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# The Role of Trapping in Black Hole Spacetimes

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## Abstract

In the here presented work we discuss a series of results that are all in one way or another connected to the phenomenon of trapping in black hole spacetimes. First we present a comprehensive review of the Kerr-Newman-Taub-NUT-de-Sitter family of black hole spacetimes and their most important properties. From there we go into a detailed analysis of the behaviour of null geodesics in the exterior region of a sub-extremal Kerr spacetime. We show that most well known fundamental properties of null geodesics can be represented in one plot. In particular, one can see immediately that the ergoregion and trapping are separated in phase space. We then consider the sets of future/past trapped null geodesics in the exterior region of a sub-extremal Kerr-Newman-Taub-NUT spacetime. We show that from the point of view of any timelike observer outside of such a black hole, trapping can be understood as two smooth sets of spacelike directions on the celestial sphere of the observer. Therefore the topological structure of the trapped set on the celestial sphere of any observer is identical to that in Schwarzschild. We discuss how this is relevant to the black hole stability problem.

In a further development of these observations we introduce the notion of what it means for the shadow of two observers to be degenerate. We show that, away from the axis of symmetry, no continuous degeneration exists between the shadows of observers at any point in the exterior region of any Kerr-Newman black hole spacetime of unit mass. Therefore, except possibly for discrete changes, an observer can, by measuring the black holes shadow, determine the angular momentum and the charge of the black hole under observation, as well as the observer's radial position and angle of elevation above the equatorial plane. Furthermore, his/her relative velocity compared to a standard observer can also be measured. On the other hand, the black hole shadow does not allow for a full parameter resolution in the case of a Kerr-Newman-Taub-NUT black hole, as a continuous degeneration relating specific angular momentum, electric charge, NUT charge and elevation angle exists in this case.

We then use the celestial sphere to show that trapping is a generic feature of any black hole spacetime.

In the last chapter we then prove a generalization of the mode stability result of [Whiting \(1989\)](#) for the Teukolsky equation for the case of real frequencies. The main result of the last chapter states that a separated solution of the Teukolsky equation governing massless test fields on the Kerr spacetime, which is purely outgoing at infinity, and purely ingoing at the horizon, must vanish. This has the consequence, that for real frequencies, there are linearly independent fundamental solutions of the radial Teukolsky equation  $R_{\text{hor}}, R_{\text{out}}$ , which are purely ingoing

at the horizon, and purely outgoing at infinity, respectively. This fact yields a representation formula for solutions of the inhomogenous Teukolsky equation, and was recently used by [Shlapentokh-Rothman \(2015\)](#) for the scalar wave equation.

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# 1. Introduction and overview

General relativity completely changed our understanding of gravity and our ideas of space and time. After its initial observational confirmation, by the measurement of the light deflection around the sun during a solar eclipse ([Dyson et al., 1920](#)), progress in observational confirmation of general relativity was scarce and until the 1960's it disappeared from most physics departments. After 1960 the number of publications related to general relativity surged and in [Hulse and Taylor \(1975\)](#) the first evidence for gravitational waves was established with the observation of the binary pulsar.

In 2015 with GW150914 the ([LIGO Scientific Collaboration and Virgo Collaboration, 2016](#)) recorded the first direct detection of gravitational waves from a binary black hole merger. This led to the award of 2017's nobel prize to Kip Thorne, Rainer Weiss and Barry Barish. Adding in the recent observation of a binary neutron star collision with optical counter-part ([LIGO Scientific Collaboration and Virgo Collaboration, 2017](#)) and the expected picture from the Event Horizon Telescope probing the strong field regime. We are living in truly exciting times for the field of general relativity.

These developments provide motivation to also explore the theoretical and mathematical aspects of general relativity in depth, besides the fact that especially on the mathematical side there exist many problems that are interesting on their own accord. The results presented in this thesis fit in nicely with the current developments on the observational side. The discussion of black hole shadows in chapter [3](#) has a direct relation to the Event Horizon Telescope, while the mode stability result is loosely related to the ring down in the binary black hole mergers. The following introduction will provide the mathematical landscape in which the novel results that are presented in this thesis live.

The local existence of solutions to the Cauchy problem for the Einstein field equations was proven by [Choquet-Bruhat \(1952\)](#). An essential step in this proof was to use the harmonic coordinate condition which reduces the Einstein field equations to a quasilinear system of wave equations. [Choquet-Bruhat and Geroch \(1969\)](#) proved that the maximal globally hyperbolic vacuum extension of a given Cauchy data set is unique. These results are among the cornerstones of the mathematical treatment of general relativity.



## 1. Introduction and overview

With this notion of dynamical evolution at hand it is possible to study the question of dynamical stability of solutions to the Einstein equations. A stationary solution is dynamically stable if the evolution of Cauchy data close to that solution settles down asymptotically to the given solution. [Christodoulou and Klainerman \(1993\)](#) proved the full non-linear stability for Minkowski space in their monumental work. The corresponding stability problem for the Kerr family of solutions is one of the most important open problems in general relativity.

Black holes play a central role in astrophysics and it is only if the Kerr family of solutions is stable that we can expect that some of the objects observed in the universe can be modeled by these solutions. For us to observe objects in the sky they either need to be very bright (super nova, gamma ray bursts) or long lived for us to come up with clever ways to observe them. For an object to be long lived its mathematical model needs to be stable against perturbations.

In order to approach the stability problem for the Kerr spacetime it is important to understand some simpler model problems. [Dafermos et al. \(2014\)](#) proved decay for the scalar wave for the full subextremal range  $|a| < M$ . The proof relies heavily on the Fourier transform of the scalar field and hence on the properties of the mode solutions. This provides the motivation for the discussion in section 2.2 and chapter 4. In two brilliant papers [Ma \(2017a,b\)](#) proved decay for the spin 1 Teukolsky equation originating from Maxwell fields and the spin 2 Teukolsky equation originating from linearized gravity for slowly rotating black holes.

The Kerr family of solutions admits only two Killing vector fields. The presence of a Killing tensor ([Carter, 1968a](#)) allows the geodesic equations to be separated. Correspondingly the associated symmetry operators make it possible to apply a separation of variables to the field equations for scalar fields (spin-0), Maxwell fields (spin-1), and linearized gravity (spin-2).

There are two phenomena that pose obstacles for the decay of spin fields. The first is trapping which is associated with the occurrence of trapped null geodesics. This is already present in Schwarzschild spacetimes. The second phenomenon is superradiance which is due to the presence of an ergoregion. This is the wave analog to the Penrose process in which energy can be extracted from a rotating black hole. Superradiance was demonstrated for the first time by [Starobinskii \(1973\)](#) using a mode analysis.

Another interesting phenomenon, obtained by a mode analysis, is the so called quasi normal ringing of the black hole. This will be discussed in section 4.1. Quasi-normal modes (QNM) are solutions with complex frequencies which are classically associated with wave equations containing a damping term. In the context of scalar fields on a black hole background these modes are associated with the phenomenon

of trapping for null geodesics. This builds the connection between trapping and mode stability.

It turns out that the question of black hole stability and black hole shadows are closely related, as the boundary of the shadow is given by the future/past trapped null geodesics. This provides the motivation for the considerations in section 3.1.

Having a thorough understanding of geodesic motion and in particular the behavior of null geodesics in Kerr spacetimes is helpful to understand many of the harder problems related to these spacetimes. Which is the motivation for our discussion in section 2.2 The Mathematica Notebook (2016) that has been developed for the lecture notes is intended to help the reader gain an intuition on the influence of various parameters on the geodesic motion.

## Overview of this Thesis

To make this thesis largely self contained we collect in chapter 2 the fundamental concepts of general relativity, which are relevant to our discussion. This includes the notion of causality, the definition of geodesics, and the Einstein field equations. In section 2.1 a number of important exact solutions to the Einstein field equations will be analyzed. In the context of the Schwarzschild solution the basic concepts of black holes will be discussed.

Then we discuss the details of null geodesic motion in the context of Kerr spacetimes in the sections 2.2 - 2.5. This includes the notions of trapping, principal null directions and  $\mathbf{T}$ -orthogonal geodesics.

In chapter 3 we introduce the notion of a black hole's shadow. We first show that in the black hole spacetimes of interest the future and past trapped set form smooth curves. Then in section 3.2 we define the notion of what it means for the shadow of two observers to be degenerate. We then analyze in section 3.3 which degeneracies exist. Finally in section 3.4 we show that the celestial sphere can be used more broadly by showing that trapping is a generic feature that exists in every black hole spacetime.

Finally in chapter 4 we first discuss how quasinormal modes relate to trapped null geodesics. Then in the sections 4.2-4.6 we prove mode stability for real frequencies for fields of arbitrary spin in subextremal Kerr spacetimes. In section 4.7 we provide some background on the charged scalar field in Reissner Nordström. Further we prove some partial results on conserved energies and explain the obstructions to obtain results that are known for the uncharged scalar field.



# 2. General relativity and Black holes

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This chapter contains a brief introduction into the fundamental concepts and equations of general relativity. Furthermore it should help to clarify the context of the discussion in chapter 3 and 4. This chapter will therefore focus especially on those fundamental properties which are relevant for the later part.

These general considerations can be found in many text books, see for example Carroll (2004). Hence it is to be understood that the material within this section

## 2. General relativity and Black holes

is all well known. The central postulate of general relativity is a 4-dimensional Lorentzian manifold equipped with a metric  $(\mathcal{M}, g)$ . The manifold  $\mathcal{M}$  is said to be time oriented if it admits a continuous, nowhere vanishing timelike vector field  $X$ . The signature of  $g$  can be chosen to be either  $(+, -, -, -)$  or  $(-, +, +, +)$ . The latter is chosen for this work. Let  $\gamma$  be a parametrized curve in the manifold. Let  $\dot{\gamma}$  be the derivative of the curve with respect to the parametrization. Then  $\gamma$  is either time-, space-, or lightlike/null depending on whether the value of the quantity

$$\epsilon = g(\dot{\gamma}, \dot{\gamma})|_p \quad (2.1)$$

is negative, positive, or zero. This property is important for the definition of causality in the context of general relativity. An event at one point on the manifold can influence events at another point on the manifold if there exists a causal, curve between the two points. A causal curve is a curve that is everywhere timelike or null. If the manifold is time oriented, the vector field  $X$  allows to further classify the causal curves into past and future directed subsets depending on the sign of  $g(X, \dot{\gamma})$ . A spacetime is then defined as a Lorentzian manifold together with a choice of time orientation. We can then make the following definition.

**Definition 2.0.1** (Chronological future/past). *Let  $S$  be a set of points in  $\mathcal{M}$ . The chronological future of  $S$  in  $\mathcal{M}$  is the set of points which can be connected to  $S$  by a future/past directed piecewise smooth timelike curve starting in  $S$ . We denote the chronological future with  $I^+(S)$  and the chronological past with  $I^-(S)$ .*

Note that  $I^\pm(S)$  are always open in  $\mathcal{M}$ .

**Definition 2.0.2** (Causal future/past). *Let  $S$  be a set of points in  $\mathcal{M}$ . The causal future of  $S$  in  $\mathcal{M}$  is the set of points which can be connected to  $S$  by a future/past directed piecewise smooth causal curve starting in  $S$ . We denote the causal future with  $J^+(S)$  and the causal past with  $J^-(S)$ .*

Note that every point on the boundary of  $J^+(S) \setminus S$  is the endpoint of a null geodesic that is either past inextendible or originates at the boundary  $\partial S$  of the set  $S$  under consideration. If  $S$  is a closed compact set, we can define a future domain of dependence  $D^+(S)$ .

**Definition 2.0.3** (Future/past domain of dependence). *The set of all points for which any past/future directed inextendible causal curve has to intersect  $S$  is called the future/past domain of dependence denoted by  $D^+(S)$  /  $D^-(S)$ .*

Suppose we have a well posed initial value problem defined on our manifold  $\mathcal{M}$ . By specifying the initial data on  $S$  the solution is uniquely determined in  $D^+(S)$ .

The boundary  $\partial D^+(S) \setminus S = H^+(S)$  of the future domain of dependence is called the future Cauchy horizon. The causal past  $J^-(S)$ , the past domain of dependence  $D^-(S)$ , and the past Cauchy horizon  $H^-(S)$  are defined analogous to their future counterparts. The causal properties of manifolds is the subject of causality theory, for a detailed discussion see for example (Beem, 2017, p.54 f). Within the present work we will only be concerned with spacetimes satisfying the strongest causality property. These are called globally hyperbolic spacetimes. A spacetime is said to be globally hyperbolic if there exists an achronal hypersurface  $\Sigma$  in  $\mathcal{M}$ , such that  $D^+(\Sigma) \cup D^-(\Sigma) = \mathcal{M}$ . Then we call  $\Sigma$  a Cauchy surface. A hypersurface is achronal if there exist no two points in  $\Sigma$  that are timelike separated.

In general relativity the force of gravity as imposed by Newtons gravity is replaced by the effect of the curvature of the manifold on the movement of unaccelerated particles therein. The parametrized curve  $\gamma(\tau)$  that describes the path of such an unaccelerated hence freely falling particle is called geodesic. The condition for the trajectory to be freely falling is that the tangent vector is parallel transported along the curve.

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \quad (2.2)$$

is called the geodesic equation. To express this equation in coordinate form we first need to introduce the Christoffel symbol

$$\Gamma_{\mu\nu}^{\beta} = \frac{1}{2} g^{\alpha\beta} (\partial_{\nu} g_{\alpha\mu} + \partial_{\mu} g_{\alpha\nu} - \partial_{\alpha} g_{\mu\nu}) \quad (2.3)$$

In coordinate form the covariant derivative of a vector field  $V^{\mu}$  is then given by

$$\nabla_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma_{\alpha\mu}^{\nu} V^{\alpha} \quad (2.4)$$

and thus the geodesic equation in coordinate form is given by

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma_{\rho\sigma}^{\mu} \frac{dx^{\rho}}{d\tau} \frac{dx^{\sigma}}{d\tau} = 0 \quad (2.5)$$

where  $x^{\mu}(\tau)$  is the parametrization of  $\gamma(\tau)$  with respect to the coordinate system. The parameter of the curve was chosen to be the proper time  $\lambda = \tau$  of the geodesic for timelike and the proper length for spacelike geodesics and any affine parametrization for null geodesics. For geodesics parametrized in this way  $\epsilon$ , defined in (2.1), is constant and equal to 1, -1, or 0 respectively. Spacelike geodesics are not considered to describe the trajectories of any physical particle. In terms of the Christoffel symbols the Riemann curvature tensor is then given by

$$R^{\mu}_{\alpha\beta\gamma} = \partial_{\beta} \Gamma_{\alpha\gamma}^{\mu} - \partial_{\gamma} \Gamma_{\alpha\beta}^{\mu} + \Gamma_{\sigma\gamma}^{\mu} \Gamma_{\beta\alpha}^{\sigma} - \Gamma_{\sigma\gamma}^{\mu} \Gamma_{\alpha\beta}^{\sigma} \quad (2.6)$$

## 2. General relativity and Black holes

The Ricci tensor is then given by  $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$  and the Ricci scalar is given by  $R = R^\alpha{}_\alpha$ . Let us now consider a one parameter family of non-intersecting geodesics  $\gamma_s(t)$ . They describe a two dimensional surface  $x^\mu(s, t)$ . This gives us two natural vector fields  $T^\mu = \partial x^\mu / \partial t$  and  $S^\mu = \partial x^\mu / \partial s$ . We can then define the “relative acceleration” between neighboring geodesics as

$$A^\mu = \frac{d^2}{dt^2} S^\mu = (\nabla_T(\nabla_T S))^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu) = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma. \quad (2.7)$$

This is known as the geodesic deviation equation. This can be interpreted as a manifestation of gravitational tidal forces.

A spacetime is considered to be physical if the metric satisfies the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}. \quad (2.8)$$

Where  $\Lambda$  is the cosmological constant and  $T_{\mu\nu}$  is the stress-energy tensor for the matter and radiation in the spacetime. Note that the equation here is in natural units with  $G = c = 1$ . We see that the curvature which effects the trajectories of particles through (2.7), couples to the energy distribution within the manifold. The stress-energy tensor  $T_{\mu\nu}$  is a symmetric 2-tensor on  $\mathcal{M}$  and describes the energy associated with a certain form of matter on  $\mathcal{M}$ . To prove properties of solutions to the equations (2.8) one can either impose a generic condition on the stress energy tensor  $T_{\mu\nu}$  or one can make a particular choice of matter fields on  $\mathcal{M}$ . The most commonly used energy conditions are the null energy condition ( $T_{\mu\nu} k^\mu k^\nu \geq 0$ , for every future pointing null vector  $k^\mu$ ), the weak energy condition ( $T_{\mu\nu} k^\mu k^\nu \geq 0$ , for every future pointing timelike vector  $k^\mu$ ), the dominant energy condition, which asserts that the weak energy condition holds and that for any future pointing causal vector  $y^\alpha$ , the vector  $-T^\beta{}_\alpha y^\alpha$  is a future pointing causal vector. Finally the strong energy condition requires that  $(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) k^\mu k^\nu \geq 0$  for every future pointing timelike vector  $k^\mu$ .

Now if we are interested in a particular matter model, then  $T_{\mu\nu}$  can be calculated as a function of the fields and the metric from the action  $S$  by variation with respect to the metric.

$$T_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \quad (2.9)$$

If we look for example at a massless scalar field  $\phi(\mathbf{x})$  on  $\mathcal{M}$  satisfying  $\square_g \phi$  the stress-energy tensor is given by

$$T_{\mu\nu}(\phi(\mathbf{x})) = \nabla_\mu\phi(\mathbf{x})\nabla_\nu\phi(\mathbf{x}) - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi(\mathbf{x})\nabla_\alpha\phi(\mathbf{x}). \quad (2.10)$$

One remarkable feature of general relativity is that, in contrast with Newtonian gravity, already the vacuum equations have non-trivial solutions. The vacuum field equations are obtained by setting  $T_{\mu\nu}$  to zero and are given by

$$R_{\mu\nu} = 0 \quad (2.11)$$

in the case of a vanishing cosmological constant.

## Overview of this section

First we will introduce the notion of Killing fields and Killing tensors in section 2.0.1. We demonstrate how one obtains conserved quantities for geodesics and conserved fluxes for fields. In section 2.1 we introduce a number of explicit solutions to Einstein's field equations. We start off with Minkowski space. In this context we introduce the conformal diagram of an asymptotically flat manifold. In section 2.1.1 we use the conformal diagram of the manifold to introduce a formal definition of a black hole. Further we will mention the photon sphere and trapping for the first time.

Then in section 2.1 we introduce the Kerr Newman Taub NUT de Sitter family of black hole spacetimes. The equations for null geodesics in these spacetimes is presented in 2.1.4.

We then continue our studies by investigating the null geodesics in Kerr spacetimes in detail. After discussing features of Kerr in section 2.3 we discuss the radial equation for null geodesics in section 2.4.1 and briefly discuss the  $\theta$  equation in section 2.4.2. In the following section 2.5 we first discuss the radially in-/out-going null geodesics in 2.5.1 then move on to the trapped set in 2.5.2 before we conclude with the  $\mathbf{T}$  orthogonal null geodesics in 2.5.3.

In section 2.6 we discuss how the insights from the previous sections can be used to understand the scalar wave equation in Kerr.

### 2.0.1. Killing Fields and Killing Tensors

Conservation laws simplify a lot of calculations in all fields of physics. In general relativity however, space and time do not exist as independent objects anymore and the classical notion of conservation of energy and momentum breaks down.



## 2. General relativity and Black holes

If our space-time contains additional structure, so called Killing fields and Killing tensors, we can find quantities which are conserved. In this section the properties of such Killing fields and Killing tensors will be discussed. A vector field  $K_\nu$  is called a Killing vector field if it satisfies the equation

$$\nabla_{(\mu}K_{\nu)} = 0. \quad (2.12)$$

The parentheses in the indices denote symmetrization. Every Killing field originates from a symmetry of the metric. It is in fact the local infinitesimal manifestation of a global symmetry. So if our spacetime contains one Killing field for example, then there exists a coordinate system in which the metric coefficients are independent of one coordinate (Carroll, 2004, pp.136). If we have multiple Killing fields there does not necessarily exist a coordinate system, in which they can all be simultaneously expressed as coordinate vector fields.

If an asymptotically flat spacetime features a Killing field which is timelike at infinity, then we call this metric stationary. We can then choose a coordinate system  $(t, x_1, x_2, x_3)$  in which the metric components are independent of  $t$  (Carroll, 2004, pp.203). If this timelike Killing field is orthogonal to a family of spacelike hypersurfaces then we call this metric static.

The above notation can be generalized to symmetric  $l$ -tensors  $K_{\nu_1 \dots \nu_l}$  in a straight forward way. The tensor  $K_{\nu_1 \dots \nu_l}$  is called a Killing tensor (Eisenhart, 2016, van Holten and Rietdijk, 1993) if it satisfies

$$\nabla_{(\mu}K_{\nu_1 \dots \nu_l)} = 0. \quad (2.13)$$

For geodesics a Killing tensor provides us with a conserved quantity if it is contracted with  $l$  copies of the four-momentum. The metric itself is a Killing tensor and the associated conserved quantity for geodesics was introduced in (2.1). Killing tensors are not directly related with symmetries of the space-time. However, they do relate to so called “hidden” symmetries Andersson and Blue (2009), which give rise to conserved currents.

### Conserved quantities and energy flows

For geodesics we can now construct a conserved quantity by contracting the Killing field  $K_\mu$  with the tangent vector of a geodesic. It is easy to show that

$$\dot{\gamma}^\mu \nabla_\mu (K_\nu \dot{\gamma}^\nu) = 0 \quad (2.14)$$

and hence this contracted quantity is conserved along the worldline of a particle moving on a geodesic. Similarly we can obtain conserved quantities from Killing

tensors by contracting with  $l$  copies of the tangent vector of a geodesic. Conserved quantities play an important role in solving certain equations on our manifold. The reason for this is that these conserved quantities, if enough of them are available, can enable us to decouple the geodesic equation.

We will now investigate the conservation laws that can be derived for scalar fields. Let us assume that we have a spacetime  $(\mathcal{M}, g)$  and a scalar function  $\phi$  that satisfies an evolution equation such that the stress-energy tensor  $T_{\mu\nu}(\phi)$  is divergence free

$$\nabla^\mu T_{\mu\nu} = 0. \quad (2.15)$$

Then we define the flux associated to any given vector field  $V^\nu$  as

$$P_\mu^V = V^\nu T_{\mu\nu}. \quad (2.16)$$

Its divergence is then simply

$$K^V = \pi(V)_{\mu\nu} T^{\mu\nu} = \nabla^\mu P_\mu^V. \quad (2.17)$$

Where  $\pi(X)$  is the deformation tensor

$$\pi(X)_{\mu\nu} = \frac{1}{2} \nabla_{(\mu} X_{\nu)} = Lie_X(g)_{\mu\nu} \quad (2.18)$$

and  $Lie_X(g)_{\mu\nu}$  is the Lie derivative of the metric along the vector field  $X$ . We see immediately that  $K^V = 0$  if  $V^\nu$  satisfies the Killing equation (2.12).

We now have to take a look at the divergence theorem for our Lorentzian manifold  $(\mathcal{M}, g)$ . Suppose  $\Sigma_0$  and  $\Sigma_1$  are two homologous spacelike, three dimensional hypersurfaces with common boundaries, bounding a spacetime region  $\mathcal{B}$ . Let  $n_0^\nu$  and  $n_1^\nu$  be the future directed, timelike, unit, normal vectors to these hyper-surfaces. Suppose  $J_\mu$  is an one-form defined on our manifold  $(\mathcal{M}, g)$ . The divergence theorem is then given by (Dafermos and Rodnianski, 2008, pp.103)

$$\int_{\Sigma_0} J_\mu n_0^\mu + \int_{\mathcal{B}} \nabla^\mu J_\mu = \int_{\Sigma_1} J_\mu n_1^\mu, \quad (2.19)$$

where the surface integrals are with respect to the induced oriented volume form on the hypersurface and the bulk integral is with respect to the volume form

$$\sqrt{-g} dx^0 \cdots dx^n. \quad (2.20)$$

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If we now take our one-form to be the vector field current  $P_\mu^V$  we get

$$\int_{\Sigma_0} P_\mu^V n_0^\mu + \int_{\mathcal{B}} \underbrace{\nabla^\mu P_\mu^V}_{=KV} = \int_{\Sigma_1} P_\mu^V n_1^\mu. \quad (2.21)$$

We see immediately from (2.18) that the vector field current is conserved if the vector field  $V^\nu$  satisfies the Killing equation. If  $V^\nu$  is a timelike vector field and  $T_{\mu\nu}$  satisfies the dominant energy condition then  $P_\mu^V n^\mu \geq 0$  for all  $n^\mu$  timelike and thus in particular  $\int_\Sigma P_\mu^V n^\mu \geq 0$  for any spacelike hypersurface  $\Sigma$ .

### 2.1. Exact Solutions to the Einstein Field Equations

The number of parametrized families of explicit solutions to the Einstein field equations is relatively small. The non-linearity of the equations makes the search for explicit solutions difficult. For the vacuum equations the important explicitly known solutions include Minkowski space, the Schwarzschild solution and the Kerr solution. In this section we will explore these solutions as well as a number of explicitly known non-vacuum solutions.

We will start with the easiest examples to introduce fundamental concepts needed for the further discussion. Minkowski space serves as reference space-time because it describes a flat manifold as is considered in classical field theories. Further the Minkowski metric serves as reference space time for asymptotic flatness. An asymptotically flat manifold is one for which the metric approaches that of Minkowski space in an appropriate sense as  $|x| \rightarrow \infty$ . The discussion here will follow Carroll (2004). In polar coordinates the Minkowski metric reads

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \quad (2.22)$$

where  $d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2$  describes the metric on a unit two sphere. The parameter range for  $\phi$  and  $\theta$  is as on the unit two sphere, while for  $r$  it is given by  $[0, \infty)$ , and for  $t$  by  $(-\infty, \infty)$ . In principle manifolds describing solutions to the Einstein field equations are non-compact. To get a better intuition, it helps to be able to draw a space time on a single piece of paper. For this purpose we introduce the so called Penrose- or conformal diagrams that help to capture the global structures of manifolds. A conformal transformation is a change of scale and can be written as

## 2.1. Exact Solutions to the Einstein Field Equations

$$\begin{aligned}\tilde{g}_{\mu\nu} &= \omega^2(x)g_{\mu\nu} \text{ or equivalently} \\ \tilde{ds}^2 &= \omega^2(x)ds^2\end{aligned}$$

where  $\omega(x)$  is a nonvanishing function. One important property of such conformal transformations is that null curves are left invariant. Hence the structure of causality is preserved.

We apply the following series of coordinate transformations to the Minkowski metric (this can be found in many text books, e.g. [Carroll \(2004\)](#))

$$\begin{array}{ll} u = t - r & -\infty < u < \infty \\ v = t + r & -\infty < v < \infty \\ \text{describing a set of null coordinates} & u \leq v, \\ U = \arctan(u) & -\frac{\pi}{2} < U < \frac{\pi}{2} \\ V = \arctan(v) & -\frac{\pi}{2} < V < \frac{\pi}{2} \\ \text{describing another set of null coordinates} & U \leq V, \\ T = V + U & \\ R = V - U & 0 \leq R < \pi \\ \text{describing a time-like and a radial coordinate} & |T| + R < \pi. \end{array}$$

The metric is then given by

$$ds^2 = \omega^{-2}(T, R)(-dT^2 + dR^2 + \sin^2(R)d\Omega^2). \quad (2.23)$$

The function  $\omega$  is given by  $\cos(T) + \cos(R)$  and the metric is thus conformally related to the unphysical metric

$$\tilde{ds}^2 = -dT^2 + dR^2 + \sin^2(R)d\Omega^2. \quad (2.24)$$

The entire Minkowski space can be represented by a subset of  $\mathbb{R} \times S^3$ . With  $S^3$  being a spacelike three sphere. The closure of the subset corresponding to Minkowski space is a compact manifold with boundaries given by

$$i^+ = \text{future time like infinity} \quad (T = \pi, R = 0),$$

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$$\begin{aligned}
 i^0 &= \text{spatial infinity} & (T = 0, R = \pi), \\
 i^- &= \text{past time like infinity} & (T = -\pi, R = 0), \\
 \mathcal{I}^+ &= \text{future null infinity} & (T = \pi - R, 0 < R < \pi), \\
 \mathcal{I}^- &= \text{past null infinity} & (T = -\pi + R, 0 < R < \pi).
 \end{aligned}$$

Together these make up the conformal infinity. We can now draw a representation of the quotient of Minkowski space over the two sphere in a simple, finite, two dimensional diagram as shown in Figure 2.1. Every point in this diagram corresponds to a two sphere with the radius given by the corresponding value of  $r$  at this point.

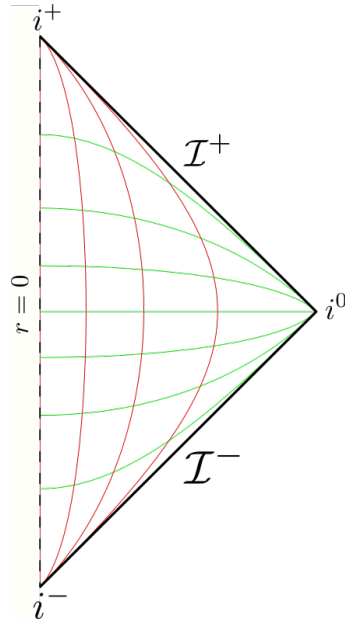


Figure 2.1.: The conformal diagram of Minkowski space. Light cones are at  $\pm 45^\circ$  throughout the diagram. The green lines correspond to hypersurfaces of constant  $t$  the red lines to hypersurfaces of constant  $r$ .

In Minkowski space all time-like geodesics will start at  $i^-$  and end at  $i^+$ . All null geodesics start somewhere on the null hypersurface  $\mathcal{I}^-$  and end on the null hypersurface  $\mathcal{I}^+$ . All radial null geodesics are at an angle of  $\pm 45^\circ$  in the diagram.

### 2.1.1. The Schwarzschild Solution and the Black Hole Concept

The Schwarzschild solution was found in [Schwarzschild \(1916\)](#) just one year after Einstein proposed his field equations. It took however almost 50 years until its properties were well understood.

We will now use the Schwarzschild space time to introduce the basic concepts relevant to understand black holes, such as horizons, singularities and trapping of null geodesics. In spherical coordinates  $(t, r, \theta, \phi)$  the metric is given by

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \quad (2.25)$$

This describes a one parameter family of solutions parametrized by  $M$ . It is straightforward to see that we recover Minkowski space when we set  $M = 0$  in this coordinate system. The parameter  $M$  is usually interpreted as the mass of the black hole. The Schwarzschild metric is static and asymptotically flat. However, one sees immediately that the metric coefficients blow up at  $r = 2M$  and at  $r = 0$ . The first one is not a true singularity of the spacetime, but rather a failure of the coordinate system to cover the whole manifold. This is analogous to the blow up of the usual spherical coordinates for  $\theta = 0, \pi$ . At  $r = 0$  on the other hand we have a true singularity of our spacetime. We see this by constructing scalar quantities from our metric such as

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48M^2}{r^6}. \quad (2.26)$$

Because of the fact that scalar quantities are coordinate invariant, this blow up is an actual feature of the manifold. The manifold is geodesically incomplete in the sense that there are geodesics which can reach the singularity at  $r = 0$  in a finite amount of their proper time for timelike geodesics or a finite amount of their affine parameter for null geodesics.

Even though the surface at  $r = 2M$  is not a singularity of this manifold, it turns out to have some interesting properties. It is the location of the event horizon. Here we only want to collect a number of facts while we will provide a proper definition of the event horizon later in the text. The coordinate vector field  $\partial_r$  becomes timelike inside that surface, whereas the coordinate vector field  $\partial_t$  becomes spacelike. On the surface itself the smooth extension of the vector field  $(\partial_t)^\mu$  becomes null. So the horizon itself is a null hypersurface with  $(\partial_t)^\mu$  tangent and normal to it. The horizon is a one way membrane in our manifold. Causal future directed curves can only pass it from the region with  $r > 2M$  to the region

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with  $r < 2M$  in the case of the future event horizon and from the region with  $r < 2M$  to the region with  $r > 2M$  in the case of the past event horizon.

We now want to understand the maximal extension of the Schwarzschild space-time.

**Definition 2.1.1** (Maximal extension). *The manifold  $(\mathcal{M}, g)$  is called inextendible as a Lorentian manifold of a certain regularity if the following holds:*

*Suppose there exists another Lorentzian manifold  $(\tilde{\mathcal{M}}, g)$  of the desired regularity together with an isometric embedding*

$$\mathbf{i} : (\mathcal{M}, g) \hookrightarrow (\tilde{\mathcal{M}}, g) \quad (2.27)$$

*then  $\mathbf{i}(\tilde{\mathcal{M}}) = \mathcal{M}$ .*

The Kruskal coordinates  $(T, R, \theta, \phi)$  (1960) which are given by

$$\begin{aligned} T &= \left( \frac{r}{2M} - 1 \right)^{1/2} e^{r/4M} \sinh \left( \frac{t}{4M} \right) \\ R &= \left( \frac{r}{2M} - 1 \right)^{1/2} e^{r/4M} \cosh \left( \frac{t}{4M} \right) \end{aligned}$$

and the usual spherical coordinates, are good coordinates throughout the maximal extension of the manifold. The Schwarzschild metric in Kruskal coordinates takes the form

$$ds^2 = \frac{32M^3}{r} e^{-r/2GM} (-dT^2 + dR^2) + r^2 d\Omega^2 \quad (2.28)$$

where  $r$  is defined implicitly by

$$T^2 - R^2 = \left( 1 - \frac{r}{2M} \right) e^{r/2M} \quad (2.29)$$

The radial null-curves are given by  $T = \pm R + \text{constant}$ . The horizon in these coordinates is given by  $T = \pm R$ . It was recently shown by Sbierski (2015) that Schwarzschild cannot be extended across  $r = 0$  as a  $C^0$  Lorentzian manifold.

To understand the black hole region and the horizon we resort again to the Penrose diagram. We can apply a similar compactification procedure as we did for Minkowski space and the Penrose diagram we obtain is then given by Figure 2.2.

## 2.1. Exact Solutions to the Einstein Field Equations

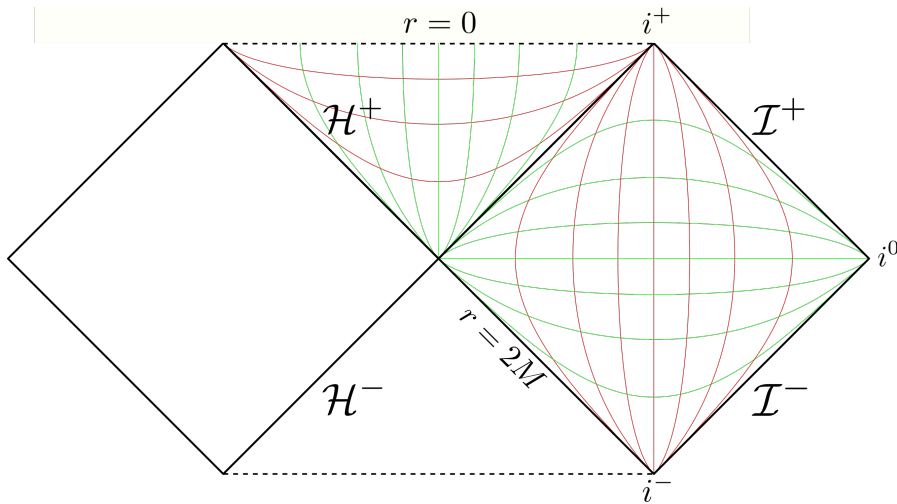


Figure 2.2.: Conformal diagram for the maximal extension of the Schwarzschild space-time. The green lines correspond to hypersurfaces of constant  $t$  the red lines to hypersurfaces of constant  $r$ .

In Figure 2.2 we can recognize four distinct regions of our manifold. The square regions correspond to two isometric copies of the exterior universe. Region upper triangle is the black hole region from where no causal curve can reach  $\mathcal{I}^+$ . the lower triangle is the white hole region from where causal future directed curves can only exit but not enter. The boundaries of the lower triangle is called the past horizon  $\mathcal{H}^-$ . The boundary of the upper triangle is called the future horizon  $\mathcal{H}^+$ . The black hole region is formally defined retroactively.

**Definition 2.1.2** (Black hole). *We define the black hole region by  $\mathcal{M} \setminus J^-(\mathcal{I}^+)$ .*

Hence the black hole consists of all the points in the manifold which do not lie within the causal past of  $\mathcal{I}^+$ . Therefore to know the location of the event horizon one needs to know the entire future development of the manifold.

Region  $I$  is given by  $J^+(\mathcal{I}^-) \cap J^-(\mathcal{I}^+)$  and is called the “exterior region”. Its closure is usually referred to as the “domain of outer communication”. In the exterior region the original Schwarzschild coordinates (2.25) are valid. As soon as objects falling into the black hole pass the horizon they can no longer causally influence processes in the domain of outer communication.

We will now have a quick look at the geodesic equations in these spacetimes as presented in Schwarzschild (1916).



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$$\mathcal{E} = V(r) + \frac{1}{2} \left( \frac{dr}{d\lambda} \right)^2, \quad (2.30)$$

with  $V(r) = -\frac{1}{2}\epsilon + \epsilon \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$ ,

and  $\mathcal{E} = \frac{1}{2}E^2$ .

Here  $\epsilon$  is given by (2.1). The quantities  $E$  and  $L$  are the conserved quantities originating from the Killing fields  $(\partial_t)^\mu$  and  $(\partial_\phi)^\mu$ . These Killing fields follow straight from the fact that the coefficients of the Schwarzschild metric are independent of these coordinates. The conserved quantities are given by

$$E = -(\partial_t)_\mu \dot{\gamma}^\mu = \left( 1 - \frac{2M}{r} \right) \frac{dt}{d\lambda} \quad L = (\partial_\phi)_\mu \dot{\gamma}^\mu = r^2 \frac{d\phi}{d\lambda} \quad (2.31)$$

and can be interpreted as the energy and the angular momentum of a particle moving on a geodesic. Every geodesic lies within a plane therefore we can always choose the  $z$ -axis to be perpendicular to the plane of motion. Therefore the  $z$ -angular momentum of the geodesic is the total angular momentum of the geodesic with respect to the black hole. Note that the expression for  $V(r)$  is exact. The last term is thus the correction from general relativity with respect to Newtonian gravity and can be used to calculate the perihelion shift of Mercury.

One important feature of black hole spacetimes is the existence of orbiting null geodesics. In the Schwarzschild solution we find circular orbits for null geodesics at a  $r = 3M$ . The sphere at  $r = 3M$  is often referred to as the photon sphere. An interesting quantity can be obtained by evaluating (2.30) at this radius. We get that the conserved quantities of every trapped null geodesic have to be in the ratio given by

$$\frac{L^2}{E^2} = 27M^2. \quad (2.32)$$

From this ratio a local observer at a point  $p$  can determine which null geodesics will approach the photon sphere asymptotically. For a Schwarzschild black hole of fixed mass  $M$  the only variable left in the virtual potential for null geodesics is the angular momentum  $L$ . The ratio above tells us nothing else but that the initial “energy” has to be equal with the maximal “energy” of the virtual potential. Only null geodesics that satisfy this condition are able to approach the photosphere asymptotically. We will explore the phenomenon of trapping in more detail in section 2.1.5, section 2.5.2 and section 3.1.

### 2.1.2. The Kerr-Newman-Taub-NUT-de-Sitter black hole spacetime

After exploiting the fundamental concepts of black hole spacetimes we now take a look at a larger family of black hole spacetimes.

In the following introduction of the spacetimes we follow closely the work of [Grenzbech et al. \(2014\)](#). However we will short cut their discussion in many places and just focus on the pieces relevant for our further considerations. The Kerr-Newman-NUT-(anti-)de Sitter space-times are stationary, axially symmetric solutions of the Einstein-Maxwell equations with a cosmological constant. [Plebański \(1975\)](#) introduced this class of spacetimes. A larger class was found by [Plebanski and Demianski \(1976\)](#). For the case without a cosmological constant, these metrics can be traced back to [Carter \(1968b\)](#).

In Boyer-Lindquist coordinates  $(t, r, \theta, \phi)$  the metric is given by ([Griffiths and Podolský, 2009](#))

$$ds^2 = \Sigma \left( \frac{1}{\Delta_r} dr^2 + \frac{1}{\Delta_\theta} d\theta^2 \right) + \frac{1}{\Sigma} \left( (\Sigma + a\chi)^2 \Delta_\theta \sin^2(\theta) - \Delta_r \chi^2 \right) d\phi^2 \\ + \frac{1}{\Sigma} (\Delta_r \chi - a(\Sigma + a\chi) \Delta_\theta \sin^2(\theta)) dt d\phi - \frac{1}{\Sigma} (\Delta_r - a^2 \Delta_\theta \sin^2(\theta)) dt^2. \quad (2.33)$$

where

$$\Sigma = r^2 + (l + a \cos \theta)^2, \quad (2.34)$$

$$\chi = a \sin^2(\theta) - 2l(\cos(\theta) + C), \quad (2.35)$$

$$\Delta = r^2 - 2Mr + a^2 - l^2 + Q^2, \quad (2.36)$$

$$\Delta_r = \Delta - \Lambda \left( (a^2 - l^2)l^2 + \left( \frac{1}{3}a^2 + 2l^2 \right) r^2 + \frac{1}{3}r^4 \right), \quad (2.37)$$

$$\Delta_\theta = 1 + \Lambda \left( \frac{4}{3}al \cos(\theta) + \frac{1}{3}a^2 \cos^2(\theta) \right). \quad (2.38)$$

The coordinates  $t$  ranges over  $(-\infty, \infty)$ , while  $\theta$  and  $\phi$  are standard coordinates on the two-sphere, for  $r$  we are only interested in the range between the horizon at  $r_H$  and infinity, or if it exists the cosmological horizon at  $r_C$ . The metric depends on five parameters, namely the mass  $M$ , the spin  $a$ , the electromagnetic charge  $Q^2 = Q_e^2 + Q_m^2$ , the NUT parameter  $l$  and the cosmological constant  $\Lambda$ . In addition, there is a parameter  $C$  that was introduced in [Manko and Ruiz \(2005\)](#) to modify the singularity that is produced by  $l$  on the  $z$  axis (see below). We will assume for physical reasons that  $M > 0$  and w.l.o.g.  $a \geq 0$  in the present work, while  $l$  and  $\Lambda$  can in principle take any value in  $\mathbb{R}$ . Note that the metric does not

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depend on the coordinates  $t$  and  $\phi$  and therefore features at least two independent Killing vector fields independent of the choice of the parameters.

We will frequently make use of the following orthonormal tetrad at a point  $p$ :

$$\begin{aligned} e_0 &= \left. \frac{(\Sigma + a\chi)\partial_t + a\partial_\phi}{\sqrt{\Sigma\Delta_r}} \right|_p, & e_1 &= \left. \sqrt{\frac{\Delta_\theta}{\Sigma}}\partial_\theta \right|_p, \\ e_2 &= \left. \frac{-(\partial_\phi + \chi\partial_t)}{\sqrt{\Sigma\Delta_\theta}\sin(\theta)} \right|_p, & e_3 &= \left. -\sqrt{\frac{\Delta_r}{\Sigma}}\partial_r \right|_p. \end{aligned} \quad (2.39)$$

This frame is a natural choice as the principal null directions can be written in the simple form  $e_0 \pm e_1$ . These generate congruences of radially outgoing and ingoing null geodesics.

The Plebański class contains the Schwarzschild ( $a = Q = l = \Lambda = 0$ ), Kerr ( $Q = l = \Lambda = 0$ ), Reissner- Nordström ( $a = l = \Lambda = 0$ ), Schwarzschild- (anti-)de Sitter ( $a = Q = l = 0$ ), Kerr-Newman ( $l = \Lambda = 0$ ), and Taub-NUT ( $a = Q = \Lambda = 0$ ) metrics as special cases.

The metric (2.33) becomes singular if,  $\Delta_r = 0$ ,  $\Delta_\theta = 0$ ,  $\sin(\theta) = 0$  or  $\Sigma = 0$ . Some of these singularities are just coordinate singularities, but some of them are true (curvature) singularities. We briefly discuss the first three cases in the following paragraphs.

1.  $\Delta_r = 0$ . Each zero of  $\Delta_r$  on the real line corresponds to a coordinate singularity indicating a horizon (excluding the case where  $l = a = 0$ ). As  $\Delta_r$  is a fourth order polynomial there can either be 4, 2 or 0 horizons. We call the horizon at the largest  $r$  coordinate the first horizon, the next the second, and so on.

If  $\Lambda \leq 0$  the second derivative of  $\Delta_r$  is strictly positive thus only two or no horizons can exist. In this case the first horizon is the black hole's event horizon. We are only interested in the exterior region in the black hole case where two horizons exist. In the case where  $\Lambda = 0$  the explicit solutions for the locations of the two horizons are given by

$$r_\pm = M \pm \sqrt{M^2 - a^2 + l^2 - Q^2} \quad (2.40)$$

and thus there is a subextremal black hole as long as  $M^2 - a^2 + l^2 + Q^2 > 0$ . If  $\Lambda > 0$ , the vectorfield  $\partial_r$  is timelike for large values of  $r$ . Therefore, the first horizon, if it exists, is called a cosmological horizon. We have a black hole if there are four horizons. The exterior region is then the region between

## 2.1. Exact Solutions to the Einstein Field Equations

the first and the second horizon which is the black hole's event horizon. Note that Descart's rule (Cohn, 1982, p.172) gives us immediately that for four horizons to exist the inequality

$$1/2 > l^2 \Lambda \quad (2.41)$$

needs to be satisfied.

In every case  $\Delta_r > 0$  is satisfied in the exterior region.

2.  $\Delta_\theta = 0$ . If it exists it indicates a horizon where the vector field  $\partial_\theta$  changes from space like to timelike. We are only interested in black holes without such horizons. Note first that  $\Delta_\theta = 0$  is only possible if  $\Lambda \neq 0$ . Solving it as a quadratic equation for  $a \cos(\theta)$  we get that

$$[a \cos(\theta)]_\pm = -2l \pm \sqrt{4l^2 - 3/\Lambda}. \quad (2.42)$$

For the cosmologically relevant case of positive  $\Lambda$  the restriction to the black hole case with four horizons guarantees that the square root is complex and the additional restriction in Grenzebach et al. (2014) is thus redundant. For negative  $\Lambda$  the square root is always real and thus the simple criterion in Grenzebach et al. (2014) is insufficient to guarantee that  $\Delta_\theta$  has no zeros. As we will not discuss the (anti-) de Sitter spacetimes in detail we will omit investigating whether this has any deeper implications.

3.  $\sin(\theta) = 0$ . The metric has a singularity on the axis, as is always the case when using spherical polar coordinates. If  $l \neq 0$ , however, this is not just a coordinate singularity but a true singularity. The parameter  $C$  induces pathologies on both pieces of the rotation axis, unless  $C = \pm 1$ . In the present work we will simply ignore these pathologies and take  $C = \pm 1$  and only consider the regular part of the rotation axis in these cases (the case  $C = -1$  corresponds to the original definition of the NUT metric Newman et al. (1963), Manko and Ruiz (2005)). For a detailed discussion of the rotation axis in the case  $l \neq 0$  and  $a \neq 0$  see Miller (1973).

For our discussion here we will assume that we stay away from such pathologies in the spacetime.

The cases without black holes we will only briefly discuss in the context of the Kerr metric. There the condition for the existence of a black hole is that  $|a| < M$  which are usually referred to as subextremal Kerr spacetimes. The case  $|a| = M$  still features an event horizon but now has zero surface gravity. This is referred to as the extremal Kerr spacetime. For  $|a| > M$  the spacetime features a naked singularity visible from infinity. This is sometimes referred to as super-extremal

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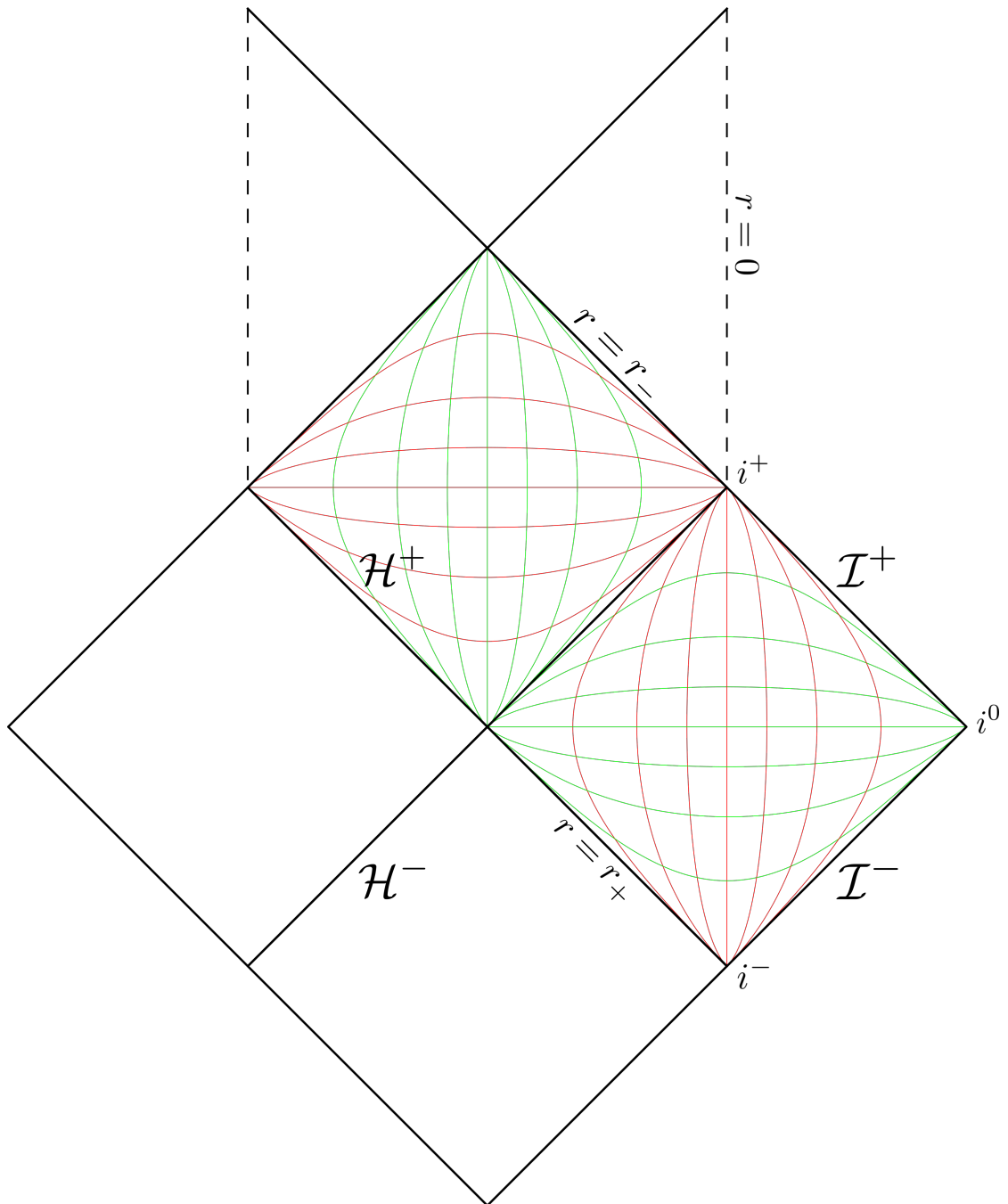


Figure 2.3.: Conformal diagram of the equatorial plane of a sub-extremal Kerr solution. The here represented region can be glued to copies of itself in an infinite sequence. The green lines correspond to hypersurfaces of constant  $t$  the red lines to hypersurfaces of constant  $r$ .

Kerr spacetime. The Penrose diagram of the maximal analytical extension of a subextremal Kerr spacetime was given in [Boyer and Lindquist \(1967\)](#).

In [Figure 2.3](#) we see an infinite number of identical copies of the exterior regions.

### 2.1.3. Open Problems in Mathematical Relativity related to Black Holes

Before going into the details of some of these black hole spacetimes we would like to present here a number of open mathematical problems related to black hole spacetimes. These open problems set the stage for our discussions in the rest of this thesis.

The **weak cosmic censorship conjecture** states that naked singularities cannot form in gravitational collapse from generic non singular initial data ([Carroll, 2004](#), pp.243). Hence that any singularity has to be hidden from an observer at infinity by an event horizon. Thus if a Kerr black hole is formed through gravitational collapse, its parameters have to lie in the subextremal range. All discussions from this point on will be limited to black hole spacetimes that satisfy this conjecture. [Thorne \(1974\)](#) showed from numerical calculations that an astrophysical black hole with an accretion disc tends towards a limiting value of  $a = 0.988$ . It can therefore not become extremal under their assumptions.

If one goes away from vacuum, there exists a counter example for the Einstein-scalar-field system in spherical symmetry given in [Christodoulou \(1994\)](#).

The **strong cosmic censorship conjecture** states that general relativity should be deterministic. In more technical terms this means that for generic initial data the Cauchy development should be locally inextendable across its boundary. [Dafermos and Luk \(2017\)](#) recently made significant progress towards a resolution of this conjecture.

The **black hole uniqueness conjecture** states that the Kerr family of solutions are the only existing solutions of the Einstein vacuum equations that are stationary asymptotically flat black hole spacetimes. This is sometimes referred to as the *no-hair conjecture* which has not yet been proven rigorously.

For the Schwarzschild solution there exists a uniqueness proof known as Birkhoff's theorem. It states that the Schwarzschild solution is the unique spherically symmetric solution to the Einstein vacuum equations. For the Kerr solution there are proven uniqueness statements under various additional assumptions. In [Hawking and Ellis \(1973\)](#), Hawking showed that there exists a second Killing vector field

## 2. General relativity and Black holes

on the event horizon of a stationary black hole spacetime. This allows to prove uniqueness under the assumption of analyticity. A similar strategy is employed by [Chruściel and Costa \(2008\)](#) again to prove a uniqueness theorem under the assumption of analyticity. In [Alexakis et al. \(2010\)](#) uniqueness of Kerr is proven close to the Kerr family of space times. One of the central obstacles to prove a global uniqueness result with this method is the possible existence of trapped null geodesics that are orthogonal to the stationary Killing vector field.

The **black hole stability conjecture** has already been mentioned in the introduction. This states that a small perturbation of Kerr should radiate away and the spacetime should asymptotically settle down to Kerr. There have been a lot of developments on this problem recently. The most important being the proof of non-linear stability of slowly rotating Kerr-de-Sitter black holes by [Hintz and Vasy \(2016\)](#).

Last but not least there is the **final state conjecture**. It states that at late time all solutions to the Einstein vacuum equations with reasonable initial data should become a number of Kerr black holes moving away from each other.

### 2.1.4. Null Geodesic Equation

We now focus our attention on null geodesics in these spacetimes. For all members of the Kerr-Newman-Taub-Nut-(anti-)De Sitter family of spacetimes there exist four linearly independent constants of motions for the geodesic equation. The norm of the tangent vector usually interpreted as the test bodie's mass

$$m = g_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu \quad (2.43)$$

which we will assume to be equal to zero from here on. The two quantities arising from the Killing vector fields  $\partial_t$ ,  $\partial_\phi$

$$E = -(\partial_t)_\mu \dot{\gamma}^\mu, \quad L_z = (\partial_\phi)_\mu \dot{\gamma}^\mu \quad (2.44)$$

which are usually interpreted as the test bodie's energy and angular momentum with respect to the axis of symmetry. The fourth constant of motion is called Carter's constant,  $K$ , and it originates from the existence of a Killing tensor, given by:

$$\sigma_{\mu\nu} = \Sigma((e_1)_\mu(e_1)_\nu + (e_2)_\mu(e_2)_\nu) - (l + a \cos \theta)^2 g_{\mu\nu}, \quad K := \sigma_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu. \quad (2.45)$$

This tensor can be obtained from the general expression of the conformal Killing-Yano tensor for the Plebański-Demiański family of solutions, as presented in [Kubizňák and Krtoš \(2007\)](#). Carter's constant corresponds somewhat loosely to the total angular momentum of the test body.

## 2.1. Exact Solutions to the Einstein Field Equations

We have mentioned before that the two Killing vector fields generate one-parameter families of isometries. It is natural to ask if the Killing tensor present in the Kerr-Newman-Taub-NUT-de-Sitter spacetimes can also be related to some sort of symmetry. This question can be answered using Hamiltonian formalism. For a Hamiltonian flow parametrized by  $\lambda$  with Hamiltonian  $H$  the derivative of any function  $f(x, p)$  is given by the Poisson bracket:

$$\frac{df}{d\lambda} = \{H, f\} \equiv \frac{\partial H}{\partial p_\mu} \frac{\partial f}{\partial x^\mu} - \frac{\partial H}{\partial x^\mu} \frac{\partial f}{\partial p_\mu}. \quad (2.46)$$

Each smooth function on phase space can be taken as a Hamiltonian and therefore gives rise to a local flow. The geodesic flow is generated by the function  $m$ .  $E$  and  $L_z$  generate translations in  $t$  and  $\phi$ . For rotating black holes the function  $K$  generates a flow that depends on fiber coordinates  $p_\mu$  and can not be projected to a symmetry of the manifold itself.

It follows from equation (2.45) that in Schwarzschild  $K$  is the square of the total angular momentum of the particle and the flow does project to a symmetry of the manifold in this case. Carter's constant is non-negative for all time like or null geodesics, which can be seen immediately from equation (2.45) and the fact that  $g_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu \leq 0$  for any future directed causal geodesic. In the case of  $a \neq 0$  it is even strictly positive for any time like geodesic.

The constants of motion can be used to write the geodesic equation as a set of four first order ODEs, cf. (Chandrasekhar, 1998, p. 242):

$$\dot{t} = \frac{\chi(L_z - E\chi)}{\Sigma\Delta_\theta \sin^2(\theta)} + \frac{(\Sigma + a\chi)((\Sigma + a\chi)E - aL_z)}{\Sigma\Delta_r}, \quad (2.47a)$$

$$\dot{\phi} = \frac{L_z - E\chi}{\Sigma\Delta_\theta \sin^2(\theta)} + \frac{a((\Sigma + a\chi)E - aL_z)}{\Sigma\Delta_r}, \quad (2.47b)$$

$$\Sigma^2 \dot{r}^2 = R(r, E, L_z, K) = ((\Sigma + a\chi)E - aL_z)^2 - \Delta_r K, \quad (2.47c)$$

$$\Sigma^2 \dot{\theta}^2 = \Theta(\theta, E, L_z, K) = \Delta_\theta K - \frac{(\chi E - L_z)^2}{\sin^2 \theta}, \quad (2.47d)$$

where the dot denotes differentiation with respect to the affine parameter  $\lambda$ . Note that the radial and the  $\theta$  equation for null geodesics are homogenous in  $E$  and thus for  $E \neq 0$

$$R(r, E, L_z, K) = E^2 R(r, 1, L_E, K_E) \quad (2.48)$$

$$\Theta(\theta, E, L_z, K) = E^2 \Theta(r, 1, L_E, K_E). \quad (2.49)$$

where  $L_E = L_z/E$  and  $K_E = K/E^2$ . This is due to the fact that an affine reparametrization  $\lambda \mapsto \alpha\lambda$ ,  $\dot{\gamma} \mapsto \alpha^{-1}\dot{\gamma}$  changes the values of  $E$ ,  $L_z$  and  $K$  while



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leaving the trajectories and the aforementioned quotients unchanged.

It turns out that for some questions there are combinations of these conserved quantities that are more convenient to work with, so we give them their own names:

$$Q = K - (aE - L_z)^2, \quad (2.50)$$

$$L^2 = L_z^2 + Q. \quad (2.51)$$

One can think of  $L^2$  as the total angular momentum of the particle, in the sense that it is this quantity that is replaced with the spheroidal eigenvalue in the potential of the wave-equation, as we will show in Section 2.6. One can then think of  $Q$  as the component of the angular momentum in a direction perpendicular to the rotation axis of the black hole.<sup>1</sup> It is important though that these interpretations should not be taken too strictly, because geodesics in Kerr spacetimes do not feature a conserved total angular momentum vector.

**Remark 2.1.1.** *In contrast to  $K$ ,  $Q$  is not positive anymore but from the equation of motion (2.47d) we get the condition that  $\Theta \geq 0$  for any geodesic to exist at a point and thus a lower bound on  $Q$*

### 2.1.5. Trapping

One of the most important features of geodesic motion in black hole spacetimes is the possibility of trapping. A geodesic is called trapped if its motion is bounded in a spatially compact region away from the horizon. In Kerr-Newman-Taub-NUT this corresponds to the geodesics motion being bounded in the  $r$  direction. In [Grenzebach et al. \(2014\)](#) it was shown that one can obtain a parametrization of the conserved quantities for trapped null geodesics in terms of their radial location:

$$K_E = \frac{16r^2\Delta(r)}{(\Delta'(r))^2} \Big|_{r=r_{trapp}}, \quad (2.52a)$$

$$aL_E = (\Sigma + a\chi) - \frac{4r\Delta(r)}{\Delta'(r)} \Big|_{r=r_{trapp}}, \quad (2.52b)$$

where  $\Delta'(r) = 2r - 2M$  is the derivative of  $\Delta(r)$  with respect to  $r$ . Further the following inequality for the area of trapping, i.e. those points in the exterior region where trapped null geodesics exist, was derived in [Grenzebach et al. \(2014\)](#):

$$(4r\Delta(r) - \Sigma\Delta'(r))^2 \leq 16a^2r^2\Delta(r)\sin^2(\theta). \quad (2.53)$$

---

<sup>1</sup>The three quantities  $K$ ,  $Q$  and  $L^2$  are often labeled differently by different authors.

Further it was shown that all trapped null geodesics at fix radius  $r$  in the exterior region are unstable and therefore no trapped null geodesics oscillating in  $r$  can exist in this region. Thus, the above set is complete, in the sense that it includes all trapped null geodesics that exist in the exterior region.

## 2.2. Null geodesics in Kerr

**Remark 2.2.1.** *The material in this section was created in joint work with Marius Oancea and Blazej Ruba. See the lecture notes [Paganini et al. \(2016\)](#).*

We are now going to use the Kerr spacetime as an example for a more detailed discussion of the properties of geodesic motion in black hole spacetimes. The restriction to Kerr is in part motivated by the fact that in recent years the Kerr spacetimes have been subject of intense investigations regarding their stability and uniqueness. Having a thorough understanding of geodesic motion and in particular the behavior of null geodesics in Kerr spacetimes is helpful to understand many of the more difficult problems related to these spacetimes. In this section we study the properties of null geodesics in the exterior region in the sub-extremal case, where  $a \in [0, M)$ . The geodesic structure of Kerr spacetimes has been subject of a lot of research. So the facts presented in this section are by no means new, however our focus is on proving that all these properties can be read off from one simple plot, thereby giving a unified and accessible presentation of the most important properties of geodesics in the Kerr spacetime, with regard to the open problems mentioned above. This presentation should make it easier for people to understand the general behaviour of null geodesics in Kerr.

The Mathematica notebook that has been developed for the lecture notes [Paganini et al. \(2016\)](#) is intended to help the reader gain an intuition on the influence of various parameters on the geodesic motion, despite the complexity of the underlying equations. The [Notebook \(2016\)](#) can be downloaded under the permanent link in the bibliography. In [Section 2.6](#) we explain where these plots give useful insights.

An extensive discussion of geodesics in Kerr spacetimes and many further references can for example be found in ([Chandrasekhar, 1998](#), p.318) and [O’Neill \(2014\)](#). See [Teo \(2003\)](#) for a nice treatment of the trapped set in Kerr including many explicit plots of trapped null geodesics at different radii. Here we focus more on global properties of the null geodesics and less on the details of motion. Analyzing the turning points for a dynamical system is a powerful tool to extract information about its global behaviours. For example, in any  $1 + 1$  dimensional

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system stable bounded orbits only exist if there exist two disjoint turning points in the spatial direction between which the system can oscillate. For geodesics in Kerr this has been studied in detail by [Wilkins \(1972\)](#). The techniques used here are very close to that paper. A different representation of the forbidden regions in phase space can be found in ([O’Neill, 2014](#), p.214) and also in [Slezakova \(2006\)](#). The presentation chosen in this section is adapted to help understand the phase space decomposition used in the proof for the decay of the scalar wave in subextremal Kerr in [Dafermos et al. \(2014\)](#).

### 2.3. Features of subextremal Kerr spacetimes

The Kerr family of spacetimes describes axially symmetric, stationary and asymptotically flat black hole solutions to the vacuum Einstein field equations. In Boyer-Lindquist (BL) coordinates  $(t, r, \phi, \theta)$  the metric is given by (2.33) with  $Q = l = \Lambda = 0$ . We will use  $e_0$  as given in (2.39) as the local time direction. Furthermore we define the local rotation frequency of the black hole to be:

$$\omega(r) = \frac{a}{r^2 + a^2}, \quad (2.54)$$

which has the rotation frequency of the horizon as a limit for  $r \searrow r_+$ :

$$\omega_H = \omega(r_+). \quad (2.55)$$

The name choice for  $\omega(r)$  is motivated by noting that a particle at rest in the local inertial frame given by the tetrad (2.39) will move in the  $\phi$  direction in Boyer-Lindquist coordinates with  $\frac{d\phi}{dt} = \omega(r)$  with respect to an observer at rest in this frame at infinity.

We now take a closer look at the Killing field  $(\partial_t)^\mu$ . It is timelike in the asymptotically flat region and it becomes spacelike in the interior of the *ergoregion*, which is defined by the inequality  $g(\partial_t, \partial_t) \geq 0$  or in terms of BL-coordinates by  $-\Delta + a^2 \sin^2 \theta \leq 0$ . The case of equality determines the boundary of the ergoregion which is often referred to as the ergosphere. Solving for the case of equality we get the radius of the ergosphere to be:

$$r_{ergo}(\theta) = M + \sqrt{M^2 - a^2 \cos^2(\theta)}. \quad (2.56)$$

At the equator the ergosphere lies at  $r = 2M$  while it corresponds to the horizon  $r = r_+$  on the rotation axis. As a consequence of the fact that  $(\partial_t)^\mu$  becomes spacelike outside of the event horizon, the “energy”  $E = -(\partial_t)_\mu \dot{\gamma}^\mu$  of a geodesic can be negative inside the ergosphere. This leads us to the Penrose process.

### 2.3. Features of subextremal Kerr spacetimes

We start initially with a an object at infinity that carries the energy  $E^{(0)} = -(\partial_t)_\mu \dot{\gamma}^{(0)\mu}$  it follows a trajectory, into the ergosphere where it decays into two objects. We have local momentum conservation as in special relativity given by

$$\dot{\gamma}^{(0)\mu} = \dot{\gamma}^{(1)\mu} + \dot{\gamma}^{(2)\mu}. \quad (2.57)$$

By contracting with the Killing field we get  $E^{(0)} = E^{(1)} + E^{(2)}$ . If we arrange the

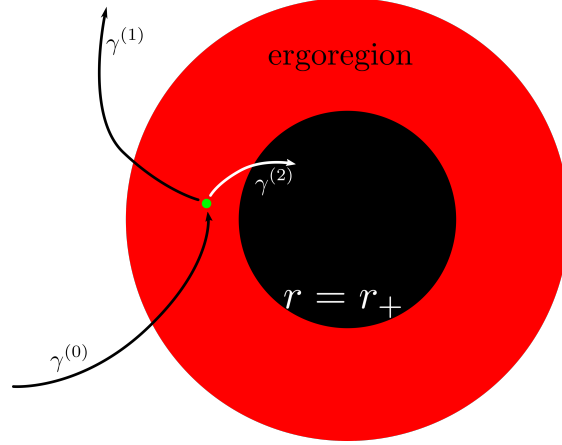


Figure 2.4.: The Penrose process (top view). A particle falls towards the black hole. Inside the ergoregion it breaks into two pieces. One of the falls into the black hole, the other escapes to infinity.

decay of our object in a way, such that  $E^{(2)} < 0$  and object 1 escapes to infinity then we get that  $E^{(1)} > E^{(0)}$ . We are thus able to extract energy from a rotating black hole. [Christodoulou \(1970\)](#) showed that the maximum amount of energy that can be extracted in this way is given by

$$M - M_{irr} = M - \frac{1}{\sqrt{2}} \left( M^2 + \sqrt{M^4 - a^2 M^2} \right)^{1/2} \quad (2.58)$$

Where  $M_{irr}$  is the irreducible mass of the black hole. This is the mass of the Schwarzschild black hole that is left over after all the rotational energy of the Kerr black hole is extracted through the Penrose process. It is given by [Christodoulou \(1970\)](#)

$$M_{irr} = \frac{A}{16\pi} \quad (2.59)$$

where  $A = 4\pi(r_+^2 + a^2)$  is the horizon area of the black hole. The horizon area theorem by [Hawking \(1971\)](#) states that the horizon area of a black hole can not

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shrink through a classical process. Therefore the above defined mass can only grow. The superradiance effect which will be briefly discussed in section 4.3.2 can be interpreted as a wave analog to the Penrose process.

### 2.4. Geodesic Equations

We now focus our attention on null geodesics. For  $L_z \neq 0$  the four equations are homogeneous in  $L_z$ . Analogous to (2.48) for the radial and the angular equations we have:

$$R(r, E, L_z, Q) = L_z^2 R(r, E_L, 1, Q_L), \quad (2.60)$$

$$\Theta(\theta, E, L_z, Q) = L_z^2 \Theta(r, E_L, 1, Q_L). \quad (2.61)$$

where  $Q_L = Q/L_z^2$  and  $E_L = E/L_z$ .

**Remark 2.4.1.** *To avoid introducing new functions whenever we change between different sets of conserved quantities we use*

$$R(r, E, L_z, Q) = R(r, E, L_z, K(Q, L_z, E)).$$

From the homogeneity of the equations of motion (2.47) we get that the only conserved quantities which affect the dynamics are conserved quotients like  $E_L$ ,  $Q_L$  or  $\frac{Q}{E^2}$  in the case of  $L_z = 0$ . The case  $L_z = 0$  can be seen as the limit of  $Q_L$  and  $E_L$  tending to infinity. In this section we will omit a separate discussion of this case as it is essentially equivalent to the Schwarzschild case and it is not needed for the understanding of the phase space decomposition in Dafermos et al. (2014). It is sufficient to consider future directed geodesics as the past directed case follows from the symmetry of the metric when replacing  $(t, \phi)$  with  $(-t, -\phi)$ . In Schwarzschild the condition  $\dot{t} > 0$  guarantees that the geodesic is future directed. For Kerr a suitable condition is to require that  $g(\dot{\gamma}, e_0) \leq 0$  is satisfied. From that we obtain the condition for causal geodesic in the exterior region to be future directed to be:

$$E \geq \omega(r)L_z. \quad (2.62)$$

In terms of the conserved quotients this condition takes the form:

$$\text{sgn}(L_z) = \begin{cases} +1 & \text{if } \mathcal{E} > \omega(r) \\ -1 & \text{if } \mathcal{E} < \omega(r) \\ \text{undet.} & \text{if } \mathcal{E} = \omega(r) \end{cases} \quad (2.63)$$

which eventually allows us to represent the pseudo potential, which will be introduced in the next subsection, for the co-rotating and the counter-rotating geodesics in the same plot.

### 2.4.1. The Radial Equation

In this section we characterize the radial motion by locating the turning points of a geodesic in  $r$  direction. Turning points are characterized by the fact that the component of the tangent vector  $\dot{\gamma}$  in the radial direction satisfies  $\dot{r} = 0$ . From equation (2.47c) we see immediately that the radial turning points are given by the zeros of the radial function  $R$ . In the following we will investigate the existence and location of these zeros. In this section we will be working with the conserved quantity  $Q$  as it is well suited to describe the phenomena we are interested here, namely the trapping and the null geodesics with negative energy.

**Lemma 2.4.1.**  $R(r, E, L_z, Q)$  is strictly positive in the exterior region for  $Q < 0$ .

*Proof.* The radial function can be written as:

$$R(r, E, L_z, Q) = E^2 r^4 + (a^2 E^2 - Q - L_z^2) r^2 + 2MKr - a^2 Q, \quad (2.64)$$

which is clearly positive for large  $r$ . For the proof we make use of the Descartes's rule, which states that if the terms of a polynomial with real coefficients are ordered by descending powers, then the number of positive roots is either equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by an even number. Powers with zero coefficient are omitted from the series. For a proof of Descartes's rule see for example (Cohn, 1982, p.172). Applied to (2.64) with  $Q < 0$  we get that for two zeros of  $R$  to exist in  $r \in (0, \infty)$  the conserved quantities of the geodesic have to satisfy the inequality:

$$a^2 E^2 - Q - L_z^2 < 0. \quad (2.65)$$

Otherwise there are no zeros at all and the proposition is true. Assume the contrary. Then for geodesics with certain parameters to exist at a given point, additionally to  $R \geq 0$  we also need to have that  $\Theta \geq 0$ . Applying this condition to equation (2.47d) and combining it with inequality (2.65) we obtain the following estimate:

$$-\cos^4 \theta a^2 E^2 \geq -Q > 0. \quad (2.66)$$

This is clearly a contradiction.  $\square$

Lemma 2.4.1 tells us that geodesics with  $Q < 0$  either come from past null infinity  $\mathcal{I}^-$  and cross the future event horizon  $\mathcal{H}^+$  or come out of the past event horizon  $\mathcal{H}^-$  and go to future null infinity  $\mathcal{I}^+$ . We will discuss the property of these geodesics in section 2.5.1. For the rest of this section we will restrict to the case of  $Q \geq 0$ . Note that even though some of the discussions and proofs might be simpler when

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working with  $K_L$  we are going to work with  $Q_L$  for all discussions that feed into the plot in section 2.6. This seems to us to be the natural choice adapted to describe trapping and the ergoregion in phase space.

To find the essential properties of the radial motion, we use pseudo potentials. The pseudo potential  $V(r, Q_L)$  is defined as the value of  $E_L$  such that the radius  $r$  is a turning point. In other words it is a solution to the equation:

$$R(r, V(r, Q_L), 1, Q_L) = \Sigma^2 \dot{r}^2 = 0. \quad (2.67)$$

This equation is quadratic in  $V(r, Q_L)$  and for non-negative  $Q_L$  there exist two real solutions at every radius, denoted by  $V_{\pm}$ . They are given by:

$$V_{\pm}(r, Q_L) = \frac{2Mar \pm \sqrt{r\Delta((1+Q_L)r^3 + a^2Q_L(r+2M))}}{r[r(r^2 + a^2) + 2Ma^2]}. \quad (2.68)$$

**Remark 2.4.2.** *The pseudo potentials should not be mistaken for potentials known from classical mechanics, where the equation of motion is given by  $\frac{1}{2}\dot{x}^2 + V(x) = E$ . However the potentials of classical mechanics can always be considered as pseudo potentials in the above sense.*

The radial function can be rewritten as:

$$R(r, \mathcal{E}, 1, Q_L) = L_z^2 r [r(r^2 + a^2) + 2Ma^2] (E_L - V_+(r, Q_L)) (E_L - V_-(r, Q_L)). \quad (2.69)$$

This form of  $R$  reveals the significance of the pseudo potentials: The only turning points that can exist for fixed  $Q_L > 0$  are those where either  $E_L = V_+(r, Q_L)$  or  $E_L = V_-(r, Q_L)$ . Analyzing the properties of the  $V_{\pm}$  allows us to extract all the information we are interested in.

First we note that for  $r$  large enough we have that  $V_+ > 0$  and  $V_- < 0$  for all  $Q_L \geq 0$ . However in the limit we have that:

$$\lim_{r \rightarrow \infty} V_{\pm} = 0. \quad (2.70)$$

At the horizon the limit of the pseudo potential and its derivative are given by:

$$\lim_{r \rightarrow r_+} V_{\pm}(r) = \omega_H \quad (2.71)$$

$$\lim_{r \rightarrow r_+} \frac{dV_{\pm}}{dr}(r) = \pm\infty. \quad (2.72)$$

**Lemma 2.4.2.** *For a fixed value of  $Q_L$  the pseudo potentials  $V_{\pm}(r, Q_L)$  have exactly one extremum as a function of  $r$  in the interval  $(r_+, \infty)$ .*

*Proof.* It is clear from the above properties that  $V_+$  ( $V_-$ ) has at least one maximum (minimum) in the DOC. From the fact that the two pseudo potentials have the same limiting value at  $\infty$  and at  $r_+$  together with (2.69) we get that in both limits we have that  $R(r, E_L, 1, Q_L) \geq 0$ . Therefore  $R(r, E_L, 1, Q_L)$  has to have an even number of zeros in the interval  $(r_+, \infty)$ . Given the fact that  $R(r, E_L, 1, Q_L)$  is a fourth order polynomial it can have at most 4 zeros. From the asymptotic behaviour of the potentials  $V_{\pm}$  we get that they need to have an odd number of extrema. Therefore if for some value of  $Q_L$  one of the potentials has more than one extremum there exists  $E_L$  such that  $R(r, E_L, 1, Q_L)$  has three zeros in  $r \in (r_+, \infty)$ . Applying Descartes' rule to (2.47c) we infer that  $R(r, E_L, 1, Q_L)$  can have at most three zeros in  $r \in [0, \infty)$ . But  $R(0, E_L, 1, Q_L) \leq 0$  and  $R(r_+, E_L, 1, Q_L) \geq 0$ . Hence there is at least one zero of  $R(r, E_L, 1, Q_L)$  in the interval  $[0, r_+]$ , so it is impossible for  $R(r, E_L, 1, Q_L)$  to have three zeros in  $(r_+, \infty)$ . From that it follows that  $V_{\pm}$  can both only have one extremum in that interval.  $\square$

Stationary points occur at the extrema of the pseudo potentials. So Lemma 2.4.2 tells us that for every fixed value of  $Q_L \geq 0$  there exist exactly two trapped geodesics with radii  $r = r_{trap}^{\pm}$  and energies  $E_{L_{trap\pm}} = V_{\pm}(r_{trap}^{\pm})$ . They will be studied in depth in section 2.5.2. Bounded geodesics with non-constant  $r$  would only be possible between two extrema of one of the pseudo potentials. These are excluded by the Lemma.

From (2.47c) we have that for any geodesics to exist we have to have  $R(r, E_L, 1, Q_L) \geq 0$ . This condition is satisfied except if  $V_-(r, Q_L) < E_L < V_+(r, Q_L)$ . This set is therefore a forbidden region in the  $(r, E_L)$  plane. Furthermore it follows that  $R(r, E_L, 1, Q_L) \leq 0$  for  $E_L = \omega(r)$  with equality only in the limits  $r \rightarrow r_+$  and  $r \rightarrow \infty$ . Therefore we have that:

$$V_-(r) \leq \omega(r) \leq V_+(r). \quad (2.73)$$

Again, the equality can occur only at the horizon and in the limit  $r \rightarrow \infty$ . This fact combined with (2.63) shows that for future pointing null geodesics

$$\text{sgn}(L_z) = \begin{cases} +1 & \text{if } E_L \geq V_+(r), \\ -1 & \text{if } E_L \leq V_-(r). \end{cases} \quad (2.74)$$

Therefore the pseudopotential  $V_+$  determines the behaviour of co-rotating null geodesics and  $V_-$  that of counter-rotating ones. Furthermore it is worth noting that  $E_L \geq V_+(r)$  implies  $E > 0$ . Finally we observe that for every fixed radius  $r \geq r_+$  we get from inspection of (2.68) that:

$$\frac{\partial V_-}{\partial Q_L} \leq 0 \leq \frac{\partial V_+}{\partial Q_L} \quad (2.75)$$



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holds. The equality in the relation occurs again only in the limits  $r \rightarrow r_+$  and  $r \rightarrow \infty$ . This means that for every radius  $r > r_+$  the range of forbidden values of  $E_L$  is strictly expanding as  $Q_L$  increases. This fact will be quite useful for the considerations in section 2.5.2.

### 2.4.2. The $\theta$ Equation

In Schwarzschild spacetimes, due to spherical symmetry the motion of any geodesic is contained in a plane. This means that for every geodesic there exists a spherical coordinate system in which it is constrained to the equatorial plane  $\theta = \frac{\pi}{2}$ . This is no longer true in Kerr spacetimes, but most geodesics are still constrained in  $\theta$  direction. The allowed range of  $\theta$  is obtained by solving the inequality  $\Theta(\theta, E, L_z, Q) \geq 0$ . After multiplication with  $\sin^2(\theta)$ ,  $\Theta(\theta, E, L_z, Q)$  can be expressed as a quadratic polynomial in the variable  $\cos^2(\theta)$ . Hence  $\Theta(\theta, E, L_z, Q) = 0$  has two solutions given by:

$$\cos^2 \theta_{turn} = \frac{a^2 E^2 - L_z^2 - Q \pm \sqrt{(a^2 E^2 - L_z^2 - Q)^2 + 4a^2 E^2 Q}}{2a^2 E^2}. \quad (2.76)$$

For  $Q > 0$  only the solution with the plus sign is relevant and the motion will always be contained in a band  $\theta_{min} < \theta_{eq} < \theta_{max}$  symmetric about the equator  $\theta_{eq} = \frac{\pi}{2}$ . As  $|L_z|$  increases, this band shrinks. In fact only in the case  $L_z = 0$  is it possible for a geodesic to reach the poles  $\theta = 0, \theta = \pi$ . Otherwise  $\Theta(\theta, E, L_z, Q)$  blows up to  $-\infty$  there. If  $Q < 0$  both solutions are positive and the inclination of the geodesic with respect to the equator is also constrained away from the equator, so either  $\theta_{eq} < \theta_{min} < \theta_{max}$  or  $\theta_{eq} > \theta_{max} > \theta_{min}$ . These trajectories will be contained in a disjoint band which intersects neither the equator nor the pole. This band can degenerate to a point, i.e. there exist null geodesics which stay at  $\theta = \text{const}$ . The relevance of these trajectories and how they are connected to the Schwarzschild case will be discussed in the next section. All possibilities for the potentials that constrain the motion in  $\theta$  direction are summarized in the Figure 2.5.

## 2.5. Special Geodesics

We will now apply the discussions of the last section to describe a number of special geodesics in Kerr geometries. All of these are in some way related to either the black hole stability problem or the black hole uniqueness problem.

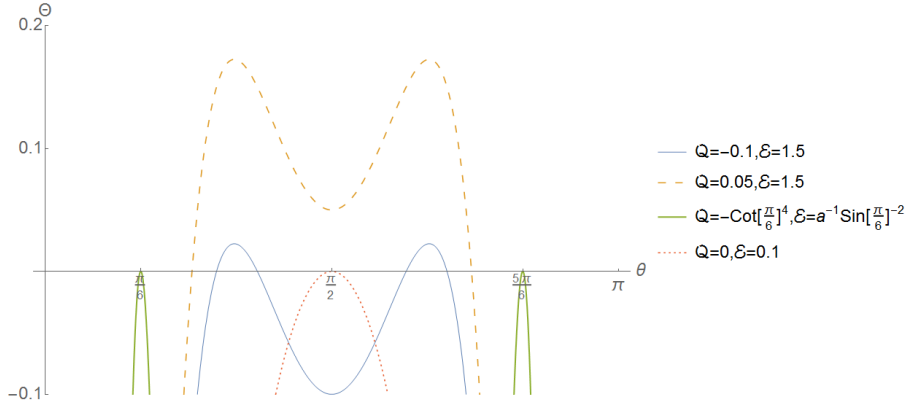


Figure 2.5.: This figure shows shapes of function  $\frac{\Theta}{L_z^2}$  for four choices of values of conserved quotients.

### 2.5.1. Radially In-/Out-going Null Geodesics

In this section we find geodesics which generalize the radially ingoing and outgoing congruences in Schwarzschild spacetimes. In section 2.4.1 we saw that the geodesics with  $Q < 0$  extend from the horizon to infinity. In section 2.4.2 we saw that  $Q < 0$  is again a special case, as these null geodesics can never intersect the equator and in the extreme case are even constrained to a fixed value of  $\theta$ . At first this behaviour seems odd, but a similar situation can be observed in Schwarzschild. If we look at geodesics which move in a plane with inclination  $\theta_0$  about the equatorial plane we see that there exists a set of null geodesics with similar properties as the ones with  $Q < 0$  in Kerr. It is clear that the radially ingoing geodesic which moves orthogonally to the axis around which the plane of motion was rotated, moves at fixed  $\theta$  value, namely that at which the plane is inclined with respect to the equatorial plane, hence  $\theta = \pi/2 \pm \theta_0$ . Furthermore some null geodesics reach the horizon before intersecting the equatorial plane. They don't necessarily move at fixed  $\theta$  but their motion in the  $\theta$  direction is still constrained away from the equatorial plane and away from the poles of the coordinate system.

Now we want to investigate the null geodesics which move at fixed  $\theta$  in Kerr. Demanding that  $\theta = \text{const.}$  is equivalent to requiring  $\Theta = \frac{d}{d\theta}\Theta = 0$ . From these conditions we obtain:

$$L_z = \pm aE \sin^2 \theta, \quad (2.77a)$$

$$Q = -a^2 E^2 \cos^4 \theta, \quad (2.77b)$$

$$K = 0. \quad (2.77c)$$

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By choosing the plus sign for  $L_z$  in the above equation, it follows from the remaining equations of motion that:

$$\frac{\dot{\phi}}{\dot{t}} = \frac{d\phi}{dt} = \omega(r), \quad (2.78a)$$

$$\frac{\dot{r}}{\dot{t}} = \frac{dr}{dt} = \pm \frac{\Delta}{r^2 + a^2}. \quad (2.78b)$$

This congruence is generated by the principal null directions  $e_0 \pm e_1$ . When choosing the minus sign for  $L_z$  in (2.77) the remaining equations of motions can not be simplified in a similar manner. In the case  $a = 0$  these are the radially in-/outgoing null geodesics. An interesting observation is that along these geodesics the inner product of the  $(\partial_t)^\mu$  vector field is monotone. A simple calculation shows that:

$$\dot{\gamma}^\mu \nabla_\mu ((\partial_t)^\nu (\partial_t)_\nu) = \dot{r} \frac{2M(r^2 - a^2 \cos^2 \theta)}{\Sigma^2} + \dot{\theta} \frac{2Ma^2 r \sin 2\theta}{\Sigma^2}. \quad (2.79)$$

For the principal null congruence we have  $\dot{\theta} = 0$ , the coefficient of  $\dot{r}$  is positive and there is no turning point in  $r$ . This property might be interesting in the context of the black hole uniqueness problem. If one could show a similar monotonicity statement for a congruence of null geodesics in general stationary black hole spacetimes, one could conclude that the ergosphere in such spacetimes has only one connected component enclosing the horizon. This is a necessary condition if one wants to show that no trapped  $\mathbf{T}$ -orthogonal null geodesics can exist in that case.

### 2.5.2. The Trapped Set in Kerr

In this section we will extend on the discussion of section 2.1.5. However we will limit our discussion to Kerr and use a different characterization in terms of conserved quantities. As mentioned in section 2.1.5 a geodesic in these spacetimes is trapped if its motion is bounded in  $r$  direction. This is only possible if  $r = \text{const.}$  or if the motion is between two turning points of the radial motion. For null geodesics in Kerr we ruled out the second option in Lemma 2.4.2. We will now discuss orbits of constant radius.<sup>2</sup> These null geodesics are stationary points of the radial motion, hence null geodesics with  $\dot{r} = \ddot{r} = 0$ . This condition is equivalent

---

<sup>2</sup>Null geodesics of constant radius are often referred to as "spherical null geodesics" but it is important to note that  $r = \text{const.}$  does not imply that the whole sphere is accessible for such geodesics.

to  $R(r) = \frac{d}{dr}R(r) = 0$ . The solutions to these equations can be parametrized explicitly by, see [Teo \(2003\)](#):

$$E_{L_{trap}}(r) = -\frac{a(r-M)}{A(r)} = \omega(r) \left(1 - \frac{2r\Delta}{A(r)}\right) \quad (2.80)$$

$$Q_{L_{trap}}(r) = -\frac{B(r)}{A^2(r)} \quad (2.81)$$

$$A(r) = r^3 - 3Mr^2 + a^2r + a^2M = (r - r_3)P_2(r) \quad (2.82)$$

$$B(r) = r^3(r^3 - 6Mr^2 + 9M^2r - 4a^2M) = (r - r_1)(r - r_2)P_4(r) \quad (2.83)$$

where  $P_2$  and  $P_4$  are polynomials in  $r$ , quadratic and quartic respectively, which are strictly positive in the DOC. The following three radii are particularly important:

$$r_1 = 2M \left(1 + \cos \left(\frac{2}{3} \arccos \left(-\frac{a}{M}\right)\right)\right) \quad (2.84)$$

$$r_2 = 2M \left(1 + \cos \left(\frac{2}{3} \arccos \left(\frac{a}{M}\right)\right)\right) \quad (2.85)$$

$$r_3 = M + 2\sqrt{M^2 - \frac{a^2}{3}} \cos \left(\frac{1}{3} \arccos \left(\frac{M(M^2 - a^2)}{(M^2 - \frac{a^2}{3})^{\frac{3}{2}}}\right)\right) \quad (2.86)$$

satisfying the inequalities:

$$M < r_+ < r_1 < r_3 < r_2 < 4M \quad (2.87)$$

for  $a \in (0, M)$ . Orbits of constant radius are allowed only inside the interval  $[r_1, r_2]$ , because outside of it  $Q$  would have to be negative. This possibility has already been excluded in [Lemma 2.4.1](#). The boundaries of the interval at  $r = r_1$  and  $r = r_2$  correspond to circular geodesics constrained to the equatorial plane with  $Q = 0$ . The trapped null geodesics at  $r = r_3$  have  $L_z = 0$  which is the reason why the functions  $E_{L_{trap}}$  and  $Q_{L_{trap}}$  blow up there. From the second representation in [\(2.80\)](#) we see that  $E_{L_{trap}}(r) - \omega(r)$  is positive in  $[r_1, r_3)$  and negative in  $(r_3, r_2]$ . Combined with [\(2.74\)](#) this implies that the stationary points in  $[r_1, r_3)$  correspond to extrema of  $V_+$  and the stationary points in  $(r_3, r_2]$  correspond to extrema of  $V_-$ . In [Lemma 2.4.2](#) we showed that  $V_+$  and  $V_-$  both have exactly one extremum. Since extrema of the pseudo potentials always correspond to orbits of constant radius, we get that the extrema of  $V_+(r, Q_L)$  and  $V_-(r, Q_L)$  have to be within the intervals  $[r_1, r_3)$  and  $(r_3, r_2]$  respectively for any value of  $Q_L$ . In [Figure 2.6](#) we plot the behaviour of these intervals as a function of  $a/M$ .

## 2. General relativity and Black holes

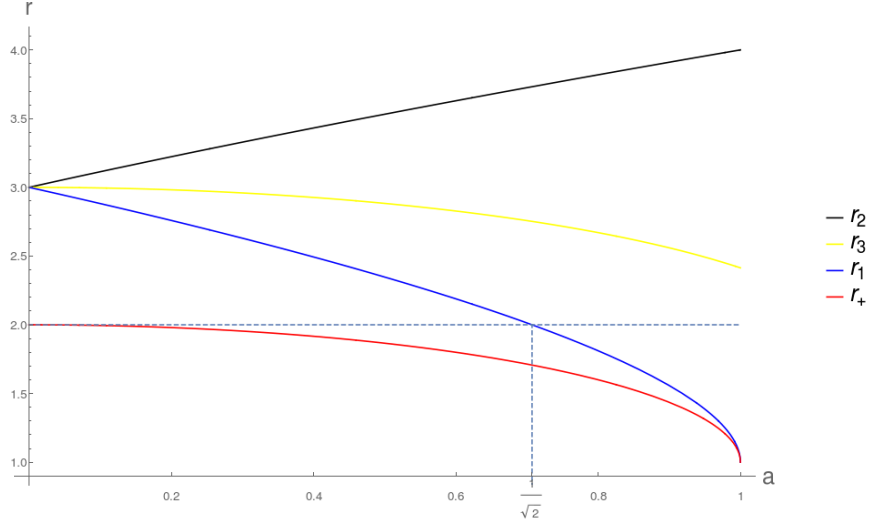


Figure 2.6.: Plot of the relation between the radius of the equatorial trapped null geodesics at  $r_1$  and  $r_2$ , the trapped null geodesic with  $L_z = 0$  at  $r_3$  and the horizon at  $r_+$  for all values of  $a$ .

We now know that the maps given by:

$$[0, \infty) \ni Q_L \mapsto r_{trap}^+ \in [r_1, r_3]$$

$$[0, \infty) \ni Q_L \mapsto r_{trap}^- \in (r_3, r_2]$$

which take  $Q_L$  into radii of trapped geodesics corresponding to the unique maximum of  $V_+(r, Q_L)$  and minimum of  $V_-(r, Q_L)$  respectively are one-to-one and therefore monotone. By using (2.81) the sign of their derivatives can be easily evaluated in some  $\epsilon$ -neighbourhood of  $r = r_3$  where the term of highest order in  $\frac{1}{r-r_3}$  dominates:

$$\frac{\partial r_{trap}^-}{\partial Q_L} < 0 < \frac{\partial r_{trap}^+}{\partial Q_L}. \quad (2.88)$$

By the equation (2.75) and the fact that radii of trapping always correspond to global extrema of the pseudo potentials we get that:

$$\frac{\partial}{\partial Q_L} E_{L_{trap}}(r_{trap}^-(Q_L)) < 0 < \frac{\partial}{\partial Q_L} E_{L_{trap}}(r_{trap}^+(Q_L)). \quad (2.89)$$

Using the chain rule and combining these two facts we obtain:

$$\frac{\partial E_{L_{trap}}}{\partial r} > 0. \quad (2.90)$$

These inequalities provide an important piece of the picture of the influence of  $Q_L$  on the trapped geodesics. We have  $Q_L = 0$  for the outermost circular geodesics and as we increase it, the radii of trapping converge towards  $r = r_3$  while  $E_L$  blows up to  $\pm\infty$ , with the sign depending on the direction from which we approach  $r_3$ . We can also describe the function  $E_{L_{trap}}(r)$ : it starts with some finite positive value at  $r = r_1$  and increases monotonically to  $+\infty$  as  $r$  approaches  $r_3$ . There it jumps to  $-\infty$  and increases again to a finite negative value at  $r = r_2$ .

It is interesting to ask what region in physical space is accessible for trapped geodesics. By plugging (2.81) and (2.80) into the equation  $\Theta = 0$  we get that for a geodesic with  $r = \text{const.}$ :

$$\cos^2 \theta_{turn} = \frac{2\sqrt{Mr^2\Delta(2r^3 - 3Mr^2 + a^2M)} - r(r^3 - 3M^2r + 2a^2M)}{a^2(r - M)^2} \quad (2.91)$$

holds. This gives two turning points in  $\theta$  direction which are symmetric about the equatorial plane. The whole region of trapping in the  $(r, \theta)$  plane is bounded by curves defined implicitly by (2.91) and  $r_1 \leq r \leq r_2$ . Figure (2.7) presents this set for a particular value of  $a$ .

**Remark 2.5.1.** *Two warnings:*

1. *One has to be careful when interpreting Figure 2.7 (and the plots in the Mathematica notebook). Despite the fact that the region in physical space occupies finite range of  $r$  values, every individual trapped null geodesic is still constrained to a fixed radius. See figure 2.8 for a schematic depiction and see Teo (2003) for an insight on what those trajectories look like in detail.*
2. *When taking  $a \rightarrow M$  in the Mathematica notebook the ergosphere appears to develop a kink on the rotation axis. This is an artifact of the coordinate system, as the ergosphere coincides with the horizon there and is thus orthogonal to itself.*

### 2.5.3. T-Orthogonal Null Geodesics

In the ergoregion there exist null geodesics with negative values of  $E$ . In physical space they are constrained to the region defined by equation (2.56). From Lemma 2.4.1 we know that geodesics with  $Q < 0$  reach either  $\mathcal{I}^+$  or come from  $\mathcal{I}^-$  and can therefore not have negative values of  $E$ . This allows us to use the pseudo potentials to give a more precise characterization of the ergoregion in phase space.

## 2. General relativity and Black holes

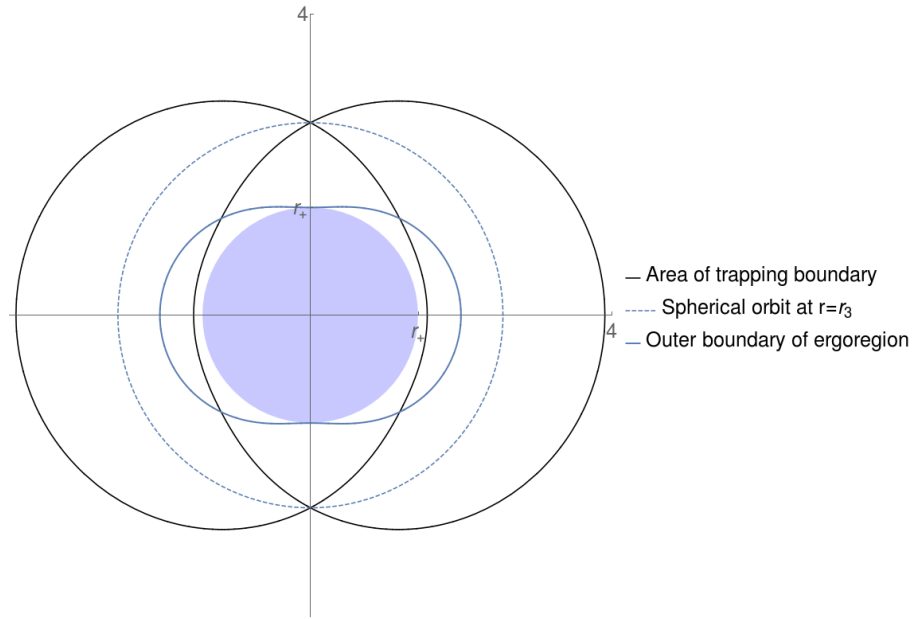


Figure 2.7.: The region accessible for trapped null geodesics for  $a = 0.902$ . The shaded region represents the black hole,  $r \leq r_+$ . The only qualitative change in this picture occurs at  $a = \frac{1}{\sqrt{2}}$  because at this value the region of trapping starts intersecting with the ergoregion.

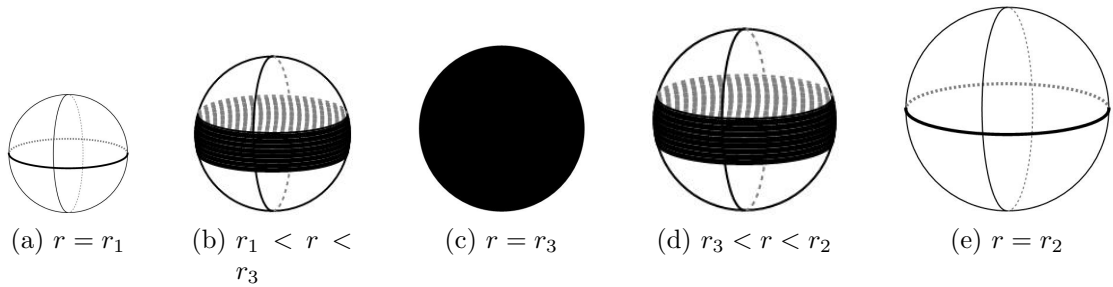


Figure 2.8.: A series of spheres at different radii  $r$  showing schematically the two dimensional bands in which a single trapped null geodesic lives. A trapped null geodesic can not reach every point in one of these bands. However for every point in this band there exists a trapped null geodesic that goes through it. The width of the band for the intermediate radii is given by the maximal angle of inclination  $\iota$ . The trapped null geodesics for (a) and (b) are prograded, the ones for (d) and (e) are retrograded. For (c) the trapped/orbiting null geodesics cross the equatorial plane at an angle of  $90^\circ$ .

## 2.6. Applications for the Virtual Potential Plot

It is located in the region where  $V_-(Q_L) > 0$ , between  $E_L = 0$  and  $V_-(Q_L)$ . An immediate consequence of that is, that all future pointing null geodesics with negative  $E_L$  begin at the past event horizon and end at the future event horizon. Furthermore they must have  $L_z < 0$ . Those null geodesics with  $E_L = 0$  can reach the boundary of the ergoregion. In this case equation (2.47d) for Kerr gives us, that:

$$Q_L = \frac{\cos^2(\theta_{max})}{\sin^2(\theta_{max})}. \quad (2.92)$$

When calculating the turning points from equation (2.47c) for Kerr we get that:

$$\sin^2(\theta_{max})R\left(r, 0, 1, \frac{\cos^2(\theta_{max})}{\sin^2(\theta_{max})}\right) = -r^2 + 2Mr - a^2\cos^2(\theta_{max}) = 0. \quad (2.93)$$

The only solution to this equation in the exterior region is:

$$r_{turn}(\theta_{max}) = M + \sqrt{M^2 - a^2\cos^2(\theta_{max})} \quad (2.94)$$

which is exactly the location of the ergosphere (2.56). So  $V_-(Q_L) > 0$  can be considered as the boundary of the ergoregion in phase space. From this considerations we see immediately that  $\mathbf{T}$ -orthogonal null geodesics are clearly non-trapped in Kerr. In fact there do not even exist any trapped null geodesics orthogonal to:

$$K^\nu = (\partial_t)^\nu + \epsilon_{min}(\partial_\phi)^\nu \quad (2.95)$$

where  $\epsilon_{min} = \min[|V_+(0, r_1)|, |V_-(0, r_2)|]$ .

## 2.6. Applications for the Virtual Potential Plot

Everything we have derived about the behaviour of null geodesics in Kerr spacetimes can be represented in a couple of simple plots. See Figure 2.9, in the [Mathematica Notebook \(2016\)](#) provided with this paper the parameters  $a/M$  and  $Q_L$  can be varied. This allows one to develop an intuitive understanding of the influence of these parameters.

Furthermore by the eikonal approximation it is clear, that a massless wave equation should relate to the null geodesic equation in the limit of high frequencies. In [Dafermos et al. \(2014\)](#) it is shown that when separating the wave equation  $\Sigma\Box\psi = 0$  the ODE for the radial function in Schrödinger form can be written as:

$$\frac{d^2u}{dr^{*2}} + \left( \frac{R(r, E = \omega, L_z = m, L^2 = \lambda_{lm})}{(r^2 + a^2)^2} - F(r) \right) u = 0 \quad (2.96)$$



## 2. General relativity and Black holes

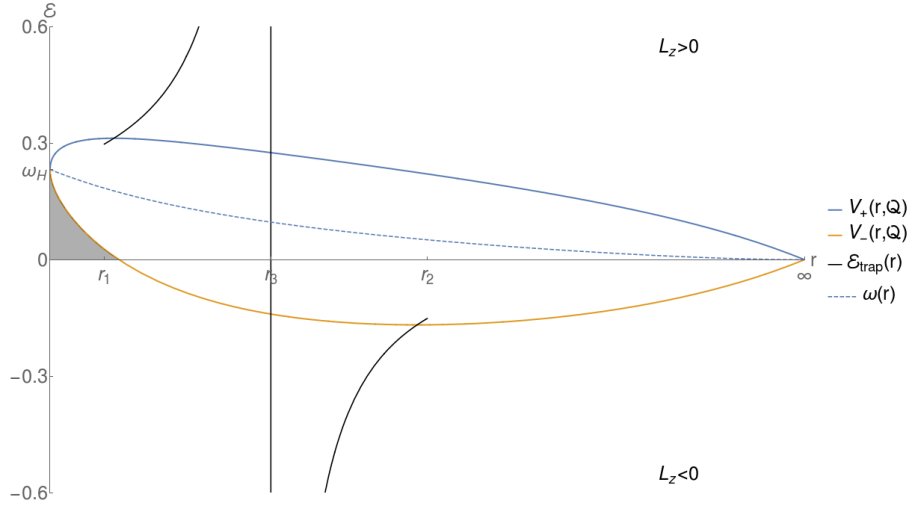


Figure 2.9.: Plot of the pseudo potentials  $V_{\pm}$  as function of a compactified radial coordinate in the exterior region for  $a = 0.764$  and  $Q_L = 0.18$ . Its qualitative features are preserved when  $a$  and  $Q_L$  are changed. The location of trapping in phase space is indicated by the function  $E_{L_{trap}}(r)$ . The extrema of the pseudo potentials are the intersection of  $V_+$  and  $V_-$  with this function. Therefore they slide on this curve as  $Q_L$  increases. The area filled in gray corresponds to geodesics with  $E < 0$ . It is clear from this plot that the regions occupied by geodesics of negative energy and trapped geodesics respectively are disjoint in phase space.

with  $F(r) = \frac{\Delta}{r^2+a^2}(a^2\Delta + 2Mr(r^2 - a^2)) \geq 0$  and hence we have the following relations:

$$\omega \sim E, \quad m \sim L_z, \quad \lambda_{lm} \sim L^2. \quad (2.97)$$

When trying to understand the different treatments of different parameter ranges in [Dafermos et al. \(2014\)](#) it is helpful to play with the parameters of the pseudo potential in the [Mathematica Notebook \(2016\)](#) provided with this thesis. The construction of the different mode currents becomes much more intuitive when thinking about where in [Figure 2.9](#) the corresponding parameters are located. Note that in the high frequency limit the pseudo potentials correspond to the location of  $\omega^2 - V(r, \omega, m, \Lambda) = 0$  and hence the location where the leading contribution to the bulk terms of the  $Q^y$  and  $Q^h$  currents change their sign.

Another interesting observation is that combining the results in [section 2.5.2](#) and [section 2.5.3](#) we can see that to separate trapping from the ergoregion in physical space it is sufficient if we restrict the null geodesics to be either co- or counter-rotating. In the co-rotating case there simply does not exist an ergoregion and

## 2.6. Applications for the Virtual Potential Plot

the statement is clear. In the counter-rotating case trapping is constrained to  $r \in (r_3, r_2]$  and  $r_3 > 2M \geq r_{ergo}$  for all Kerr spacetimes with  $a < M$ . In this direction particularly interesting might be the potential functions  $\Psi_{\pm}$  in [Hasse and Perlick \(2006\)](#) which have interesting properties in physical space.



# 3. Black Hole Shadows

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The trapped directions on the light cone are closely related to the notion of black hole shadows. The shadow of the black hole is defined as the innermost trajectory on which light from a background source passing a black hole can reach the observer. The past trapped set of null geodesics through a point thus corresponds to the boundary of the black hole shadow. The first discussion of the shadow in Schwarzschild spacetimes can be found in [Synge \(1966\)](#), and, for extremal Kerr at infinity, it was later calculated in [Bardeen \(1973\)](#). Analyzing the shadows of black holes is of direct physical interest as there is hope for the Event Horizon Telescope to be able to resolve the black hole in the center of the Milky Way (Sgr A\*) well enough so that one can compare it to the predictions from theoretical calculations, see for example [Doeleman et al. \(2008\)](#) or more recently [Fish et al. \(2016\)](#). This perspective has led to a number of advancements in the theoretical treatment of black hole shadows in recent years [Cunha et al. \(2016\)](#), [Grenzebach \(2015\)](#), [Grenzebach et al. \(2014, 2015\)](#), [Hioki and Maeda \(2009\)](#), [Li and Bambi \(2014\)](#), [Bardeen \(1973\)](#), [Schee and Stuchlik \(2009\)](#), [Takahashi and Takahashi \(2010\)](#).

#### Overview of this section

In section 3.1.1 we introduce the framework for the discussion of the black hole shadows, in particular the notion of the celestial sphere. In section 3.2.1 we discuss the shadow for observers at points of symmetry. We use this context to introduce the formal definition of degeneracies and how they arise. In section 3.1.2 we recall the explicit form of the shadow in the spacetimes we consider. We then proceed to show that this parametrization actually defines a smooth curve for the past and the future trapped set. In section 3.3 we introduce the recipe of the search for continuous degeneracies. Finally, in section 3.3.3 we present the proof for the main result of this chapter.

Appendix A.2 is devoted to deriving several results on Möbius transformations needed in the main text and a list of somewhat long, explicit expressions have been shifted to Appendix A.3. In section 3.4 we discuss the use of the celestial sphere as a tool to approach a broader spectrum of problems.

## 3.1. Smoothness of the future and past trapped sets in Kerr-Newman-Taub-NUT spacetimes

**Remark 3.1.1.** *The material in this section was created in joint work with Marius Oancea see [Paganini and Oancea \(2017\)](#).*

The novel insights in this section are contained in the proof of Theorem 3.1.1 which is the first rigorous proof of the observations in [Grenzebach et al. \(2014\)](#). The significance of Theorem 3.1.1 is that we prove that for any subextremal Kerr-Newman-Taub-NUT spacetime, including Schwarzschild, the past and the future trapped sets at any regular point in the exterior region are smooth closed curves on the celestial sphere of any observer. We would like to stress that Theorem 3.1.1 therefore describes a property of trapping which does not change when going from Schwarzschild to Kerr-Newman-Taub-NUT.

Beyond its relevance for the discussion of black hole shadows the result is also of interest with respect to decay estimates for fields in the exterior region of Kerr black holes. Trapping is one of the biggest obstacle to prove such decay results. The area of trapping<sup>1</sup> changes substantially when going from Schwarzschild, where it is restricted to one fix radius, to Kerr where the area of trapping covers a finite range of radii. This makes it a lot harder to prove decay in Kerr than it is for

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<sup>1</sup>In other publications the area of trapping is referred to as "photon region".

### 3.1. Smoothness of the future and past trapped sets in Kerr-Newman-Taub-NUT spacetimes

Schwarzschild. This difficulty was only recently overcome in [Dafermos et al. \(2014\)](#) for all subextremal Kerr space times. To study the decay of fields in Kerr, it is thus important to understand which properties of trapping survive when going from Schwarzschild to Kerr. In this section we will provide yet another argument that the ergoregion and trapping are best to be understood in phase space.

#### 3.1.1. Trapping as a Set of Directions

In this section we will introduce a formal framework for our discussion. This allows us to give a more technical discussion of the trapped sets in Kerr-Newman-Taub-NUT spacetimes.

##### Framework

First we have to introduce the basic framework and notations. Let  $\mathcal{M}$  be a smooth manifold with Lorenzian metric  $g$ . At any point  $p$  in  $\mathcal{M}$  it is possible to find an orthonormal basis  $(e_0, e_1, e_2, e_3)$  for the tangent space, with  $e_0$  being the timelike direction. It is sufficient to treat only future directed null geodesics as the past directed ones are identical up to a sign flip in the parametrization and we are only interested in global properties of the null geodesics. The tangent vector to any future pointing null geodesic can be written as:

$$\dot{\gamma}(k|_p)|_p = \alpha(e_0 + k_1e_1 + k_2e_2 + k_3e_3) \quad (3.1)$$

where  $\alpha = -g(\dot{\gamma}, e_0)$  and  $k = (k_1, k_2, k_3)$  satisfies  $|k|^2 = 1$ , hence  $k \in S^2$ . The geodesic is independent of the scaling of the tangent vector as this corresponds to an affine reparametrization for the null geodesic. We will therefore set  $\alpha = 1$  in the following discussion.

**Remark 3.1.2.** *We will refer to the  $S^2(e_0)$  as the celestial sphere of a timelike observer at  $p$ , whose tangent vector is given by  $e_0$ , along the lines of e.g. ([Penrose and Rindler, 1987, p.8](#)).*

Given we fixed a starting point  $p$  and a tangent vector (3.1) by choosing  $k$  and  $\alpha$ , there exists a unique solution to the geodesic equation with this initial data. Thus, we make the following definition:

**Definition 3.1.1.** *Let  $\gamma(k|_p)$  denote a null geodesic through  $p$  whose tangent vector at  $p$  is given by equation (3.1).*

### 3. Black Hole Shadows

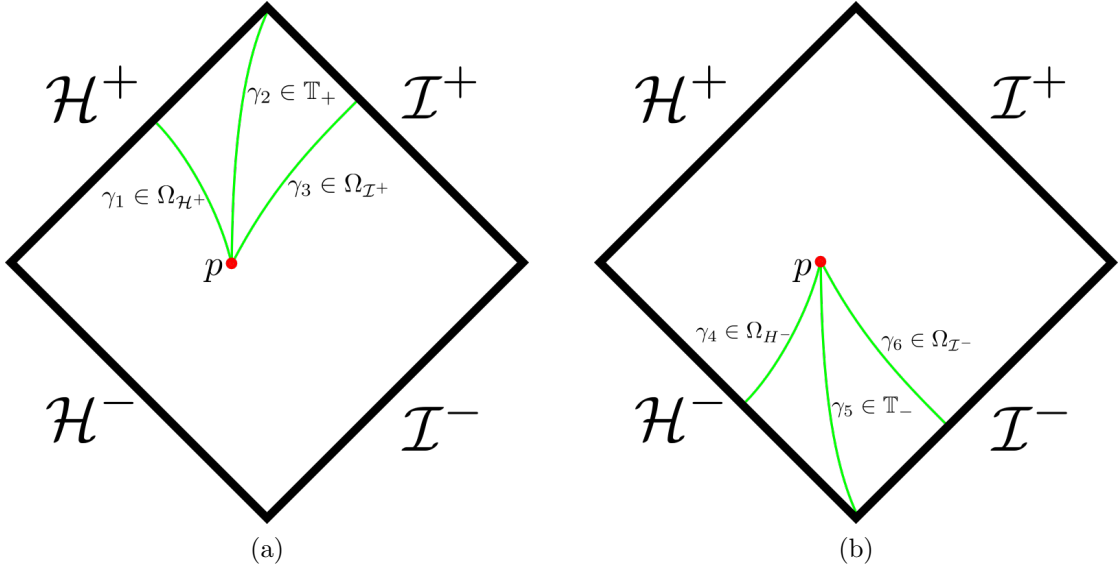


Figure 3.1.: Conformal diagrams giving a schematic representation of elements of the sets in Definition 3.1.2.

Suppose now that  $\mathcal{M}$  is the exterior region of a black hole spacetime with a complete future and past null infinity  $\mathcal{I}^\pm$  and a boundary given by the future and past event horizon  $\mathcal{H}^+ \cup \mathcal{H}^-$ . We can then define the following sets on  $S^2$  at every point  $p$ .

**Definition 3.1.2.** *The future infalling set:*  $\Omega_{\mathcal{H}^+}(p) := \{k \in S^2 | \gamma(k|_p) \cap \mathcal{H}^+ \neq \emptyset\}$ .  
*The future escaping set:*  $\Omega_{\mathcal{I}^+}(p) := \{k \in S^2 | \gamma(k|_p) \cap \mathcal{I}^+ \neq \emptyset\}$ .  
*The future trapped set:*  $\mathbb{T}_+(p) := \{k \in S^2 | \gamma(k|_p) \cap (\mathcal{H}^+ \cup \mathcal{I}^+) = \emptyset\}$ .  
*The past infalling set:*  $\Omega_{\mathcal{H}^-}(p) := \{k \in S^2 | \gamma(k|_p) \cap \mathcal{H}^- \neq \emptyset\}$ .  
*The past escaping set:*  $\Omega_{\mathcal{I}^-}(p) := \{k \in S^2 | \gamma(k|_p) \cap \mathcal{I}^- \neq \emptyset\}$ .  
*The past trapped set:*  $\mathbb{T}_-(p) := \{k \in S^2 | \gamma(k|_p) \cap (\mathcal{H}^- \cup \mathcal{I}^-) = \emptyset\}$

We further define the trapped set to be:

**Definition 3.1.3.** *The trapped set:*  $\mathbb{T}(p) := \mathbb{T}_+(p) \cap \mathbb{T}_-(p)$ .

The region of trapping in the manifold  $\mathcal{M}$  is then given by:

**Definition 3.1.4.** *Region of trapping:*  $\mathcal{A} := \{p \in \mathcal{M} | \mathbb{T}(p) \neq \emptyset\}$ .

**Definition 3.1.5.** *We refer to the set  $\Omega_{\mathcal{H}^-}(p) \cup \mathbb{T}_-(p)$  as the shadow of the black hole.*

### 3.1. Smoothness of the future and past trapped sets in Kerr-Newman-Taub-NUT spacetimes

Note that light from a background source, i.e. not in between the black hole and the observer and sufficiently far away, can only reach the observer in the set  $\Omega_{\mathcal{T}^-}(p)$  and hence the shadow will be black. For any practical purposes one can only extract information about the boundary of the shadow from an observation. In the following we are going to show that for the Kerr-Newman-Taub-NUT black hole the boundary of the shadow is given by the set  $\mathbb{T}_-(p)$  and that this set consists of those directions that asymptote to the trapped null geodesics in the past.

#### The trapped sets

We will now discuss the properties of the sets  $\mathbb{T}_{\pm}(p)$  in Kerr-Newman-Taub-NUT. Note that the equations of motion for  $r$  (2.47c) and  $\theta$  (2.47d) have two solutions that differ only by a sign for a fixed combination of  $E, L_z, K$ . This leads to the important observation, already made in Grenzebach et al. (2014), that the shadow for the standard observer, i.e. we work with the tetrad given in (2.39), is symmetric on the celestial sphere with respect to the  $k_1 = 0$  plane (i.e. the great circle in the celestial sphere defined by the meridians  $\psi = \pi/2$  and  $\psi = -\pi/2$  in the coordinate system we will define below for the celestial sphere). Therefore if  $(k_1, k_2, k_3) \in \mathbb{T}_-(p)$  then we always have that  $(-k_1, k_2, k_3) \in \mathbb{T}_-(p)$ . Further note that from the radial equation (2.47c) we get immediately that if  $k = (k_1, k_2, k_3) \in \mathbb{T}_+(p)$  then  $k = (k_1, k_2, -k_3) \in \mathbb{T}_-(p)$ .

We start by analyzing the sets for points of symmetry. First we have a detailed look at the situation in Schwarzschild. An explicit formula for the shadow of a Schwarzschild black hole was first given in Synge (1966). We present the argument here for completeness. In Schwarzschild the orthonormal tetrad (2.39) reduces to:

$$\begin{aligned} e_0 &= \frac{1}{\sqrt{1 - 2M/r}} \partial_t, & e_1 &= \frac{1}{r} \partial_\theta, \\ e_2 &= \frac{1}{r \sin \theta} \partial_\phi, & e_3 &= \sqrt{1 - 2M/r} \partial_r. \end{aligned} \quad (3.2)$$

To determine the structure of  $\mathbb{T}_{\pm}(p)$  in Schwarzschild it is sufficient to consider  $p$  in the equatorial plane and  $k = (0, \sin \Psi, \cos \Psi)$  with  $\Psi \in [0, \pi]$ . The entire sets  $\mathbb{T}_{\pm}(p)$  are then obtained by rotating around the  $e_1$  direction. Note that from the tetrad it is obvious that  $E(k) = E(r)$  is independent of  $\Psi$ . On the other hand  $L_z(k)$  is zero for  $\Psi = \{0, \pi\}$  and maximal for  $\Psi = \pi/2$ . Away from that maximum,  $L_z$  is a monotone function of  $\Psi$ . Note that the geodesic that corresponds to  $\Psi = \pi/2$



### 3. Black Hole Shadows

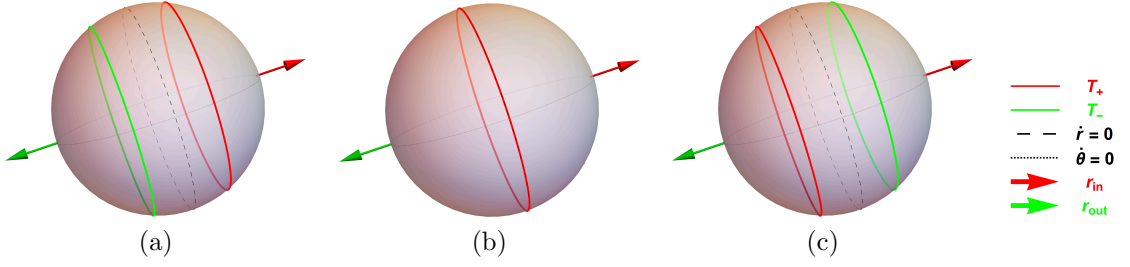


Figure 3.2.: The trapped set on the celestial sphere of a standard observer at different radial location in a Schwarzschild DOC. Observer (a) is located outside the photon sphere at  $r = 3.9M$ , observer (b) is located on the photon sphere at  $r = 3M$  and finally observer (c) is located between the horizon and the photonsphere at  $r = 2.5M$ . One can see that the future trapped set moves from the ingoing hemisphere in (a) to the outgoing hemisphere in (c) as one varies the location of the observer. The future and past trapped set coincide on the  $\dot{r} = 0$  line when the observer is located on the photon sphere at  $r = 3M$  in (b)

has  $k_1 = 0$  and thus a radial turning point. Thus the  $E/L_z$  value of this geodesic corresponds to the minimum value any geodesic can have at this point in the manifold. For  $r \neq 3M$  this is smaller than the value of trapping and thus there exist two  $k$  with the property that  $E/L_z(k) = 1/\sqrt{27M^2}$ . One of them has  $k_1 > 0$  and therefore  $\dot{r} > 0$  and one has  $\dot{r} < 0$ . For  $r > 3M$  the first corresponds to  $\mathbb{T}_-(p)$  and the second corresponds to  $\mathbb{T}_+(p)$ . For  $2M < r < 3M$  the roles are reversed. For  $r = 3M$  we have  $\mathbb{T}_+(p) = \mathbb{T}_-(p) = (0, k_2, k_3)$ . In Figure 3.2 we depict these three cases for some fixed radii. To conclude we see that  $\mathbb{T}_+(p)$  and  $\mathbb{T}_-(p)$  are circles on the celestial sphere independent of the value of  $r(p)$ .

**Definition 3.1.6.** *A Point of symmetry is a point for which there exists a one parameter family of isometries with closed orbits, which all leave the point itself invariant.*

**Lemma 3.1.1.** *The sets  $\mathbb{T}_+(p)$  and  $\mathbb{T}_-(p)$  are circles<sup>2</sup> on the celestial sphere of any timelike observer at any regular point of symmetry in the exterior region of any subextremal Kerr-Newman-Taub-NUT spacetime.*

*Proof.* To determine the structure of  $\mathbb{T}_\pm(p)$  we observe that when we pick a future/past trapped direction and apply the diffeomorphism, the spacial directions of  $TM|_p$  are rotated around the vector pointing along the axis left invariant by

<sup>2</sup>With circle we mean here the intersection between a sphere and a plane.

### 3.1. Smoothness of the future and past trapped sets in Kerr-Newman-Taub-NUT spacetimes

the diffeomorphism. Therefore the future/past trapped direction traces proper circles on the celestial sphere. Therefore the future and past trapped set at such a point  $p$  always correspond to a collection of circles independent of the details of the manifold or the location of  $p$  therein. We are now going to show that in the spacetimes under consideration here  $\mathbb{T}_\pm(p)$  consist of exactly one circle.

For spherically symmetric spacetimes this is a well known fact. For Schwarzschild we presented the detailed argument above. For the other spherically symmetric black hole spacetimes the argument works essentially the same.

For an observer located at a regular point on the rotation axis of Kerr-Newman-Taub-NUT black holes we can apply the following argument. Note that for  $l \neq 0$  and  $a \neq 0$  we have to choose  $C = \pm 1$  for the procedure to apply to the regular part of the rotation axis in these cases. For all other values of  $C$  both parts of the rotation axis are singular. Hence the discussion here does not apply to those cases.

From equation (2.47d) it is clear that null geodesics that can reach the rotation axis have to have  $L_z = 0$ . For the case  $L_z = 0$  it is clear that there exists only one value of  $K_E^{trapp}(L_z = 0)$  and  $r_{trapp}(L_z = 0)$ . To treat an observer on the rotation axis we need to introduce a new coordinate system which covers the axis. We will use Cartesian coordinates  $(t, x, y, z)$ , which are related to the Boyer-Lindquist coordinates  $(t, r, \theta, \phi)$  by the following relations:

$$\begin{aligned} t &= t \\ x &= r \sin(\theta) \cos(\phi) \\ y &= r \sin(\theta) \sin(\phi) \\ z &= r \cos(\theta) \end{aligned} \tag{3.3}$$

Then the following set is an orthonormal tetrad on the rotation axis ( $x = y = 0$ ):

$$\begin{aligned} \tilde{e}_0 &= -\sqrt{\frac{z^2 + (a+l)^2}{z^2 - 2mz + a^2 + Q^2 - l^2}} \partial_t \Big|_p, & \tilde{e}_1 &= \frac{z}{\sqrt{z^2 + (a+l)^2}} \partial_x \Big|_p, \\ \tilde{e}_2 &= \frac{z}{\sqrt{z^2 + (a+l)^2}} \partial_y \Big|_p, & \tilde{e}_3 &= \sqrt{\frac{z^2 - 2mz + a^2 + Q^2 - l^2}{z^2 + (a+l)^2}} \partial_z \Big|_p. \end{aligned} \tag{3.4}$$

As  $\tilde{e}_3$  points along the rotation axis and is thus left invariant under a rotation of the manifold, we know that along the trapped set  $k_x^2 + k_y^2 = const.$  will be satisfied. Calculating Carter's constant from the tangent vector on the rotation axis we see that it is given by:

$$K = \sigma_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu = (1 - k_z^2)((a+l)^2 + z^2) \tag{3.5}$$

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In the above expression,  $\sigma_{\mu\nu}$  is the Killing tensor expressed in Cartesian coordinates, on the rotation axis ( $x = y = 0$ ). We see immediately that on the celestial sphere there exists at most two values of  $k_z$  such that:

$$K_E(k_z)|_p = \frac{K(k_z)}{E^2(\tilde{e}_0)} \Big|_p = K_E^{trapp}(L_z = 0) \quad (3.6)$$

These correspond to the future and the past trapped set. If the two solutions coincide we are at  $z = r_{trapp}(L_z = 0)$  and the directions are both future and past trapped. It remains to show that there will always be at least one value of  $k_z$ , such that the condition for future/past trapping is satisfied. We know that  $k_z = -1$  always hits the horizon, while  $k_z = 1$  always escapes to infinity and the infalling and outgoing sets are open due to the continuous dependence on initial data for solutions to the geodesic equation. Therefore, the trapped sets have to be non-empty.  $\square$

#### 3.1.2. Parametrization of the Shadow for generic observers

The parametrization of the shadow at any point in the exterior region of a Kerr-Newman-Taub-NUT spacetime has been explicitly obtained in [Grenzebach et al. \(2014\)](#). In [Grenzebach et al. \(2014\)](#) the parametrization of the shadow curve was in fact obtained for the more general case of the Kerr-Newman-Taub-NUT-(anti-)de Sitter spacetime family. It was further extended in [Grenzebach et al. \(2015\)](#) to the full Plebański-Demiański class. In the following we will prove that this parametrization actually describes a smooth curve everywhere in the exterior region of a Kerr-Newman-Taub-NUT spacetime. From here on for the rest of the paper we will always assume that  $a \neq 0$  as  $a = 0$  has been treated in the previous section. Fixing the orthonormal tetrad (2.39) to which we will refer as “standard observer”, the celestial sphere can be coordinated by standard spherical coordinates  $\rho \in [0, \pi]$  and  $\psi \in [0, 2\pi)$  so that (3.1) can be written as:

$$\hat{\gamma}(\rho, \psi)|_p = \alpha(-e_0 + e_1 \sin \rho \cos \psi + e_2 \sin \rho \sin \psi + e_3 \cos \rho). \quad (3.7)$$

The principal null direction towards the black hole is given by  $\rho = \pi$ . Following [Grenzebach et al. \(2014\)](#) one finds the following parametrization of the celestial sphere in terms of constants of motion:

$$\sin(\psi) = \frac{\tilde{L}_E + a \cos^2(\theta) + 2l \cos(\theta)}{\sqrt{K_E} \sin(\theta)} \Big|_{\theta(p)} \quad (3.8a)$$

$$\sin(\rho) = \frac{\sqrt{\Delta K_E}}{r^2 + l^2 - a \tilde{L}_E} \Big|_{r(p)} \quad (3.8b)$$

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where

$$\tilde{L}_E = L_E - a + 2lC \quad (3.9)$$

By plugging the relations (2.52) in the above equations we obtain a parametrization of the future and past trapped set in terms of the radius to which a particular future trapped direction is approaching. We use  $x$  to parametrize the trapped sets with  $x = r_{trapp}$ . The parameter  $x$  corresponds to the asymptotic value of  $r$  along the past null geodesic with initial tangent vector along the direction defined by  $\{\rho(x), \psi(x)\}$ . We will show in the following proof that the parameter  $x$  is restricted to the interval  $[r_{min}(\theta), r_{max}(\theta)]$ . Here  $r_{min}(\theta)$  and  $r_{max}(\theta)$  are given as the intersection of a cone of constant  $\theta$  with the boundary of the area of trapping. See Grenzebach et al. (2014), the parametrization of the trapped sets is then given by:

$$f(x) := \sin(\psi) = \frac{\Delta'(x)\{x^2 + (l + a \cos[\theta(p)])^2\} - 4x\Delta(x)}{4ax\sqrt{\Delta(x)}\sin(\theta(p))} \quad (3.10a)$$

$$h(x) := \sin(\rho) = \frac{4x\sqrt{\Delta(r(p))\Delta(x)}}{\Delta'(x)(r(p)^2 - x^2) + 4x\Delta(x)} \quad (3.10b)$$

Note that the shadow curve is independent of the Manko-Ruiz parameter  $C$ . We are now ready to prove our main Theorem.

**Theorem 3.1.1.** *The sets  $\mathbb{T}_+(p)$  and  $\mathbb{T}_-(p)$  are simple, closed, smooth curves on the celestial sphere of any timelike observer at any regular point in the exterior region of any subextremal Kerr-Newman-Taub-NUT spacetime.*

*Proof.* We start by analyzing the right-hand side of (3.10a):

$$\frac{df(x)}{dx} = \frac{\{x^2 + (l + a \cos[\theta])^2\}((M - x)^3 - M(M^2 - a^2 - Q^2 + l^2))}{2ax^2\Delta^{3/2}\sin(\theta)} \quad (3.11)$$

which is strictly negative for  $x \in (r_+, \infty)$ . Further the limit of the right hand side of (3.10a) is  $\infty$  for  $x \rightarrow r_+$  and  $-\infty$  for  $x \rightarrow \infty$ . Therefore, the function  $f$  is strictly monotone in the interval  $x \in (r_+, \infty)$  and, hence invertible. Then,  $x(\psi) = f^{-1}(\sin(\psi))$  is a smooth function of  $\psi$  with extrema at the extremal points of  $\sin(\psi)$ . As was observed in Grenzebach et al. (2014) the minimum  $x = r_{min}(\theta(p))$  at  $\psi = \pi/2$  and the maximum of  $x = r_{max}(\theta(p))$  at  $\psi = 3\pi/2$  correspond exactly to the intersections of a cone with constant  $\theta$  with the boundary of the region of trapping. This can be seen by setting the left hand side of (3.10a) equal to  $\pm 1$ ,

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and comparing to (2.53). So we have that  $x(\psi) \in [r_{min}(\theta(p)), r_{max}(\theta(p))]$  for all values of  $\theta(p)$ . Now we take a look at the right hand side of equation (3.10b):

$$\frac{dh(x)}{dx} = \frac{2(r^2 - x^2)\Delta(r)((x - M)^3 + M(M^2 - a^2 - Q^2 + l^2))}{\sqrt{\Delta(x)\Delta(r)}((r^2 - x^2)\frac{\Delta'(x)}{2} + 2x\Delta(x))^2}. \quad (3.12)$$

This is positive when  $x < r(p)$  and negative when  $x > r(p)$ . The denominator never vanishes for  $x \in (r_+, \infty)$  because:

$$(4x\Delta(x) + (r(p)^2 - x^2)\Delta'(x))|_{\{r(p)=r_+, x=r_+\}} = 0 \quad (3.13)$$

and

$$\frac{d}{dx}(4x\Delta(x) + (r(p)^2 - x^2)\Delta'(x)) = 2(3x^2 - 6Mx + 2(a^2 - l^2 + Q^2) + r(p)^2) > 0, \quad (3.14)$$

$$\frac{d}{dr(p)}(4x\Delta(x) + (r(p)^2 - x^2)\Delta'(x)) = 2r(p)\Delta'(x) > 0, \quad (3.15)$$

where we used  $r(p) > r_+ > M > \sqrt{a^2 - l^2 + Q^2}$  in (3.14).

If we set  $x = r(p)$  in (3.10b) then the right-hand side is equal to 1. Furthermore in any of the limits  $r(p) \rightarrow r_+$ ,  $r(p) \rightarrow \infty$ ,  $x \rightarrow r_+$ , and as  $x \rightarrow \infty$  it goes to zero.

**Case 1.** If  $p \notin \mathcal{A}$  hence if  $r(p) \notin [x_{min}(\theta(p)), x_{max}(\theta(p))]$  then the two functions

$$\rho_1(\psi) = \arcsin(h(x(\psi))) : [0, 2\pi) \rightarrow [\rho_{1min}, \rho_{1max}] \subset \left(0, \frac{\pi}{2}\right) \quad (3.16)$$

$$\rho_2(\psi) = \pi - \arcsin(h(x(\psi))) : [0, 2\pi) \rightarrow [\rho_{2min}, \rho_{2max}] \subset \left(\frac{\pi}{2}, \pi\right) \quad (3.17)$$

are both smooth with  $\rho_1(0) = \rho_1(2\pi)$  and  $\rho_2(0) = \rho_2(2\pi)$ . If  $p$  is between the region of trapping and the asymptotically flat end, the function  $\rho_2(\psi)$  corresponds to  $\mathbb{T}_+(p)$  and  $\rho_1(\psi)$  corresponds to  $\mathbb{T}_-(p)$ . Because  $(\pi/2, \pi]$  corresponds to the geodesic with  $\dot{r} < 0$ . If  $p$  is between the region of trapping and the horizon then the role of  $\rho_1(\psi)$  and  $\rho_2(\psi)$  are switched.

**Case 2.** If  $p \in \mathcal{A}$  we need to do some extra work. For simplicity we only consider the interval  $\psi \in [\pi/2, 3\pi/2]$  as the rest follows by symmetry of  $\sin(\psi)$  in  $[0, \pi]$  across  $\pi/2$  and in  $[\pi, 2\pi]$  across  $3\pi/2$ . We define:

$$\psi_0(r(p)) = \pi - \arcsin\left(\frac{\Delta'(r(p))\{r(p)^2 + (l + a \cos[\theta(p)])^2\} - 4r(p)\Delta(r(p))}{4r(p)\sqrt{\Delta(r(p))}a \sin(\theta(p))}\right). \quad (3.18)$$

### 3.1. Smoothness of the future and past trapped sets in Kerr-Newman-Taub-NUT spacetimes

The two functions

$$\rho_3(\psi) = \begin{cases} \arcsin(h(x(\psi))) & \text{if } \psi \in [\pi/2, \psi_0(r(p))] \\ \pi - \arcsin(h(x(\psi))) & \text{if } \psi \in (\psi_0(r(p)), 3\pi/2] \end{cases} \quad (3.19)$$

$$\rho_4(\psi) = \begin{cases} \pi - \arcsin(h(x(\psi))) & \text{if } \psi \in [\pi/2, \psi_0(r(p))] \\ \arcsin(h(x(\psi))) & \text{if } \psi \in (\psi_0(r(p)), 3\pi/2] \end{cases} \quad (3.20)$$

are then smooth on  $[\pi/2, 3\pi/2]$ . For a proof see Appendix A.1 and note that at  $\psi_0$ ,  $h(x(\psi))$  satisfies the conditions required in the appendix. Since  $p \in \mathcal{A}$  we have that  $x_{\min}(\theta(p)) < r(p) < x_{\max}(\theta(p))$ . Therefore the geodesic on the celestial sphere parametrized by  $x_{\max}(\theta(p))$  has to have  $\dot{r} > 0$  and thus has to be in  $[0, \pi/2)$ . On the other hand the geodesic on the celestial sphere parametrized by  $x_{\min}(\theta(p))$  has to have  $\dot{r} < 0$  and thus has to be in  $(\pi/2, \pi]$ . In fact by the monotonicity of the right hand side of (3.10a) and the fact that  $x(\psi_0) = r(p)$  we know that for  $\psi \in [\pi/2, \psi_0)$  we have  $x(\psi) < r(p)$  and for  $\psi \in (\psi_0, 3\pi/2]$  we have  $x(\psi) > r(p)$ . Thus we can conclude that for  $p \in \mathcal{A}$ ,  $\rho_4$  corresponds to  $\mathbb{T}_+(p)$  and  $\rho_3$  corresponds to  $\mathbb{T}_-(p)$  and thus both sets are smooth.

**Case 3.** In the special case when  $r(p) = x_{\max}(\theta(p))$  or  $r(p) = x_{\min}(\theta(p))$  the functions  $\rho_1$  and  $\rho_2$ , which describe  $\mathbb{T}_\pm(p)$ , do reach  $\rho = \pi/2$  at  $\psi = 3\pi/2$  (for  $r(p) = x_{\max}(\theta(p))$ ) or  $\psi = \pi/2$  (for  $r(p) = x_{\min}(\theta(p))$ ) respectively. However since in these cases we have that

$$\frac{d^2}{d\psi^2}(h(x(\psi))) = 0 \quad (3.21)$$

the two sets meet at this point tangentially and do not cross over into the other hemisphere.

Together with Lemma 3.1.1 this concludes the proof.  $\square$

**Remark 3.1.3.** In *Grenzebach et al. (2014)* it was observed that  $\rho_{\max}$  of  $\mathbb{T}_+(p)$  always corresponds to the trapped geodesic with  $x_{\min}(\theta(p))$  and  $\rho_{\min}$  of  $\mathbb{T}_+(p)$  always corresponds to the trapped geodesic with  $x_{\max}(\theta(p))$ . When  $p$  is outside the region of trapping  $h(x)|_{x_{\max}}$  is a local maximum of  $h(x(\psi))$  (as a function of  $\psi$ ) and  $h(x)|_{x_{\min}}$  is a local minimum of  $h(x(\psi))$ . When  $p$  is between the region of trapping and the horizon  $h(x)|_{x_{\max}}$  is a local minimum of  $h(x(\psi))$  and  $h(x)|_{x_{\min}}$  is a local maximum of  $h(x(\psi))$ . Since outside  $\mathbb{T}_+(p)$  is always described by  $\rho_2(\psi)$  and inside by  $\rho_1(\psi)$ ,  $\rho_{\min}$  then always corresponds to  $x_{\min}$  and  $\rho_{\max}$  always corresponds to  $x_{\max}$ . This also holds for  $p \in \mathcal{A}$ . For  $\mathbb{T}_-(p)$  the correspondence is switched.

**Remark 3.1.4.** We have only proved Theorem 3.1.1 for one standard observer at any particular point. However since any other observer at this point is related to

### 3. Black Hole Shadows

the standard observer by a Lorentz transformation and the Lorentz transformations act as conformal transformations on the celestial sphere (Penrose and Rindler, 1987, p.14), the Theorem indeed holds for any observer. In Grenzebach (2015) the quantitative effect on the shape of the shadow of boosts in different directions are discussed.

**Remark 3.1.5.** *The parametrization for the trapped set on the celestial sphere of any standard observer in Grenzebach et al. (2014, 2015) was derived for a much more general class of spacetimes. Therefore Theorem 3.1.1 might actually hold for these cases as well. However this is beyond the scope of this thesis.*

From Theorem 3.1.1 we immediately get the following Corollary:

**Corollary 3.1.1.** *For any observer at any regular point  $p$  in the exterior region of a subextremal Kerr-Newman-Taub-NUT spacetime away from the axis of symmetry we have that for any  $k \in \mathbb{T}_+(p)$  and any  $\epsilon > 0$*

- $B_\epsilon(k) \cap \Omega_{\mathcal{H}^+}(p) \neq \emptyset$
- $B_\epsilon(k) \cap \Omega_{\mathcal{I}^+}(p) \neq \emptyset$ .

So if we interpret the celestial sphere as initial data space for null geodesics starting at  $p$ , the Corollary is a coordinate independent formulation of the fact that trapping in the exterior region of subextremal Kerr-Newman-Taub-NUT black holes is unstable.

See Figure 3.3 as an example on how the trapped sets change under a variation of the radial location of the observer in a Kerr spacetime.

In the Mathematica Notebook (2016) provided with this thesis the parameters  $a/M$  and  $\mathcal{Q}$  as well as the location of the observer  $\{r(p), \theta(p)\}$  can be varied to generate the Figures 3.2 and Figures 3.3 .

### 3.1. Smoothness of the future and past trapped sets in Kerr-Newman-Taub-NUT spacetimes

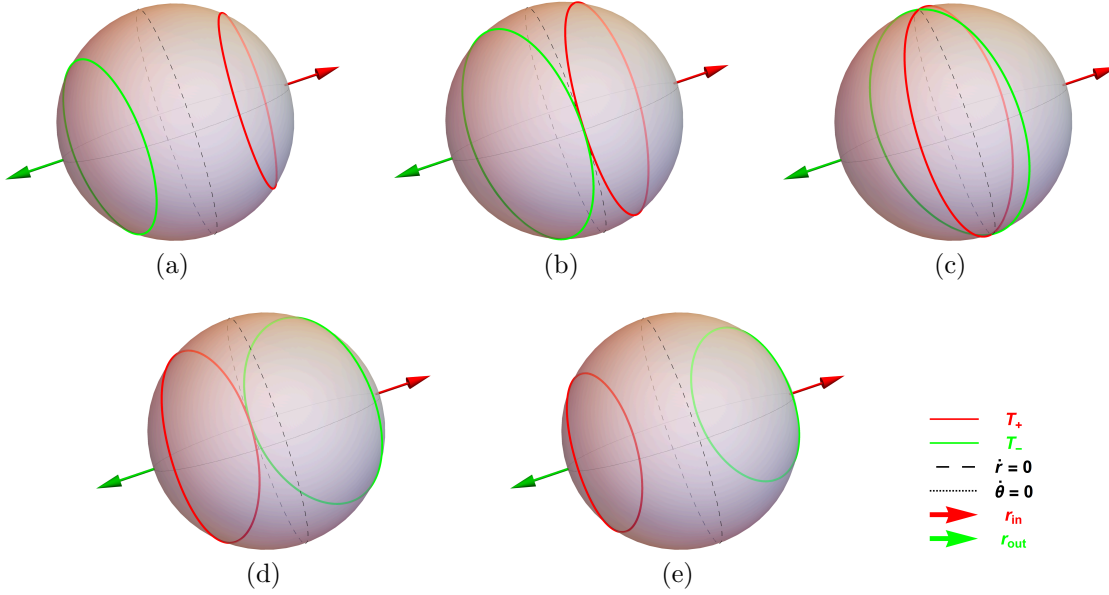


Figure 3.3.: The trapped set on the celestial sphere of a standard observer at different radial locations in the equatorial plane of the exterior region of a Kerr black hole with  $a = 0.9$ . Observer (a) is located outside the region of trapping at  $r = 5M$ . Observer (b) is located on the outer boundary of the region of trapping at  $r = r_2 = 3.535M$ . Observer (c) is located inside the region of trapping at  $r = r_3 = 2.56M$ . Observer (d) is located on the inner boundary of the area of trapping at  $r = r_1 = 1.73M$ . Finally observer (e) is located between the horizon and the area of trapping at  $r = 1.59M$ . Again one can observe how the two trapped sets move in opposite directions on the celestial sphere as the observer approaches the black hole. In (a) the future trapped set is on the ingoing hemisphere and the past trapped set is on the outgoing hemisphere. In (b) they meet in one point tangentially but are still entirely in one hemisphere except for that one point. In (c) the trapped sets intersect in two points and both have parts in both hemispheres. In (d) they only meet at one point tangentially again (now on the "other" side of the celestial sphere) and finally in (e) the future trapped set is entirely in the outgoing hemisphere and the past trapped set is entirely in the ingoing hemisphere.



## 3.2. Shadows and their degeneracies

**Remark 3.2.1.** *The material in section 3.2 and 3.3 was created in joint work with Marius Oancea and Marc Mars et al. (2017)*

In Hioki and Maeda (2009), Li and Bambi (2014) possible ways to extract the black hole parameters from the observation of the shadow have been explored. However, a strict treatment of the question "How much information about the black hole is there in the shape of the shadow?" has, to our knowledge, not been carried out. The only work we are aware of, that takes into account the fact that an observer cannot a priori know what his detailed motion with respect to the manifold is, can be found in Grenzebach (2015). However, the focus in that work is more on the explicit deformation due to different velocities rather than a systematic study on how the freedom of picking any observer at a point influences the possibility of extracting information about the black hole from the shape of the shadow.

The most important conceptual idea introduced in this section is the notion of what it means for the shadows at two points to be degenerate. In the case of degeneracy there exist two distinct observers for which the shadow is absolutely identical. Consequently, an observer cannot distinguish - from shape and size of the shadow alone - between the two situations.

We will put the concept of degeneracy to work in this section by proving the existence of two continuous degeneracies, one parameter curves in the parameter space of observers in the exterior region of Kerr-Newman-Taub-NUT spacetimes. Beyond that, we show that there are no further continuous degeneracies.

The discussion will again be limited to Kerr-Newman-Taub-NUT black holes. This has two reasons, first for observations within our galaxy the cosmological constant should be negligible, second including the cosmological constant increases the complexity of the arguments without providing additional insight.

### 3.2.1. Degeneration for observers located on an axis of symmetry

The following discussions applies for observers located on an axis of symmetry, i.e. an observer located at any regular point  $p$  in the exterior region of a black hole spacetime, for which there exists a one parameter family of isometries with closed orbits that leave  $p$  invariant. This includes in particular any point in the exterior region of a spherically symmetric black hole spacetime, as well as observers located

### 3.3. Which degeneracies exist?

on the rotation axis of e.g. Kerr. The discussion for points of symmetry, is here treated in a separate section to introduce several important concepts needed for our main theorem. Points of symmetry are special with respect to degeneracies as it was shown in section 3.1 that the shadow for observers at regular points of symmetry in the exterior of Kerr-Newman-Taub-NUT black holes is circular.

It is well-known (see e.g. (Penrose and Rindler, 1987, p.14)) that a change of observer (i.e. an orthochronous Lorentz transformation of the tetrad) corresponds to a conformal transformation on the celestial sphere, and vice versa. Restricting oneself to orientation preserving transformations, they are isomorphic to Möbius transformations. A fundamental property of conformal transformations on  $\mathbb{S}^2$  is that they map circles into circles. As a consequence if  $p_1$  and  $p_2$  are points in (possibly different) spacetimes in the family under consideration, and both lie on an axis of symmetry then, upon identification of the two celestial spheres by a respective choice of time oriented orthonormal basis, there exists a Lorentz transformation<sup>3</sup> ( $\mathbf{LT}$ ) of the observer such that  $\mathbb{T}_-(p_1) = \mathbf{LT}[\mathbb{T}_-(p_2)]$ . This concept is central to our argument.

**Definition 3.2.1.** *The shadows at two points  $p_1, p_2$  are called degenerate if, upon identification of the two celestial spheres by the orthonormal basis, there exists an element of the conformal group on  $\mathbb{S}^2$  that transforms  $\mathbb{T}_-(p_1)$  into  $\mathbb{T}_-(p_2)$ .*

**Remark 3.2.2.** *The shadow at two points  $p_1, p_2$  being degenerate implies that for every observer at  $p_1$  there exists an observer at  $p_2$  for which the shadow on  $\mathbb{S}^2$  is identical. Because this notion compares structures on  $\mathbb{S}^2$ , the two points need not be in the same manifold for their shadows to be degenerate. Just from the shadow alone an observer can not distinguish between these two configurations.*

Combining the discussion above with the shape of the shadow at points off the axis when  $a \neq 0$  described in the next subsection, we conclude that the only reliable information that an observer known to live in the exterior of a Kerr-Newman-Taub-NUT black hole can extract from observing a circular shadow is that he/she is on an axis of the symmetry of the black hole.

### 3.3. Which degeneracies exist?

The fact that a reflection about the  $k_3 = 0$  plane maps  $\mathbb{T}_+(p)$  to  $\mathbb{T}_-(p)$  implies that the properties of the past and the future trapped sets are equivalent. In particular this implies that if there exists a conformal map  $\Psi$  from  $\mathbb{T}_-(p_1)$  to  $\mathbb{T}_-(p_2)$  then

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<sup>3</sup>By this we always mean an orthochronous Lorentz transformation.

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there exists another conformal map, related to  $\Psi$  by conjugation with a reflection about the  $k_3 = 0$  plane, that maps  $\mathbb{T}_+(p_1)$  to  $\mathbb{T}_+(p_2)$ .

An observer can only see the past, hence  $\mathbb{T}_-(p)$ , so we will concentrate on this curve in the search of degeneracies. However the fact that the properties  $\mathbb{T}_-(p)$  and  $\mathbb{T}_+(p)$  are equivalent tells us that our results will hold true for both. Here we would like to remind the reader that  $\mathbb{T}_-(p)$  is symmetric with respect to a reflection about the  $k_1 = 0$  plane.

The question one would like to answer is, which observers can be fully distinguished based on the shape of the shadow they observe. A quick inspection of the equations (3.10) shows that the shadow is invariant up to a reparametrization  $x \rightarrow x/M$  as long as the following quantities are constant:

$$\theta, \frac{r}{M}, \frac{a}{M}, \frac{Q}{M}, \frac{l}{M}. \quad (3.22)$$

With this degeneracy we see that the shape of the shadow can only be affected by the change of dimensionless parameters. There is a discrete degeneracy for two observers with:

$$M_1 = M_2 \quad l_1 = -l_2 \quad r_1 = r_2 \quad a_1 = a_2 \quad Q_1 = Q_2 \quad \theta_1 = \pi - \theta_2 \quad (3.23)$$

In the case  $l = 0$  this corresponds to a reflection of the observers position with respect to the equatorial plane, while when  $l \neq 0$  the spacetime itself changes. In either case, two observers related by this transformation are fully indistinguishable from the observation of the shadow.

The comparison for the shadows of two arbitrary observers is a difficult problem and it is unclear to the authors how to determine all possible degeneracies. We will therefore restrict ourselves to continuous degeneracies. Hence a family of observers who form a  $C^1$  curve in the space of parameters for which the shadows are indistinguishable.

In the following we will introduce a method to systematically search for continuous degeneracies and to prove when such degeneracies do not exist. We will heavily rely on the fact that we have an explicit parametrization  $c(x; r, \theta, M, a, Q, l)$  for the curve defining the boundary of the shadow at a point  $p$  with coordinates  $r, \theta$  in the exterior region of a Kerr-Newman-Taub-NUT spacetime with parameters  $(M, a, Q, l)$  (and curve parameter  $x$ ).

To studying continuous degenerations we impose that the first variation of the curve is zero. From here on we will look at the shadow as a curve in the complex plane which is obtained from the parametrization (3.10) by stereographic projection of the celestial sphere (Penrose and Rindler, 1987, p.10):

$$c(x) = \frac{X(x) + iY(x)}{1 - Z(x)}, \quad (3.24a)$$

### 3.3. Which degeneracies exist?

$$X(x) = \sin(\rho) \sin(\psi) = h(x) \cdot f(x), \quad (3.24b)$$

$$Y(x) = \sin(\rho) \cos(\psi) = \pm h(x) \cdot \sqrt{1 - f^2(x)}, \quad (3.24c)$$

$$Z(x) = \cos(\rho) = -\text{sgn} \left( \frac{\partial h}{\partial x} \right) \sqrt{1 - h^2(x)}. \quad (3.24d)$$

The freedom of sign choice in (3.24c) comes from that fact that upon stereographic projection the symmetry with respect to the  $k_1 = 0$  plane on the celestial sphere becomes a reflection symmetry with respect to the real axis for the curve in the complex plane. The sign in  $Z$  makes the curve  $C^1$  and is the right choice to describe  $\mathbb{T}_-$ . If we were to describe  $\mathbb{T}_+$  instead, the global minus sign in front would have to be dropped. Outside the area of trapping  $\text{sgn} \left( \frac{\partial h}{\partial x} \right)$  has a fixed sign. Inside the area of trapping it changes sign when  $x = r$ , where  $h = 1$  and thus  $Z = 0$ .

From the definition of degeneracies for black hole shadows it follows that any degeneracy is characterized by a change in parameters together with a Möbius transformation on the shadow (as the Möbius transformation on the complex plane are equivalent to the orientation preserving conformal transformations on the Riemann sphere). Therefore when searching for continuous degeneracies we have to take the Möbius transformation into consideration. The limitation of our result to continuous degeneracies arises from the fact that we analyze small perturbations, hence we linearize the problem.

The first order of the action of any member of the conformal group on  $\mathbb{S}^2$  on a curve is given by:

$$\Psi_\epsilon(c) = c(x) + \epsilon \vec{\xi}|_{c(x)} + \mathcal{O}(\epsilon^2), \quad (3.25)$$

where  $\epsilon$  is a small parameter and where  $\xi$  is a conformal Killing vector field on  $\mathbb{S}^2$ . The first variation of the curve with respect to a parameter  $p$  is given by:

$$c(x; p + dp) = c(x, p) + \vec{V}_p dp + \mathcal{O}(dp^2), \quad (3.26)$$

where  $dp$  is an infinitesimal change of the parameter and  $V_p$  is given by  $\partial_p c(x, p)$ . The most generic variation vector for a curve is then:

$$\vec{V} = \sum_{p \in \mathcal{P} = \{r, \theta, M, a, Q, l\}} \vec{V}_p dp + \sum_{\xi \in \text{Lie}(Mb)} \vec{\xi}|_{c(x)} \epsilon_\xi. \quad (3.27)$$

We can now formulate a necessary and sufficient condition for the curve to be invariant under a continuous deformation. This is the case if there exists a nontrivial combination of  $dp$  and  $\epsilon_\xi$  such that  $V$  is tangential to the curve. Letting  $n$  be the normal to the curve  $c(x, p)$ , the condition is that:

$$\vec{V} \cdot \vec{n} \equiv 0 \quad (3.28)$$

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has a nontrivial solution in terms of  $dp$  and  $\epsilon_\xi$ . Here we did not yet restrict the vector field  $\xi$ , however as we will discuss next, there are a priori restrictions on the most general conformal Killing vector capable of compensating the deformations induced by the change in parameters.

#### 3.3.1. Vector Fields from Möbius Transformations

By the definition of degeneracies for every observer at point  $p_1$  there exists an observer at point  $p_2$  who observes the exact same shadow. For our purpose we can reformulate this statement the following way: If the shadows at two points are degenerate, then there exists a Möbius transformation that maps the stereographic projection of the shadow of a standard observer at point  $p_1$  to the stereographic projection of the shadow of the shadow of the standard observer at point  $p_2$ .

As we have observed in section 3.1.2 the stereographic projection of the shadow of any standard observer is reflection symmetric with respect to the real line. A rather involved argument (which may be of independent interest) is needed to show that only those conformal transformation that preserve the reflection symmetry can be used to “counter” the deformation from the change in parameters (as those correspond to a change between standard observers). The detailed proof is given in Appendix A.2.

One finds that the most general such conformal Killing vector is an arbitrary linear combination of the three linearly independent vector fields given by:

$$\vec{\xi}_1 = \partial_x, \quad \vec{\xi}_2 = x\partial_y + y\partial_x, \quad \vec{\xi}_3 = (x^2 - y^2)\partial_x + 2xy\partial_y, \quad (3.29)$$

in terms of Cartesian coordinates  $\{x, y\}$  on the complex plane, i.e.  $z = x + iy$ .

#### 3.3.2. Conditions for Continuous Degeneracies

We now start with the explicit calculations. Most of them are by no means difficult, but they are lengthy and have thus been performed mostly in Mathematica. Here we will describe the essential steps involved. From here on we will restrict to a domain of  $x$  such that  $\text{sgn}\left(\frac{\partial h}{\partial x}\right)$  does not change. This does not restrict our argument, as our aim is to prove that a certain quantity is zero independent of  $x$ . So it is equivalent to consider the problem in an open and dense interval. With:

$$\vec{V}_p = \begin{pmatrix} \frac{d(\text{Re}(c(x,p)))}{dp} \\ \frac{d(\text{Im}(c(x,p)))}{dp} \end{pmatrix}, \quad (3.30)$$

### 3.3. Which degeneracies exist?

and with the normal vector to a curve parametrized by  $x$  in two dimensions given by:

$$\vec{n} = \pm \left( \begin{array}{c} \frac{d(\text{Im}(c(x)))}{dx} \\ -\frac{d(\text{Re}(c(x)))}{dx} \end{array} \right), \quad (3.31)$$

we can calculate the various terms that show up in (3.28). Note here that the sign choice in the definition of the normal vector corresponds to the choice between the inward and the outward pointing normal to the curve. Because we want to find curves with  $V \cdot n = 0$  it doesn't matter which orientation or normalization we choose for  $n$  as long as we choose it consistently, hence we pick the plus sign. From equation (3.24a) we directly get:

$$\text{Re}(c(x)) = \frac{X(f(x), h(x))}{1 - Z(h(x))}, \quad (3.32)$$

$$\text{Im}(c(x)) = \frac{Y(f(x), h(x))}{1 - Z(h(x))}. \quad (3.33)$$

Plugging everything in, we obtain the following result in terms of  $f(x)$  and  $h(x)$ :

$$\vec{V}_p \cdot \vec{n} = \frac{h(x) \left( \frac{\partial f(x,p)}{\partial x} \frac{\partial h(x,p)}{\partial p} - \frac{\partial f(x,p)}{\partial p} \frac{\partial h(x,p)}{\partial x} \right)}{\sqrt{1 - f^2(x)} \sqrt{1 - h^2(x)} (1 - \sqrt{1 - h^2(x)})^2}, \quad (3.34)$$

$$\vec{\xi}_1 \cdot \vec{n} = \frac{\sqrt{1 - h^2(x)} f(x) h(x) \frac{\partial f(x)}{\partial x} + (1 - f^2(x)) \frac{\partial h(x)}{\partial x}}{\sqrt{1 - f^2(x)} \sqrt{1 - h^2(x)} (1 - \sqrt{1 - h^2(x)})^2}, \quad (3.35)$$

$$\vec{\xi}_2 \cdot \vec{n} = \frac{h^2(x) \frac{\partial f(x)}{\partial x}}{\sqrt{1 - f^2(x)} (1 - \sqrt{1 - h^2(x)})^2}, \quad (3.36)$$

$$\begin{aligned} \vec{\xi}_3 \cdot \vec{n} = & \frac{h^2(x) (1 - \sqrt{1 - h^2(x)}) \left( f(x) h(x) \frac{\partial f(x)}{\partial x} + \frac{\partial h(x)}{\partial x} - f^2(x) \frac{\partial h(x)}{\partial x} \right)}{\sqrt{1 - f^2(x)} \sqrt{1 - h^2(x)} (1 - \sqrt{1 - h^2(x)})^4} \\ & - \frac{f(x) h^4(x) \frac{\partial f(x)}{\partial x}}{\sqrt{1 - f^2(x)} \sqrt{1 - h^2(x)} (1 - \sqrt{1 - h^2(x)})^4}. \end{aligned} \quad (3.37)$$

At this point it is important to note that  $f(x, \theta, M, a, Q, l)$ ,  $h(x, r, M, a, Q, l)$  and all their partial derivatives are rational functions in  $x$  after multiplication with  $\sqrt{\Delta(x)}$ . For a list of all partial derivatives of  $f(x, \theta, M, a, Q, l)$  and  $h(x, r, M, a, Q, l)$  see Appendix A.3. Hence any product of  $f$ ,  $h$  and their derivatives which contain an even number of factors is a rational function in  $x$ , while any such product with an odd number of factors is a rational function in  $x$  after multiplication with  $\Delta_r(x)$ . (i.e.  $h(x) f(x) \frac{\partial f(x,p)}{\partial x} \frac{\partial h(x,p)}{\partial p}$  and  $h^3(x) \left( \frac{\partial f(x,p)}{\partial x} \right)^2 \sqrt{\Delta(x)}$  are both rational functions in  $x$ ).

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Further we notice that away from the real axis we have  $f^2(x) < 1$  and outside the area of trapping we always have  $h^2(x) < 1$ .

**Definition 3.3.1.** *A degeneration is called intrinsic when there is no need to act with a Möbius transformation to counter the deformation in the shadow due to the change in parameters.*

The condition for an intrinsic degeneracy of the shadow is then the existence of a non-trivial value of  $dp$  such that the following linear combination vanishes:

$$\sum_{p \in \mathcal{P}} \left( \frac{\partial f(x, p)}{\partial x} \frac{\partial h(x, p)}{\partial p} - \frac{\partial f(x, p)}{\partial p} \frac{\partial h(x, p)}{\partial x} \right) dp \equiv 0, \quad (3.38)$$

where  $\mathcal{P}$  is the set of parameters within which we are searching for degeneracies of the shadow. If we now write down the general linear combination that we required to be zero in condition (3.28):

$$\beta \vec{\xi}_1 \cdot \vec{n} + \alpha \vec{\xi}_2 \cdot \vec{n} + \gamma \vec{\xi}_3 \cdot \vec{n} + \sum_{p \in \mathcal{P}} \vec{V}_p \cdot \vec{n} dp \equiv 0, \quad (3.39)$$

we get one set of terms which are products of  $f$ ,  $h$  and their derivatives with an odd number of total powers and another set of terms with an odd number of total powers but an additional factor of  $\sqrt{1 - h^2(x)}$ . Now if  $\sqrt{1 - h^2(x)}$  is not a rational function (showing this will be part of our program), then for the above condition to be true, both sets of terms have to be equal to zero on their own, as adding a rational and an irrational function can never be equal to zero unless both functions themselves are equal to zero on their own. This gives us a system of two equations that we can solve for  $\beta$  and  $\gamma$ :

$$\beta = \frac{\sum_{p \in \mathcal{P}} h(x) \left( \frac{\partial f(x, p)}{\partial x} \frac{\partial h(x, p)}{\partial p} - \frac{\partial f(x, p)}{\partial p} \frac{\partial h(x, p)}{\partial x} \right) dp}{2 \left( (1 - h^2) f(x) h(x) \frac{\partial f(x)}{\partial x} - (1 - f^2(x)) \frac{\partial h(x)}{\partial x} \right)} \quad (3.40)$$

$$\begin{aligned} & + \alpha \frac{h^2(x) \frac{\partial f(x)}{\partial x}}{2 \left( f(x) h(x) \frac{\partial f(x)}{\partial x} - (1 - f^2(x)) \frac{\partial h(x)}{\partial x} \right)}, \\ \gamma = & \frac{\sum_{p \in \mathcal{P}} h(x) \left( \frac{\partial f(x, p)}{\partial x} \frac{\partial h(x, p)}{\partial p} - \frac{\partial f(x, p)}{\partial p} \frac{\partial h(x, p)}{\partial x} \right) dp}{2 \left( (1 - h^2) f(x) h(x) \frac{\partial f(x)}{\partial x} - (1 - f^2(x)) \frac{\partial h(x)}{\partial x} \right)} \quad (3.41) \\ & - \alpha \frac{h^2(x) \frac{\partial f(x)}{\partial x}}{2 \left( f(x) h(x) \frac{\partial f(x)}{\partial x} - (1 - f^2(x)) \frac{\partial h(x)}{\partial x} \right)}. \end{aligned}$$

### 3.3. Which degeneracies exist?

Now we know that  $\beta$  and  $\gamma$  both are constants. With the above result this is only possible if both terms are independent of  $x$  individually. Adding them and noticing that  $\alpha = 0$  is always a possibility, we conclude that the condition for the existence of a degeneracy of the shadow within a certain set of parameters  $\mathcal{P}$  is the existence of a non-trivial  $dp$  satisfying:

$$\frac{\partial}{\partial x} \left( \frac{\sum_{p \in \mathcal{P}} h(x) \left( \frac{\partial f(x,p)}{\partial x} \frac{\partial h(x,p)}{\partial p} - \frac{\partial f(x,p)}{\partial p} \frac{\partial h(x,p)}{\partial x} \right) dp}{2 \left( (1-h^2)f(x)h(x) \frac{\partial f(x)}{\partial x} - (1-f^2(x)) \frac{\partial h(x)}{\partial x} \right)} \right) \equiv 0 \quad (3.42)$$

has a non-trivial solution in terms of the  $dp$ . If only the trivial solution exists than there exists no continuous degeneracy within the parameter set  $\mathcal{P}$ . We can elaborate further on the role of  $\alpha$  as follows: either

$$\frac{\partial}{\partial x} \left( \frac{h^2(x) \frac{\partial f(x)}{\partial x}}{2 \left( f(x)h(x) \frac{\partial f(x)}{\partial x} - (1-f^2(x)) \frac{\partial h(x)}{\partial x} \right)} \right) \equiv 0, \quad (3.43)$$

and  $\alpha$  can take any value but with the the only effect of modifying both  $\beta$  and  $\gamma$ , or (3.43) does not hold, and we must take  $\alpha = 0$ . In no case the validity of (3.43) affects the existence of a degeneracy. In fact, one can show that the above condition can never be satisfied but since this is of no relevance to our argument, we will omit the proof here and just assume  $\alpha$  to be zero. For the actual proof this leaves us with the following strategy:

1. Check whether or not intrinsic degeneracies exist using condition (3.38).
2. Check whether eventual intrinsic degeneracies can be used to eliminate parameters from the set within which one has to search for degeneracies.
3. Check that  $\sqrt{1-h^2(x)}$  is an irrational function for all possible combinations of the remaining parameters in  $\mathcal{P}$ .
4. Check that the denominator of the first term in (3.40) is not equivalent to zero for all possible combinations of parameters.
5. Check whether there exist any non-trivial solutions to (3.42) for all possible combinations of the remaining parameters.

Note that wherever in these steps we have to show that something is either equivalent to zero or not equivalent to zero the expressions we have to check are polynomials. Hence the condition is that the coefficients for every order of  $x$  have to be equal to zero simultaneously, which leaves us with a system of equations that has to be satisfied. These system of equations in the steps above are of different complexities, however for most steps too involved to be solved by hand. Note that



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the derivation until here is independent of the detailed form of  $f(x)$  and  $h(x)$  and hence in principle valid for any black hole spacetime where the parametrization (3.10) exists, hence given the results in Grenzebach et al. (2015) the following analysis can in principle be carried out for the entire Plebański-Demiański class of black hole spacetimes.

#### 3.3.3. Continuous Degeneracies

We now apply the above recipe to the Kerr-Newman-Taub-NUT family, which we are interested in the present work, hence our set of parameters is given by  $\mathcal{P} = \{M, a, Q, l, r, \theta\}$  for this section. We will here re-derive the degeneracy already mentioned in (3.22) to illustrate the way the method works. We start with the first point in the list, the search for intrinsic degeneracies.

**Lemma 3.3.1.** *There are two intrinsic degeneracies given by:*

$$\frac{a}{M} = C_1, \quad \frac{r}{M} = C_2, \quad \frac{Q}{M} = C_3, \quad \frac{l}{M} = C_4, \quad \theta = C_5. \quad (3.44)$$

and

$$\begin{aligned} a \sin \theta &= C_1, & l + a \cos \theta &= C_2, & Q + 2a \cos \theta (l + a \cos \theta) &= C_3, \\ r &= C_4, & M &= C_5. \end{aligned}$$

*Proof.* The derivatives in Appendix A.3 are written such that every term in (3.38):

$$\begin{aligned} &\left( \frac{\partial f}{\partial x} \frac{\partial h}{\partial a} - \frac{\partial f}{\partial a} \frac{\partial h}{\partial x} \right) da + \left( \frac{\partial f}{\partial x} \frac{\partial h}{\partial M} - \frac{\partial f}{\partial M} \frac{\partial h}{\partial x} \right) dM + \left( \frac{\partial f}{\partial x} \frac{\partial h}{\partial r} \right) dr \\ &+ \left( \frac{\partial f}{\partial x} \frac{\partial h}{\partial Q} - \frac{\partial f}{\partial Q} \frac{\partial h}{\partial x} \right) dQ + \left( \frac{\partial f}{\partial x} \frac{\partial h}{\partial l} - \frac{\partial f}{\partial l} \frac{\partial h}{\partial x} \right) dl - \left( \frac{\partial f}{\partial \theta} \frac{\partial h}{\partial x} \right) d\theta \equiv 0 \end{aligned} \quad (3.45)$$

has the same denominator. The numerator in the above equation is a polynomial in  $x$  of order 11. Hence this condition gives us a system of 11 equation. Solving this system leaves us with two degrees of freedom. One of the solution is given by  $dl = ldM/M$ . Insewrting this yields the following set of ODEs:

$$\frac{da}{a} = \frac{dM}{M}, \quad \frac{dr}{r} = \frac{dM}{M}, \quad \frac{dQ}{Q} = \frac{dM}{M}, \quad \frac{dl}{l} = \frac{dM}{M}, \quad d\theta = 0, \quad (3.46)$$

which can be integrated to give:

$$\frac{a}{M} = C_1, \quad \frac{r}{M} = C_2, \quad \frac{Q}{M} = C_3, \quad \frac{l}{M} = C_4, \quad \theta = C_5. \quad (3.47)$$

### 3.3. Which degeneracies exist?

where  $C_1, C_2, C_3, C_4$  and  $C_5$  are integration constants. We now explain a method that will be used several times below. A degeneration involving four integration constants means that (locally) the parameter space is threaded by a congruence of curves, with any two points along the same curve having identical shadows. Consider now another degeneracy, independent of the previous one. This means that the vector field tangent to the new congruence of curves is linearly independent of the previous one. Consider a point  $p$  in parameter space where the two vectors are linearly independent. At that point, and in fact in an open neighbourhood thereof, the two congruences of curves are nowhere tangent to each other. It follows that the shadow at any point in this open set is now invariant under a two parameter family of transformations, i.e. a two-dimensional surface in parameter space. Consider a hypersurface passing through  $p$  and transverse to the first congruence. The intersection of this hypersurface with the invariant two-dimensional surface is necessarily a non-trivial degeneration curve, which obviously does not belong to the first congruence. This means that we can look for linearly independent degenerations by restricting the problem to a hypersurface transverse to the original one. This greatly simplifies the computations. Geometrically, the procedure is analogous to performing a gauge fixing. In summary, the idea is to use the existing degenerations to reduce the order of the problem. Point (2) in the strategy outlined above refers precisely to this “gauge fixing” procedure.

Applying this strategy, the second degeneracy condition can be found without loss of generality by setting  $dM = 0$  (the foliation by hypersurfaces is given by  $M = \text{const}$ , which indeed is transverse to the congruence of curves defined by (3.47)). Solving the set of equations obtained from (3.45) with  $dM = 0$  yields:

$$d\theta = \frac{\sin \theta}{a} dl, \quad da = -\cos \theta dl, \quad dQ = 2(l + a \cos \theta) dl, \quad dr = 0, \quad dM = 0, \quad (3.48)$$

and can be integrated to yield:

$$a \sin \theta = C_1, \quad l + a \cos \theta = C_2, \quad Q + 2a \cos \theta (l + a \cos \theta) = C_3, \quad r = C_4, \quad M = C_5, \quad (3.49)$$

where  $C_1, C_2, C_3, C_4$  and  $C_5$  are again integration constants.  $\square$

The first degeneracy can be “gauge fixed” immediately and globally by fixing  $M = \text{const}$  and restricting the whole problem to this lower dimensional parameter space. We want to exploit in a similar way the second degeneracy and reduce the problem further. The vector field along the second degeneration can be read off

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directly from (3.48) and it always has a non-zero component along the  $l$  direction. Thus, a suitable family of transverse hypersurfaces is  $l = \text{const}$ .

Now we want to prove that if we set  $M = \text{const}$  and  $l = \text{const}$  then there is no further degeneracy in  $\mathcal{P} = \{a, r, Q, \theta\}$ . We start with point (3) of the recipe in the previous section.

**Lemma 3.3.2.** *The function  $\sqrt{1 - h^2(x)}$  is irrational.*

*Proof.* We need to prove that

$$[\Delta'(x)(r^2 - x^2) + 4x\Delta(x)]^2 - 16x^2\Delta(r)\Delta(x) = P(x)^2 \quad (3.50)$$

admits no solution where  $P(x)$  is a polynomial on  $x$ . The leading term in the right-hand side is  $4x^6$ , which combined with the fact that a global sign in  $P(x)$  is irrelevant, shows that  $P(x)$  must be of the form  $P(x) = 2x^3 + K_2x^2 + K_1x + K_0$ . The zero, first and fifth order coefficients in (3.50) are immediately solved to give:

$$K_2 = -6M, \quad K_1 = -2\epsilon(r^2 + 2\beta), \quad K_0 = 2\epsilon Mr^2,$$

where  $\epsilon = \pm 1$  and  $\beta := a^2 - l^2 + q^2$ . The choice  $\epsilon = -1$  makes  $P(x) \equiv \Delta'(x)(r^2 - x^2) + 4x\Delta(x)$  and equation (3.50) becomes  $16x^2\Delta(r)\Delta(x) = 0$ , which is impossible for  $r$  in the exterior region. For the choice  $\epsilon = 1$  the coefficients in  $x^4$  and  $x^3$  in (3.50) impose, respectively:

$$\begin{aligned} 2Mr + \beta &= 0, \\ -2(2Mr + \beta) - \Delta(r) + \beta &= 0. \end{aligned}$$

Since in the exterior region  $r > r_+ > 0$ , the first requires  $\beta < 0$  and the second  $\Delta(r) = \beta < 0$ , which is impossible.

We conclude that  $\sqrt{1 - h^2(x)}$  is an irrational function. □

Next we check that the denominator in (3.42) is non-trivial for all allowed parameter combinations.

**Lemma 3.3.3.**

$$\left( (1 - h^2)f(x)h(x)\frac{\partial f(x)}{\partial x} - (1 - f^2(x))\frac{\partial h(x)}{\partial x} \right) \neq 0 \quad (3.51)$$

### 3.3. Which degeneracies exist?

*Proof.* Plugging in the parametrization (3.10) we get:

$$\frac{\sqrt{\Delta(r)}\{-x(\Delta'(x))^2 - 2\Delta(x)(\Delta'(x) - x\Delta''(x))\}g_1(x)}{16a^2x\sqrt{\Delta(x)}\sin^2\theta(4x\Delta(x) + (r^2 - x^2)\Delta'(x))^3} \neq 0, \quad (3.52)$$

where  $g_1(x)$  is given by the following polynomial of order six:

$$\begin{aligned} g_1(x) = & 16x [x^2 + (l + a \cos \theta)^2] \Delta(r) (2 [x^2 + (l + a \cos \theta)^2] \Delta'(x) - 8x\Delta(x)) - \\ & (4x\Delta(x) + (r^2 - x^2)\Delta'(x)) (32a^2x(r^2 - x^2) \sin^2 \theta \\ & + (r^2 + (l + a \cos \theta)^2) (-32x\Delta(x) + 8(x^2 + (l + a \cos \theta)^2)\Delta'(x))). \end{aligned} \quad (3.53)$$

The first factor in the numerator of (3.52) is clearly non-zero for an observer in the exterior region. The second factor is a polynomial in  $x$  with leading term  $-4x^3$ , hence non-identically zero. The zeroth order coefficient for  $g_1(x)$  is:

$$-32M^2r^2(l + a \cos \theta)^2(r^2 + (l + a \cos \theta)^2), \quad (3.54)$$

thus the only way this can be zero for an observer in the exterior region is if  $l = -a \cos \theta$ . Plugging that in for the other coefficient we get that the first order coefficient is given by  $-64MQ^2r^4$  and the fifth order coefficient is given by  $-192M(-Q^2 + 2Mr)$ . Those can never be equal to zero at the same time which finishes the proof for this lemma.  $\square$

When we plug the parametrization into the numerator inside the parenthesis in (3.42) we get this is equal to:

$$\frac{\{-x(\Delta'(x))^2 - 2\Delta(x)(\Delta'(x) - x\Delta''(x))\}g_2(x)}{2a^2\sqrt{\Delta(x)}\sin\theta(4x\Delta(x) + (r^2 - x^2)\Delta'(x))^3}, \quad (3.55)$$

where  $g_2(x)$  is given by the following polynomial of order five:

$$\begin{aligned} g_2(x) = & 2a [x^2 + (l + a \cos \theta)^2] (4x\Delta(x) + (r^2 - x^2)\Delta'(x)) \cdot \\ & (2QdQ + 2ada + (2r - 2M)dr) - \\ & \left\{ 16x(x^2 - r^2)(da + a \cot \theta d\theta)\Delta(x) + 16aQx [r^2 + (l + a \cos \theta)^2] dQ + \right. \\ & 16a^2x [r^2 + (l + a \cos \theta)^2] da + (2x - 2M) \cdot \\ & \left[ 8ar [x^2 + (l + a \cos \theta)^2] dr + 4(r^2 - x^2)(x^2 + l^2 - a^2 \cos^2 \theta)da \right. \\ & \left. \left. + \frac{4a(r^2 - x^2)(2al + \cos \theta(x^2 + l^2 + a^2 + a^2 \sin^2 \theta))}{\sin \theta} d\theta \right] \right\} \Delta(r). \end{aligned} \quad (3.56)$$

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We can now plug (3.52) and (3.55) into condition (3.42) to obtain:

$$\frac{\partial}{\partial x} \left( \frac{8x \sin \theta g_2(x)}{\sqrt{\Delta(r)} g_1(x)} \right) \equiv 0. \quad (3.57)$$

At this point we introduce the notion of a restricted degeneracy.

**Definition 3.3.2.** *A restricted degeneracy is one where a combination of parameters has to be zero instead of just being constant.*

Since a degeneracy is defined by a curve in parameter space, two things may happen. Either the curve is tangent to the submanifold of parameter space defined by the restricted degeneracy, or it is transverse to it. In the latter case, the curve leaves immediately the submanifold, and hence the degeneration curve must exist away from the submanifold. It follows that the only degeneracies that one could be missing by the general analysis are those satisfying not only that the parameters are zero, but also that their variation is zero, so that the curve is tangent to the restricted submanifold.

An example of a restricted degeneracy is  $\sin \theta = 0$ , under which condition (3.57) is obviously satisfied. By the argument above, the corresponding degeneration curves must satisfy  $d\theta = 0$ . The other parameters can vary arbitrarily in this case. Thus, with a slight abuse, we recover the degeneracy on the rotation axis. Of course, the argument in this case is not fully sound since it ignores the fact that the coordinate system and the shadow parametrization breaks down on the axis. This argument just serves the purpose to illustrate the concept of a restricted degeneracy. The situation on the rotation axis was treated properly in section 3.2.1. In the following we will always assume that  $\sin \theta \neq 0$ .

**Lemma 3.3.4.**

$$\frac{\partial}{\partial x} \left( \frac{8x \sin \theta g_2(x)}{\sqrt{\Delta(r)} g_1(x)} \right) \equiv 0 \quad \implies \quad g_2(x) \equiv 0. \quad (3.58)$$

*Proof.* First note that (3.57) can only be true if either  $g_1(x) = Bxg_2(x)$  for some non-zero constant  $B$ , or if  $g_2(x) \equiv 0$ . We will now exclude the first possibility. Note that the zeroth order coefficient of  $xg_2(x)$  is zero and with the zeroth order coefficient for  $g_1(x)$  given in (3.54). Thus, the only chance for the two to be proportional is if:

$$l = -a \cos \theta. \quad (3.59)$$

We fixed  $l$  this requires for  $d\theta = \cot \theta a^{-1} da$  to hold. However plugging these two condition into  $g_2(x)$  we get that now its zeroth order coefficient also vanishes. Thus

### 3.3. Which degeneracies exist?

not only the zeroth, but also the first term in  $g_1(x)$  must be zero. Plugging (3.59) into  $g_1(x)$  it follows that its first order coefficient is given by  $-64MQ^2r^4$ , which vanishes only if  $Q = 0$ . Setting  $Q = 0$  and  $dQ = 0$  everywhere, the first order coefficient in  $g_2(x)$  zero, while the second order coefficient of  $g_1(x)$  is  $96M^2r^4$ . This is manifestly non-zero and we reach a contradiction. Thus, the only possibility is  $g_2(x) = 0$  and the Lemma is proved.  $\square$

From this lemma, the remaining task is to show that there exists no non-trivial solution for  $g_2(x) \equiv 0$  which is equivalent to condition (3.38). We emphasize, in particular, that the previous lemma already implies that all degenerations of the shadow must be intrinsic.

The next Theorem, which is the main result of this paper, proves that there are no more degeneracies than those already found.

**Theorem 3.3.1.** *The only continuous degenerations of the black hole shadow for observers located at coordinate position  $r, \theta$  in the exterior region of Kerr-Newman-Taub-NUT black holes with parameters  $M, a, Q$  and  $l$  are given for observers such that their parameters have the same value for all the following functions:*

$$\frac{a}{M} = C_1, \quad \frac{r}{M} = C_2, \quad \frac{Q}{M} = C_3, \quad \frac{l}{M} = C_4, \quad \theta = C_5. \quad (3.60)$$

or

$$a \sin \theta = C_1, \quad l + a \cos \theta = C_2, \quad Q + 2a \cos \theta (l + a \cos \theta) = C_3, \quad r = C_4 M = C_5. \quad (3.61)$$

*Proof.* The two degeneracies have already been derived in Lemma 3.3.1. Given Lemma 3.3.4 we know that the condition for degeneracies to exist is given by (3.57). The only thing remaining to show is that  $g_2(x) \equiv 0$  has no non-trivial solutions. The highest order coefficient is given by:

$$aQdQ + a(r - M)dr + (a^2 - \Delta(r))da - \frac{a \cos \theta \Delta(r)}{\sin \theta} d\theta = 0. \quad (3.62)$$

We solve this for  $dr$  and substitute back into  $g_2$ . This leads to a a third order polynomial in  $x$ , i.e.  $g_2 = \sum_{i=0}^3 w_i x^i$ , and each coefficient  $w_i$  must vanish. The combination  $Mw_3 + w_2$  is very simple:

$$Mw_3 + w_2 = -\frac{16\Delta(r)M(r^2 + (l + a \cos \theta)^2)}{\sin \theta} (a \cos \theta d\theta + \sin \theta da) = 0.$$

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The first factor is nowhere zero in the exterior region, so we can solve for  $da$ :

$$da = -\frac{a \cos \theta}{\sin \theta} d\theta, \quad (3.63)$$

and substitute back into  $g_2(x)$ , which factorizes as:

$$g_2(x) = \frac{16a\Delta(r)(r-x)}{r-M} g_3(x),$$

where  $g_3(x)$  is a quadratic polynomial in  $x$ . Obviously,  $g_2$  is identically zero only if  $g_3 \equiv 0$ . The highest order term of  $g_3$  is:

$$-rQdQ + \frac{a}{\sin \theta} (l(M-r) + aM \cos \theta) d\theta = 0. \quad (3.64)$$

At this point we need to split the treatment in two cases depending on whether  $Q \equiv 0$  or not.

For the case with  $Q \neq 0$ , we solve (3.64) for  $dQ$  and substitute back into  $g_3$  to obtain:

$$g_3(x) = \frac{aM(r-M)(r^2 + (l + a \cos \theta)^2)}{r \sin \theta} (l + a \cos \theta) d\theta.$$

Thus  $g_3(x) \equiv 0$  can only happen if  $l + a \cos \theta = 0$ . Taking its differential and inserting  $da$  from (3.63) yields  $-a(\sin \theta)^{-1} d\theta = 0$ , hence  $d\theta = da = dr = dQ = 0$  and we have no continuous degeneration.

The remaining case is when  $Q = 0$  and  $dQ = 0$ . We want to impose  $g_3(x) \equiv 0$ , so that in particular it must be that  $g_3(x = M) = 0$ . Evaluating:

$$g_3(x = M) = \frac{Ma^2 \cos \theta d\theta (r^2 + (l + a \cos \theta)^2)}{\sin \theta},$$

which implies  $\cos \theta d\theta = 0$ , and hence  $d\theta = 0$  (if  $d\theta \neq 0$  it must be  $\theta = \pi/2$  so that  $d\theta = 0$  anyway). Consequently  $d\theta = da = dr = dQ = 0$ , which finishes the proof.  $\square$

## 3.4. The Celestial Sphere as a Tool

The following section contains partial results towards resolving the question of whether there can exist trapped null geodesics orthogonal to the Killing vector field  $T$  in general smooth stationary black hole space times with positive surface

### 3.4. The Celestial Sphere as a Tool

gravity. The results in this section do not allow a conclusive answer however they might turn out to be useful tools on the way towards an actual proof.

In the following we will make heavy use of the continuous dependence on initial data for the geodesic equation. We will use the following theorem <sup>4</sup>

**Theorem 3.4.1.** *Let  $\mathcal{M}$  be a  $C^1$  spacetime. Let  $p$  and  $q$  be two points on the same geodesic  $\gamma$ , with the tangent vector at  $p$  pointing towards  $q$ . Let  $p$  and  $q$  be separated by a finite affine parameter. Let  $\mathcal{U}(q)$  be an open neighbourhood of  $q$  in  $\mathcal{M}$  then there exists an open neighbourhood  $T\mathcal{U}(p, \dot{\gamma}|_p)$  of  $(p, \dot{\gamma}|_p)$  in  $T\mathcal{M}$  such that any geodesic in this neighbourhood intersects  $\mathcal{U}(q)$ .*

*Proof.* Let  $\mathcal{U}_n$ ,  $n \in [0, N]$  be a finite sequence of normal neighbourhoods that cover  $\gamma$  between  $p$  and  $q$  with  $p \in \mathcal{U}_0$  and  $q \in \mathcal{U}_N$ . Let  $\Psi_n$  be the coordinate chart that belongs to the normal neighbourhood  $\mathcal{U}_n$ .

Now we pick a sequence of points  $p_n$  such that  $p_n \in \mathcal{U}_n \cap \mathcal{U}_{n-1}$ .

For  $p_N$  the theorem is true by the continuous dependence on initial data for ODEs in  $\mathbb{R}^n$ , for a proof see (Hartman, 2002, p.95).

For  $p_{N-1}$  we have to do a little extra work. We have to show that there exists a neighbourhood  $T\mathcal{U}(p_{N-1}, \dot{\gamma}|_{p_{N-1}})$  such that for any  $\tilde{\gamma}(\lambda)$  that is a solution to the geodesic equation with initial data in  $T\mathcal{U}(p_{N-1}, \dot{\gamma}|_{p_{N-1}})$  there exists a parameter  $\lambda_0$  such that  $(\tilde{\gamma}(\lambda_0), \dot{\tilde{\gamma}}(\lambda_0)) \in T\mathcal{U}(p_N, \dot{\gamma}|_{p_N})$ . This is true by applying the continuous dependence on initial data for ODEs in  $\mathcal{U}_{N-1}$  and observing that for points in  $\mathcal{U}_N \cap \mathcal{U}_{N-1}$  the determinant of the Jacobian of the map  $\Psi_N \circ (\Psi_{N-1})^{-1}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  is bounded from above and below.

Now if it is true for  $p_{N-1}$  it is clear that it is true for any  $p_n$  and by that also for  $p$  itself.  $\square$

For our further argument we will only need the following corollary of Theorem 3.4.1.

**Corollary 3.4.1.** *Let  $\mathcal{M}$  be a  $C^1$  spacetime. Let  $p$  and  $q$  be two points on the same null geodesic with the tangent vector  $\dot{\gamma}(k|_p)$  at  $p$  pointing towards  $q$ . Let  $p$  and  $q$  be separated by a finite affine parameter. Let  $e_0$  be a unit timelike vector and  $S^2(e_0)$  the associated celestial sphere. Let  $\mathcal{U}(q)$  be an open neighbourhood of  $q$  in  $\mathcal{M}$  then there exists an open neighbourhood  $B_\epsilon(k)$  of  $k$  on  $S^2(e_0)$  such that any null geodesic  $\gamma(\tilde{k}|_p)$  with  $\tilde{k} \in B_\epsilon(k)$  intersects  $\mathcal{U}(q)$ .*

Where  $\gamma(\tilde{k}|_p)$  is given by definition 3.1.1.

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<sup>4</sup>Despite the fact that this is a rather basic statement, we were unable to find a source for this Theorem in the form we intend to use here. Nevertheless it is to be expected that this result is in fact well known.



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#### 3.4.1. Existence of Trapping in General Black Hole Spacetimes

In the following we will show that the existence of trapping is a generic feature of black hole spacetimes. Recall that  $\mathcal{H}^\pm = \overline{J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)} \setminus J^\mp(\mathcal{I}^\pm)$ . In Kerr, Schwarzschild and Minkowski  $\mathcal{I}^\pm$  are smooth. The various stability proofs for Minkowski space (Christodoulou and Klainerman, 1993, Hintz and Vasy, 2017, Bieri, 2009) come to different conclusions on the regularity of  $\mathcal{I}^\pm$  in these more generic settings depending on the choice of initial data. For the arguments in this section it would of course be easiest to assume that  $\mathcal{I}^\pm$  are smooth, however for the simple argument presented here it is sufficient to assume that they are in  $C^1$ .

**Lemma 3.4.1.** *Let  $\mathcal{M}$  be a  $C^1$  spacetime. Let  $\mathcal{M}$  be compactifiable with complete  $\mathcal{I}^\pm$ . Let  $p$  be any point in  $J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)$ . If the sets  $\Omega_{\mathcal{H}^\pm}(p)$  are non empty then they are open in  $S^2(e_0)$*

*Proof.* Let  $k$  be in  $\Omega_{\mathcal{H}^+}(p)$ . Let  $q_0$  be the point where  $\gamma(k|_p)$  intersects the event horizon  $\mathcal{H}^+$ . Since  $q_0$  is a regular point in  $\mathcal{M}$ ,  $\gamma(k|_p)$  can be extended across  $q_0$  and hence the affine parameter between  $p$  and  $q_0$  has to be finite. Let  $q$  be a point on  $\gamma(k|_p)$  in the future of  $q_0$  and  $\mathcal{U}(q)$  an open neighbourhood of  $q$  in  $\mathcal{M}$  with  $\mathcal{U}(q) \cap \overline{J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)} = \emptyset$  then Corollary 3.4.1 guarantees that the Lemma is true, because any null geodesic that intersects  $\mathcal{U}(q)$  has to intersect the horizon.  $\square$

The proof for  $\Omega_{\mathcal{H}^-}$  is identical.

**Lemma 3.4.2.** *Let  $\mathcal{M}$  be a  $C^1$  spacetime. Let  $\mathcal{M}$  be compactifiable with complete  $\mathcal{I}^\pm$  in  $C^1$ . Let  $p$  be any point in  $J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)$ . Then  $\Omega_{\mathcal{I}^\pm}(p)$  are open sets in  $S^2$ .*

*Proof.* Let  $k$  be in  $\Omega_{\mathcal{I}^+}(p)$ . Let  $\tilde{M}$  be the closure of the compactification of  $M$ . Let  $q_0$  be the point where  $\gamma(k|_p)$  intersects  $\mathcal{I}^+$ . The spacetime and therefore also the null geodesic  $\gamma(k|_p)$  can be extended across the conformal boundary of  $\tilde{M}$ . Note that for the following it is not relevant that this extension is not unique but it has to be in  $C^1$ . Now  $q_0$  is a regular point in this extension and the affine parameter between  $p$  and  $q_0$  is finite in the compactification. Let  $q$  be a point on  $\gamma(k|_p)$  in the future of  $q_0$  and  $\mathcal{U}(q)$  an open neighbourhood of  $q$  in the extension of  $\tilde{M}$  with  $\mathcal{U}(q) \cap \overline{J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)} = \emptyset$  then Corollary 3.4.1 guarantees that the Lemma is true, because any null geodesic that intersects  $\mathcal{U}(q)$  has to intersect  $\mathcal{I}^+$ .  $\square$

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The proof for  $\Omega_{\mathcal{I}^-}$  is identical. Note that the regularity assumptions on  $\mathcal{I}^\pm$  in this Lemma can quite likely be relaxed and replaced by a sufficiently fast fall off in the requirements of asymptotic flatness. The argument would go along the lines that in Minkowski space null geodesics can only have outward turning points. For Schwarzschild and Kerr this is true far enough from the black hole. This is expected to be true far enough out for any asymptotically flat spacetime. Therefore once the outgoing null geodesic enters this region it can only move further away. This will be true for the ones close by as well. Thereby establishing the openness of the sets without going all the way out to  $\mathcal{I}^\pm$  and therefore independently of the fact whether the manifold can be extended across  $\mathcal{I}^\pm$  in any regularity.

These Lemmas allow us to prove the main theorem in this section.

**Theorem 3.4.2.** *Let  $\mathcal{M}$  be a  $C^1$  spacetime. Let  $\mathcal{M}$  be compactifiable with complete  $\mathcal{I}^\pm$  in  $C^1$ . Let  $p$  be any point in  $J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)$ . If  $\Omega_{\mathcal{H}^\pm}(p)$  is non-empty then*

- $\mathbf{T}_\pm(p)$  is non-empty
- let  $w(\lambda)$  be any continuous path on  $S^2(p)$  with  $\lambda \in [0, 1]$  such that  $w(0) \in \Omega_{\mathcal{H}^\pm}(p)$  and  $w(1) \in \Omega_{\mathcal{I}^\pm}(p)$ , we have that  $w(\lambda) \cap \mathbf{T}_\pm(p) \neq \emptyset$ .

*Proof.* By completeness of  $\mathcal{I}^\pm$  and the fact that  $p \in J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)$  we have that  $\Omega_{\mathcal{I}^\pm}(p)$  are non-empty. By definition we have  $\Omega_{\mathcal{H}^\pm}(p) \cap \Omega_{\mathcal{I}^\pm}(p) = \emptyset$ . By Lemma 3.4.1 and Lemma 3.4.2 the sets are open and the theorem follows directly.  $\square$

#### Arrival Time Function on $S^2(p)$

In this subsection we will introduce a function on the celestial sphere that might be useful for further studies. We will assume  $\mathcal{I}^+$  and  $\mathcal{H}^+$  to be smooth and of topology  $S^2 \times \mathbb{R}$ . Let  $v$  be the coordinate on the  $\mathbb{R}$  part of  $\mathcal{H}^+$ . Let  $u$  be the coordinate on the  $\mathbb{R}$  part of  $\mathcal{I}^+$ . As before let  $p$  be a point in the exterior region. For the following we assume  $\Omega_{\mathcal{H}^+}(p)$  and  $\Omega_{\mathcal{I}^+}(p)$  to be non-empty. We denote by  $v[\gamma(k|_p) \cap \mathcal{H}^+]$  the coordinate value where a null geodesics intersects the horizon and by  $u[\gamma(k|_p) \cap \mathcal{I}^+]$  the coordinate value where a null geodesics intersects  $\mathcal{I}^+$ . For any point we can shift the coordinates in such a way that these values are always positive as there exists an earliest possible arrival time. Now we can define the arrival time function on  $S^2(p)$

$$f(k) = \begin{cases} \frac{1}{v[\gamma(k|_p) \cap \mathcal{H}^+]} & \text{if } k \in \Omega_{\mathcal{H}^+}(p) \\ 0 & \text{if } k \in \mathbf{T}_+(p) \\ \frac{-1}{u[\gamma(k|_p) \cap \mathcal{I}^+]} & \text{if } k \in \Omega_{\mathcal{I}^+}(p). \end{cases} \quad (3.65)$$

We now propose the following conjecture

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**Conjecture 3.4.1.** *For  $p \in J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-) \subset \mathcal{M}$  where  $\mathcal{M}$  is a smooth compactifiable spacetime the function  $f(k)$  is continuous on  $S^2(p)$ .*

We will now present the sketch of an argument why one should expect this to be true.

Inside  $\Omega_{\mathcal{H}^+}(p)$  and inside  $\Omega_{\mathcal{I}^+}(p)$  this is clear by Corollary 3.4.1. Hence the Lemma reduces to the following claims:

1. Let  $w(\lambda)$  be any continuous path on  $S^2(p)$  with  $\lambda \in [0, 1]$  such that for  $\lambda \in [0, 1)$ ,  $w(\lambda) \in \Omega_{\mathcal{H}^+}(p)$  and for  $\lambda = 1$ ,  $w(1) \in \partial\Omega_{\mathcal{H}^+}(p)$  then we have that  $\lim_{\lambda \rightarrow 1} f(w(\lambda)) = 0$ .
2. Let  $w(\lambda)$  be any continuous path on  $S^2(p)$  with  $\lambda \in [0, 1]$  such that for  $\lambda \in [0, 1)$ ,  $w(\lambda) \in \Omega_{\mathcal{I}^+}(p)$  and for  $\lambda = 1$ ,  $w(1) \in \partial\Omega_{\mathcal{I}^+}(p)$  then we have that  $\lim_{\lambda \rightarrow 1} f(w(\lambda)) = 0$ .

A proof for these could be reached by contradiction. For case 1. assume that there exists such a continuous path  $w(\lambda)$  such that there exists a  $v_{max}$  such that for any  $k \in \{w(\lambda) \cap \Omega_{\mathcal{H}^+}(p)\}$ , we have that  $v[\gamma(k|_p) \cap \mathcal{H}^+] \leq v_{max}$ . One has to show that this implies that there exists  $\tau_{max}$  such that for any  $k \in \{w(\lambda) \cap \Omega_{\mathcal{H}^+}(p)\}$  we have that  $\tau_{\mathcal{H}^+}(k) := \{\tau | \gamma(k|_p)[\tau] \in \mathcal{H}^+\}$  satisfies  $\tau_{\mathcal{H}^+}(k) \leq \tau_{max}$ . Further one has to show that this implies that by the considerations of Lemma 3.4.1 there exists  $\epsilon_{min}(\tau_{max}) > 0$  such that for any  $k \in \{w(\lambda) \cap \Omega_{\mathcal{H}^+}(p)\}$  we have that  $B_{\epsilon_{min}(\tau_{max})}(k) \subset \Omega_{\mathcal{H}^+}(p)$ . Therefore in particular also  $w(1) \in \Omega_{\mathcal{H}^+}(p)$  in contradiction to the assumption that  $w(1) \in \partial\Omega_{\mathcal{H}^+}(p)$  and the fact that  $\Omega_{\mathcal{H}^+}(p)$  is open.

For case 2. assume that there exists such a continuous path  $w(\lambda)$  such that there exists a  $u_{max}$  such that for any  $k \in \{w(\lambda) \cap \Omega_{\mathcal{I}^+}(p)\}$ , we have that  $u[\gamma(k|_p) \cap \mathcal{I}^+] \leq u_{max}$ . In the following the parameter  $\tau$  is to be considered the affine parameter of the null geodesic in the compactified manifold. One has to show that this implies that there exists  $\tau_{max}$  such that for any  $k \in \{w(\lambda) \cap \Omega_{\mathcal{I}^+}(p)\}$  we have that  $\tau_{\mathcal{I}^+}(k) := \{\tau | \gamma(k|_p)[\tau] \in \mathcal{I}^+\}$  satisfies  $\tau_{\mathcal{I}^+}(k) \leq \tau_{max}$ . Therefore by the considerations of Lemma 3.4.2 there exists  $\epsilon_{min}(\tau_{max}) > 0$  such that for any  $k \in \{w(\lambda) \cap \Omega_{\mathcal{I}^+}(p)\}$  we have that  $B_{\epsilon_{min}(\tau_{max})}(k) \subset \Omega_{\mathcal{I}^+}(p)$ . Therefore in particular also  $w(1) \in \Omega_{\mathcal{I}^+}(p)$  in contradiction to the assumption that  $w(1) \in \partial\Omega_{\mathcal{I}^+}(p)$  and the fact that  $\Omega_{\mathcal{I}^+}(p)$  is open.

#### 3.4.2. The Celestial Sphere and T-Orthogonal Trapping

In the following we will show that under the assumption that trapping is unstable we can show that no trapped null geodesics with negative energy can exist. It

is conceivable that this result can be strengthened to show that instability of trapping actually implies that all trapped null geodesics have to have positive energy. However at present times we failed with all attempts to do so.

**Lemma 3.4.3.** *Let  $\mathcal{M}$  be a smooth stationary Lorentzian manifold with Killing vector field  $T$  with one asymptotically flat end. Further assume future and past trapping to be unstable in the sense that for any observer at any point  $p$  in the exterior region we have that for any  $k \in \mathbb{T}_+(p)$  for any  $\epsilon > 0$*

- $B_\epsilon(k) \cap \Omega_{\mathcal{H}^+}(p) \neq \emptyset$
- $B_\epsilon(k) \cap \Omega_{\mathcal{I}^+}(p) \neq \emptyset$ .

then any trapped null geodesic  $\gamma$  in the exterior region satisfies  $g(T, \dot{\gamma}) \geq 0$ .

*Proof.* Note that for every  $k \in \Omega_{\mathcal{I}^+}(p)$  we have  $-\dot{\gamma}^\mu(k|_p)T_\mu > 0$ . Due to the instability condition we can choose a convergent sequence  $q_i \in \Omega_{\mathcal{I}^+}(p)$  with  $\lim_{i \rightarrow \infty} q_i = k$  for any  $k \in \mathbb{T}_+(p)$ . We then have that

$$E(k) = -\dot{\gamma}^\mu(k|_p)T_\mu = \lim_{i \rightarrow \infty} -\dot{\gamma}^\mu(q_i|_p)T_\mu \geq 0 \quad (3.66)$$

The statement then follows from the fact that the trapped set is given by  $\mathbb{T}(p) := \mathbb{T}_+(p) \cap \mathbb{T}_-(p)$ .  $\square$

## Conclusion

As this chapter is the only one containing results of direct physical interest we add here a discussion of the presented results.

Despite the fact that trapping in a Kerr-Newman-Taub-NUT spacetime is much more complicated than in Schwarzschild, we showed in this chapter that the topological structure of the future and past trapped set at any point in the exterior region in Kerr-Newman-Taub-NUT is in fact simple and identical to the situation in Schwarzschild. Even though the qualitative features of  $\mathbb{T}_\pm(p)$  do not change under a change of parameters, the quantitative features do.

We then showed that there exist only two continuous degeneracies for the shadow of any observer in the exterior region of a Kerr-Newman-Taub-NUT spacetime. In particular when one focuses on the physically relevant case of Kerr-Newman (hence  $l = 0$ ) the only continuous degeneracy is given by scaling of all parameters with the mass. If one assumes that, apart from the discrete spacetime isometries, no discrete degeneracies exist, then the result presented in this paper suggests that in principle an observer in the exterior region of a Kerr-Newman spacetime could

### 3. *Black Hole Shadows*

extract the relative angular momentum  $a/M$  of the black hole, as well as the relative charge  $Q/M$ , the relative distance  $r/M$ , and the angle of observation relative to the rotation axis of the black hole. Additionally, one could extract how fast one is moving in comparison to a standard observer at that point in the manifold. It is interesting however, that from an observation of the shadow alone an observer can never conclude that the Taub-NUT charge must vanish.

Note that if one looks at the projection of the shadows of the standard observers on the complex plane and chooses a parabolic and a hyperbolic Möbius transformation (see Appendix A.2) for each standard observer such that all shadows intersect the real axis at  $+1$  and  $-1$ , it turns out that the changes of the shape due to variations of  $r/M$  and  $Q/M$  are extremely small. Hence reading off these parameters from the shadow would require a very precise measurement of the shadow curve. Adding in the fact that the light sources can be rather messy, the observational task is certainly formidable, so that at least in the foreseeable future there is little hope that from the shape of the shadows alone one can extract in practice more than a rough estimate on  $a/M$ . However, from a theoretical point of view it seems plausible (and our results are a strong indication in this direction) that one can extract very detailed information about a black hole just by looking at it.

Finally we showed that the celestial sphere is also of interest as a mathematical tool to prove properties of trapping in general black hole spacetimes.

# 4. Mode Stability

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In this chapter we will move away from geodesics on to fields in black hole spacetimes. However these considerations are closely related. We will use the following pages to elaborate on this relation, before going into the proof of the main theorem of this chapter. We will now discuss the quasinormal modes (QNM) that will serve as a bridge between trapping and mode stability. We start with some general remarks

A partial differential equation (PDE) is called separable, if there exists a coordinate system in which a product Ansatz can be chosen such that the PDE turns into a set of ordinary differential equations (ODE) for each factor. The factors each only depend on one variable. For a wave equation the solutions described by the product of the solutions of these ODEs are called wave modes, or simply modes. We will use the wave equation for the massless scalar field on Kerr for illustration. We will consider the rescaled D'Alembertian given by  $\square = \Sigma \square_g$ . This form of

#### 4. Mode Stability

the wave equation in Boyer-Lindquist coordinates can be found in many papers [Andersson and Blue \(2009\)](#) [Finster et al. \(2005\)](#) and is given by

$$\square\psi = \left[ \frac{\partial}{\partial r} \Delta \frac{\partial}{\partial r} + \frac{1}{\Delta} R(r; a, M; \partial_t, \partial_\phi, Q(\partial_\phi, \partial_\theta)) \right] \psi = 0 \quad (4.1)$$

where the function  $R$  is the same as in the geodesic equation.

The partial derivatives associated with the two Killing vectorfields are symmetry operators of the wave equation. We want our mode solutions to be eigenfunctions of these symmetry operator. Thus

$$\partial_t \psi(t, r, \phi, \theta) = -i\omega \psi(t, r, \phi, \theta), \quad (4.2)$$

$$\partial_\phi \psi(t, r, \phi, \theta) = im \psi(t, r, \phi, \theta), \quad (4.3)$$

$$(4.4)$$

and therefore the wave mode has to be of the form  $\psi(t, r, \phi, \theta) = T(t)P(\phi)\tilde{\psi}(r, \theta)$  with

$$T(t) = e^{-i\omega t}, \quad (4.5)$$

$$P(\phi) = e^{im\phi}. \quad (4.6)$$

The orbits in  $\phi$  are closed hence  $P(\phi) = P(\phi + 2\pi)$  and therefore  $m$  has to be an integer. The orbit in  $t$  is unlimited and thus there are no such restrictions on  $\omega$ . It can be any complex number. If we now plug this Ansatz into equation(4.1) we are left with a PDE for  $\tilde{\psi}(r, \theta)$ . This can then be separated into two ordinary differential equations for  $S_{ml}^\omega(\theta)$  and  $R_{ml}^\omega(r)$ . So we can write our general mode solution as a product of these functions which each depend on one variable only

$$\psi(t, r, \theta, \phi) = T(t)R_{ml}^\omega(r)S_{ml}^\omega(\theta)P(\phi). \quad (4.7)$$

We are now interested in the properties of these mode solutions.

### Overview of this chapter

In section 4.1 we discuss the notion of quasinormal modes and how they relate to trapped null geodesics. In section 4.3 we collect some background on the radial Teukolsky equation and discuss the asymptotic behavior of its solutions in section 4.3.1. Lemma 4.3.1 collects the facts about solutions with no incoming radiation which we shall need for the proof of our main result. The phenomenon of superradiance is reviewed in section 4.3.2. This analysis yields the previously

#### 4.1. Quasi Normal Modes and their location in Phase-Space

known fact that for non-superradiant frequencies and for half-integer spins, mode stability holds. Section 4.4 introduces the integral transformation which will be used along the lines of Whiting (1989) to transform the radial Teukolsky equation to a non-superradiant equation. This transformation is the essential step in the proof of mode stability. The limiting behavior of the transformed radial function is analyzed in section 4.5, and the proof of the main theorem is given in section 4.6.

In section 4.7 we provide some background on the charged scalar field in Reissner Nordström.

### 4.1. Quasi Normal Modes and their location in Phase-Space

The quasi normal modes correspond to the free oscillations of a field. This is a field that is purely outgoing on all boundaries of the patch of the manifold under consideration. It turns out they are closely related to the trapped orbits in the geometric optics approximation. The ray-approximation is valid for high frequencies  $\omega$ . Bekenstein (1973a) argues that the ray approximation is valid when the characteristic frequency of the beam  $\omega_c$  is much larger than the inverse of the characteristic scale of the black hole, hence  $\omega_c \gg M^{-1}$ . In this section we will first introduce the general form of quasi normal modes. We will discuss their origin and interpretation and how they relate to the quasi normal modes on a Kerr geometry.

Most systems in physics, can oscillate in some way. These oscillation occur usually at characteristic frequencies which are depending on the parameters of the system. If we look at the strings of a guitar for example, the wavelength of the oscillations have to be a proper fraction of twice the strings length. In a perfect system there does not exist any energy dissipation and the oscillations, once excited, can go on forever. However, as soon as we go to a real system we have to take dissipation into account. This leads to a fall-off in the amplitude of the oscillation over time as the system loses energy. The characteristic wavelength does not change, because the scale of the system is still the same. Such damped oscillations can be approximated by quasi normal modes. These are mode solutions  $\exp(i\sigma t)$  with complex frequencies  $\sigma = \omega + i\Gamma$ . The complex part in the frequency leads to the exponential decay. The different modes in a mechanical system usually have different rates of decay, thus the late time behavior is usually dominated by a single mode.



#### 4. Mode Stability

The following discussion about the quasi normal modes will focus on the location of the real part of their frequency. The analysis of the quasi normal modes can in principle be generalized to all integer spin field but to build the bridge between trapping and QNMs the case of a scalar field is sufficient. The complex frequencies of the quasi normal modes can be calculated in a number of different, but equivalent, ways. [Detweiler \(1977\)](#) showed that calculating the frequencies in the picture of free oscillations of the perturbation field is equivalent to calculating the resonance frequencies in the scattering picture. [Mashhoon \(1985\)](#) calculated the quasi normal frequencies through a perturbation of the trapped prograded equatorial null rays. [Hod \(2009\)](#) showed that these frequencies calculated by Mashhoon are equivalent to the frequencies calculated in the picture of free oscillations for the high  $a$  limit, with  $(M^2 - a^2) \ll a^2$  and  $|a| < M$ . In [Yang et al. \(2012\)](#) it is shown that the leading order terms of the wave parameters for the quasi normal modes correspond to the conserved quantities for trapped null-geodesics. Similar to the discussion in section 2.6 the real part of the frequency  $\omega$  corresponds to the energy  $E$ , the index  $m$  corresponds to the  $z$ -angular-momentum  $L_z$  and the real part of  $\lambda_{lm}$  corresponds to  $L^2$ .

[Mashhoon \(1985\)](#) calculated the quasi normal frequencies from the ray approximation in the high  $a$  limit to be

$$\sigma_n = m\omega_+ - i \left( n + \frac{1}{2} \right) \beta\omega_+. \quad (4.8)$$

The real frequency is simply given by the  $z$ -angular-momentum number of the mode times the Kepler frequency of the null ray at  $r_1$  given by

$$\omega_+ \equiv \frac{M^{1/2}}{r_1^{3/2} + aM^{1/2}}. \quad (4.9)$$

The function  $\beta$  is given by [Mashhoon \(1985\)](#)

$$\beta = \frac{(12M)^{1/2}(r_1 - r_+)(r_1 - r_-)}{r_1^{3/2}(r_1 - M)}. \quad (4.10)$$

[Hod \(2009\)](#) showed that in the extremal limit these quasi normal modes can be rewritten in terms of the Bekenstein-Hawking temperature of the black-hole

$$T_{BH} = \frac{(M^2 - a^2)^{1/2}}{4\pi M[M + (M^2 - a^2)^{1/2}]}. \quad (4.11)$$

For the limit  $T_{BH} \rightarrow 0$  we get

$$\sigma_n = m\omega_H + 2\pi T_{BH} \left( 1 - \frac{\sqrt{3}}{2} \right) m - i2\pi T_{BH} \left( n + \frac{1}{2} \right) + \mathcal{O}(MT_{BH}^2) \quad (4.12)$$

## 4.2. Mode Stability on the real axis

and the horizon frequency given in (2.55) expands as

$$\omega_H = \frac{1}{2M} - 2\pi T_{BH} + \mathcal{O}(MT_{BH}^2). \quad (4.13)$$

For the Schwarzschild black hole Press (1971) found that the real part of the quasi normal frequencies of a Schwarzschild black hole are given by

$$\omega = l \frac{1}{27^{1/2}M}. \quad (4.14)$$

This ratio is well known from trapping (2.32). Goebel (1972) obtained the same result from considerations similar to the ones of Mashhoon presented above.

The above discussion covers only a small fraction of the properties and interpretations of quasi normal modes. The quasi normal modes are often interpreted as the vibrations of the black hole (Press (1971)) however Goebel (1972) argued these can not be “excited” in the usual sense as for example the vibrations of molecules in a gas. Furthermore they do not provide a source for fields. They only provide a temporary storage for high-frequency modes, originating from other physical processes in the spacetime such as the collision of two black holes. The quasi normal modes are strongly dependent on the black hole parameters. Hence if a precise measurement of the ring down after the black hole merger is obtained in LIGO/VIRGO it allows for a detailed characterization of the parameters of the black hole formed after the merger.

In the following section we will focus on the location of frequencies of the QNMs in the complex plane. In particular we will expand the mode stability result by Whiting (1989) to the real axis.

## 4.2. Mode Stability on the real axis

**Remark 4.2.1.** *The material in section 4.2 through 4.6 was created in joint work with Lars Andersson, Siyuan Ma and Bernard Whiting 2017*

The field equations on the Kerr spacetime for massless test fields with spins  $\mathfrak{s}$  between 0 and 2 imply that the scalar components with extreme spin weights  $s = \pm\mathfrak{s}$  solve the Teukolsky Master Equation (TME) (Teukolsky, 1973), a separable, spin-weighted wave equation. Let

$$\mathbf{L} = \partial_r \Delta \partial_r - \frac{1}{\Delta} \{ (r^2 + a^2) \partial_t + a \partial_\phi - (r - M) s \}^2$$

#### 4. Mode Stability

$$\begin{aligned}
& -4s(r + ia \cos \theta) \partial_t + \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta \\
& + \frac{1}{\sin^2 \theta} \{a \sin^2 \theta \partial_t + \partial_\phi + is \cos \theta\}^2
\end{aligned}$$

where  $(t, r, \theta, \phi)$  are Boyer-Lindquist coordinates and  $\Delta = r^2 - 2Mr + a^2$ . Then (Whiting, 1989)

$$\mathbf{L}\Phi = 0 \tag{4.15}$$

is a form of the TME on the Kerr exterior background with mass  $M$  and angular momentum per unit mass  $a$ . The parameter  $s$  is the spin weight of the field  $\Phi$ .

For completeness, we recall the definition of the fields  $\Phi$  solving the TME. In order to do this, we make the spin weight explicit as a subindex. For  $s = 0$ , the TME is equivalent to the scalar wave equation  $\nabla^a \nabla_a \Phi_0 = 0$ . For spins  $s = 1/2, 1, 3/2, 2$  the field equations are Dirac-Weyl, Maxwell, Rarita-Schwinger, and linearized gravity, respectively. For spins  $0, 1/2, 1, 2$ , see Teukolsky (1973), for the spin-3/2 case, Torres del Castillo and Silva-Ortigoza (1992), see also Silva-Ortigoza (1995). Working in the Kinnersley principal tetrad, let  $\phi_0, \phi_2$  be the Newman-Penrose scalars of spin weights  $1, -1$  for a Maxwell test field on the Kerr background, and let  $\dot{\Psi}_0, \dot{\Psi}_4$  denote the linearized Weyl scalars of spin weights  $2, -2$  for a solution of the linearized vacuum Einstein equations on the Kerr background, see Aksteiner and Andersson (2011) for details. Let the scalar  $\zeta$  be chosen so that  $\zeta \propto \Psi_2^{-1/3}$ , where  $\Psi_2$  is the spin weight zero Weyl scalar. In Boyer-Lindquist coordinates, we can take  $\zeta = r - ia \cos \theta$ . The scalar fields  $\Phi_s$  for integer  $s$  are defined by setting

$$\Phi_{-2} = \Delta^{-1} \zeta^4 \dot{\Psi}_4, \quad \Phi_{-1} = \Delta^{-1/2} \zeta^2 \phi_2, \tag{4.16a}$$

$$\Phi_1 = \Delta^{1/2} \phi_0, \quad \Phi_2 = \Delta \dot{\Psi}_0 \tag{4.16b}$$

Similarly, let  $\chi_0, \chi_1$  denote the scalars of spin weights  $\pm 1/2$  for a Dirac-Weyl test field, and  $H_0, H_3$  the scalars of spin weights  $\pm 3/2$  for a Rarita-Schwinger test field. We define

$$\Phi_{-3/2} = \Delta^{-3/4} \zeta^3 H_3, \quad \Phi_{-1/2} = \Delta^{-1/4} \zeta \chi_1, \tag{4.16c}$$

$$\Phi_{1/2} = \Delta^{1/4} \chi_0, \quad \Phi_{3/2} = \Delta^{3/4} H_0 \tag{4.16d}$$

The TME admits separated solutions of the form

$$\Phi = e^{-i\omega t} e^{im\phi} S(\theta) R(r). \tag{4.17}$$

where  $\omega, m$  are the frequencies corresponding to the Killing vector fields  $(\partial_t)^a, (\partial_\phi)^a$ . Let

$$K = (r^2 + a^2)\omega - am \tag{4.18}$$

Then with

$$\mathbf{R} = \partial_r \Delta \partial_r + \frac{K^2 - 2iK(r-M)s - (r-M)^2 s^2}{\Delta} + 4sir\omega - \Lambda \quad (4.19)$$

$$\begin{aligned} \mathbf{S} = & \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta - \frac{m^2}{\sin^2 \theta} + a^2 \cos^2 \theta \omega^2 - 2a\omega s \cos \theta - \frac{2ms \cos \theta}{\sin^2 \theta} - s^2 \cot^2 \theta \\ & + \Lambda + 2a\omega m - a^2 \omega^2, \end{aligned} \quad (4.20)$$

where  $\Lambda$  is a separation constant, which can be assumed to be real for real  $\omega$ , we have after making the substitutions  $\partial_t \leftrightarrow -i\omega$ ,  $\partial_\phi \leftrightarrow im$ ,

$$\mathbf{L} = \mathbf{R} + \mathbf{S},$$

and

$$[\mathbf{R}, \mathbf{S}] = 0.$$

In particular,  $\mathbf{R}, \mathbf{S}$  are commuting symmetry operators for  $\mathbf{L}$ . It follows from the above that for separated waves of the form (4.17), (4.15) is equivalent to the equations  $\mathbf{R}R = 0$ ,  $\mathbf{S}S = 0$ . We shall refer to the equations

$$\mathbf{R}R = 0 \quad (4.21a)$$

$$\mathbf{S}S = 0 \quad (4.21b)$$

as the radial and angular Teukolsky equations, respectively. As for the treatment of the real frequency case by [Shlapentokh-Rothman \(2015\)](#), we shall not be concerned with the analysis of the angular Teukolsky equation here, but point out that  $\mathbf{S}$  is formally self-adjoint on  $[0, \pi]$  with respect to  $\sin \theta d\theta$ . Imposing the condition that the solutions correspond to regular spin-weighted functions fixes the boundary conditions at  $\theta = 0, \pi$  and equation (4.21b) becomes a Sturm-Liouville problem which has a discrete, real spectrum; see [Leaver \(1986\)](#) for more details. The separation constant used here is related to that used in [Teukolsky and Press \(1974\)](#) by  $\Lambda + 2a\omega m - a^2 \omega^2 = E - s^2$ , and to the one used in [Whiting \(1989\)](#) and [Teukolsky \(1972\)](#) by  $\Lambda + 2a\omega m - a^2 \omega^2 = A + s$ .

For fields of non-zero spin, the TME does not admit a real action, and hence standard arguments do not yield energy conservation and dispersive estimates. This is an obstacle to proving stability for the test fields with non-zero spin on the Kerr exterior spacetime, which would be an important step towards proving non-linear stability of the Kerr black hole, i.e. that a Kerr black hole with  $|a| < M$  is dynamically stable as a solution to the vacuum Einstein field equations, in the sense that the maximal development of a sufficiently small perturbation of the Kerr solution is asymptotic in the future to a member of the Kerr family.

#### 4. Mode Stability

In [Whiting \(1989\)](#), one of the authors gave a proof of mode stability. In particular, the TME has no separated wave solutions, or modes, which are such that the frequency  $\omega$  has positive imaginary part, and which have no incoming radiation in the sense that the wave is outgoing at infinity, and ingoing at the horizon, see section 4.3.1. The main result of [Whiting \(1989\)](#) states that the TME admits no exponentially growing solutions without incoming radiation. In the case of  $\Im\omega > 0$ , the condition of no incoming radiation can be restated as saying that the solution has support only on the future horizon and null infinity. On the other hand, there do exist mode solutions with no incoming radiation for certain frequencies with negative imaginary part. This case corresponds to quasi-normal modes ([Kokkotas and Schmidt, 1999](#)), which are exponentially decaying in time.

It is known that exponentially growing modes must arise by quasi-normal frequencies passing from the lower half plane through the real axis into the upper half plane as  $a$  is changed from zero. This was argued heuristically by Press and Teukolsky (1973, p. 651) and later shown by Hartle and Wilkins (1974), see also [Teukolsky and Press \(1974, p. 452\)](#). For this reason, the mode stability problem can be reduced to considering the case of real frequencies.

Recently, the mode stability argument has been revisited for the case of real frequencies, restricting to the spin-0 case ([Shlapentokh-Rothman, 2015](#)). In the case of real frequencies, the mode stability result states that restricting to separated waves with no incoming radiation in the above sense, the radial Teukolsky equation has no non-trivial solutions. This has the consequence that there are linearly independent solutions  $R_{\text{hor}}, R_{\text{out}}$  which are purely ingoing at the horizon and outgoing at infinity, respectively, a fact which plays a central role in the proof of boundedness and decay for scalar waves on sub-extreme Kerr exterior spacetimes with  $|a| < M$  ([Dafermos et al., 2014](#)), in particular it is used to treat the superradiant range of frequencies; see section 4.3.2 for background on superradiance. We mention here a recent paper ([Finster and Smoller, 2016](#)) which also discusses the stability problem for the Teukolsky equation.

Motivated by the relevance of the TME for the black hole stability problem we here give a proof of mode stability on the real axis for fields with arbitrary spin. Our main result is the following, cf. Theorem 4.6.1 below.

**Theorem 4.2.1** (Mode stability on the real axis). *Let  $\Phi$  be a separated solution to the TME with  $\omega > 0$  for the sub-extreme Kerr black hole. Assume that  $\Phi$  has purely ingoing radiation at the horizon and purely outgoing radiation at infinity. Then  $\Phi = 0$ .*

**Remark 4.2.2.** *A classical scattering argument can be used to show mode stability on the real axis for half-integer spins, or for frequencies outside of the superradiant*

### 4.3. The radial Teukolsky equation

range, see equation (4.26) below. The proof of mode stability on the real axis presented in this section is independent of that scattering argument.

The fact that there are no solutions to the TME with no incoming radiation has the important consequence that the radial Teukolsky equation has two fundamental solutions  $R_{\text{hor}}$  and  $R_{\text{out}}$  which are ingoing at the horizon, and outgoing at infinity, respectively, and are linearly independent, with non-vanishing Wronskian. This implies that one can construct solutions of the inhomogenous Teukolsky equation using the method of variation of the parameter, see Remark 4.6.1 below for details. The properties of the solutions  $R_{\text{hor}}$  and  $R_{\text{out}}$  can be used to estimate the solution of the inhomogenous Teukolsky equation.

Unless otherwise stated, we shall in the rest of the section restrict to positive frequency,  $\omega > 0$ . Note that the substitution  $(\omega, m, s) \rightarrow (-\omega, -m, -s)$  maps solutions of TME to solutions.

In case  $\omega = 0$ , (4.17) represents a time independent solution of the TME. In this case, the radial Teukolsky equation becomes a hypergeometric equation with three regular singular points  $r_-, r_+, \infty$  which requires a separate discussion. It has been pointed out that this equation does not have solutions which are well-behaved at the horizon and at infinity, see Teukolsky (1972, p. 1117), see also Press and Teukolsky (1973, p. 651).

## 4.3. The radial Teukolsky equation

The radial Teukolsky equation is a second order ordinary differential equation with rational coefficients, and an analysis of its singular points yields that it is a confluent Heun equation (Slavyonov and Lay, 2000, Ronveaux, 1995), with regular singular points at  $r_-, r_+$  and an irregular singular point of rank 1 at  $\infty$ . Here we use the notion of rank following Erdélyi (1956, p. 60). The s-rank of  $r = \infty$  as in (Slavyonov and Lay, 2000) is 2. In this context it is natural to consider the radial Teukolsky equation on the complex  $r$ -plane. In a neighborhood of each regular singular point, the general solution is a linear combination of the fundamental Frobenius solutions, while at the irregular singular point, one considers the Thomé, or normal, solutions. These are formal solutions but in each Stokes sector, a pair of solutions can be found which are asymptotically represented by the normal solutions, cf. Erdélyi (1956, Chapter III) for details, see also Olver (1997).

Equation (4.21a) with  $\mathbf{R}$  given by (4.19) is in self-adjoint form. The form of the equation and its solutions can be changed by transformations of the independent

#### 4. Mode Stability

variable  $r$  (eg. Möbius transformations), s-homotopic transformations (rescalings), and integral transformations of the dependent variable. In the rest of this section, as a preparation for the integral transformations considered in section 4.4, we will state a few basic facts about the radial Teukolsky equation and its solutions.

The rotation speeds  $\omega_{\pm}$  and surface gravities  $\kappa_{\pm}$  of the outer and inner horizons located at  $r_{\pm} = M \pm \sqrt{M^2 - a^2}$  are given by

$$\omega_{\pm} = a/(2Mr_{\pm}), \quad (4.22)$$

$$\kappa_{\pm} = \pm \frac{r_+ - r_-}{4Mr_{\pm}} \quad (4.23)$$

Associated to null generators of the horizons,

$$\chi_{\pm}^a = (\partial_t)^a + \omega_{\pm}(\partial_{\phi})^a, \quad (4.24)$$

we have for a wave with time frequency  $\omega$  and azimuthal frequency  $m$ , the frequencies

$$k_{\pm} = \omega - \omega_{\pm}m. \quad (4.25)$$

The superradiant range of frequencies is characterized, independent of the signs of  $\omega$ ,  $m$  and  $a$ , by

$$\omega k_+ < 0 \quad (4.26)$$

The tortoise coordinate  $r_*$  is defined by

$$dr_* = \frac{r^2 + a^2}{\Delta} dr \quad (4.27)$$

We have

$$r_* \sim \frac{1}{2\kappa_+} \ln(r - r_+), \quad \text{for } r \rightarrow r_+ \quad (4.28)$$

$$r_* \sim r + 2M \ln(r), \quad \text{for } r \rightarrow \infty \quad (4.29)$$

Let  $r_{\sharp}$  be defined by

$$dr_{\sharp} = \frac{a}{\Delta} dr$$

and set  $u_{\pm} = t \pm r_*$  and  $\phi_{\pm} = \phi \pm r_{\sharp}$ . Then  $(u_+, r, \theta, \phi_+)$  and  $(u_-, r, \theta, \phi_-)$  are the ingoing and outgoing Eddington-Finkelstein coordinates, respectively. The ingoing coordinates are regular on the future horizon, while the outgoing ones are regular on the past horizon.

For later use, it will be convenient to introduce the quantities

$$\xi = \frac{i(am - 2Mr_+\omega)}{r_+ - r_-} = -i \frac{k_+}{2\kappa_+} \quad (4.30a)$$

$$\eta = \frac{-i(am - 2Mr_-\omega)}{r_+ - r_-} = -i \frac{k_-}{2\kappa_-} \quad (4.30b)$$

### 4.3.1. Asymptotics

We shall now discuss the asymptotic behavior of the solutions of the radial Teukolsky equation (4.21a) at the singular points. The radial function  $R$  used here corresponds to the function  $R$  used in (Teukolsky, 1973) and (Teukolsky and Press, 1974), multiplied by a factor  $\Delta^{s/2}$ . The asymptotic behavior of  $R$  which we state below can be read off from (Teukolsky and Press, 1974, Table 1) after taking the factor  $\Delta^{s/2}$  into account, see also Castro et al. (2013) for discussion of asymptotics. For completeness we shall indicate the derivation of these results.

#### Asymptotics at $r = \infty$

In order to analyze the possible asymptotic behavior of solutions to the radial Teukolsky equation at infinity, it is convenient to transform the equation to normal form. Write (4.21a) as

$$\partial_r \Delta \partial_r R + V_0 R = 0 \quad (4.31)$$

and let  $G = \Delta^{-1/2}$ . The rescaling  $R = yG$  transforms (4.31) to the form

$$\partial_r^2 y + Qy = 0 \quad (4.32)$$

with

$$Q = -\frac{1}{2} \partial_r \frac{\partial_r \Delta}{\Delta} - \frac{1}{4} \left( \frac{\partial_r \Delta}{\Delta} \right)^2 + \frac{V_0}{\Delta}$$

The leading terms in  $Q$  at  $r = \infty$  are

$$Q = \omega^2 + \frac{2i\omega s + 4\omega^2 M}{r} + O(r^{-2}).$$

From this we can determine following Erdélyi (1956, Chapter III) that the two normal solutions to the radial Teukolsky equation (4.21a) near  $r = \infty$  have the asymptotic forms

$$R \sim e^{\pm i\omega r} r^{\pm 2iM\omega r \mp s - 1} \quad (4.33)$$

with the upper sign corresponding to outgoing waves. Due to the fact that the singular point at  $r = \infty$  is irregular, of rank 1, we shall need some further details concerning the Stokes phenomenon. As mentioned above, the rank of the irregular singular point  $r = \infty$  is 1. Following Erdélyi (1956, Chapter III.5) we find that the Stokes line is the real line  $\Im r = 0$  in the complex  $r$ -plane. The Stokes line decomposes the complex  $r$ -plane into two Stokes sectors, and in each Stokes sector, one of the two normal solutions is exponentially increasing, and one is exponentially decreasing. These are referred to as the *dominant* and *recessive* solutions.



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In particular, if we consider a sectorial region  $S$ ,  $|r| > A$ ,  $\alpha < \text{Arg } r < \beta$ , contained in a Stokes sector, the asymptotic expansions of the two normal solutions hold uniformly in  $\text{Arg } r$ , for  $r \in S$ . For later use, we note here that for a sectorial region  $S$  in the upper half plane, with  $0 < \alpha < \pi/2 < \beta < \pi$  as in figure 4.1, the outgoing condition at  $r = \infty$ , i.e. the upper sign choice in (4.33), corresponds to the recessive normal solution in  $S$ .

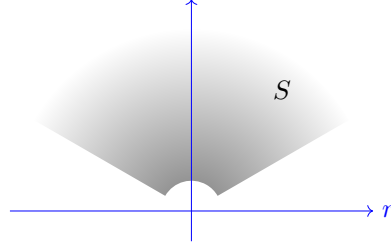


Figure 4.1.: The sectorial region  $S$

#### Asymptotics at $r_{\pm}$

The characteristic exponents for regular singular points  $r_+$ ,  $r_-$  of the radial Teukolsky equation are

$$\{\xi - s/2, -\xi + s/2\} \quad \text{for } r_+ \quad (4.34a)$$

$$\{\eta - s/2, -\eta + s/2\} \quad \text{for } r_- \quad (4.34b)$$

Let  $r_0$  be one of the regular singular points, with characteristic exponents  $\rho_j$ ,  $j = 1, 2$ , which we order so that  $\Re \rho_1 \leq \Re \rho_2$ . We first consider the non-resonant case where  $\rho_1 - \rho_2$  is not an integer. In this case the Frobenius solutions at  $r_0$  are of the form

$$R_j(r) = (r - r_0)^{\rho_j} R_{0,j}(r), \quad j = 1, 2, \quad (4.35)$$

where  $R_{0,j}(r)$  are analytic in a neighborhood of  $r_0$  of radius  $r_+ - r_-$ , cf. [Slavyonov and Lay \(2000, §1.1.4\)](#), see also [Erdélyi \(1956, p. 60\)](#). We may normalize the solutions so that  $R_{0,j}(r_+) = 1$ .

We next consider the case of resonance at  $r_+$ , i.e.  $\xi = 0$  and  $s$  an integer. Note that  $\xi = 0$  corresponds to the upper bound of the range of superradiant frequencies, see section 4.3.2. In this case, the characteristic exponents  $\rho_j$  take values  $-s/2, s/2$ , and the Frobenius solutions of (4.21a) corresponding to  $\rho_1$  contains a logarithmic term,

$$R_1(r) = (r - r_+)^{\rho_1} R_{0,1} + A_2 (r - r_+)^{\rho_2} R_{0,2}(r) \ln(r - r_+). \quad (4.36)$$

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Here, for  $R_{0,1}(r_+) = 1$ ,  $A_2$  is a non-zero constant that can be computed. In the resonant case, we choose  $R_{0,1}$  so that  $R_{0,1} = \sum_{n=0}^{\infty} c_n (r - r_+)^n$  with  $c_{|s|} = 0$ , cf. [Slavyonov and Lay \(2000, Theorem 1.3\)](#). The case of resonance at  $r_-$  is similar.

#### Waves with no incoming radiation

We shall say that waves which are outgoing at infinity and ingoing at the horizon, i.e.  $r_+$ , satisfy the no incoming radiation condition. A discussion of the boundary conditions for the radial Teukolsky equation can be found in ([Teukolsky, 1973, §V](#)), where also the notion of ingoing and outgoing waves is defined, see also ([Press and Teukolsky, 1973, p. 653, eq. \(2.10\)](#)).

As discussed by Penrose ([1965](#)), an analysis of the asymptotic behavior of massless fields at null infinity leads, upon taking into account the scaling properties of the tetrad components of the field, to specific rates of fall-off depending on the spin weight of the field. This is known as the peeling property; see [Mason and Nicolas \(2012\)](#), [Frauendiener et al. \(1996\)](#), [Andersson et al. \(2014\)](#), [Hinder et al. \(2011\)](#) for discussions of various aspects of peeling. The peeling property can be summarized by saying that for a scalar component  $\varphi_s$  of spin weight  $s$ , defined with respect to the Kinnersley tetrad, we have  $\varphi_s = O(r^{-3-s})$ . Taking into account the rescalings given in ([4.16](#)) we find for that the peeling behavior of the solution of the TME is

$$\Phi_s = O(r^{-1-s}), \quad \text{as } r \rightarrow \infty. \quad (4.37)$$

In order to analyze the behavior of spinning fields at the horizon, a tetrad which is well behaved at the horizon must be used. Following Teukolsky and Press ([1974, §IV](#)), see also [Hawking and Hartle \(1972\)](#) and [Znajek \(1977\)](#), one finds that the fields  $\Delta^{s/2}\Phi_s$  are regular on the horizon.

Outgoing waves at  $r = \infty$  must have an asymptotic form compatible with peeling, as discussed above, and should have positive radial group velocity, while ingoing waves at the horizon as seen by a physically well-behaved observed must be *non-special* (i.e. neither vanishing nor singular on the horizon) and should have negative radial group velocity.

Based on the above discussion of Frobenius and Thomé solutions, we are led to the following definition.

**Definition 4.3.1** (No incoming radiation condition). *Let  $R$  be a solution of the radial Teukolsky equation. Then we shall say that  $R$  has no incoming radiation provided*

$$R(r) \sim e^{i\omega r} r^{2iM\omega} r^{-s-1} \quad \text{at } r = \infty, \quad (4.38)$$

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$$R(r) \sim (r - r_+)^{\xi - s/2} \quad \text{at } r = r_+. \quad (4.39)$$

In particular, we shall require that  $R(r)$  is equal to the Frobenius solution with exponent  $\xi - s/2$  at  $r = r_+$  and equal near infinity  $r = \infty$  to the normal solution which is recessive in the upper half plane.

Specializing the discussion in this section to the case of waves with no incoming radiation, we can state the following lemma, which summarizes the properties that we shall make use of. Note that we here view  $R$  as a solution of (4.21a) in the complex  $r$ -plane. The results stated in the lemma are direct consequences of the discussion in this section and the references given there.

**Lemma 4.3.1.** *Let  $R$  be a solution to the radial Teukolsky equation with no incoming radiation. Then the following hold:*

1. *If  $\xi \neq 0$  or if  $s$  is not a positive integer, then near  $r_+$ ,*

$$R = (r - r_+)^{\xi - s/2} R_{+,1}(r), \quad (4.40)$$

*where  $R_{+,1}$  has a power series expansion in  $r - r_+$  which converges in the disk  $|r - r_+| < r_+ - r_-$ .*

2. *If  $\xi = 0$  and  $s$  is a positive integer, then near  $r_+$ ,*

$$R = (r - r_+)^{-s/2} R_{+,1}(r) + A(r - r_+)^{s/2} \ln(r - r_+) R_{+,2}(r) \quad (4.41)$$

*where  $R_{+,1}, R_{+,2}$  have power series expansions in  $r - r_+$  which converge in the disk  $|r - r_+| < r_+ - r_-$ . Here  $A$  is a constant which can be calculated from  $R_{+,1}(r_+)$ .*

3. *Let  $S$  be a sectorial region in the upper half  $r$ -plane, of the form  $|r| > r_0$ ,  $\alpha < \text{Arg } r < \beta$  with  $0 < \alpha < \beta < \pi$ . Then  $R$  has an asymptotic expansion*

$$R \sim e^{i\omega r} r^{2iM\omega} r^{-s-1} \sum_{n=0}^{\infty} c_n r^{-n}, \quad \text{as } r \rightarrow \infty, \quad (4.42)$$

*which is valid uniformly in  $S$ . In particular, the estimate*

$$e^{-i\omega r} R = O(r^{-s-1}) \quad (4.43)$$

*is valid in  $S$ .*

### 4.3. The radial Teukolsky equation

**Remark 4.3.1.** 1. It follows from the properties of the Frobenius solutions, cf. [Slavyonov and Lay \(2000, Theorem 1.3\)](#), that in the non-resonant case, if  $R_{+,1}(r_+) = 0$  then  $R \equiv 0$ . To see this, the coefficients in the expansion of  $R_{+,1}$  are determined by  $R_{+,1}(r_+)$ , and hence if  $R_{+,1}(r_+) = 0$ , then  $R$  vanishes in a neighborhood of  $r_+$  and hence must be identically zero. In the resonant case with  $s = 0$  and  $\xi = 0$ , the logarithmic solution is excluded by condition [\(4.39\)](#). Finally, in the resonant case with  $s > 0$  and  $\xi = 0$ , if  $R_{+,1}(r_+) = 0$ , we find that also  $A = 0$  and hence it follows that  $R(r) \equiv 0$ , see also the discussion in section [4.3.1](#).

2. The estimate in [\(4.43\)](#) can be rephrased as saying that there is a constant  $C$  such that in the sectorial region  $S$ ,

$$|e^{-i\omega r} R| \leq C|r^{-s-1}|$$

The constant  $C$  depends on the parameters of the system, and the limit

$$\lim_{r \rightarrow +\infty} |r^{s+1} R(r)|$$

where the limit is taken along the positive real line. By [\(Olver, 1997, Chapter 7, Theorem 2.2\)](#), the asymptotic form [\(4.42\)](#) of  $R$  is valid in a circular sector  $|\text{Arg}(-iz)| < 3\pi/2 - \delta$  for  $\delta > 0$ , and this is the maximal sector of validity.

3. For completeness, we record that the asymptotic representation along the real line can be stated in terms of the tortoise coordinate as

$$R \sim \begin{cases} e^{i\omega r} r^{2iM\omega} r^{-s-1} & \sim e^{i\omega r_*} r^{-s-1} & \text{as } r \rightarrow \infty \\ (r - r_+)^{\xi-s/2} & \sim \Delta^{-s/2} e^{-ik_+ r_*} & \text{as } r \rightarrow r_+ \end{cases} \quad (4.44)$$

which, as mentioned above, after taking into account the rescaling by  $\Delta^{s/2}$ , agrees with the asymptotic form stated in [\(Teukolsky and Press, 1974, Table 1\)](#).

### 4.3.2. Superradiance

In this subsection we shall review the classical scattering analysis for spinning fields following [Teukolsky and Press \(1974\)](#), see also [Chandrasekhar \(1998\)](#). The results that we present here are not new, however, to the best of our knowledge the fact that superradiance does not happen for the spin- $\frac{3}{2}$  case has not been discussed before, see [Remark 4.3.3](#) below. We make the dependence of the spin weight explicit by a subindex  $s$ . Let  $R_s$  be a solution of the radial Teukolsky equation [\(4.21a\)](#) with spin weight  $s$ . The rescaling  $v_s = (r^2 + a^2)^{1/2} R_s$  transforms the radial

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Teukolsky equation to an equation with independent variable  $r_* \in (-\infty, \infty)$ , of the form

$$\frac{d^2}{dr_*^2} v_s + V_s v_s = 0 \quad (4.45)$$

with

$$V_s = \frac{\Delta}{(r^2 + a^2)^2} V_{0,s} - V_{\text{resc}}$$

where

$$V_{\text{resc}} = \frac{1}{(r^2 + a^2)} \frac{d^2}{dr_*^2} (r^2 + a^2)^{1/2} = \frac{\Delta}{(r^2 + a^2)^4} (a^2 \Delta + 2Mr(r^2 - a^2)). \quad (4.46)$$

We shall refer to (4.45) as the Schrödinger form of the radial Teukolsky equation. We have

$$\lim_{r \rightarrow r_+} V_s = (i\kappa_+ s + k_+)^2, \quad (4.47a)$$

$$\lim_{r \rightarrow \infty} V_s = \omega^2. \quad (4.47b)$$

In particular, for  $s \neq 0$ , the potential  $V_s$  is complex and

$$\bar{V}_s = V_{-s}, \quad (4.48)$$

where  $\bar{V}_s$  denotes the complex conjugate. Let  $v_s, v_{-s}$  be (a priori independent) solutions of (4.45) with spin weight  $s, -s$  respectively, and define the Wronskian

$$W[v_s, \bar{v}_{-s}] = v'_s \bar{v}_{-s} - \bar{v}'_{-s} v_s$$

where we have used a  $'$  to denote  $d/dr_*$ . Due to (4.48), both  $v_s$  and  $\bar{v}_{-s}$  solve the same equation, and hence the Wronskian is conserved,

$$W[v_s, \bar{v}_{-s}]' = 0. \quad (4.49)$$

We now make a scattering ansatz for  $R_s(r)$  which is purely ingoing at the horizon and a superposition of an ingoing and an outgoing part at infinity,

$$R_s(r) \sim \begin{cases} Y_{\text{hole},s} \Delta^{-s/2} e^{-ik_+ r_*}, & \text{at } r = r_+ \\ Y_{\text{in},s} e^{-i\omega r_*} r^{s-1} + Y_{\text{out},s} e^{i\omega r_*} r^{-s-1}, & \text{at } r = \infty. \end{cases} \quad (4.50)$$

Here we have intentionally left the normalization of the ingoing mode free. The fact that the Wronskian is conserved gives the identity

$$-4Mr_+(ik_+ + \mathfrak{s}\kappa_+) Y_{\text{hole},s} \overline{Y_{\text{hole},-s}} = -2i\omega Y_{\text{in},s} \overline{Y_{\text{in},-s}} + 2i\omega Y_{\text{out},s} \overline{Y_{\text{out},-s}}. \quad (4.51)$$

### 4.3. The radial Teukolsky equation

Following Teukolsky and Press (1974), see also Starobinsky and Churilov (1974), we now use the Teukolsky-Starobinsky Identities (TSI) to establish a relation between the fields with spin weights  $s, -s$ . Define the operators  $\mathcal{D}, \mathcal{D}^\dagger$  by  $\mathcal{D} = (d/dr - iK/\Delta)$  and  $\mathcal{D}^\dagger = \mathcal{D}(-\omega, -m)$ . For real  $\omega$  the dagger operation is identical with a complex conjugation. The TSI for the solutions of the radial Teukolsky equation are Chandrasekhar (1998), see also Kalnins et al. (1989), Fiziev (2009),

$$\Delta^{s/2} \mathcal{D}^{2s} \Delta^{s/2} R_{-s} = C_s R_s \quad (4.52a)$$

$$\Delta^{s/2} \mathcal{D}^{\dagger 2s} \Delta^{s/2} R_s = \overline{C_s} R_{-s} \quad (4.52b)$$

where  $C_s$  are constants depending on the parameters  $s, a, M, m, \omega$ , and the separation constant  $\Lambda$ .

**Remark 4.3.2.** *The TSI can be understood as saying that the operator  $\Delta^{s/2} \mathcal{D}^{2s} \Delta^{s/2}$  applied to  $R_{-s}$  is proportional to a solution  $R_s$  of the radial Teukolsky equation with spin weight  $s$  and vice versa. In applying the TSI we thus restrict to solutions  $R_s, R_{-s}$  satisfying this condition. It can be shown, cf. Aksteiner et al. (2016) and references therein, that if the spin-weighted fields  $R_s, R_{-s}$  are the radial functions corresponding to the components with extreme spin weight of a field satisfying the spin- $s$  test field equations on the Kerr background, then the TSI hold.*

Applying (4.52) to the asymptotic solutions of  $R_{\pm s}$  at the horizon and at infinity, and comparing leading order terms gives relations between  $Y_{\text{hole}, \pm s}$ ,  $Y_{\text{in}, \pm s}$ , and  $Y_{\text{out}, \pm s}$ . Some calculations give the identity

$$A_s |Y_{\text{hole}, s}|^2 = |Y_{\text{in}, s}|^2 - B_s |Y_{\text{out}, s}|^2 \quad (4.53)$$

where

$$B_s = \frac{(2\omega)^{4s}}{|C_s|^2}, \quad B_{-s} = B_s^{-1} \quad (4.54)$$

and  $A_s$  are given by

$$A_{-s} = \begin{cases} \left(\frac{2Mr_+}{\omega}\right)^{2s+1} \frac{\prod_{n=0}^s (k_+^2 + (s-n)^2 \kappa_+^2)}{k_+} & \text{if } s = 0, 1, 2, \dots \\ \left(\frac{2Mr_+}{\omega}\right)^{2s+1} \prod_{n=0}^{\lfloor s \rfloor} (k_+^2 + (s-n)^2 \kappa_+^2) & \text{if } s = \frac{1}{2}, \frac{3}{2}, \dots \end{cases} \quad (4.55)$$

$$A_s = \left(\frac{2Mr_+}{\omega}\right)^2 \frac{(k_+^2 + s^2 \kappa_+^2)}{A_{-s}}$$

where  $\lfloor s \rfloor$  is the integer part of  $s$ , i.e. the largest integer less than or equal to  $s$ . For the current considerations, equation (4.55) has been checked up to  $s = 2$  but

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it can be expected that the relation holds for all half-integer spins  $\mathfrak{s} \geq 0$ . The reflection and transmission coefficients  $\mathcal{R}_\mathfrak{s}, \mathcal{T}_\mathfrak{s}$  are defined as

$$\mathcal{R}_\mathfrak{s} = B_\mathfrak{s} \frac{|Y_{\text{out},\mathfrak{s}}|^2}{|Y_{\text{in},\mathfrak{s}}|^2} = B_{-\mathfrak{s}} \frac{|Y_{\text{out},-\mathfrak{s}}|^2}{|Y_{\text{in},-\mathfrak{s}}|^2}, \quad \mathcal{T}_\mathfrak{s} = A_\mathfrak{s} \frac{|Y_{\text{hole},\mathfrak{s}}|^2}{|Y_{\text{in},\mathfrak{s}}|^2} = A_{-\mathfrak{s}} \frac{|Y_{\text{hole},-\mathfrak{s}}|^2}{|Y_{\text{in},-\mathfrak{s}}|^2}$$

Then  $\mathcal{R}_\mathfrak{s}$  represents the fraction of the ingoing wave energy which is reflected out to infinity, while  $\mathcal{T}_\mathfrak{s}$  represents the fraction which is transmitted into the black hole. By construction we have the conservation law  $\mathcal{R}_\mathfrak{s} + \mathcal{T}_\mathfrak{s} = 1$ .

From the values of the coefficients  $A_\mathfrak{s}$  given in (4.55) we see that for integer spins,  $A_\mathfrak{s}$  changes sign with  $\omega k_+$ , while for half-integer spins,  $A_\mathfrak{s}$  is positive. This means that when  $\omega < m\omega_+$  (recall we are considering the case  $\omega > 0$ ), the reflection coefficient for a field with integer spin will be greater than unity. This phenomenon is known as superradiance.

If  $A_\mathfrak{s} > 0$ , i.e. for the non-superradiant frequencies or for the half-integer spins, then  $\mathcal{R}_\mathfrak{s} < 1$ . In particular, this implies that if  $Y_{\text{in},\mathfrak{s}} = 0$ , then also  $Y_{\text{out},\mathfrak{s}} = 0$  so that a solution with no incoming radiation must be zero. Thus, in these cases mode stability holds for real frequencies, and the classical scattering analysis given here is sufficient to prove Theorem 4.2.1.

**Remark 4.3.3.** *The fact that the coefficient  $A_{\pm\frac{1}{2}}$  is positive also for superradiant frequencies is known and follows from the fact that the spin- $\frac{1}{2}$  field admits a future directed conserved current; see [Mason and Nicolas \(1998\)](#) concerning the spin-3/2 case.*

It remains to determine the square modulus of the constants  $C_\mathfrak{s}$  in (4.52). For this purpose we apply  $\Delta^{\mathfrak{s}/2} \mathcal{D}^{\dagger 2\mathfrak{s}} \Delta^{\mathfrak{s}/2}$  to (4.52a) and use (4.52b) to get

$$\Delta^{\mathfrak{s}/2} \mathcal{D}^{\dagger 2\mathfrak{s}} \Delta^{\mathfrak{s}/2} R_{-\mathfrak{s}} = |C_\mathfrak{s}|^2 R_{-\mathfrak{s}} \quad (4.56)$$

The left hand side can now be evaluated using the TME for  $R_{-\mathfrak{s}}$ .

One finds

$$|C_{\frac{1}{2}}|^2 = \Lambda + \frac{1}{2} \quad (4.57a)$$

$$|C_1|^2 = (\Lambda + 1)^2 + 4am\omega - 4a^2\omega^2 \quad (4.57b)$$

$$|C_{\frac{3}{2}}|^2 = \left(\Lambda + \frac{3}{2}\right)^3 + \left(\Lambda + \frac{3}{2}\right)^2 + 16\left(\Lambda + \frac{3}{2}\right)(a^2\omega^2 - am\omega) - 16a^2\omega^2 \quad (4.57c)$$

$$|C_2|^2 = (\Lambda + 2)^4 + 4(\Lambda + 2)^3 + 4(\Lambda + 2)^2(1 + 10am\omega - 10a^2\omega^2) + 48(\Lambda + 2)(am\omega + a^2\omega^2) + 144\omega^2(a^2m^2 + M^2 - 2a^3m\omega + a^4\omega^2) \quad (4.57d)$$

For  $\mathfrak{s} = 1$  and  $\mathfrak{s} = 2$  this agrees with the expressions obtained in (Teukolsky and Press, 1974) and (Chandrasekhar, 1998) after taking into account the different conventions used there.

## 4.4. Integral transformations

As in (Shlapentokh-Rothman, 2015), it is convenient to transform the radial Teukolsky equation to its canonical form before introducing the integral transform that shall be used. The rescaling

$$R(r) = (r - r_-)^{\eta-s/2} (r - r_+)^{\xi-s/2} e^{-i\omega r} g(r) \quad (4.58)$$

puts the radial Teukolsky equation in canonical form. Letting

$$\gamma = 2\eta - s + 1 \quad (4.59a)$$

$$\delta = 2\xi - s + 1 \quad (4.59b)$$

$$p = -2i\omega \quad (4.59c)$$

$$\alpha = 1 - 2s \quad (4.59d)$$

$$\sigma = -\Lambda - 2ir_-(1 - 2s)\omega - s \quad (4.59e)$$

we have that  $\mathbf{R}R = 0$  is equivalent to

$$T_r g(r) = 0 \quad (4.60)$$

where

$$\begin{aligned} T_r h(r) = & (r - r_-)(r - r_+) \frac{d^2 h}{dr^2} \\ & + (\gamma(r - r_+) + \delta(r - r_-) + p(r - r_-)(r - r_+)) \frac{dh}{dr} + (\alpha p(r - r_-) + \sigma) h \end{aligned} \quad (4.61)$$

is a Heun operator in canonical form, with parameters  $\gamma, \delta, p, \alpha, \sigma$ .

Let  $\tilde{T}$  be a new Heun operator with different parameters given by

$$\tilde{\gamma} := \alpha = 1 - 2s \quad (4.62a)$$

$$\tilde{\delta} := \gamma + \delta - \alpha = 1 - 4iM\omega \quad (4.62b)$$

$$\tilde{p} := p \quad (4.62c)$$

$$\tilde{\alpha} := \gamma = 1 - s + 2\eta \quad (4.62d)$$

$$\tilde{\sigma} := \sigma \quad (4.62e)$$



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and let  $f(x, r)$  be defined as

$$f(x, r) = e^{-p \frac{(x-r_-)(r-r_-)}{r_+ - r_-}}. \quad (4.63)$$

With the above choice of parameters for  $\tilde{T}$  we have that

$$(\tilde{T}_x - T_r)f(x, r) = 0. \quad (4.64)$$

As we shall see, this means that we can use  $f(x, r)$  as the kernel for an integral transformation between solutions of these two Heun equations. Let a contour  $\mathcal{C}$  in the complex  $r$ -plane be given and let  $g(r)$  be a solution to (4.61) with parameters as in (4.59). Defining  $\tilde{g}(x)$ , following Whiting (1989), by

$$\tilde{g}(x) = \int_{\mathcal{C}} f(x, r)(r - r_-)^{\gamma-1}(r - r_+)^{\delta-1} e^{pr} g(r) dr, \quad (4.65)$$

we have that

$$\begin{aligned} \tilde{T}_x \tilde{g}(x) &= \int_{\mathcal{C}} \tilde{T}_x f(x, r)(r - r_-)^{\gamma-1}(r - r_+)^{\delta-1} e^{pr} g(r) dr \\ &= \int_{\mathcal{C}} T_r f(x, r)(r - r_-)^{\gamma-1}(r - r_+)^{\delta-1} e^{pr} g(r) dr \\ &= (r - r_-)^{\gamma}(r - r_+)^{\delta} e^{pr} \left( \frac{df(x, r)}{dr} g(r) - f(x, r) \frac{dg(r)}{dr} \right) \Big|_{\partial \mathcal{C}} \\ &\quad + \int_{\mathcal{C}} f(x, r)(r - r_-)^{\gamma-1}(r - r_+)^{\delta-1} e^{pr} T_r g(r) dr. \end{aligned} \quad (4.66)$$

The last step is an integration by parts. Note that the expression in the last line vanishes identically, because  $g(r)$  satisfies (4.61). Hence, provided the integral in (4.65) converges and the boundary condition

$$(r - r_-)^{\gamma}(r - r_+)^{\delta} e^{pr} \left( \frac{df(x, r)}{dr} g(r) - f(x, r) \frac{dg(r)}{dr} \right) \Big|_{\partial \mathcal{C}} = 0 \quad (4.67)$$

is satisfied, we see that  $\tilde{g}$  satisfies the transformed equation

$$\tilde{T}_x \tilde{g}(x) = 0 \quad (4.68)$$

Using the parameters (4.59) in equation (4.65) and using the relation (4.58) we can write  $\tilde{g}$  in the form

$$\tilde{g}(x) = \int_{\mathcal{C}} e^{2i\omega \frac{(x-r_-)(r-r_-)}{r_+ - r_-}} (r - r_-)^{2\eta-s} (r - r_+)^{2\xi-s} e^{-2i\omega r} g(r) dr \quad (4.69a)$$

$$= \int_{\mathcal{C}} e^{2i\omega \frac{(x-r_-)(r-r_-)}{r_+ - r_-}} (r - r_-)^{\eta-s/2} (r - r_+)^{\xi-s/2} e^{-i\omega r} R(r) dr \quad (4.69b)$$

**Remark 4.4.1.** Assuming no incoming radiation for  $R$ , we have

$$(r - r_-)^{\eta-s/2} (r - r_+)^{\xi-s/2} e^{-i\omega r} R(r) \sim \begin{cases} (r - r_+)^{2\xi-s} & \text{for } r \rightarrow r_+ \\ r^{-2s-1} & \text{for } r \rightarrow \infty \end{cases} \quad (4.70)$$

### 4.4.1. Transforming to self-adjoint and Schrödinger form

We now transform (4.68) to self-adjoint form, by the s-homotopic transformation

$$u(x) = (x - r_-)^{-s}(x - r_+)^{-2iM\omega} e^{-i\omega x} \tilde{g}(x) \quad (4.71)$$

Then  $u$  satisfies the equation

$$\partial_x \Delta \partial_x u(x) + \tilde{V}_0(x) u(x) = 0 \quad (4.72)$$

where

$$\tilde{V}_0(x) = -\Lambda + \Delta \omega^2 \left( \frac{x + r_-}{x - r_+} \right)^2 - 4i\omega(x - r_-)\eta - \frac{x - r_+}{x - r_-} s^2 \quad (4.73)$$

Let  $x_*$  be the tortoise coordinate corresponding to  $x$ ,

$$dx_* = \frac{x^2 + a^2}{\Delta} dx$$

Then, writing  $' = d/dx_*$ , and defining

$$U = (x^2 + a^2)^{1/2} u, \quad (4.74)$$

we have the Schrödinger form of the transformed equation,

$$U'' + \tilde{V}U = 0, \quad (4.75)$$

where now

$$\tilde{V} = \frac{\Delta}{(x^2 + a^2)^2} \tilde{V}_0 - V_{\text{resc}} \quad (4.76)$$

with  $V_{\text{resc}}$  as in (4.46).

**Remark 4.4.2.** For  $\omega \in \mathbb{R}$ , the potential  $\tilde{V}$  given by (4.76) is real. Hence if  $U$  is a solution to (4.75) the Wronskian  $W[U, \bar{U}] = U'\bar{U} - \bar{U}'U$  is conserved,  $W' = 0$ , where  $\bar{U}$  denotes the complex conjugate of  $U$ .

We have

$$\tilde{V}|_{x=r_+} = \frac{(r_+ - r_-)^2}{r_+^2} \omega^2, \quad \lim_{x \rightarrow \infty} \tilde{V}(x) = \omega^2$$

**Remark 4.4.3.** In the paper (Shlapentokh-Rothman, 2015) where the problem of mode stability on the real axis is considered for the case  $s = 0$ , the integral transform (4.69) is applied with the contour  $\mathcal{C}$  consisting of the real half-line starting at  $r_+$ ,  $\mathcal{C} = [r_+, \infty)$ .

#### 4. Mode Stability

### 4.5. Limits

Recall (Montgomery and Vaughan, 2006, Theorem C.2) that for  $\Re\zeta > 0$ ,

$$\int_0^\infty e^{-x} x^{\zeta-1} dx = \Gamma(\zeta), \quad (\text{Euler's integral}). \quad (4.77)$$

where  $\Gamma(\zeta)$  is the Gamma function. The Gamma function extends to a meromorphic function on the complex plane with simple poles at the non-positive integers. Let  $\rho > 0$  and let  $\mathcal{H}$  be the contour in the complex  $z$ -plane which consists of the half line from  $-\infty - i\rho$  to  $-i\rho$ , the semi-circle of radius  $\rho$  connecting  $-i\rho$  with  $i\rho$  and the half line from  $i\rho$  to  $i\rho - \infty$ , see figure 4.2. Then for  $\zeta \in \mathbb{C}$  we have

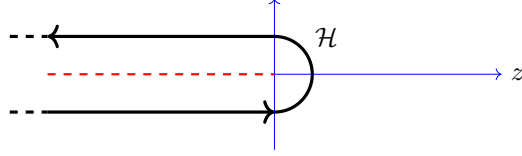


Figure 4.2.: Hankel's contour

(Montgomery and Vaughan, 2006, Theorem C.3),

$$\frac{1}{2\pi i} \int_{\mathcal{H}} e^z z^{-\zeta} dz = \frac{1}{\Gamma(\zeta)}, \quad (\text{Hankel's integral}). \quad (4.78)$$

Among the many relations known for the Gamma function, we also recall the product formula (Montgomery and Vaughan, 2006, Eq. (C.6))

$$\Gamma(\zeta)\Gamma(1-\zeta) = \frac{\pi}{\sin(\pi\zeta)}$$

We now consider the integral transform (4.69) for a contour  $\mathcal{C}$ . We restrict to the case

$$\omega > 0.$$

and to contours  $\mathcal{C}$  such that the integral (4.69) converges, and the boundary condition (4.67) is satisfied.

Let

$$\nu = 2\omega(x - r_-)/(r_+ - r_-),$$

and define

$$h(r) = (r - r_-)^{\eta-s/2} (r - r_+)^{-\xi+s/2} e^{-i\omega r} R(r) \quad (4.79)$$

We note that in view of (4.70), we have

$$h(r) \sim \begin{cases} 1 & \text{for } r \rightarrow r_+ \\ r^{-2\xi-s-1} & \text{for } r \rightarrow \infty \end{cases} \quad (4.80)$$

We have the following corollary to Lemma 4.3.1.

**Corollary 4.5.1.** *Let  $h$  be given by (4.79). Then,  $h$  is analytic on the complex plane except at the singular points  $r_-$ ,  $r_+$  of the radial Teukolsky equation, where  $h$  may have branch points. Further, it holds that*

1. *In the non-resonant case,  $\xi \neq 0$  or  $s$  not a positive integer,  $h(r)$  is analytic at  $r_+$ .*
2. *Let  $S$  be the circular sector defined in Lemma 4.3.1. Then*

$$h(r) = O(r^{-s-1})$$

*holds in  $S$ .*

**Remark 4.5.1.** *By Lemma 4.3.1,  $h$  is analytic for  $|r - r_+| < |r_+ - r_-|$  if  $s \leq 0$ .*

A calculation shows

$$\lim_{r \rightarrow r_+} |h(r)|^2 = |r_+ - r_-|^{-2s} \lim_{r \rightarrow r_+} \Delta^s |R(r)|^2 \quad (4.81)$$

For a given  $\mathcal{C}$ , we define, after choosing a suitable branch of  $h$  if necessary,

$$I(\nu, \alpha) = \int_{\mathcal{C}} e^{i\nu(r-r_+)} (r - r_+)^{\alpha} h(r) dr \quad (4.82)$$

Then

$$\tilde{g}(x) = e^{i\nu(x)(r_+ - r_-)} I(\nu(x), 2\xi - s)$$

In the rest of this section, we shall calculate the limit  $\lim_{\nu \rightarrow \infty} \nu^{\alpha+1} I(\nu, \alpha)$ . This argument is closely related to the proof of Watson's Lemma, cf. (Wang and Guo, 1989, §1.9), which can be used to derive an expansion at  $\nu = \infty$  of this expression. See also Shlapentokh-Rothman (2015) for  $\mathfrak{s} = 0$ .

## 4. Mode Stability

### 4.5.1. The case $r = \infty$ , $s > 0$

Let  $\rho_0 > 0$  be small. We choose  $\mathcal{C}$  to be the rotated Hankel type contour in the complex  $r$ -plane which consists of the half line from  $i\infty + r_+ - \rho_0$  to  $r_+ - \rho_0$ , the semicircle of radius  $\rho_0$  connecting  $r_+ - \rho_0$  with  $r_+ + \rho_0$ , and the half line from  $r_+ + \rho_0$  to  $i\infty + r_+ + \rho_0$ , see figure 4.3. Using this contour in the definition of  $\tilde{g}(x)$ , we find that due to the exponential decay of the kernel  $e^{i\nu(r-r_+)}$  for  $\Im r \nearrow \infty$ , the boundary condition (4.67) is satisfied with this choice of  $\mathcal{C}$  and hence  $\tilde{g}(x)$  is a solution to the transformed equation (4.68).

We calculate

$$I(\nu, \alpha) = \int_{\mathcal{C}} e^{i\nu(r-r_+)} (r - r_+)^{\alpha} h(r) dr$$

substitute  $\nu(r - r_+) = -it$

$$= (-i)^{\alpha+1} \nu^{-\alpha-1} \int_{\mathcal{C}_t} e^{t^\alpha} h(r_+ - \nu^{-1}it) dt \quad (4.83)$$

where  $\mathcal{C}_t$  coincides with the Hankel contour with  $\rho = \nu\rho_0$ .

A limiting argument together with Hankel's integral formula (4.78) now yields the following result.

**Lemma 4.5.1.** *Let  $\mathcal{C}$  be the contour as in figure 4.3, and let  $h$  satisfy the conclusions of Corollary 4.5.1. Then it holds that*

$$\lim_{\nu \rightarrow \infty} \nu^{\alpha+1} I(\nu, \alpha) = (-i)^{\alpha+1} \frac{2\pi i}{\Gamma(-\alpha)} h(r_+)$$

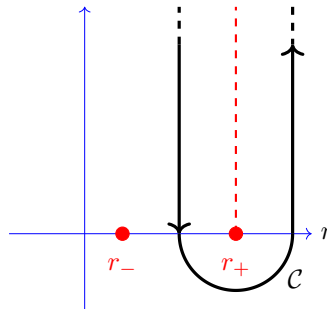


Figure 4.3.: The contour used in Lemma 4.5.1

*Proof.* If  $r_+$  is a branch point for  $h$ , we choose a branch of  $h$  by cutting the complex plan along the half line in the imaginary direction starting at  $r_+$ , see figure 4.3. In view of its definition, the integral  $I(\nu, \alpha)$  is independent of  $\rho_0$ . Hence we can set  $\rho_0 = \nu^{-1}\rho$  so that  $\mathcal{C}_t$  coincides with the Hankel contour. Starting from (4.83) we have,

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \nu^{\alpha+1} I(\nu, \alpha) &= \lim_{\nu \rightarrow \infty} (-i)^{\alpha+1} \int_{\mathcal{C}_t} e^{t^\alpha} h(r_+ - \nu^{-1}it) dt \\ &= (-i)^{\alpha+1} h(r_+) \int_{\mathcal{H}} e^{t^\alpha} dt \\ &= (-i)^{\alpha+1} \frac{2\pi i}{\Gamma(-\alpha)} h(r_+) \end{aligned}$$

where in the last step we used (4.78).  $\square$

**Corollary 4.5.2.** *Assume that  $R$  is a solution of the radial Teukolsky equation with  $\omega > 0$ , and with no incoming radiation. Let  $U$  be defined via (4.74) and the integral transform (4.69) with the contour  $\mathcal{C}$  as in figure 4.3. Then  $U$  solves (4.75) and satisfies*

$$\lim_{x \rightarrow \infty} |U(x)| = C \lim_{r \rightarrow r_+} \Delta^{s/2} |R(r)| \quad (4.84)$$

where

$$C = \left( \frac{r_+ - r_-}{2\omega} \right)^{-s+1} \frac{2\pi}{|\Gamma(s - 2\xi)|} |r_+ - r_-|^{-s} \quad (4.85)$$

and the limit on the left hand side of equation (4.84) is taken along the real axis.

*Proof.* Due to the exponential decay of the kernel as  $\Im r \rightarrow \infty$ , the integral (4.69) converges, and the boundary condition (4.67) is satisfied. Therefore  $\tilde{g}$  solves (4.68) and  $U$  solves (4.75). In order to evaluate the limit of  $U$  at  $x = \infty$ , we use Lemma 4.5.1. We have

$$\begin{aligned} \lim_{x \rightarrow \infty} |U(x)| &= \lim_{x \rightarrow \infty} |x^{-s+1} \tilde{g}(x)| \\ &= \left( \frac{r_+ - r_-}{2\omega} \right)^{-s+1} \lim_{\nu \rightarrow \infty} \nu^{-s+1} |I(\nu, 2\xi - s)| \\ &= \left( \frac{r_+ - r_-}{2\omega} \right)^{-s+1} \frac{2\pi}{|\Gamma(s - 2\xi)|} |r_+ - r_-|^{-s} \lim_{r \rightarrow r_+} \Delta^{s/2} |R(r)| \end{aligned}$$

$\square$

**Remark 4.5.2.** *With the above choice of contour, Corollary 4.5.2 is valid for arbitrary  $s$ . For  $s > 0$ ,  $C$  defined by (4.85) is bounded and non-zero. However, if  $\xi = 0$ , then with  $s$  a non-positive integer, the constant  $C$  will vanish. Therefore, we shall in the next subsection consider a different contour which is more suitable for the case  $s \leq 0$ .*

#### 4. Mode Stability

##### 4.5.2. The case $s \leq 0$

We consider the contour  $\mathcal{C}$  consisting of the half line connecting  $r_+$  with  $i\infty$ , i.e.

$$\mathcal{C} = \{r \in [r_+, i\infty)\}, \quad (4.86)$$

see figure 4.4. Due to the exponential decay of the kernel  $e^{i\nu(r-r_+)}$  as  $\Im r \rightarrow \infty$ ,

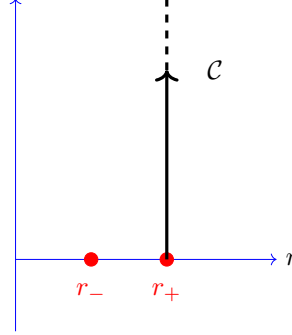


Figure 4.4.: A contour of Euler type

the boundary condition at  $i\infty$  is automatically satisfied. Further, due to  $g(r) \sim 1$  at  $r = r_+$ , the boundary condition at  $r = r_+$  is satisfied. Thus, with this choice of contour we have that  $\tilde{g}(z)$  is a solution to the transformed equation (4.68).

Starting with  $I(\nu, \alpha)$  defined by (4.82), we calculate

$$I(\nu, \alpha) = \int_{\mathcal{C}} e^{i\nu(r-r_+)} (r - r_+)^{\alpha} h(r) dr$$

substitute  $\nu(r - r_+) = it$

$$= i^{\alpha+1} \nu^{-\alpha-1} \int_0^{\infty} e^{-t} t^{\alpha} h(r_+ + \nu^{-1}it) dt$$

A limiting argument together with Euler's integral formula (4.77) now yields the following result which is the analog of Lemma 4.5.1.

**Lemma 4.5.2.** *Assume that  $h$  satisfies the conclusions of Corollary 4.5.1. Then it holds that*

$$\lim_{\nu \rightarrow \infty} \nu^{\alpha+1} I(\nu, \alpha) = i^{\alpha+1} \Gamma(\alpha + 1) h(r_+)$$

Using the definitions we obtain the following analog of Corollary 4.5.2

#### 4.6. Mode stability on the real axis

**Corollary 4.5.3.** *Assume that  $R$  is a solution of the radial Teukolsky equation with  $\omega > 0$  and with no incoming radiation. Assume  $s \leq 0$  and let  $U$  be given by (4.74), and defined via the integral transform (4.69) with the contour  $C$  given by (4.86). Then  $U$  solves (4.75), and we have*

$$\lim_{x \rightarrow \infty} |U(x)| = C \lim_{x \rightarrow r_+} \Delta^{s/2} |R(r)| \quad (4.87)$$

where

$$C = \left( \frac{r_+ - r_-}{2\omega} \right)^{-s+1} |\Gamma(2\xi - s + 1)| |r_+ - r_-|^{-s} \quad (4.88)$$

*Proof.* The fact that  $U$  solves (4.75) follows as in Corollary 4.5.2 due to the exponential decay of the kernel as  $\Im r \rightarrow \infty$ . Lemma 4.5.2 yields

$$\begin{aligned} \lim_{x \rightarrow \infty} |U(x)| &= \lim_{x \rightarrow \infty} |x^{-s+1} \tilde{g}(x)| \\ &= \left( \frac{r_+ - r_-}{2\omega} \right)^{-s+1} \lim_{\nu \rightarrow \infty} \nu^{-s+1} |I(\nu, 2\xi - s)| \\ &= \left( \frac{r_+ - r_-}{2\omega} \right)^{-s+1} |\Gamma(2\xi - s + 1)| |r_+ - r_-|^{-s} \lim_{r \rightarrow r_+} (\Delta^{s/2} |R(r)|) \end{aligned}$$

□

**Remark 4.5.3.** *For  $s \leq 0$ ,  $C$  defined by (4.88) is bounded and non-zero.*

## 4.6. Mode stability on the real axis

We are now ready to prove our main result.

**Theorem 4.6.1.** *Assume that  $\omega > 0$  and that  $R$  is a solution of the radial Teukolsky equation with no incoming radiation. Then  $R = 0$ .*

*Proof.* We first consider the case  $s \leq 0$ . Let  $U(x)$  be given by (4.74) and constructed via the integral transform (4.69) as explained above. By corollary 4.5.3,  $U(x)$  solves (4.75). By Remark 4.4.2 the Wronskian  $W[U, \bar{U}] = U'\bar{U} - \bar{U}'U$  is conserved,  $W' = 0$ . From the definition of  $U$ , we can write it as

$$U(x) = Z(x)I(\nu(x), 2\xi - s).$$

where

$$Z(x) = (x^2 + a^2)^{1/2} (x - r_-)^{-s} (x - r_+)^{-2iM\omega} e^{-i\omega x} e^{i\nu(x)(r_+ - r_-)} \quad (4.89)$$



#### 4. Mode Stability

We have

$$W[U, \bar{U}] = W[Z, \bar{Z}]I\bar{I} + W[I, \bar{I}]Z\bar{Z} \quad (4.90)$$

From (4.82) we have by differentiating under the integral sign,

$$\frac{d}{dx}I(\nu, \alpha) = i\frac{d\nu}{dx}I(\nu, \alpha + 1) \quad (4.91)$$

where

$$\frac{d\nu}{dx} = \frac{2\omega}{r_+ - r_-} \quad (4.92)$$

Write  $Z' = dZ/dx_*$  as in section 4.4. For  $x \rightarrow r_+$ , we have

$$\begin{aligned} Z' &= \frac{\Delta}{x^2 + a^2} \left( \frac{-2iM\omega}{x - r_+} \right) Z(x) + O(x - r_+) \\ &= \frac{r_+ - r_-}{r_+^2 + a^2} (-2iM\omega) Z(x) + O(x - r_+) \\ &= -4iM\kappa_+\omega Z(x) + O(x - r_+) \end{aligned}$$

This gives

$$W[Z, \bar{Z}](r_+) = -16iM^2\kappa_+\omega r_+(r_+ - r_-)^{-2s}$$

In view of the discussion above  $I(\nu(x), \alpha)$  and  $dI(\nu(x), \alpha)/dx$  have well defined limits at  $x = r_+$ , and hence

$$\lim_{x \rightarrow r_+} \frac{dI(\nu(x), \alpha)}{dx_*} = 0$$

This gives

$$W[U, \bar{U}](r_+) = W[Z, \bar{Z}](r_+)I(\nu(r_+), 2\xi - s)\bar{I}(\nu(r_+), 2\xi - s)$$

In particular,

$$iW[U, \bar{U}](r_+) > 0 \quad (4.93)$$

for  $|a| < M$ . We now consider the limit  $x \rightarrow \infty$ . Equations (4.89) and (4.92) give for large  $x$ ,

$$Z' = (i\omega + O(1/x))Z(x) \quad (4.94)$$

which yields

$$Z'(x)\bar{Z}(x) = (i\omega + O(1/x))Z(x)\bar{Z}(x)$$

From (4.91) and corollary 4.5.3, we have that

$$\frac{dI(\nu(x), \alpha)}{dx_*} = O(1/x)I(\nu(x), \alpha)$$

for large  $x$ . This means that

$$\begin{aligned} \lim_{x \rightarrow \infty} W[U, \bar{U}](x) &= \lim_{x \rightarrow \infty} W[Z, \bar{Z}](x) |I(\nu(x), 2\xi - s)|^2 \\ &= 2i\omega \lim_{x \rightarrow \infty} |Z(x)|^2 |I(\nu(x), 2\xi - s)|^2 \\ &= 2i\omega \lim_{x \rightarrow \infty} |U(x)|^2 \end{aligned}$$

Write  $|U(\infty)|^2 = \lim_{x \rightarrow \infty} |U(x)|^2$  and  $|U(r_+)|^2 = \lim_{x \rightarrow r_+} |U(x)|^2$ . The conservation property of the Wronskian gives for  $\omega > 0$  and  $|a| < M$

$$\begin{aligned} 0 &= i(W(\infty) - W(r_+)) \\ &= -2\omega |U(\infty)|^2 - iW(r_+) \end{aligned}$$

In view of (4.93) this can hold only if  $|U(\infty)|^2 = 0$ . Taking into account the definition of  $U(x)$ , and Corollaries 4.5.2 and 4.5.3, this implies that  $R_{+,1}(r_+) = 0$ . Hence in view of point 1 of Remark 4.3.1, we have that  $R(r) \equiv 0$ .

For the case  $s = \mathfrak{s} > 0$  we shall present two alternative approaches. Let  $R_{\mathfrak{s}}$  denote a solution to the radial Teukolsky equation with no incoming radiation. It is straightforward to check that the TSI relation (4.52b) yields a solution  $R_{-\mathfrak{s}}$  with no incoming radiation. In order to demonstrate that  $R_{\mathfrak{s}} = 0$  it suffices to show that the solution  $R_{-\mathfrak{s}}$  defined by

$$R_{-\mathfrak{s}} = \Delta^{\mathfrak{s}/2} \mathcal{D}^{\dagger 2\mathfrak{s}} \Delta^{\mathfrak{s}/2} R_{\mathfrak{s}} \quad (4.95)$$

is non-vanishing. This follows due to the asymptotic form of  $R_{\mathfrak{s}}$  given by (4.44), and the fact that  $\mathcal{D}^{\dagger}$  is to leading order

$$\mathcal{D}^{\dagger} = \frac{d}{dr} + i\omega + O(1/r)$$

Arguing as in the first part of the proof, we find that  $R_{-\mathfrak{s}}$  must vanish, and hence also  $R_{\mathfrak{s}}$ . The inference we want to draw from equation (4.95) is that if  $R_{-\mathfrak{s}} = 0$  then  $R_{\mathfrak{s}} = 0$  must also hold. The argument for this fails at algebraically special modes (Chandrasekhar, 1984). However, this fact is not an obstacle to our inference since algebraically special modes do not have no incoming radiation in the sense of definition 4.3.1 and occur in case  $C_{\mathfrak{s}}$  vanishes, which does not happen for real frequencies.

A second, alternate argument for the case  $s > 0$  can be given as follows. For the non-resonant case, with  $s > 0$  we can argue along exactly the same lines as in the first part of the proof, but with corollary 4.5.2 playing the role of corollary

## 4. Mode Stability

**4.5.3.** Finally, for the resonant case, with  $\xi = 0$  and  $s$  a positive integer, we can apply the scattering relation (4.53). In the resonant case, we have  $k_+ = 0$  which yields that the transmission coefficient vanishes,  $\mathcal{T}_s = 0$ . Assuming no incoming radiation, this yields

$$Y_{\text{in},s} = 0$$

We now show that for spins  $s = 1, 2$ , the TSI constant  $C_s$  given in equation (4.57) is non-vanishing in the case of resonant frequencies,  $k_+ = 0$ , where  $k_+$  is given by (4.25). If  $k_+ = 0$ , then  $\omega = am/(2Mr_+)$ . For  $s = 1$ , this gives

$$|C_1|^2 = (\Lambda + 1)^2 + 4am\omega - 4a^2\omega^2 = (\Lambda + 1)^2 + (8Mr_+ - 4a^2)\omega^2 > 0.$$

For  $s = 2$ ,  $\Im C_2 = 12M\omega$ , cf. Teukolsky and Press (1974, Eq. (3.25)), see also Chandrasekhar (1998, p. 462-463). This shows that  $|C_2|^2 > 0$ . Hence,  $R_s = 0$ , which completes the proof of Theorem 4.6.1.  $\square$

**Remark 4.6.1.** *The radial Teukolsky equation (4.21a) has conserved Wronskian*

$$W_{\mathbf{R}}[R_1, R_2] = \Delta \partial_r R_1 R_2 - \Delta R_1 \partial_r R_2,$$

*i.e.  $\partial_r W_{\mathbf{R}}[R_1, R_2] = 0$  if  $R_1, R_2$  solve (4.21a). Let  $R_{hor}$  and  $R_{out}$  be solutions of the radial Teukolsky equation which are ingoing at the horizon and outgoing at infinity, respectively. Theorem 4.6.1 implies that  $W_{\mathbf{R}}[R_{hor}, R_{out}]$  is non-vanishing.*

*Consider an inhomogenous version of the radially Teukolsky equation,*

$$\mathbf{R}R = F \tag{4.96}$$

*In view of the above, we can use the method of variation of parameter to find a particular solution to (4.96),*

$$R(r) = \frac{1}{W_{\mathbf{R}}[R_{out}, R_{hor}]} \left( R_{hor}(r) \int_r^\infty R_{out}(t) F(t) dt + R_{out}(r) \int_{r_+}^r R_{hor}(t) F(t) dt \right)$$

*Due to the regular dependence of  $W_{\mathbf{R}}$  on  $\omega$  this can in principle be used to estimate the solution of the inhomogenous Teukolsky equation. This fact is related to the so-called quantitative mode stability, cf. Shlapentokh-Rothman (2015).*

## 4.7. Charged scalar field in Reissner Nordström

In this section we will discuss the charged massive and massless scalar field on a Reissner-Nordström geometry. There has been a good amount of work devoted

#### 4.7. Charged scalar field in Reissner Nordström

to this problem, see for example [Hod and Piran \(1997, 1998a,b\)](#), [Hartman et al. \(2010\)](#), [Hod \(2005, 2013, 2010\)](#). One can hope for it to serve as a toy model to understand superradiance. It has the great advantage that we are back to spherical symmetry which should simplify the analysis. However as we will discuss in this section the slow fall-off of the vector potential provides an obstacle for standard techniques to work.

Within this section we assume both, the background metric and the background electro-magnetic fields to be fixed. First we introduce the modified electro-magnetic connection

$$D^\mu = \nabla^\mu + iqA^\mu \quad (4.97)$$

where  $q$  denotes the charge coupling constant. The action for the charged scalar field is given by

$$S_q(\varphi) = \int \underbrace{(g^{\mu\nu} \overline{D_\mu \varphi} D_\nu \varphi + m^2 \varphi \overline{\varphi})}_{=\mathcal{L}} \sqrt{-g} d^n x, \quad (4.98)$$

where  $\varphi$  is a complex scalar field and  $g$  denotes the determinant of the metric. Let  $\chi$  be a scalar function on the manifold. Then the action is invariant under a simultaneous replacement

$$\tilde{\varphi} \rightarrow e^{iq\chi} \varphi, \quad (4.99)$$

$$\tilde{A}_\mu \rightarrow A_\mu + \nabla_\mu \chi. \quad (4.100)$$

The charged scalar field equation, which is the Euler-Lagrange equation for (4.98), is given by

$$[D^\mu D_\mu - m^2] \varphi = 0. \quad (4.101)$$

The symmetric stress energy tensor is then given by

$${}^S T_{\mu\nu}(\varphi) = 2 \frac{1}{\sqrt{-g}} \frac{\delta S_q(\varphi)}{\delta g^{\mu\nu}} = \overline{D_\mu \varphi} D_\nu \varphi + \overline{D_\nu \varphi} D_\mu \varphi - g_{\mu\nu} (\overline{D^\rho \varphi} D_\rho \varphi + m^2 \varphi \overline{\varphi}). \quad (4.102)$$

The Noether or canonical stress energy is given by

$${}^C T^\mu{}_\nu(\varphi) = \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \varphi)} \nabla_\nu \varphi + \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \overline{\varphi})} \nabla_\nu \overline{\varphi} - \delta^\mu_\nu \mathcal{L} \quad (4.103)$$

Further we can define the charge current

$$J^\mu = \frac{\delta S_q(\varphi)}{\delta A_\mu} = iq(\overline{\varphi} D^\mu \varphi - \varphi \overline{D^\mu \varphi}) \quad (4.104)$$

Which is divergence free if  $\varphi$  satisfies (4.101)

$$\nabla_\mu J^\mu = 0 \quad (4.105)$$

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on the other hand the symmetric stress energy is not divergence free in this case. In fact we get

$$\nabla^\mu T_{\mu\nu} = F_{\mu\nu} J^\mu \quad (4.106)$$

where  $F_{\mu\nu}$  is the electro magnetic field strength tensor

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu. \quad (4.107)$$

Hence the relation (4.106) tells us that the change in energy is given by the interaction of the fields charge with the background electro-magnetic field. The electro-magnetic field strength tensor can also be interpreted as an additional curvature term of the modified electro-magnetic connection as compared to the standard connection

$$[D_\mu, D_\nu]\varphi = F_{\mu\nu}\varphi. \quad (4.108)$$

A straight forward calculation then gives us the following relationship between the canonical and the symmetric stress energy.

$${}^C T_{\mu\nu}(\varphi) = {}^S T_{\mu\nu}(\varphi) + J_\mu A_\nu = {}^S T_{\mu\nu}(\varphi) + A_\nu \frac{\delta S_q(\varphi)}{\delta A^\mu} \quad (4.109)$$

This suggests that the symmetric stress energy can be interpreted as the fields internal energy while the canonical stress energy is the sum of the internal energy plus the potential energy (charge times electro-magnetic potential) and therefore represents the total energy of the field.

Now let  $K^\nu$  be an arbitrary vector field on the manifold, then the associated momentum flux is given by <sup>1</sup>

$$P^K_\mu = K^\nu [{}^C T_{\mu\nu}(\varphi)]. \quad (4.110)$$

A straight forward calculation shows that the divergence of this flux is given by

$$\nabla^\mu P^K_\mu = Lie_K(g)_{\mu\nu} T^{\mu\nu} + Lie_K(A)_\mu J^\mu \quad (4.111)$$

or written in a more suggestive way

$$\nabla^\mu P^K_\mu = Lie_K(g)_{\mu\nu} 2 \frac{1}{\sqrt{-g}} \frac{\delta S_q(\varphi)}{\delta g_{\mu\nu}} + Lie_K(A)_\mu \frac{\delta S_q(\varphi)}{\delta A_\mu} \quad (4.112)$$

We see that if the vector field  $K$  is Killing and the vector potential is invariant under the flow of the Killing vector field then we get the conserved fluxes found in Tóth (2017), hence

$$\nabla^\mu P^K_\mu = 0. \quad (4.113)$$

---

<sup>1</sup>Note since  ${}^C T_{\mu\nu}$  is not symmetric it matters with which index we contract the vector field here.

#### 4.7. Charged scalar field in Reissner Nordström

Note that the above conditions is nothing but the statement that  $K$  is the generator of a symmetry of the action (4.98) and hence  $P^K$  is the conserved Noether flux associated with that symmetry. From the here presented considerations, especially the relation (4.109) and (4.112) hold in a more general situation.

**Conjecture 4.7.1.** *Let  $S(\xi, F^i)$  be an action for the field  $\xi$  (which can in principal be a tensor field) with  $F^i$  being an arbitrary number of background fields (which are not scalar) that couple to  $\xi$ . Then a relation of the form*

$${}^C T_{\mu\nu}(\xi) = {}^S T_{\mu\nu}(\xi) + \sum_i F_{\nu\alpha_1\dots\alpha_n}^i \frac{\delta S_q(\xi)}{\delta (F^i)_{\alpha_1\dots\alpha_n\mu}} \quad (4.114)$$

and for

$$P^K_{\mu} = K^{\nu} [{}^C T_{\mu\nu}(\xi)]. \quad (4.115)$$

the divergence is roughly of the form

$$\nabla^{\mu} P^K_{\mu} = Lie_K(g)_{\mu\nu} 2 \frac{1}{\sqrt{-g}} \frac{\delta S_q(\varphi)}{\delta g_{\mu\nu}} + \sum_i Lie_K(F^i)_{\mu\alpha_1\dots\alpha_n} \frac{\delta S_q(\varphi)}{\delta F^i_{\mu\alpha_1\dots\alpha_n}} \quad (4.116)$$

Note that there is a substantial chance that the conjecture above has already been proven somewhere in the literature however we were unable to find this statement anywhere. Further more it is expected but remains to be shown that the [Belinfante \(1940\)](#)-[Rosenfeld \(1940\)](#) symmetrisation procedure can be expressed in the above form as well.

Now lets go to the specific setting of a charged scalar field in Reissner Nordström where the metric is given by (2.33) with  $a = \Lambda = l = 0$  and the the electro magnetic potential can be chosen as  $A_{\nu} = -\delta_{\nu}^0 Q/r$ . If we choose  $K^{\mu} = (\partial_t)^{\mu}$  we get a conserved flux  $P^K$ . If we integrate the momentum flux across the three dimensional space like hypersurface  $\Sigma_{\tau} := \{p \in \mathcal{M} | t(p) = \tau\}$ , which is a Cauchy surface for the exterior region, we get

$$\begin{aligned} \int_{\Sigma_{\tau}} P^K_{\mu} n^{\mu} d\sigma &= \int_{r_+}^{\infty} \int_{\Omega} \left( \frac{r^2}{\Delta} |\partial_t \varphi|^2 + \left( m^2 - \frac{q^2 Q^2}{\Delta} \right) |\varphi|^2 + \frac{\Delta}{r^2} |\partial_r \varphi|^2 \right. \\ &\quad \left. + \frac{1}{r^2} \left( |\partial_{\theta} \varphi|^2 + \frac{1}{\sin^2 \theta} |\partial_{\phi} \varphi|^2 \right) \right) r^2 \sin^2 \theta dr d\theta d\phi \end{aligned} \quad (4.117)$$

For  $m = 0$  we would need to prove the following Hardy type estimate

$$\int_{r_+}^{\infty} (r - r_+)(r - r_-) |\partial_r \varphi|^2 - \frac{r^2 q^2 Q^2}{(r - r_+)(r - r_-)} |\varphi|^2 dr > 0 \quad (4.118)$$

#### 4. Mode Stability

which is roughly equivalent to proving

$$\int_0^\infty r |\partial_r \varphi|^2 - \frac{C}{r} |\varphi|^2 dr \quad (4.119)$$

to be positive. However this is the critical case for the standard Hardy estimate and does not hold. However the degeneracy of the energy at  $r_+$  originates from the failure of the coordinate system, so there is a chance that a similar result as we will discuss below for Minkowsky can be obtained when working in coordinates that are regular at the horizon. The only case one can prove positivity of the energy (4.117) known to us is if  $m^2 > q^2 Q^2 / r_+$ . If we apply a gauge transformation to the vector potential such that it vanishes at the horizon instead of infinity  $\tilde{A}_\nu = -\delta_\nu^0 (Q/r - Q/r_+)$  one can show that the coefficient of the  $|\varphi|^2$  term is always positive and thus is the energy. This has already been shown in [Bachelot \(2004\)](#). On the other hand we can define the dyadosphere as the region in the manifold where  $m^2 - \frac{q^2 Q^2}{\Delta} \leq 0$ . The boundary is then given by

$$r_D = M + \sqrt{M^2 + \left(\frac{q^2}{m^2} - 1\right) Q^2} > r_+. \quad (4.120)$$

Note that in the limit of the massless charged field  $m \rightarrow 0$  we have that  $r_D \approx M + \left|\frac{q}{m} Q\right|$  going to infinity. Therefore in contrast to the ergosphere in Kerr the dyadosphere is not compact in the massless limit. Time and again the slow fall-off of the electro magnetic potential towards infinity will prove as major obstacle to prove any of the known results in the case of mass less fields of Kerr. In return it allows for some results in the case of massive fields that do not hold true in Kerr. Interesting enough, when we take Minkowski space with a point charge at the origin as a background we get the energy to be

$$\begin{aligned} \int_{\Sigma_\tau} P^K{}_\mu n^\mu d\sigma = \int_0^\infty \int_\Omega \left( |\partial_t \varphi|^2 + \left(m^2 - \frac{q^2 Q^2}{r^2}\right) |\varphi|^2 + |\partial_r \varphi|^2 \right. \\ \left. + \frac{1}{r^2} \left( |\partial_\theta \varphi|^2 + \frac{1}{\sin^2 \theta} |\partial_\phi \varphi|^2 \right) \right) r^2 \sin^2 \theta dr d\theta d\phi. \end{aligned} \quad (4.121)$$

The relevant Hardy estimate for the massless case is now

$$\int_0^\infty r^2 |\partial_r \varphi|^2 - q^2 Q^2 |\varphi|^2 dr \geq 0 \quad (4.122)$$

which is true as long as  $q^2 Q^2 \leq 1/4$ . It would be interesting to check how this bound relates to known physical systems such as the hydrogen atom.

### 4.7.1. Mode analysis

We will now take a look at the mode solutions of the form

$$\varphi = e^{-i\omega t} Y_{lm}(\phi, \theta) R(r) \quad (4.123)$$

where  $Y_{lm}$  are the standard spherical harmonics with eigenvalue  $l(l+1)$ . The radial equation is then given by

$$\left( \frac{\partial}{\partial r} \Delta \frac{\partial}{\partial r} + \underbrace{\frac{r^4}{\Delta} \left( \omega + \frac{qQ}{r} \right)^2 - l(l+1) - r^2 m^2}_{=V_q(r)} \right) R(r) = 0 \quad (4.124)$$

Along the lines of the previous section it can be shown that for  $m = 0$  the super-radiant scattering occurs in the frequency range [Bekenstein \(1973b\)](#)

$$\omega \left( \omega + q \frac{Q}{r_+} \right) < 0 \quad (4.125)$$

This is similar with the Kerr case where  $\omega_H$  is the corresponding black hole property at the horizon and  $m$  gives the coupling of a mode to this black hole property. Here however the coupling  $q$  is not a parameter of the mode solution and can thus be chosen freely. [Hod \(2013, 2015\)](#) showed a no bomb theorem for this equation by proving that for any combination of  $m$ ,  $M$ ,  $q$  and  $Q$  the potential  $V_q(r)$  can have at max two zeros in the interval  $(r_+, \infty)$ . When we calculate the charge density for modes in  $\Sigma_\tau$  we get

$$\int_{\Sigma_\tau} J_\mu n^\mu d\sigma = \int_{r_+}^{\infty} \frac{2r^3 q Q}{\Delta} \left( \omega + \frac{qQ}{r} \right) |R(r)|^2 dr. \quad (4.126)$$

This integral is of course not well defined, but we see that the superradiant modes are those for which the charge density changes sign when going from the horizon to infinity. Hence a negative charge flux across the horizon requires the total positive charge flux at infinity to be outgoing. [Di Menza and Nicolas \(2015\)](#) showed that it is possible to find numerically a finite energy solution on a fixed background that does extract energy from the black hole. [Baake and Rinne \(2016\)](#) showed that one can numerically find solutions to the full coupled Einstein-Maxwell-charged-scalar-field system in spherical symmetry where superradiant energy extraction can be observed.

We will now try to repeat the procedure of section 4.4 for the mode stability for the massless charged scalar field. The rescaling

$$R(r) = (r - r_-)^\eta (r - r_+)^\xi e^{-i\omega r} g(r) \quad (4.127)$$



#### 4. Mode Stability

with  $\xi = \frac{ir_+(qQ - \omega r_+)}{r_+ - r_-}$  and  $\eta = \frac{ir_-(qQ - \omega r_-)}{r_+ - r_-}$  puts equation (4.124) in canonical form. Letting

$$\gamma = 2\eta + 1 \tag{4.128a}$$

$$\delta = 2\xi + 1 \tag{4.128b}$$

$$p = -2i\omega \tag{4.128c}$$

$$\alpha = 1 - 2iqQ + 4iM\omega \tag{4.128d}$$

$$\sigma = -l(l+1) - iqQ - 4qQ\omega r_- + 4\omega^2 r_-^2 + 2i\omega r_+ \tag{4.128e}$$

we have that  $\mathbf{R}R = 0$  is equivalent to

$$T_r g(r) = 0 \tag{4.129}$$

where  $T_r$  is again given by (4.61) with parameters  $\gamma, \delta, p, \alpha, \sigma$ . Let  $\tilde{T}$  be a new Heun operator with different parameters given by

$$\tilde{\gamma} := \alpha \tag{4.130a}$$

$$\tilde{\delta} := \gamma + \delta - \alpha \tag{4.130b}$$

$$\tilde{p} := p \tag{4.130c}$$

$$\tilde{\alpha} := \gamma \tag{4.130d}$$

$$\tilde{\sigma} := \sigma. \tag{4.130e}$$

Now even if the integral transform could be shown to work exactly as it does for Teukolsky in Kerr, it wouldn't help much. If we go from  $\tilde{T}_x \tilde{g}(x) = 0$  to the Schrödinger form, one finds that in this case we still do have superradiance for the interval where

$$\omega \left( \omega + q \frac{Q}{2M} \right) < 0. \tag{4.131}$$

So we would in fact gain a small stable frequency range for the frequencies where (4.125) is satisfied but (4.131) is not. Now if we artificially replace the vector potential by  $A_\nu^* = -\delta_\nu^0 Q / r^2$  and apply the procedure, then the transformed equation  $\tilde{T}_x^* \tilde{g}(x) = 0$  does not have any superradiance anymore and the mode stability proof should go through.

#### Conclusion

Despite the simplification due to spherical symmetry the charged scalar field on Reissner Nordström poses substantial difficulties which we were not able to overcome within the scope of this work. The difficulties mostly seem to originate in the slow fall-off of the vector potential. The problem seems worthy of further investigations.

# A. Appendix

## A.1.

Let  $f(x)$  be a smooth function on  $[-1, 1]$  vanishing at the boundary points with a unique maximum with value 1 at zero. Furthermore, we consider that  $f(0) = 1$ ,  $f'(0) = 0$  and  $f''(0) < 0$ . Then, we define:

$$g_{1a}(x) = \arcsin(f(x)) : \quad [-1, 0) \rightarrow [0, \pi/2) \quad (\text{A.1})$$

$$g_{2a}(x) = \pi - \arcsin(f(x)) : \quad [-1, 0) \rightarrow (\pi/2, \pi] \quad (\text{A.2})$$

$$g_{1b}(x) = \arcsin(f(x)) : \quad (0, 1] \rightarrow [0, \pi/2) \quad (\text{A.3})$$

$$g_{2b}(x) = \pi - \arcsin(f(x)) : \quad (0, 1] \rightarrow (\pi/2, \pi] \quad (\text{A.4})$$

Note that  $g'_{1a/b}(x) = -g'_{2a/b}(x)$ . By standart analytic arguments one gets that  $d/dx(\arcsin(f(x)))|_{x=0} = \sqrt{-f''(0)}$ . Note that on  $[-1, 0)$  the derivative of  $g_1(x)$  is positive while on  $(0, 1]$  it is negative. Together, this gives us that the function

$$g(x) = \begin{cases} g_{1a}(x) & \text{if } x \in [-1, 0) \\ \pi/2 & \text{if } x = 0 \\ g_{2b}(x) & \text{if } x \in (0, 1] \end{cases} \quad (\text{A.5})$$

is smooth at  $x = 0$  and therefore on  $[-1, 1]$ , with  $d/dx(g(x))|_{x=0} = \sqrt{-f''(0)}$ .

## A.2. Möbius transformation

The Riemann sphere  $\mathbb{S}^2$  can be globally parametrized by stereographic projection by means of  $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ . A Möbius transformation is a map

$$\chi : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}},$$

$$c \longmapsto \chi(z) := \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1.$$

## A. Appendix

The set of all Möbius transformations define a group, denoted by Mb.

This group is isomorphic to the set of positively oriented conformal maps of  $\mathbb{S}^2$  endowed with the standard round metric.

In this appendix we prove the following theorem.

**Theorem A.2.1.** *Let  $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$  be the Riemann sphere and Mb the set of Möbius transformations. Let  $c : \mathbb{S}^1 \rightarrow \overline{\mathbb{C}}$  be an embedding. If there exists  $\chi \neq \text{Id}_{\overline{\mathbb{C}}}$  that leaves  $c$  invariant as a set, then  $c$  is a generalized circle (i.e. a circle or a straight line with the point at infinity attached), or there exists  $n \in \mathbb{N}$  such that  $\chi^n = \text{Id}_{\overline{\mathbb{C}}}$  and  $c$  is conjugate to a closed curve invariant under rotations of angle  $\frac{2\pi m}{n}$ ,  $m \in \mathbb{Z}$  around the origin of  $\mathbb{C}$ .*

By “invariant as a set” we mean that there is a diffeomorphism  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  such that  $\chi \circ c = c \circ f$  (the image of  $c$  and  $\chi \circ c$  are obviously the same). A closed embedded curve  $c$  is *conjugate* to another closed embedded curve  $c_1$  if there exists  $\chi_1 \in \text{Mb}$  such that  $\chi_1 \circ c = c_1$ .

*Proof:* We first note that the problem is invariant under conjugation: for any  $\xi \in \text{Mb}$  the conjugate curve  $c_\xi := \xi \circ c$  is invariant (as a set) under the conjugate transformation  $\chi_\xi := \xi \cdot \chi \cdot \xi^{-1}$ , as it is obvious from:

$$\chi_\xi \circ c_\xi = (\xi \circ \chi \circ \xi^{-1}) \circ (\xi \circ c) = \xi \circ (\chi \circ c) = \xi \circ c \circ f = c_\xi \circ f.$$

It is well-known that all Möbius transformations (different from the identity) can be classified by conjugation into four disjoint classes: parabolic, elliptic, hyperbolic or loxodromic. Each class admits a canonical representative, in the sense that any element in the class is conjugate to this representative. The representatives can be chosen as follows:

$$\begin{array}{ll} \text{Parabolic:} & \chi_P(z) = \frac{z}{1+z} \\ \text{Elliptic:} & \chi_E(z) = e^{i\theta} z, \quad 0 \neq \theta \in \mathbb{R}_{\text{mod } 2\pi} \\ \text{Hyperbolic:} & \chi_H(z) = e^\lambda z \quad \lambda \in \mathbb{R} \setminus \{0\} \\ \text{Loxodromic:} & \chi_L(z) = kz \quad k \in \mathbb{C} \setminus \{\mathbb{R}\} \quad \text{and} \quad |k| \neq 1 \end{array} \quad (\text{A.6})$$

Thus, we may assume without loss of generality that the transformation  $\chi$  leaving  $c$  invariant is one of these canonical transformations. Obviously  $\chi^m$ ,  $m \in \mathbb{Z}$  also leaves  $c$  invariant. The action of  $\chi^m$  is immediate to obtain in the elliptic, hyperbolic and loxodromic canonical cases. In the parabolic case, a simple inductive argument shows that:

$$\chi_P^m(z) = \frac{z}{mz + 1}, \quad m \in \mathbb{Z}.$$

Thus, it follows easily that the cyclic group  $\{\chi^n; n \in \mathbb{Z}\}$  is finite (i.e.  $\chi^m = \text{Id}$  for some  $m \in \mathbb{Z}$ ) if and only if  $\chi$  is elliptic and  $\frac{\theta}{2\pi} \in \mathbb{Q}_{\text{mod } 1}$ .

Let us consider first the loxodromic, hyperbolic and parabolic cases. We start by showing that the embedded loop  $c$  must pass through the origin  $z = 0$  of the complex plane. Let  $0 \neq z_0 \in \mathbb{C}$  be any point on the curve, i.e.  $z_0 \in \text{Im}(c)$  and define, for each  $m \in \mathbb{Z}$ ,  $z_m := \chi^m(z_0) \in \overline{\mathbb{C}}$ . From invariance of the curve under  $\chi$ , all points in the sequence  $\{z_m\}$  lie on the image of the curve. From compactness of  $\text{Im}(c) \subset \overline{\mathbb{C}}$  it follows that the set of accumulation points of  $\{z_m\}$  is non-empty and a subset of  $\text{Im}(c)$ .

When  $\chi$  is hyperbolic or loxodromic, the canonical form is  $\chi_k(z) := kz$  with  $|k| \neq 1$ . The sequences are now  $z_m := \chi_k^m(z_0) = k^m z_0$ . If  $|k| > 1$ , the sequence converges to  $z = 0$  as  $m \rightarrow -\infty$ . If  $|k| < 1$ , the sequence converges to  $z = 0$  as  $m \rightarrow \infty$ . In either case  $z = 0$  is an accumulation point, so the loop  $c$  passes through  $z = 0$ . When  $\chi$  is parabolic, the sequence is  $z_m = \frac{z_0}{mz_0+1}$  which converges to  $z = 0$  as  $|m| \rightarrow \infty$ , and we reach the same conclusion.

We can now show that a loxodromic Möbius transformation does not leave any closed embedded loop invariant. Let us take differentials in the invariance equation  $\chi \circ c = c \circ f$  and evaluate at the invariant point  $p := \{z = 0\}$ :

$$d\chi|_p(\dot{c}) = \dot{f}|_{c^{-1}(p)}\dot{c},$$

which simply states the fact that the differential map of  $\chi_p$  must preserve the direction of  $\dot{c}|_p$  (it may change its scale, but not the direction). The differential of  $\chi(z) = kz$  at  $z = 0$  is  $d\chi|_{z=0} = k$ . Thus, this differential acts on a vector  $v$  by scaling with  $|k|$  and rotating by  $\arg(k)$ . When  $k$  is not real, all vectors  $v \neq 0$  change direction and we reach a contradiction. Thus, no embedded loop is invariant under a loxodromic Möbius transformation.

We next consider the hyperbolic case. The canonical representative is now  $\chi = \chi_H$ . Let  $\xi$  be a rotation of the form  $\xi(z) = e^{i\alpha}z$   $\alpha \in \mathbb{R}$ . Upon conjugation with  $\xi$ , the map  $\chi_H$  remains unchanged. The conjugate curve  $\xi \circ c$  passes through  $z = 0$ , and the parameter  $\alpha$  can be adjusted so that its tangent vector there points along the real axis  $x$ . Since  $c$  is an embedded curve, there is a neighbourhood  $U$  of  $z = 0$  such that  $U \cap c$  is connected and in fact a graph over the real axis. After restricting  $U$  if necessary we may assume that  $U$  is an open disk centered at  $z = 0$ . We consider the curve  $c_U := c \cap U$  from now on. This curve can be parametrized by  $x$ , i.e.  $c(x) = x + iy(x)$  where  $y(x)$  is a smooth function of  $x \in (-\epsilon, \epsilon)$ . The parameter  $\lambda$  in the definition of  $\chi_H$  can be assumed to be negative (if it were positive simply replace  $\chi_H$  by  $\chi_H^{-1}$ ). Then  $\chi_H$  maps  $U$  into itself, and leaves the

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curve  $c_U$  invariant. So, it must be the case that, for all  $x \in (-\epsilon, \epsilon)$ :

$$e^\lambda(x + iy(x)) = x'(x) + iy(x'(x)) \iff y(e^\lambda x) = e^\lambda y(x). \quad (\text{A.7})$$

where  $x'(x)$  indicates the reparametrization of the curve induced by the Möbius transformation  $\chi_H$ . Define the function  $P(u) := e^{-\lambda u} y(e^{\lambda u})$ . By construction,  $P(u)$  is smooth on  $(-\infty, \lambda^{-1} \ln \epsilon)$ . In terms of  $P$ , the function  $y(x)$  restricted to  $x > 0$  takes the form  $y(x) = xP(\lambda^{-1} \ln x)$ . The invariance property (A.7) becomes, when applied at the point  $x = e^{\lambda u}$ :

$$P(u + 1) = e^{-\lambda u} e^{-\lambda} y(e^{\lambda u} e^\lambda) = e^{-\lambda u} y(e^{\lambda u}) = P(u).$$

So  $P(u)$  is a periodic function of period one. We can now compute the derivative of  $y(x)$  (prime denotes derivative with respect to  $u$ ):

$$\frac{dy(x)}{dx} = P(\lambda^{-1} \ln x) + \lambda^{-1} P'|_{\lambda^{-1} \ln x}.$$

If  $P(u)$  is not a constant function the combination  $P(u) + \lambda^{-1} P'(u)$  does not converge as  $u \rightarrow -\infty$ . To show this, take the sequence  $u_n = u_0 - n$  with  $u_0 \in [-1, 0)$  defined by the condition that  $P(u_0)$  attains the supremum of  $P(u)$  and another sequence  $u'_n = u_1 - n$  where  $u_1 \in [-1, 0)$  is the value where  $P(u)$  attains the infimum. By periodicity, the sequences  $P(u_n)$  and  $P(u'_n)$  are both constant. Moreover,  $P'$  vanishes on all points  $u_n$  and  $u'_n$ . Thus, the sequences  $\{P(u_n) + \lambda^{-1} P'(u_n)\}$  and  $\{P(u'_n) + \lambda^{-1} P'(u'_n)\}$  converge to the same limit if and only if  $P(u_0) = P(u_1)$ , i.e. if the function  $P(u)$  is constant, as claimed. As a consequence,  $\frac{dy}{dx}$  converges as  $x \rightarrow 0^+$  if and only if  $P(u) = a$  for some constant  $a$ , or equivalently iff  $y(x) = ax$ . Since, in our setup,  $\frac{dy}{dx} = 0$  at  $x = 0$  we conclude that  $y(x) = 0$ . We have proved this fact in a neighbourhood  $U$  of 0, but this extends to the whole loop  $c$  by applying repeatedly the transformation  $\chi_H$ . In summary, we have shown that the only embedded loops invariant under the canonical representative  $\chi_H$  of hyperbolic Möbius transformations is the line  $(x, y = 0)$ , and arbitrary rotations thereof around the origin. We now use the property that Möbius transformations map generalized circles into generalized circles, and conclude that an embedded loop which is not a generalized circle can never be invariant under a hyperbolic Möbius transformation.

We want to use a similar argument for the parabolic case. To that aim, it is preferable to use a different representative. More precisely, recall that for  $\chi = \chi_P$  given in (A.6) the invariant embedded loop  $c$  necessarily passes through  $z = 0$ . Let us apply a conjugation with the inversion map  $\hat{\xi}(z) = -1/z$ . The conjugate  $\hat{\chi}_P = \hat{\xi} \circ \chi_P \circ \hat{\xi}^{-1}$  is given by  $\hat{\chi}_P(z) = z - 1$  and the conjugate loop  $\hat{c} := \hat{\xi} \circ c$  passes through the point at infinity. Consider the vector field:

$$\zeta = z^2 \partial_z + \bar{z}^2 \partial_{\bar{z}}.$$

## A.2. Möbius transformation

This field is smooth in a neighbourhood of the point at infinity. Indeed, the vector field  $\partial_{x'} = \partial_{z'} + \partial_{\bar{z}'}$  is clearly smooth in a neighbourhood of zero. The inversion map  $z' = -\frac{1}{z}$  transforms this neighbourhood of zero into a neighbourhood of infinity and transforms the vector field  $\partial_{x'}$  into  $\zeta$ , from which smoothness follows. In the coordinates  $\{x, y\}$  defined by  $z = x + iy$  we have:

$$\zeta = (x^2 - y^2) \partial_x + 2xy \partial_y.$$

The property of invariance of an embedded loop under a Möbius transformation is preserved by reparametrizations of the curve, so we are free to choose the parametrization of  $\hat{c}$ . However, we must make sure that the parameter is smooth everywhere, including a neighbourhood of infinity. To that aim we choose to parametrize  $\hat{c}$  with arc length  $s$  with respect to the round sphere metric:

$$ds^2 = \frac{1}{\left(1 + \frac{1}{4}(x^2 + y^2)\right)^2} (dx^2 + dy^2), \quad (\text{A.8})$$

which extends smoothly to the point at infinity. As before, let  $0 \neq z_0 = \hat{c}(s_0) = (x_0, y_0) \in \mathbb{C}$  be a point on the curve. From the condition that the tangent vector  $T|_p$  of the curve is unit with respect to (A.8), there exists  $\alpha \in [0, 2\pi)$  such that:

$$T|_p = F|_p (\cos \alpha \partial_x + \sin \alpha \partial_y),$$

with  $F|_p$  determined by:

$$F|_p = 1 + \frac{1}{4} (x^2 + y^2) \Big|_{(x_0, y_0)}.$$

We compute the scalar product with the vector  $\zeta$  to find:

$$\langle T|_p, \zeta|_p \rangle = \frac{\cos \alpha (x^2 - y^2) + 2 \sin \alpha xy}{1 + \frac{1}{4} (x^2 + y^2)} \Big|_{(x_0, y_0)}.$$

Consider now the sequence of points  $\{z_m = (x_0 - m, y_0)\}$ . From invariance under  $\widehat{\chi}_P$ , they also lie on the curve  $\hat{c}$ . In fact, the set  $\text{Im}(\hat{c})$  defines a periodic submanifold, in the sense that a unit translation along the  $x$  axis leaves it invariant. As a consequence, all the tangent vectors  $T|_{p_m}$  of the curve at each point  $z_m$  must be parallel to each other (in the natural euclidean sense of the term). Hence  $\alpha$  is the same for all  $z_m$ . Let us compute the limit along the sequence of the scalar product  $\langle T|_{p_m}, \zeta|_{p_m} \rangle$ :

$$\lim_{m \rightarrow \infty} \langle T|_{p_m}, \zeta|_{p_m} \rangle = \lim_{m \rightarrow \infty} \frac{\cos \alpha ((x_0 - m)^2 - y_0^2) + 2 \sin \alpha (x_0 - m) y_0}{1 + \frac{1}{4} ((x_0 - m)^2 + y_0^2)} = 4 \cos \alpha.$$

## A. Appendix

Given that the curve is smooth everywhere, including infinity, and that the sequence  $\{z_m\}$  converges to the point at infinity, it follows that all the tangent vectors  $T|_{p_m}$  must converge, namely to the unit tangent vector  $T_\infty$  to the curve there. The scalar products above must then converge to a single finite value, and this must happen independently of the initial point  $z_0$ . Since the limit depends on  $\alpha$  we conclude that  $\alpha$  must be the same for all points along the curve. If  $\alpha = \frac{\pi}{2}$  or  $\alpha = \frac{3\pi}{2}$  then the curve would be an infinite collection of vertical lines in the  $\{x, y\}$  plane, all of them passing through the point at infinity and the curve  $\hat{c}$  would not be embedded. Thus the tangent vector  $T_p$  must have a non-zero component along the  $x$  axis everywhere along the curve. This implies that it can be described as a graph  $y(x)$  on the  $x$  axis. Since  $y(x)$  must reach a local maximum and  $\alpha$  vanishes there we conclude that  $\alpha = 0$  at all points, and hence that  $y = y_0 = \text{const}$ . So, the embedded loop  $\hat{c}$  must be the straight line  $y = y_0$ . This claim is for embedded curves invariant under the parabolic transformation  $z \rightarrow z - 1$ . Upon conjugation, and using again that Möbius transformations map generalized circles into generalized circles, we conclude that the only embedded closed loops invariant under a parabolic transformation are generalized circles.

It only remains to consider the elliptic case, i.e.  $\chi = \chi_E$ . Since  $\chi_E$  is a rotation of angle  $\theta$  of the complex plane around its origin, the invariant embedded loop  $c$  defines a figure invariant under a rotation of angle  $\theta \neq 2\pi k$ ,  $k \in \mathbb{Z}$ . Consider the set of all angles  $\beta \in (0, 2\pi)$  under which this figure is invariant and let  $\beta_0$  be its infimum. If  $\beta_0 = 0$ , the curve must be a circle. If  $\beta_0$  is different from zero, then there must exist  $n \in \mathbb{N}$  such that  $\beta_0 = \frac{2\pi}{n}$  (if such  $n$  did not exist, define  $n \in \mathbb{N}$  by  $n\beta_0 < 2\pi < (n+1)\beta_0$ , the angle  $(n+1)\beta_0 - 2\pi$  is positive, smaller than  $\beta_0$  and belongs to the set of rotation angles that leave the figure invariant, which is a contradiction.) Thus  $\beta_0 = \frac{2\pi}{n}$  and in fact all other symmetry angles must be a multiple of this (by a similar argument as before). The number  $n$  is called the *order of symmetry* of the figure. In summary, the closed embedded loop  $c$  is invariant under  $\chi_E$  if and only if it is a circle centered at zero, or a figure with a discrete rotational symmetry of order  $n$ . The statement of the theorem then follows once again from the fact that the collection of generalized circles is preserved under Möbius transformations.  $\square$

As discussed in the main text, the shadow curve for suitable chosen observers at any point in the class of black hole spacetimes under consideration here has the property of being reflection symmetric. In precise terms, let the map  $r : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  be defined by reflection with respect to the real axis  $y = 0$ , i.e.  $r(z) = \bar{z}$ . A closed embedded loop  $c : \mathbb{S}^1 \rightarrow \overline{\mathbb{C}}$  is **reflection symmetric** if there exists a smooth map  $f_1 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  such that  $r \circ c = c \circ f_1$ . One checks immediately that  $f_1$  is a

diffeomorphism of  $\mathbb{S}^1$  (in fact an orientation reversing diffeomorphism). Our aim is to determine which elements  $\chi \in \text{Mb}$  have the property that the conjugate curve  $\chi \circ c$  is also reflection symmetric. Thus, we want to impose the condition that there exists a diffeomorphism  $f_2 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  such that  $r \circ \chi \circ c = \chi \circ c \circ f_2$ , which in turn is equivalent to  $\chi^{-1} \circ r \circ \chi \circ r^{-1} \circ c \circ f_1 = c \circ f_2$ , i.e. to:

$$\chi^{-1} \circ r \circ \chi \circ r^{-1} \circ c = c \circ f,$$

where  $f := f_2 \circ f_1^{-1}$  is an orientation preserving diffeomorphism of  $\mathbb{S}^1$ . The map  $\tilde{\chi} := \chi^{-1} \circ r \circ \chi \circ r^{-1}$  is by construction an element of the Möbius group, and leaves the loop defined by  $c$  invariant (as a submanifold). From Theorem A.2.1 it follows that  $\tilde{\chi}$  is the identity map, unless either  $\text{Im}(c)$  is conjugate to a figure with discrete rotational symmetry of order  $n$  and, in addition,  $\tilde{\chi}$  is conjugate to  $\chi_{m,n} := z \rightarrow e^{i\frac{2\pi m}{n}} z$  for some integer  $m$  between  $-n$  and  $n$ , or else  $c$  is a generalized circle.

In this paper we are interested in Möbius transformations sufficiently close to the identity that map reflection symmetric curves into reflection symmetric curves. Since, for fixed  $n$   $\{\xi_{m,n}; -n < m < n\}$  is discrete, it is disjoint to a sufficiently small neighbourhood of the identity map  $\text{Id}_{\mathbb{C}}$ , and we can ignore the case of discrete rotational symmetry of order  $n$ . Also, we restrict ourselves to non-degenerate spacetimes points, where the shadow curve is not a generalized circle (for simplicity we call such curves “non-circular”). So, we conclude that  $\tilde{\chi}$  must be the identity map, i.e.:

$$\chi^{-1} \circ r \circ \chi \circ r^{-1} = \text{Id}_{\mathbb{C}} \quad \iff \quad r \circ \chi \circ r^{-1} = \chi.$$

Letting  $\chi$  correspond to the  $\text{SL}(2, \mathbb{C})$  matrix:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

it is immediate to compute that  $r \circ \chi \circ r^{-1}$  corresponds to the  $\text{SL}(2, \mathbb{C})$  matrix:

$$\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}.$$

Thus, if  $\chi$  is sufficiently close to the identity map and the reflection symmetric curve  $c$  is non-circular, it must be the case that  $\chi \in \text{SL}(2, \mathbb{R})$ , i.e. all  $\alpha, \beta, \gamma, \delta$  are real parameters.

Our second aim is to identify the infinitesimal transformations with generate this subgroup of Möbius transformations. Consider a one parameter subgroup



## A. Appendix

$\tau : \mathbb{R} \rightarrow \text{SL}(2, \mathbb{C})$  of  $\text{SL}(2, \mathbb{C})$  and denote by  $\chi_{\tau(s)}$ ,  $s \in \mathbb{R}$  the corresponding curve in the Möbius group. A straightforward computation gives, for each  $z \in \mathbb{C}$ :

$$\frac{d\chi_{\tau(s)}(z)}{ds} = \beta_0 + (\alpha_0 - \delta_0) - \gamma_0 z^2,$$

where  $\alpha_0 = \left. \frac{d\alpha(s)}{ds} \right|_{s=0}$ ,  $\beta_0 = \left. \frac{d\beta(s)}{ds} \right|_{s=0}$ ,  $\gamma_0 = \left. \frac{d\gamma(s)}{ds} \right|_{s=0}$ ,  $\delta_0 = \left. \frac{d\delta(s)}{ds} \right|_{s=0}$ . The condition that the curve  $\tau(s)$  takes values in  $\text{SL}(2, \mathbb{C})$  requires that  $\delta_0 = -\alpha_0$ . Thus, the infinitesimal generator of this one-parameter subgroup is:

$$\xi = (\beta_0 + 2\alpha_0 z - \gamma_0 z^2) \partial_z + (\overline{\beta_0} + 2\overline{\alpha_0} \bar{z} - \overline{\gamma_0} \bar{z}^2) \partial_{\bar{z}}.$$

Thus if we restrict ourselves to the subgroup of transformation preserving the reflection symmetry of a non-circular curve  $c$ , the generators are:

$$\xi = \beta_0 (\partial_z + \partial_{\bar{z}}) + 2\alpha_0 (z\partial_z + \bar{z}\partial_{\bar{z}}) - \gamma_0 (z^2\partial_z + \bar{z}^2\partial_{\bar{z}}), \quad \alpha_0, \beta_0, \gamma_0 \in \mathbb{R}.$$

In terms of Cartesian coordinates  $\{x, y\}$  on the complex plane, i.e.  $z = x + iy$ , this vector field becomes:

$$\xi = \beta_0 \partial_x + 2\alpha_0 (x\partial_x + y\partial_y) - \gamma_0 ((x^2 - y^2) \partial_x + 2xy\partial_y).$$

So, the three generators of Möbius transformations preserving reflection symmetry turn out to be the translations along the  $x$  axis  $\xi_1 = \partial_x$ , the dilations about the origin  $\xi_2 = x\partial_y + y\partial_x$  and a third conformal Killing vector given by  $\xi_3 = (x^2 - y^2)\partial_x + 2xy\partial_y$ . These vector fields generate a Lie algebra with structure constants:

$$[\xi_1, \xi_2] = \xi_1, \quad [\xi_1, \xi_3] = 2\xi_2, \quad [\xi_2, \xi_3] = \xi_3.$$

Note that the subset of reflection symmetric transformations that leave the origin  $\{x = 0, y = 0\}$  invariant is generated by  $\{\xi_2, \xi_3\}$ , which is, naturally, a two-dimensional subalgebra. Another observation is that the only element in  $\{\xi_1, \xi_2, \xi_3\}$  which is a Killing vector of  $\mathbb{C} \cup \{\infty\}$  endowed with the spherical metric  $ds^2 = (1 + \frac{1}{4}(x^2 + y^2))^{-2} (dx^2 + dy^2)$ , is  $4\xi_1 + \xi_3$  (and its constant multiples). This Killing field corresponds to rotations of the sphere leaving invariant the poles with corresponding equator mapping onto the real axis by stereographic projection.

### A.3. Partial derivatives of $f$ and $h$

$$\frac{\partial f}{\partial a} = \frac{4x\Delta^2 - a^2\{x^2 + (l + a \cos \theta)^2\}\Delta' - \Delta(4a^2x + (x^2 + l^2 - a^2 \cos^2 \theta)\Delta')}{4a^2x\Delta^{3/2} \sin \theta} \quad (\text{A.9a})$$

A.3. Partial derivatives of  $f$  and  $h$

$$\frac{\partial f}{\partial l} = \frac{l\{x^2 + (l + a \cos \theta)^2\}\Delta' - 2\Delta(2lx + (l + a \cos \theta)\Delta')}{4ax\Delta^{3/2} \sin \theta} \quad (\text{A.9b})$$

$$\frac{\partial f}{\partial M} = -\frac{\{x^2 + (l + a \cos \theta)^2\}(2\Delta + x\Delta') + 4x^2\Delta}{4ax\Delta^{3/2} \sin \theta} \quad (\text{A.9c})$$

$$\frac{\partial f}{\partial Q} = -\frac{Q(\Delta'\{x^2 + (l + a \cos \theta)^2\} + 4x\Delta)}{4ax\Delta^{3/2} \sin \theta} \quad (\text{A.9d})$$

$$\frac{\partial f}{\partial \theta} = -\frac{2a(l + a \cos \theta)\Delta' \sin^2 \theta + \cos \theta(\Delta'\{x^2 + (l + a \cos \theta)^2\} - 4x\Delta)}{4ax\Delta^{1/2} \sin^2 \theta} \quad (\text{A.9e})$$

$$\frac{\partial f}{\partial x} = \frac{\{x^2 + (l + a \cos \theta)^2\}((M - x)^3 - M(M^2 - a^2 - Q^2 + l^2))}{2ax^2\Delta^{3/2} \sin \theta} \quad (\text{A.9f})$$

$$\frac{\partial h}{\partial a} = \frac{4ax(-8x\Delta(r)\Delta(x) + (\Delta(r) + \Delta(x))((r^2 - x^2)\Delta'(x) + 4x\Delta(x)))}{\sqrt{\Delta(x)\Delta(r)}((r^2 - x^2)\Delta'(x) + 4x\Delta(x))^2} \quad (\text{A.10a})$$

$$\frac{\partial h}{\partial Q} = \frac{4Qx(-8x\Delta(r)\Delta(x) + (\Delta(r) + \Delta(x))((r^2 - x^2)\Delta'(x) + 4x\Delta(x)))}{\sqrt{\Delta(x)\Delta(r)}((r^2 - x^2)\Delta'(x) + 4x\Delta(x))^2} \quad (\text{A.10b})$$

$$\frac{\partial h}{\partial l} = \frac{4lx(8x\Delta(r)\Delta(x) - (\Delta(r) + \Delta(x))((r^2 - x^2)\Delta'(x) + 4x\Delta(x)))}{\sqrt{\Delta(x)\Delta(r)}((r^2 - x^2)\Delta'(x) + 4x\Delta(x))^2} \quad (\text{A.10c})$$

$$\frac{\partial h}{\partial M} = \frac{4x(r\Delta(x)(-4x\Delta(x) + (x^2 - r^2)\Delta'(x)) + \Delta(r)(2(r^2 + x^2)\Delta(x) + x(x^2 - r^2)\Delta'(x)))}{\sqrt{\Delta(x)\Delta(r)}((r^2 - x^2)\Delta'(x) + 4x\Delta(x))^2} \quad (\text{A.10d})$$

$$\frac{\partial h}{\partial r} = \frac{2x\Delta(x)(4x\Delta(x)\Delta'(r) + (\Delta'(r)(r^2 - x^2) - 4r\Delta(r))\Delta'(x))}{\sqrt{\Delta(x)\Delta(r)}((r^2 - x^2)\Delta'(x) + 4x\Delta(x))^2} \quad (\text{A.10e})$$

$$\frac{\partial h}{\partial x} = \frac{2(r^2 - x^2)\Delta(r)((x - M)^3 + M(M^2 - a^2 - Q^2 + l^2))}{\sqrt{\Delta(x)\Delta(r)}((r^2 - x^2)\frac{\Delta'(x)}{2} + 2x\Delta(x))^2}. \quad (\text{A.10f})$$



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