the fact that the differentiations $\partial^j_{x_n'}$ applied to $\mathrm{Op}^+(b_0)$ can be translated to $\mathrm{Op}^+(i\xi_n)^j$ and, since $(i\xi_n)^j$ are minus-and plus-symbols at the same time, we have

$$\begin{aligned} \partial^j_{x_n'}\mathrm{Op}^+(b_0) &= \mathrm{Op}^+((i\xi_n)^j)\mathrm{Op}^+(b_0) = \mathrm{Op}^+\big((i\xi_n)^j b_0(\xi_n)\big) \\ &= \mathrm{Op}^+(b_0)\mathrm{Op}^+((i\xi_n)^j) = \mathrm{Op}^+(b_0)\partial^j_{x_n'}. \end{aligned} \tag{2.3.66}$$

This explains the right hand side of (2.3.65). It remains to recognize that the composition between a green operator of type zero, represented by the Schwartz kernels $c_j(x_n, x_n')$, composed with $\mathrm{Op}^+(b_0)$ from the right, is again a Green operator, with such a kernel. Here we apply Theorem 2.2.7. Summing up we characterized the first operator in (2.3.62). Concerning the second one it is simply a complex number, using that $g_{22}h_{22}$ is the multiplication of complex numbers which $g_{21}h_{12}$ is the composition of operators

$$h_{12} : \mathbb{C} \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+), \quad g_{21} : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \mathbb{C},$$

i.e.,

$$g_{21}h_{12} : \mathbb{C} \longrightarrow \mathbb{C}$$

by a complex number. □

Proposition 2.3.19. (i) *Let $a(\xi_n) \in S^{\mu}_{\mathrm{tr}}(\mathbb{R}), s \in \mathbb{N}$, and consider the operator*

$$\mathrm{Op}^+(a) : H^s(\mathbb{R}_+) \longrightarrow H^{s-\mu}(\mathbb{R}_+). \tag{2.3.67}$$

Then we have

$$\boldsymbol{a} := \mathrm{r}^+\mathrm{Op}(l_-^{s-\mu})\mathrm{e}^+_{s-\mu}\mathrm{Op}^+(a)\mathrm{Op}^+(l_-^{-s}) = \mathrm{Op}^+(l_-^{-\mu}a) + g_0 \tag{2.3.68}$$

for a $g_0 \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$. For $s - \mu \geq 0$ the operator $\boldsymbol{a}$ coincides with

$$\mathrm{Op}^+(l_-^{s-\mu})\mathrm{Op}^+(a)\mathrm{Op}^+(l_-^{-s}).$$

(ii) *For $g \in \mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+), s \in \mathbb{N}, s > e - 1/2$, we have*

$$g_1 := \mathrm{Op}^+(l_-^{s-\mu})g\,\mathrm{Op}^+(l_-^{-s}) \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+).$$

Proof.

(i) We have

$$\begin{aligned} &\mathrm{r}^+\mathrm{Op}(l_-^{s-\mu})\mathrm{e}^+_{s-\mu}\mathrm{r}^+\mathrm{Op}(a)\mathrm{e}^+ \\ &= \mathrm{r}^+\mathrm{Op}(l_-^{s-\mu}a)\mathrm{e}^+ - \mathrm{r}^+\mathrm{Op}(l_-^{s-\mu})(1 - \mathrm{e}^+_{s-\mu}\mathrm{r}^+)\mathrm{Op}(a)\mathrm{e}^+. \end{aligned} \tag{2.3.69}$$

By virtue of $(1 - \mathrm{e}^+_{s-\mu}\mathrm{r}^+)\mathrm{Op}(a)\mathrm{e}^+ \in H_0^{s-\mu}(\overline{\mathbb{R}}_-)$ the right hand side of (2.3.69) is equal to $\mathrm{r}^+\mathrm{Op}(l_-^{s-\mu}a)\mathrm{e}^+$, cf. Theorem 2.3.8. For $s - \mu \geq 0$ we may replace $\mathrm{e}^+_{s-\mu}$ by e^+ with the same result. From Theorem 2.3.18 we obtain

$$\boldsymbol{a} = \mathrm{Op}^+(l_-^{s-\mu}a)\mathrm{Op}(l_-^{-s}) = \mathrm{Op}^+(l_-^{-\mu}a) + g_0$$

for a $g_0 \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$. Assertion (ii) is a consequence of Proposition 2.3.15. □

Theorem 2.3.20. *Let $\boldsymbol{a} \in \mathcal{B}^{\mu,e}(\overline{\mathbb{R}}_+), s \in \mathbb{N}, s \geq \max\{e, \mu\}$. Then*

$$R : \boldsymbol{a} \longmapsto \mathrm{Op}^+(l_-^{s-\mu})\boldsymbol{a}\mathrm{Op}^+(l_-^{-s}) =: \boldsymbol{b} \tag{2.3.70}$$

induces an isomorphism

$$R : \mathcal{B}^{\mu,e}(\overline{\mathbb{R}}_+) \longrightarrow \mathcal{B}^{0,0}(\overline{\mathbb{R}}_+) \tag{2.3.71}$$

with the inverse

$$R^{-1} : \boldsymbol{b} \longmapsto \mathrm{Op}^+(l_-^{-s+\mu})\boldsymbol{b}\mathrm{Op}^+(l_-^{s}). \tag{2.3.72}$$

Proof. The map (2.3.71) is a consequence of Proposition 2.3.19. In order to show $RR^{-1} = \mathrm{id}$ we observe that

$$RR^{-1}\boldsymbol{b} = \mathrm{Op}^+(l_-^{s-\mu})\mathrm{Op}^+(l_-^{-s+\mu})\boldsymbol{a}\mathrm{Op}^+(l_-^{s})\mathrm{Op}^+(l_-^{-s}) = \boldsymbol{a}$$

because of $\mathrm{Op}^+(l_-^{-s+\mu})\mathrm{Op}^+(l_-^{s-\mu}) = 1$, $\mathrm{Op}^+(l_-^{s})\mathrm{Op}^+(l_-^{-s}) = 1$, cf. Proposition 2.3.11. Relation $RR^{-1} = \mathrm{id}$ can be proved in an analogous manner.

□

Remark 2.3.21. *Similarly as Propositions* 2.2.11 *and* 2.2.12 *for Green and trace operators of some type $e \in \mathbb{N}$ we have unique representations of* (2.3.2) *and* (2.3.3) *in the form*

$$g_{11} = \sum_{j=0}^{e-1} k_j \circ \gamma^j + g_0 \tag{2.3.73}$$

and

$$g_{21} = \sum_{j=0}^{e-1} c_j \gamma^j + x_{n0} \tag{2.3.74}$$

for suitable potential operators k_j, Green and trace operators g_0 and x_{n0} of type 0.

Lemma 2.3.22. *Assume $g \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$ and let*

$$1 + g : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+)$$

be an isomorphism. Then there is an $h \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$ such that

$$(1 + g)^{-1} = 1 + h. \tag{2.3.75}$$

Proof. By virtue of $(1 + g)^{-1} \in \mathcal{L}(L^2(\mathbb{R}_+))$ we have

$$h := (1 + g)^{-1} - 1 \in \mathcal{L}(L^2(\mathbb{R}_+)).$$

According to Theorem 2.2.7 we have to show that

$$h, h^* : L^2(\mathbb{R}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+)$$

are continuous, using a similar property of g, g^*. From (2.3.75) it follows that $1 = (1+g)(1+h)$, i.e., $0 = g + h + gh$, i.e., $h = -g(1 + h)$. Thus,

$$h : L^2(\mathbb{R}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+)$$

is continuous, since $1 + h \in \mathcal{L}(L^2(\mathbb{R}_+))$. Moreover, we have $1 = (1 + h)(1 + g)$, i.e.,

$$1 = (1 + g^*)(1 + h^*) = 1 + g^* + h^* + g^*h^*$$

which entails $h^* = -g^*(1 + h^*)$ and hence the continuity of

$$h^* : L^2(\mathbb{R}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+).$$

□

Proposition 2.3.23. *Let $\boldsymbol{g} \in \mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+; j, j)$ and assume that*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \boldsymbol{g} : \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^j \end{matrix} \longrightarrow \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^j \end{matrix} \tag{2.3.76}$$

is an isomorphism for some $s = s_0 > e - \frac{1}{2}$. Then (2.3.76) *is an isomorphism for every $s > e - \frac{1}{2}$, and the inverse has analogous form, namely,*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \boldsymbol{k} \tag{2.3.77}$$

for some $\boldsymbol{k} \in \mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+; j, j)$.

Proof. Let us write the operator $\boldsymbol{g}$ in (2.3.76) form

$$\boldsymbol{g} = \begin{pmatrix} f & k \\ b & q \end{pmatrix}. \tag{2.3.78}$$

Then for (2.3.76) we obtain

$$\begin{pmatrix} 1+f & k \\ b & q \end{pmatrix} \tag{2.3.79}$$

which is invertible by assumption. We now use the fact that in the space of all $j \times j$ matrices (identified with $\mathbb{R}^{j^2}$) the invertible elements are open and dense. Therefore, there is a $j \times j$ matrix which is invertible and $\operatorname{dist}(q, r) < \varepsilon$ for a given $\varepsilon > 0$. We now employ the fact that then also

$$\begin{pmatrix} 1+f & k \\ b & r \end{pmatrix} \tag{2.3.80}$$

is invertible when $\varepsilon > 0$ is sufficiently small. This can be proved by applying the Neumann series. In order to express the inverse of (2.3.79) we first assume that we already constructed the inverse of (2.3.80), cf. the expression below. Setting

$$\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} := \begin{pmatrix} 1+f & k \\ b & r \end{pmatrix}^{-1} \tag{2.3.81}$$

we obtain

$$\begin{pmatrix} 1+f & k \\ b & q \end{pmatrix} \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ n & m \end{pmatrix} \tag{2.3.82}$$

where the right-hand side is invertible since both factors on the left-hand side are invertible. Since the right-hand side is a triangular matrix the operator m is an invertible $j \times j$-matrix, and we have $n := bd_{11} + qd_{21}$. Then, since

$$\begin{pmatrix} 1 & 0 \\ n & m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -m^{-1}n & m^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.3.83}$$

it follows that

$$\begin{pmatrix} 1+f & k \\ b & q \end{pmatrix}^{-1} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -m^{-1}n & m^{-1} \end{pmatrix}. \tag{2.3.84}$$

The invertibility of the matrix r allows us to form

$$\begin{pmatrix} 1 & -kr^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+f & k \\ b & r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -r^{-1}b & r^{-1} \end{pmatrix} = \begin{pmatrix} 1+g & 0 \\ 0 & 1 \end{pmatrix} \tag{2.3.85}$$

for $g := f - kr^{-1}b \in \mathcal{B}^{\mathrm{e}}_{\mathrm{G}}(\overline{\mathbb{R}}_+)$, becomes of $f \in \mathcal{B}^{\mathrm{e}}_{\mathrm{G}}(\overline{\mathbb{R}}_+)$and $-kr^{-1}b \in \mathcal{B}^{\mathrm{e}}_{\mathrm{G}}(\overline{\mathbb{R}}_+)$, cf. the first operator in the (2.3.61). Then (2.3.85) gives us

$$\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -r^{-1}b & r^{-1} \end{pmatrix} \begin{pmatrix} (1+g)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -kr^{-1} \\ 0 & 1 \end{pmatrix}, \tag{2.3.86}$$

and the next step is to compute $(1+g)^{-1}$. From (2.3.86) we see that

$$1 + g = 1 + f - kr^{-1}b : H^s(\mathbb{R}_+) \longrightarrow H^s(\mathbb{R}_+) \tag{2.3.87}$$

is isomorphism. By virtue of (2.3.73) we can write

$$1 + g = 1 + g_0 + \sum_{i=0}^{e-1} k_i \circ \gamma^i$$

for unique $g_0 \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$ and potential operators $k_i, i = 0, \dots, d-1$. We now choose some $g_1 \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$ such that

$$1 + g_0 + g_1 : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+) \tag{2.3.88}$$

is an isomorphism. The construction of g_1 is as follows. Then, using generalities on Fredholm operators and since g_0 is a compact operator in $L^2(\mathbb{R}_+)$ we observe that

$$1 + g_0 : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+)$$

is Fredholm of index 0. The spaces

$$V := \ker(1 + g_0), \quad W := \ker(1 + g_0^*)$$

which are of the same finite dimension e and $V, W \subset \mathcal{S}(\overline{\mathbb{R}}_+)$. This allows us to form an isomorphism

$$\begin{pmatrix} 1 + g_0 & w \\ v & 0 \end{pmatrix} : \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^e \end{matrix} \longrightarrow \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^e \end{matrix} \tag{2.3.89}$$

wher $v : V \longrightarrow \mathbb{C}^e$ and $w : \mathbb{C}^e \longrightarrow W$. Thus for some sufficiently small $\varepsilon > 0$, the operator

$$\begin{pmatrix} 1 + g_0 & w \\ v & \varepsilon\, \mathrm{id}_{\mathbb{C}^e} \end{pmatrix} : \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^e \end{matrix} \longrightarrow \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^e \end{matrix} \tag{2.3.90}$$

is also an isomorphism. In a similar manner as we saw that (2.3.87) is an isomorphism we obtain an isomorphism (2.3.88) for $g_1 = -w\varepsilon^{-1}\mathrm{id}_{\mathbb{C}^e}\, v$. From Lemma 2.3.22 we obtain

$$(1 + g_0 + g_1)^{-1} = 1 + h$$

for some $h \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$. Therefore, writing

$$1 + g = 1 + g_0 + g_1 - g_1 + \sum_{i=0}^{d-1} k_i \circ \gamma^i$$

gives us

$$(1 + h)(1 + g) = 1 + (1 + h)(-g_1 + \sum_{i=0}^{d-1} k_i \circ \gamma^i) : H^s(\mathbb{R}_+) \longrightarrow H^s(\mathbb{R}_+). \tag{2.3.91}$$

By construction the operator $-g_1$ is of finite rank, namely,

$$-g_1 = \sum_{i=1}^{e} l_i m_i$$

for some potential operators

$$l_i : \mathbb{C} \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+)$$

and trace operators

$$m_i : L^2(\overline{\mathbb{R}}_+) \longrightarrow \mathbb{C}$$

of type 0. Then

$$(1+h)(-g_1 + \sum_{i=0}^{d-1} k_i \gamma^i) = \sum_{i=1}^{e} (1+h) l_i m_i + \sum_{i=0}^{d-1} (1+h) k_i \gamma^i \tag{2.3.92}$$

where $(1+h)l_i$ and $(1+h)k_i$ are potential operators. This allows us to write (2.3.92) in the form

$$(1+h)(1+g) = 1 + \sum_{j=1}^{e+d} p_j s_j = 1 + \mathcal{PS} : H^s(\mathbb{R}_+) \longrightarrow H^s(\mathbb{R}_+) \tag{2.3.93}$$

for the vectors of operators $\mathcal{P} := (p_1, \ldots, p_{e+d})$, $\mathcal{S} := (s_1, \ldots, s_{e+d})$, and $1 + \mathcal{PS}$ is invertible. Consider the operators

$$1 + \mathcal{SP} : \mathbb{C}^{e+d} \longrightarrow \mathbb{C}^{e+d}. \tag{2.3.94}$$

Let us show that (2.3.94) is invertible if and only if $1 + \mathcal{PS}$ is invertible. In fact writing

$$\mathfrak{P} := \begin{pmatrix} 1 & \mathcal{P} \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{S} := \begin{pmatrix} 1 & 0 \\ -\mathcal{S} & 1 \end{pmatrix}, \quad \mathfrak{F} := \begin{pmatrix} 1 & -\mathcal{P} \\ \mathcal{S} & 1 \end{pmatrix}$$

with 1 denoting the respective identity maps, we have

$$\mathfrak{PFS} = \begin{pmatrix} 1+\mathcal{PS} & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{SFP} = \begin{pmatrix} 1 & 0 \\ 0 & 1+\mathcal{SP} \end{pmatrix}.$$

Simple computations show that

$$(1 - \mathcal{SP})^{-1} = 1 - \mathcal{S}(1+\mathcal{PS})^{-1}\mathcal{P}$$

and

$$(1+\mathcal{PS})^{-1} = 1 - \mathcal{P}(1+\mathcal{SP}),$$

where

$$g_3 := -\mathcal{P}(1+\mathcal{SP})^{-1}\mathcal{S} \in \mathcal{B}_{\mathrm{G}}^d(\overline{\mathbb{R}}_+).$$

Thus the inverse of (2.3.92), or equivalently the inverse of (2.3.93) has the form

$$1 + g_3 = \big((1+h)(1+g)\big)^{-1} = (1+g)^{-1}(1+h)^{-1}$$

which gives us

$$(1+g)^{-1} = (1+h)(1+g_3) = 1 + g_4$$

for $g_4 \in \mathcal{B}_{\mathrm{G}}^d(\overline{\mathbb{R}}_+)$. Using Remark 2.3.2 the operator

$$g : H^s(\mathbb{R}_+) \longrightarrow H^s(\mathbb{R}_+) \tag{2.3.95}$$

is compact and induces a continuous operator

$$g : H^s(\mathbb{R}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+). \tag{2.3.96}$$

Thus $1+g$ is Fredholm in $H^s(\mathbb{R}_+)$ of index 0. From (2.3.96) we obtain $\ker(1+g) \subset \mathcal{S}(\overline{\mathbb{R}}_+)$. Because $(1+g)u = 0$, $u \in H^s(\mathbb{R}_+)$, entails $u \in \mathcal{S}(\overline{\mathbb{R}}_+)$ and $\ker(1+g)$ is independent of s. Since $1+g$ is an isomorphism in $H^s(\mathbb{R}_+)$ we have $\ker(1+g) = \{0\}$; then the coker is also trivial, and hence the computation is independent of s. □

Definition 2.3.24. (i) *An* $a(\xi_n) \in S^{\mu}_{\mathrm{tr}}(\mathbb{R})$ *is called elliptic, if* $a(\xi_n) \neq 0$ *for all* $\xi_n \in \mathbb{R}$ *and* $a_{(\mu)}(\xi_n) \neq 0$ *for* $\xi_n \neq 0$.

(ii) *An operator*

$$\boldsymbol{a} := \begin{pmatrix} \mathrm{Op}^+(a) + g & k \\ b & q \end{pmatrix} \in \mathcal{B}^{\mu,e}(\overline{\mathbb{R}}_+; j_1, j_2) \tag{2.3.97}$$

(*cf. notation* (2.3.8) *and* (2.3.1) *where* g *play the role of* g_{11}) *is called elliptic, if* $a(\xi_n)$ *is elliptic as in* (i).

(iii) *An operator*

$$\boldsymbol{p} := \begin{pmatrix} \mathrm{Op}^+(p) + h & c \\ w & m \end{pmatrix} \in \mathcal{B}^{-\mu,h}(\overline{\mathbb{R}}_+; j_2, j_1) \tag{2.3.98}$$

for some type $h \in \mathbb{N}$ *is called a parametrix of* $\boldsymbol{a} \in \mathcal{B}^{\mu,e}(\overline{\mathbb{R}}_+; j_1, j_2)$ *if*

$$\boldsymbol{pa} = \mathrm{diag}(1,0) + \boldsymbol{g}_{\mathrm{L}}, \quad \boldsymbol{ap} = \mathrm{diag}(1,0) + \boldsymbol{g}_{\mathrm{R}} \tag{2.3.99}$$

for certain $\boldsymbol{g}_{\mathrm{L}} \in \mathcal{B}^{e_{\mathrm{L}}}_{\mathrm{G}}(\overline{\mathbb{R}}_+; j_1, j_1)$, $\boldsymbol{g}_{\mathrm{R}} \in \mathcal{B}^{e_{\mathrm{R}}}_{\mathrm{G}}(\overline{\mathbb{R}}_+; j_2, j_2)$ *for some resulting types* $e_{\mathrm{L}} = \max\{\mu, e\}$, $e_{\mathrm{R}} = \max\{e - \mu\}$.

Proposition 2.3.25. *Let* $b(\xi_n) \in S^0_{\mathrm{tr}}(\mathbb{R})$, $k \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$, *and interpret* $\mathrm{Op}^+(b) + k$ *either as a continuous operator*

$$\boldsymbol{b} : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+) \tag{2.3.100}$$

or

$$\boldsymbol{b}_{\mathcal{S}} : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+). \tag{2.3.101}$$

If $b(\xi_n)$ *is elliptic in the sense of Definition* 2.3.24 (i), *then* (2.3.100), (2.3.101) *are both Fredholm operators, and there is a parametrix* $\boldsymbol{q} := \mathrm{Op}^+(b^{-1})$ *such that*

$$\boldsymbol{qb} = 1 + k_{\mathrm{L}}, \quad \boldsymbol{bq} = 1 + k_{\mathrm{R}} \tag{2.3.102}$$

for some $k_{\mathrm{L}}, k_{\mathrm{R}} \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$. *There are finite-dimensional subspaces* $K, L \subset \mathcal{S}(\overline{\mathbb{R}}_+)$ *such that*

$$K = \ker \boldsymbol{b},\ L \cap \mathrm{im}\, \boldsymbol{b} = \{0\},\ L + \mathrm{im}\, \boldsymbol{b} = L^2(\mathbb{R}_+), \tag{2.3.103}$$

$$K = \ker \boldsymbol{b}_{\mathcal{S}},\ L \cap \mathrm{im}\, \boldsymbol{b}_{\mathcal{S}} = \{0\},\ L + \mathrm{im}\, \boldsymbol{b}_{\mathcal{S}} = \mathcal{S}(\overline{\mathbb{R}}_+). \tag{2.3.104}$$

Thus $\mathrm{ind}\, \boldsymbol{b} = \mathrm{ind}\, \boldsymbol{b}_{\mathcal{S}}$. *In particular,* (2.3.100) *is an isomorphism if and only if* (2.3.101) *is an isomorphisim, and we have in this case* $\boldsymbol{b}^{-1} \in \mathcal{B}^{0,0}(\overline{\mathbb{R}}_+)$.

Proof. If b is elliptic, $\boldsymbol{b} = \mathrm{Op}^+(b) + k$, and $\boldsymbol{q} := \mathrm{Op}^+(b^{-1})$, then we have relations (2.3.102) for some $k_{\mathrm{L}}, k_{\mathrm{R}} \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$, cf. Theorem (2.3.16) (i). Since Green operators are compact in $L^2(\mathbb{R}_+)$, cf. Remark 2.3.2, the operator (2.3.100) is Fredholm. This is a general information on Fredholm operators in Hilbert spaces. In particular, there are finite-dimensional spaces $K, L \subset L^2(\mathbb{R}_+)$ such that $K = \ker \boldsymbol{b}$, $L \oplus \mathrm{im}\, \boldsymbol{b} = L^2(\mathbb{R}_+)$. We have $K \subset \mathcal{S}(\overline{\mathbb{R}}_+)$ since $u \in \ker \boldsymbol{b}$ implies $\mathrm{Op}^+(b)u = -ku \in \mathcal{S}(\overline{\mathbb{R}}_+)$. Moreover,

$$\mathrm{Op}^+(b^{-1})\mathrm{Op}^+(b)u = (1 + k_{\mathrm{L}})u$$

for some $k_{\mathrm{L}} \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$, and $\mathrm{Op}^+(b)u = -ku$ yields

$$\mathrm{Op}^+(b^{-1})(-ku) = (1 + k_{\mathrm{L}})u \tag{2.3.105}$$

and hence, since $\mathrm{Op}^+(b^{-1}) - k := k_1 \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$ it follows that

$$k_1 u = (1 + k_{\mathrm{L}})u.$$

Thus

$$u = (k_1 - k_{\mathrm{L}})u \in \mathcal{S}(\overline{\mathbb{R}}_+),$$

and we see that $K \subset \mathcal{S}(\overline{\mathbb{R}}_+)$. For the cokernel which we may idintify with L we can proceed in analogous manner by passing to adjoint operators. In other words we conclude $L \subset \mathcal{S}(\overline{\mathbb{R}}_+)$. Thus together with (2.3.103) we obtain relations (2.3.104), and hence $\mathrm{ind}\boldsymbol{b} = \mathrm{ind}\boldsymbol{b}_{\mathcal{S}}$. In particular, we see (2.3.100) is an isomorphism if and only if (2.3.101) is an isomorphism. Let us show that the invertibility of (2.3.100) has the consequence that $\boldsymbol{b}^{-1} \in \mathcal{B}^{0,0}(\overline{\mathbb{R}}_+)$. From the considerations before we know in this case that $\mathrm{ind}\,\boldsymbol{b} = 0$. This implies, because of

$$\boldsymbol{q}_1 := \mathrm{Op}^+(b^{-1})\mathrm{Op}^+(b) = 1 + k_{\mathrm{L}}$$

for some $k_{\mathrm{L}} \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$ that $\mathrm{ind}\,\mathrm{Op}^+(b) = 0$. In fact, we have $\mathrm{ind}\,(\mathrm{Op}^+(b) + k) = 0$, and since $k \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$ is compact in $L^2(\mathbb{R}_+)$ also $\mathrm{ind}\,\mathrm{Op}^+(b) = 0$. Thus also $\mathrm{Op}^+(b^{-1})$ has index 0, since

$$\begin{aligned}\mathrm{ind}\,\mathrm{Op}^+(b^{-1})\mathrm{Op}^+(b) &= \mathrm{ind}(1 + k_{\mathrm{L}}) = 0\\ &= \mathrm{ind}\,\mathrm{Op}^+(b^{-1}) + \mathrm{ind}\,\mathrm{Op}^+(b)\end{aligned}$$

i.e.,

$$\mathrm{ind}\,\mathrm{Op}^+(b^{-1}) = \mathrm{ind}(1 + k_{\mathrm{L}}) - \mathrm{ind}\,\mathrm{Op}^+(b).$$

From the considerations before in $L^2(\mathbb{R}_+)$, now applied to the elliptic symbol b^{-1}, we have finite-dimentional subspaces $K_1, L_1 \subset \mathcal{S}(\overline{\mathbb{R}}_+)$ such that

$$K_1 = \ker \mathrm{Op}^+(b^{-1}), \quad L_1 \cap \mathrm{im}\,\mathrm{Op}^+(b^{-1}) = 0, \tag{2.3.106}$$

$$L_1 + \mathrm{im}\,\mathrm{Op}^+(b^{-1}) = L^2(\mathbb{R}_+). \tag{2.3.107}$$

The next step of the proof is to find an $l \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$ such that

$$\mathrm{Op}^+(b^{-1}) + l : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+)$$

is an isomorphism. This is done as follows. We set $m = \dim K_1 = \dim L_1$ and choose isomorphisms

$$k_1 : \mathbb{C}^m \longrightarrow L_1, \quad d_1 : K_1 \longrightarrow \mathbb{C}^m$$

and then

$$\begin{pmatrix} \mathrm{Op}^+(b^{-1}) & k_1 \\ d_1 & 0 \end{pmatrix} : \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^m \end{matrix} \longrightarrow \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^m \end{matrix} \tag{2.3.108}$$

is an isomorphism. In fact, (2.3.107) shows that

$$\begin{pmatrix} \mathrm{Op}^+(b^{-1}) & k_1 \end{pmatrix} : \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^m \end{matrix} \longrightarrow L^2(\mathbb{R}_+) \tag{2.3.109}$$

is surjective. Moreover, we see that

$$(\mathrm{Op}^+(b^{-1}))u + k_1 v = 0$$

implies $\mathrm{Op}^+(b^{-1})u = -k_1 v$ which entails that both sides are zero, cf. the second identity of (2.3.106). Then, since k_1 is injective, we conclude $v = 0$, thus $\mathrm{Op}^+(b^{-1})u = 0$ gives us $u \in K_1$. Then, (2.3.108) is an isomorphism if and only if the second row induces an isomorphism

$$(d_1 \quad 0) : \ker(\mathrm{Op}^+(b^{-1}) \quad k_1) \longrightarrow \mathbb{C}^m.$$

Which is equivalent to the isomorphism

$$d_1 : K_1 \longrightarrow \mathbb{C}^m.$$

Here we applied Lemma 2.1.4. Using openess of isomorphisms in a Hilbert space we find an $\varepsilon > 0$ such that also

$$\begin{pmatrix} \mathrm{Op}^+(b^{-1}) & k_1 \\ d_1 & \varepsilon \end{pmatrix} : \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^m \end{matrix} \longrightarrow \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^m \end{matrix} \tag{2.3.110}$$

is an isomorphism. Let us now pass to the isomorphism

$$\begin{aligned} \begin{pmatrix} 1 & -k_1\varepsilon^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Op}^+(b^{-1}) & k_1 \\ d_1 & \varepsilon \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\varepsilon^{-1}d_1 & 1 \end{pmatrix} \\ = \begin{pmatrix} \mathrm{Op}^+(b^{-1}) - k_1\varepsilon^{-1}d_1 & 0 \\ 0 & \varepsilon \end{pmatrix} \end{aligned}$$

on the left-hand side we have a composition of isomorphisms; so the right-hand side is an isomorphism as well. Thus, since $\varepsilon : \mathbb{C}^m \longrightarrow \mathbb{C}^m$ is an isomorphism also

$$\mathrm{Op}^+(b^{-1}) - k_1\varepsilon^{-1}d_1 : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+)$$

is an isomorphism. Here $g := -k_1\varepsilon^{-1}d_1 \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$. Now, since $\boldsymbol{b} = \mathrm{Op}^+(b) + k$ is an isomorphism also

$$(\mathrm{Op}^+(b^{-1}) + g)(\mathrm{Op}^+(b) + k) = 1 + h + g\mathrm{Op}^+(b) + = \mathrm{Op}^+(b^{-1})k = 1 + h_1$$

for $h_1 = h + g\mathrm{Op}^+(b) + \mathrm{Op}^+(b^{-1})k \in \mathcal{B}^0_{\mathrm{G}}(\mathbb{R}_+)$ is an isomorphism. Now it remains to recall that we have $(1 + h_1)^{-1} = 1 + h_2$ cf. Lemma 2.3.22. Thus we obtain

$$\begin{aligned} (1 + h_2)(\mathrm{Op}^+(b^{-1}) + g) &= \mathrm{Op}^+(b^{-1}) + h_2\mathrm{Op}^+(b^{-1}) + g + h_2 g \\ &= \mathrm{Op}^+(b^{-1}) + h_3 = (\mathrm{Op}^+(b) + k)^{-1}, \end{aligned}$$

for $h_3 \in \mathcal{B}^0_{\mathrm{G}}(\overline{\mathbb{R}}_+)$. □

Proposition 2.3.26. *Let $\boldsymbol{b} := \mathrm{Op}^+(b) + k \in \mathcal{B}^{0,0}_{\mathrm{G}}(\overline{\mathbb{R}}_+)$ as in Proposition 2.3.25 be ellipticity, and realize $\boldsymbol{b}$ as a continuous operator*

$$\boldsymbol{b} : H^s(\mathbb{R}_+) \longrightarrow H^s(\mathbb{R}_+) \tag{2.3.111}$$

$s \in \mathbb{R}, s > -1/2$. Then (2.3.111) is a Fredholm operator, and $\boldsymbol{q} := \mathrm{Op}^+(b^{-1})$ is a parametrix of $\boldsymbol{b}$, also in the sense of a map (2.3.111). Moreover, The finite-dimensional subspaces $K, L \subset \mathcal{S}(\overline{\mathbb{R}}_+)$ of Proposition 2.3.25 have analogous properties with respect to (2.3.111) namely,

$$K = \ker \boldsymbol{b}, \quad L \cap \mathrm{im}\, \boldsymbol{b} = \{0\}, \quad L + \mathrm{im}\, \boldsymbol{b} = H^s(\mathbb{R}_+) \tag{2.3.112}$$

and we als have relations (2.3.101) is an isomorphism.

Proof. As we know from Proposition 2.3.10 the operator $\mathrm{Op}^+(b) + k$ induces a continuous operator (2.3.111). If $b(\xi_n)$ is elliptic then (2.3.111) is Fredholm. In fact, $\boldsymbol{q}$ is a parametrix and the remainders are the same as in Proposition 2.3.25, namely, k_{L} and k_{R}. Those are compact in $H^s(\mathbb{R}_+)$ for $s > -1/2$, see also Remark 2.3.2. Similarly as in the proof of Proposition 2.3.25 it follows that $K = \ker \boldsymbol{b} \subset \mathcal{S}(\overline{\mathbb{R}}_+)$, and we obtain the first relation of (2.3.112). The second one follows by passing to the formal adjoint, namely, $(\mathrm{Op}^+(b) + k)^* = \mathrm{Op}^+(\overline{b}) + k^*$ which is of the same structure as the original operator. In particular, we see that (2.3.111) is an isomorphism exactly if (2.3.101) is an isomorphism. Moreover, the inverse of (2.3.112) is of the same shape as the one computed in the proof of Proposition 2.3.25.

□

Proposition 2.3.27. *Let $a(\xi_n) \in S^{\mu}_{\mathrm{tr}}(\mathbb{R}), g \in \mathcal{B}^{e}_{\mathrm{G}}(\overline{\mathbb{R}}_+)$ and interpret $\mathrm{Op}^+(a) + g$ either as a continuous operator*

$$\boldsymbol{a} : H^s(\mathbb{R}_+) \longrightarrow H^{s-\mu}(\mathbb{R}_+) \tag{2.3.113}$$

for $s > \max\{\mu, e\} - 1/2$, or

$$\boldsymbol{a}_{\mathcal{S}} : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+). \tag{2.3.114}$$

If $a(\xi_n)$ is elliptic in the sense of Definition 2.3.24 (i), then both (2.3.113) and (2.3.114) are Fredholm operators, and there is a parametrix $\boldsymbol{p} := \mathrm{Op}^+(a^{-1})$ such that

$$\boldsymbol{pa} = 1 + g_{\mathrm{L}}, \quad \boldsymbol{ap} = 1 + g_{\mathrm{R}} \tag{2.3.115}$$

for some $g_{\mathrm{L}} \in \mathcal{B}^{e_{\mathrm{L}}}_{\mathrm{G}}(\overline{\mathbb{R}}_+), g_{\mathrm{R}} \in \mathcal{B}^{e_{\mathrm{R}}}_{\mathrm{G}}(\overline{\mathbb{R}}_+)$ for $e_{\mathrm{L}} = \max\{\mu, e\}, e_{\mathrm{R}} = \max\{e - \mu, 0\}$. There are finite-dimensional subspaces $V, W \subset \mathcal{S}(\overline{\mathbb{R}}_+)$ such that

$$V = \ker \boldsymbol{a}, \quad W \cap \operatorname{im} \boldsymbol{a} = \{0\}, \quad W + \operatorname{im} \boldsymbol{a} = H^{s-\mu}(\mathbb{R}_+), \tag{2.3.116}$$

$$V = \ker \boldsymbol{a}_{\mathcal{S}}, \quad W \cap \operatorname{im} \boldsymbol{a}_{\mathcal{S}} = \{0\}, \quad W + \operatorname{im} \boldsymbol{a}_{\mathcal{S}} = \mathcal{S}(\overline{\mathbb{R}}_+). \tag{2.3.117}$$

Thus $\operatorname{ind} \boldsymbol{a} = \operatorname{ind} \boldsymbol{a}_{\mathcal{S}}$. In particular, (2.3.113) is an isomorphism if and only if (2.3.114) is an isomorphism, and in that case we have $\boldsymbol{a}^{-1} \in \mathcal{B}^{-\mu,h}(\overline{\mathbb{R}}_+)$ for $h = \max\{e - \mu, 0\}$.

Proof. Using Theorem 2.3.20 we pass to the operator

$$\begin{aligned} \boldsymbol{b}_{s_1} &:= \mathrm{Op}^+(l_-^{s_1-\mu})\boldsymbol{a}\mathrm{Op}^+(l_-^{-s_1}) \in \mathcal{B}^{0,0}(\overline{\mathbb{R}}_+), \\ \boldsymbol{b}_{s_1} &: H^{s_0}(\mathbb{R}_+) \longrightarrow H^{s_0}(\mathbb{R}_+),\ s_0 > -1/2 \end{aligned}$$

for $s_1 := \max\{\mu, e\}$, $s_0 = s - s_1 > -1/2$, according to the assumption of Proposition 2.3.26, for

$$b_{s_1}(\xi_n) = l_-^{s_1-\mu}(\xi_n)a(\xi_n)l_-^{-s_1}(\xi_n)$$

which is elliptic. We then obtain a parametrix of $\boldsymbol{b}_{s_1}$ of the form $\boldsymbol{q} = \mathrm{Op}^+(b_{s_1}^{-1})$ such that

$$\boldsymbol{q}\boldsymbol{b}_{s_1} = 1 + k_{\mathrm{L}}, \quad \boldsymbol{b}_{s_1}\boldsymbol{q} = 1 + k_{\mathrm{R}}.$$

This means

$$\begin{aligned} \boldsymbol{q}\mathrm{Op}^+(l_-^{s_1-\mu})\boldsymbol{a}\mathrm{Op}^+(l_-^{-s_1}) &= 1 + k_{\mathrm{L}}, \\ \mathrm{Op}^+(l_-^{s_1-\mu})\boldsymbol{a}\mathrm{Op}^+(l_-^{-s_1})\boldsymbol{q} &= 1 + k_{\mathrm{R}}. \end{aligned}$$

For $\boldsymbol{p} := \mathrm{Op}^+(l_-^{-s_1})\boldsymbol{q}\mathrm{Op}^+(l_-^{s_1-\mu})$, we thus obtain

$$\begin{aligned}\boldsymbol{pa} &= \mathrm{Op}^+(l_-^{-s_1})\boldsymbol{q}\mathrm{Op}^+(l_-^{s_1-\mu})\boldsymbol{a}\\ &= \mathrm{Op}^+(l_-^{-s_1})\boldsymbol{q}\mathrm{Op}^+(l_-^{s_1-\mu})\mathrm{Op}^+(l_-^{-s_1+\mu})\boldsymbol{b}_{s_1}\mathrm{Op}^+(l_-^{s_1})\\ &= \mathrm{Op}^+(l_-^{-s_1})\boldsymbol{q}\boldsymbol{b}_{s_1}\mathrm{Op}^+(l_-^{s_1}) = \mathrm{Op}^+(l_-^{-s_1})(1+k_{\mathrm{L}})\mathrm{Op}^+(l_-^{s_1})\\ &= 1+\mathrm{Op}^+(l_-^{-s_1})k_{\mathrm{L}}\mathrm{Op}^+(l_-^{s_1}) = 1+g_{\mathrm{R}}\end{aligned}$$

and, analogously,

$$\begin{aligned}\boldsymbol{ap} &= \mathrm{Op}^+(l_-^{-s_1+\mu})\boldsymbol{b}_{s_1}\mathrm{Op}^+(l_-^{s_1})\mathrm{Op}^+(l_-^{-s_1})\boldsymbol{q}\mathrm{Op}^+(l_-^{s_1-\mu})\\ &= \mathrm{Op}^+(l_-^{-s_1+\mu})\boldsymbol{b}_{s_1}\boldsymbol{q}\mathrm{Op}^+(l_-^{s_1-\mu}) = \mathrm{Op}^+(l_-^{-s_1+\mu})(1+k_{\mathrm{R}})\mathrm{Op}^+(l_-^{s_1-\mu})\\ &= 1+\mathrm{Op}^+(l_-^{-s_1+\mu})k_{\mathrm{R}}\mathrm{Op}^+(l_-^{s_1-\mu}) = 1+g_{\mathrm{R}}.\end{aligned}$$

Here we employed relations of the type (2.3.36). From Theorem 2.3.18 we obtain $g_{\mathrm{L}} \in \mathcal{B}_{\mathrm{G}}^{e_{\mathrm{L}}}(\overline{\mathbb{R}}_+), g_{\mathrm{R}} \in \mathcal{B}_{\mathrm{G}}^{e_{\mathrm{R}}}(\overline{\mathbb{R}}_+)$ for $e_{\mathrm{L}} = \max\{\mu, e\}, e_{\mathrm{R}} = \max\{e-\mu, 0\}$. Finally we have isomorphisms

$$\begin{aligned}\mathrm{Op}^+(l_-^{-s_1}) &: H^{s_0}(\mathbb{R}_+) \longrightarrow H^{s}(\mathbb{R}_+),\\ \mathrm{Op}^+(l_-^{-s_1+\mu}) &: H^{s_0}(\mathbb{R}_+) \longrightarrow H^{s-\mu}(\mathbb{R}_+)\end{aligned}$$

for $s = s_0 + s_1$. Thus

$$\boldsymbol{a} = \mathrm{Op}^+(l_-^{-s_1+\mu})\boldsymbol{b}_{s_1}\mathrm{Op}^+(l_-^{s_1}),$$

defines a Fredholm operators (2.3.113) and (2.3.114). At the same time we obtain relations (2.3.116), (2.3.117) for the spaces

$$V := \mathrm{Op}^+(l_-^{-s_1})K, \quad W := \mathrm{Op}^+(l_-^{-s_1+\mu})L$$

are an immediated consequence. In particular, the operator (2.3.114) is an isomorphism if and only (2.3.115) is an isomorphis. We then have

$$\boldsymbol{a}^{-1} = \mathrm{Op}^+(l_-^{-s_1})\boldsymbol{b}_{s_1}\mathrm{Op}^+(l_-^{s_1-\mu})$$

and from Theorem 2.3.18 we obtain the types in $\boldsymbol{a}^{-1} \in \mathcal{B}^{-\mu,h}(\overline{\mathbb{R}}_+)$ for $h = \max\{e-\mu, 0\}$. □

Theorem 2.3.28. *Let $\boldsymbol{a} \in \mathcal{B}^{0,0}(\overline{\mathbb{R}}_+; j_1, j_2)$, be interpreted as a continuous operator*

$$\boldsymbol{a} : \begin{array}{c} L^2(\mathbb{R}_+)\\ \oplus\\ \mathbb{C}^{j_1}\end{array} \longrightarrow \begin{array}{c} L^2(\mathbb{R}_+)\\ \oplus\\ \mathbb{C}^{j_2}\end{array}, \tag{2.3.118}$$

and let $\boldsymbol{a}^*$ be the adjoint of $\boldsymbol{a}$ in the sense of the identity

$$(\boldsymbol{a}u, v)_{L^2(\mathbb{R}_+)\oplus\mathbb{C}^{j_2}} = (u, \boldsymbol{a}^*v)_{L^2(\mathbb{R}_+)\oplus\mathbb{C}^{j_1}} \tag{2.3.119}$$

for all $u \in L^2(\mathbb{R}_+) \oplus \mathbb{C}^{j_1}, v \in L^2(\mathbb{R}_+) \oplus \mathbb{C}^{j_2}$. Then we have $\boldsymbol{a}^* \in \mathcal{B}^{0,0}(\overline{\mathbb{R}}_+; j_2, j_1)$.

Proof. Is straightforward. □

Corollary 2.3.29. *For $\boldsymbol{a} \in \mathcal{B}^{0,0}(\overline{\mathbb{R}}_+)$ we have $\boldsymbol{a}^* \in \mathcal{B}^{0,0}(\overline{\mathbb{R}}_+)$. Moreover, writing*

$$\boldsymbol{a} = \begin{pmatrix} a & k \\ t & q \end{pmatrix} \tag{2.3.120}$$

the operators t^* and k^* are potential and trace operators, respectively.

Definition 2.3.30. *By $\mathcal{B}^{\mu,e}(\overline{\mathbb{R}}_+; j_1, j_2)$ we denote the space of all block-matrix operators*

$$\boldsymbol{a} = \begin{pmatrix} b+g & k \\ t & q \end{pmatrix} \tag{2.3.121}$$

for $b := \mathrm{Op}^+(a), a(x_n, \xi_n) \in S^\mu_{\mathrm{tr}}(\overline{\mathbb{R}}_+ \times \mathbb{R})$, $a(x_n, \xi_n)$ independent of x_n for large x_n, and

$$\begin{pmatrix} g & k \\ t & q \end{pmatrix} \in \mathcal{B}^e_{\mathrm{G}}(\overline{\mathbb{R}}_+; j_1, j_2). \tag{2.3.122}$$

Definition 2.3.30 extends Definition 2.3.24 to the case of symbols $a(x_n, \xi_n)$ with non-constant coefficients. The above-mentioned result can be used to investigate the space of Definition 2.3.30 with respect to its mapping properties, symbolic structure, ellipticity, etc. .

First it is clear that any $\boldsymbol{a} \in \mathcal{B}^{\mu,e}(\overline{\mathbb{R}}_+; j_1, j_2)$ induces continuous operators

$$\boldsymbol{a} : \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_1} \end{matrix} \longrightarrow \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_2} \end{matrix} \tag{2.3.123}$$

for any $s > \max\{\mu, e\} - \frac{1}{2}$. In particular, we have continuity

$$\boldsymbol{a} : \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_1} \end{matrix} \longrightarrow \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_2} \end{matrix} . \tag{2.3.124}$$

Definition 2.3.31. (i) *An $\boldsymbol{a} \in \mathcal{B}^{\mu,e}(\overline{\mathbb{R}}_+)$ is called elliptic if $a(x_n, \xi_n) \in S^\mu_{\mathrm{tr}}(\overline{\mathbb{R}}_+ \times \mathbb{R})$ is elliptic in the sense $a_{(\mu)}(x_n, \xi_n) \neq 0$ for all $x_n \in \overline{\mathbb{R}}_+, \xi_n \in \mathbb{R} \setminus \{0\}$, and if $a(0, \xi_n)$ is elliptic in the sense of Definition* 2.3.24 (i).

(ii) *A $\boldsymbol{P} \in \mathcal{B}^{-\mu,h}(\overline{\mathbb{R}}_+, j_2, j_1)$ of the form* (2.3.98) *for some type $h \in \mathbb{N}$ is called a parametrix of $\boldsymbol{a}$ if relations* (2.3.99) *hold for some types e_{L} and $e_{\mathrm{R}} \in \mathbb{N}$.*

Theorem 2.3.32. *An elliptic opeartore $\boldsymbol{a} \in \mathcal{B}^{\mu,e}(\overline{\mathbb{R}}_+, j_1, j_2)$ has a parametrix $\boldsymbol{P} \in \mathcal{B}^{-\mu,h}(\overline{\mathbb{R}}_+, j_2, j_1)$ for $h = \max\{e - \mu, 0\}$.*

2.4 Complete symbolic structures

We now investigate both interior symbols with the transmission property at the boundary and complete operator-valued symbols describing boundary value problems close to the boundary. In this section we focus on a neighbourhood of a point on the boundary of the form

$$(x', x_n) \in \Omega \times \overline{\mathbb{R}}_+ \tag{2.4.1}$$

for an open set $\Omega \subseteq \mathbb{R}^{n-1}$. In notation from the beginning of Section 2.1 the set Ω represents a chart on the boundary ∂X, while $x_n \in \overline{\mathbb{R}}_+$ comes from the splitting of variables close to the boundary direction. In our consideration we take into account that the normal half-axis has a negative counterpart, and similarly as in the preceding section we observe distributions as well as symbols also in terms of the x_n-axis. The notation refers to variables $x \in \mathbb{R}^n$, and (2.4.1) means $x = (x', x_n)$, with covariable $\xi = (\xi', \xi_n)$. In the following definition symbols $a(x, \xi) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ will be specified in terms of what we call the transmission property at the boundary, expressed in terms of the sequence of homogeneous components $a_{(\mu-j)}(x, \xi), j \in \mathbb{N}$.

Definition 2.4.1. *A symbol $a(x', x_n, \xi', \xi_n) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R} \times \mathbb{R}^n_{\xi',\xi_n})$ for $\mu \in \mathbb{Z}$ is said to have the transmission property at $x_n = 0$ if*

$$\big(D^k_{x_n} D^\alpha_{\xi'} a_{(\mu-j)}\big)(x', 0, 0, 1) = (-1)^{\mu-j}\big(D^k_{x_n} D^\alpha_{\xi'} a_{(\mu-j)}\big)(x', 0, 0, -1) \tag{2.4.2}$$

for all $x' \in \Omega, k, j \in \mathbb{N}$. Let

$$S^\mu_{\mathrm{tr}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$$

denote the space of all symbols with the transmission property. Set

$$S^\mu_{\mathrm{tr}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n) := \{a|_{\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n} : a \in S^\mu_{\mathrm{tr}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)\}. \tag{2.4.3}$$

By $S^\mu_{\mathrm{tr}}(\mathbb{R}^n)$ we denote the subspace of all $a(x, \xi) \in S^\mu_{\mathrm{tr}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ that are independent of x.

Remark 2.4.2. *The space $S^\mu_{\mathrm{tr}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ is closed in $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ in the corresponding Fréchet topology of classical symbols; the same is true of $S^\mu_{\mathrm{tr}}(\mathbb{R}^n_\xi)$ with respect to the space $S^\mu_{\mathrm{cl}}(\mathbb{R}^n_\xi)$ of symbols with constant coefficients.*

Remark 2.4.3. *The space $S^\mu_{\mathrm{tr}}(\Omega_{x'} \times \mathbb{R}_{x_n} \times \mathbb{R}^n_{\xi',\xi_n})$ can be specialized for the case $n = 1$, i.e., symbols $a(x_n, \xi_n) \in S^\mu_{\mathrm{tr}}(\mathbb{R} \times \mathbb{R})$. The latter space can be identified with $C^\infty(\mathbb{R}_{x_n}, S^\mu_{\mathrm{tr}}(\mathbb{R}))$ where $S^\mu_{\mathrm{tr}}(\mathbb{R})$ has been defined after relation* (2.3.12).

Let us also discuss the coordinate invariance, to be used later on, concerning operators globally on a manifold with boundary. Let

$$\chi = (\chi', \mathrm{id}) : \Omega \times \mathbb{R} \longrightarrow \tilde{\Omega} \times \mathbb{R} \tag{2.4.4}$$

be a diffeomorphism, where $\mathrm{id} : \mathbb{R} \longrightarrow \mathbb{R}$ is the identity operator and $\chi' : \Omega \longrightarrow \tilde{\Omega}$ a diffeomorphism. In a similar sense we employ notation

$$\chi = (\chi', \mathrm{id}) : \Omega \times \overline{\mathbb{R}}_+ \longrightarrow \tilde{\Omega} \times \overline{\mathbb{R}}_+ \tag{2.4.5}$$

with χ' as before and id is the identity map. Then, according to general transformation rules of symbols we have a symbol push forward

$$\begin{aligned} \chi_* &: S^\mu_{\mathrm{tr}}(\Omega \times \mathbb{R} \times \mathbb{R}^n) \longrightarrow S^\mu_{\mathrm{tr}}(\tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n), \\ \chi_* &: S^\mu_{\mathrm{tr}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n) \longrightarrow S^\mu_{\mathrm{tr}}(\tilde{\Omega} \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n). \end{aligned} \tag{2.4.6}$$

Let us associate with symbols $a(x', x_n, \xi', \xi_n) \in S^\mu_{\mathrm{tr}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ operator-valued symbols which contribute later on to the upper left corners of boundary symbols.

The constructions refer to what we call decoupling of symbols which makes sense also in a slightly more general context, namely, when we replace $\mathbb{R}$ by $\mathbb{R}^d$ for a d-dimensional variable x_n with the corresponding covariable $\xi_n \in \mathbb{R}^d$. Starting with

$$a(x', x_n, \xi', \xi_n) \in S^\mu(\Omega \times \mathbb{R}^d \times \mathbb{R}^{n-1+d}_{\xi',\xi_n}) \tag{2.4.7}$$

we form

$$\boldsymbol{a}(x', x_n, \xi', \xi_n) := a(x', \langle\xi'\rangle^{-1}x_n, \xi', \langle\xi'\rangle\xi_n) \tag{2.4.8}$$

referred to as the decoupled symbol. For decoupling of symbols there are well-known theorems which we use for the moment without proof; the details will be given later on. In the following theorem we employ symbols taking values in Fréchet spaces, cf. (1.4.1), (1.4.3).

Theorem 2.4.4. *The decoupling map*

$$a(x', x_n, \xi', \xi_n) \longrightarrow \boldsymbol{a}(x', x_n, \xi', \xi_n) \tag{2.4.9}$$

induces continuous operators

$$S^\mu(\Omega \times \mathbb{R}^d \times \mathbb{R}^{n-1+d}_{\xi',\xi_n}) \longrightarrow S^\mu(\Omega_{x'} \times \mathbb{R}^{n-1}_{\xi'}, S^\mu(\mathbb{R}^d_{x_n} \times \mathbb{R}^d_{\xi_n})) \tag{2.4.10}$$

as well as

$$S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^d \times \mathbb{R}^{n-1+d}_{\xi',\xi_n}) \longrightarrow S^\mu_{\mathrm{cl}}(\Omega_{x_n} \times \mathbb{R}^{n-1}_{\xi'}, S^\mu_{\mathrm{cl}}(\mathbb{R}^d_{x_n} \times \mathbb{R}^d_{\xi'})). \tag{2.4.11}$$

Note that Theorem 2.4.4 gives rise to some particular symbol spaces, e,g.,

$$S^\mu(\mathbb{R}^{n-1}_{\xi'}, C^\infty(\mathbb{R}^d_{x_n})) \tag{2.4.12}$$

as a special case of the right-hand side of (2.4.10) where the elements do not depend on x' and on ξ_n.

With (2.4.8) we associate the family of operators $\mathrm{Op}_{x_n}(\boldsymbol{a})(x', \xi')$ by forming

$$\mathrm{Op}_{x_n}(\boldsymbol{a})(x', \xi')v(x_n) = \iint \mathrm{e}^{i(x_n - x_n')\xi_n}\boldsymbol{a}(x', x_n, \xi', \xi_n)v(x_n')\,dx_n' đ\xi_n. \tag{2.4.13}$$

In the following consideration we assume that the x_n-dependence of $a(x', x_n, \xi', \xi_n)$ for large $|x_n|$ is specified in such a way that

$$\mathrm{Op}_{x_n}(\boldsymbol{a})(x', \xi') : H^s(\mathbb{R}^d) \longrightarrow H^{s-\mu}(\mathbb{R}^d) \tag{2.4.14}$$

is continuous for every $s \in \mathbb{R}$ and $(x', \xi') \in \Omega \times \mathbb{R}^{n-1}$. For our purposes it suffices to assume that $a(x', x_n, \xi', \xi_n)$ is independent of x_n for $|x_n| \geq C$ for some $C > 0$. Then (2.4.14) represents an element

$$\mathrm{Op}_{x_n}(\boldsymbol{a})(x', \xi') \in C^\infty(\Omega \times \mathbb{R}^{n-1}, \mathcal{L}(H^s(\mathbb{R}^d), H^{s-\mu}(\mathbb{R}^d))) \tag{2.4.15}$$

for every $s \in \mathbb{R}$.

Lemma 2.4.5. *We have*

$$\mathrm{Op}_{x_n}(\boldsymbol{a})(x', \xi') = \kappa^{-1}_{\langle\xi'\rangle}\mathrm{Op}_{x_n}(a)\kappa_{\langle\xi'\rangle}. \tag{2.4.16}$$

Proof. We have

$$\begin{aligned}
\kappa_\delta^{-1}\mathrm{Op}_{x_n}(a)\kappa_\delta &= \kappa_\delta^{-1}\iint \mathrm{e}^{i(x_n-x_n')\xi_n} a(x',x_n,\xi',\xi_n)\kappa_\delta u(x_n')\,dx_n' đ\xi_n \\
&= \iint \mathrm{e}^{i(\delta^{-1}x_n-x_n')\xi'} a(x',\delta^{-1}x_n,\xi',\xi_n)u(\delta x_n')\,dx_n' đ\xi_n \\
&= \iint \mathrm{e}^{i\delta^{-1}(x_n-\delta x_n')\xi_n} a(x',\delta^{-1}x_n,\xi',\xi_n)u(\delta x_n')\,dx_n' đ\xi_n \\
&= \iint \mathrm{e}^{i(x_n-\tilde{x}_n)\tilde{\xi}_n} a(x',\delta^{-1}x_n,\xi',\delta\tilde{\xi}_n)u(\tilde{x}_n)\,d\tilde{x}_n\, đ\tilde{\xi}_n.
\end{aligned}$$

The proof is complete when we insert $\delta = \langle\xi'\rangle$. □

Lemma 2.4.6. *Let $a(x',x_n,\xi',\xi_n) \in S^\mu(\Omega\times\mathbb{R}^d\times\mathbb{R}^{n-1+d}_{\xi',\xi_n})$ be independent of x_n for large $|x_n|$. Then we have*

$$\mathrm{Op}_{x_n}(a)(x',\xi') \in S^\mu(\Omega\times\mathbb{R}^{n-1}; H^s(\mathbb{R}^d), H^{s-\mu}(\mathbb{R}^d))$$

for all $s\in\mathbb{R}$. In addition if $a = a(x',\xi',\xi_n)$ is independent of x_n and $a(x',\xi',\xi_n) \in S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^{n-1+d})$ then

$$\mathrm{Op}_{x_n}(a)(x',\xi') \in S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^{n-1}; H^s(\mathbb{R}^d), H^{s-\mu}(\mathbb{R}^d)). \tag{2.4.17}$$

Proof. Let us first assume a to be independent of x_n. From the symbolic estimate

$$|a(x',\xi',\xi_n)| \le \langle\xi',\xi_n\rangle^\mu$$

for $x'\in K \Subset \Omega$, where $c = c(K) > 0$ is a constant it follows that

$$|a(x',\xi',\langle\xi'\rangle\xi_n)| \le c\langle\xi',\langle\xi'\rangle\xi_n\rangle^\mu = c\langle\xi'\rangle^\mu\langle\xi_n\rangle^\mu$$

using that $\langle\xi',\langle\xi'\rangle\xi_n\rangle = \langle\xi'\rangle\langle\xi_n\rangle$. Thus, by virtue of Lemma 2.4.5 we have

$$\begin{aligned}
&\|\kappa^{-1}_{\langle\xi'\rangle}\mathrm{Op}_{x_n}(a)(x',\xi')\kappa_{\langle\xi'\rangle}v\|^2_{H^{s-\mu}(\mathbb{R}^d)} = \|\mathrm{Op}_{x_n}(\boldsymbol{a})(x',\xi')v\|^2_{H^{s-\mu}(\mathbb{R}^d)} \\
&= \int \langle\xi_n\rangle^{2(s-\mu)}|\boldsymbol{a}(x',\xi',\xi_n)\hat{v}(\xi_n)|^2\,đ\xi_n \le c\int\langle\xi_n\rangle^{2(s-\mu)}\langle\xi'\rangle^{2\mu}\langle\xi_n\rangle^{2\mu}|\hat{v}(\xi_n)|^2\,đ\xi_n \\
&= c\langle\xi'\rangle^{2\mu}\int\langle\xi_n\rangle^{2s}|\hat{v}(\xi_n)|^2\,đ\xi_n = c\langle\xi'\rangle^{2\mu}\|v\|^2_{H^s(\mathbb{R}^d)}.
\end{aligned}$$

This implies

$$\|\kappa^{-1}_{\langle\xi'\rangle}\mathrm{Op}_{x_n}(a)(x',\xi')\kappa_{\langle\xi'\rangle}\|_{\mathcal{L}(H^s(\mathbb{R}^d),H^{s-\mu}(\mathbb{R}^d))} \le c\langle\xi'\rangle^\mu.$$

In an analogous manner we get estimates

$$\|\kappa^{-1}_{\langle\xi'\rangle}D^\alpha_{x'}D^\beta_{\xi'}\big(\mathrm{Op}_{x_n}(a)(x',\xi')\big)\kappa_{\langle\xi'\rangle}\|_{\mathcal{L}(H^s(\mathbb{R}^d),H^{s-\mu}(\mathbb{R}^d))} \le c\langle\xi'\rangle^{\mu-|\beta|}.$$

If a depends on x_n, because of the first part of the proof we may assume that a vanishes for $|x_n|\ge R$ for some $R>0$. Then it follows that

$$a(x',x_n,\xi',\xi_n) \in C_0^\infty(\mathbb{R}^d_{x_n})_R\hat{\otimes}_\pi S^\mu(\Omega\times\mathbb{R}^{n-1+d}_{\xi',\xi_n})$$

where $C_0^\infty(\mathbb{R}^d_{x_n})_R$ is Fréchet space of all $\varphi\in C^\infty(\mathbb{R}^d_{x_n})$ vanishing for $|x_n|\ge R$. By virtue of Theorem 1.7.1 we can write the symbol as a convergent sum

$$a(x',x_n,\xi',\xi_n) = \sum_{j=0}^\infty \lambda_j a_j(x',\xi',\xi_n)c_j(x_n) \tag{2.4.18}$$

for $\lambda_j \in \mathbb{C}, \sum |\lambda_j| < \infty$, $a_j \in S^\mu(\Omega \times \mathbb{R}^{n-1+d}_{\xi',\xi_n})$, $c_j \in C_0^\infty(\mathbb{R}^d_{x_n})_R$, tending to 0 as $j \to \infty$ in the respective spaces. From (2.4.18) we obtain

$$a(x', \langle\xi'\rangle^{-1}x_n, \xi', \langle\xi'\rangle\xi_n) = \sum_{j=0}^{\infty} \lambda_j a_j(x', \xi', \langle\xi'\rangle\xi_n) c_j(\langle\xi'\rangle^{-1}x_n) \tag{2.4.19}$$

which is first a formal conclusion, but we will obtain convergence. From the continuity of

$$a(x', \xi', \xi_n) \longrightarrow \boldsymbol{a}(x', \xi', \xi_n)$$

stated in Theorem 2.4.4 we see that

$$a_j(x', \xi', \xi_n) \longrightarrow 0$$

in $S^\mu(\Omega \times \mathbb{R}^{n-1+d})$ entails

$$\boldsymbol{a}_j(x', \xi', \xi_n) \longrightarrow 0$$

in $S^\mu(\Omega \times \mathbb{R}^{n-1}, S^\mu(\mathbb{R}^d_{\xi_n}))$ as $j \to 0$. In addition it can easily be proved that $c \in C_0^\infty(\mathbb{R}^d_{x_n})_R$ gives rise to $c(\langle\xi'\rangle^{-1}x_n) \in S^0(\mathbb{R}^{n-1}_{\xi'}, C^\infty(\mathbb{R}^d_{x_n}))$ and $c_j(x_n) \longrightarrow 0$ in $C_0^\infty(\mathbb{R}^d_{x_n})_R$ entails $c(\langle\xi'\rangle^{-1}x_n) \longrightarrow 0$ in $S^0(\mathbb{R}^{n-1}_{\xi'}, C^\infty(\mathbb{R}^d_{x_n}))$ as $j \to \infty$. Thus (2.4.19) converges in the space (2.4.10) of decoupled symbols.

Now let us pass to the second statement of Lemma 2.4.6. If $a = a(x', \xi', \xi_n)$ for $\xi' \neq 0$

$$\begin{aligned}
\kappa_\delta^{-1}\mathrm{Op}_{x_n}(a_{(\mu)})(x', \xi')\kappa_\delta &= \kappa_\delta^{-1} \iint e^{i(x_n - x_n')\xi_n} a_{(\mu)}(x', \xi', \xi_n)\kappa_\delta u(x_n')\, dx_n' đ\xi_n \\
&= \iint e^{i(\delta^{-1}x_n - x_n')\xi_n} a_{(\mu)}(x', \xi', \xi_n) u(\delta x_n')\, dx_n' đ\xi_n \\
&= \iint e^{i\delta^{-1}(x_n - \delta x_n')\xi_n} a_{(\mu)}(x', \delta^{-1}\xi', \xi_n) u(\delta x_n')\, dx_n' đ\xi_n \\
&= \delta^\mu \iint e^{i(x_n - \tilde{x}_n)\tilde{\xi}_n} a_{(\mu)}(x', \delta^{-1}\xi', \delta\tilde{\xi}_n) u(\tilde{x}_n)\, d\tilde{x}_n\, đ\xi_n.
\end{aligned}$$

which yields

$$\kappa_\delta^{-1}\mathrm{Op}_{x_n}(a_{(\mu)})(x', \xi')\kappa_\delta = \delta^\mu \mathrm{Op}_{x_n}(a_{(\mu)})(x', \delta^{-1}\xi')$$

so

$$\delta^\mu \kappa_\delta^{-1}\mathrm{Op}_{x_n}(a_{(\mu)})(x', \xi')\kappa_\delta = \mathrm{Op}_{x_n}(a_{(\mu)})(x', \delta\xi',)$$

In similar manner we can proceed with $a_{(\mu-j)}$ for all j. □

Note that we can also analyze the x_n-dependent case by applying the Taylor expansion

$$a(x', x_n, \xi', \xi_n) = \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \left(\frac{\partial^\alpha}{\partial {x_n}^\alpha} a \right)(x', 0, \xi', \xi_n) x_n^\alpha + r_{N+1}(a)(x', x_n, \xi', \xi_n) \tag{2.4.20}$$

where we have $r_{N+1}(a)(x', x_n, \xi', \xi_n) \in S^\mu(\Omega \times \mathbb{R}^d \times \mathbb{R}^{n-1+d}_{\xi',\xi_n})$ as a difference of symbols of that kind.

Expression (2.4.20) will be a part of characterizing (2.4.17) as a classical operator-valued symbol also when $a = a(x', x_n, \xi', \xi_n)$ depends on x_n. Assume for simplicity $d = 1$.

First, using the form of the remainder in (2.4.17) we have

$$r_{N+1}(a)(x', x_n, \xi', \xi_n) = x_n^{N+1}\tilde{r}_{N+1}(a)(x', x_n, \xi', \xi_n) \tag{2.4.21}$$

for a symbol $\tilde{r}_{N+1}(a)(x', x_n, \xi', \xi_n) \in S^\mu(\Omega \times \mathbb{R}^d \times \mathbb{R}^{n-1+d}_{\xi',\xi_n})$. Then using the first statement of Lemma 2.4.6 we have

$$\mathrm{Op}_{x_n}(\tilde{r}_{N+1}(a)(x', \xi')) \in S^\mu(\Omega \times \mathbb{R}^{n-1}, H^s(\mathbb{R}^d), H^{s-\mu}(\mathbb{R}^d)). \tag{2.4.22}$$

Another Lemma tells us that

$$\begin{aligned} &\mathrm{Op}_{x_n}(x_n^{N+1}\tilde{r}_{N+1}(a)(x', \xi') = \\ &\mathrm{Op}_{x_n}(r_{N+1})(x', \xi') \in S^{\mu-(N+1)}(\Omega \times \mathbb{R}^{n-1}, H^s(\mathbb{R}^d), H^{s-\mu}(\mathbb{R}^d)). \end{aligned} \tag{2.4.23}$$

Thus it remains to look at the finite sum on the right of (2.4.20) and to show that it consists of homogeneous terms. Let us look at $a_{(\mu)}$ rather than a and determine the twisted homogeneity of $\mathrm{Op}_{x_n}(a^{(\alpha)}_{(\mu)}x_n^\alpha)(x', \xi')$ for

$$a^{(\alpha)}_{(\mu)}(x', 0, \xi', \xi_n) := \frac{\partial^\alpha}{\partial {x_n}^\alpha} a_{(\mu)}(x', 0, \xi', \xi_n).$$

Then (for $\xi' \neq 0$) an elementary computation gives us

$$\begin{aligned} \mathrm{Op}_{x_n}(a^{(\alpha)}_{(\mu)}x_n^\alpha)(x', \delta\xi')u(x_n) &= \iint \mathrm{e}^{i(x_n-x_n')\xi_n} a^{(\alpha)}_{(\mu)}(x', 0, \delta\xi', \xi_n) x_n^\alpha u(x_n')\, dx_n' đ\xi_n \\ &= \delta^\mu \iint \mathrm{e}^{i(x_n-x_n')\xi_n} a^{(\alpha)}_{(\mu)}(x', 0, \xi', \delta^{-1}\xi_n) x_n^\alpha u(x_n')\, dx_n' đ\xi_n \\ &= \delta^\mu \iint \mathrm{e}^{i(x_n-x_n')\delta\xi_n} a^{(\alpha)}_{(\mu)}(x', \xi') x_n^\alpha u(x_n')\, dx_n' \delta\, đ\xi_n \\ &= \delta^\mu \iint \mathrm{e}^{i(x_n-\delta^{-1}\tilde{x}_n)\delta\xi_n} a^{(\alpha)}_{(\mu)}(x', \xi') x_n^\alpha u(\delta^{-1}\tilde{x}_n)\, d\tilde{x}_n\, đ\xi_n \\ &= \delta^\mu \iint \mathrm{e}^{i(\delta x_n-\tilde{x}_n)\xi_n} a^{(\alpha)}_{(\mu)}(x', \xi',)\delta^{-|\alpha|}(\delta x_n)^\alpha u(\delta^{-1}\tilde{x}_n)\, d\tilde{x}_n\, đ\xi_n \\ &= \delta^{\mu-|\alpha|}\kappa_\delta \mathrm{Op}_{x_n}(a^{(\alpha)}_{(\mu)}x_n^\alpha)\kappa_\delta^{-1}u(x_n). \end{aligned} \tag{2.4.24}$$

Remark 2.4.7. *The map $a \longrightarrow \mathrm{Op}_{x_n}(a)(x', \xi')$ induces a continuous operator*

$$S^\mu(\Omega \times \mathbb{R}^d \times \mathbb{R}^{n-1+d}) \longrightarrow S^\mu(\Omega \times \mathbb{R}^{n-1}; H^s(\mathbb{R}^d), H^{s-\mu}(\mathbb{R}^d)). \tag{2.4.25}$$

For classical a which is independent of x_n we obtain a continuous operator

$$S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^d \times \mathbb{R}^{n-1+d}) \longrightarrow S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1}; H^s(\mathbb{R}^d), H^{s-\mu}(\mathbb{R}^d)). \tag{2.4.26}$$

In the following theorem we employ edge spaces introduced in Definition 1.5.1 and their local versions $\mathcal{W}^s_{\mathrm{comp}}(\Omega, H), \mathcal{W}^s_{\mathrm{loc}}(\Omega, H)$, and we refer to Theorem 1.5.9. In the present case we consider the spaces $H^s(\mathbb{R})$ or $H^s(\mathbb{R}_+)$ with group action $(\kappa_\delta u)(x_n) = \delta^{1/2}u(\delta x_n), \delta \in \mathbb{R}_+$. Then, using Theorem 1.5.9 every

$$a(x', \xi') \in S^\mu(\Omega \times \mathbb{R}^{n-1}; H^s(\mathbb{R}), H^{s-\mu}(\mathbb{R}))$$

induces a continuous operator

$$\mathrm{Op}(a) : H^s_{\mathrm{comp}(x')}(\Omega \times \mathbb{R}) \longrightarrow H^{s-\mu}_{\mathrm{loc}(x')}(\Omega \times \mathbb{R}) \tag{2.4.27}$$

where

$$H^s_{\mathrm{comp}(x')}(\Omega \times \mathbb{R}) = \big\{u \in H^s(\mathbb{R}^{n-1} \times \mathbb{R}) : u(x', x_n) = 0 \text{ for } x' \in \mathbb{R}^{n-1} \setminus K \text{ for some } K \Subset \Omega\big\}$$

$$H^s_{\mathrm{loc}(x')}(\Omega \times \mathbb{R}) = \big\{u \in H^s_{\mathrm{loc}(x')}(\Omega \times \mathbb{R}) : \varphi u \in H^s_{\mathrm{comp}(x')}(\Omega \times \mathbb{R}) \text{ for every } \varphi \in C_0^\infty(\Omega)\big\}.$$

From Theorem 1.5.9 for $H = H^s(\mathbb{R}), \tilde{H} = H^{s-\mu}(\mathbb{R})$ and $a(x', \xi') \in S^\mu(\Omega \times \mathbb{R}^{n-1}; H, \tilde{H})$ we obtain continuity of $\mathrm{Op}_{x'}(.)$ as operators

$$\mathrm{Op}_{x'}(a) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, H^s(\mathbb{R})) \longrightarrow \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, H^{s-\mu}(\mathbb{R})) \tag{2.4.28}$$

or equivalently,

$$\mathrm{Op}_{x'}(a) : H^s_{\mathrm{comp}(x')}(\Omega \times \mathbb{R}) \longrightarrow H^{s-\mu}_{\mathrm{loc}(x')}(\Omega \times \mathbb{R}). \tag{2.4.29}$$

Let us set $H^s_{\mathrm{comp}(x')}(\Omega \times \mathbb{R}_+) := \big\{u|_{\Omega\times\mathbb{R}_+} : u \in H^s_{\mathrm{comp}(x')}(\Omega \times \mathbb{R})\big\}$ and $H^s_{\mathrm{loc}(x')}(\Omega \times \mathbb{R}_+) := \big\{u|_{\Omega\times\mathbb{R}_+} : u \in H^s_{\mathrm{loc}(x')}(\Omega \times \mathbb{R})\big\}$.

Theorem 2.4.8. *For every $a(x', x_n, \xi', \xi_n) \in S^\mu_{\mathrm{tr}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ such that a is independent of x_n for $|x_n| > C$ for some $C > 0$, we have*

$$\mathrm{Op}^+(a)(x', \xi') := \mathrm{r}^+\mathrm{Op}_{x_n}(a)(x', \xi')\mathrm{e}^+ \in S^\mu(\Omega \times \mathbb{R}^{n-1}; H^s(\mathbb{R}_+), H^{s-\mu}(\mathbb{R}_+)). \tag{2.4.30}$$

These symbols induce continuous operators

$$\mathrm{Op}_{x'}\big(\mathrm{r}^+\mathrm{Op}_{x_n}(a)(x', \xi')\mathrm{e}^+\big) : H^s_{\mathrm{comp}(x')}(\Omega \times \mathbb{R}_+) \longrightarrow H^{s-\mu}_{\mathrm{loc}(x')}(\Omega \times \mathbb{R}_+) \tag{2.4.31}$$

for every $s \in \mathbb{R}, s > -1/2$.

Proof. Use Remark 1.5.7. □

Theorem 2.4.9. *Let $a(x', x_n, \xi', \xi_n) \in S^\mu_{\mathrm{tr}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ be as in Theorem 2.4.8. Then we have*

$$\mathrm{r}^+\mathrm{Op}_{x_n}(a)\mathrm{e}^+(x', \xi') \in S^\mu(\Omega \times \mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+), \mathcal{S}(\overline{\mathbb{R}}_+))$$

and continuous operators

$$\mathrm{Op}_{x'}\big(\mathrm{r}^+\mathrm{Op}_{x_n}(a)\mathrm{e}^+(x', \xi')\big) : C_0^\infty(\Omega, \mathcal{S}(\overline{\mathbb{R}}_+)) \longrightarrow C^\infty(\Omega, \mathcal{S}(\overline{\mathbb{R}}_+)). \tag{2.4.32}$$

Remark 2.4.10. *As a byproduct of Theorems 2.4.8, 2.4.9 we obtain continuous operators*

$$S^\mu_{\mathrm{tr}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)_C \longrightarrow S^\mu(\Omega \times \mathbb{R}^{n-1}; H^s(\mathbb{R}_+), H^{s-\mu}(\mathbb{R}_+)) \tag{2.4.33}$$

and

$$S^\mu_{\mathrm{tr}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)_C \longrightarrow S^\mu(\Omega \times \mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+), \mathcal{S}(\overline{\mathbb{R}}_+)). \tag{2.4.34}$$

Here subscript C indicates spaces of symbols which are independent of x_n for $|x_n| > C$.

We set

$$L^\mu_{\mathrm{tr}}(\Omega \times \overline{\mathbb{R}}_+) = \{\mathrm{Op}^+(a) : a(x', x_n, \xi', \xi_n) \in S^\mu_{\mathrm{tr}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)\}. \tag{2.4.35}$$

Theorem 2.4.11. *The spaces $L^{\mu}_{\mathrm{tr}}(\Omega \times \overline{\mathbb{R}}_+), \mu \in \mathbb{Z}$, are coordinate invariant under diffeomorphisms* (2.4.5) *and the operator push forward induces isomorphisms*

$$\chi_* : L^{\mu}_{\mathrm{tr}}(\Omega \times \overline{\mathbb{R}}_+) \longrightarrow L^{\mu}_{\mathrm{tr}}(\tilde{\Omega} \times \overline{\mathbb{R}}_+) \tag{2.4.36}$$

and we have

$$\chi_*(\mathrm{Op}^+(a)) = \mathrm{Op}^+(\chi_*(a)) \tag{2.4.37}$$

with obvious meaning of notation, cf. formula (1.1.34).

Definition 2.4.12. *We define*

$$R^{\mu,e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2) \tag{2.4.38}$$

for $\mu \in \mathbb{Z}, e \in \mathbb{N}$, as the space of all operator families

$$\boldsymbol{a}(x', \xi') := \begin{pmatrix} \mathrm{Op}^+(a)(x', \xi') & 0 \\ 0 & 0 \end{pmatrix} + \boldsymbol{g}(x', \xi') \tag{2.4.39}$$

for arbitrary $a(x', x_n, \xi', \xi_n) \in S^{\mu}_{\mathrm{tr}}(\Omega \times \mathbb{R} \times \mathbb{R}^n_{\xi',\xi_n})$, cf. Definition 2.4.1, *and $\boldsymbol{g}(x', \xi') \in R^{\mu,e}_{\mathrm{G}}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2)$, cf. Definition* 2.2.13. *The elements of* (2.4.38) *are called boundary amplitude functions. In addition by*

$$R^{\mu,e}_{\mathrm{G}}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2) \tag{2.4.40}$$

we denote the set of all $\boldsymbol{g}(x', \xi')$ in formula (2.4.39), *called Green symbols. The space of Green symbols of order $\mu \in \mathbb{Z}$ and type $e \in \mathbb{N}$ is denoted by $R^{\mu,e}_{\mathrm{G}}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2)$.*

We define

$$\mathcal{B}^{\mu,e}_{\mathrm{G}}(\Omega \times \overline{\mathbb{R}}_+; j_1, j_2) := \{\mathrm{Op}_{x'}(\boldsymbol{g}) : \boldsymbol{g}(x', \xi') \in R^{\mu,e}_{\mathrm{G}}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2)\}. \tag{2.4.41}$$

Recall that this notation is introduced in Definition 2.4.12.

Theorem 2.4.13. *The spaces $\mathcal{B}^{\mu,e}(\Omega \times \overline{\mathbb{R}}_+; j_1, j_2)$ is coordinate invariant under diffeomorphisms* (2.4.5) *and under operator push forward χ_* induces isomorphisms*

$$\chi_* : \mathcal{B}^{\mu,e}_{\mathrm{G}}(\Omega \times \overline{\mathbb{R}}_+; j_1, j_2) \longrightarrow \mathcal{B}^{\mu,e}_{\mathrm{G}}(\tilde{\Omega} \times \overline{\mathbb{R}}_+; j_1, j_2), \tag{2.4.42}$$

where

$$\chi_*(\mathrm{Op}_{x'}(\boldsymbol{g})) = \mathrm{Op}_{\tilde{x}'}(\chi_* \boldsymbol{g}) \tag{2.4.43}$$

with obvious meaning of notation.

Remark 2.4.14. *As a consequence of Theorem* 2.4.8, 2.4.9, *and the properties of Green, trace and potential entries in $\boldsymbol{a}(x', \xi') \in R^{\mu,e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2)$, cf. Remark* 2.2.14, *we have*

$$R^{\mu,e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2) \subset S^{\mu}(\Omega \times \mathbb{R}^{n-1}; H_1, H_2)$$

for

$$H_1 := \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_1} \end{matrix}, \quad H_2 := \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_2} \end{matrix} \tag{2.4.44}$$

for any $s > e - 1/2$ as well as

$$R^{\mu,e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2) \subset S^{\mu}(\Omega \times \mathbb{R}^{n-1}; E_1, E_2)$$

for

$$E_1 := \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_1} \end{matrix}, \quad E_2 := \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_2} \end{matrix}. \tag{2.4.45}$$

The operator functions $\boldsymbol{a}(x', \xi')$ in (2.4.38) have a principal symbolic hierarchy

$$\sigma(\boldsymbol{a}) := \big(\sigma_\psi(\boldsymbol{a}), \sigma_\partial(\boldsymbol{a})\big) \tag{2.4.46}$$

consisting of the interior principal symbol

$$\sigma_\psi(\boldsymbol{a})(x', x_n, \xi', \xi_n) := a_{(\mu)}(x', x_n, \xi', \xi_n),$$

for $(\xi', \xi_n) \neq 0$, the homogeneous principal component of $a(x', x_n, \xi', \xi_n)$ in the usual sense, and

$$\sigma_\partial(\boldsymbol{a})(x', \xi') := \begin{pmatrix} \sigma_\partial(\mathrm{Op}^+(a))(x', \xi') & 0 \\ 0 & 0 \end{pmatrix} + \boldsymbol{g}_{(\mu)}(x', \xi') \tag{2.4.47}$$

for $\xi' \neq 0$ where

$$\sigma_\partial(\mathrm{Op}^+(a))(x', \xi') := \mathrm{Op}^+(a|_{x_n=0})(x', \xi')$$

and $\boldsymbol{g}_{(\mu)}(x', \xi')$ is the principal part of $\boldsymbol{g}$ cf. Remark 2.2.14. Observe that

$$\sigma_\partial(\boldsymbol{a})(x', \delta\xi') = \delta^\mu \begin{pmatrix} \kappa_\delta & 0 \\ 0 & \mathrm{id}_{\mathbb{C}^{j_2}} \end{pmatrix} \sigma_\partial(\boldsymbol{a})(x', \xi') \begin{pmatrix} \kappa_\delta^{-1} & 0 \\ 0 & \mathrm{id}_{\mathbb{C}^{j_1}} \end{pmatrix} \tag{2.4.48}$$

for every $\delta \in \mathbb{R}_+$, cf. also relation (2.2.42). The group action

$$(\kappa_\delta u)(x_n) = \delta^{1/2} u(\delta x_n), \delta \in \mathbb{R}_+,$$

refers either to $u \in H^s(\mathbb{R}_+)$ for $s > e - 1/2$, or $u \in \mathcal{S}(\overline{\mathbb{R}}_+)$. In other words the boundary symbol of $\boldsymbol{a} \in R^{\mu,e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2)$ is interpreted either as a family of operators

$$\sigma_\partial(\boldsymbol{a})(x', \xi') : \begin{array}{c} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_1} \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_2} \end{array}, \tag{2.4.49}$$

or

$$\sigma_\partial(\boldsymbol{a})(x', \xi') : \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_1} \end{array} \longrightarrow \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_2} \end{array}. \tag{2.4.50}$$

,

Definition (2.4.46) gives rise to the principal symbolic map

$$R^{\mu,e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2) \longrightarrow \mathrm{symb}\, R^{\mu,e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2), \tag{2.4.51}$$

$$\boldsymbol{a}(x', \xi') \longmapsto \sigma(\boldsymbol{a}) = (\sigma_\psi(\boldsymbol{a}), \sigma_\partial(\boldsymbol{a})).$$

The map (2.4.51) is well-defined if we can produce the components from $\boldsymbol{a}$. The reconstruction procedure for the first component can be performed in terms of

$$\iint e^{i(x'-x'')\xi'} \Big\{ \mathrm{r}^+ \iint e^{i(x_n - x_n')\xi_n} a(x', x_n, \xi', \xi_n) u(x_n', x'')\, dx_n' đ\xi_n \mathrm{e}^+ \Big\} dx'' đ\xi'$$

which represents an element $A \in L^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}_+)$. According to (1.1.20) we have a decomposition $A = A_0 + C$ wher A_0 is properly supported, C is smoothing operator. Then Remark 1.1.6

allows us to produce a unique element $a_0 \in S^{\mu}_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n_{\xi',\xi_n})$ such that $A_0 = \mathrm{Op}_{x',x_n,}(a_0)$. Then we obtain

$$a_{(\mu)}(x', x_n, \xi', \xi_n) = \lim_{\delta \to \infty} \delta^{-\mu} a_0(x', x_n, \delta\xi', \delta\xi_n).$$

In a similar manner we obtain $\sigma_\partial(\boldsymbol{a})(x', \xi')$ by a limit

$$\sigma_\partial(\boldsymbol{a})(x', \xi') = \lim_{\delta \to \infty} \delta^{-\mu} \begin{pmatrix} \kappa_\delta^{-1} & 0 \\ 0 & \mathrm{id}_{\mathbb{C}^{j_2}} \end{pmatrix} \boldsymbol{a}(x', \delta\xi') \begin{pmatrix} \kappa_\delta & 0 \\ 0 & \mathrm{id}_{\mathbb{C}^{j_1}} \end{pmatrix}.$$

In other words (2.4.51) is well-defined.

Let us set for the moment

$$\sigma^\mu(\boldsymbol{a}) := \sigma(\boldsymbol{a}),\ \sigma^\mu_\psi(\boldsymbol{a}) = \sigma_\psi(\boldsymbol{a})\,, \sigma^\mu_\partial(\boldsymbol{a}) = \sigma_\partial(\boldsymbol{a}).$$

Remark 2.4.15. *For $\boldsymbol{a} \in R^{\mu,e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2)$ satisfying $\sigma^\mu(\boldsymbol{a}) = 0$ it follows that*

$$\boldsymbol{a} \in R^{\mu-1,e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2).$$

Applying the reproducing process of symbols again we can determine

$$\sigma^{\mu-1}(\boldsymbol{a}) := (\sigma^{\mu-1}_\psi(\boldsymbol{a}), \sigma^{\mu-1}_\partial(\boldsymbol{a})),$$

and then, successively, for

$$R^{\mu-j,e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2) := \{\boldsymbol{a} \in R^{\mu-(j-1),e}(\Omega \times \mathbb{R}^q; j_1, j_2) : \sigma^{\mu-(j-1)}(\boldsymbol{a}) = 0\}, \tag{2.4.52}$$

$j \geq 1$, we obtain $\sigma^{\mu-j}(\boldsymbol{a})$ for every $\boldsymbol{a} \in R^{\mu-j,e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2)$.

Note that

$$a_{(\mu)}(x', x_n, \delta\xi', \delta\xi_n) = \delta^\mu a_{(\mu)}(x', x_n, \xi', \xi_n)$$

for $\delta = |\xi', \xi_n|^{-1}$ gives us

$$a_{(\mu)}(x', x_n, \xi', \xi_n) = |\xi', \xi_n|^\mu a_{(\mu)}\big(x', x_n, \frac{\xi'}{|\xi', \xi_n|}, \frac{\xi_n}{|\xi', \xi_n|}\big), \tag{2.4.53}$$

i.e., $a_{(\mu)}$ is determined by its values on the unit cosphere bundle

$$S^*(\Omega \times \overline{\mathbb{R}}_+) := \big\{(x', x_n, \xi', \xi_n) \in \Omega \times \overline{\mathbb{R}}_+ \times (\mathbb{R}^{n-1}_{\xi'} \times \mathbb{R}_{\xi_n}) : |\xi', \xi_n| = 1\big\}.$$

Relation (2.4.53) can be regarded as an extension by homogeneity μ of

$$a_{(\mu)}|_{S^*(\Omega \times \overline{\mathbb{R}}_+)} \in C^\infty(S^*(\Omega \times \overline{\mathbb{R}}_+)) \tag{2.4.54}$$

to $\Omega \times \overline{\mathbb{R}}_+ \times (\mathbb{R}^{n-1} \times \mathbb{R} \setminus \{0\})$. The subspace of $C^\infty(S^*(\Omega \times \overline{\mathbb{R}}_+))$ determined by restrictions $a_{(\mu)}|_{S^*(\Omega \times \overline{\mathbb{R}}_+)}$ for symbols

$$a(x', x_n, \xi', \xi_n) \in S^\mu_{\mathrm{tr}}(\Omega \times \mathbb{R} \times \mathbb{R}^n_{\xi',\xi_n})$$

closed in $C^\infty(S^*(\Omega \times \overline{\mathbb{R}}_+))$ and hence is a Fréchet subspace. If $\pi_{\psi,j}, j \in \mathbb{N}$, is the semi-norm system of their space, then

$$\pi_{\psi,j}(\sigma^\mu_\psi(\boldsymbol{a})|_{S^*(\Omega \times \overline{\mathbb{R}}_+)}) =: p^\mu_{\psi,j}(\boldsymbol{a})$$

is a semi-norm on the space $R^{\mu,e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2)$.

Moreover, we have

$$\sigma_\partial^\mu(\boldsymbol{a})(x', \delta\xi') = \delta^\mu \begin{pmatrix} \kappa_\delta & 0 \\ 0 & 1 \end{pmatrix} \sigma_\partial^\mu(\boldsymbol{a})(x', \xi') \begin{pmatrix} \kappa_\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix} \tag{2.4.55}$$

(with 1 being the respective identity maps) and hence, for $\delta = |\xi'|^{-1}$ we obtain

$$\sigma_\partial^\mu(\boldsymbol{a})(x', \xi') = |\xi'|^\mu \begin{pmatrix} \kappa_{|\xi'|}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \sigma_\partial^\mu(\boldsymbol{a})(x', \frac{\xi'}{|\xi'|}) \begin{pmatrix} \kappa_{|\xi'|} & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.4.56}$$

Relation (2.4.56) is an extension by twisted homogeneity μ of

$$\sigma_\partial^\mu(\boldsymbol{a})|_{S^*\Omega} \in C^\infty(S^*\Omega, \mathcal{B}^{\mu,e}(\overline{\mathbb{R}}_+; j_1, j_2))$$

to $\Omega \times (\mathbb{R}^{n-1} \setminus \{0\})$, where $\mathcal{B}^{\mu,e}(\overline{\mathbb{R}}_+; j_1, j_2))$ defined in Definition 2.3.13 is equipped with its natural Fréchet topology, and

$$S^*\Omega = \{(x', \xi') \in \Omega \times \mathbb{R}^{n-1} : |\xi'| = 1\}$$

is the unit cosphere bundle of Ω. Then, if $\pi_{\partial,j}, j \in \mathbb{N}$, is the system of semi-norms of the Fréchet topology of $C^\infty(S^*\Omega, \mathcal{B}^{\mu,e}(\overline{\mathbb{R}}_+; j_1, j_2))$, then

$$\pi_{\partial,j}(\sigma_\partial^\mu(\boldsymbol{a})|_{S^*\Omega}) =: p_{\partial,j}^\mu(\boldsymbol{a}). \tag{2.4.57}$$

We now obtain on the space $R^{\mu,e}(\Omega \times \mathbb{R}^{n-1})$ a semi-norm system (2.4.57). Applying this construction also to all lower order symbolic components we obtain in an analogous manner semi-norms

$$p_{\psi,j}^{\mu-k}(\boldsymbol{a}),\ p_{\partial,j}^{\mu-k}(\boldsymbol{a}), \quad j \in \mathbb{N}, k \in \mathbb{N}. \tag{2.4.58}$$

This yields altogether a semi-norm system on the space $R^{\mu,e}(\Omega \times \mathbb{R}^{n-1})$ which turns it to a Fréchet space.

Theorem 2.4.16. *Let $\boldsymbol{a}_j(x', \xi') \in R^{\mu-j,e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2)$ be an arbitrary sequence. Then there is an $\boldsymbol{a}(x', \xi') \in R^{\mu,e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2)$ such that*

$$\boldsymbol{a}(x', \xi') - \sum_{j=0}^N \boldsymbol{a}_j(x', \xi') \in R^{\mu-(N+1),e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2)$$

for every $N \in \mathbb{N}$, and $\boldsymbol{a}$ is unique modulo

$$R^{-\infty,e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2) := \bigcap_{j \in \mathbb{N}} R^{\mu,e}(\Omega \times \mathbb{R}^{n-1}; j_1, j_2).$$

We then write

$$\boldsymbol{a} \sim \sum_{j=0}^\infty \boldsymbol{a}_j$$

and call $\boldsymbol{a}$ an asymptotic sum of the $\boldsymbol{a}_j$.

Remark 2.4.17. *Note that analogously as in Remark* 1.1.4 *we can produce an asymptotic sum of the* $\boldsymbol{a}_j$ *by the convergent sum in the Fréchet space* $R^{\mu,e}(\Omega\times\mathbb{R}^{n-1};j_1,j_2)$

$$\boldsymbol{a}(x',\xi')=\sum_{j=0}^{\infty}\chi(\frac{\xi'}{c_j})\boldsymbol{a}_j(x',\xi')$$

where $\chi(\xi')$ *is any excision function and* c_j *a sequence of positive numbers increasing sufficiently fast as* $j\to\infty$.

Theorem 2.4.18. $\boldsymbol{a}(x',\xi')\in R^{\mu,d}(\Omega\times\mathbb{R}^{n-1};j_0,j_2)$, $\boldsymbol{b}(x',\xi')\in R^{\nu,e}(\Omega\times\mathbb{R}^{n-1};j_1,j_0)$ *implies*

$$(\boldsymbol{ab})(x',\xi')\in R^{\mu+\nu,h}(\Omega\times\mathbb{R}^{n-1};j_1,j_2) \tag{2.4.59}$$

for $h=\max\{d+\nu,e\}$ *and we have*

$$\sigma_\psi(\boldsymbol{ab})=\sigma_\psi(\boldsymbol{a})\sigma_\psi(\boldsymbol{b}),\quad \sigma_\partial(\boldsymbol{ab})=\sigma_\partial(\boldsymbol{a})\sigma_\partial(\boldsymbol{b})$$

with componentwise composition, cf. analogously, Theorem 2.3.18.

Remark 2.4.19. *Observe that differentiations in* x',ξ' *give rise to linear operators*

$$D_{x'}^{\alpha}D_{\xi'}^{\beta}:R^{\mu,d}(\Omega\times\mathbb{R}^{n-1};j_1,j_2)\longrightarrow R^{\mu-|\beta|,d}(\Omega\times\mathbb{R}^{n-1};j_1,j_2) \tag{2.4.60}$$

for every $\alpha,\beta\in\mathbb{N}^{n-1}$.

An example of asymptotic summation is the Leibniz product between symbols $\boldsymbol{a}(x',\xi')\in R^{\mu,d}(\Omega\times\mathbb{R}^{n-1};j_0,j_2),\boldsymbol{b}(x',\xi')\in R^{\nu,e}(\Omega\times\mathbb{R}^{n-1};j_1,j_0)$,

$$\boldsymbol{a}(x',\xi')\#\boldsymbol{b}(x',\xi')\in R^{\mu+\nu,h}(\Omega\times\mathbb{R}^{n-1};j_1,j_2) \tag{2.4.61}$$

for $h=\max\{d+\nu,e\}$, defined by twisted homogeneity of order μ.

$$\boldsymbol{a}(x',\xi')\#\boldsymbol{b}(x',\xi')\sim\sum_{\alpha\in\mathbb{N}^{n-1}}\frac{1}{\alpha!}\partial_{\xi'}^{\alpha}\boldsymbol{a}(x',\xi')D_{x'}^{\alpha}\boldsymbol{b}(x',\xi'). \tag{2.4.62}$$

The asymptotic summation makes sense because of

$$\partial_{\xi'}^{\alpha}\boldsymbol{a}(x',\xi')D_{x'}^{\alpha}\boldsymbol{b}(x',\xi')\in R^{\mu+\nu-|\beta|,h}(\Omega\times\mathbb{R}^{n-1};j_1,j_2) \tag{2.4.63}$$

cf. Theorem 2.4.18. and Remark 2.4.19.

Theorem 2.4.20. $\boldsymbol{a}(x',\xi')\in R^{0,0}(\Omega\times\mathbb{R}^{n-1};j_1,j_2)$ *implies* $\boldsymbol{a}^*(x',\xi')\in R^{0,0}(\Omega\times\mathbb{R}^{n-1};j_2,j_1)$ *in the sense of*

$$(\boldsymbol{a}(x',\xi')f,g)_{L^2(\mathbb{R}_+)\oplus\mathbb{C}^{j_2}}=(f,\boldsymbol{a}^*(x',\xi')g)_{L^2(\mathbb{R}_+)\oplus\mathbb{C}^{j_1}} \tag{2.4.64}$$

for all

$$f\in C_0^\infty(\mathbb{R}_+)\oplus\mathbb{C}^{j_1},\quad g\in C_0^\infty(\mathbb{R}_+)\oplus\mathbb{C}^{j_2} \tag{2.4.65}$$

and we have

$$\sigma_\psi(\boldsymbol{a}^*)=\sigma_\psi(\boldsymbol{a})^*,\quad \sigma_\partial(\boldsymbol{a}^*)=\sigma_\partial(\boldsymbol{a})^* \tag{2.4.66}$$

where

$$\sigma_\psi(\boldsymbol{a})^*(x',x_n,\xi',\xi_n)=\sigma_\psi(\overline{\boldsymbol{a}})(x',x_n,\xi',\xi_n)$$

and

$$(\sigma_\partial(\boldsymbol{a}(x',\xi')f,g)_{L^2(\mathbb{R}_+)\oplus\mathbb{C}^{j_2}}=(f,\sigma_\partial(\boldsymbol{a})^*(x',\xi')g)_{L^2(\mathbb{R}_+)\oplus\mathbb{C}^{j_1}} \tag{2.4.67}$$

$(x',\xi')\in\Omega\times(\mathbb{R}^{n-1}\setminus\{0\})$, *for all* f,g *as in* (2.4.65).

The operator-valued symbols $\boldsymbol{g}(x',\xi') \in R_{\mathrm{G}}^{\mu,e}(\Omega\times\mathbb{R}^{n-1};j_1,j_2)$ take values in block-matrices of the form

$$\boldsymbol{g}(x',\xi') = \begin{pmatrix} g & k \\ b & q \end{pmatrix}(x',\xi') \in \mathcal{B}_{\mathrm{G}}^{\mu,e}(\overline{\mathbb{R}}_+;j_1,j_2) \tag{2.4.68}$$

cf. notation in (2.3.97).

2.5 Local operators of Boutet de Monvel's type

We now turn to pseudo-differential boundary value problems referring to $\Omega\times\overline{\mathbb{R}}_+, \Omega\subseteq\mathbb{R}^{n-1}$ open, which corresponds to a chart on a smooth manifold with boundary. We will give a definition of $\mathcal{B}^{\mu,e}(\Omega\times\overline{\mathbb{R}}_+;j_-,j_+)$ which are 2×2-block matrices.

Let us first define the class $\mathcal{B}^{-\infty,e}(\Omega\times\overline{\mathbb{R}}_+;j_-,j_+)$ of smoothing operators

$$\mathcal{C} = \begin{pmatrix} C_{11}+G_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} : \begin{matrix} C_0^\infty(\Omega\times\overline{\mathbb{R}}_+) \\ \oplus \\ C_0^\infty(\Omega,\mathbb{C}^{j_-}) \end{matrix} \longrightarrow \begin{matrix} C^\infty(\Omega\times\overline{\mathbb{R}}_+) \\ \oplus \\ C^\infty(\Omega,\mathbb{C}^{j_+}) \end{matrix}.$$

Later on, in order to simplify formulas we consider the case $j_- = j_+ = 1$; the extension of notation to arbitrary $j_-, j_+ \in\mathbb{N}$ is straightforward. Smoothing operators of the kind C_{11} are defined by kernels

$$c_{11}(x',x_n,x'',x_n') \in C^\infty(\Omega\times\overline{\mathbb{R}}_+\times\Omega\times\overline{\mathbb{R}}_+)$$

where

$$C_{11}u(x',x_n) = \int_\Omega\int_{\overline{\mathbb{R}}_+} c_{11}(x',x_n,x'',x_n')u(x'',x_n')\,dx''dx_n' \tag{2.5.1}$$

for $u\in C_0^\infty(\Omega\times\overline{\mathbb{R}}_+)$. Moreover, we define smoothing Green operators G_{11} of type $e\in\mathbb{N}$ by

$$G_{11}u(x',x_n) = \sum_{k=0}^{e}\int_\Omega\int_{\overline{\mathbb{R}}_+} g_{11,k}(x',x_n,x'',x_n')\frac{\partial^k}{\partial {x_n'}^k}u(x'',x_n')\,dx''dx_n' \tag{2.5.2}$$

where $g_{11,k}(x',x_n,x'',x_n')\in C^\infty(\Omega\times\overline{\mathbb{R}}_+\times\Omega\times\overline{\mathbb{R}}_+)$ for $k=0,\dots,e$. A smoothing trace operator C_{21} of type e is defined as a column vector of operators $C_{21} = {}^{\mathrm{t}}(C_{21}^1,\dots,C_{21}^l)$ by

$$(C_{21}^l u)(x') = \sum_{k=0}^{e}\int_\Omega\int_{\overline{\mathbb{R}}_+} c_{21,k}^l(x',x'',x_n')\frac{\partial^k}{\partial {x_n'}^k}u(x'',x_n')\,dx''dx_n', \tag{2.5.3}$$

where $l=1,\dots,j_+$, $c_{21,k}^l(x',x'',x_n')\in C^\infty(\Omega\times\Omega\times\overline{\mathbb{R}}_+), k=0,\dots,e$. Moreover, a smoothing potential operator C_{12} is defined as a row vector of operators $(C_{12}^1,\dots,C_{12}^{j_-})$ by

$$(C_{12}^m v)(x',x_n) = \sum_{m=1}^{j_-}\int_\Omega c_{12}^m(x',x_n,x'')v(x'')\,dx'' \tag{2.5.4}$$

where $m=1,\dots,j_-$, $c_{12}^m(x',x_n,x'')\in C^\infty(\Omega\times\overline{\mathbb{R}}_+\times\Omega)$, $v := {}^{\mathrm{t}}(v_1,\dots,v_{j_-})$, $v\in C_0^\infty(\Omega), m = 1,\dots,j_-$. Finally C_{22} is a standard smoothing operator, namely, a $j_+\times j_-$ matrix

$$(C_{22}^l v)(x') = \sum_{m=1}^{j_-}\int_\Omega c_{22}^{lm}(x',x'')v(x'')\,dx'',\ l=1,\dots,j_+ \tag{2.5.5}$$

for kernels $c_{22}^{lm}(x',x'')\in C^\infty(\Omega\times\Omega)$, $l=1,\dots,j_+$, $m=1,\dots,j_-$.

Definition 2.5.1. *For every $\mu \in \mathbb{Z}, e \in \mathbb{N}$, we define $\mathcal{B}^{\mu,e}(\Omega \times \overline{\mathbb{R}}_+; j_-, j_+)$ as the space of all operators of the form*

$$\mathcal{A} = \mathrm{Op}_{x'}(\boldsymbol{a}) + \mathcal{C} \tag{2.5.6}$$

for arbitrary $\boldsymbol{a}(x', \xi') \in R^{\mu,e}(\Omega \times \mathbb{R}^{n-1}; j_-, j_+)$ and $\mathcal{C} \in \mathcal{B}^{-\infty,e}(\Omega \times \overline{\mathbb{R}}_+; j_-, j_+)$. Operators of this kind are called smoothing operators of Boutet de Monvel's calculus of order μ and type e.

In local considerations over $\Omega \times \overline{\mathbb{R}}_+$ for convenience we impose some assumptions on the behaviour of kernels involved in $\mathcal{C} \in \mathcal{B}^{-\infty,e}(\Omega \times \overline{\mathbb{R}}_+; j_-, j_+)$ for large x_n or x'_n. We also impose a similar assumption on $a(x', \xi') \in R^{\mu,e}(\Omega \times \overline{\mathbb{R}}_+; j_-, j_+)$. In order to make such precautions easy in future we tacitly assume that the upper left corner (2.4.39) is defined for a symbol $a(x', x_n, \xi', \xi_n) \in \mathcal{S}(\mathbb{R}_{x_n}, S^{\mu}_{\mathrm{tr}}(\Omega \times \mathbb{R}^n_{\xi',\xi_n}))$, i.e., with a strong decay for $|x_n| \to \infty$. The other ingredients of Green, trace or potential operators in $\mathcal{B}^{\mu,e}(\Omega \times \overline{\mathbb{R}}_+; j_-, j_+)$ have such a strong decay for large $|x_n|$ by definition. Concerning the kernels involved in $\mathcal{C}$ we also assume the Schwartz property in x_n, x'_n tending to infinity. Later on, in the global calculus of operators on a smooth manifold with boundary such a behaviour will be automatic, since localizations by multiplications by factors from a partition of unity, etc. will cause the desired properties.

Theorem 2.5.2. *Every $\mathcal{A} \in \mathcal{B}^{\mu,e}(\Omega \times \overline{\mathbb{R}}_+; j_-, j_+)$, first realized as a continuous operator*

$$\mathcal{A} : \begin{matrix} C_0^\infty(\Omega \times \overline{\mathbb{R}}_+) \\ \oplus \\ C_0^\infty(\Omega, \mathbb{C}^{j_-}) \end{matrix} \longrightarrow \begin{matrix} C^\infty(\Omega \times \overline{\mathbb{R}}_+) \\ \oplus \\ C^\infty(\Omega, \mathbb{C}^{j_+}) \end{matrix} \tag{2.5.7}$$

extends to a continuous operator

$$\mathcal{A} : \begin{matrix} H^s_{\mathrm{comp}}(\Omega \times \mathbb{R}_+) \\ \oplus \\ H^s_{\mathrm{comp}}(\Omega, \mathbb{C}^{j_-}) \end{matrix} \longrightarrow \begin{matrix} H^{s-\mu}_{\mathrm{loc}}(\Omega \times \mathbb{R}_+) \\ \oplus \\ H^{s-\mu}_{\mathrm{loc}}(\Omega, \mathbb{C}^{j_+}) \end{matrix} \tag{2.5.8}$$

for every $s \in \mathbb{R}, s > e - 1/2$.

Let us now formulate the principal symbols

$$\sigma(\mathcal{A}) := (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A})) \tag{2.5.9}$$

of operators $\mathcal{A} \in \mathcal{B}^{\mu,e}(\Omega \times \overline{\mathbb{R}}_+; j_-, j_+)$. Writing $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,2}$, and $\mathcal{A}_{11} = A + G$ for $A = \mathrm{Op}^+(a), a \in S^{\mu}_{\mathrm{tr}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)$, and $G \in \mathcal{B}^{\mu,e}_{\mathrm{G}}(\Omega \times \overline{\mathbb{R}}_+)$, we define

$$\sigma_\psi(\mathcal{A}) := \sigma_\psi(A)$$

which is just the homogeneous principal symbol of $a(x', x_n, \xi', \xi_n)$ as a classical symbol of order μ. Moreover, looking at (2.5.6) we have an $\boldsymbol{a}(x', \xi') \in R^{\mu,e}(\Omega \times \mathbb{R}^{n-1}; j_-, j_+)$ which is of the form (2.4.39), where $\boldsymbol{g}(x', \xi') \in R^{\mu,e}_{\mathrm{G}}(\Omega \times \mathbb{R}^{n-1}_+; j_-, j_+)$ is an element of $S^{\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1}; H_1, H_2)$ for the spaces (2.4.44), and we have the corresponding principal symbol $\boldsymbol{g}_{(\mu)}(x', \xi')$ of twisted homogeneity μ, for $T^*\Omega \setminus 0$. Moreover, we define

$$\sigma_\partial(\mathrm{Op}^+(a))(x', \xi') := \mathrm{r}^+\mathrm{Op}(a|_{x_n=0})\mathrm{e}^+(x', \xi'), \tag{2.5.10}$$

i.e., in terms of $a(x', x_n, \xi', \xi_n)$, frozen at $x_n = 0$. We set

$$\sigma_\partial(\mathcal{A})(x', \xi') := \begin{pmatrix} \sigma_\partial(\mathrm{Op}^+(a))(x', \xi') & 0 \\ 0 & 0 \end{pmatrix} + \boldsymbol{g}_{(\mu)}(x', \xi'). \tag{2.5.11}$$

From this definition we easily see that

$$\sigma_\partial(\mathcal{A})(x',\delta\xi') = \delta^\mu \begin{pmatrix} \kappa_\delta & 0 \\ 0 & \mathrm{id}_{\mathbb{C}^{j_+}} \end{pmatrix} \sigma_\partial(\mathcal{A})(x',\xi') \begin{pmatrix} \kappa_\delta^{-1} & 0 \\ 0 & \mathrm{id}_{\mathbb{C}^{j_-}} \end{pmatrix} \tag{2.5.12}$$

for every $\delta \in \mathbb{R}_+, (x',\xi') \in T^*\Omega \setminus 0$.

For any $\omega(x',x'') \in C^\infty(\Omega\times\Omega)$ such that $\operatorname{supp}\omega$ is proper (i.e., $\operatorname{supp}\omega \cap \{(x',x'') \in \Omega\times\Omega : x' \in M\}$ and $\operatorname{supp}\omega \cap \{(x',x'') \in \Omega\times\Omega : x'' \in M'\}$ are compact sets for arbitrary $M \Subset \Omega, M' \Subset \Omega$) and $\operatorname{supp}\omega$ contains $\operatorname{diag}(\Omega\times\Omega)$ in its open interior, we can represent any $\mathcal{A} \in \mathcal{B}^{\mu,e}(\Omega\times\overline{\mathbb{R}}_+; j_-, j_+)$ in the form

$$\operatorname{Op}(\boldsymbol{a}) = \operatorname{Op}(\omega\boldsymbol{a}) + \operatorname{Op}((1-\omega)\boldsymbol{a}) \tag{2.5.13}$$

modulo $\mathcal{B}^{-\infty,e}(\Omega\times\overline{\mathbb{R}}_+; j_-, j_+)$. Then $\operatorname{Op}(\omega\boldsymbol{a})$ is properly supported in the sense that its distributional kernel has proper support in a similar sense as indicated before in connection with ω. From (2.5.13) we see that every $\mathcal{A} \in \mathcal{B}^{\mu,e}(\Omega\times\overline{\mathbb{R}}_+; j_-, j_+)$ can be decomposed in the form

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{C},$$

where $\mathcal{A}_0 \in \mathcal{B}^{\mu,e}(\Omega\times\overline{\mathbb{R}}_+; j_-, j_+)$ is properly supported and $\mathcal{C} \in \mathcal{B}^{-\infty,e}(\Omega\times\overline{\mathbb{R}}_+; j_-, j_+)$.

Theorem 2.5.3. *Let $\mathcal{A} \in \mathcal{B}^{\mu,e}(\Omega\times\overline{\mathbb{R}}_+; j_0, j_2)$, $\mathcal{B} \in \mathcal{B}^{\nu,l}(\Omega\times\overline{\mathbb{R}}_+; j_1, j_0)$ and assume that either $\mathcal{A}$ or $\mathcal{B}$ is properly supported. Then we have $\mathcal{AB} \in \mathcal{B}^{\mu+\nu,h}(\Omega\times\overline{\mathbb{R}}_+; j_1, j_2)$ for $h = \max\{e+\nu, l\}$, and*

$$\sigma_\psi(\mathcal{AB}) = \sigma_\psi(\mathcal{A})\sigma_\psi(\mathcal{B}), \quad \sigma_\partial(\mathcal{AB}) = \sigma_\partial(\mathcal{A})\sigma_\partial(\mathcal{B}). \tag{2.5.14}$$

If $\mathcal{A}$ or $\mathcal{B}$ belongs to $\mathcal{B}_{\mathrm{G}}^{\mu,e}$ then so is the composition $\mathcal{AB}$.

2.6 Global calculus and ellipticity

In the global formulation of pseudo-differential BVPs we assumed that X is a smooth and not necessarily compact manifold with boundary ∂X. Then, as is well-known, that ∂X has a collar neighbourhood V in X which can be identified with $V = \partial X \times [0,1)$ in the splitting of variables (x', x_n) and with covariables (ξ', ξ_n). We choose an open covering $(U_1,\dots,U_N,U_{N+1},\dots,U_L)$ by coordinate neighbourhoods and charts

$$\begin{aligned} &\chi_j : U_j \longrightarrow \mathbb{R}^{n-1}\times\overline{\mathbb{R}}_+, \quad j = 1,\dots,N, \\ &\chi_l : U_l \longrightarrow \mathbb{R}^n, \quad l = N+1,\dots,L \end{aligned} \tag{2.6.1}$$

and a subordinate partition of unity $(\varphi_1,\dots,\varphi_N,\varphi_{N+1},\dots,\varphi_L)$ where for functions $\varphi_j := \varphi'_j\omega$, for a partition of unity $(\varphi'_1,\dots,\varphi'_N)$ on ∂X subordinate to the open covering $(U'_1,\dots,U'_N)$ for $U'_j = U_j \cap V$ and a cut-off function $\omega(x_n)$ (i.e., $\omega \in C_0^\infty([0,1)), \omega(x_n) \equiv 1$ close to $x_n = 0$). For $l = N+1,\dots,L$ we simply assume $\varphi_l \in C_0^\infty(U_l)$ such that $(\varphi_1,\cdots,\varphi_N,\varphi_{N+1},\dots,\varphi_L)$ are a subordinate partition of unity to $(U_1,\dots,U_L)$. Note that every $U \in (U_1,\dots,U_N)$ has the form $U' \times [0,1)$ charts for $U' = U \cap V$ and admits

$$(\chi', \chi'') : U' \times [0,1) \longrightarrow \mathbb{R}^{n-1}\times\overline{\mathbb{R}}_+ \tag{2.6.2}$$

where $\chi' : U' \longrightarrow \mathbb{R}^{n-1}$ is a chart on the boundary and $\chi'' : [0,1) \longrightarrow \overline{\mathbb{R}}_+$ a diffeomorphism which is the identity on $[0, \frac{1}{2})$.

Let M be a closed smooth manifold, for simplicity arc-wise connected. Then by $\mathrm{Vect}(M)$ we denote the set of smooth complex vector bundles over M. Concerning generalities we refer to Atiyah's exposition [3] or any other text book such as Husemoller [22]. We then have a canonical projection $p : E \to M$ of the total space of the bundle, again denoted by E to the base manifold which projects fiberwise to a corresponding point of M, i.e., $p^{-1}(m) \cong \mathbb{C}^k$, and $k \in \mathbb{N}$ is the fiber dimension. Clearly, compactness of M is not necessary for the definition, and we also have the case of real vector bundles, i.e., with $\mathbb{C}$ replaced by $\mathbb{R}$. In particular, tangent and cotangent bundles TM and T^*M of M are real vector bundles, and the canonical projection π projects the real fibers to corresponding base points. From now freely employ here such standard notation; if necessary we recall some aspects. If we have to point out complex or real fibers we also write $\mathrm{Vect}_{\mathbb{C}}(M)$ and $\mathrm{Vect}_{\mathbb{R}}(M)$ for the respective sets of complex or real vector bundles, otherwise we also drop $\mathbb{C}$. Several generalizations will occur in the present exposition. For instance, if X is a smooth manifold with boundary from the above consideration we have the double $2X$ which is closed and then $\mathrm{Vect}(X)$ means the set of all $\tilde{E}|_X$ with $\tilde{E} \in \mathrm{Vect}(2X)$ in the former sense, where $|_X$ means the restriction of the corresponding bundle to X. In particular, we employ such notation for different base manifolds, e,g., $\mathrm{Vect}(\partial X)$. In the following definition for abbreviation we write $\boldsymbol{v} := (E, F; J_-, J_+)$ for any $E, F \in \mathrm{Vect}(X)$, $J_-, J_+ \in \mathrm{Vect}(\partial X)$. First we briefly say what are smoothing operators $\mathcal{B}_{\mathrm{G}}^{-\infty,0}\begin{pmatrix} X \\ \times & ;\boldsymbol{v} \\ \partial X \end{pmatrix}$ of type 0, namely, 2×2-block matrix operators

$$\mathcal{G} : \begin{array}{c} C^\infty(X, E) \\ \oplus \\ C^\infty(\partial X, J_-) \end{array} \longrightarrow \begin{array}{c} C^\infty(X, F) \\ \oplus \\ C^\infty(\partial X, J_+) \end{array} \tag{2.6.3}$$

with smooth kernels. Let us illustrate the kernels first for the case of trivial bundles of fiber dimension 1, where we also omit E, J_-, etc. in the notation (2.6.3) and write in this case $\boldsymbol{v} := \boldsymbol{v}_1$, $\boldsymbol{v}_1 = (1, 1; 1, 1)$.

As at the very beginning we fix Riemannian metrics g_X and $g_{\partial X}$ on X and ∂X, respectively, where a collar neighbourhood V of ∂X in X can be identified with $\partial X \times [0, 1)$ with $g_{\partial X}$ being the restriction of g_X to the boundary. In that case, as is common in pseudo-differential operators we can identify smoothing operators with kernels, but here we control the smoothness up to ∂X. The space $\mathcal{B}_{\mathrm{G}}^{-\infty,0}\begin{pmatrix} X \\ \times & ;\boldsymbol{v}_1 \\ \partial X \end{pmatrix}$ is defined as the set of all 2×2-block matrices

$$\mathcal{G} := \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \tag{2.6.4}$$

where the interior part G_{11} is asked to have a kernel $g_{11} \in C^\infty(X \times X)$ and with smoothness in x, x' up to ∂X, i.e.,

$$G_{11}u(x) = \int_X g_{11}(x, x')u(x')\,dx', \tag{2.6.5}$$

$u(x') \in C^\infty(X)$ with dx' being the measure associated with g_X. Similarly, the potential part G_{12} has a kernel $g_{12} \in C^\infty(X \times \partial X)$ and

$$G_{12}v(x) = \int_X g_{12}(x, x''')v(x''')\, dx''', \tag{2.6.6}$$

where dx''' is associated with $g_{\partial X}$. Similarly, the smoothing trace part has a kernel g_{21} in $C^\infty(\partial X \times X)$ and the right lower corner G_{22} has a kernel in $C^\infty(\partial X \times \partial X)$. In the case of $e \in \mathbb{N}, e \neq 0$, we define the space $\mathcal{B}_{\mathrm{G}}^{-\infty,e}\begin{pmatrix} X \\ \times & ; \boldsymbol{v}_1 \\ \partial X \end{pmatrix}$ of elements of order $-\infty$ and type e to be the space of all operators

$$\mathcal{G} := \sum_{j=0}^{e} \mathcal{G}_j \operatorname{diag}(D^j, 0) \tag{2.6.7}$$

for any $\mathcal{G}_j \in \mathcal{B}_{\mathrm{G}}^{-\infty,0}\begin{pmatrix} X \\ \times & ; \boldsymbol{v}_1 \\ \partial X \end{pmatrix}$ and D^j differential operators of order j supported close to ∂X and of the form $D^j v = \frac{\partial^j}{\partial x_n^j} v$, $v \in C^\infty(\partial X)$.

The case $\mathcal{B}_{\mathrm{G}}^{-\infty,e}\begin{pmatrix} X \\ \times & ; \boldsymbol{v} \\ \partial X \end{pmatrix}$ of smoothing operators referring to arbitrary $v = (E, F; J_-, J_+)$ is straightforward. Nevertheless, because of construction below we sketch a few points. We fix Hermitean matrices in E, F over X and in J_-, J_+ over ∂X and express kernels in terms of external products of the involved bundles. In the case of smoothing operators between distributional sections or, in particular, smooth sections as in (2.6.7) of bundles we employ an analoguous definition as (2.6.7).

Definition 2.6.1. *The space* $\mathcal{B}_{\mathrm{G}}^{-\infty,e}\begin{pmatrix} X \\ \times & ; \boldsymbol{v} \\ \partial X \end{pmatrix}$ *of global smoothing operators referring to* $\boldsymbol{v} = (E, F; J_-, J_+)$ *of type* $e \in \mathbb{N}$ *consists of all operators of the form* (2.6.7) *where* $\mathcal{G}_j$ *and* D^j *have the following new interpretation,*

$$\begin{aligned} g_{11} &\in C^\infty(X \times X, F \boxtimes E^*), \quad g_{12} \in C^\infty(X \times \partial X, F \boxtimes J_-^*) \\ g_{21} &\in C^\infty(\partial X \times X, J_+ \boxtimes E^*), \quad g_{22} \in C^\infty(\partial X \times \partial X, J_+ \boxtimes J_-^*) \end{aligned}. \tag{2.6.8}$$

The meaning of $\boxtimes$ *is the respective exterior tensor product between liftings of bundles of one factor of the Cartesian product to the Cartesian product itself, where upper star denotes Hermitean adjoints which determines sesquilinear pairing*

$$(E, E^*) = E \times E^* \longrightarrow \mathbb{C}, \tag{2.6.9}$$

etc. which extent to exterior tensor products between lifted bundles, such as

$$F \boxtimes E^* := \pi_2^* F \otimes \pi_1^* E^* \tag{2.6.10}$$

and the integration in upper left corners has the interpretation

$$G_{11}u = \int (g_{11}(x, x'), u)_{(E^*,E)}\, dx' \in C^\infty(X, F) \tag{2.6.11}$$

for $u \in C^\infty(X,E)$, *with* (2.6.10) *being interpreted as the standard tensor product between bundles over* $X \times X$, *obtained by lifting from*

$$\pi_1 : X \times X \longrightarrow X, \quad \pi_2 : X \times X \longrightarrow X$$

from $X \times X$ *to the first/second factor. In an analogous manner the other entries are defined.*

Every $\mathcal{A} \in \mathcal{B}^{\mu,e}\begin{pmatrix} X & \\ \times & ;\boldsymbol{v} \\ \partial X & \end{pmatrix}$ for $\boldsymbol{v} = (E,F;J_-,J_+)$, $E,F \in \mathrm{Vect}(X)$, $J_-,J_+ \in \mathrm{Vect}(\partial X)$, has a properly supported representative, i.e., every $\mathcal{A} \in \mathcal{B}^{\mu,e}\begin{pmatrix} X & \\ \times & ;\boldsymbol{v} \\ \partial X & \end{pmatrix}$ admits a representation

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{G}$$

for a properly supported $\mathcal{A}_0 \in \mathcal{B}^{\mu,e}\begin{pmatrix} X & \\ \times & ;\boldsymbol{v} \\ \partial X & \end{pmatrix}$ and $\mathcal{G} \in \mathcal{B}^{-\infty,e}\begin{pmatrix} X & \\ \times & ;\boldsymbol{v} \\ \partial X & \end{pmatrix}$, where by properly supported we understand proper support of the kernel of the upper left corner $\in L^\mu_{\mathrm{cl}}(\mathrm{int}X;E,F)$ and the operator-valued distributional kernel of $\mathrm{Op}_{x'}(\boldsymbol{a})$ for $\boldsymbol{a}(x',\xi') \in R^{\mu,e}(\Omega \times \mathbb{R}^{n-1};\boldsymbol{v})$ close to the boundary is asked to have a proper support.

Theorem 2.6.2. *Every* $\mathcal{A} \in \mathcal{B}^{\mu,e}\begin{pmatrix} X & \\ \times & ;\boldsymbol{v} \\ \partial X & \end{pmatrix}$ *for* $\boldsymbol{v} = (E,F;J_-,J_+)$ *induces continuous operators*

$$\mathcal{A} : \begin{matrix} H^s_{\mathrm{comp}}(\mathrm{int}\,X,E) \\ \oplus \\ H^s_{\mathrm{comp}}(\partial X,J_-) \end{matrix} \longrightarrow \begin{matrix} H^{s-\mu}_{\mathrm{loc}}(\mathrm{int}\,X,F) \\ \oplus \\ H^{s-\mu}_{\mathrm{loc}}(\partial X,J_+) \end{matrix} \tag{2.6.12}$$

for every $s > \max\{\mu,e\} - \frac{1}{2}$. *Moreover, if* $\mathcal{A}$ *is properly supported then in* (2.6.12) *we may write* "comp" *or* "loc" *on both sides.*

Operators $\mathcal{A} \in \mathcal{B}^{\mu,e}\begin{pmatrix} X & \\ \times & ;\boldsymbol{v} \\ \partial X & \end{pmatrix}$ for $\boldsymbol{v} = (E,F;J_-,J_+)$ have a pair of principal symbols

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}))$$

where

$$\sigma_\psi(\mathcal{A}) : \pi_X^* E \longrightarrow \pi_X^* F \tag{2.6.13}$$

for the canonical projection

$$\pi_X : T^*X \setminus 0 \longrightarrow X$$

only depends on the upper left corner $\sigma_\psi(A)$, and

$$\sigma_\partial(\mathcal{A}) : \pi_{\partial X}^* \begin{pmatrix} H^s(\mathbb{R}_+,E') \\ \oplus \\ J_- \end{pmatrix} \longrightarrow \pi_{\partial X}^* \begin{pmatrix} H^{s-\mu}(\mathbb{R}_+,F') \\ \oplus \\ J_+ \end{pmatrix} \tag{2.6.14}$$

the boundary symbol, $s > \max\{\mu, e\} - \frac{1}{2}$, where $E' := E|_{\partial X}, F' := F|_{\partial X}$, and

$$\pi_{\partial X} : T^*\partial X \setminus 0 \longrightarrow \partial X,$$

is the canonical projection. Alternatively, we can identify $\sigma_\partial(\mathcal{A})$ with a bundle morphism

$$\sigma_\partial(\mathcal{A}) : \pi^*_{\partial X} \begin{pmatrix} \mathcal{S}(\overline{\mathbb{R}}_+, E') \\ \oplus \\ J_- \end{pmatrix} \longrightarrow \pi^*_{\partial X} \begin{pmatrix} \mathcal{S}(\overline{\mathbb{R}}_+, F') \\ \oplus \\ J_+ \end{pmatrix}. \tag{2.6.15}$$

Theorem 2.6.3. *Let* $\mathcal{A} \in \mathcal{B}^{\mu,e} \begin{pmatrix} X & \\ \times & ; \boldsymbol{a} \\ \partial X & \end{pmatrix}$, *for* $\boldsymbol{a} := (E_0, F; J_0, J_2)$, $\mathcal{B} \in \mathcal{B}^{\nu,l} \begin{pmatrix} X & \\ \times & ; \boldsymbol{b} \\ \partial X & \end{pmatrix}$ *for* $\boldsymbol{b} := (E, E_0; J_1, J_0)$. *If* $\mathcal{A}$ *or* $\mathcal{B}$ *be properly supported. Then we have*
$\mathcal{AB} \in \mathcal{B}^{\mu+\nu,h} \begin{pmatrix} X & \\ \times & ; \boldsymbol{a} \circ \boldsymbol{b} \\ \partial X & \end{pmatrix}$ *for* $h = \max(\nu + e, l)$

$$\boldsymbol{a} \circ \boldsymbol{b} = (E, F; J_1, J_2)$$

and

$$\sigma_\psi(\mathcal{AB}) = \sigma_\psi(\mathcal{A})\sigma_\psi(\mathcal{B}), \quad \sigma_\partial(\mathcal{AB}) = \sigma_\partial(\mathcal{A})\sigma_\partial(\mathcal{B}). \tag{2.6.16}$$

If $\mathcal{A}$ or $\mathcal{B}$ belongs to subclass with subscript "G" then also $\mathcal{AB}$ is Green in such a sense.

Definition 2.6.4. *An operator* $\mathcal{A} \in \mathcal{B}^{\mu,e}(X; \boldsymbol{v})$ *is called elliptic if* (2.6.13) *and* (2.6.14) *are isomorphism. Observe that* (2.6.14) *is isomorphism if and only if* (2.6.15) *is an isomorphism.*

The isomorphism (2.6.13) is also called interior ellipticity of $\mathcal{A}$ and (2.6.14)
Shapiro-Lopatinskij condition of the contributions from the boundary with respect to the interior, see also Agmon, Douglis, Nirenberg [1] for a differential operators A and Boutet de Monvel [7] for pseudo-differential operators.

Remark 2.6.5. *Interior ellipticity, together with standard homogeneity*

$$\sigma_\psi(A)(x, \delta\xi) = \delta^\mu \sigma_\psi(A)(x, \xi) \tag{2.6.17}$$

for $\delta > 0$ *remains preserved under reduction of orders, up to shifting the order, cf. Example* 2.6.7 *below. So we may conclude, by reducing with order* $-\mu$ *that the resulting order is zero, cf. the first relation of* (2.6.16) *we may look at the case of order zero. Let* A_0 *the respective operator of order zero. Then the transmission property means*

$$\sigma_\psi(A_0)(x, \xi) = \sigma_\psi(A_0)(x, -\xi)$$

for any $\xi \neq 0$. *Together with* (2.6.17) *in this case that means, when* $\sigma_\psi(A_0)(x, 0)$ *is also an isomorphisms*

$$E' \longrightarrow F'. \tag{2.6.18}$$

Thus in ellipticity of BVPs we may assume, without loss of generality, property (2.6.18).

This observation and more details may be found in [7] and also [48].

Theorem 2.6.6. *Let* $\mathcal{A} \in \mathcal{B}^{\mu,e}\begin{pmatrix} X \\ \times \\ \partial X \end{pmatrix}; \boldsymbol{v}$ *be elliptic. Then* $\mathcal{A}$ *has a parametrix* $\mathcal{P} \in \mathcal{B}^{-\mu,l}\begin{pmatrix} X \\ \times \\ \partial X \end{pmatrix}; \boldsymbol{v}^{-1}$ *for* $\boldsymbol{v}^{-1} := (F, E; J_+, J_-), l = \max\{e-\mu, 0\}$*, and*

$$\sigma_\psi(\mathcal{P}) = \sigma_\psi(\mathcal{A}^{-1}), \quad \sigma_\partial(\mathcal{P}) = \sigma_\partial(\mathcal{A}^{-1}). \tag{2.6.19}$$

The information of Chapter 2 is necessary for treating BVPs on manifolds in this exposition from Chapter 4 on. Although the material in Chapter 2 is entirely classical it is, as we shall see, a challenge to formulate the right approach on singular manifolds, from the conical case on. In addition it is important to recall that classical examples, such as Dirichlet or Neumann problems for Laplace operators and other BVPs with Shapiro-Lopatinskij-ellipticity are covered by the calculus. Most of those examples show that the ellipticity of homogeneous boundary symbols require the indicated extra entries, i.e., at least J_- or J_+ are of fiber dimension $\neq 0$, one of them may vanish. However, for elliptic operators on X it happens that both dimensions vanish, i.e., those Shapiro-Lopatinskij-elliptic ellements only consist of left upper corners. Since we need this case below we briefly formulate such order reducing examples. For references below, and in order to complete the construction in Remark 2.6.5 we explicitely give the corresponding.

Example 2.6.7. *For every* $\mu \in \mathbb{Z}$ *and any* $E, F \in \mathrm{Vect}(X)$*, there exists Shapiro-Lopatinskij-elliptic elements of upper left corner form*

$$R_E^\mu \in \mathcal{B}^{\mu,0}\begin{pmatrix} X \\ \times \\ \partial X \end{pmatrix}; \boldsymbol{v}$$

for $\boldsymbol{v} = (E, F; 0, 0)$ *such that*

$$R_E^\mu : H^s(X, E) \longrightarrow H^{s-\mu}(X, E) \tag{2.6.20}$$

induces isomorphisms for all $s > -1/2$*.*

This example goes back to Grubb [19] also studied in detail in Harutyunyan [20, Section 4.1]. Operators with the required properties are based on local symbols which are close to ∂X of the form

$$r_-^\mu(\xi', \xi_n) = \left(\varphi\left(\frac{\xi_n}{C\langle\xi'\rangle}\right)\langle\xi'\rangle - i\xi_n\right)^\mu \tag{2.6.21}$$

for the splitting $\xi = (\xi', \xi_n)$ of the covariable ξ close to the boundary, $C > 0$ is a constant sufficiantly large, while far from ∂X the respective operator is glued to the one with symbol (2.6.21) with an elliptic operator with symbol $(1+|\xi|^2)^{\frac{\mu}{2}}$. Clearly, there are involved localizing factors from a partition of unity on X, and this creates lower order terms. This concerns trivial bundles E of fiber dimension 1. The general case is obtained in a similar manner as Boutet de Monvel in [7] constructed his reductions of orders, but with symbols which are not precisely as (2.6.21). In a final step of the construction we replace ξ' by (ξ', ζ) for an extra covariable $\zeta \in \mathbb{R}^d$. Then we obtain parameter-dependent ellipticity, in $\mathcal{B}^{\mu,e}$, and then, by taking $|\zeta|$ sufficiently large and fixed, we obtain indeed the claimed order-reducing property (2.6.20). In [62] it is proved in detail that symbols like (2.6.21) are also classical.

2.7 Ellipticity and Fredholm property

We now discuss the aspect of whether or not a σ_ψ-elliptic operator $A \in \mathcal{B}^{\mu,e}(X; E, F; 0, 0)$ admits an element $\mathcal{A} \in \mathcal{B}^{\mu,e}(X; \boldsymbol{v})$ for $\boldsymbol{v} = (E, F; J_1, J_2)$ for suitable $J_1, J_2 \in \mathrm{Vect}\,(\partial X)$ such that $\mathcal{A}$ is elliptic in the sense of Definition 2.6.4.

Remark 2.7.1. *If $A \in \mathcal{B}^{\mu,e}(X; E, F; 0, 0)$ is σ_ψ-elliptic, then*

$$\sigma_\partial(A) : \pi^*_{\partial X} H^s(\mathbb{R}_+, E') \longrightarrow \pi^*_{\partial X} H^{s-\mu}(\mathbb{R}_+, E'), \tag{2.7.1}$$

is a family of Fredholm operators for $s > e - \mu - \frac{1}{2}$, parametrized by $(x', \xi') \in T^*\partial X \setminus 0$, i.e.,

$$\sigma_\partial(A)(x', \xi') : H^s(\mathbb{R}_+, E'_{x'}) \longrightarrow H^{s-\mu}(\mathbb{R}_+, F'_{x'}) \tag{2.7.2}$$

is Fredholm for every $x' \in \partial X$ and $\xi' \in T^*_{x'}\partial X \setminus 0$. We have

$$\sigma_\partial(A)(x', \delta\xi') = \delta^\mu \kappa_\delta \sigma_\partial(A)(x', \xi') \kappa_\delta^{-1} \tag{2.7.3}$$

for all $\delta \in \mathbb{R}_+$.

The situation of elliptic boundary symbols in block matrix form is close to a possible definition of the $K =$ functor for a compact topological space X. The situation is as follows. Given such an X and a family of Fredholm operators

$$a(x) : H \longrightarrow \tilde{H} \tag{2.7.4}$$

between Hilbert spaces $H, \tilde{H}$ continuously depending on $x \in X$. In the case of Remark 2.7.1 we can assume $X = S^*\partial X$ which is the unit cosphere bundle induced by $T^*\partial X \setminus 0$, equipped with a Riemannian metric. In the above-mentioned Dirichlet or Neumann problems the boundary symbols are surjective. This is a relevant case.

Lemma 2.7.2. *A ssume that the Fredholm operators* (2.7.4) *are surjective for all $x \in X$. Then*

$$\{\ker a(x) : x \in X\}$$

is a (*locally trivial continuous*) *complex vector bundle.*

Proof. see the book [20]. □

Lemma 2.7.3. *For every family* (2.7.4) *of Fredholm operators there is an $N \in \mathbb{N}$ and a map*

$$\begin{pmatrix} a(x) & k \end{pmatrix} : \begin{matrix} H \\ \oplus \\ \mathbb{C}^N \end{matrix} \longrightarrow \tilde{H} \tag{2.7.5}$$

which is surjective for every $x \in X$.

Proof. Let $x_0 \in X$. Then there is an $N(x_0) \in \mathbb{N}$ and a

$$k(x_0) : \mathbb{C}^{N(x_0)} \longrightarrow \tilde{H}$$

such that

$$(a(x) \quad k(x_0)) : \begin{array}{c} H \\ \oplus \\ \mathbb{C}^{N(x_0)} \end{array} \longrightarrow \tilde{H} \tag{2.7.6}$$

is surjective for $x = x_0$. Then there is an open neighbourhood $U(x_0)$ of x_0 such that (2.7.6) is surjective for all $x \in U(x_0)$. This construction can be carried on for all $x_0 \in X$. This gives us an open covering of X by corresponding neighbourhoods $U(x_0)$. Thus there is a finite subcovering $(U(x_0), U(x_1), \dots, U(x_m))$ of X, since X is compact. Then the block matrix

$$(a(x) \quad k(x_0) \quad k(x_1) \cdots \quad k(x_m)) : \begin{array}{c} H \\ \oplus \\ \mathbb{C}^{N(x_0)} \\ \oplus \\ \mathbb{C}^{N(x_1)} \\ \oplus \\ \vdots \\ \oplus \\ \mathbb{C}^{N(x_m)} \end{array} \longrightarrow \tilde{H} \tag{2.7.7}$$

is surjective for all $x \in X$. Thus it suffices to set

$$k := \oplus_{j=0}^{m} k(x_j).$$

□

Combining Lemma 2.7.2 and 2.7.3 gives us a family of isomorphisms

$$\boldsymbol{a} := \begin{pmatrix} a(x) & k(x) \\ t(x) & q(x) \end{pmatrix} : \begin{array}{c} H \\ \oplus \\ G_{1,x} \end{array} \longrightarrow \begin{array}{c} \tilde{H} \\ \oplus \\ G_{2,x} \end{array} \tag{2.7.8}$$

for every Fredholm family (2.7.4) and some $G_1, G_2 \in \mathrm{Vect}(X)$. We call

$$\mathrm{ind}_X a := [G_2] - [G_1] \in K(X). \tag{2.7.9}$$

For $X = S^*\partial X$ as noted before we have the canonical projection

$$\pi_{\partial X} : S^*\partial X \longrightarrow \partial X.$$

Then the pull back of $J \in \mathrm{Vect}(\partial X)$ to $\pi^*_{\partial X} J \in \mathrm{Vect}(S^*\partial X)$ extends to a group homomorphism

$$\pi^*_{\partial X} : K(\partial X) \longrightarrow K(S^*\partial X).$$

The condition

$$\mathrm{ind}_{S^*\partial X} \sigma_\partial(A) \in \pi^*_{\partial X} K(\partial X) \tag{2.7.10}$$

is called Atiyah-Bott obstruction.

Theorem 2.7.4. *The $\sigma_\psi-$elliptic operator $A \in \mathcal{B}^{\mu,e}(X; E, F; 0, 0)$ admits an $(\sigma_\psi, \sigma_\partial)-$elliptic operator $\mathcal{A} \in \mathcal{B}^{\mu,e}(X; \boldsymbol{v})$ for suitable $\boldsymbol{v} = (E, F; J_1, J_2)$ if and only if condition (2.7.10) is satisfied.*

Chapter 3

Operators on Manifolds with Edge

3.1 Manifolds with higher singularities

Manifolds with edge are defined as specific stratified spaces of singularity order 1. That means the respective topological space M is written as a disjoint union of subspaces

$$M = s_0(M) \bigcup s_1(M)$$

where $s_j(M)$ are smooth manifolds of dimension

$$\dim s_1(M) < \dim s_0(M).$$

Moreover, $s_1(M)$ has a neighbourhood V in M which has the structure of a locally trivial X^Δ-bundle over $s_1(M)$ for some closed manifold X of dimension n. Here

$$X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X), \tag{3.1.1}$$

is a regular cone with base X, and $\{0\} \times X$ in the quotient space (3.1.1) is collapsed to a point, the vertex of the cone. An example is the wedge

$$M := X^\Delta \times \Omega$$

for an open set $\Omega \subseteq \mathbb{R}^q$. In this case we have

$$s_0(M) = M \setminus s_1(M), \quad \text{for} \quad s_1(M) = \Omega. \tag{3.1.2}$$

For the stratification of M we also write

$$s(M) = (s_0(M), s_1(M)). \tag{3.1.3}$$

Operators on a general manifold M with edge $Y := s_1(M)$ will be locally near Y expressed as operators on open stretched wedges

$$X^\wedge \times \Omega, \quad X^\wedge := \mathbb{R}_+ \times X,$$

coming from (3.1.2) with corresponding local splitting of variables

$$(r, x, y) \in X^\wedge \times \Omega.$$

By $\mathrm{Diff}^{\nu}(\cdot)$ we denote the Fréchet space of all differential operators of order $\nu \in \mathbb{N}$ with smooth coefficients on the respective open smooth manifold, indicated by dot. First a differential operator $A \in \mathrm{Diff}^{\mu}(s_0(M)), \mu \in \mathbb{N}$, is called edge-degenerate if close to Y in the above-mentioned splitting of variables $(r, x, y) \in \mathbb{R}_+ \times X \times \Omega$ it has the form

$$A = r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y)\Big(-r\frac{\partial}{\partial r}\Big)^{j}\big(rD_y\big)^{\alpha} \tag{3.1.4}$$

for coefficients

$$a_{j\alpha} \in C^{\infty}(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X)).$$

Examples of such operators appear when we introduce polar coordinates with respect to $\tilde{x} \in \mathbb{R}^{n+1}_{\tilde{x}} \setminus \{0\}$ into a differential operator

$$\tilde{A} = \sum_{|\gamma| \leq \mu} c_{\gamma}(\tilde{x}, y) D^{\gamma}_{\tilde{x},y}, \tag{3.1.5}$$

$c_{\gamma}(\tilde{x}, y) \in C^{\infty}(\mathbb{R}^{n+1} \times \Omega)$, in $(\tilde{x}, y) \in \mathbb{R}^{n+1}_{\tilde{x}} \times \Omega$. Then the respective bijection

$$(\mathbb{R}^{n+1}_{\tilde{x}} \setminus \{0\}) \times \Omega \longrightarrow \mathbb{R}_+ \times S^n \times \Omega \tag{3.1.6}$$

turns $\tilde{A}|_{(\mathbb{R}^{n+1}\setminus\{0\})\times\Omega}$ to an edge-degenerate operator (4.4.5), where $X = S^n$. Other important examples of operator (3.1.4) of order $\mu = 2$ Laplace-Beltrami operators to Riemannian metrics of the form

$$dr^2 + r^2 g_X + dy^2 \tag{3.1.7}$$

where g_X is a Riemannian metric on X. Parallel to the stratification (3.1.3) for operators (3.1.4) we have a principal symbolic hierarchy

$$\sigma(A) = (\sigma_0(A), \sigma_1(A)), \tag{3.1.8}$$

where $\sigma_0(A)$ is the standard homogeneous principal symbol of A over $s_0(M)$ living on $T^*s_0(M) \setminus 0$. Moreover,

$$\sigma_1(A)(y, \eta) := r^{-\mu} \sum_{j+|\alpha| \leq M} a_{j\alpha}(0, y)\Big(-r\frac{\partial}{\partial r}\Big)^{j}\big(r\eta\big)^{\alpha} \tag{3.1.9}$$

is called the homogeneous edge symbol, $(y, \eta) \in T^*Y \setminus 0$ regarded as a family of

$$\sigma_1(A)(y, \eta) := \mathcal{K}^{s,\gamma}(X^{\wedge}) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}) \tag{3.1.10}$$

between weighted Kegel-spaces, to be defined below. For the moment we can also look at operators

$$\sigma_1(A)(y, \eta) : C^{\infty}(X^{\wedge}) \longrightarrow C^{\infty}(X^{\wedge}), \tag{3.1.11}$$

(or also between spaces $C^{\infty}_0(X^{\wedge})$). Homogeneity in operator- valued symbols means

$$\sigma_1(A)(y, \delta\eta) = \delta^{\mu}\kappa_{\delta}\sigma_1(A)(y, \eta)\kappa_{\delta}^{-1}, \tag{3.1.12}$$

where

$$(\kappa_{\delta}u)(r, x) := \delta^{(n+1)/2}u(\delta r, x), \tag{3.1.13}$$

$\delta \in \mathbb{R}_+, n = \dim X$.

Note that conical singularities also may be subsumed under edge singularities when the edge is of dimension 0. Instead of (3.1.4) we then talk about operators of Fuchs type. Those are of the form

$$A = r^{-\mu} \sum_{j=0}^{\mu} a_j(r) \Big(-r \frac{\partial}{\partial r} \Big)^j \tag{3.1.14}$$

for coefficients $a_j(r) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}(X))$, for a closed manifold X as before. The stratification of a manifold B with conical singularities

$$s(B) = (s_0(B), s_1(B))$$

in this case means, that $s_1(B)$ consists of finitely many points and $s_0(B) = B \setminus s_1(B)$ is an open manifold. The analogue of (3.1.8), i.e., the symbols to be controlled, are

$$\sigma(A) = (\sigma_0(A), \sigma_{\mathrm{c}}(A))$$

with $\sigma_0(A)$ being the homogeneous principal symbol of A over $s_0(M)$, while $\sigma_{\mathrm{c}}(A)$ will be called here the principal conormal symbol,

$$\sigma_{\mathrm{c}}(A)(v) = \sum_{j=0}^{\mu} a_j(0) v^j : H^s(X) \longrightarrow H^{s-\mu}(X) \tag{3.1.15}$$

which is a family of continuous operators between standard Sobolev spaces over X, depending on the complex Mellin covariable v, often specified for

$$v \in \Gamma_{\frac{n+1}{2}-\gamma} = \big\{ v \in \mathbb{C} : \mathrm{Re}\, v = \frac{n+1}{2} - \gamma \big\}$$

for a weight $\gamma \in \mathbb{R}$ and $n = \dim X$. Note that in contrast to (3.1.9) we drop here the weight factor $r^{-\mu}$ in front of the expression which is in this connection more convenient. The situation becomes more clear if we include spaces with higher singularities $k \in \mathbb{N} = \{0, 1, 2, \dots\}$ into the consideration. It is obvious that simple constructions which generate cones or wedges, or other set-theoretical manipulations such as Cartesian products give rise to higher singularities.

Examples are, for instance, spaces like

$$X^\Delta \times X^\Delta \times \dots X^\Delta \quad (k \text{ factors})$$

for a closed manifold X. We can also look at the case when the spaces X depend on the respective factor. The following definition singles out a category of so-called stratified spaces of singularity order k where pseudo-differential operator theories similar to those for conical or edge singularities are motivated by numerous applications to physics and engineering. We denote the category of those topological spaces by $\mathfrak{M}_k$ and the definition will be iterative, starting from spaces in $\mathfrak{M}_0$ ($k = 0$ means smoothness), and repeatedly forming cones and wedges, followed by some globalization and then passing again to higher cones and wedges, using the spaces defined in the step before. In this scheme the above-mentioned spaces with conical singularities and edges belong to $\mathfrak{M}_1$. The formal definition of $\mathfrak{M}_k$ for $k \geq 1$ is as follows.

The category $\mathfrak{M}_k$ of topological spaces M (with some basic properties concerning paracompactness, etc.) is characterized by the following properties. There is singled out a subspace $s_k(M) \in \mathfrak{M}_0$ such that $M \setminus s_k(M) \in \mathfrak{M}_{k-1}$, and $s_k(M)$ has a neighbourhood V in M with the structure of a locally trivial $B^{\triangle}_{k-1}$-bundle over V for a base $B_{k-1} \in \mathfrak{M}_{k-1}$. The transition maps of fibers $B^{\triangle}_{k-1}$ inductively use isomorphisms in $\mathfrak{M}_{k-1}$. For $B^{\triangle}_{k-1}$ they are defined via isomorphisms

$$\mathbb{R}_+ \times B_{k-1} \longrightarrow \mathbb{R}_+ \times B_{k-1} \tag{3.1.16}$$

in $\mathfrak{M}_{k-1}$ which are well-defined on the level of singularity $k-1$, such that (3.1.16) is a restriction of an isomorphism

$$\mathbb{R} \times B_{k-1} \longrightarrow \mathbb{R} \times B_{k-1}$$

to $\mathbb{R}_+ \times B_{k-1}$. After having transition maps between the fibers we can easily define transitions of local wedges, say

$$\mathbb{R}_+ \times B_{k-1} \times \mathbb{R}^{q_k} \longrightarrow \mathbb{R}_+ \times B_{k-1} \times \mathbb{R}^{q_k} \tag{3.1.17}$$

in $\mathfrak{M}_{k-1}$. In a next step we can repeat the requirements, i.e., for $M \setminus s_k(M)$ there exists an $s_{k-1}(M \setminus s_k(M)) := s_{k-1}(M) \in \mathfrak{M}_0$ such that

$$(M \setminus s_k(M)) \setminus s_{k-1}(M) \in \mathfrak{M}_{k-2}. \tag{3.1.18}$$

Moreover, (3.1.18) has close to $s_{k-1}(M)$ a $B^{\triangle}_{k-2}$-bundle structure for $B_{k-2} \in \mathfrak{M}_{k-2}$.

After finitely many steps we arrive at a sequence of strata in $\mathfrak{M}_0$

$$s(M) := (s_0(M), s_1(M), \ldots, s_k(M)) \tag{3.1.19}$$

of dimensions

$$\dim s_0(M) > \dim s_1(M) > \cdots > \dim s_k(M). \tag{3.1.20}$$

Only $\dim s_k(M)$ may be zero; then it represents a corner singularity, otherwise $s_k(M)$ is an edge singularity of singularity order k. From the construction it follows that $M = \bigcup_{j=0}^{k} s_j(M)$ is a disjoint union.

The spaces $M \in \mathfrak{M}_k$, $k \geq 1$, admit a natural definition of spaces $\mathrm{Diff}^{\mu}_{\mathrm{deg}}(M)$ of degenerate differential operators, similarly as those described in formulas (3.1.4) or (3.1.14). In the case of singularity k where M locally close to $s_k(M)$ is modeled on $B^{\triangle} \times \mathbb{R}^q$ for $q := \dim s_k(M)$ and $B \in \mathfrak{M}_{k-1}$ we say that an $A \in \mathrm{Diff}^{\mu}(s_0(M))$ belongs to $\mathrm{Diff}^{\mu}_{\mathrm{deg}}(M)$ if in the splitting (r, x, y) of variables in $\mathbb{R}_+ \times B_k \times \mathbb{R}^q$ the operator A has the form (3.1.4) where now the coefficients a_{jk} belong to

$$C^{\infty}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, \mathrm{Diff}^{\mu-(j+|\alpha|)}_{\mathrm{deg}}(B)) \tag{3.1.21}$$

where $\mathrm{Diff}^{\mu-(j+|\alpha|)}_{\mathrm{deg}}(B)$ is inductively defined in the iteration step before, including its natural Fréchet topology which makes it possible to talk about smoothness in (3.1.21) with respect to (r, y) up to $r = 0$. Then in terms of an expression similarly as in (3.1.9) we get the "most singular" component $\sigma_k(A)(y, \eta), \eta \neq 0$ which is the kth component of the principal symbolic hierarchy

$$\sigma(A) := (\sigma_0(A), \sigma_1(A), \ldots, \sigma_k(A)). \tag{3.1.22}$$

In the next step we interpret A as an element of $\mathrm{Diff}^{\mu}_{\mathrm{deg}}(M \setminus s_k(M))$ where $M \setminus s_k(M)$ has the main stratum $s_{k-1}(M)$ and apply the same process. This gives us $\sigma_{k-1}(A)$ in the sequence

(3.1.22). After finitely many steps we arrive at $\sigma_0(A)$ which is the standard scalar homogeneous principal symbol of $A \in \mathrm{Diff}^\mu(s_0(M))$. In this way the components of (3.1.22) are associated with the strata in (3.1.19) in a natural way. This scheme of relations is reproduced in an analogous manner in the corresponding pseudo-differential concept.

In order to illustrate the situation we can look at the unit cube M in $\mathbb{R}^3$ which belong to $\mathfrak{M}_3$. Then $s_3(M)$ consists of the 8 corner points, $s_2(M)$ is the disjoint union of 12 one-dimentsional open edges, $s_1(M)$ is the disjoint union of 6 two-dimensional faces, and $s_0(M)$ is the open interior of the cube.

In the consideration of operators on singular spaces it is often useful to look at two copies of $M \setminus s_k(M)$, locally close to $s_k(M)$ identified with $\mathbb{R}_- \times B_{k-1} \times \mathbb{R}^{q_k}$ and $\mathbb{R}_+ \times B_{k-1} \times \mathbb{R}^{q_k}$ and then to glue these spaces together along the intersection

$$\overline{\mathbb{R}}_- \times B_{k-1} \times \mathbb{R}^{q_k} \bigcap \overline{\mathbb{R}}_+ \times B_{k-1} \times \mathbb{R}^{q_k} = \{0\} \times B_{k-1} \times \mathbb{R}^{q_k}$$

which yields $\mathbb{R} \times B_{k-1} \times \mathbb{R}^{q_k}$ in $\mathfrak{M}_{k-1}$. This can be done in an invariant manner, and globally we obtain a space $2\mathbb{M} \in \mathfrak{M}_{k-1}$ for the stretched space $\mathbb{M}$ of M which is locally identified with $\overline{\mathbb{R}}_+ \times B_{k-1} \times \mathbb{R}^{q_k}$.

The above-mentioned categories $\mathfrak{M}_k$ of spaces with singularities of order k can be studied in more detail under different aspects, e.g., analogous of relations from differential geometry. This seems to be not completely done in topological investigations on stratified spaces. Moreover, the description of degenerate differential operators on $M \in \mathfrak{M}_k$ opens the program of studying ellipticity in terms of suitable Fredholm or bijectivity conditions of the symbols containing (or not containing) extra trace and potential conditions and Green operators referring to the above lower-dimensional strata contained in (3.1.19). As soon as we have established such a program of ellipticity it is adequate to ask parametrices of elliptic differential operators, also having corresponding principal symbolic hierarchies. This program should be pursued along a functional-analytic description of adequate distribution spaces on M as well on its lower-dimensional strata, and we should observe Fredholm property, index, and other interesting features which extend corresponding observations on ellipticity on smooth manifolds. A particularly interesting aspect in this framework is elliptic regularity of solutions in weighted distribution spaces with asymptotics. For the case of boundary value problems for differential operators there is a comprehensive monograph of Nazarov and Plamenevskij [45], devoted to the problem of computing asymptotics of solutions. Remember that parametrices should belong to operator algebras with corresponding symbolic structures. A particularly important feature of such theories, parallel to the geometric process of forming cones and wedges, is the construction of higher (here called) Kegel spaces and also of subspaces with different kind of asymptotics (discrete and continuous). This is well-motivated for the case of a base manifold with or without boundary which, also concerns the material in Chapter 3. These working directions form a huge bunch of investigations, which are also to a large extent topic of the present thesis. Chapter 3 is devoted to basic knowledge on pseudo-differential theories for $\mathfrak{M}_1$, i.e., conical or edge singularities.

3.2 Edge-degenerate pseudo-differential operators

The calculus of operators on a manifold M with edge Y where $\dim Y = q$ is aimed at constructing an algebra of edge-degenerate pseudo-differential operators containing the operators (3.1.4) together with the parametrices of elliptic elements. Ellipticity will be a property of both components of the principal symbolic hierarchy. Similarly as in boundary value problems we also need pseudo-differential trace and potential operators. Edge-degenerate pseudo-differential operators in $X^\wedge \times \Omega$ are defined in terms of operator-valued symbols

$$p(r, y, \rho, \eta) := \tilde{p}(r, y, r\rho, r\eta) \tag{3.2.1}$$

for families

$$\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})). \tag{3.2.2}$$

X is a closed manifold of dimension n. In this formulation we employ the fact that $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$ is a Fréchet space in a natural way. Then the associated pseudo-differential operator, first based on the Fourier transform, takes the form

$$\begin{aligned} Au &= r^{-\mu} \iint e^{i(r-r')\rho + i(y-y')\eta} p(r, y, \rho, \eta) u(r', y')\, dr' dy' đ\rho\, đ\eta \\ &= r^{-\mu} \mathrm{Op}_y(\mathrm{Op}_r(p)(y, \eta)) u. \end{aligned} \tag{3.2.3}$$

Here

$$\mathrm{Op}_r(p)(y, \eta) v(r) = \iint e^{i(r-r')\rho} p(r, y, \rho, \eta) v(r')\, dr' đ\rho. \tag{3.2.4}$$

The operator functions (3.2.2) for edge-degenerate differential operators (3.1.4) have the form

$$\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}) = \sum_{j+|\alpha| \le \mu} a_{j\alpha}(r, y)(-i\tilde{\rho})^j \tilde{\eta}^\alpha.$$

3.3 Mellin transform and weighted spaces

If we consider operators of the form $\mathrm{Op}_r(p)(y, \eta)$ or $\mathrm{Op}_y(\mathrm{Op}_r(p))$ with p characterized by (3.2.1), (3.2.2) we have continuity like

$$\mathrm{Op}_r(p)(y, \eta) : H^s_{\mathrm{comp}}(\mathbb{R}_+ \times X) \longrightarrow H^{s-\mu}_{\mathrm{loc}}(\mathbb{R}_+ \times X) \tag{3.3.1}$$

for every fixed (y, η) and

$$\mathrm{Op}_y(\mathrm{Op}_r(p)(y, \eta)) : H^s_{\mathrm{comp}}(\mathbb{R}_+ \times X \times \Omega) \longrightarrow H^{s-\mu}_{\mathrm{loc}}(\mathbb{R}_+ \times X \times \Omega), \tag{3.3.2}$$

cf. Theorem 1.1.9. However, in order to control continuity up to $r = 0$ we pass to representations of our operators based on the Mellin transform in r. Let us first formulate some tools around the Mellin transform on the half-axis $\mathbb{R}_+$,

$$Mu(v) = \int_0^\infty r^v u(r) \frac{dr}{r}. \tag{3.3.3}$$

For $u \in C_0^\infty(\mathbb{R}_+)$ we have $Mu(v) \in \mathcal{A}(\mathbb{C})$; here $\mathcal{A}(\mathbb{C})$ means the space of entire functions in $\mathbb{C}$ in the Fréchet topology of uniform convergence on compact subsets. More precisely, setting

$$\Gamma_\beta := \{v \in \mathbb{C} : \mathrm{Re}\, v = \beta\} \tag{3.3.4}$$

for any $\beta \in \mathbb{R}$, we have

$$Mu|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta) \tag{3.3.5}$$

for every β, uniformly on compact intervals. That means, that $Mu(\beta + i\rho)$ for $v = \beta + i\rho$ is a bounded family in $\mathcal{S}(\mathbb{R}_\rho)$ for $c \le \beta \le c', c \le c'$. Observe that

$$(Mr^\beta f)(v) = Mf(v + \beta) \tag{3.3.6}$$

for any $f \in C_0^\infty(\mathbb{R}_+), \beta \in \mathbb{R}$. We often call

$$M_\gamma : u \longrightarrow Mu|_{\Gamma_{\frac{1}{2}-\gamma}}$$

the weighted Mellin transform for the weight γ. It can be proved that

$$M_\gamma : C_0^\infty(\mathbb{R}_+) \longrightarrow \mathcal{S}(\Gamma_{\frac{1}{2}-\gamma})$$

extends by continuity to an isomorphism

$$M_\gamma : r^\gamma L^2(\mathbb{R}_+) \longrightarrow L^2(\Gamma_{\frac{1}{2}-\gamma}),$$

with the inverse

$$M_\gamma^{-1} g(r) = \int_{\Gamma_{\frac{1}{2}-\gamma}} r^{-v} g(v)\, đv,$$

$đv := (2\pi i)^{-1} dv, g(v) \in L^2(\Gamma_{\frac{1}{2}-\gamma})$.

Let us now turn to pseudo-differential operators based on the Mellin transform. In order to give a motivation we first observe that

$$-r\frac{\partial}{\partial r} = M^{-1} v M$$

when M is interpreted as M_0, or, more generally, $(-r\frac{\partial}{\partial r})^j = M^{-1} v^j M$, for every $j \in \mathbb{N}$, cf., analogously, relation (1.1.1). Considering Mellin amplitude functions $f(r, r', v) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, S^\mu_{(\mathrm{cl})}(\Gamma_{\frac{1}{2}-\gamma}))$, cf. , analogously, relation (1.1.10), pseudo-differential operators with respect to the weighted Mellin transform are defined as

$$\mathrm{Op}_M^\gamma(f)u(r) = \iint \left(\frac{r}{r'}\right)^{-(\frac{1}{2}-\gamma+i\rho)} f(r, r', \frac{1}{2} - \gamma + i\rho) u(r') \frac{dr'}{r'}\, đ\rho, \tag{3.3.7}$$

$đ\rho = (2\pi i)^{-1} d\rho$. From

$$\left(\frac{r}{r'}\right)^{-(\frac{1}{2}-\gamma+i\rho)} = \mathrm{e}^{-i\rho(\log r - \log r')} \left(\frac{r}{r'}\right)^{-\frac{1}{2}+\gamma}$$

and comparing the substitution in Theorem 1.1.11 of coordinates we see that Mellin pseudo-differential operators are standard pseudo-differential operators, however, with another phase function, here $-\rho(\log r - \log r')$, i.e., $\mathrm{Op}_M^\gamma(f) \in L^\mu_{(\mathrm{cl})}(\mathbb{R}_+)$. This is associated with the diffeomorphism

$$\chi^{-1} : \mathbb{R}_+ \longrightarrow \mathbb{R}, \quad \chi^{-1} : r \longrightarrow -\log r.$$

Let us now define Sobolev spaces based on the Mellin transform on $\mathbb{R}_+ \times X$. First consider $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)$ for $s, \gamma \in \mathbb{R}$, defined as the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \langle v, \xi \rangle^{2s} |(M_{r\to v} F_{x\to\xi} u)(v,\xi)|^2 \, đv \, đ\xi \right\}^{\frac{1}{2}}. \tag{3.3.8}$$

Moreover, for a smooth closed manifold X we define $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $X^\wedge = \mathbb{R}_+ \times X$ in terms of localizations and charts $\chi_j : U_j \longrightarrow \mathbb{R}^n$ on X, where $(U_1, \ldots, U_N)$ is an open covering of X by coordinate neighbourhoods. Let $(\varphi_1, \ldots, \varphi_N)$ is a subordinate partition of unity we form

$$\|u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} := \left\{ \sum_{j=1}^N \|\varphi_j u \circ (\mathrm{id}_{\mathbb{R}_+} \times \chi_j^{-1})\|^2_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)} \right\}^{1/2}. \tag{3.3.9}$$

The spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ have natural properties such as

$$r^\beta \mathcal{H}^{s,\gamma}(X^\wedge) = \mathcal{H}^{s,\gamma+\beta}(X^\wedge) \tag{3.3.10}$$

for arbitrary $\gamma, \beta \in \mathbb{R}$.

For the edge calculus we need a modification of $\mathcal{H}^{s,\gamma}(X^\wedge)$ for large r. To this end we first formulate spaces $H^s_{\mathrm{cone}}(X^\wedge)$ for $s \in \mathbb{R}$, defined as the set of all

$$u \in H^s_{\mathrm{loc}}(\mathbb{R} \times X)|_{\mathbb{R}_+\times X}$$

such that for any cut-off function ω on the r half-axis and any $\varphi \in C_0^\infty(U)$ for a coordinate neighbourhood U on X and a diffeomorphism

$$\kappa : \mathbb{R}_+ \times U \longrightarrow \Gamma_V \subset \mathbb{R}^{1+n} \tag{3.3.11}$$

induced by a diffeomorphism $\chi : U \longrightarrow V$ for a coordinate neighbourhood V on S^n (the unit sphere in $\mathbb{R}^{1+n}$) and

$$\kappa(r, x) := r\chi(x)$$

we have

$$\kappa_*(1-\omega)\varphi u \in H^s(\mathbb{R}^{1+n}) \tag{3.3.12}$$

with $\kappa_* = (\kappa^{-1})^*$ being the push forward under κ. Then $H^s_{\mathrm{cone}}(X^\wedge)$ is a Hilbert space as well, and we then define the weighted Kegel space $\mathcal{K}^{s,\gamma}(X^\wedge)$ as

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{\omega u_0 + (1-\omega)u_\infty : u_0 \in \mathcal{H}^{s,\gamma}(X^\wedge),\ u_\infty \in H^s_{\mathrm{cone}}(X^\wedge)\} \tag{3.3.13}$$

for any cut-off function ω. The space (3.3.13) is well-defined, i.e., independent of the choice of ω. For purposes below we also form spaces with weight $e \in \mathbb{R}$ at ∞, namely,

$$\mathcal{K}^{s,\gamma;e}(X^\wedge) := [r]^{-e} \mathcal{K}^{s,\gamma}(X^\wedge).$$

The spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ and $H^s_{\mathrm{cone}}(X^\wedge)$ are Hilbert spaces, and also (3.3.13) can be equipped with the Hilbert space structure of a non-direct sum cf. Subsection 1.7.

Applying this to $E_0 = \mathcal{H}^{s,\gamma}(X^\wedge), E_1 = H^s_{\mathrm{cone}}(X^\wedge)$, we have the Hilbert spaces

$$[\omega]\mathcal{H}^{s,\gamma}(X^\wedge),\ [1-\omega]H^s_{\mathrm{cone}}(X^\wedge), \tag{3.3.14}$$

and

$$\mathcal{K}^{s,\gamma}(X^\wedge) = [\omega]\mathcal{H}^{s,\gamma}(X^\wedge) + [1-\omega]H^s_{\mathrm{cone}}(X^\wedge) \tag{3.3.15}$$

as the non-direct sum is a Hilbert space. Setting

$$(\kappa_\delta u)(r,x) := \delta^{(n+1)/2}u(\delta r, x) \tag{3.3.16}$$

for $\delta \in \mathbb{R}_+$, $u \in \mathcal{K}^{s,\gamma}(X^\wedge)$, $n = \dim X$, the space $\mathcal{K}^{s,\gamma}(X^\wedge)$ turns to a Hilbert space with group action, cf. the terminology at the beginning of Section 1.4.

3.4 Kernel cut-off and Mellin quantization

The operator functions (3.2.1), (3.2.2) give rise to symbols in the sense of Definition 1.4.4. In this context we apply a Mellin quantization, i.e., we turn the action (3.2.4) which employs the Fourier transform in r to an action based on the Mellin transform. To this end we first introduce suitable spaces of operator-valued Mellin symbols.

Definition 3.4.1. *For $\mu \in \mathbb{R}$ by*

$$M^\mu_{\mathcal{O}}(X;\mathbb{R}^l_\lambda) \tag{3.4.1}$$

we denote the set of all

$$h(v,\lambda) \in \mathcal{A}(\mathbb{C}, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^l_\lambda)) \tag{3.4.2}$$

such that

$$h(\beta + i\rho, \lambda) \in L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+l}_{\rho,\lambda}) \tag{3.4.3}$$

for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals.

Note that the class (3.4.1) admits a natural notion of ellipticity which means that (3.4.3) for some fixed $\beta \in \mathbb{R}$ is parameter-dependent elliptic of order μ, i.e., its parameter-dependent homogeneous principal symbol never vanishes for $(\rho,\lambda) \neq 0$. This condition is independent of β.

In (3.4.2) we employ the fact that $L^\mu_{\mathrm{cl}}(X;\mathbb{R}^l_\lambda)$ is a Fréchet space. Concerning holomorphic functions with values in a Fréchet space, see also the monograph of Jarchow [26].

We write $M^\mu_{\mathcal{O}}(X)$ when $l = 0$. In addition let

$$M^{-\infty}_{\mathcal{O}}(X;\mathbb{R}^l) := \bigcap_{\mu\in\mathbb{R}} M^\mu_{\mathcal{O}}(X;\mathbb{R}^l). \tag{3.4.4}$$

In the special case $\dim X = 0$ we simply write $M^\mu_{\mathcal{O}}(\mathbb{R}^l)$ for (3.4.1) and $M^m_{\mathcal{O}}$. For $l > 0$ we talk about classical symbols with the covariable $\lambda \in \mathbb{R}^l$ rather than parameter-dependent operator functions. Let us give an impression on how reach the spaces of Mellin symbols of (3.4.1). Here we apply a process, called kernel cut-off, which produces symbols which are holomorphic in covariables.

In order to give a transparent description we first consider symbols with constant coefficients

$$a(\xi) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^n_\xi). \tag{3.4.5}$$

Because of $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^n_\xi) \subset \mathcal{S}'(\mathbb{R}^n_\xi)$ we can restrict the Fourier transform

$$F : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

to $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^n)$. Thus, setting

$$k(a)(\zeta) := \int \mathrm{e}^{i\zeta\xi} a(\xi)\, đ\xi \tag{3.4.6}$$

for (3.4.5) we obtain an element of $\mathcal{S}'(\mathbb{R}^n_\zeta)$.

Lemma 3.4.2. *For* (3.4.5) *we have*

$$\operatorname{sing\,supp} k(a)(\zeta) \subseteq \{0\} \tag{3.4.7}$$

and for any excision function $\chi(\zeta)$ in $\mathbb{R}^n$ (i.e., $\chi \in C^\infty(\mathbb{R}^n), \chi(\zeta) = 0, \text{for} |\zeta| < \varepsilon_0, \chi(\zeta) = 1$ *for* $|\zeta| > \varepsilon_1$, *for some* $0 < \varepsilon_0 < \varepsilon_1$) *we have*

$$\chi(\zeta) k(a)(\zeta) \in \mathcal{S}(\mathbb{R}^n_\zeta). \tag{3.4.8}$$

Proof. Applying $D^\alpha_\xi e^{i\zeta\xi} = (i\zeta)^\alpha \mathrm{e}^{i\zeta\xi}$ for $\alpha \in \mathbb{N}^n$ in (3.4.6) we obtain for $\zeta \neq 0$

$$k(a)(\zeta) = -|\zeta|^{-2N} \int \Delta^N_\xi e^{i\zeta\xi} a(\xi)\, đ\xi \tag{3.4.9}$$

with Δ_ξ being the Laplace operator in the variable ξ. Integration by parts on the right-hand side of (3.4.9) yields

$$k(a)(\zeta) = -|\zeta|^{-2N} \int e^{i\zeta\xi} \Delta^N_\xi a(\xi)\, đ\xi. \tag{3.4.10}$$

By taking $N \in \mathbb{N}$ large enough we see that $k(a)(\zeta) \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Applying now $D^\beta_\zeta \mathrm{e}^{i\zeta\xi} = \xi^\beta \mathrm{e}^{i\zeta\xi}$ for any $\beta \in \mathbb{N}^n$ we obtain

$$(1 - \Delta_\zeta)^M \mathrm{e}^{i\zeta\xi} = (1 + |\xi|^2)^M \mathrm{e}^{i\zeta\xi} \tag{3.4.11}$$

for any $M \in \mathbb{N}$. Thus

$$\begin{aligned}(1 - \Delta_\zeta)^M k(a)(\zeta) &= \int \mathrm{e}^{i\zeta\xi} (1 + |\xi|^2)^M a(\xi)\, đ\xi \\ &= -|\zeta|^{-2N} \int \mathrm{e}^{i\zeta\xi} \Delta^N_\xi (1 + |\xi|^2)^M a(\xi)\, đ\xi.\end{aligned} \tag{3.4.12}$$

Here we choose again N so large that the integral on the right converges. At the same time we see that

$$\sup_{|\zeta| \geq c} |(1 - \Delta_\zeta)^M |\zeta|^{2N} k(a)(\zeta)| < \infty \tag{3.4.13}$$

for any $c > 0$. $\square$

By using Lemma 3.4.2 we can write

$$k(a)(\zeta) = \chi(\zeta)k(a)(\zeta) + \psi(\zeta)k(a)(\zeta) \tag{3.4.14}$$

for $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\chi := 1 - \psi$ which is a cut-off function, and then

$$\begin{aligned} a(\xi) &= F_{\zeta\to\xi}\big(\chi(\zeta)k(a)(\zeta)\big) + F_{\zeta\to\xi}\big(\psi(\zeta)k(a)(\zeta)\big) \\ &= c(\xi) + h(\xi) \end{aligned} \tag{3.4.15}$$

where

$$c(\xi) \in S^{-\infty}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n), \tag{3.4.16}$$

while $h(\xi) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^n_\xi)$. Since $\operatorname{supp}\big(\psi(\zeta)k(a)(\zeta)\big)$ is compact we have $h(\xi) \in \mathcal{A}(\mathbb{C}^n_\xi)$. Since

$$\psi(\zeta) := 1 - \chi(\zeta)$$

which is a cut-off function, i.e., $\psi \in C_0^\infty(\mathbb{R}^n)$, $\psi(\zeta) \equiv 1$ in a neighbourhood of $\zeta = 0$. we call

$$V_F(\psi)a(\xi) := F_{\zeta\to\xi}(\psi k(a))(\xi) \tag{3.4.17}$$

a kernel cut-off operator and $V_F(\chi)$ a kernel excision operator.

Proposition 3.4.3. *The operator* (3.4.17) *induces a continuous map*

$$V_F(\psi) : S^\mu_{(\mathrm{cl})}(\mathbb{R}^n) \longrightarrow S^\mu_{(\mathrm{cl})}(\mathbb{R}^n), \tag{3.4.18}$$

where

$$V_F(\psi)a(\xi) = a(\xi) \bmod S^{-\infty}(\mathbb{R}^n). \tag{3.4.19}$$

Proof. We have

$$k(a)(\zeta) = \psi(\zeta)k(a)(\zeta) + \chi(\zeta)k(a)(\zeta). \tag{3.4.20}$$

Then (3.4.19) follows from (3.4.8). Because of

$$a(\xi) = V_F(\psi)a(\xi) + V_F(\chi)a(\xi)$$

and the continuity of $V_F(\chi) : S^\mu_{(\mathrm{cl})}(\mathbb{R}^n) \longrightarrow S^{-\infty}(\mathbb{R}^n)$ which is a consequence of Lemma 3.4.2 we obtain the continuity of (3.4.18). □

We have $\psi(\zeta)k(a)(\zeta) \in \mathcal{S}'(\mathbb{R}^n)$ and

$$\operatorname{supp}(\psi k(a)) \text{ compact}. \tag{3.4.21}$$

Thus $V_F(\psi)a(\xi)$ admits the interpretation of an operator $V_F(\psi) : S^\mu_{(\mathrm{cl})}(\mathbb{R}^n) \longrightarrow \mathcal{A}(\mathbb{C}^n_\theta)$ for $\theta := \xi + i\delta \in \mathbb{C}^n$, such that $V_F(\psi)a|_{\operatorname{Im}\theta=0} = V_F(\psi)a(\xi)$. We shall see below that

$$V_F(\psi)a(\xi + i\delta) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^n_\xi) \quad \text{and} \quad V_F(\psi)a(\xi + i\delta) = a(\xi) \bmod S^{\mu-1}_{(\mathrm{cl})}(\mathbb{R}^n) \tag{3.4.22}$$

for every fixed $\delta \in \mathbb{R}^n$, uniformly in compact sets in $\mathbb{R}^n_\delta$. It is desirable to extend the definition of $V_F(\cdot)$ to arbitrary $\varphi \in C_0^\infty(\mathbb{R}^n)$ and even to φ in the space

$$C^\infty(\mathbb{R}^n)_{\mathrm{b}} := \{\varphi \in C^\infty(\mathbb{R}^n) : \sup_{\zeta\in\mathbb{R}^n} |D^\alpha_\zeta \varphi(\zeta)| < \infty \text{ for all } \alpha \in \mathbb{N}^n\}.$$

Later on we also employ spaces $C^\infty(\mathbb{R}, E)_{\mathrm{b}}$ for a Fréchet space E, defined in a similar manner. Clearly, for $\varphi \in C^\infty(\mathbb{R}^n)_{\mathrm{b}}$ in general we lose the existence of holomorphic extensions to the complex space $\mathbb{C}^n$.

Observe that $V_F(\varphi)a(\xi) = F_{\zeta\to\xi}(\varphi)k(a))(\xi)$, can be written as an oscillatory integral

$$V_F(\varphi)a(\xi) = \int \mathrm{e}^{-i\zeta\xi}\varphi(\zeta)\Big\{\int \mathrm{e}^{i\zeta\xi'}a(\xi')\,đ\xi'\Big\}\,đ\zeta = \iint \mathrm{e}^{-i\zeta\tilde{\xi}}\varphi(\zeta)a(\xi-\tilde{\xi})\,d\zeta đ\tilde{\xi}. \tag{3.4.23}$$

Concerning oscillatory integral techniques, see, e.g., Shubinś book [69], or, with other methods, Kumano-go [34]. In order to unify the terminology we call $V_F(\varphi)$ a kernel cut-off operators also when φ is more general than a cut-off function. This concerns again an extension of $V_F(\varphi)a(\xi)$ for $\varphi \in C_0^\infty(\mathbb{R}^n)$ into the complex space $\theta \in \mathbb{C}^n$, $\theta := \xi + i\rho$ which defines an operator

$$V_F(\varphi) : S^\mu_{(\mathrm{cl})}(\mathbb{R}^n_\xi) \longrightarrow \mathcal{A}(\mathbb{C}^n_\theta) \tag{3.4.24}$$

such that $V_F(\varphi)a|_{\operatorname{Im}\theta=0} = V_F(\varphi)a(\xi)$.

Note that the kernel cut-off process also makes sense for parameter-dependent symbols

$$a(\xi,\lambda) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^{n+l}_{\xi,\lambda}),$$

$l \in \mathbb{N}$. For $k(a)(\zeta,\lambda) := \int \mathrm{e}^{i\zeta\xi}a(\xi,\lambda)\,đ\xi$, then we have

$$k(a)(\zeta,\lambda) \in \mathcal{S}'(\mathbb{R}^{n+l}_{\xi,\lambda}), \quad \text{and} \quad \chi(\zeta)k(a)(\zeta,\lambda) \in \mathcal{S}(\mathbb{R}^{n+l}_{\xi,\lambda}), \tag{3.4.25}$$

i.e., for an excision function $\chi(\zeta)$

$$V_F(\chi)a(\xi,\lambda) = F_{\zeta\to\xi}(\chi(\zeta)k(a))(\xi,\lambda) \in \mathcal{S}(\mathbb{R}^{n+l}_{\xi,\lambda}) = S^{-\infty}(\mathbb{R}^{n+l}_{\xi,\lambda}). \tag{3.4.26}$$

Then the kernel cut-off operator in the parameter-dependent is

$$V_F(\psi)a(\xi,\lambda) = a(\xi,\lambda) \bmod S^{-\infty}(\mathbb{R}^{n+l}). \tag{3.4.27}$$

For $\varphi \in C^\infty(\mathbb{R}^n)_b$ we form

$$V_F(\varphi)a(\xi,\lambda) = \iint \mathrm{e}^{-i\zeta\tilde{\xi}}\varphi(\zeta)a(\xi-\tilde{\xi},\lambda)\,d\theta đ\tilde{\xi} \tag{3.4.28}$$

when $\varphi \in C_0^\infty(\mathbb{R}^n)$ it follows that $V_F(\varphi)a(\xi,\lambda)$ extends to a function $V_F(\varphi)a(\theta,\lambda)$ which is holomorphic in $\theta \in \mathbb{C}^n$, as we shall show

$$V_F(\varphi)a(\xi+i\delta,\lambda) = V_F(\varphi)a(\xi,\lambda) \bmod S^{\mu-1}_{(\mathrm{cl})}(\mathbb{R}^{n+l}),$$

for every $\delta \in \mathbb{R}^n$. Our main application concerns the case $n = 1$. This will be assumed from now on.

Theorem 3.4.4. *The kernel cut-off operator $V_F : (\varphi, a) \longrightarrow V_F(\varphi)a$ defines a bilinear continuous mapping*

$$V_F : C^\infty(\mathbb{R})_{\mathrm{b}} \times S^\mu_{(\mathrm{cl})}(\mathbb{R}^{1+l}_{\xi,\lambda}) \longrightarrow S^\mu_{(\mathrm{cl})}(\mathbb{R}^{1+l}_{\xi,\lambda}), \tag{3.4.29}$$

and $V_F(\varphi)a(\xi,\lambda)$ admits an asymptotic expansion

$$V_F(\varphi)a(\xi,\lambda) \sim \sum_{k=0}^\infty \frac{(-1)^k}{k!} D^k_\zeta\varphi(0)\partial^k_\xi a(\xi,\lambda). \tag{3.4.30}$$

Proof. The mapping $C^\infty(\mathbb{R}_\zeta)_\mathrm{b} \times S^\mu_{(\mathrm{cl})}(\mathbb{R}^{1+l}_{\xi,\lambda}) \longrightarrow C^\infty(\mathbb{R}^{1+l}_{\xi,\lambda}, S^\mu_{(\mathrm{cl})}(\mathbb{R}_\zeta \times \mathbb{R}_{\tilde{\xi}})_\mathrm{b})$ for $S^\mu_{(\mathrm{cl})}(\mathbb{R}_\zeta \times \mathbb{R}_{\tilde{\xi}})_\mathrm{b} := C^\infty(\mathbb{R}_\zeta, S^\mu_{(\mathrm{cl})}(\mathbb{R}_{\tilde{\xi}}))_\mathrm{b}$, defined by $(\varphi, a) \to \varphi(\zeta)a(\eta - \tilde{\xi}, \lambda)$, is bilinear and continuous. In order to show the continuity of (3.4.29) it suffices to verify that $V_F(\varphi)a \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^{1+l}_{\xi,\lambda})$ and to apply the closed graph theorem. Since $D^\beta_{\xi,\lambda} V_F(\varphi)a(\xi,\lambda) = V_F(\varphi)(D^\beta_{\xi,\lambda} a(\xi,\lambda))$ for any $\beta \in \mathbb{N}^{1+l}$, it suffices to show

$$|V_F(\varphi)a(\xi,\lambda)| \le c\langle \xi,\lambda\rangle^\mu \tag{3.4.31}$$

for all $(\mu,\lambda) \in \mathbb{R}^{1+l}$, for a constant $c > 0$. We regularise the oscillatory integral (3.4.28) as follows:

$$V_F(\varphi)a(\eta,\lambda) = \iint \mathrm{e}^{-i\zeta\tilde{\xi}} \langle\zeta\rangle^{-2}\{(1-\partial_\zeta^2)^N \varphi(\zeta)\} a_N(\xi,\tilde{\xi},\lambda)\, d\zeta đ\tilde{\xi},$$

where

$$a_N(\xi,\tilde{\xi},\lambda) = (1-\partial^2_{\tilde{\xi}})\{\langle\tilde{\xi}\rangle^{-2N} a(\xi-\tilde{\xi},\lambda)\} \tag{3.4.32}$$

for $N \in \mathbb{N}$ sufficiently large. The function (3.4.32) is a linear combination of terms $(\partial^j_{\tilde{\xi}}\langle\tilde{\xi}\rangle^{-2N})(\partial^k_\xi a)(\xi-\tilde{\xi},\lambda)$ for $0 \le j,k \le 2$. For the following conclusions we recall Peetre's inequality $\langle\xi'+\xi''\rangle^s \le c^{|s|}\langle\xi'\rangle^{|s|}\langle\xi''\rangle^s$ for all $\xi',\xi'' \in \mathbb{R}$, $s \in \mathbb{R}$. We have

$$\langle\xi-\tilde{\xi},\lambda\rangle^\mu \le C\langle\tilde{\xi}\rangle^{|\mu|}\langle\xi,\lambda\rangle^\mu, \tag{3.4.33}$$

when we write $(\xi-\tilde{\xi},\lambda) = (\xi,\lambda) - (\tilde{\xi},0)$. It follows that

$$\begin{aligned} |\partial^j_{\tilde{\xi}}\langle\tilde{\xi}\rangle^{-2N}(\partial^k_\xi a)(\xi-\tilde{\xi},\lambda)| &\le |\partial^j_{\tilde{\xi}}\langle\tilde{\xi}\rangle^{-2N}\|(\partial^k_\xi a)(\xi-\tilde{\xi},\lambda)\| \\ &\le c\langle\tilde{\xi}\rangle^{-2N}\langle(\xi-\tilde{\xi},\lambda)^\mu \le c\langle\tilde{\xi}\rangle^{|\mu|-2N}\langle\xi,\lambda\rangle^\mu \end{aligned}$$

for some $c > 0$. This implies analogous estimates for the function (3.4.32). For N so large that $\mu - 2N \le 0$ we obtain the estimate (3.4.31). In order to show (3.4.30) we employ the Taylor expansion

$$\varphi(\zeta) = \sum_{k=0}^N \frac{1}{k!}(\partial^k_\zeta\varphi)(0)\zeta^k + \zeta^{N+1}\varphi_{N+1}(\zeta),$$

for

$$\varphi_{N+1}(\zeta) = \frac{1}{N!}\int_0^1 (1-t)^N(\partial^{N+1}_\zeta\varphi)(t\zeta)\,dt. \tag{3.4.34}$$

The function $\varphi_{N+1}(\zeta)$ belongs to $C^\infty(\mathbb{R})_\mathrm{b}$. □

Corollary 3.4.5. *Let $\psi(\zeta) \in C^\infty_0(\mathbb{R})$ be a cut-off function on $\mathbb{R}$ (i.e., $\psi \equiv 1$ in a neighbourhood of 0), and set $\psi_\varepsilon(\zeta) = \psi(\varepsilon\zeta)$ for $0 < \varepsilon \le 1$. Then for every $a(\xi,\lambda) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^{1+l})$ we have*

$$\lim_{\varepsilon\to 0} V_F(\psi_\varepsilon)a(\xi,\lambda) = a(\xi,\lambda) \tag{3.4.35}$$

in the topology of $S^\mu_{(\mathrm{cl})}(\mathbb{R}^{1+l})$.

In fact, we have $\lim_{\varepsilon\to 0}\psi_\varepsilon = 1$ in the topology of $C^\infty(\mathbb{R})_\mathrm{b}$; then (3.4.35) is a consequence of the continuity of (3.4.29) for a fixed symbol on the left hand side of (3.4.29).

The kernel cut-off process can be extended from symbols in $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{1+l}_{\xi,\lambda})$ to other situations, e.g., operator families

$$f(v,\lambda) \in L^{\mu}_{\mathrm{cl}}(X;\Gamma_\beta \times \mathbb{R}^l_\lambda)$$

for some fixed $\beta \in \mathbb{R}$. Here the covariable ξ is substituted by $\operatorname{Im} v$ for $v \in \Gamma_\beta$. We then have the kernel cut-off operator as a map

$$V_F(\psi) : L^{\mu}_{\mathrm{cl}}(X;\Gamma_\beta \times \mathbb{R}^l_\lambda) \longrightarrow M^{\mu}_{\mathcal{O}}(X;\mathbb{R}^l_\lambda) \tag{3.4.36}$$

with analogous properties as before, in particular, setting

$$h(v,\lambda) := (V_F(\psi)f)(v,\lambda)$$

for $f(v,\lambda) \in L^{\mu}_{\mathrm{cl}}(X;\Gamma_\beta \times \mathbb{R}^l)$ we have

$$h(v,\lambda)|_{\Gamma_\beta \times \mathbb{R}^l} - f(v,\lambda) \in L^{-\infty}(X;\Gamma_\beta \times \mathbb{R}^l).$$

This shows that the space $M^{\mu}_{\mathcal{O}}(X;\mathbb{R}^l)$ is "nearly" as rich as the space $L^{\mu}_{\mathrm{cl}}(X;\Gamma_\beta \times \mathbb{R}^l)$ itself. The Mellin kernel cut-off is also involved in the Mellin quantization, that turns Fourier-based pseudo-differential actions connected with edge-degenerate families (3.2.1) to Mellin-based actions with holomorphic Mellin symbols of the kind (3.4.1). There are different variants of such Mellin quantizations. The basic information is contained in the following theorem.

Theorem 3.4.6. *For every edge-degenerate operator family* (3.2.1) *with the property* (3.2.2) *there is an*

$$\tilde{h}(r,y,v,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M^{\mu}_{\mathcal{O}}(X;\mathbb{R}^q_{\tilde{\eta}}))$$

such that for

$$h(r,y,v,\eta) := \tilde{h}(r,y,v,r\eta) \tag{3.4.37}$$

we have

$$\operatorname{Op}_r(p)(y,\eta) = \operatorname{Op}^{\gamma}_{\mathrm{M}}(h)(y,\eta) \bmod C^\infty(\Omega, L^{-\infty}(X^\wedge;\mathbb{R}^q_\eta)) \tag{3.4.38}$$

for any γ and $\tilde{h}(r,y,v,\tilde{\eta})$ is uniquely determined modulo $C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M^{-\infty}_{\mathcal{O}}(X;\mathbb{R}^q_{\tilde{\eta}}))$.

Similar constructions in the sense of kernel cut-off work when we replace the Fourier transform by the Mellin transform. In this context we also have weights but a translation in the complex Mellin plane allows us to focus on the weight $\gamma = \frac{1}{2}$ where the weight line is Γ_0. As we saw in the context of the Fourier transform the kernel cut-off operators only act on covariables, i.e., the symbols may depend on variables or on extra parameters. Thus, the Mellin- analogue of (3.4.6) may be started at once for symbols $a(r,r',v)$ with variable coefficients. The symbol may also be operator-valued, e.g., $a(r,r',v) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, L^\mu(X;\Gamma_0))$, and then, instead of (3.4.6) we look at

$$\begin{aligned} k(a)(r,r',v) &= \int_{\Gamma_0} e^{-v} a(r,r',v)\, đv \\ &= (M^{-1}_{\frac{1}{2},v\to\rho} a)(r,r',\rho). \end{aligned} \tag{3.4.39}$$

The Mellin-distributional kernel of $\operatorname{Op}^{1/2}_M(a)$ is just related to the expression

$$\operatorname{Op}^{1/2}_M(a) = \int_0^\infty k(a)(r,r',\frac{r}{r'})u(r')\,\frac{dr'}{r'}, \tag{3.4.40}$$

i.e., $k(a)(r, r', \frac{r}{r'})$ is the operator-valued kernel. If we consider for the moment the case with constant coefficients and look at the symbol $a(v) \in L^\mu_{\mathrm{cl}}(X; \Gamma_0)$, i.e., the Mellin covariable v is varying on Γ_0, then the Mellin kernel cut-off operator acting on $a(v)$ (for the weight $\gamma = 0$) takes the form

$$(H(\psi)a)(v) = M_{\frac{1}{2}, \theta \to v}(\psi(\theta)k(a)(\theta)) \tag{3.4.41}$$

for a cut-off function $\psi(\theta)$ on the positive θ half-axis which is of compact support and $\equiv 1$ in a neighbourhood of $\theta = 1$. Here we take into account that the singular support of $k(v)$ as an operator-valued distribution lies at $\theta = 1$, which corresponds to r/r', i.e., the diagonal $r = r'$. We do not recall here all properties of the kernel cut-off in the Mellin variant but tacitly employ the relevant properties. More details may be found in [53, Subsection 2.2.2].

3.5 The edge algebra

Let us first recall that $\mathfrak{M}_k$ means the category of spaces of singularity of order k, cf. Section 3.1. We understand a topological space M which is stratified, i.e., has a stratification indicated by a sequence of subspaces

$$s(M) = (s_0(M), s_1(M), \ldots, s_k(M)) \tag{3.5.1}$$

cf. formula (3.1.19).

The analysis of pseudo-differential operators on an $M \in \mathfrak{M}_k$ is induced by fixing a class of typical differential operators A with a hierarchy of symbols

$$\sigma(A) = (\sigma_0(A), \sigma_1(A), \ldots, \sigma_k(A)) \tag{3.5.2}$$

associated with the stratification (3.5.1) which determines ellipticity such that elliptic elements have parametrices in our operator class. A classical example is the case of a manifold M with smooth boundary ∂M. In this case $M \in \mathfrak{M}_1$ and

$$s_0(M) = \operatorname{int} M, \quad s_1(M) = \partial M.$$

Concerning differential operators and symbols the notation is as follows.

For $M \in \mathfrak{M}_k$ for fixed $k \in \mathbb{N}, k > 0$, by $\mathrm{Diff}^\mu_{\deg}(M)$ we denote the space of all differential operators over $s_0(M)$ which are close to $s_j(M)$, $1 \le j \le k$, of the form in the "stretched" splitting $(r, x, y) \in B^\wedge_{j-1} \times s_j(M)$ of the form

$$A = r^{-\mu} \sum_{l+|\alpha| \le \mu} a_{l\alpha}(r, y) \Big(-r \frac{\partial}{\partial r} \Big)^l (rD_y)^\alpha \tag{3.5.3}$$

when $\dim s_j(M) > 0$ for coefficients

$$a_{l\alpha}(r, y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega_j, \mathrm{Diff}^{\mu-(l+|\alpha|)}_{\deg}(B_{j-1})) \tag{3.5.4}$$

for some $B_{j-1} \in \mathfrak{M}_{j-1}, \Omega_j \subseteq \mathbb{R}^{q_j}$ open, and where $q_j = \dim s_j(M)$

$$A = r^{-\mu} \sum_{l \le \mu} a_l(r) \Big(-r \frac{\partial}{\partial r} \Big)^l \tag{3.5.5}$$

when $\dim s_j(M) = 0$ (which only may the case if $j = k$), where M is locally close to $s_k(M)$ a cone $B_{k-1}^{\triangle}$ for some $B_{k-1} \in \mathfrak{M}_{k-1}$ for coefficients

$$a_l(r) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}_{\deg}^{\mu-l}(B_{k-1})). \tag{3.5.6}$$

This definition it iterative, and it suffices to make the definition for $j = k$, then to make it for $j = k-1$, etc., where $\mathrm{Diff}_{\deg}^{\mu}(M) := \mathrm{Diff}^{\mu}(M)$ for $M \in \mathfrak{M}_0$. Then, because of the hierarchy of typical differential operators we can successively define the hierarchy of principal symbols, beginning with $\sigma_k(A)$ as

$$\sigma_k(A)(y,\eta) = r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(0,y)\Big(-r\frac{\partial}{\partial r}\Big)^j (r\eta)^\alpha, \eta \neq 0, \tag{3.5.7}$$

continuous in so-called weighted Kegel-spaces

$$\sigma_k(A)(y,\eta) : \mathcal{K}^{s,\gamma}(B_{k-1}^{\wedge}) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(B_{k-1}^{\wedge}) \tag{3.5.8}$$

for $s \in \mathbb{R}, \gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{R}^k$, when $\dim s_k(M) > 0$ and

$$\sigma_k(A)(v) = \sum_{l=0}^{\mu} a_l(0) v^l \tag{3.5.9}$$

when $\dim s_k(M) = 0$ for the above-mentioned coefficients, frozen at $r = 0$. The variable v is the complex covariable from the Mellin transform but it will be often regarded for $v \in \Gamma_{(b+1)/2-\gamma}$ for $b := \dim B_{k-1}$.

Let us turn to the edge calculus when the dimension of the edge is > 0 and $k = 1$; this case is typical in mixed boundary value problems to be interpreted in terms of the edge calculus of boundary value problems. Then B_{k-1} is simply a smooth compact manifold X. The main motivation of the edge calculus is to express parametrices of edge-degenerate differential operators in the framework of a corresponding pseudo-differential calculus. Differential operators can be written both in terms of the Fourier and the Mellin transform. For parametrices with a control up to $r = 0$ it is more convenient to employ the Mellin representation in r-direction. However, in the pseudo-differential case the way from the Fourier to the Mellin representation is not so obvious and referred to as a Mellin quantization. The Fourier representation of edge-degenerate operators concerns operators

$$r^{-\mu}\mathrm{Op}_y \mathrm{Op}_r(p) =: D \tag{3.5.10}$$

for $p(r,y,\rho,\eta) := \tilde{p}(r,y,r\rho,r\eta)$ and

$$\tilde{p}(r,y,\tilde{\rho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L_{\mathrm{cl}}^{\mu}(X;\mathbb{R}_{\tilde{\rho},\tilde{\eta}}^{1+q})).$$

Then we can pass to an amplitude function

$$r^{-\mu}\omega\mathrm{Op}_r(p)(y,\eta)\omega' \tag{3.5.11}$$

for some cut-off functions $\omega \prec \omega'$. It is convenient to modify (3.5.11) as follows. We write (3.5.11) in the form

$$r^{-\mu}\omega\{\tilde{\omega}_\eta \mathrm{Op}_{F_r}(p)(y,\eta)\tilde{\omega}'_\eta + (1-\tilde{\omega}_\eta)\mathrm{Op}_{F_r}(p)(y,\eta)(1-\tilde{\omega}''_\eta)\}\omega' \tag{3.5.12}$$

for $\tilde{\omega}_\eta(r) := \tilde{\omega}(r[\eta])$, etc., and cut-off functions $\omega'' \prec \tilde{\omega} \prec \tilde{\omega}'$. Then we obtain (3.5.11)= (3.5.12) modulo an (y,η)-dependent smoothing remainder which we do not specify. Applying kernel cut-off there is an $h(r,y,v,\eta)$ for $h(r,y,v,\eta) := \tilde{h}(r,y,v,r\eta), \tilde{h}(r,y,v,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, M^\mu_{\mathcal{O}_v}(X;\mathbb{R}^q_{\tilde{\eta}}))$ such that

$$\mathrm{Op}_M^{\gamma-n/2}(h)(y,\eta) - \mathrm{Op}_{F_r}(p)(y,\eta) \in C^\infty(\Omega, L^{-\infty}(X;\mathbb{R}^q_\eta)) \tag{3.5.13}$$

for every (y,η). Thus, forming

$$r^{-\mu}\omega\{\tilde{\omega}_\eta \mathrm{Op}_M^{\gamma-n/2}(h)(y,\eta)\tilde{\omega}'_\eta + (1-\tilde{\omega}_\eta)\mathrm{Op}_{F_r}(p)(y,\eta)(1-\tilde{\omega}''_\eta)\}\omega' \tag{3.5.14}$$

where for $u \in C_0^\infty(\mathbb{R}_+)$

$$\mathrm{Op}_{F_r}(p)(y,\eta)u(r) := \mathrm{Op}_r(p)(y,\eta)u(r) = \iint \mathrm{e}^{i(r-r')\rho}p(r,r',y,\rho,\eta)u(r')\,dr'đ\rho, \tag{3.5.15}$$

we have $(3.5.11) - (3.5.14) \in C^\infty(\Omega, L^{-\infty}(X;\mathbb{R}^q_\eta))$ for every (y,η), and assuming without loss of generality that p is of bounded support in r then

$$a(y,\eta) := r^{-\mu}\omega\{\tilde{\omega}_\eta \mathrm{Op}_M^{\gamma-n/2}(h)\tilde{\omega}'_\eta + (1-\tilde{\omega}_\eta)\mathrm{Op}_{F_r}(p)(1-\tilde{\omega}''_\eta)\}\omega' \tag{3.5.16}$$

induces a family of maps

$$a(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{3.5.17}$$

for every $s \in \mathbb{R}, n = \dim X$ is involved in the convention of weights for normalization reasons. The weighted Kegel spaces involved in (3.5.17) for a smooth closed manifold X and any real r can be defined as

$$\begin{aligned}\mathcal{K}^{s,\gamma}(X^\wedge) =&\{u \in H^s_{\mathrm{loc}}(\mathbb{R}_+ \times X) : u = \omega u_0 + (1-\omega)u_\infty \\ &\text{for}\, u_0 \in \mathcal{H}^{s,\gamma}(X^\wedge), u_\infty \in H^s_{\mathrm{cone}}(X^\wedge)\},\end{aligned} \tag{3.5.18}$$

cf. also relation (3.3.13). Here $\mathcal{H}^{s,\gamma}(X^\wedge)$ is the weighted Mellin Sobolev space of smoothness $s \in \mathbb{R}$ and weight $\gamma \in \mathbb{R}$, according to formulas (3.3.9), (3.3.10), and $H^s_{\mathrm{cone}}(X^\wedge)$ is defined by (3.3.11), (3.3.12). If is an easy task to verify that the group of transformations $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$,

$$(\kappa_\delta u)(r,x) := \delta^{(n+1)/2}u(\delta r, x),\ \delta \in \mathbb{R}_+, \tag{3.5.19}$$

induces a group action on the Hilbert space $\mathcal{K}^{s,\gamma}(X^\wedge)$ which is unitary for $s = \gamma = 0$. In this case we have

$$\mathcal{K}^{0,0}(X^\wedge) = \mathcal{H}^{0,0}(X^\wedge) = r^{-n/2}L^2(\mathbb{R}_+ \times X)_{drdx}. \tag{3.5.20}$$

Proposition 3.5.1. *We have* $a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$.

Proof. Let $\chi(\eta)$ be an exision function, and write

$$a(y,\eta) = \chi(\eta)a(y,\eta) + (1-\chi(\eta))a(y,\eta).$$

For convenience, we assume $a(y,\eta)$ to be independent of y. The generalization is straightforword. Then, since

$$\chi(\eta)a(\eta) \in C_0^\infty(\mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)))$$

we have $\chi(\eta)a(\eta) \in S^{-\infty}(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$ which is contained in the claimed symbol space for any $s \in \mathbb{R}$. Thus it suffices to look at $(1-\chi(\eta))a(\eta)$ which has the property to vanish in a neighbourhood of $\eta = 0$. Then the assertion can be verified by applying tensor product representations

$$h(r,v,\eta) = \sum_{j=1}^{\infty} \lambda_j \varphi_j(r) h_j(v,\eta), \quad h_j(v,\eta) = \tilde{h}(v,\tilde{\eta}),$$

for $\tilde{h}(v, r\eta) \in M^{\mu}_{\mathcal{O}_v}(X; \mathbb{R}^q_{\tilde{\eta}}))$,

$$\tilde{p}(r,\tilde{\rho},\tilde{\eta}) \sim \sum_{j=1}^{\infty} \lambda_j \psi_j(r) \tilde{p}(\tilde{\rho},\tilde{\eta}) \tag{3.5.21}$$

where $\varphi_j \in C^\infty(\mathbb{R}_+)$ tends to zero in this space for $j \to \infty$, $\tilde{h}(v,\tilde{\eta}) \in M^{\mu}_{\mathcal{O}_v}(X; \mathbb{R}^q_{\tilde{\eta}}))$ tends to zero as $j \to \infty$, $\psi_j(r) \in C_0^\infty(\mathbb{R}_+), \psi_j(r) \to 0$ and $\tilde{p}_j(\tilde{\rho},\tilde{\eta}) \in L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{q+1}_{\tilde{\rho},\tilde{\eta}})$, $\tilde{p}_j \to 0$ as $j \to \infty$. Setting $h_0(r,v,\eta) := \tilde{h}(0,v,r\eta)$,

$$p_0(r,\rho,\eta) := \tilde{p}(0, r\rho, r\eta)$$

then we have smoothing Mellin and pseudo-differential objects, where r is only involved in combination with covariables. $\square$

Thus, when we define the edge symbol

$$\sigma_1(a)(y,\eta) := r^{-\mu}\{\tilde{\omega}_{|\eta|}\mathrm{Op}_M^{\gamma-n/2}(h_0)(y,\eta)\tilde{\omega}'_{|\eta|} + (1-\tilde{\omega}_{|\eta|})\mathrm{Op}_{F_r}(p_0)(y,\eta)(1-\tilde{\omega}''_{|\eta|})\}. \tag{3.5.22}$$

for $\eta \neq 0$, $\omega_{|\eta|} := \omega(r|\eta|)$, etc., and we obtain homogeneity

$$\sigma_1(a)(y,\delta\eta) = \delta^\mu \kappa_\delta \sigma_1(a)(y,\eta)\kappa_\delta^{-1} \tag{3.5.23}$$

as an operator function

$$\sigma_1(a)(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge). \tag{3.5.24}$$

Relation (3.5.23) is also-called twisted homogeneity of order μ. Expression (3.5.22) is motivated by relation

$$\sigma_1(a)(y,\eta) = \lim_{\delta\to\infty} \delta^{-\mu}\kappa_\delta^{-1} a(y,\delta\eta)\kappa_\delta \tag{3.5.25}$$

for the amplitude function (3.5.16). By virtue of

$$a(y,\eta) \in C^\infty(\Omega, L^{\mu}_{\mathrm{cl}}(X;\mathbb{R}^q)) \tag{3.5.26}$$

we have parameter-dependent homogeneous principal symbol, namely, $\sigma_0(a)(r,x,y,\xi,\rho,\eta)$ of standard homogeneity of order μ for $(\xi,\rho,\eta) \neq 0$. In addition we define a subordinate symbol comming from the interpretation of (3.5.24) as an operator in the cone algebra over $X^\wedge$, called the principal conormal symbol

$$\sigma_{\mathrm{c}}(\sigma_1(a)(y,\eta))(v) := h_{0,0}(y,v) := \tilde{h}(0,y,v,0).$$

Let us give some further comment on the Mellin-edge quantization of operator families (3.5.11). The point is that, besides the degenerate behaviour of symbols there is no straightforward control up to $r = 0$ of the associated mapping property in Sobolev spaces (the nature of which also belong to the results of the edge calculus, see material below). The situation reminds of the transmission property which ensures via truncation quantization (i.e., in terms of operators $\mathrm{e}^+, \mathrm{r}^+$) which entails continuity of operators in standard Sobolev spaces. Note that a smooth manifold with boundary is a special case of a manifold with edge. The inner normal $\overline{\mathbb{R}}_+$ may be regarded as a cone, also referred to as the model cone of local wedges. The edge case is geometrically characterized by model cones X^Δ rather than $\overline{\mathbb{R}}_+$ where X is a smooth closed manifold of dimension $n > 0$. Moreover, the edge case is dealing with edge-degenerate symbols. If turned out (from the "early" formulations of edge calculus on) that the method of truncation quantization does not work and "usual" Sobolev spaces with a control up to $r = 0$ are not the right choice. So there was to be invented another quantization, i.e., a replacement of (3.5.10) in r-direction (modulo smoothing remainders) which works in alternative adequate scales of spaces. The answer is given in stretched variables where the cone X^Δ was blown up to $X^\wedge = \mathbb{R}_+ \times X$ (the open stretched cone) and the quantization in r-direction is formulated by the Mellin transform on $\mathbb{R}_+$. This is just the motivation of Mellin quantization in the cone calculus, i.e., over $X^\wedge$, which is organized by a suitable replacement of the (degenerate) Fourier-symbol p by a Mellin symbol h such that relation (3.5.13) holds, where the edge-calculus also contributes dependence on (y, η). Expression (3.5.14) is now another modification which is a mix between Mellin action close to $r = 0$ and the Fourier action far from $r = 0$ which once again leaves smoothing remainders compared with what we have on the left hand side of (3.5.13). In this case the decision was to employ η-depending cut-offs because of some twisted homogeneity properties of the involved operator functions. At the same time the operator functions (3.5.15) became operator-valued symbols, see Proposition 3.5.1, with Kegel spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ which are important for this approach. Let us postpone for the moment the role of those spaces for the future wedge spaces. In any case, at this point we see that the spaces $H^s(\mathbb{R}_+)$ from the boundary symbolic calculus of Chapter 2 are replaced in the edge symbolic calculus by weighted Sobolev spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$, where the choice of weights $\gamma \in \mathbb{R}$ is a consequence of requiring invertibility of conormal symbols in ellipticity of the cone calculus over X^Δ. Now another step in the development was established in the paper [18], where the Mellin-edge quantization (3.5.14) was replaced by

$$r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(h)(y, \eta),$$

at least, what concerned the edge symbolic calculus, where the complicated expressions (3.5.22) could be replaced by

$$\sigma_1(a)(y, \eta) = r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(h_0)(y, \eta) \tag{3.5.27}$$

with the same h_0 as before. The surprising aspect here is that (3.5.27) is a family of continuous maps like (3.5.24) although the Mellin action is containing a weight shift from γ to $\gamma - \mu$ up to $r = \infty$, which seems to be a contradiction to the nature of Kegel spaces which are equal to $H^s_{\mathrm{cone}}(X^\wedge)$, and the latter ones do not refer to any weight. The explanation in this case is that different ingredients are interplaying in the right way, namely, the holomorphy of Mellin symbols in v, the edge-degenerate parameter $r\eta$, where $\eta \neq 0$, and the weight factor $r^{-\mu}$. All this together shows as is done in [18], that the Fourier-part of (3.5.14) and a similar operator function which replaces $r^{-\mu}\mathrm{Op}_{F_r}(p)(y, \eta)$ (together with the cut-offs) by $r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(h)(y, \eta)$

only causes a difference by a Green symbol which is flat at $r = 0$ of infinite order. This explains why the weight effect at $r = \infty$ is ignored by (3.5.27) since Green symbols are Schwartz functions in r at infinity. These constructions were helpful also in the proof of the algebra property of the edge calculus. Nevertheless the role of $r^{-\mu}\mathrm{Op}_{F_r}(p)(y,\eta)$ played an important role since a pure Mellin description of operators did not suggest at once that (and why) weight effects at $r = \infty$ could be ignored.

Subsequent applications in the context of parabolicity, where the η-dependent cut-offs were a really irritating point, showed together with the above-mentioned insight, that another edge-quantization is easier manageable, namely, after Mellin-quantization which produces $h(r,y,v,\eta) = \tilde{h}(r,y,v,r\eta)$ for

$$\tilde{h}(r,y,v,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, M^\mu_{\mathcal{O}_v}(X;\mathbb{R}^q_{\tilde{\eta}})) \tag{3.5.28}$$

from $p(r,y,\rho,\eta) = \tilde{p}(r,y,r\rho,r\eta)$ and

$$\tilde{p}(r,y,\tilde{\rho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}))$$

we may replace (3.5.14) by

$$r^{-\mu}\{\omega\mathrm{Op}_M^{\gamma-n/2}(h)(y,\eta)\omega' + (1-\omega)\mathrm{Op}_r(p)(y,\eta)(1-\omega'')\}, \tag{3.5.29}$$

see, [40] concerning parabolicity in the edge calculus, which deals with anisotropic versions of the symbol class of Proposition 3.5.1 and which reflects the original operator modulo a smoothing one over $X^\wedge$.

3.6 The principal symbolic hierarchy and the edge calculus

As we noted before the amplitude function (3.5.16) as well as other variants which we discussed are the result of a so-called Mellin-edge quantization of the operator function (3.5.11), motivated by the shape of edge-degenerate differential operators of the form (3.5.3) for $M = \mathfrak{M}_1$ for $\dim s_1(M) = q > 0$, and $X \in \mathfrak{M}_0$ and by the program to express parametrices of elliptic elements within the edge calculus. We systematically refer to the Mellin-edge quantization, simpler than (3.5.16), namely, we write

$$r^{-\mu}\omega\mathrm{Op}_M^{\gamma-n/2}(h)(y,\eta)\omega' \tag{3.6.1}$$

for $h(r,y,v,\eta) := \tilde{h}(r,y,v,r\eta)$ for some $\tilde{h}(r,y,v,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M^\mu_{\mathcal{O}_v}(X;\mathbb{R}^q_{\tilde{\eta}}))$. Let us now generate step by step the remaining ingredients of the operators of the edge calculus.

Remark 3.6.1. *Similarly as Proposition* 3.5.1 *we have for the operator function* (3.6.1) *the relation*

$$a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)). \tag{3.6.2}$$

Definition 3.6.2. *Let M be a manifold with edge $Y = s_1(M)$, fix cut-off function $\omega'' \prec \omega \prec \omega'$, and set, as usual $s_0(M) = M \setminus s_1(M)$. The edge pseudo-differential algebra is furnished by spaces*

$$L^\mu(M, \boldsymbol{g}) \tag{3.6.3}$$

for weight data $\boldsymbol{g} := (\gamma, \gamma - \mu, \Theta)$ *with* $\Theta := (-(l+1), 0]$ *being a weight interval. The space* (3.6.3) *for any* $\mu \in \mathbb{R}$ *is defined as the set of operators*

$$A := H + M + G + A_{\mathrm{int}} + C \tag{3.6.4}$$

where $M + G$ *are so-called smoothing Mellin plus Green operators, belonging to* $L^{\mu}_{M+G}(M, \boldsymbol{g})$ *and* $C \in L^{-\infty}(M, \boldsymbol{g})$*, to be defined below, moreover,* $A_{\mathrm{int}} \in (1-\omega)L^{\mu}_{\mathrm{cl}}(s_0(M))(1-\omega'')$ *and* H *is a sum over* φ, φ' *of operators of the form*

$$\varphi \mathrm{Op}_y\{r^{-\mu}\omega \mathrm{Op}_{M_r}^{\gamma-n/2}(h)(y,\eta)\omega'\}\varphi' \tag{3.6.5}$$

for $\varphi, \varphi' \in C_0^{\infty}(\mathbb{R}^q)$*, referring to an open covering of* Y *by coordinate neighbourhoods* $U \cong \mathbb{R}^q$ *and functions* φ *from a subordinate partition of unity and other localizing functions* φ' *of compact support, where* $\varphi \prec \varphi'$*, and* $h(r, y, v, \eta) = \tilde{h}(r, y, v, r\eta)$ *for elements* $\tilde{h}(r, y, v, \tilde{\eta}) \in C^{\infty}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, M^{\mu}_{\mathcal{O}_v}(X; \mathbb{R}^q_{\tilde{\eta}}))$*.*

Definition 3.6.2 implies

$$L^{\mu}(M, \boldsymbol{g}) \subset L^{\mu}_{\mathrm{cl}}(s_0(M)).$$

Therefore, for every $A \in$ (3.6.3) we have a homogeneous principal symbol in the standard sense $\sigma_0(A)$ as an invariantly defined function $T^*(s_0(M)) \setminus 0$, and the so-called edge symbol

$$\sigma_1(A)(y,\eta) = \sigma_1(H)(y,\eta) + \sigma_1(M+G)(y,\eta) \tag{3.6.6}$$

for $h_0(r, y, v, \eta) := \tilde{h}(0, y, v, r\eta)$

$$\sigma_1(H)(y,\eta) = r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(h_0)(y,\eta) \tag{3.6.7}$$

for $\eta \neq 0$ and the Mellin plus Green contribution $M + G$ contained in (3.6.4). We easily see that (3.6.7) satisfies the homogeneity relation

$$\sigma_1(A)(y, \delta\eta) = \delta^{\mu}\kappa_{\delta}\sigma_1(A)(y,\eta)\kappa_{\delta}^{-1} \tag{3.6.8}$$

for all $\delta \in \mathbb{R}_+$.

Summing up operators $A \in L^{\mu}(M, \boldsymbol{g})$ have a pair of principal symbols

$$\sigma(A) := (\sigma_0(A), \sigma_1(A)) \tag{3.6.9}$$

where $\sigma_0(A)$ is the scalar homogeneous principal symbol over $M \setminus Y$, mentioned before, and

$$\sigma_1(A)(y,\eta) \in S^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); \mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge})) \tag{3.6.10}$$

defined by (3.6.2). Summing up the operator $H + M + G$ may be locally in $\mathbb{R}^q$ written as $\mathrm{Op}_y(a)$ modulo a smoothing remainder for an amplitude function $a(y, \eta) \in S^{\mu}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}))$ with principal part (3.6.10) in the sense of relation (3.6.1). The symbol property for $a(y, \eta)$ gives rise to continuity of operators $\mathrm{Op}_y(a)$ in weighted edge Sobolev spaces.

Let us first recall that if H is a separable Hilbert space with group action we have the space

$$\mathcal{W}^s(\mathbb{R}^q, H)$$

with the norm

$$||u||_{\mathcal{W}^s(\mathbb{R}^q,H)} = \left\{ \int \langle\eta\rangle^{2s} ||\kappa^{-1}_{\langle\eta\rangle}(Fu)(\eta)||^2_H \, đ\eta \right\}^{1/2}. \tag{3.6.11}$$

For $H := \mathcal{K}^{s,\gamma}(X^\wedge)$ this gives rise to the spaces $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$ for any $s, \gamma \in \mathbb{R}$. Using coordinate invariance in variables y and a partition of unity on Y we can introduce spaces

$$\mathcal{W}^s(Y, \mathcal{K}^{s,\gamma}(X^\wedge)).$$

Those have the property

$$\varphi \mathcal{W}^s(Y, \mathcal{K}^{s,\gamma}(X^\wedge)) \subset H^s_{\mathrm{loc}}(M \setminus Y),$$

for any $\varphi \in C^\infty_0(M \setminus Y)$ see [53, Proposition 3.1.21]. Thus it makes sense to define for any compact manifold M with edge Y the spaces

$$H^{s,\gamma}(M) := \{\omega u + (1-\omega)v : u \in \mathcal{W}^s(Y, \mathcal{K}^{s,\gamma}(X^\wedge)), v \in H^s_{\mathrm{loc}}(M \setminus Y)\} \tag{3.6.12}$$

where ω is any smooth function on $M \setminus Y$ which is identically to 1 close to Y and vanishes outside some neighbourhood of Y. In the non-compact case we have comp/loc-analogues of the spaces (3.6.12).

Theorem 3.6.3. *Any $A \in L^\mu(M, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$ and compact M induces continuous operators*

$$A : H^{s,\gamma}(M) \longrightarrow H^{s-\mu,\gamma-\mu}(M) \tag{3.6.13}$$

for any $s \in \mathbb{R}$. In the non-compact case we have analogues of such mapping properties between corresponding comp/loc-*spaces.*

Theorem 3.6.4. *Let $A \in L^\mu(M, \boldsymbol{a})$, $B \in L^\nu(M, \boldsymbol{b})$ for a compact space M with edge, where*

$$\boldsymbol{a} = (\gamma, \gamma-\mu, \Theta), \quad \boldsymbol{b} = (\gamma-\mu, \gamma-(\mu+\nu), \Theta). \tag{3.6.14}$$

Then we have $AB \in L^{\mu+\nu}(M, \boldsymbol{a} \circ \boldsymbol{b})$ for $\boldsymbol{a} \circ \boldsymbol{b} = (\gamma, \gamma-(\mu+\nu), \Theta)$, where

$$\sigma_i(AB) = \sigma_i(A)\sigma_i(B) \tag{3.6.15}$$

for $i = 0, 1$. Moreover, if A or B belongs to the corresponding L^μ_{G}- or $L^\mu_{\mathrm{M+G}}$-subclass then the product has this property as well.

We can (and will) also employ an alternative definition of Kegel spaces, namely

$$\mathcal{K}^{s,\gamma}(X^\wedge) = r^s \mathrm{Op}_M^{-n/2}(h)(\eta)\big(\mathcal{K}^{0,\gamma-s}(X^\wedge)\big) \tag{3.6.16}$$

$\eta \neq 0$, fixed where $\mathcal{K}^{0,\gamma-s}(X^\wedge) = \mathbb{K}^{\gamma-s}\mathcal{K}^{0,0}(X^\wedge)$ for any strictly positive function on $\mathbb{R}_+$ which is equal to $r^{\gamma-s}$ for $0 < r < \varepsilon_0$ and 1 for $r > \varepsilon_1$, for some $0 < \varepsilon_0 < \varepsilon_1$, where $h(r, v, \eta) = \tilde{h}(v, r\eta)$ is defined by a suitable parameter-dependent elliptic element $\tilde{h}(v, \tilde{\eta}) \in M^\mu_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}})$ of the restriction of Γ_β with respect to v, which is an element of $L^\mu_{\mathrm{cl}}(X; \Gamma_\beta \times \mathbb{R}^q)$ cf. formula (3.4.3) and the comment after Definition 3.4.1. It is independent of $\beta \in \mathbb{R}$, i.e., the above-mentioned notion of ellipticity of $h(v, \tilde{\eta})$ is independent of β. Applying this procedure to $h(v, \tilde{\eta}) := \tilde{h}(0, y, v, \tilde{\eta})$ for some fixed $y \in Y$ we obtain a notion of ellipticity of the operator

$\sigma_1(A)(y,\eta)$ for any fixed y, and $\eta \neq 0$. The y-independent Mellin symbol in relation (3.6.16) is defined by $h_0(r,v,\eta) := \tilde{h}(0,v,r\eta)$. The function

$$\sigma_c(\sigma_1(A))(y,v) \in L^\mu_{cl}(X;\Gamma_\beta)$$

for $v \in \Gamma_{\frac{n+1}{2}-\gamma}$ is called the principal conormal symbol of the operator A, and

$$\sigma_1(A)(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{3.6.17}$$

is Fredholm if and only if the above-mentioned $\tilde{h}(y,v,\tilde{\eta})$ is elliptic in $M^\mu_{\mathcal{O}}(X;\mathbb{R}^q_{\tilde{\eta}})$ and

$$\sigma_c(\sigma_1(A))(y,v) : H^s(X) \longrightarrow H^{s-\mu}(X) \tag{3.6.18}$$

for $v \in \Gamma_{\frac{n+1}{2}-\gamma}$ is bijective for $|\operatorname{Im} v|$ sufficiently large. The principal conormal symbols are y-dependent holomorphic operator functions with values in $L^\mu_{cl}(X)$. For instance, if our operator A is edge-degenerate, i.e.,

$$A = r^{-\mu} \sum_{j+|\alpha|\leq\mu} a_{j\alpha}(r,y)\Big(-r\frac{\partial}{\partial r}\Big)^j (rD_y)^\alpha \tag{3.6.19}$$

then the amplitude function (3.6.1) which is producing A by $A = \operatorname{Op}_{y\to\eta}(a)$ contains the Mellin symbol

$$h(r,y,v,\eta) = \tilde{h}(r,y,v,r\eta)$$

for $\tilde{h}(r,y,v,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, M^\mu_{\mathcal{O}_v}(X;\mathbb{R}^q_{\tilde{\eta}}))$ where

$$\tilde{h}(r,y,v,r\eta) = \sum_{j+|\alpha|\leq\mu} a_{j\alpha}(r,y)v^j(r\eta)^\alpha. \tag{3.6.20}$$

In the context of ellipticity we have invertiblity of (3.6.18) for large $|\operatorname{Im} v|$ and for any fixed $y \in \mathbb{R}^q$, and $\operatorname{Re} v = \beta \in \mathbb{R}$ uniformly in compact intervals. In the present case we have

$$\sigma_c(\sigma_1(\cdot))(y,v) = \sum_{j=0}^{\mu} a_{j0}(0,y)v^j. \tag{3.6.21}$$

Applying kernel cut-off to $\sigma_c(\sigma_1(\cdot))^{-1}(y,v)$ gives us a holomorphic family again, modulo y-dependent meromorphic remainders which are smoothing, i.e., take values in $M^{-\infty}_{\mathcal{R}}(X)$ for some pattern of y-dependent poles r_j including multiplicities m_j+1 see Definition 3.7.4 (i) below for

$$\mathcal{R} = \{(r_j,m_j)\}_{j\in\mathbb{I}} \subset \mathbb{C}\times\mathbb{N}. \tag{3.6.22}$$

In the simplest case we assume in this construction that the non-bijectivity points of (3.6.18) including their multiplicities are y-independent. Then the components of (3.6.22) are y-independent as well. We then call (3.6.22) a constant discrete Mellin asymptotic type.

3.7 The asymptotic part of the edge calculus

In order to explain the global smoothing operators C in formula (3.6.4) of the edge algebra we establish weighted spaces with asymptotics, for the moment with constant discrete asymptotics, i.e., constant with respect to the variable y along the edge.

Definition 3.7.1. *A sequence*

$$\mathcal{P} := \{(p_j, m_j)\}_{j\in\mathbb{J}} \subset \mathbb{C} \times \mathbb{N} \tag{3.7.1}$$

for an index set $\mathbb{J} \subseteq \mathbb{N} \cup \{+\infty\}$ *is said to be a discrete asymptotic type associated with weight data* (γ, Θ) *for a weight* $\gamma \in \mathbb{R}$ *and a weight interval* $\Theta = (\vartheta, 0]$, *for some* $\vartheta \in \mathbb{R}_- \cup \{-\infty\}$, *if*

$$\pi_{\mathbb{C}}\mathcal{P} = \{p_j\}_{j\in\mathbb{J}} \tag{3.7.2}$$

is finite for $\vartheta > -\infty$ *and*

$$\pi_{\mathbb{C}}\mathcal{P} \subset \{v \in \mathbb{C} : \frac{n+1}{2} - \gamma + \vartheta < \operatorname{Re} v < \frac{n+1}{2} - \gamma\} \tag{3.7.3}$$

for $n = \dim X$ *while for* $\vartheta = -\infty$ *we ask* $\operatorname{Re} p_j \to -\infty$ *as* $j \to \infty$, *when* $\pi_{\mathbb{C}}\mathcal{P}$ *is infinite.*

A $\mathcal{P}$ as in Definition 3.7.1 is said to satisfy the shadow condition if $(p, m) \in \mathcal{P}$ implies $(p - l, m) \in \mathcal{P}$ for every $l \in \mathbb{N}$ such that

$$\operatorname{Re} p - l > \frac{n+1}{2} - \gamma + \vartheta.$$

Later on, in applications, our asymptotic types will automatically satisfy the shadow condition. Therefore, in order to simplify the formalism, from now on we impose the shadow condition as an assumption. Then $\pi_{\mathbb{C}}\mathcal{P}$ will be infinite for $\Theta = (-\infty, 0]$, while for finite Θ it suffices to take $\Theta = (-(\theta + 1), 0]$ for some $\theta \in \mathbb{N}$. Singular functions on $X^\wedge$ for finite Θ will be formulated in terms of the space

$$\mathcal{E}_{\mathcal{P}}(X^\wedge) := \{\omega(r) \sum_{j\in\mathbb{J}} \sum_{l=0}^{m_j} c_{jl}(x) r^{-p_j} \log^l r : c_{jl} \in C^\infty(X)\} \tag{3.7.4}$$

for some cut-off function ω on the $\mathbb{R}_+$ half-axis. We consider (3.7.4) in its natural Fréchet topology of finitely many copies of $C^\infty(X)$. We may and will replace $\sum_{j\in\mathbb{J}}$ in (3.7.4) by $\sum_{j=0}^N$ for some $N \in \mathbb{N} \cap \{+\infty\}$. Note that

$$\mathcal{E}_{\mathcal{P}}(X^\wedge) \subset \mathcal{K}^{s,\gamma}(X^\wedge)$$

for every $s \in \mathbb{R}$, i.e.,

$$\mathcal{E}_{\mathcal{P}}(X^\wedge) \subset \mathcal{K}^{\infty,\gamma}(X^\wedge).$$

Also

$$\mathcal{K}^{s,\gamma}_{\Theta}(X^\wedge) := \varprojlim_{\epsilon>0} \mathcal{K}^{s,\gamma+\theta+1-\epsilon}(X^\wedge) \tag{3.7.5}$$

is Fréchet as a projective limit of Fréchet spaces, and (3.7.5) is direct to (3.7.4). Then

$$\mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge) := \mathcal{K}^{s,\gamma}_{\Theta}(X^\wedge) + \mathcal{E}_{\mathcal{P}}(X^\wedge) \tag{3.7.6}$$

is Fréchet in the topology of the direct sum. For $\Theta = (-\infty, 0]$ and $\mathcal{P}$ associated with (γ, Θ) we first form

$$\mathcal{P}_k := \{(p,m) \in \mathcal{P} : \frac{n+1}{2} - \gamma - (k+1) < \operatorname{Re} p < \frac{n+1}{2} - \gamma\} \tag{3.7.7}$$

which is finite. Therefore, we have the spaces $\mathcal{K}^{s,\gamma}_{\mathcal{P}_k}(X^\wedge)$ and we set

$$\mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge) = \varprojlim_{k\in\mathbb{N}} \mathcal{K}^{s,\gamma}_{\mathcal{P}_k}(X^\wedge).$$

For any $e \in \mathbb{R}$ we set

$$\begin{aligned} \mathcal{K}^{s,\gamma;e}(X^\wedge) &:= [r]^{-e}\mathcal{K}^{s,\gamma}(X^\wedge), \\ \mathcal{K}^{s,\gamma;e}_{\mathcal{P}}(X^\wedge) &:= [r]^{-e}\mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge). \end{aligned} \tag{3.7.8}$$

, Given $\mathcal{P}$ we form weighted edge spaces

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge))$$

using the fact that $\mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)$ is a Fréchet space with group action. The space $\mathcal{E}_{\mathcal{P}}(X^\wedge)$ is not group invariant. Nevertheless we can form spaces

$$H^s(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}}(X^\wedge))$$

and, because of

$$T : \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)) \longrightarrow H^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)) \tag{3.7.9}$$

for the operator $T(\eta) := F^{-1}\kappa^{-1}_{[\eta]}F$, generated by a reformulation of norms

$$\left\{ \int [\eta]^{2s} ||\kappa^{-1}_{[\eta]}(Fu)(\eta)||^2_H \, đ\eta \right\}^{1/2} = \left\{ \int [\eta]^{2s} ||F(F^{-1}\kappa^{-1}_{[\eta]}F)u||^2_H \, đ\eta \right\}^{1/2}, \tag{3.7.10}$$

cf. Remark 1.5.3, we obtain $\mathcal{W}^s(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}}(X^\wedge))$ as a Fréchet subspace of $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$ for any $s \in \mathbb{R}$. Thus we can define $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge))$ also as a non-direct sum

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)) = \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\Theta}(X^\wedge)) + \mathcal{W}^s(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}}(X^\wedge)). \tag{3.7.11}$$

Both summands refer to the group action $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ which is on the right-hand side of (3.7.11) we have a representation of any $u \in \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge))$ into

$$u = u_\Theta + u_{\mathcal{P}} \tag{3.7.12}$$

where $u_\Theta \in \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\Theta}(X^\wedge))$ is the flat part of u with respect to the weight $\gamma \in \mathbb{R}$ and $u_{\mathcal{P}} \in \mathcal{W}^s(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}}(X^\wedge))$ are the singular functions of constant $T^{-1}H^s(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}}(X^\wedge))$ discrete edge asymptotics. In other words the elements $u_{\mathcal{P}} \in F^{-1}\kappa_{[\eta]}FH^s(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}}(X^\wedge)) = F^{-1}\kappa_{[\eta]}FH^s(\mathbb{R}^q)\hat{\otimes}_\pi\mathcal{E}_{\mathcal{P}}(X^\wedge)$ take the form of

$$u_{\mathcal{P}}(r,y) = F^{-1}\omega(r[\eta])[\eta]^{\frac{n+1}{2}} \sum_{j=0}^{N}\sum_{l=0}^{m_j} c_{jl}(x)(r[\eta])^{-p_j}\log^l(r[\eta])\hat{v}(\eta) \tag{3.7.13}$$

for $v \in H^s(\mathbb{R}^q)$. Thus the asymptotic terms of the edge calculus are specific classical pseudo-differential operators of potential type

$$K_{\mathcal{P}} : H^s(\mathbb{R}^q) \longrightarrow \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)) \tag{3.7.14}$$

$K_{\mathcal{P}} = F^{-1}_{\eta \to y} f(\eta) F_{y \to \eta}$ with $f(\eta)$ being a linear combination of symbols of

$$f(\eta)_{jl} := \kappa_{[\eta]}(\omega(r) c_{jl}(x) r^{-p_j} \log^l r) = \mathrm{Op}_y(f) \tag{3.7.15}$$

for

$$f(\eta) : c \longrightarrow \sum \sum \omega(r[\eta])[\eta]^{\frac{n+1}{2}} c_{jl}(x)(r[\eta])^{-p_j} \log^l (r[\eta])c. \tag{3.7.16}$$

for $c \in \mathbb{C}$. Remember that $H^s(\mathbb{R}^q) = \mathcal{W}^s(\mathbb{R}^q, \mathbb{C})$. An alternative definition of $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge))$ is also

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)) = \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\Theta}(X^\wedge)) + \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{\infty,\gamma}_{\mathcal{P}}(X^\wedge)) \tag{3.7.17}$$

as a non-direct sum, using the group action (3.5.19) on

$$\mathcal{K}^{\infty,\gamma}_{\mathcal{P}}(X^\wedge) = \mathcal{K}^{\infty,\gamma}_{\Theta}(X^\wedge) + \mathcal{E}_{\mathcal{P}}(X^\wedge). \tag{3.7.18}$$

The definition of spaces $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge))$ has also a global meaning on a compact smooth manifold Y, locally modeled on $\mathbb{R}^q$. In other words we obtain the spaces $\mathcal{W}^s(Y, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge))$ and $\mathcal{W}^s(Y, \mathcal{K}^{s,\gamma}(X^\wedge))$, using a partition of unity on Y and corresponding charts to $\mathbb{R}^q$. On a manifold M with edge Y we also have the spaces $H^s_{\mathrm{loc}}(M \setminus Y)$ and the definition of the spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ shows that

$$\varphi \mathcal{W}^s(Y, \mathcal{K}^{s,\gamma}(X^\wedge)) = \varphi H^s_{\mathrm{loc}}(M \setminus Y) \tag{3.7.19}$$

for any $\varphi \in C_0^\infty(M \setminus Y)$. Thus for any cut-off function ω on M it makes sense to define the spaces

$$H^{s,\gamma}(M) := \omega \mathcal{W}^s(Y, \mathcal{K}^{s,\gamma}(X^\wedge)) + (1-\omega) H^s_{\mathrm{loc}}(M \setminus Y) \tag{3.7.20}$$

as a non-direct sum which is independent of the choice of ω. Analogously we set

$$H^{s,\gamma}_{\mathcal{P}}(M) := \omega \mathcal{W}^s(Y, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)) + (1-\omega) H^s_{\mathrm{loc}}(M \setminus Y) \tag{3.7.21}$$

for any discrete asymptotic type $\mathcal{P}$.

Definition 3.7.2. *An operator C belong to $L^{-\infty}(M, \boldsymbol{g})$ if C induces continuous operators*

$$\begin{aligned} C &: H^{s,\gamma}(M) \longrightarrow H^{\infty,\gamma-\mu}_{\mathcal{P}}(M), \\ C^* &: H^{s,-\gamma+\mu}(M) \longrightarrow H^{\infty,-\gamma}_{\mathcal{Q}}(M) \end{aligned} \tag{3.7.22}$$

for C-depending asymptotic types $\mathcal{P}$ and $\mathcal{Q}$, associated with $(\gamma-\mu, \Theta)$ and $(-\gamma, \Theta)$, respectively.

Let us now pass to smoothing Mellin plus Green operators of the edge calculus by introducing corresponding operator-valued symbols.

Definition 3.7.3. *A* $g(y,\eta) \in S^{\nu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$ *for some* $s \in \mathbb{R}$ *is called a Green symbol if*

$$\begin{aligned} g(y,\eta) &\in \bigcap_{s,e\in\mathbb{R}} S^{\nu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma;e}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu;\infty}_{\mathcal{P}}(X^\wedge)), \\ g^*(y,\eta) &\in \bigcap_{s,e\in\mathbb{R}} S^{\nu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,-\gamma+\mu;e}(X^\wedge), \mathcal{K}^{\infty,-\gamma;\infty}_{\mathcal{Q}}(X^\wedge)) \end{aligned} \tag{3.7.23}$$

for every $s \in \mathbb{R}$*, and asymptotic types* $\mathcal{P}, \mathcal{Q}$ *associated with* $(\gamma-\mu, \Theta)$ *and* $(-\gamma, \Theta)$*, respectively.*

Let

$$R^{\nu}_{\mathrm{G}}(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g}) \tag{3.7.24}$$

denote the space of all $g(y,\eta)$ with the indicated properties. Those are called Green amplitude functions of order ν. The weight shift μ is not affected by ν.

Green symbols allow us to define Green operators of the class $L^{\nu}_{\mathrm{G}}(M, \boldsymbol{g})$ for any $\nu \in \mathbb{R}$ while μ in $\boldsymbol{g}$ is given independently. Such an operator G has a kernel in $C^\infty(s_0(M) \times s_0(M))$ such that for any open covering of Y by coordinate neighbourhoods $(U_1, \dots, U_N)$ and "charts"

$$\chi_j : U_j \times X^\wedge \longrightarrow \mathbb{R}^q \times X^\wedge$$

a subordinate partition of unity $(\varphi_1, \dots, \varphi_N)$ and a system of functions $(\varphi'_1, \dots, \varphi'_N)$, $\varphi_j \prec \varphi'_j \in C^\infty_0(U_j)$ we have

$$G = \sum_{j=1}^{N} \varphi_j (\chi_j^{-1})_* \operatorname{Op}_y(g_j)\, \varphi'_j + C \tag{3.7.25}$$

for some Green symbols $g_j(y,\eta), j = 1, \dots, N$, and a $C \in L^{-\infty}(M, \boldsymbol{g})$.

Let us now introduce smoothing Mellin operators of the edge calculus. For this and we first define the space $M^{-\infty}_{\mathcal{R}}(X)$ for some Mellin asymptotic type $\mathcal{R}$ which is a sequence of pairs

$$\mathcal{R} := \{(r_j, n_j)\}_{j\in\mathbb{I}} \subset \mathbb{C} \times \mathbb{N} \tag{3.7.26}$$

for any index set $\mathbb{I} \subseteq \mathbb{Z} \cup \{-\infty\} \cup \{+\infty\}$ such that $\pi_{\mathbb{C}}\mathcal{R} := \{r_j\}_{j\in\mathbb{I}}$ intersect every strip

$$\{v \in \mathbb{C} : |\operatorname{Re} v| \le C\}$$

in a finite set, for any $C > 0$.

Definition 3.7.4. (i) *The space* $M^{-\infty}_{\mathcal{R}}(X)$ *is defined as the set of all meromorphic functions*

$$f(v) \in \mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}}\mathcal{R}, L^{-\infty}(X))$$

which have poles at $r_j \in \pi_{\mathbb{C}}\mathcal{R}$ *of multiplicity* $m_j + 1$*, where the Laurent coefficients at* r_j *are operators of finite rank, and for any* $\pi_{\mathbb{C}}\mathcal{R}$*-excision function* χ *(i.e.,* $\chi(v) \in C^\infty(\mathbb{C}), \chi(v) = 0$ *for* $\operatorname{dist}(v, \pi_{\mathbb{C}}\mathcal{R}) < \varepsilon_0, \chi(v) = 1$ *for* $\operatorname{dist}(v, \pi_{\mathbb{C}}\mathcal{R}) > \varepsilon_1$ *for some* $0 < \varepsilon_0 < \varepsilon_1 < \infty$*) we have*

$$\chi(v) f(v)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta, L^{-\infty}(X)) \tag{3.7.27}$$

for every $\beta \in \mathbb{R}$*, uniformly in compact* β*-intervals.*

(ii) *The space* $M^\mu_{\mathcal{R}}(X)$ *is defined as the non-direct sum*

$$M^\mu_{\mathcal{O}}(X) + M^{-\infty}_{\mathcal{R}}(X) \tag{3.7.28}$$

for any Mellin asymptotic type $\mathcal{R}$ *and* $\mu \in \mathbb{R}$, *cf. Definition* 3.4.1

Note that (3.7.28) is non-direct; since

$$M^\mu_{\mathcal{O}}(X) \cap M^{-\infty}_{\mathcal{R}}(X) = M^{-\infty}_{\mathcal{O}}(X).$$

Smoothing Mellin symbols are the raw material of some operator-valued symbols of the kind

$$m(y,\eta) := r^{-\mu}\omega_\eta \sum_{j=0}^{\theta} r^j \sum_{|\alpha|\le j} \mathrm{Op}_M^{\gamma_{j\alpha}-n/2}(f_{j\alpha})(y)\eta^\alpha \omega'_\eta \tag{3.7.29}$$

where $\Theta = ((-\theta+1),0]$ for a sequence of Mellin symbols $f_{j\alpha}(y,v) \in C^\infty(\Omega, M^{-\infty}_{\mathcal{R}_{j\alpha}}(X))$ and $\omega_\eta(r) = \omega([\eta]r), \omega'_\eta(r) = \omega'([\eta]r)$ for some cut-off functions ω, ω'. In order that (3.7.29) is well-defined as a family of continuous operators

$$m(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge) \tag{3.7.30}$$

we assume that the weights $\gamma_{j\alpha} \in \mathbb{R}$ satisfy the conditions $\gamma - j \le \gamma_{j\alpha} \le \gamma$ and $\pi_{\mathbb{C}} R_{j\alpha} \cap \Gamma_{\frac{n+1}{2}-\gamma_{j\alpha}} = \emptyset$ for all $j \in \mathbb{N}, \alpha \in \mathbb{N}^q$. Then

$$r^{-\mu} r^j \mathrm{Op}_M^{\gamma_{j\alpha}-n/2} : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow r^j \mathcal{K}^{\infty,\gamma_{j\alpha}-\mu}(X^\wedge) \hookrightarrow \mathcal{K}^{\infty,\gamma_{j\alpha}+j-\mu}(X^\wedge) \hookrightarrow \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge). \tag{3.7.31}$$

Note that when $\tilde{m}(y,\eta)$ is another expression like (3.7.29) but for modified $\tilde{\gamma}_{j\alpha}$ satisfying the above conditions and unchanged Mellin symbols $f_{j\alpha}(y)$ or alternative cut-off functions $\tilde{\omega}, \tilde{\omega}'$, then

$$m(y,\eta) - \tilde{m}(y,\eta)$$

is Green symbol in the sense of the above definition. In any case we have

$$m(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge)) \tag{3.7.32}$$

or

$$m(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu}_{\mathcal{Q}}(X^\wedge)) \tag{3.7.33}$$

for every asymptotic type $\mathcal{P}$ and some resulting $\mathcal{Q}$. The reason for the latter relations is the rule

$$\kappa_\delta \mathrm{Op}_M^{\gamma-n/2}(f)\kappa_\delta^{-1} = \mathrm{Op}_M^{\gamma-n/2}(f) \tag{3.7.34}$$

for any $\delta \in \mathbb{R}_+$ and

$$\kappa_\delta r^{-\mu} \kappa_\delta^{-1} = \delta^{-\mu} r^{-\mu}. \tag{3.7.35}$$

Let us come back to Definition 3.6.2 and add more details on the operator classes $L^\mu_{\mathrm{M+G}}(M, \boldsymbol{g})$ which can be characterized close to $Y := s_1(M)$ in local variables $y \in \mathbb{R}^q$ by amplitude functions

$$m(y,\eta) + g(y,\eta) \in R^\mu_{\mathrm{M+G}}(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g}), \tag{3.7.36}$$

cf. formula (3.7.24), and $m(y,\eta)$ is of the form (3.7.29). Definition 3.7.3 and relations (3.7.32) show that for any fixed (y,η) the operators in (3.7.36) take values in smoothing Mellin plus Green operators of the cone calculus over the infinite stretched cone $X^\wedge$, cf. the considerations below which are to some extent devoted to a special case of edge calculus when $\dim s_1(M) = 0$. In this case $s_1(M)$ represents a conical singularity. It is important to emphasize this case, in order to illustrate more features of the edge calculus.

3.8 Ellipticity, parametrices, Fredholm property

Let us introduce the full spaces

$$R^{\mu}(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g}) \tag{3.8.1}$$

of the edge amplitude functions, containing (3.7.36) together with non-smoothing Mellin amplitude functions belonging to operators H in (3.6.4) which are up to globalization contained in (3.6.5), namely,

$$k(y, \eta) := r^{-\mu} \omega \mathrm{Op}_M^{\gamma - n/2}(h)(y, \eta) \omega'. \tag{3.8.2}$$

In other words, (3.8.1) is furnished by operator functions

$$k(y, \eta) + m(y, \eta) + g(y, \eta) \tag{3.8.3}$$

for such $k(y, \eta)$ defined by arbitrary $h(r, y, v, \eta) = \tilde{h}(r, y, v, r\eta)$ with $\tilde{h}$ being given in (3.5.28) and $m(y, \eta) + g(y, \eta)$ defined as before. These ingrediants come from the parts of the operators (3.6.4) close to the edge $s_1(M)$ and, as we indicated, determine the twisted homogeneous principal symbol $\sigma_1(A)(y, \eta), \eta \neq 0$.

For purposes below we make some further comment about $L^{\mu}_{\mathrm{M+G}}$-spaces and their symbolic structure. As noted before the edge calculus $L^{\mu}(M, \boldsymbol{g})$, cf. Definition 3.6.2, contains the subclasses L^{μ}_{G} and $L^{\mu}_{\mathrm{M+G}}$, and the corresponding amplitude functions (3.7.36) take values in operators on $X^{\wedge}$, acting between $\mathcal{K}^{s,\gamma}$-spaces and subspaces with asymptotics. (y, η)-wise they are also closed under algebraic operations, and in particular, there are composition rules. In addition, there are operations with the symbols, themselves, e.g., asymptotic summations. This material will tacitly be used; details may be found in [53]. For understanding parametrices it is instructive to realize operators of order zero of the form

$$1 + M + G \in 1 + L^0_{\mathrm{M+G}}(M, \boldsymbol{g}) \tag{3.8.4}$$

as a subcalculus of $L^0(M, \boldsymbol{g})$. Similarly as Theorem 3.6.4 we observe the following specialization.

Theorem 3.8.1. *Given operators $A \in 1 + L^0_{\mathrm{M+G}}(M, \boldsymbol{a}), B \in 1 + L^0_{\mathrm{M+G}}(M, \boldsymbol{b})$ over a compact space M with edge where $\boldsymbol{a}, \boldsymbol{b}$ are defined as in (3.6.14), then we have $AB \in 1 + L^0_{\mathrm{M+G}}(M, \boldsymbol{a} \circ \boldsymbol{b})$, and we can express*

$$\sigma_1(AB) = \sigma_1(A)\sigma_1(B)$$

in terms of the Mellin expressions involved in A and B, respectively. If $A \in 1 + L^0_{\mathrm{G}}$ or $B \in 1 + L^0_{\mathrm{G}}$, then the same is true of the composition.

The latter aspect has been mentioned already in Theorem 3.6.4, i.e., L_{G}- or $L_{\mathrm{M+G}}$-classes form subideals in the full edge calculus and hence also in $L_{\mathrm{M+G}}$. According to the width Θ of the weight interval smoothing Mellin operators also may play the role of Green operators. When we diminish $\Theta = ((-\theta + 1), 0]$ to $\Theta' := ((-\theta' + 1), 0]$ for $\theta' < \theta$, then we can split up the sum (3.7.29) into

$$m(y, \eta) = m'(y, \eta) + m''(y, \eta) \tag{3.8.5}$$

where m' indicates the sum up to θ' and m'' the one from $\theta' + 1$ to θ. The remark is then that $m''(y, \eta) \in R^{\mu}_{\mathrm{G}}(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g})$. In other words the smoothing Mellin part turns to a Green symbol when in the involved factors $(r')^j$ j becomes large enough. Also other observations are interesting, e.g., the classes (3.7.36) are closed under differentiations with respect to y or η, where

$$D_y^\alpha D_\eta^\beta R^\mu_{\mathrm{M+G}}(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g}) \subseteq R^{\mu-|\beta|}_{\mathrm{M+G}}(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g}) \tag{3.8.6}$$

and we have

$$D_\eta^\beta R^\mu_{\mathrm{G}}(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g}) \subseteq R^{\mu-|\beta|}_{\mathrm{G}}(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g}) \tag{3.8.7}$$

when $|\beta|$ is large enough. Another important point is to consider the conormal symbolic structure of Mellin plus Green symbols. We set

$$\sigma_{\mathrm{c}}^{-\mu+j}(m)(y, v, \eta) := \sum_{|\alpha|=j} f_{j\alpha}(y, v)\eta^\alpha, j = 0, \ldots, \theta. \tag{3.8.8}$$

The conormal symbols (3.8.8) are polynomials in η of order j, and hence they are independent of η for $j = 0$. $\sigma_{\mathrm{c}}^{-\mu}(m)(y, v)$ is called the principal conormal symbol of $m(y, \eta)$ in the sense of the subordinate cone calculus over $X^\wedge$. We do not list all properties in this connection, but we keep in mind that $m(y, \eta)$ is uniquely determined by the sequence of conormal symbols for $j = 0, \ldots, \theta$, modulo Green symbols, despite of the presence of cut-off factors $\omega_\eta(r), \omega'_\eta(r)$ or the choice of $\gamma_{j\alpha}$ with the properties after relation (3.7.30).

Later on in parametrix constructions in the edge calculus for elliptic edge operators, locally close to the edge written $\mathrm{Op}_y(a)$ for $a \in R^\mu(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g})$, cf. (3.8.1), we separately look at the subalgebra of operators (3.8.4).

Definition 3.8.2. *An operator* (3.8.4) *in* $1 + L^0_{\mathrm{M+G}}(M, \boldsymbol{g})$ *is called elliptic if*

$$1 + \sigma_{\mathrm{c}}^0(m)(y, v) \neq 0 \tag{3.8.9}$$

for all $v \in \Gamma_{\frac{n+1}{2}-\gamma}$ *and all* y.

Another aspect are then parametrices within

$$1 + R^0_{\mathrm{M+G}}(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g}) \tag{3.8.10}$$

where we need the composition result of Theorem 3.8.1 and the typical Mellin-Leibniz translation product. This will enable us below to establish Neumann series arguments for some step in parametrix constructions for the full edge calculus.

Since (y, η)-wise the operators in (3.8.1) are cone operators we can also fix (y, η) and formulate ellipticity of such operators of the form $1 + m + g$ in the sense of operators over $X^\wedge$. The corresponding cone algebra will be denoted by $L^\mu(X^\wedge, \boldsymbol{g}), L^\mu_{\mathrm{M+G}}(X^\wedge, \boldsymbol{g})$, etc., and in this particular situation below we also employ notation A, M, G when the objects in that sense are interpreted as cone operators, cf. notation (3.8.15) below. In particular, ellipticity of $1 + M + G \in 1 + L^0_{\mathrm{M+G}}(X^\wedge, \boldsymbol{g})$ is again given by Definition 3.8.2.

Let us formulate more details from this program. Roughly speaking, similarly as in parametrix constructions of standard elliptic pseudo-differential operators on a smooth manifold, cf. Section 1.1, here we refer to the symbolic structure of operators in $1 + L^0_{\mathrm{M+G}}(M, \boldsymbol{g})$ which is here equipped with some "floors", the one for the model cone and then for the associated wedge with cone-operator-valued symbols in $(y, \eta) \in \mathbb{R}^q \times \mathbb{R}^q$, cf. (3.8.10). A general insight in all contexts which are discussed in the parametrix construction set-up is that the structures in consideration are closed under multiplication and formal adjoint. In other words we need algebras, and our composition rules are doing exactly this. What concerns the level of cones, one composition observation is that the system of spaces $M_{\mathcal{R}}^{-\infty}(X)$ of Definition 3.7.4 is closed under composition of the corresponding meromorphic operator functions, i.e.,

$$f(v) \in M_{\mathcal{R}}^{-\infty}(X), h(v) \in M_{\mathcal{Q}}^{-\infty}(X) \tag{3.8.11}$$

for arbitrary Mellin asymptotic types implies

$$f(v)h(v) \in M_{\mathcal{S}}^{-\infty}(X) \tag{3.8.12}$$

for some resulting Mellin asymptotic type $\mathcal{S}$. A similar relation is true of operator functions containing the identity 1 as a summand, i.e.,

$$(1 + f(v))(1 + h(v)) = 1 + p(v) \tag{3.8.13}$$

for some $p(v) \in M_{\mathcal{P}}^{-\infty}(X)$ in obvious notation. There is a remarkable result as follows.

Proposition 3.8.3. *For every $f(v) \in M_{\mathcal{R}}^{-\infty}(X)$ there is an $l(v) \in M_{\mathcal{Q}}^{-\infty}(X)$ such that*

$$1 + l(v) = (1 + f(v))^{-1} \tag{3.8.14}$$

in the sense of the multiplications.

A proof may be found in [53], and also in the Ph-D thesis of Krainer [33]. Another important ingredient of constructing a parametrix in $1 + L^0_{\mathrm{M+G}}(M, \boldsymbol{g})$ is the so-called Mellin translation product which can be formulated for compositions mentioned in Theorem 3.6.4. In this connection we introduce the sequence of conormal symbols of operators in $L^\mu(M, \boldsymbol{g})$ in general. If we forget for the moment the edge then the amplitude functions (3.7.36) take values in the cone calculus, here over $X^\wedge$. The corresponding cone operators are of similar form as (3.6.4). Those form space $L^\mu(X^\wedge, \boldsymbol{g})$ of operators

$$A = H + M + G, \tag{3.8.15}$$

where in this case

$$H = r^{-\mu}\omega \mathrm{Op}_M^{\gamma - \frac{n}{2}}(h)\omega' \tag{3.8.16}$$

for an $h(r, v) \in C^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^\mu(X))$, and cut-off functions $\omega(r), \omega'(r)$ and M is a sum

$$M = r^{-\mu}\omega \sum_{j=0}^{\theta} r^j \mathrm{Op}_M^{\gamma_j - \frac{n}{2}}(f_j)\omega' \tag{3.8.17}$$

for Mellin symbols $f_j(v) \in M_{\mathcal{R}_j}^{-\infty}(X)$, $j = 1, \dots, \theta$, with the same conditions on γ_j as in (3.7.29) and Green operators G in this case satisfy the conditions

$$\begin{aligned} G &\in \bigcap_{s,e \in \mathbb{R}} \mathcal{L}(\mathcal{K}^{s,\gamma;e}(X^\wedge), \mathcal{K}_{\mathcal{P}}^{\infty,\gamma-\mu;\infty}(X^\wedge)) \\ G^* &\in \bigcap_{s,e \in \mathbb{R}} \mathcal{L}(\mathcal{K}^{s,-\gamma+\mu;e}(X^\wedge), \mathcal{K}_{\mathcal{Q}}^{\infty,-\gamma;\infty}(X^\wedge)) \end{aligned} \tag{3.8.18}$$

for G-dependent asymptotic types $\mathcal{P}$, $\mathcal{Q}$, similarly as in Definition 3.7.3. The operator A_{int} in this case may be dropped and also the smoothing operators C which are included in the class of Green operators.

Now we have the conormal symbols

$$\sigma_{\mathrm{c}}^{\mu-j}(A)(v) := \frac{\partial^j}{\partial r^j} h(r,v)|_{r=0} + f_j(v) \tag{3.8.19}$$

for $j = 0, \ldots, \theta$, and those, under compositions of cone operators satisfy the announced Mellin translation product.

Theorem 3.8.4 ([52]). *The sequences of conormal symbols of operators A and $\tilde{A}$ in the above-mentioned cone algebras,*

$$\sigma_{\mathrm{c}}(A)(v) := (\sigma_{\mathrm{c}}^{\mu-j}(A)(v))_{j=0,\ldots,\theta} \tag{3.8.20}$$

and

$$\sigma_{\mathrm{c}}(\tilde{A})(v) := (\sigma_{\mathrm{c}}^{\tilde{\mu}-k}(\tilde{A})(v))_{k=0,\ldots,\theta} \tag{3.8.21}$$

satisfy under compositions the following Mellin translation product:

$$\sigma_{\mathrm{c}}^{(\mu+\tilde{\mu})-i}(A\tilde{A})(v) = \sum_{j+k=i} T^{-(\tilde{\mu}-k)}(\sigma_{\mathrm{c}}^{\mu-j}(A)(v))\sigma_{\mathrm{c}}^{\tilde{\mu}-k}(\tilde{A})(v) \tag{3.8.22}$$

where $(T^\beta f)(v) := f(v+\beta)$ is the Mellin translation in the complex plane, such that

$$\sigma_{\mathrm{c}}(A\tilde{A})(v) = (\sigma_{\mathrm{c}}^{(\mu+\tilde{\mu})-i}(A\tilde{A})(v))_{i=0,\ldots,\theta}. \tag{3.8.23}$$

Let us apply the conormal symbolic structure in the algebra $1 + L^0_{\mathrm{M+G}}(X^\wedge, \boldsymbol{g})$ where the conormal symbols (3.8.19) have the special form

$$\sigma_{\mathrm{c}}^{-j}(A)(v) := 1 + f_j(v), \quad j = 0, \ldots, \theta. \tag{3.8.24}$$

For $j = 0$ the respective Mellin term only consist of a single f_0. As noted before ellipticity in $1 + L^0_{\mathrm{M+G}}(X^\wedge, \boldsymbol{g})$ means $1 + f_0(v) \neq 0$ for all $v \in \Gamma_{\frac{n+1}{2}-\gamma}$.

Theorem 3.8.5. *Let $1 + A \in 1 + L^0_{\mathrm{M+G}}(X^\wedge, \boldsymbol{g})$ be elliptic then there is a parametrix $1 + P \in 1 + L^0_{\mathrm{M+G}}(X^\wedge, \boldsymbol{g}^{-1})$ such that*

$$(1+P)(1+A) = 1 - G_{\mathrm{L}} \tag{3.8.25}$$

for a Green operator $G_{\mathrm{L}} \in L_{\mathrm{G}}(X^\wedge, \boldsymbol{g}_{\mathrm{L}})$, and

$$(1+A)(1+P) = 1 - G_{\mathrm{R}} \tag{3.8.26}$$

for a $G_{\mathrm{R}} \in L_{\mathrm{G}}(X^\wedge, \boldsymbol{g}_{\mathrm{R}})$.

Proof. Since $f_0(v) \in M^{-\infty}_{\mathcal{R}_0}(X)$ for some Mellin asymptotic type $\mathcal{R}_0$ we can apply Proposition 3.8.3, and we find an $l_0(v) \in M^{-\infty}_{Q_0}(X)$ such that

$$(1 + l_0(v))(1 + f_0(v)) = 1. \tag{3.8.27}$$

Then the Mellin translation product allows us to formally invert the sequence (3.8.24), i.e., knowing $l_0(v)$ we can successively determine all components for a parametrix $1+P$, i.e., except for (3.8.27) we compute

$$l_i(v) := \sigma_{\mathrm{c}}^{-i}(1+P)(v), \quad i = 1, \dots, \theta \tag{3.8.28}$$

using the rules (3.8.22), namely,

$$\sigma_{\mathrm{c}}^{0}(1+P)(v+1)\sigma_{\mathrm{c}}^{-1}(A)(v) + \sigma_{\mathrm{c}}^{-1}(P)(v)\sigma_{\mathrm{c}}^{0}(1+A)(v) \tag{3.8.29}$$

which means

$$-\sigma_{\mathrm{c}}^{0}(1+P)(v+1)\sigma_{\mathrm{c}}^{-1}(A)(v) = \sigma_{\mathrm{c}}^{-1}(P)(v)\sigma_{\mathrm{c}}^{0}(1+A)(v) \tag{3.8.30}$$

i.e.,

$$\sigma_{\mathrm{c}}^{-1}(P)(v) = -\sigma_{\mathrm{c}}^{0}(1+P)(v+1)\sigma_{\mathrm{c}}^{-1}(A)(v)(\sigma_{\mathrm{c}}^{0}(1+A)(v))^{-1}. \tag{3.8.31}$$

Clearly we can replace P in all those relations by $1+P$. This process can be continued. The next step comes from (3.8.22) for $i=2$ and by the known conormal symbols of $1+P$ for $i=0,1$ we can express $\sigma_{\mathrm{c}}^{-2}(1+P)(v)$ by terms which are computed before. In this process we employ that the operator functions in $M_{\mathcal{R}}^{-\infty}(X)$ for any Mellin asymptotic type $\mathcal{R}$ can be translated by shifts in the complex Mellin plane, and then $\pi_{\mathbb{C}}\mathcal{R}$ is also translated by the corresponding shift, but the quality of spaces remains preserved. In addition we employ relation (3.8.11), i.e., v-wise compositions (3.8.11) yield analogous classes (3.8.12). After finitely (namely, θ) steps we can determine the full sequence (3.8.28) i.e., we can form

$$P := \sum_{i=0}^{\theta} r^i \mathrm{Op}_M^{\gamma_i - \frac{n}{2}}(l_i) \tag{3.8.32}$$

where $1+P$ is a left parametrix of $1+A$ in the sense

$$(1+P)(1+A) = 1 - G \tag{3.8.33}$$

for a Green operator in the cone calculus, cf. relations (3.8.18). As noted before this result just provides the final step in constructing parametrices in the cone calculus, modulo $1-G$ for Green remainders. Another part inversion procedure has to be activated for the corresponding Mellin-edge calculus. $\square$

Summing up the amplitude functions $a(y,\eta)$ in (3.8.1) describe the specific edge part of operators in (3.6.3). In other words we mainly look at the contributions which are new in this calculus and localized close to the edge, except for the smoothing operator C, cf. formula (3.6.4), which are defined globally on M. The interior part A_{int} is also a global information but non-smoothing and belonging to the pseudo-differential calculus on $s_0(B)$ as a smooth manifold. The situation is analogous to boundary value problems, where the boundary plays the role of the edge, and $\overline{\mathbb{R}}_+$ is the model cone of the wedge which describes in this case a collar neighbourhood of the boundary. Also for boundary value problems, in general without the transmission property Definition 3.6.2 is valid, but when the transmission property is satisfied, as in Boutet de Monvel's algebra, then the amplitude functions in (3.8.1) do not contain any Mellin terms. Those operators $M+G$ are only G appear in parametrix constructions, also when the original elliptic operators do not contain such terms, like in (degenerate) differential operators.

Let us briefly comment the interplay between edge symbolic calculus which takes place on the (infinite stretched) cone, and the edge calculus itself. As we observed in the case of Boutet de Monvel's calculus the edge symbols (analogous of boundary symbols) are operator-valued and form themselves a calculus, here the cone calculus over $X^\wedge$. In this case it is essential to look at the infinite cone with control up to $r = \infty$, which we recognize as a conical exit of the model cone to infinity, partly motivated by twisted homogeneity with respect to the rescaling group $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$. On the other hand cone calculus is often considered on a compact manifold M with conical singularity $s_1(M)$. So there are different versions of cone algebras, the one on $X^\wedge$ where upper $^\wedge$ indicates stretched variables, and another one on compact M. Let us focus on the $X^\wedge$-case. Here the motivation is similar as in the edge case for a higher-dimensional edge, namely, to create an operator calculus which contains all Fuchs type differential operators together with parametrices of elliptic elements, also controlled by a principal symbolic hierarchy, in this case, again with two components, the interior symbol and the principal conormal symbol, cf. the discussion around formula (3.6.18). However, over $X^\wedge$ we also have to take care of the conical exit to infinity which incorporates a quite independent symbolic structure with its own conditions of ellipticity, cf. the material of [30].

Let us stop here the discussion of the cone calculus, since we mainly focus on the higher-dimensional edges. However, we tacitly employ the properties of the cone algebra over $X^\wedge$, similarly as the boundary symbolic calculus in Chapter 2. Let us finally develop the concept of ellipticity in the edge calculus.

Definition 3.8.6. *An $A \in L^\mu(M, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$ is called elliptic if*

(i) *A as an element of $L^\mu_{\mathrm{cl}}(s_0(M))$ is elliptic in the standard sense,*

(ii) *close to $s_1(M)$, the edge, in the local splitting of variables into $(r, x, y) \in X^\wedge \times \mathbb{R}^q$ and covariables (ρ, ξ, η) the reduced interior symbol*
$$\tilde{\sigma}_0(A)(r, x, y, \rho, \xi, \eta)$$
is asked to be non-vanishing for all $(\rho, \xi, \eta) \neq 0$ and all (r, x, y) including $r = 0$,

(iii) *the twisted homogeneous principal edge symbol*
$$\sigma_1(A)(y, \eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{3.8.34}$$
defines a family of isomorphisms for all $s \in \mathbb{R}$, and $\eta \neq 0$.

Remark 3.8.7. *Condition* (iii) *in Definition* 3.8.6 *can be generalized to the property that* (3.8.34) *is only Fredholm rather than to be bijective which is the case in both cases for all $s \in \mathbb{R}$ at the same time and is satisfied when it holds for one $s \in \mathbb{R}$. In the non-bijective case the associated operators in the edge calculus have to be equipped with extra conditions of trace and potential type; similarly as in Boutet de Monvel's calculus. However, in the present exposition we try to keep the formalism as concise as possible and ignore the case of Fredholm families. Concerning more information of K-theoretic character in this context, see the monograph* [20].

Moreover, observe that both ellipticity and later on parametrices of A do not require the assumption of compactness of M, though X is always assumed to be compact.

Definition 3.8.8. *Let $A \in L^{\mu}(M, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$. An operator $P \in L^{-\mu}(M, \boldsymbol{g}^{-1})$ for $\boldsymbol{g}^{-1} := (\gamma - \mu, \gamma, \Theta)$ is called a parametrix of A if relations*

$$PA = 1 - C_{\mathrm{L}}, \quad AP = 1 - C_{\mathrm{R}} \tag{3.8.35}$$

hold for remainders $C_{\mathrm{L}} \in L^{-\infty}(M, \boldsymbol{g}_{\mathrm{L}})$ and $C_{\mathrm{R}} \in L^{-\infty}(M, \boldsymbol{g}_{\mathrm{R}})$, for $\boldsymbol{g}_{\mathrm{L}} = (\gamma, \gamma, \Theta)$ and $\boldsymbol{g}_{\mathrm{R}} = (\gamma - \mu, \gamma - \mu, \Theta)$, respectively.

Theorem 3.8.9. *An elliptic $A \in L^{\mu}(M, \boldsymbol{g})$, $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$ has a parametrix $P \in L^{-\mu}(M, \boldsymbol{g}^{-1})$, $\boldsymbol{g}^{-1} := (\gamma - \mu, \gamma, \Theta)$.*

Remark 3.8.10. *In Theorem* 3.8.9 *we tacitly assumed that asymptotic data involved in A are constant under varying y along the edge. By that we mean the Mellin asymptotic types in the smoothing Mellin symbols $f_{j\alpha}$ in* (3.7.29)*, the asymptotic types $\mathcal{P}, Q$ in the Green symbols of Definition* 3.7.3 *as well as those in the smoothing operators in Definition* 3.7.2 *and finally those which appear by the inverses of non-smoothing Mellin symbols $h_0(r, y, v, \eta) = \tilde{h}(0, y, v, r\eta)$ occurring in formula* (3.6.7)*. There are different ways to get rid of these assumptions. One of the methods is to observe the asymptotic effects only in a small neighbourhood of the reference weight lines $\Gamma_{\frac{n+1}{2}-\gamma}$ and $\Gamma_{\frac{n+1}{2}-(\gamma-\mu)}$ in the complex Mellin plane; this was systematically in Seiler's thesis* [68]*. Another method is to generalize discrete to continuous asymptotics, as is done, e.g., in* [52] *or* [53]*, see also the special sections of* [30]*. In both cases we obtain parametrices in the respective operator classes, and also the Fredholm property of elliptic operators* (3.6.13)*. In any case the Fredholm property requires compactness M, and the theorem is as follow.*

Proof of Theorem 3.8.9. Let us sketch the main steps of the proof of Theorem 3.8.9. First it is clear that the ellipticity condition (i) entails the existence of a parametrix P_{int} of $A_{\mathrm{int}} := A|_{s_0(M)}$, where without loss of generality we may assume A_{int} to be properly supported on the open manifold $s_0(M)$. The idea of constructing P is to cover the stretched manifold $\mathbb{M}$ associated with M by open sets

$$U_0 \cup U_1$$

where U_0 can be taken as $s_0(M)$ and U_1 is a kind of caller neighbourhood of $\partial\mathbb{M}$, in local coordinates described by splitting of variables into $(r, x, y) \in \overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q$. If (φ_0, φ_1) is a subordinate partition of unity with respect to (U_0, U_1) and if $P_{\partial\mathbb{M}}$ is a parametrix over U_1 then P itself may be found in the form

$$P = \varphi_0 \, P_{\mathrm{int}} \, \varphi_0' + \varphi_1 \, P_{\partial\mathbb{M}} \, \varphi_1' \tag{3.8.36}$$

for other functions $\varphi_0 \prec \varphi_0'$ in $C_0^{\infty}(U_0)$ and $\varphi_1 \prec \varphi_1'$ in $C_0^{\infty}(U_1)$ (clearly the support close to $\partial\mathbb{M} \cap U_1$) goes up to $r = 0$ as is the case for global cut-off functions. Similarly as in other cases (3.8.36) does not depend on the auxiliary data φ_0, φ_1, etc., modulo smoothing operators in the edge calculus. The bijectivity condition in (ii) for reduced symbols is covered by the condition that the Mellin symbols

$$\tilde{h}(r, y, v, \tilde{\eta}) \in C^{\infty}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, M^{\mu}_{\mathcal{O}_v}(X; \mathbb{R}^q_{\tilde{\eta}})) \tag{3.8.37}$$

in Definition 3.6.2 are parameter-dependent elliptic with respect to the parameter $\tilde{\eta}$, for all r, y, uniformly up to $r = 0$. Then, as we see how operator-valued symbols in (y, η) are involved in (3.6.5) we first establish the symbol inverses in terms of the Leibniz inversion which yield an operator function

$$h^{(-1)}(r,y,v,\eta)=\tilde{h}^{(-1)}(r,y,v,r\eta) \tag{3.8.38}$$

for an

$$\tilde{h}^{(-1)}(r,y,v,\tilde{\eta})\in C^\infty(\overline{\mathbb{R}}_+\times\mathbb{R}^q,M^{-\mu}_{\mathcal{O}_v}(X;\mathbb{R}^q_{\tilde{\eta}})) \tag{3.8.39}$$

and then we construct a preliminary parametrix $P^0_{\partial\mathbb{M}}$ of $A_{\partial\mathbb{M}}:=H$, cf. notation of formula (3.6.4). Before we establish $P^0_{\partial\mathbb{M}}$, we first sketch the last step of the parametrix construction. Similarly as (3.8.36) we can form

$$P^0:=\varphi_0\,P_{\mathrm{int}}\,\varphi_0'+\varphi_1\,P_{\partial\mathbb{M}}\,\varphi_1'\in L^{-\mu}(M,\boldsymbol{g}), \tag{3.8.40}$$

and then, since P^0 coincides with the non-smoothing part of the desired parametrix, we obtain

$$P^0A=1-C \tag{3.8.41}$$

for a remainder $C\in L^0_{\mathrm{M+G}}(M,\boldsymbol{g}_{\mathrm{L}})$. Then in another step of the full parametrix we can find a $D\in L^0_{\mathrm{M+G}}(M,\boldsymbol{g}_{\mathrm{L}})$ such that

$$(1-D)(1-C)=1 \mod L^0_{\mathrm{G}}(M,\boldsymbol{g}_{\mathrm{L}}). \tag{3.8.42}$$

In constructing $1-D$ we can employ results of Theorem 3.8.5, i.e., we may focus on the smoothing Mellin plus Green structure of the operators near the edge, because off some neighbourhood of the edge the occurring remainders have smooth kernels. At the present step we still have in the remainders full Mellin plus Green amplitude functions. Those belong to (3.7.36) for $\boldsymbol{g}:=\boldsymbol{g}_{\mathrm{L}}$. Since all our manipulations before took place with different kinds of elliptic operators within the edge calculus. We now observe that the twisted homogeneous principal edge symbols of P^0 and A are multiplicative. The one of A just occurs in the ellipticity condition (3.8.34) and is bijective. The one of P^0 is generated automatically and is not necessary bijective. However, we have

$$\sigma_1(P^0A)(y,\eta)=\sigma_1(P^0)(y,\eta)\,\#_y\,\sigma_1(A)(y,\eta) \tag{3.8.43}$$

with # being the Leibniz multiplication of the respective symbols; note that in this operator-valued set-up the covariable $\eta=0$ is not excluded, although ellipticity of these symbols only concerns $\eta\neq0$. Under (3.8.43) the principal conormal symbols are y-wise multiplicative, up to the translation indicated in Theorem 3.8.4. Applying Proposition 3.8.3 y-wise we find an element $l_{00}(y,v)\in C^\infty(\mathbb{R}^q,M^{-\infty}_{\mathcal{Q}}(X))$ such that

$$(1-T^\mu l_{00}(y,v))(1-f_{00}(y,v))=1 \tag{3.8.44}$$

where f_{00} is by notation the principal conormal symbol of the Mellin plus Green part of $\sigma_1(P^0A)(y,\eta)$. Let us set

$$P^1:=(1-\mathrm{Op}_M(l_{00})(y))P^0.$$

Then the principal conormal symbol of $(1-\mathrm{Op}_M(l_{00})(y))P^0$ is just the inverse of the one of $\sigma_1(A)(y,\eta)$, and hence we found a modification of P^0 to P^1 such that

$$P^1A=1-C^1$$

where

$$C^1 \in L^{-1}_{\mathrm{M+G}}(M, \boldsymbol{g}_{\mathrm{L}}) + L^0_{\mathrm{G}}(M, \boldsymbol{g}_{\mathrm{L}}).$$

We now can apply Theorem 3.8.4 to remove by multiplying from the left an element $1-(M^1+G^1) \in L^{-1}_{\mathrm{M+G}} + L^0_{\mathrm{G}}$ to P^1 to get a P^2 such that

$$\sigma_1(P^2)(y,\eta) = \sigma_1^{-1}(A)(y,\eta)$$

for all (y,η). Thus we obtain altogether an operator P^3 such that

$$P^3 A = 1 - C$$

holds where C belong to $L^0_{\mathrm{G}}(M, \boldsymbol{g}_{\mathrm{L}})$. In order to improve P^3 to the desired parametrix P_L with remainder in $L^{-\infty}_{\mathrm{G}}(M, \boldsymbol{g}_{\mathrm{L}})$ we can add to $\sigma_1(C)(y,\eta)$ a finite rank Green symbol to obtain a bijective operator family $1-\sigma_1(C)(y,\eta)$. Then the Leibniz inverse of this gives by

$$P_{\mathrm{L}} := \mathrm{Op}_y(1-\sigma_1(C)(y,\eta)^{\#-1})P^3.$$

In a similar manner we can construct P_{R} as asserted.

$\square$

Theorem 3.8.11. *Let M be a compact manifold with edge. For an operator $A \in L^\mu(M, \boldsymbol{g})$, $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$, the following conditions are equivalent:*

(i) *A is elliptic*

(ii)

$$A : H^{s,\gamma}(M) \longrightarrow H^{s-\mu,\gamma-\mu}(M) \tag{3.8.45}$$

is a Fredholm operator for some $s = s_0 \in \mathbb{R}$. This is equivalent with the Fredholm property of (3.8.45) *for all $s \in \mathbb{R}$.*

In addition parametrices in (3.8.35) *can be chosen in such a way that C_{L} and C_{R} are projections to finite dimensional subspaces of $H^{\infty,\gamma}(M)$ and $H^{\infty,\gamma-\mu}(M)$, respectively.*

Proof. The proof of (ii) $\Rightarrow$ (i) is complicated and we will drop it, cf. [6]. This assertion just means that ellipticity is a necessary condition for the Fredholm property. Relation (i) $\Rightarrow$ (ii), i.e., that ellipticity is sufficient, is a consequence of Theorem 3.8.9. In fact the operators C_{L} and C_{R} are compact in the respective weighted Sobolev spaces and then relations (3.8.35) are equivalent for the Fredholm property for any s. This is a general functional analytic fact, see any text book, say, [69]. In particular, $\ker A \subset H^{\infty,\gamma}(M)$ is of finite dimension, where A is interpreted as (3.8.45) for some s_0, where $\ker A$ in that sense is independent of s_0, and $\operatorname{coker} A$ can be represented by a finite-dimensional subspace $W \subset H^{\infty,\gamma-\mu}(M)$ complementary to $\operatorname{Im} A$, i.e.,

$$\operatorname{Im} A \oplus W = H^{s-\mu,\gamma-\mu}(M)$$

for all $s = s_0$; in other words W is independent of s_0. The properties concerning projections can be obtained by the following abstract arguments.

Consider scales of Hilbert spaces, briefly denoted by $(H^s)_{s\in\mathbb{R}}$ and $(\tilde{H}^s)_{s\in\mathbb{R}}$, and

$$A : H^s \longrightarrow \tilde{H}^{s-\mu}$$

a Fredholm operator for all s, then there are operators of finite rank

$$K : \mathbb{C}^{n_-} \longrightarrow W, \quad T : \ker A \longrightarrow \mathbb{C}^{n_+} \tag{3.8.46}$$

which are bijections between $\mathbb{C}^{n_-}$ to a $W \subset H^\infty$ of A which is of dimension n_- and an isomorphism of $\ker A$ to $\mathbb{C}^{n_+}$ such that

$$\mathcal{A} := \begin{pmatrix} A & K \\ T & 0 \end{pmatrix} : \begin{matrix} H^s \\ \oplus \\ \mathbb{C}^{n_-} \end{matrix} \longrightarrow \begin{matrix} \tilde{H}^{s-\mu} \\ \oplus \\ \mathbb{C}^{n_+} \end{matrix} \tag{3.8.47}$$

is an isomorphism. For the parametrix of A which is of order μ and denoted in this substract by P (and in our concrete application also belongs to the calculus of order $-\mu$) we can construct an isomorphism

$$\mathcal{P} := \begin{pmatrix} P & C \\ B & 0 \end{pmatrix} : \begin{matrix} \tilde{H}^{s-\mu} \\ \oplus \\ \mathbb{C}^{m_-} \end{matrix} \longrightarrow \begin{matrix} H^s \\ \oplus \\ \mathbb{C}^{m_+} \end{matrix} \tag{3.8.48}$$

of a similar structure such that $m_- - m_+ = -(n_+ - n_-)$ since

$$n_+ - n_- = \operatorname{ind} A = -\operatorname{ind} P,$$

in other words $m_- - n_- = m_+ - n_+ := k$ for some $k \in \mathbb{Z}$. Then we can modify $P \oplus I$ where I is a bijection between k-dimensional subspaces $\tilde{W} \subset \tilde{H}^\infty$ and $W \subset H^\infty$. Such that $\ker P \cap \tilde{W} = \{0\}$ and $\operatorname{coker} P \cap W = \{0\}$. If $k \leq 0$ then we replace P by $P \oplus N$ where N is a $k \times k$-block matrix with entries $\equiv 0$, representing the zero map $\tilde{W} \longrightarrow W$. These modifications do not destroy the property of a parametrix of A in the upper left corner but after this modification of P which we (now without loss of generality) denote by the same letter, we have in new notation of $m_\pm$ that $n_- = m_+, n_+ = m_-$. We then obtain in new notation

$$\mathcal{P} := \begin{pmatrix} P & C \\ B & 0 \end{pmatrix} : \begin{matrix} \tilde{H}^{s-\mu} \\ \oplus \\ \mathbb{C}^{n_+} \end{matrix} \longrightarrow \begin{matrix} H^s \\ \oplus \\ \mathbb{C}^{n_-} \end{matrix}. \tag{3.8.49}$$

Since our 2×2-block matrix operators are isomorphisms we have $\mathcal{A}^{-1} = \mathcal{P}$ i.e.,

$$\mathcal{P} := \begin{pmatrix} P & C \\ B & 0 \end{pmatrix} \begin{pmatrix} A & K \\ T & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which means $PA + CT = 1$ and similarly,

$$\begin{pmatrix} A & K \\ T & 0 \end{pmatrix} \begin{pmatrix} P & C \\ B & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

i.e., $AP + KB = 1$. Since CT is projection $H^s \longrightarrow \ker A$ and KB a projection $\tilde{H}^{s-\mu} \longrightarrow \ker P \cong \operatorname{coker} A$ we proved that our remainders are just projections, namely,

$$C_{\mathrm{L}} = CT, \quad C_{\mathrm{R}} = KB.$$

□

Chapter 4

Edge Boundary Value Problems

4.1 The approach via the Mellin transform

In the present Chapter N is a manifold with edge Y and boundary ∂N. We do not necessarily assume that N is compact, i.e., when we consider an infinite cone X^Δ or an infinite wedge $X^\Delta \times \mathbb{R}^q$ for a manifold X with boundary. Those spaces are examples of singular spaces with boundary, and they are smooth but non-compact off the respective singularities (i.e., the tip of the cone X^Δ or the edge $\mathbb{R}^q$ of $X^\Delta \times \mathbb{R}^q$). When we remove those singularities $s_1(X^\Delta)$ or $s_1(X^\Delta \times \mathbb{R}^q)$ we get smooth non-compact manifolds with boundary, and the operator classes from Chapter 2 are equipped with some bundle information $\boldsymbol{v} = (E, F; J_-, J_+)$. Looking at the above singular spaces themselves we have to explain the nature of the bundles close to $s_1(X^\Delta)$ or $s_1(X^\Delta \times \mathbb{R}^q)$. The canonical answer would be that they are coming from the respective stretched manifolds which are defined for any singular manifold. In this case the stretched manifold for X^Δ is equal to the closed cylinder $\overline{\mathbb{R}}_+ \times X$ and for $X^\Delta \times \mathbb{R}^q$ the Cartesian product $\overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q$. The ellipticity condition (apart from the parameter-dependence) will contain bijectivities of operator functions in variables/covariables on the respective stretched manifolds, depending (according to a notation of Melrose [43]) on compressed coordinates on the cotangent bundles of the corresponding stretched manifolds. In other words the bundles E, F are basically living on the respective stretched manifolds and deserve notation like $\mathbb{E}, \mathbb{F}$, etc. Compared with [43] the situation here is slightly more complicated, since X in our case is a smooth manifold with boundary and the stretched manifolds require a careful definition, because they already have corners which can be easily demonstrated by cutting away from a manifold with smooth boundary like a cylinder $\mathbb{R} \times X$ (where X just has a boundary) the negative half cylinder $\mathbb{R}_- \times X$. Then there remains $\overline{\mathbb{R}}_+ \times X$ which is a manifold with corner, i.e., we have an independent half-axis variables, one from $\overline{\mathbb{R}}_+$, the other one from the inner normal to ∂X. Under such circumstances it becomes complicated to alwayes indicate precise definitions of the involved vector bundles, and to keep in mind various liftings, etc. Therefore, from now on we drastically simplify notation and indicate everywhere trivial bundles of fiber dimension 1, both on X as well as on ∂X, and also on the liftings to $\overline{\mathbb{R}}_+ \times X$ and $\overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q$ with respect to canonical projections $\overline{\mathbb{R}}_+ \times X \longrightarrow X$ and $\overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q \longrightarrow X$, respectively. So when we employ Boutet de Monvel's calculus from Chapter 2 the bundles E, F are both replaced by $X \times \mathbb{C}$, whereas J_-, J_+ are both replaced by $\partial X \times \mathbb{C}$. The problem left to the reader is to imagine to what extent bundles on cylinders like $\overline{\mathbb{R}}_+ \times X$ or $\overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q$ are pull backs of bundles on the respective bottoms. In addition we mainly talk about $\mathbb{R}_+ \times X$ rather than $\overline{\mathbb{R}}_+ \times X$, since main attention concerns the open half axis and the use of the Mellin transform, but some weight data close to $r = 0$ in operators of Boutet de Monvel's calculus remain under control.

Our manifold N with edge and boundary is locally close to Y modeled on $X^\Delta \times \mathbb{R}^q$. The analysis will be formulated on $N \setminus Y$, represented by the splitting of variables (r, x, y), referring to a chart $U \to \mathbb{R}^q$ on Y and a stretched singular chart mapping $N \setminus Y$ close to points $(\cdot, y_0)$ to $X^\wedge = \mathbb{R}_+ \times X$ for any fixed y_0. On the open stretched cone we have weighted Kegel spaces

$$\mathcal{K}^{s,\gamma}(X^\wedge) \tag{4.1.1}$$

of smoothness $s \in \mathbb{R}$ and weight $\gamma \in \mathbb{R}$. Since X has a smooth boundary we often consider the double $2X$ obtained by gluing together two copies $X := X_+$ and X_- of X along the common boundary ∂X, and (4.1.1) is defined as $\mathcal{K}^{s,\gamma}(2X)^\wedge|_{(\operatorname{int} X)^\wedge}$, using a known definition of such spaces $\mathcal{K}^{s,\gamma}(B^\wedge)$ for a closed smooth manifold B. For the Mellin approach it is important to employ a definition of such spaces purely based on Mellin operators taking values in (classical) parameter-dependent pseudo-differential operators in $L^\mu(B; \mathbb{R}^d_\zeta)$ which have been used in Chapter 2. Recall cf. formula (3.3.13), that $\mathcal{K}^{s,\gamma}(B^\wedge)$ can be written in the form

$$\mathcal{K}^{s,\gamma}(B^\wedge) = \{\omega u_0 + (1-\omega)u_\infty : u_0 \in \mathcal{H}^{s,\gamma}(B^\wedge), u_\infty \in H^s_{\mathrm{cone}}(B^\wedge)\} \tag{4.1.2}$$

where $\omega = \omega(r)$ is a cut-off function on the r half axis and $\mathcal{H}^{s,\gamma}(B^\wedge)$ is defined via local Fourier-Mellin Sobolev spaces with the norm

$$||u||_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)} = \left\{ \int_{\mathbb{R}^n} \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \langle \xi, v\rangle^{2s} |(F_{x\to\xi} M_{r\to v} u)(\xi, v)|^2 \, đ\xi đv \right\}^{1/2} \tag{4.1.3}$$

for

$$\Gamma_\beta := \{v \in \mathbb{C} : \operatorname{Re} v = \beta\}$$

for any real $\beta, đ\xi = (2\pi)^{-n} d\xi, đv := (2\pi i)^{-1} dv$ and complex integration along $\Gamma_{\frac{n+1}{2}-\gamma}$ from $\operatorname{Im} v = -\infty$ to $\operatorname{Im} v = +\infty$. The space $H^s_{\mathrm{cone}}(B^\wedge)$ is a Fourier-based Sobolev space which corresponds to $H^s(\mathbb{R}^{1+n}_{\tilde{x}})$ in Euclidean variables $\tilde{x}$ up to ∞ when $B = S^n$ is the unit sphere in $\mathbb{R}^{1+n}$. For B in general there is an invariant definition with such an "Euclidean" behavior at ∞ on conical subsets V defined by $\{\tilde{x} \in \mathbb{R}^{1+n} \setminus \{0\} : \frac{\tilde{x}}{|\tilde{x}|} \in V_1\}$ for some coordinate neighborhood V_1 on S^n. The Mellin transform gives rise to weighted Mellin pseudo-differential operators locally in $\mathbb{R}_+ \times \mathbb{R}^n$ given by

$$\begin{aligned} &\operatorname{Op}_M^{\gamma-n/2}(f)u(r,x) := \\ &\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\frac{r}{r'}\right)^v e^{i(x-x')\xi} f(r, r', x, x', v, \xi) u(r', x') \, dx' đ\xi \frac{dr'}{r'} đv \end{aligned} \tag{4.1.4}$$

for some Mellin-Fourier symbols $f(r, r', x, x', v, \xi) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, S^\mu_{\mathcal{O}_v}(\mathbb{R}^n_\xi))$. Expression (4.1.4) can be seen as a weighted Mellin operator, similarly as (3.3.7), here with operator-valued symbol, acting on the base of the cone. Here $S^\mu_{\mathcal{O}_v}(\mathbb{R}^n_\xi)$ is the space of all $\mathcal{A}(\mathbb{C}_v, S^\mu_{\mathrm{cl}}(\mathbb{R}^n_\xi))$ such that $f(\lambda + i\rho, \xi) \in S^\mu_{\mathrm{cl}}(\mathbb{R}_\rho \times \mathbb{R}^n_\xi)$ for every $\lambda \in \mathbb{R}$, uniformly in compact λ-intervals.

Globally along B we can also define spaces of Mellin symbols of order μ

$$M^{\mu}_{\mathcal{O}_v}(B)$$

as the set of all $f(v) \in \mathcal{A}(\mathbb{C}_v, L^{\mu}_{\mathrm{cl}}(B))$ such that $f(\lambda + i\rho) \in L^{\mu}_{\mathrm{cl}}(B; \Gamma_\lambda)$ for every $\lambda \in \mathbb{R}$, uniformly in compact λ-intervals. Here the parameter space Γ_λ is identified with $\{\rho \in \mathbb{R} : \operatorname{Im} v = \rho\}$. It is also very instructive to define all those symbol and operator classes including a parameter $\zeta \in \mathbb{R}^d$. Then we obtain the space $M^{\mu}_{\mathcal{O}_v}(B; \mathbb{R}_\zeta)$. The edge calculus, concerning degenerate operators requires r-dependent Mellin symbols with involved parameters ζ

$$f(r, v, \zeta) := \tilde{f}(r, v, r\zeta)$$

for $\tilde{f}(r, v, \tilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+, M^{\mu}_{\mathcal{O}_v}(B; \mathbb{R}^d_{\tilde{\zeta}}))$. We then employ a result of the boundary edge calculus, namely, that for any $f(r, v, \zeta)$ of that kind the weighted Mellin operator

$$r^{-\mu}\mathrm{Op}_M^{\gamma - n/2}(f)(\zeta) : \mathcal{K}^{s,\gamma}(B^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(B^\wedge) \tag{4.1.5}$$

is continuous for every $s \in \mathbb{R}$ and weights $\gamma \in \mathbb{R}$, varying in any finite weight interval and that we find elements f such that (4.1.5) induces isomorphisms for sufficiently large $|\zeta|$. It is essential in this context that such Mellin symbols f of order $-s$ induce an isomorphism

$$r^{-s}\mathrm{Op}_M^{\gamma - s - n/2}(f)(\zeta) : \mathcal{K}^{0,\gamma-s}(B^\wedge) \longrightarrow \mathcal{K}^{s,\gamma}(B^\wedge). \tag{4.1.6}$$

This allows us, starting from the simple space $\mathcal{K}^{0,\gamma-s}(B^\wedge)$ to define $\mathcal{K}^{s,\gamma}(B^\wedge)$ in an intrinsic manner by Mellin operators, though the original definition of weighted Kegel spaces employ the Fourier transform at ∞. In the case with boundary we set

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \mathcal{K}^{s,\gamma}(B^\wedge)|_{(\mathrm{int\ X})^\wedge}. \tag{4.1.7}$$

4.2 The asymptotic content of the edge calculus

We now formulate the asymptotic part $(M + G)(\zeta)$ of the edge-calculus of boundary value problems $\zeta \in \mathbb{R}^d$. The approach is similar to the case without boundary cf. Section 3.7. We employ discrete asymptotic types associated with weight data (γ, Θ) for $\Theta = (-(\theta + 1), 0]$, $\theta \in \mathbb{N} \setminus \{0\}$,

$$\mathcal{P} = \{(p_j, m_j)\}_{j=0,\dots,N} \subset \mathbb{C} \times \mathbb{N} \tag{4.2.1}$$

for an $N \in \mathbb{N} \cup \{+\infty\}$, where $\pi_{\mathbb{C}}\mathcal{P} := (p_j)_{j=0,\dots,N}$ is finite, otherwise $\pi_{\mathbb{C}}\mathcal{P}$ may be infinite and then $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$. In any case we require that

$$\pi_{\mathbb{C}}\mathcal{P} \subset \{v \in \mathbb{C} : \frac{n+1}{2} - \gamma + (\theta + 1) < \operatorname{Re} v < \frac{n+1}{2} - \gamma\}. \tag{4.2.2}$$

If Θ is finite, the space $\mathcal{E}_{\mathcal{P}}(X^\wedge)$ of singular functions of discrete asymptotics of type $\mathcal{P}$ on the open stretched cone is defined as

$$\mathcal{E}_{\mathcal{P}}(X^\wedge) = \left\{ \sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk}(x) r^{-p_j} \log^k r : c_{jk} \in C^\infty(X) \right\} \tag{4.2.3}$$

which is Fréchet in topology of a corresponding direct sum of copies of $C^\infty(X)$, where the number of copies is determined by the number of elements of $\pi_{\mathbb{C}}\mathcal{P}$ including the multiplicities m_j+1.
For convenience we assume that asymptotic type $\mathcal{P}$ satisfies the shadow condition, cf. Section 3.7. The notation in (4.2.3) is valid for a smooth compact manifold X of dimension n with boundary. We intend to apply it in analogous form for ∂X, and then denote corresponding asymptotic types by $\mathcal{P}_\partial$. However, for some normalizing reasons we adapt the dependence on dimensions in relation (4.2.2). We will use the same rule on the position of points p_j both for $\mathcal{P}$ and $\mathcal{P}_\partial$. Later on, we fix the normalizing factor $\delta^{\frac{n+1}{2}}$ in the rescaling group in spaces (4.1.7) , i.e.,

$$\kappa_\delta u(r,x)=\delta^{\frac{n+1}{2}}u(\delta r,x) \tag{4.2.4}$$

when x varies on X. Moreover, we use a modified group action κ'_δ when x varies on ∂X where only the normalizing factor $\delta^{\frac{n+1}{2}}$ is shifted, according to the order of trace operators.

We define flat functions relative to the weight γ as

$$\mathcal{K}^{s,\gamma}_\Theta(X^\wedge)=\varprojlim_{\epsilon>0}\mathcal{K}^{s,\gamma-\theta-1-\epsilon}(X^\wedge) \tag{4.2.5}$$

which is a Fréchet space as well, where

$$\mathcal{E}_\mathcal{P}(X^\wedge)\cap\mathcal{K}^{s,\gamma}_\Theta(X^\wedge)=\{0\}.$$

Such a relation is valid also when the boundary of X is empty. It comes from the fact that the coefficients c_{jk} in (4.2.3) are uniquely determined by the functions themselves. We then set

$$\mathcal{K}^{s,\gamma}_\mathcal{P}(X^\wedge):=\mathcal{E}_\mathcal{P}(X^\wedge)+\mathcal{K}^{s,\gamma}_\Theta(X^\wedge) \tag{4.2.6}$$

in the Fréchet topology of the direct sum. If Θ is infinite we form

$$\mathcal{P}_l:=\{(p,m)\in\mathcal{P}:\operatorname{Re}p>\gamma-l\}$$

for any $l\in\mathbb{N}$. Then we have the spaces $\mathcal{K}^{s,\gamma}_{\mathcal{P}_l}(X^\wedge)$, and we set

$$\mathcal{K}^{s,\gamma}_\mathcal{P}(X^\wedge):=\varprojlim_{l\in\mathbb{N}}\mathcal{K}^{s,\gamma}_{\mathcal{P}_l}(X^\wedge). \tag{4.2.7}$$

There are some other notations, derived from this kind of spaces. In particular, we set

$$\mathcal{K}^{s,\gamma;g}(X^\wedge):=[r]^{-g}\mathcal{K}^{s,\gamma}(X^\wedge),\quad \mathcal{K}^{s,\gamma;g}_\mathcal{P}(X^\wedge):=[r]^{-g}\mathcal{K}^{s,\gamma}_\mathcal{P}(X^\wedge) \tag{4.2.8}$$

for any fixed strictly positive function $[r]$ such that $[r]=r$ for $|r|\geq c$ for some $c>0$.

Definition 4.2.1. *A family of operators*

$$g(y,\eta,\zeta)\in C^\infty\Big(\mathbb{R}^q\times\mathbb{R}^{q+d}_{\eta,\zeta},\mathcal{L}\Big(\begin{array}{c}\mathcal{K}^{s,\gamma;g}(X^\wedge)\\ \oplus\\ \mathcal{K}^{s,\gamma;g}((\partial X)^\wedge)\\ \oplus\\ \mathbb{C}\end{array},\begin{array}{c}\mathcal{K}^{s-\mu,\gamma-\mu;g}(X^\wedge)\\ \oplus\\ \mathcal{K}^{s-\mu,\gamma-\mu;g}((\partial X)^\wedge)\\ \oplus\\ \mathbb{C}\end{array}\Big)\Big), \tag{4.2.9}$$

continuous for all $s, g \in \mathbb{R}$, is called a (local) Green symbol of order μ, first of type 0*, associated with weight data $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, in the edge calculus of BVPs with constant discrete asymptotics (constant with respect to y) if*

$$g(y,\eta,\zeta) \in S^{\mu}_{\mathrm{cl}}\left(\mathbb{R}^q \times \mathbb{R}^{q+d}_{\eta,\zeta}; \begin{array}{c}\mathcal{K}^{s,\gamma;g}(X^\wedge)\\ \oplus \\ \mathcal{K}^{s,\gamma;g}((\partial X)^\wedge)\\ \oplus \\ \mathbb{C}\end{array}, \begin{array}{c}\mathcal{K}^{\infty,\gamma-\mu;\infty}_{\mathcal{P}}(X^\wedge)\\ \oplus \\ \mathcal{K}^{\infty,\gamma-\mu;\infty}_{\mathcal{P}_\partial}((\partial X)^\wedge)\\ \oplus \\ \mathbb{C}\end{array}\right) \tag{4.2.10}$$

and

$$g^*(y,\eta,\zeta) \in S^{\mu}_{\mathrm{cl}}\left(\mathbb{R}^q \times \mathbb{R}^{q+d}_{\eta,\zeta}; \begin{array}{c}\mathcal{K}^{s,-\gamma+\mu;g}(X^\wedge)\\ \oplus \\ \mathcal{K}^{s,-\gamma+\mu;g}((\partial X)^\wedge)\\ \oplus \\ \mathbb{C}\end{array}, \begin{array}{c}\mathcal{K}^{\infty,-\gamma;\infty}_{\mathcal{Q}}(X^\wedge)\\ \oplus \\ \mathcal{K}^{\infty,-\gamma;\infty}_{\mathcal{Q}_\partial}((\partial X)^\wedge)\\ \oplus \\ \mathbb{C}\end{array}\right) \tag{4.2.11}$$

Recall that the 3×3 diagonal matrix of group actions in (4.2.10) or (4.2.11), which determine twisted orders in the block matrices of symbols correspond to $\mathrm{diag}(\kappa_\delta, \kappa'_\delta, \kappa''_\delta)$ for suitable powers of δ, similarly as in (4.2.4). Examples for such effects may be found also in [20, page 276].
A Green symbol of order μ and type $e \in \mathbb{N}$ is defined as

$$g(y,\eta,\zeta) = g_0(y,\eta,\zeta) + \sum_{j=1}^{e} g_j(y,\eta,\zeta)\mathrm{diag}(\partial^j_{x_n}, 0, 0) \tag{4.2.12}$$

for Green symbols $g_j(y,\eta,\zeta)$ in the former sense, $j = 0, \dots, e$. Here $\mathrm{diag}\,(\partial^j_{x_n}, 0, 0)$ means the diagonal matrix with corresponding entries, where x_n indicates a global normal variable to the boundary ∂X. Let $R^{\mu,e}_{\mathrm{G}}(\mathbb{R}^q_y \times \mathbb{R}^{q+d}_{\eta,\zeta}, \boldsymbol{g})$ denote the space of such Green symbols.

Remark 4.2.2. *Let us point out once again the meaning of notation in* Definition 4.2.1. *As noted at the beginning of* Section 4.1*, the occurring Kegel spaces over $X^\wedge$ and $(\partial X)^\wedge$, respectively, are spaces of distributional sections over the respective spaces. In elliptic theories we assume the fibre dimension of bundles in the upper left corners of operator block matrices to be equal. However, in the operator algebra itself those dimensions may be arbitrary, but also zero. The same is true for the other entries, including those in lower right corners. E.g., the components $\mathbb{C}$ represent fibres of vector bundles on Y. In precise descriptions one of those or both may also vanish. Phenomena are similar to what we know from Boutet de Monvel's calculus over X, outlined in* Chapter 2.

Similarly as (4.2.1) we now formulate discrete asymptotic types for Mellin symbols. Those are defined as sequences

$$\mathcal{R} = \{(r_j, n_j)\}_{j \in \mathbb{I}} \subset \mathbb{C} \times \mathbb{N} \tag{4.2.13}$$

for some index set $\mathbb{I} \subseteq \mathbb{Z} \cup \{-\infty\} \cup \{+\infty\}$ such that $\pi_{\mathbb{C}}\mathcal{R} = \{r_j\}_{j \in \mathbb{I}}$ intersects every finite strip

$$\{v \in \mathbb{C} : c \le \mathrm{Re}\, v \le c'\}$$

in a finite set. We now introduce the space

$$\mathcal{B}_{\mathcal{R}}^{-\infty,e}\begin{pmatrix} X \\ \times \\ \partial X \end{pmatrix} \tag{4.2.14}$$

of smoothing Mellin symbols $f(v) := (f_{ij}(v))_{i,j=1,2}$ with asymptotics of types $\mathcal{R} := (\mathcal{R}_{ij})_{i,j=1,2}$. The elements $f(v)$ of (4.2.14) are assumed to belongs to

$$\mathcal{A}(\mathbb{C}_v \setminus \pi_{\mathbb{C}}\mathcal{R}, \mathcal{B}^{-\infty,e}\begin{pmatrix} X \\ \times \\ \partial X \end{pmatrix}) \tag{4.2.15}$$

such that $f(v)$ is meromorphic with poles at $r_l \in \pi_{\mathbb{C}}\mathcal{R}$, more precisely, the respective entry f_{ij} has poles at $r_{l,ij}$ of multiplicity $n_{l,ij}+1$, with finite rank Laurent coefficients belonging to the ij-entry of $\mathcal{B}^{-\infty,e}\begin{pmatrix} X \\ \times \\ \partial X \end{pmatrix}$, and for any $\pi_{\mathbb{C}}\mathcal{R}$-excision function $\chi(v)$ we have

$$\chi f|_{\Gamma_\lambda} \in \mathcal{S}(\Gamma_\lambda, \mathcal{B}^{-\infty,e}\begin{pmatrix} X \\ \times \\ \partial X \end{pmatrix}).$$

Clearly these formulations contain some abbreviations. We could speak about the entries separately and refer to the individual Mellin asymptotic types, including $r_{l,ij}$ and $n_{l,ij}+1$. However, as far we have several asymptotic types there is always a larger one, in fact, more crude, which contains the given ones in an evident way. Corresponding refinements of formulations are left to the reader. In the following definition we employ such a simplified style of notions.

Definition 4.2.3. *An operator function*

$$m(y,\eta,\zeta) = \omega_{\eta,\zeta} r^{-\mu} \sum_{j=0}^{\theta} r^j \sum_{|\alpha|\le j} \mathrm{Op}_M^{\gamma_{j\alpha}-n/2}(f_{j\alpha})(y)(\eta,\zeta)^\alpha \omega'_{\eta,\zeta} \tag{4.2.16}$$

for elements $f_{j\alpha}(y,v) \in C^\infty(\mathbb{R}^q, \mathcal{B}_{\mathcal{R}_{j\alpha}}^{-\infty,e}\begin{pmatrix} X \\ \times \\ \partial X \end{pmatrix})$ *and weights* $\gamma - j \le \gamma_{j\alpha} \le \gamma$ *such that*

$$\Gamma_{\frac{n+1}{2}-\gamma_{j\alpha}} \cap \pi_{\mathbb{C}}\mathcal{R}_{j\alpha} = \emptyset$$

for all $j = 0, \dots, \theta$, $|\alpha| \le j$ *is called a smoothing Mellin operator family of the cone calculus.*

Remark 4.2.4. *The notation Green operators is inherited by a similar notion in Boutet de Monvel's calculus, where we have* $\overline{\mathbb{R}}_+$ *rather than* $X^\wedge$ *and more specific asymptotic types, namely, coming from Taylor expansions of smooth functions. The associated Green operators are related to Green's function of boundary value problems.*

Let $R^{\mu,e}_{\mathrm{M+G}}(\mathbb{R}^q_y \times \mathbb{R}^{q+d}_{\eta,\zeta}, \boldsymbol{g})$ denote the space of all so-called smoothing Mellin plus Green symbols $(m+g)(y,\eta,\zeta)$ of the parameter-dependent edge calculus of boundary value problems. By construction they are 2×2-block matrix functions of operators, where the various entries have the interpretation either of upper left corner or of operators of trace and potential type.

We can specialize the definition to the case of (y,η)-independent operator functions, where only $\zeta \in \mathbb{R}^d$ remains. Then we obtain operators of the calculus over the infinite (stretched) cone. The corresponding operator class will be denoted by $L^{\mu}_{\mathrm{M+G}}\begin{pmatrix} X \\ \times \\ \partial X\end{pmatrix}$, and we treat the elements as operators rather than amplitude functions.

Example 4.2.5. *A family of 2×2-block matrices of operators $h_{\mathrm{G}}(r,y,\eta,\zeta) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q_y, \mathcal{B}^{\mu,0}_{\mathcal{O}_v,\mathrm{G}}\begin{pmatrix} X & \\ \times & ;\mathbb{R}^{q+d}_{\tilde\eta,\tilde\zeta} \\ \partial X & \end{pmatrix})$ determines Green and smoothing Mellin edge symbols if*

$$\omega_{\eta,\zeta} r^{-\mu}\mathrm{Op}^{\gamma-n/2}_{M_r}(h_{\mathrm{G}})(y,\eta,\zeta)\omega'_{\eta,\zeta} : \begin{array}{c} \mathcal{K}^{s,\gamma;g}(X^\wedge) \\ \oplus \\ \mathcal{K}^{s,\gamma;g}((\partial X)^\wedge) \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{\infty,\gamma-\mu;\infty}_{\mathcal{P}}(X^\wedge) \\ \oplus \\ \mathcal{K}^{\infty,\gamma-\mu;\infty}_{\mathcal{P}_\partial}((\partial X)^\wedge) \end{array} \tag{4.2.17}$$

and the (y,η,ζ)-wise formal adjoint with respect to the $\mathcal{K}^{0,0}(X^\wedge)\oplus\mathcal{K}^{0,0}((\partial X)^\wedge)$-scalar products

$$\omega'_{\eta,\zeta} r^{-\mu}\mathrm{Op}^{\gamma-n/2}_{M_r}(h^*_{\mathrm{G}})(y,\eta,\zeta)\omega''_{\eta,\zeta} : \begin{array}{c} \mathcal{K}^{s,-\gamma+\mu;g}(X^\wedge) \\ \oplus \\ \mathcal{K}^{s,-\gamma+\mu;g}((\partial X)^\wedge) \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{\infty,-\gamma;\infty}_{\mathcal{Q}}(X^\wedge) \\ \oplus \\ \mathcal{K}^{\infty,-\gamma;\infty}_{\mathcal{Q}_\partial}((\partial X)^\wedge) \end{array} \tag{4.2.18}$$

are continuous for all $s,g \in \mathbb{R}$, for Green symbol h^*_{G} pointwise coming from Boutet de Monvel's calculus and for some discrete asymptotic types $(\mathcal{P},\mathcal{P}_\partial)$ and $(\mathcal{Q},\mathcal{Q}_\partial)$, respectively.

Recall that in symbolic estimates of corresponding classical symbols we work with $\mathrm{diag}(\kappa_\delta,\kappa'_\delta)$ as group actions for suitably modified powers of δ.
The relationship between (4.2.17) and the form of smoothing Mellin plus Green operators which has been introduced before can be illustrated by applying a Taylor expansion of $h(r,y,\eta,\zeta)$ at $r=0$. In fact, writing

$$h_{\mathrm{G}}(r,y,\eta,\zeta) = \sum_{j=0}^{\theta} r^j \frac{\partial^j}{\partial r^j} h_{\mathrm{G}}(r,y,\eta,\zeta)|_{r=0} + R(r,y,\eta,\zeta) \tag{4.2.19}$$

where the differentiation on the right-hand side of (4.2.19) is to be applied to the variable $r \in \overline{\mathbb{R}}_+$ from the assumed smooth dependence and by the chain rule to r which is involved in the arguments $r\eta, r\zeta$. In the latter case we produce powers of various components of η and ζ. Thus, when we multiply by $r^{-\mu}$ and by the cut-off functions $\omega_{\eta,\zeta}$ and $\omega'_{\eta,\zeta}$ we exactly produce an expression as in (4.2.16). The remainder $R(r,y,\eta,\zeta)$ contains an r- power of exponent$>\theta$ and then we obtain a Green symbol in the edge framework on our manifold with edge, which is at the same time Green operator-valued in close to the boundary of N.

In other words, Example 4.2.5 shows that the expression (4.2.17) belongs to $R^{\mu,g}_{\mathrm{M+G}}(\mathbb{R}_y \times \mathbb{R}^{q+d}_{\eta,\zeta}, \boldsymbol{g})$ for any $\boldsymbol{g}$, i.e., for $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$ for Θ of arbitrary length. In the example we discussed the 2×2 operator matrix case. Components in third rows and columns are automatically contained in Definition 4.2.1. This class of examples can also be extended to functions $h_{\mathrm{G}}(r, y, \eta, \zeta)$ taking values in $\mathcal{B}_{\mathcal{R}}^{-\infty,e}\left(\begin{matrix} X \\ \times \\ \partial X \end{matrix}; \mathbb{R}^{q+d}_{\tilde{\eta},\tilde{\zeta}}\right)\Big|_{\tilde{\eta}=r\eta,\tilde{\zeta}=r\zeta}$ for some constant Mellin asymptotic type $\mathcal{R}$. Then the same procedure with Taylor expansions gives us more general examples of smoothing Mellin plus Green symbols in the edge calculus of boundary value problems.

Note that Green operators admit alternative descriptions in terms of kernels, see analogously the paper of Seiler [67] in the case without boundary.

With amplitude functions

$$(m+g)(y, \eta, \zeta) \in R^{\mu,e}_{\mathrm{M+G}}(\mathbb{R}^q_y \times \mathbb{R}^{q+d}_{\eta,\zeta}, \boldsymbol{g}) \tag{4.2.20}$$

we can associate a similar analysis as in the boundaryless case, cf. [53]. For instance, when we change the cut-off functions involved in (4.2.16) or the weights $\gamma_{j\alpha}$ under preserving the $f_{j\alpha}$, we only obtain Green remainders. Another typical observation is that the summand for $j = 0$ is independent of (η, ζ), except for the dependence of the cut-off functions on (η, ζ). Another important aspect is that (4.2.20) is contained in

$$S^{\mu}_{\mathrm{cl}}\left(\mathbb{R}^q \times \mathbb{R}^{q+d}_{\eta,\zeta}; \begin{matrix} \mathcal{K}^{s,\gamma}(X^\wedge) \\ \oplus \\ \mathcal{K}^{s,\gamma}((\partial X)^\wedge) \\ \oplus \\ \mathbb{C} \end{matrix}, \begin{matrix} \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge) \\ \oplus \\ \mathcal{K}^{\infty,\gamma-\mu}((\partial X)^\wedge) \\ \oplus \\ \mathbb{C} \end{matrix}\right) \tag{4.2.21}$$

for $s > \max\{\mu, e\} - \frac{1}{2}$ and similarly, when we insert in both spaces on the right-hand side subspaces with asymptotics. The latter relations allow us to pass to continuous operators $\mathrm{Op}_y(m+g)$ between corresponding weighted edge spaces and subspaces with asymptotics.

Proposition 4.2.6. *Let $a(y, \eta, \zeta)$ in (4.2.21) be a sequence of smoothing Mellin plus Green symbols, $j \in \mathbb{N}$, and assume that the asymptotic types contained in Green parts are independent of j. Moreover, let $a_j(y, \eta, \zeta)$ be of order $\mu - j$. Then there is an $a(y, \eta, \zeta)$ in (4.2.21) such that for every N we have*

$$a(y, \eta, \zeta) = \sum_{j=0}^{N} \chi(\eta, \zeta) a_j(y, \eta, \zeta) + r(y, \eta, \zeta) \tag{4.2.22}$$

for some smoothing Mellin plus Green symbol of order $\mu - (N+1)$ and $a(y, \eta, \zeta)$ is unique modulo a corresponding symbol of that kind of order $-\infty$ and type e.

The proof is similar to the case of Mellin plus Green operators on manifolds with edge without boundary, see, [53, Proposition 3.3.11]. In fact, smoothing Mellin symbols become Green as soon as their orders are $\leq N$ for some suitable finite N. Therefore, there are only finitely many a_j in Proposition 4.2.6 not Green, which proves the statement.

4.3 Composition of smoothing Mellin plus Green operators

The property (4.2.21) also allows us to form homogeneous components with respect to twisted homogeneity. We mainly consider the case of (y, η, ζ)-dependent symbols. (y, η)-independent families are an obvious special case. If $g(y, \eta, \zeta)$ belonging to (4.2.21) is Green, then the main symbolic information is $g_{(\mu)}(y, \eta, \zeta)$, the homogeneous principal symbol of order μ which is also of 3×3-block matrix form.

concerning (4.2.16) we form the homogeneous principal part of order μ by

$$m_{(\mu)}(y, \eta, \zeta) := \omega_{|\eta,\zeta|} r^{-\mu} \sum_{j=0}^{\theta} r^j \sum_{|\alpha|=j} \mathrm{Op}_M^{\gamma_{j\alpha}-n/2}(f_{j\alpha})(y)(\eta, \zeta)^\alpha \omega'_{|\eta,\zeta|}. \tag{4.3.1}$$

Theorem 4.3.1. *Let $(m+g)(y, \eta, \zeta), (l+h)(y, \eta, \zeta)$ be two elements in the class of smoothing Mellin plus Green symbols of order μ and ν, and with weight data $(-\nu, \gamma - (\mu+\nu), \Theta)$ and $(\gamma, \gamma-\nu, \Theta)$, respectively. Then the composition $(m+g)(l+h)(y, \eta, \zeta)$ is again smoothing Mellin plus Green, of order $\mu + \nu$, and with weight data $(\gamma, \gamma - (\mu + \nu), \Theta)$, and the homogeneous principal symbols multiplicatively. If one of the factors is Green, then so is the composition.*

Remark 4.3.2. *The composition of smoothing Mellin plus Green symbols also entails a composition between the sequences of associated conormal symbols, via an analogoue of the Mellin translation product, known from the calculus in the case without boundary.*

4.4 The edge algebra of BVPs

In the preceding subsections we introduce the material on smoothing Mellin plus Green operators in 3×3-block matrix form

$$L^{\mu,e}_{\mathrm{M+G}}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d) \tag{4.4.1}$$

for $\mathcal{N}$ is a manifold N with edge Y and boundary, with weight data $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$. The definition of (4.4.1) is based on locally near Y representing operators on $X^\wedge \times \mathbb{R}^q$ in terms of Mellin plus Green symbols, while far from Y we define

$$L^{-\infty,e}_{\mathrm{M+G}} \begin{pmatrix} N & & \\ \times & ,\boldsymbol{g} & ;\mathbb{R}^d_\zeta \\ \partial N & & \end{pmatrix} \Bigg|_{N\setminus Y} = \mathcal{B}^{-\infty,e} \begin{pmatrix} N \setminus Y & & \\ \times & ,\boldsymbol{g} & ;\mathbb{R}^d_\zeta \\ \partial N \setminus Y & & \end{pmatrix}. \tag{4.4.2}$$

By definition we have similarly as (4.4.2) the property

$$L^{\mu,e} \begin{pmatrix} N & & \\ \times & ,\boldsymbol{g} & ;\mathbb{R}^d_\zeta \\ \partial N & & \end{pmatrix} \Bigg|_{N\setminus Y} = \mathcal{B}^{\mu,e} \begin{pmatrix} N \setminus Y & & \\ \times & ,\boldsymbol{g} & ;\mathbb{R}^d_\zeta \\ \partial N \setminus Y & & \end{pmatrix}. \tag{4.4.3}$$

Since extra entries corresponding to matrix elements $(m+g)_{13}, (m+g)_{23}, (m+g)_{33}$ or $(m+g)_{31}, (m+g)_{32}$ only concern the smoothing Mellin plus Green part, in the consideration of non-smoothing elements of the full calculus it suffices to look at 2×2-upper left corners. Let us now pass to defining the block matrix spaces

$$L^{\mu,e}\begin{pmatrix} N \\ \times \\ \partial N \end{pmatrix}, \boldsymbol{g} \; ; \mathbb{R}^d_\zeta \Bigg), \quad L^{\mu,e}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d_\zeta) \tag{4.4.4}$$

for $N \times \partial N \times Y$ of edge BVPs on N with non-smoothing ingredients. The elements of the space on the right-hand side of (4.4.4) formally have the form

$$\mathcal{A}(\zeta) = (A_{ij}(\zeta))_{i,j=1,2,3}$$

where the 2×2-upper left corners $(A_{ij})_{i,j=1,2}$ are just (by definition) constituted by the operator space on the left of (4.4.4), the ingredient of which have the structure as the set of all operator functions

$$\mathcal{A}(\zeta) = H(\zeta) + (M+G)(\zeta) + A_{\text{int}}(\zeta) + C(\zeta) \tag{4.4.5}$$

where $\mathcal{C}(\zeta) \in L^{-\infty,e}\left(\begin{matrix} N \\ \times \\ \partial N \end{matrix}, \boldsymbol{g} \; ; \mathbb{R}^d_\zeta\right)$ are smoothing operators, $A_{\text{int}}(\zeta)$ in $\mathcal{B}^{\mu,e}\left(\begin{matrix} N \\ \times \\ \partial N \end{matrix} ; \mathbb{R}^d_\zeta\right)$, up to some cut-off factors $(1-\omega)$ from the left, $(1-\omega'')$ from the right, and $(M+G)(\zeta) \in L^{\mu,e}_{\text{M+G}}\left(\begin{matrix} N \\ \times \\ \partial N \end{matrix}, \boldsymbol{g} \; ; \mathbb{R}^d_\zeta\right)$ in 2×2-block matrix form. The elements referring to Y with entries in the third row or column are defined by the corresponding rows and columns of $\mathcal{G}+\mathcal{C}$, where $\mathcal{G}$ is a 3×3-block matrix with symbols as in Definition 4.2.1, namely Green, and $\mathcal{C}$ is a smoothing operator of the calculus, defined at the end of Section 4.5. Since it makes sense to formulate contributions close to Y and to refere to local coordinates $\mathbb{R}^q$ on Y we focus on corresponding local expressions. The operators $H(\zeta)$ are based on holomorphic Mellin symbols, according to the following notation. By

$$\mathcal{B}^{\mu,e}_{\mathcal{O}_v}\left(\begin{matrix} X \\ \times \\ \partial X \end{matrix} ; \mathbb{R}^{q+d}_{\eta,\zeta}\right) \tag{4.4.6}$$

denote the space of all $h(v,\eta,\zeta) \in \mathcal{A}(\mathbb{C}_v, \mathcal{B}^{\mu,e}\left(\begin{matrix} X \\ \times \\ \partial X \end{matrix} ; \mathbb{R}^{q+d}_{\eta,\zeta}\right))$ such that

$$h(v,\eta,\zeta)|_{\Gamma_\lambda \times \mathbb{R}^d_\zeta} \in \mathcal{B}^{\mu,e}\left(\begin{matrix} X \\ \times \\ \partial X \end{matrix} ; \Gamma_\lambda \times \mathbb{R}^{q+d}_{\eta,\zeta}\right)$$

for every $\lambda \in \mathbb{R}$, uniformly in compact λ intervals. The space (4.4.6) is Fréchet in a natural way. This allows us to define the space of all $h(r,y,v,\eta,\zeta) = \tilde{h}(r,y,v,r\eta,r\zeta)$ for

$$\tilde{h}(r,y,v,\tilde{\eta},\tilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, \mathcal{B}^{\mu,e}_{\mathcal{O}_v}\left(\begin{matrix} X \\ \times \\ \partial X \end{matrix} ; \mathbb{R}^{q+d}_{\tilde{\eta},\tilde{\zeta}}\right)). \tag{4.4.7}$$

We then form for any such h the operator

$$H(\zeta) = \mathrm{Op}_y\{r^{-\mu}\omega \mathrm{Op}_{M_r}^{\gamma-n/2}(h)(y,\eta,\zeta)\omega'\} \tag{4.4.8}$$

for cut-off functions $\omega \prec \omega'$. The representation (4.4.5) of operators in Boutet de Monvel's calculus on a manifold with edge and parameters ζ is justified by the results of [29] which extends corresponding results of [18] to the present situation of boundary value problems. According to this viewpoint we also define an edge symbol

$$\sigma_1(H(\cdot))(y,\eta,\zeta) := r^{-\mu}\mathrm{Op}_{M_r}^{\gamma-n/2}(h_0)(y,\eta,\zeta) \tag{4.4.9}$$

for $h_0(r,y,v,\eta,\zeta) := \tilde{h}(0,y,v,r\eta,r\zeta)$, and $(\eta,\zeta)\neq 0$. Then

$$\sigma_1(H(\cdot))(y,\eta,\zeta) : \begin{array}{c}\mathcal{K}^{s,\gamma}(X^\wedge)\\ \oplus\\ \mathcal{K}^{s,\gamma}((\partial X)^\wedge)\end{array} \longrightarrow \begin{array}{c}\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)\\ \oplus\\ \mathcal{K}^{s-\mu,\gamma-\mu}((\partial X)^\wedge)\end{array}. \tag{4.4.10}$$

Moreover, for

$$(M+G)(\zeta) \in L^{\mu,e}_{\mathrm{M+G}}\left(\begin{array}{c}N\\ \times\\ \partial N\end{array}, \boldsymbol{g} \;;\mathbb{R}^d_\zeta\right), \tag{4.4.11}$$

locally close to Y represented by $\mathrm{Op}_y(m+g)(\zeta)$. In other words, up to smoothing operators, to be defined below, we introduced the 3×3-block matrix operator space, i.e., the second space in formula (4.4.4). This will be often abbreviated by $L^{\mu,e}(\mathcal{N},\boldsymbol{g};\mathbb{R}^d_\zeta)$.

For elements of smoothing Mellin plus Green type, using charts and a partition of unity, etc., we define

$$\sigma_1((M+G)(\cdot))(y,\eta,\zeta) := m_{(\mu)}(y,\eta,\zeta) + g_{(\mu)}(y,\eta,\zeta) \tag{4.4.12}$$

for $(\eta,\zeta)\neq 0$, where $m_{(\mu)}$ is defined by (4.3.1) and $g_{(\mu)}$ is the homogeneous principal symbol of $g(y,\eta,\zeta)$, cf. Definition 4.2.1. Both $m_{(\mu)}$ and $g_{(\mu)}$ may be interpreted as 3×3 matrices by filling up the 2×2 matrix $m_{(\mu)}$ by zeros in the entries contained in the third row or column. A similar interpretation holds for the operator matrices themselves. The remaining summands on the right-hand side of (4.4.5), namely, $A_{\mathrm{int}}(\zeta)$ and $C(\zeta)$ do not contribute to $\sigma_1(\mathcal{A}(\cdot))(y,\eta,\zeta)$. However, we have a 2-component principal symbolic hierarchy

$$\sigma(\mathcal{A}(\cdot)) = (\sigma_0(\mathcal{A}(\cdot)), \sigma_1(\mathcal{A}(\cdot))) \tag{4.4.13}$$

where $\sigma_0(\mathcal{A}(\cdot))$ splits up to the interior part

$$\sigma_{0,\psi}(\mathcal{A}(\cdot))(x,\xi;r,y,\rho,\eta,\zeta) \tag{4.4.14}$$

which is the parameter-dependent homogeneous principal symbol containing all variables x,r,y and covariables ξ,ρ,η,ζ only depending on the upper left corner $A_{11}(\zeta)$ of the 3×3-operator block matrix

$$\mathcal{A}(\zeta) = \begin{pmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}(\zeta) \tag{4.4.15}$$

and

$$\sigma_{0,\partial}(\mathcal{A}(\cdot))(x',\xi';r,y,\rho,\eta,\zeta), \tag{4.4.16}$$

the boundary part which only depends on the upper left 2×2-corner of $\mathcal{A}(\zeta)$. Note that both $\sigma_{0,\psi}(\mathcal{A}(\cdot))(x,\xi;r,y,\rho,\eta,\zeta)$ and $\sigma_{0,\partial}(\mathcal{A}(\cdot))(x',\xi';r,y,\rho,\eta,\zeta)$, contain the weight factor $r^{-\mu}$. Moreover, we have a variant of so-called reduced symbol without this factor, namely, by

$$\tilde{\sigma}_{0,\psi}(\mathcal{A}(\cdot))(x,\xi;r,y,\rho,\eta,\zeta) := r^{\mu}\sigma_{0,\psi}(\mathcal{A}(\cdot))(x,\xi;r,y,r^{-1}\rho,r^{-1}\eta,r^{-1}\zeta) \tag{4.4.17}$$

and

$$\tilde{\sigma}_{0,\partial}(\mathcal{A}(\cdot))(x',\xi';r,y,\rho,\eta,\zeta) := r^{\mu}\sigma_{0,\partial}(\mathcal{A}(\cdot))(x',\xi';r,y,r^{-1}\rho,r^{-1}\eta,r^{-1}\zeta) \tag{4.4.18}$$

which are both smooth in r up to $r = 0$. Similarly as (4.4.14) and (4.4.16) the reduced symbols only depend on A_{11} and on the upper left 2×2-corner of (4.4.15).

Remark 4.4.1. *As is standard in symbolic hierarchies we have natural compatibility relations between the components of* (4.4.13)*, including between the ψ-and the ∂-part of σ_0.*

4.5 Continuity in weighted spaces

Similarly as in the edge calculus without boundary we have weighted spaces on N and ∂N, respectively, which are defined by local spaces of the kind

$$\mathcal{W}^s(\mathbb{R}^q, H) \tag{4.5.1}$$

where H is a Hilbert or Fréchet space with group action. In our application those spaces are Kegel spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ and $\mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)$ with asymptotics of type $\mathcal{P}$ as well as corresponding spaces referring to the boundary. Spaces over Y which are involved in operators of lower right corners are simply standard Sobolev spaces $H^s(Y)$, $s \in \mathbb{R}$. Recall that (4.5.1) is equipped with the norm

$$||u||_{\mathcal{W}^s(\mathbb{R}^q,H)} = \left\{ \int_{\mathbb{R}^q} [\eta]^{2s} ||\kappa^{-1}_{[\eta]}(Fu)(\eta)||^2_H \, d\!\!\!{}^-\eta \right\}^{1/2} \tag{4.5.2}$$

where a group action $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ on the Hilbert space H is determined as a family of isomorphisms

$$\kappa_\delta : H \longrightarrow H,$$

$\delta \in \mathbb{R}_+$, such that $\kappa_\delta\kappa_{\delta'} = \kappa_{\delta\delta'}, \kappa_1 = \mathrm{id}_H$, and $\delta \to \kappa_\delta h$ an element in $C(\mathbb{R}_+, H)$ for every $h \in H$. Such a definition extends to any Frèchet space E written as a projective limit of

$$E = \varprojlim_{j \in \mathbb{N}} E^j$$

embedded in E^0 for all j such that E^0 is equipped with a group action κ in the former sense, and the group action on E^j is the restriction of κ from E^0 to E^j. Our spaces $\mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)$ with asymptotics are of this kind. In the present situation $N \in \mathfrak{N}_1$ is a manifold with edge Y, with boundary. We employ notation as at the beginning of Section 4.1, namely, the process of doubling up a given space with boundary. In other words , for two copies of our compact manifold $N = N_+$ with edge and boundary and its negative counterpart N_- we form $2N := N_- \cup_\partial N_+$ where $\cup_\partial$ indicates gluing together N_+ and N_- along the common boundary. Then $B := 2N$ is a manifold with edge Y and without boundary, and B allows us to form a stretched manifold $\mathbb{B}$ which is locally close to Y modeled on $\overline{\mathbb{R}}_+ \times 2X \times \mathbb{R}^q$. Then the double $2\mathbb{B}$ of $\mathbb{B}$ is locally identified with $\mathbb{R}\times 2X\times\mathbb{R}^q$ with, i.e., by identifying the boundary of $\overline{\mathbb{R}}_+\times 2X\times\mathbb{R}^q$ with that of the negative counterpart $\overline{\mathbb{R}}_-\times 2X\times\mathbb{R}^q$ along $\{0\}\times 2X\times\mathbb{R}^q$. This common boundary is in turn corresponds to a locally trivial $2X$-bundle over Y. Now concerning concrete weighted edge spaces we first form

$$\mathcal{W}^s(\mathbb{R}^q,\mathcal{K}^{s,\gamma}((2X)^\wedge))$$

with respect to the group action κ on $H=\mathcal{K}^{s,\gamma}((2X)^\wedge))$ defined by

$$(\kappa_\delta u)(r,x):=\delta^{(n+1)/2}u(\delta r,x). \tag{4.5.3}$$

Globally on Y by using a convenient atlas of charts $U_j\to\mathbb{R}^q, j\in\mathbb{N}$, and a subordinate partition of unity we form the spaces

$$\mathcal{W}^s(Y,\mathcal{K}^{s,\gamma}((2X)^\wedge)),$$

which are subspaces of $H^s_{\mathrm{loc}}(s_0(2N))$. Thus, using a cut-off function ω on $2N$ ($\equiv 1$ close to Y, $\equiv 0$ off some small neighbourhood of Y) it makes sense to set

$$H^{s,\gamma}(2N):=\mathcal{W}^s(Y,\mathcal{K}^{s,\gamma}((2X)^\wedge))+(1-\omega)H^s_{\mathrm{loc}}(s_0(2N)). \tag{4.5.4}$$

We can restrict the spaces to $\operatorname{int}N_+$, by using that $2N=N_-\cup_\partial N_+$. This gives us

$$H^{s,\gamma}(N):=\omega\mathcal{W}^s(Y,\mathcal{K}^{s,\gamma}(X^\wedge_+))+(1-\omega)H^s_{\mathrm{loc}}(s_0(N_+)). \tag{4.5.5}$$

The space $\mathcal{W}^s(Y,\mathcal{K}^{s,\gamma}(X^\wedge))$ can also be understood in terms of $\mathcal{W}^s(\mathbb{R}^q,\mathcal{K}^{s,\gamma}(X^\wedge))$ in the sense of (4.5.2) for $H=\mathcal{K}^{s,\gamma}(X^\wedge)$ endowed with the group action (4.5.3) induced in an obvious manner from the case over $(2X)^\wedge$. The reason for the definition via doubles is that $X^\wedge$ has a second order corner, and operators have to be controlled close to such a singularity, which is done here thanks to the transmission property of our pseudo-differential operators with the $\mathrm{r}^+/\mathrm{e}^+$ convention close to the boundary. So the calculus in different singular directions, first $x_n\in\mathbb{R}_+$, the inner normal to the boundary and $r\in\mathbb{R}_+$, the cone axis variable for the open stretched cone $X^\wedge$ with the "right" Mellin operator convention will be possible, since formally we pretend X or N to behave like a boundaryless configuration. This will be done first in local terms on

$$\mathbb{R}_+\times\overline{\mathbb{R}}^n_+\times\mathbb{R}^q\ni(r,x,y), \tag{4.5.6}$$

for $x=(x',x_n)$ and then globally on N, using a natural transition behaviour between representations for different charts. Recall from Chapter 2 , given a symbol $a(x,\xi)$ on $\mathbb{R}^n$ with the transmission property at $x_n=0$ we define

$$\mathrm{Op}^+(a)(x',\xi')=\mathrm{r}^+\mathrm{Op}(a)(x',\xi')\mathrm{e}^+ \tag{4.5.7}$$

and then

$$\mathrm{Op}^+_x(a)=\mathrm{Op}_{x'}(\mathrm{Op}^+(a)(x',\xi')) \tag{4.5.8}$$

with (4.5.7) being interpreted as an operator-valued symbol in variables $x'\in\mathbb{R}^{n-1}$ and covariables ξ'. We envoke the concept of sleeping variables and covariables which are contained in the symbol a, namely, we have for the cone theory over $X^\wedge$ for $x=(x',x_n)$, cf. (4.5.6),

$$a(r,x,\rho,\xi)=r^{-\mu}a(r,x,r\rho,\xi) \tag{4.5.9}$$

or, more generally, in variables of (4.5.6)

$$a(r,x,y,\rho,\xi,\eta) = r^{-\mu}a(r,x,r\rho,\xi,r\eta) \tag{4.5.10}$$

for

$$\tilde{a}(r,x,y,\tilde{\rho},\xi,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}^n_+ \times \mathbb{R}^q, S^\mu_{\mathrm{tr}}(\mathbb{R}_{\tilde{\rho}} \times \mathbb{R}^n_\xi \times \mathbb{R}^q_{\tilde{\eta}})). \tag{4.5.11}$$

The operator convention (4.5.10), (4.5.11), for edge-degenerate symbols with transmission property will be combined with other elements of the edge calculus, namely, the kernel cut-off in Mellin terms, denoted by

$$V_\psi : S^\mu_{\mathrm{tr}}(\mathbb{R}_{\tilde{\rho}} \times \mathbb{R}^n_\xi \times \mathbb{R}^q_{\tilde{\eta}}) \longrightarrow S^\mu_{\mathcal{O}_v,\mathrm{tr}}(\mathbb{R}^n_\xi \times \mathbb{R}^q_{\tilde{\eta}}) \tag{4.5.12}$$

or, more generally,

$$V_\psi : C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}^n_+ \times \mathbb{R}^q, S^\mu_{\mathrm{tr}}(\mathbb{R}_{\tilde{\rho}} \times \mathbb{R}^n_\xi \times \mathbb{R}^q_{\tilde{\eta}})) \longrightarrow C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}^n_+ \times \mathbb{R}^q, S^\mu_{\mathcal{O}_v,\mathrm{tr}}(\mathbb{R}^n_\xi \times \mathbb{R}^q_{\tilde{\eta}})). \tag{4.5.13}$$

Applying these maps to symbols (4.5.11) gives us Mellin symbols

$$\tilde{h}_{11}(r,x,y,v,\xi,\tilde{\eta}) \in \mathcal{A}(\mathbb{C}_v, C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}^n_+ \times \mathbb{R}^q, S^\mu_{\mathrm{tr}}(\mathbb{R}^n_\xi \times \mathbb{R}^q_{\tilde{\eta}}))), \tag{4.5.14}$$

and our local $\mathrm{r}^+/\mathrm{e}^+$ operator convention combined with the Mellin convention gives rise to (y,η)-dependent families of operators

$$\tilde{h}_{11}(r,x,y,v,\xi,r\eta) := h_{11}(r,x,y,v,\xi,\eta) \tag{4.5.15}$$

$$\omega r^{-\mu}\mathrm{Op}^{\gamma-n/2}_{M_r}(\mathrm{Op}^+_x(h_{11}))(y,\eta)\omega' : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge), \tag{4.5.16}$$

for cut-off functions $\omega \prec \omega'$ in the axial variable r. Expression (4.5.16) contains some assumptions on how we interpret notation. From the construction on a manifold N with edge and boundary it is, up to an expression of local representatives interpreted in terms of (stretched) wedges $\mathbb{R}^q \times X^\wedge$ where because of subsequent summations we may assume that our functions in (4.5.14) have compact support in coordinate neighbourhoods on Y and X, respectively, and the functions are expressed in local coordinates in $\overline{\mathbb{R}}^n_+$ or $\mathbb{R}^q$. Clearly, up to some diffeomorphisms of the respective local representations in $x \in \overline{\mathbb{R}}^n_+$ and $y \in \mathbb{R}^q$ to X and Y, respectively, and multiplications by localizing factors $\psi_j \prec \psi'_j$, in y and $\varphi_l \prec \varphi'_l$ in x (expressed either on Y ($\mathbb{R}^q$) or on X ($\overline{\mathbb{R}}^n_+$)), where $(\psi_j)_{j=1,\dots,J}$ represents a partition of unity on Y subordinate to the chosen charts, and similarly, $(\varphi_l)_{l=1,\dots,L}$ on X, where the supports of ψ'_j and φ'_l are compact as well, the operators (4.5.16) contain $\psi_j\varphi_l \cdots \varphi'_i\psi'_j$, and finally the global operators are obtained by taking sums over $j = 1,\dots,J$ and $l = 1,\dots,L$. For brevity we omit this complicated notation but only preserve the cut-off functions ω, ω' multiplied from the respective sides which is a typical effect in our degenerate operators close to the edge. Possible problems with the pseudo-locality of operators will be cutted away by using a partition of unity for corresponding global actions on $X^\wedge$. In other words, in globalized form the Mellin operator families on $X^\wedge$ are mappings

$$\omega r^{-\mu}\mathrm{Op}^{\gamma-n/2}_{M_r}(f)(y,\eta)\omega' : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{4.5.17}$$

for $f(r,y,v,\eta) = \tilde{f}(r,y,v,r\eta)$,

$$\tilde{f}(r,y,v,\tilde{\eta}) = (\mathrm{Op}^+_x(h_{11}))(r,y,v,\eta)$$

belongs to $C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, \mathcal{B}^{\mu;e}_{\mathcal{O}_v}(X;\mathbb{R}^q_{\tilde{\eta}}))$. The operator families (4.5.16) can be generalized to the ζ-dependent case, assuming

$$\tilde{h}_{11}(r,x,y,v,\xi,\tilde{\eta},\tilde{\zeta}) \in \mathcal{A}(\mathbb{C}_v, C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}^n_+ \times \mathbb{R}^q, S^\mu_{\mathrm{tr}}(\mathbb{R}^n_\xi \times \mathbb{R}^q_{\tilde{\eta}} \times \mathbb{R}^d_{\tilde{\zeta}}))), \tag{4.5.18}$$

rather than (4.5.14). Other details on ζ-dependence are straightforward and then the context agrees with the one of the preceding section. When we write h_{11} instead of $r^{-\mu}\mathrm{Op}^{\gamma-n/2}_{M_r}$, then the globalized operator (4.5.17) represents an element

$$r^{-\mu}h_{11}(y,\eta,\zeta) = r^{-\mu}\mathrm{Op}^{\gamma-n/2}_{M_r}(f)(y,\eta,\zeta) \in S^\mu(\mathbb{R}^q \times \mathbb{R}^{q+d}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)). \tag{4.5.19}$$

Recall that the order μ is induced by the weight factor $r^{-\mu}$. Another argument for (4.5.19) is a standard tensor product argument. From this property we deduce the following.

Theorem 4.5.1. *The operator*

$$\varphi\mathrm{Op}_y\{r^{-\mu}\mathrm{Op}^{\gamma-n/2}_{M_r}(h_{11})(y,\eta,\zeta)\omega'\}\varphi' \tag{4.5.20}$$

induces a continuous operator

$$H_{11}(\zeta) : H^{s,\gamma}(N) \longrightarrow H^{s-\mu,\gamma-\mu}(N) \tag{4.5.21}$$

for every $s \in \mathbb{R}$.

Proof. The result is a consequence of (4.5.19), i.e., the continuity of pseudo-differential operators with symbols of twisted homogeneity μ. □

Similar continuity results hold for the remaining entries of (4.4.8) as well as for the contributions from (4.4.11), which yield, together with $A_{\mathrm{int}}(\zeta)$ and $C(\zeta)$. In fact, the entries $(H_{ij}(\zeta))_{i,j=1,2}$ are determined by complete symbols $h_{ij}(y,\eta,\zeta)$, more precisely,

$$h_{ij}(y,\eta,\zeta)_{i,j=1,2}(y,\eta,\zeta) \in S^\mu\left(\mathbb{R}^q \times \mathbb{R}^{q+d}; \begin{array}{c}\mathcal{K}^{s,\gamma}(X^\wedge)\\ \oplus \\ \mathcal{K}^{s,\gamma}((\partial X)^\wedge)\end{array}, \begin{array}{c}\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))\\ \oplus \\ \mathcal{K}^{s-\mu,\gamma-\mu}((\partial X)^\wedge)\end{array}\right), \tag{4.5.22}$$

then we obtain the following result.

Theorem 4.5.2. *The* 2×2*-upper left corners of operators* (4.4.5) *induce continuous operators*

$$\begin{array}{c}H^{s,\gamma}(N)\\ \oplus \\ H^{s,\gamma}(\partial N)\end{array} \longrightarrow \begin{array}{c}H^{s-\mu,\gamma-\mu}(N)\\ \oplus \\ H^{s-\mu,\gamma-\mu}(\partial N)\end{array} \tag{4.5.23}$$

for every $s \in \mathbb{R}$. *The other entries of* (4.4.5) *referring to the third components are completing the operators to continuous maps*

$$\begin{array}{c}H^{s,\gamma}(N)\\ \oplus \\ H^{s,\gamma}(\partial N)\\ \oplus \\ H^s(Y)\end{array} \longrightarrow \begin{array}{c}H^{s-\mu,\gamma-\mu}(N)\\ \oplus \\ H^{s-\mu,\gamma-\mu}(\partial N)\\ \oplus \\ H^{s-\mu}(Y)\end{array} \tag{4.5.24}$$

Proof. A similar result holds with respect to subspaces with asymptotics. The main contribution comes from a description of operators in terms of symbols in

$$S^{\mu}\left(\mathbb{R}^q \times \mathbb{R}^{q+d}; \begin{array}{c} \mathcal{K}^{s,\gamma}(X^{\wedge}) \\ \oplus \\ \mathcal{K}^{s,\gamma}((\partial X)^{\wedge}) \\ \oplus \\ \mathbb{C} \end{array}, \begin{array}{c} \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge})) \\ \oplus \\ \mathcal{K}^{s-\mu,\gamma-\mu}((\partial X)^{\wedge}) \\ \oplus \\ \mathbb{C} \end{array}\right). \tag{4.5.25}$$

The smoothing operator $C(\zeta)$ is continuous by definition, given below, and the interior part yields continuity in the 2×2-upper left corner, known from the boundaryless case. □

Remark 4.5.3. *Recall that all spaces contain general abbreviations in the sense that we consider spaces of distributional sections in vector bundles cf. the remarks at the beginning of Section* 4.1. *This holds both for the symbols and for the final operators. Although the meaning is straightforward, details require some voluminous notation.*

Remark 4.5.4. *There is an analogue of Theorem* 4.5.2 *which states continuity between corresponding subspaces with* (*discrete*) *asymptotics. This is employed later on for establishing elliptic regularity with asymptotics. In this framework we also employ continuity of a Green operators in our calculus from a space without asymptotics to someone with asymptotics.*

It remains to define the space

$$L^{-\infty,e}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d_{\zeta}) \tag{4.5.26}$$

of smoothing operators. Those can be defined, first for $e = 0$, starting with continuous operators

$$\begin{array}{c} H^{s,\gamma}(B) \\ \oplus \\ H^{s,\gamma}(\partial N) \\ \oplus \\ H^{s}(Y) \end{array} \longrightarrow \begin{array}{c} H^{\infty,\gamma-\mu}_{\mathcal{P}}(B) \\ \oplus \\ H^{\infty,\gamma-\mu}_{\mathcal{P}_{\partial}}(\partial N) \\ \oplus \\ H^{\infty}(Y) \end{array} \tag{4.5.27}$$

consisting of vectors over different manifolds B and ∂N with edge plus a smooth manifold Y. Here we have a natural local scalar product $(\cdot,\cdot)_0$

$$H^{0,0}(B) \oplus H^{0,0}(\partial N) \oplus H^{0}(Y).$$

For compact $B, \partial N, Y$ this allows us to define a non-degenerate sesquilinear pairing

$$(\cdot,\cdot)_0 : H^{s,\gamma}(B) \oplus H^{s,\gamma}(\partial N) \oplus H^{s}(Y) \times H^{-s,-\gamma}(B) \oplus H^{-s,-\gamma}(\partial N) \oplus H^{-s}(Y) \longrightarrow \mathbb{C} \tag{4.5.28}$$

which gives rise to formal adjoints $\mathcal{C}^*(\zeta)$ of continuous operators like (4.5.27). Then we first obtain

$$L^{-\infty,0}\begin{pmatrix} B & \\ \times & \\ \partial N & , \boldsymbol{g} \\ \times & \\ Y & \end{pmatrix} \tag{4.5.29}$$

as the set of all $\mathcal{C}$ of the kind (4.5.27) for all $s \in \mathbb{R}$ and for suitable asymptotic types $\mathcal{P}, \mathcal{P}_{\partial}$, such that $\mathcal{C}^*$ is of analogous structure, with other asymptotic types Q, Q_{∂}, depending on $\mathcal{C}$.

Such operators are well-known from the edge calculus without boundary, and those operator can be understood in terms of kernels. Concerning the ζ-dependent case we simply form

$$L^{-\infty,0}\begin{pmatrix} B & & \\ \times & & \\ \partial N & ,\boldsymbol{g} & ;\mathbb{R}^d_\zeta \\ \times & & \\ Y & & \end{pmatrix} := \mathcal{S}(\mathbb{R}^d_\zeta, L^{-\infty,0}\begin{pmatrix} B & \\ \times & \\ \partial N & ,\boldsymbol{g} \\ \times & \\ Y & \end{pmatrix}). \tag{4.5.30}$$

In order to get spaces of smoothing operators over $N \times \partial N \times Y$ we restrict the kernels associated with (4.5.30) from $B = 2N$ to $\operatorname{int} N$ where $N = N_+$ is the positive part of N contained in the double. This gives us the modification for

$$L^{-\infty,0}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d_\zeta) \tag{4.5.31}$$

non-trivial $e \in \mathbb{N}$ is similar to (4.2.12). We form sums

$$\mathcal{C}(\zeta) = \mathcal{C}_0(\zeta) + \sum_{j=1}^{e} \mathcal{C}_j(\zeta)\operatorname{diag}(\partial^j_{x_n}, 0, 0) \tag{4.5.32}$$

for $\mathcal{C}_0(\zeta)$ and $\mathcal{C}_j(\zeta)$ belonging to (4.5.31).

All operators in $L^{\mu,e}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d_\zeta)$ which are defined so far, except for $L^{-\infty,e}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d_\zeta)$ can be defined in terms of locally finite sums of local expressions, compactly supported in coordinate neighborhoods on the respective manifolds. Such operators are called properly supported. In other words, we have

Proposition 4.5.5. *Every $\mathcal{A}(\zeta) \in L^{\mu,e}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d_\zeta)$ can be written in the form*

$$\mathcal{A}(\zeta) = \mathcal{A}_0(\zeta) + \mathcal{C}(\zeta) \tag{4.5.33}$$

for a properly supported $\mathcal{A}_0(\zeta) \in L^{\mu,e}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d_\zeta)$ and some $\mathcal{C}(\zeta) \in L^{-\infty,e}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d_\zeta)$.

4.6 Compositions

Similarly as Boutet de Monvel's calculus on a smooth manifold with boundary we have the algebra property in which the symbols are multiplicative. This requires to note that the space $\Sigma^{\mu,e}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d)$ of parameter-dependnet symbols belonging to elements in $L^{\mu,e}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d_\zeta)$ itself has a rich structure. In particular, the components of

$$\sigma(\mathcal{A}(\cdot)) = (\sigma_0(\mathcal{A}(\cdot)), \sigma_1(\mathcal{A}(\cdot))) \tag{4.6.1}$$

satisfy natural compatibility conditions. Moreover, the map

$$\sigma : L^{\mu,e}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d_\zeta) \longrightarrow \Sigma^{\mu,e}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d) \tag{4.6.2}$$

is well-defined by a reproducing process of symbols when the elements $\mathcal{A}(\zeta)$ are given. For instance, if $\mathcal{G}(\zeta) \in L^{\mu,e}_{\mathrm{G}}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d_\zeta)$ we find the associated twisted homogeneous principal symbol by a limit

$$g_{(\mu)}(y,\eta,\zeta) = \sigma^{\mu}(\mathcal{G}(\cdot))(y,\eta,\zeta) =$$

$$\lim_{\delta\to\infty} \begin{pmatrix} \kappa_\delta & & 0 \\ & \kappa'_\delta & \\ 0 & & \delta^{\frac{n+1}{2}} \end{pmatrix}^{-1} \delta^{-\mu} g(y,\delta\eta,\delta\zeta) \begin{pmatrix} \kappa_\delta & & 0 \\ & \kappa'_\delta & \\ 0 & & \delta^{\frac{n+1}{2}} \end{pmatrix} \tag{4.6.3}$$

for the above-mentioned meaning of κ'_δ, where we apply a symbolic map $\mathcal{G}(\zeta) \to g(y,\eta,\zeta)$ according to the general rule to producing from a properly supported representative of $\mathcal{G}(\zeta)$ such that $\mathcal{G}(\zeta) = \mathrm{Op}_y(g)(\zeta)$ modulo $L^{-\infty,e}(\mathcal{N},\boldsymbol{g};\mathbb{R}^d_\zeta)$.

Thus, setting

$$L^{\mu-1,e}(\mathcal{N},\boldsymbol{g};\mathbb{R}^d_\zeta) := \{\mathcal{A} \in L^{\mu,e}(\mathcal{N},\boldsymbol{g};\mathbb{R}^d_\zeta) : \sigma^\mu(\mathcal{A}) = 0\}$$

we have an exact sequence

$$0 \longrightarrow L^{\mu-1,e}(\mathcal{N},\boldsymbol{g};\mathbb{R}^d_\zeta) \xrightarrow{\ \iota\ } L^{\mu,e}(\mathcal{N},\boldsymbol{g};\mathbb{R}^d_\zeta) \xrightarrow{\ \sigma\ } \Sigma^{\mu,e}(\mathcal{N},\boldsymbol{g};\mathbb{R}^d) \tag{4.6.4}$$

with ι being the canonical embedding, and the sequence splits, similarly as in the standard scalar pseudo-differential calculus.

For $L^{\mu,e}_{\mathrm{M+G}}$ and $L^{\mu,e}$ itself we have other adequate rules of the relationship between operators and principal symbols. Similarly as the above notation we successively form

$$L^{\mu-2,e}(\mathcal{N},\boldsymbol{g};\mathbb{R}^d_\zeta) := \{\mathcal{A} \in L^{\mu-1,e}(\mathcal{N},\boldsymbol{g};\mathbb{R}^d_\zeta) : \sigma^{\mu-1}(\mathcal{A}) = 0\}$$

using the fact that $L^{\mu-1,e}(\mathcal{N},\boldsymbol{g};\mathbb{R}^d_\zeta)$ has a principal symbol of order $\mu-1$, denoted by $\sigma^{\mu-1}$. By continuity this process we define

$$L^{\mu-(N+1),e}(\mathcal{N},\boldsymbol{g};\mathbb{R}^d_\zeta) := \{\mathcal{A} \in L^{\mu-N,e}(\mathcal{N},\boldsymbol{g};\mathbb{R}^d_\zeta) : \sigma^{\mu-N}(\mathcal{A}) = 0\}. \tag{4.6.5}$$

Theorem 4.6.1. *Let $\mathcal{A}_j(\zeta) \in L^{\mu-j,e}(\mathcal{N},\boldsymbol{g};\mathbb{R}^d_\zeta), j \in \mathbb{N}$, be an arbitrary sequence where the involved asymptotic types in Green operators as well as the Boutet de Monvel types e are independent of j. Then there is an $\mathcal{A}(\zeta) \in L^{\mu-j,e}(\mathcal{N},\boldsymbol{g};\mathbb{R}^d_\zeta)$ such that for every $N \in \mathbb{N}$*

$$\mathcal{A}(\zeta) - \sum_{j=0}^{N} \mathcal{A}_j(\zeta) \in L^{\mu-(N+1),e}(\mathcal{N},\boldsymbol{g};\mathbb{R}^d_\zeta), \tag{4.6.6}$$

and $\mathcal{A}(\zeta)$ is unique modulo $L^{-\infty,e}(\mathcal{N},\boldsymbol{g};\mathbb{R}^d_\zeta)$.

The proof is a straightforward consequence of the definition.

Theorem 4.6.2. *Given operators families $\mathcal{A}(\zeta) \in$ (4.5.26) for $\boldsymbol{g}_1 := (\gamma-\nu,\nu-(\mu+\nu),\Theta)$ and $\mathcal{B}(\zeta)$ of analogous structure, of order ν and type $\tilde{e}$ with weight data $\boldsymbol{g}_2 := (\gamma,\gamma-\nu,\Theta)$ we have*

$$\mathcal{A}(\zeta)\mathcal{B}(\zeta) \in L^{\mu+\nu,l}(\mathcal{N},\tilde{\boldsymbol{g}};\mathbb{R}^d_\zeta) \tag{4.6.7}$$

and

$$\sigma(\mathcal{A}(\cdot))\sigma(\mathcal{B}(\cdot)) = \sigma(\mathcal{A}(\cdot)\mathcal{B}(\cdot)) \tag{4.6.8}$$

for $\tilde{\boldsymbol{g}} = (\gamma,\gamma-(\mu+\nu),\Theta)$ and $l = \max\{\nu+e,\tilde{e}\}$, with component-wise composition. If $\mathcal{A}(\zeta)$ or $\mathcal{B}(\zeta)$ belongs to the respective operator spaces with subscript $M+G$ or G, then the same is true of the composition. In such compositions we either assume that N,Y are compact, or localized by compactly supported functions, or properly supported.

The proof is a generalization of the composition behaviour in edge algebras of the standard edge calculus, see, the methods of [18], [68].

4.7 Ellipticity

Ellipticity of operators in $\mathcal{A}(\zeta)$ in the space

$$L^{\mu,e}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d_\zeta) \tag{4.7.1}$$

is defined in terms of the symbols (4.6.1). First we have the symbols (including their reduced variants) $\sigma_0, \tilde{\sigma}_0, \sigma_1, \tilde{\sigma}_1$ from the parameter-dependent Boutet de Monvel's calculus, where for the moment we treat σ_0 and $\tilde{\sigma}_0$ as scalar symbols

$$\sigma_0(\mathcal{A}(\cdot))(\boldsymbol{x}, \boldsymbol{\xi}, \zeta) \tag{4.7.2}$$

where $\boldsymbol{x}$ is the variable running over $\operatorname{int} N$ and $\boldsymbol{\xi}$ the associated covariable. Close to ∂N we have a splitting $\boldsymbol{x} = (\boldsymbol{x}', \boldsymbol{x}_n)$ with covariables $\boldsymbol{\xi} = (\boldsymbol{\xi}', \boldsymbol{\xi}_n, \zeta)$ which gives us the operator-valued boundary symbol

$$\sigma_{0,\partial}(\mathcal{A}(\cdot))(\boldsymbol{x}', \boldsymbol{x}_n, \boldsymbol{\xi}', \boldsymbol{\xi}_n) : \begin{array}{c} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{array}. \tag{4.7.3}$$

Close to the edge Y we have a splitting of variables and covariables into $\boldsymbol{x} = (r, x_{\text{int}}, y)$ and covariables $\boldsymbol{\xi} = (\rho, \xi_{\text{int}}, \eta)$ for variables/covariables $(x_{\text{int}}, \xi_{\text{int}}, \zeta)$ globally over X, which gives us

$$\sigma_{0,\text{int}}(\mathcal{A}(\cdot))(r, x_{\text{int}}, y, \rho, \xi_{\text{int}}, \eta, \zeta), \tag{4.7.4}$$

and the reduced version

$$\tilde{\sigma}_{0,\text{int}}(\mathcal{A}(\cdot))(r, x_{\text{int}}, y, \rho, \xi_{\text{int}}, \eta, \zeta) = r^\mu \sigma_{0,\text{int}}(\mathcal{A}(\cdot))(r, x_{\text{int}}, y, r^{-1}\rho, r^{-1}\xi_{\text{int}}, r^{-1}\eta, r^{-1}\zeta). \tag{4.7.5}$$

Close to the edge Y and close to the boundary we can replace $(x_{\text{int}}, \xi_{\text{int}})$ by (x', x_n, ξ', ξ_n) which gives us

$$\tilde{\sigma}_{0,\partial}(\mathcal{A}(\cdot))(r, x', y, \rho, \xi', \eta, \zeta) = r^\mu \sigma_{0,\partial}(\mathcal{A}(\cdot))(r, x', y, r^{-1}\rho, r^{-1}\xi', r^{-1}\eta, r^{-1}\zeta). \tag{4.7.6}$$

Concerning the σ_1-level we may focus on a neighbourhood of Y and represent symbols by charts on the edge to $\mathbb{R}^q$ which gives us, using the splitting of variables/covariables into

$$(\boldsymbol{x}, \boldsymbol{\xi}, \zeta) = (r, x, y, \rho, \xi, \eta, \zeta)$$

$\sigma_1(\mathcal{A}(\cdot))(y, \eta, \zeta)$ which is a 3×3-block matrix with 2×2-upper left corner taking values in $\mathcal{B}^{\mu,e}\left(\begin{array}{c} X \\ \times \\ \partial X \end{array}; \mathbb{R}^d_\zeta\right)$-valued operator families. We extend this 2×2-family to a 3×3- family by adding zeros at all entries in the third row and column. Let us denote this "artificial" symbol by

$$\sigma^{\mathcal{B}}(\mathcal{A}(\cdot))(y, \eta, \zeta) \tag{4.7.7}$$

we then form

$$\begin{aligned} \sigma_1(\mathcal{A}(\cdot))(y, \eta, \zeta) =& \sigma_1^{\mathcal{B}}(\mathcal{A}(\cdot))(y, \eta, \zeta) + g_{(\mu)}(y, \eta, \zeta) \\ &: \begin{array}{c} \mathcal{K}^{s,\gamma}(X^\wedge) \\ \oplus \\ \mathcal{K}^{s,\gamma}((\partial X)^\wedge) \\ \oplus \\ \mathbb{C} \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \\ \oplus \\ \mathcal{K}^{s-\mu,\gamma-\mu}((\partial X)^\wedge). \\ \oplus \\ \mathbb{C} \end{array} \end{aligned} \tag{4.7.8}$$

Remember that all these constructions for symbols are valid for covariables $\neq 0$.

Definition 4.7.1. *An operator family*

$$\mathcal{A}(\zeta) \in L^{\mu,e}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d_\zeta) \tag{4.7.9}$$

is called elliptic if it is firstly

(i) σ_0- *elliptic, i.e.,* $\sigma_{0,\psi}$ *is non-vanishing for all* $(\boldsymbol{\xi}, \zeta) \neq 0$ *and the reduced interior symbol* $\tilde{\sigma}_{0,\psi}$ *is asked to be non-vanishing for all covectors* $\neq 0$ *and all variables including* $r = 0$*, Moreover,* $\sigma_{0,\partial}$ *as well as* $\tilde{\sigma}_{0,\partial}$ *are families of bijections for all* $(\rho, \xi', \eta, \zeta) \neq 0$*, in the reduced case also up to* $r = 0$*.*

(ii) *Secondly* σ_1- *elliptic, i.e.,* (4.7.8) *is a family of bijections between the respective spaces for all* $(\eta, \zeta) \neq 0$ *and all* $s > \max\{\mu, e\} - \frac{1}{2}$*.*

Theorem 4.7.2. *An* (*parameter-dependent*) *elliptic operator* $\mathcal{A}(\zeta)$ *in* (4.7.1) *has a* (*parameter-dependent*) *properly supported parametrix* $\mathcal{P}(\zeta)$ *in* $L^{-\mu,h}$ *over* $\mathcal{N} = N \times \partial N \times Y$ *with weight data* $\boldsymbol{g}^{-1} = (\gamma - \mu, \gamma, \Theta)$*, and* $h = \max\{e - \mu, 0\}$*, i.e.,*

$$\mathcal{P}(\zeta)\mathcal{A}(\zeta) = \mathcal{I} - \mathcal{C}_{\mathrm{L}}(\zeta), \quad \mathcal{A}(\zeta)\mathcal{P}(\zeta) = \mathcal{I} - \mathcal{C}_{\mathrm{R}}(\zeta) \tag{4.7.10}$$

for smoothing families $\mathcal{C}_{\mathrm{L}}(\zeta)$ *and* $\mathcal{C}_{\mathrm{R}}(\zeta)$*, associated with the weight data* $\boldsymbol{g}_{\mathrm{L}} = (\gamma, \gamma, \Theta)$ *and* $\boldsymbol{g}_{\mathrm{R}} = (\gamma - \mu, \gamma - \mu, \Theta)$*.*

Proof. Ellipticity of $\mathcal{A}(\zeta)$ guarantees invertibility or bijectivity of $\sigma(\mathcal{A})$. Thus Theorem 4.6.2 combined with (4.6.4) gives us a properly supported $\mathcal{P}_1(\zeta) = \mathrm{Op}(\sigma^{-1})$ such that

$$\mathcal{P}_1(\zeta)\mathcal{A}(\zeta) = \mathcal{I} - \mathcal{R}(\zeta)$$

where $\mathcal{R}(\zeta) \in L^{-1,e_l}$ over $\mathcal{N}$, for $e_l = \max\{\mu, e\}$ with weight data $\boldsymbol{g}_{\mathrm{L}} = (\gamma, \gamma, \Theta)$. Applying a finite analogoue of the Neumann series gives us an $\mathcal{M}(\zeta) \in L^{-1,e_l}$ and a refined parametrix $B_{\mathrm{M}}(\zeta) \in L^{-\mu,e_l}$ such that

$$B_{\mathrm{M}}(\zeta)\mathcal{A}(\zeta) = \mathcal{I} + \mathcal{M}(\zeta).$$

We also can apply an infinite formal Neumann series, using Theorem 4.6.1 such that we find a properly supported $\mathcal{D}(\zeta) \in L^{-1,e_l}$ such that

$$(\mathcal{I} + \mathcal{D}(\zeta))(\mathcal{I} - \mathcal{R}(\zeta)) = \mathcal{I} + \mathcal{C}_{\mathrm{L}}(\zeta)$$

for some $\mathcal{C}_{\mathrm{L}} \in L^{-\infty,e_l}$. Then we may set $\mathcal{P}(\zeta) = (\mathcal{I} + \mathcal{D}(\zeta))\mathcal{P}_1(\zeta)$. In a similar manner we can proceed from the right and a standard algebraic argument shows that $\mathcal{P}(\zeta)$ is at the same time a right parametrix. □

4.8 Fredholm property and invertibility

The following consideration refers to the case that N is compact.

Theorem 4.8.1. *An elliptic element* $\mathcal{A}(\zeta) \in L^{\mu,e}(\mathcal{N}, \boldsymbol{g}; \mathbb{R}^d_\zeta)$ *induces a family of Fredholm operators*

$$\mathcal{A}(\zeta) : \begin{array}{c} H^{s,\gamma}(N) \\ \oplus \\ H^{s,\gamma}(\partial N) \\ \oplus \\ H^{s}(Y) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu,\gamma-\mu}(N) \\ \oplus \\ H^{s-\mu,\gamma-\mu}(\partial N), \\ \oplus \\ H^{s-\mu}(Y) \end{array} \tag{4.8.1}$$

for all $s \in \mathbb{R}$. This holds including $d = 0$, i.e., the case without parameters. For $d > 0$ there is a constant $C > 0$ independent of s such that for $|\zeta| > C$ the operators (4.8.1) *are isomorphisms. There is then a family $\mathcal{P}(\zeta) \in L^{-\mu,h}(\mathcal{N}, \boldsymbol{g}^{-1}; \mathbb{R}^d_\zeta)$, for $h = \max\{e - \mu, 0\}$, such that for any fixed $|\zeta| > C$ we have $\mathcal{P}(\zeta) = \mathcal{A}^{-1}(\zeta)$.*

Remark 4.8.2. *The invertibility within our operator algebra also holds without parameters. In the "simplest form" it holds when $\mathcal{A}(\zeta) \in L^{\mu,e}(\mathcal{N}; \mathbb{R}^d_\zeta)$ is elliptic. Then, if $\mathcal{A}$ is invertible between the spaces in* (4.8.1) *for some fixed $s_0 \in \mathbb{R}$, then $\mathcal{A}(\zeta)$ is invertible for all $s \in \mathbb{R}$, and we have $\mathcal{A}^{-1}(\zeta) \in L^{-\mu,h}(\mathcal{N}; \mathbb{R}^d_\zeta)$. This property is also-called a "strong form" such a property holds without explicitly requiring ellipticity. In this frame ellipticity is a necessary consequence of the Fredholm property or invertibility. It is highly likely that this is true in the present context, though a proof might be voluminous.*

Proof of Theorem 4.8.1. First, from Theorem 4.7.2 we know that there is a parameter-dependnet parametrix $L^{-\mu,h}(\mathcal{N}, \boldsymbol{g}^{-1}; \mathbb{R}^d_\zeta)$. That means we have relations (4.7.10). In the compact case the respective operator families represent families of continuous operator in the weighted spaces occurring in (4.8.1) , cf. the second statement of Theorem 4.5.2, for any $s \in \mathbb{R}$ which is suitable in Boutet de Monvel's framework. Since the remainders in (4.7.10) are compact, the general criterion for the Fredholm property works, i.e., $\mathcal{A}(\zeta)$ is Fredholm for every ζ. At the same time, since the operator norms both of $\mathcal{C}_{\mathrm{L}}$ and $\mathcal{C}_{\mathrm{R}}$ tend to zero as $|\zeta| \to \infty$, we can invert $\mathcal{I} - \mathcal{C}_{\mathrm{L}}$ and $\mathcal{I} - \mathcal{C}_{\mathrm{R}}$ and from (4.7.10) we obtain

$$(\mathcal{I} - \mathcal{C}_{\mathrm{L}})^{-1}\mathcal{P}(\zeta)\mathcal{A}(\zeta) = \mathcal{I},$$

and similarly from the other side. Thus

$$\mathcal{A}^{-1}(\zeta) = (\mathcal{I} - \mathcal{C}_{\mathrm{L}})^{-1}\mathcal{P}(\zeta) \tag{4.8.2}$$

left inverse of $\mathcal{A}(\zeta)$ and Theorem 4.6.2 tells us that this belongs to $L^{-\mu,h}(\mathcal{N}, \boldsymbol{g}^{-1}; \mathbb{R}^d_\zeta)$, when $|\zeta|$ is sufficiently large. In a similar way we find a right inverse. Thus (4.8.2) is a two-sided inverse, by virtue of a standard algebraic argument.

□

4.9 Examples and applications

These final notes indicate the context where BVPs on a manifold with edge appear in concrete examples. Let us mention in this connection the monographs [20] and [30] with numerous models on mixed, transmission and crack problems, see also other references to investigations in more classical context, see also [24], [25]. Another investigation [9] concerns a characterization of the well-known Zaremba problem in connection with the edge calculus on a specific, relatively simple, manifold with edge and boundary. The latter references show how concrete examples may open specific voluminous investigations.

Index

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