The Cauchy problem for the linearised Einstein equation and the Goursat problem for wave equations

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Abstract

In this thesis, we study two initial value problems arising in general relativity. The first is the Cauchy problem for the linearised Einstein equation on general globally hyperbolic spacetimes, with smooth and distributional initial data. We extend well-known results by showing that given a solution to the linearised constraint equations of arbitrary real Sobolev regularity, there is a globally defined solution, which is unique up to addition of gauge solutions. Two solutions are considered equivalent if they differ by a gauge solution. Our main result is that the equivalence class of solutions depends continuously on the corresponding equivalence class of initial data. We also solve the linearised constraint equations in certain cases and show that there exist arbitrarily irregular (non-gauge) solutions to the linearised Einstein equation on Minkowski spacetime and Kasner spacetime.

In the second part, we study the Goursat problem (the characteristic Cauchy problem) for wave equations. We specify initial data on a smooth compact Cauchy horizon, which is a lightlike hypersurface. This problem has not been studied much, since it is an initial value problem on a non-globally hyperbolic spacetime. Our main result is that given a smooth function on a non-empty, smooth, compact, totally geodesic and non-degenerate Cauchy horizon and a so called admissible linear wave equation, there exists a unique solution that is defined on the globally hyperbolic region and restricts to the given function on the Cauchy horizon. Moreover, the solution depends continuously on the initial data. A linear wave equation is called admissible if the first order part satisfies a certain condition on the Cauchy horizon, for example if it vanishes. Interestingly, both existence of solution and uniqueness are false for general wave equations, as examples show. If we drop the non-degeneracy assumption, examples show that existence of solution fails even for the simplest wave equation. The proof requires precise energy estimates for the wave equation close to the Cauchy horizon. In case the Ricci curvature vanishes on the Cauchy horizon, we show that the energy estimates are strong enough to prove local existence and uniqueness for a class of non-linear wave equations. Our results apply in particular to the Taub-NUT spacetime and the Misner spacetime. It has recently been shown that compact Cauchy horizons in spacetimes satisfying the null energy condition are necessarily smooth and totally geodesic. Our results therefore apply if the spacetime satisfies the null energy condition and the Cauchy horizon is compact and non-degenerate.

Zusammenfassung

In der vorliegenden Arbeit werden zwei Anfangswertsprobleme aus der Allgemeinen Relativitätstheorie betrachtet. Das erste ist das Cauchyproblem für die linearisierte Einsteingleichung auf allgemeinen global hyperbolischen Raumzeiten mit glatten und distributionellen Anfangsdaten. Wir verallgemeinern bekannte Ergebnisse, indem wir zeigen, dass für jede gegebene Lösung der linearisierten Constraintgleichungen mit reeller Sobolevregularität eine global definierte Lösung existiert, die eindeutig ist bis auf Addition von Eichlösungen. Zwei Lösungen sind äquivalent falls sie sich durch eine Eichlösung unterscheiden. Unser Hauptergebnis ist, dass die Äquivalenzklasse der Lösungen stetig von der zugehörigen Äquivalenzklasse der Anfangsdaten abhängt. Wir lösen auch die linearisierten Constraintgleichungen in Spezialfällen und zeigen, dass beliebig irreguläre (nicht Eich-) Lösungen der linearisierten Einsteingleichungen auf der Minkowski-Raumzeit und der Kasner-Raumzeit existieren.

Im zweiten Teil betrachten wir das Goursatproblem (das charakteristische Cauchyproblem) für Wellengleichungen. Wir geben Anfangsdaten auf einem Cauchyhorizont vor, der eine lichtartige Hyperfläche ist. Dieses Problem wurde bisher noch nicht viel betrachtet, weil es ein Anfangswertproblem auf einer nicht global hyperbolischen Raumzeit ist. Unser Hauptergebnis ist: Gegeben eine glatte Funktion auf einem nicht-leeren glatten, kompakten, totalgeodätischen und nicht-degenerierten Cauchyhorizont und eine so genannte zulässige Wellengleichung, dann existiert eine eindeutige Lösung, die auf dem global hyperbolischen Gebiet definiert ist und deren Einschränkung auf dem Cauchyhorizont die gegebene Funktion ist. Die Lösung hängt stetig von den Anfangsdaten ab. Eine Wellengleichung heißt zulässig, falls der Teil erster Ordnung eine gewisse Bedingung am Cauchyhorizont erfüllt, zum Beispiel falls er gleich Null ist. Interessant ist, dass Existenz der Lösung und Eindeutigkeit falsch sind für allgemeine Wellengleichungen, wie Beispiele zeigen. Falls wir die Bedingung der Nichtdegeneriertheit weglassen, ist Existenz von Lösungen falsch sogar für die einfachste Wellengleichung. Der Beweis benötigt genaue Energieabschätzungen für die Wellengleichung nahe am Cauchyhorizont. Im Fall, dass die Ricci-Krümmung am Cauchyhorizont verschwindet, zeigen wir, dass die Energieabschätzungen stark genug sind, um lokale Existenz und Eindeutigkeit für eine Klasse von nicht-linearen Wellengleichungen zu zeigen. Unser Ergebnis ist zum Beispiel auf der Taub-NUT-Raumzeit oder der Misner-Raumzeit gültig. Es wurde vor kurzem gezeigt, dass kompakte Cauchyhorizonte in Raumzeiten, die die Nullenergiebedingung erfüllen, notwendigerweise glatt und totalgeodätisch sind. Unsere Ergebnisse sind deshalb auf Raumzeiten gültig, die die Nullenergiebedingung erfüllen, wenn der Cauchyhorizont kompakt und nicht-degeneriert ist.

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Contents

1. Introduction

Initial value problems play an important role in general relativity. The fundamental question is: if we have all the information about the universe at a certain time, can we predict the future and reconstruct the past? In this thesis, we consider two initial value problems of very different nature. The first one is the Cauchy problem for the linearised Einstein equation, describing gravitational waves, where we give initial data on a spacelike Cauchy hypersurface. The second is the Goursat problem for wave equations, where we specify initial data on a lightlike hypersurface. The lightlike hypersurfaces that we consider will be compact Cauchy horizons.

The purpose of the first part is to give a systematic study of the Cauchy problem for the linearised Einstein vacuum equation on globally hyperbolic spacetimes. The main application is to the theoretical study of gravitational waves, which are often modelled as solutions of the linearised Einstein equation.

The study of the linearised Einstein equation on special backgrounds is subject to current research, in particular in connection to the stability problem for black holes. See [14] and references therein. Despite this, very little is published on the Cauchy problem on general globally hyperbolic spacetimes. Many basic results are considered "well-known", but not much is written up. In a recent paper [16, Thm. 3.1, Thm. 3.3], Fewster and Hunt show that given a *smooth* solution to the linearised constraint equation, there is a globally defined smooth solution to the linearised Einstein vacuum equation. The solution is unique up to addition of a "linearised isometry", i.e. a Lie derivative of the background metric. A proof of these statements was also sketched in [17, Thm. 4.5]. In this thesis we extend these results by considering initial data of arbitrary real Sobolev regularity, see Theorem 4.1.2. This is not obvious, since one needs to choose the "gauge" in a way that works in low regularity. We also show that the solution is unique up to adding a Lie derivative of the background metric (a gauge solution) of corresponding regularity, see Theorem 4.2.1.

The natural question to ask now is whether the solution depends continuously on the initial data. For this, one needs a well-defined "solution map", mapping initial data to the corresponding solution. Since there exist more than one solution to given initial data, the question of continuous dependence on initial data does, *a priori*, not make sense. In order to get a well-defined bijection between initial data and solutions, one needs to quote out the space of gauge solutions. We prove that the quotient vector space obtained by quoting out the space of gauge solutions from the space of solutions is a well-defined topological vector space. Similarly, we prove that the quotient vector space obtained by quoting out gauge producing initial data from the space of initial data is a well-defined topological vector space. We show that solving the Cauchy problem gives a topological isomorphism between these spaces. Leaving the precise statement for Theorem 4.4.1, let us state the result informally here. For a Cauchy hypersurface Σ in a globally hyperbolic spacetime M, the solution map

$$Solve : \text{Initial data on } \Sigma \,\big/ \text{Gauge producing i.d.} \to \text{Global solutions on } M \,\big/ \text{Gauge sol.}$$

is an isomorphism of topological vector spaces. An immediate consequence is that the solution map

$$\widetilde{Solve} : \text{Initial data on } \Sigma \to \text{Global solutions on } M \,\big/ \text{Gauge solutions}$$

is continuous and surjective.

It is expected that the above results can be generalised to various models with matter, using the methods presented here, but we will for simplicity restrict to the vacuum case.

We also discuss how to solve the linearised constraint equations. On the Minkowski space, it is well-known that the so called transverse-traceless (TT-)tensors are a natural "gauge choice" for the initial data. In Theorem 5.1.2, we prove a generalisation of this result to the linearised constraint equations on general closed manifolds with vanishing scalar curvature. We also give examples of arbitrarily irregular (non-gauge) solutions to the linearised Einstein equation on Kasner spacetime and on Minkowski spacetime. This is done by constructing irregular initial data and solving the Cauchy problem.

In the second part of the thesis, we study scalar wave equations with initial data on compact Cauchy horizons. Cauchy horizons lie in the boundary of a globally hyperbolic region. Examples of spacetimes containing compact Cauchy horizons include the Misner spacetime and the Taub-NUT spacetime. We study the following problem: given a smooth function defined on a smooth and compact Cauchy horizon, does there exist a smooth solution to the wave equation on the globally hyperbolic part, which extends smoothly to the horizon such that the restriction is the given initial data? There are not many results on this question in the literature, since it is an initial problem on a non-globally hyperbolic spacetime.

A smooth Cauchy horizon is a lightlike hypersurface of the spacetime. Initial value problems for wave equations, where the initial data is specified on lightlike hypersurfaces are called *Goursat problems* or *characteristic Cauchy problems*. One difference to the Cauchy problem is that one only specifies the function as initial data and not the normal derivative of the function. There are many results on the Goursat problem, also for non-linear wave equations, when the lightlike hypersurface is a light cone or two intersecting lightlike hypersurfaces (see e.g. [13], [15] and [27] and references therein). However, such lightlike hypersurfaces cannot be compact Cauchy horizons. For linear wave equations, the Goursat problem has been studied for more general lightlike hypersurfaces by Bär and Tagne Wafo in [7], generalising results of Hörmander in [20]. They consider lightlike hypersurfaces that are subsets of a globally hyperbolic spacetime, which we do not. In fact, if a spacetime contains a smooth compact Cauchy horizon, it has to contain closed or almost closed lightlike curves. Therefore a spacetime containing a smooth compact Cauchy horizon *cannot* be globally hyperbolic.

To assume that the compact Cauchy horizon is smooth could seem like a strong restriction at first. However, it was recently proven independently in [22] and [23] that compact Cauchy horizons in spacetimes satisfying the null energy condition are necessarily smooth and totally geodesic lightlike hypersurfaces of the spacetime. The null energy condition is satisfied by definition if and only if $\mathrm{ric}(V, V) \geq 0$ for any lightlike vector V. Since the null energy condition is commonly satisfied by models in general relativity, it is natural to consider smooth, compact and totally geodesic Cauchy horizons.

Our main result is that if a non-empty, smooth, compact and totally geodesic Cauchy horizon is in addition *non-degenerate*, then "most" of the linear wave equations are uniquely solvable on the globally hyperbolic region for given initial data on the horizon. The solution depends continuously on initial data. We give the precise formulation in Theorem 6.2.2. For example, wave equations of the form

$$\Box u + \alpha u = f,$$

where α and f are smooth functions on the spacetime, are always uniquely solvable. General linear wave equations are of the form

$$\Box u + \partial_W u + \alpha u = f,$$

where W is a smooth vector field. In Theorem 6.2.2 we prove well-posedness assuming that the vector field W, restricted to the horizon, *is nowhere pointing out of the globally hyperbolic region*. This is a condition on the vector field on the Cauchy horizon only. Remarkably, both existence of solution and uniqueness are false if we drop this assumption, as simple examples show. This is different from both the Cauchy problem and the Goursat problem studied in [7], where global existence and uniqueness holds for any wave equation. If we drop the assumption that the Cauchy horizon is non-degenerate, not even the wave equation $\Box u = 0$ is solvable for general initial data, as examples show. Both Cauchy horizons in the Taub-NUT spacetime and the Cauchy horizon in the Misner spacetime are non-degenerate, so Theorem 6.2.2 applies.

The proof of Theorem 6.2.2 proceeds in several steps. We construct a "null time function" in a small neighbourhood of the Cauchy horizon. This enables us to formulate energies and prove precise energy estimates for the wave equation close to the horizon. From the energy estimates, uniqueness follows. For the existence, we first show that it is possible to construct an asymptotic solution to the wave equation on the horizon. We finish the proof by using the energy estimates to construct an actual solution from the asymptotic solution.

If we assume that the Ricci curvature of the spacetime vanishes at the Cauchy horizon, we are able to improve the energy estimates. The improvement is strong enough to conclude local existence and uniqueness for non-linear wave equations. More precisely, we show that if the Ricci curvature of the spacetime vanishes at the Cauchy horizon and if $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, the non-linear wave equation

$$\Box u = f(u),$$

is uniquely solvable in a neighbourhood of the Cauchy horizon, given any initial data. See Theorem 7.1.1 for the precise statement. Again this result applies in particular to the Taub-NUT spacetime and the Misner spacetime.

Compact Cauchy horizons are in several ways different from (partial) Cauchy hypersurfaces. The domain of dependence of a compact Cauchy horizon will in general only be the Cauchy horizon itself and nothing else. Hence Theorem 6.2.2 gives control of the solution outside of the domain of dependence of the initial hypersurface, which is usually not the case for the Cauchy problem. Moreover, we give examples of compact Cauchy horizons such that for any point on the Cauchy horizon, the causal future of the point is the full globally hyperbolic region. This shows that there is *no finite speed of propagation*, when specifying the initial data on a compact Cauchy horizon. Solving the Goursat problem with initial data on a compact Cauchy horizon is therefore a global problem and cannot be localised to one coordinate patch at the time.

Wave equations with initial data on compact Cauchy horizons have not been studied other than in special cases. In [25], Moncrief and Isenberg solved a linear wave equation with initial data on Cauchy horizons. They appealed to the Cauchy-Kowaleskaya theorem under the assumption that the spacetime was analytic. The same wave equation was studied by Friedrich, Rácz and Wald in [19], without the analyticity assumption. However, their methods require that the horizon has closed generators and that the initial data are invariant under the flow of the generators. In this thesis, we neither assume any invariance on the initial data nor analyticity of the spacetime metric.

A question that is related to this thesis, is to determine the asymptotics of the Einstein equations, using methods for so called Fuchsian equations. The purpose is to prescribe the asymptotic behaviour at a timelike singularity and show the existence of a solution to the Einstein equation with these asymptotics. Solutions with asymptotics that stay bounded towards the initial singularity could be interpreted as solutions of certain wave equations with initial data on a compact Cauchy horizon. Most of the results are done in the analytic setting, see [3] and references therein. Some more recent results dropped the assumption of analyticity, see for example [2], [28], [10] and [30] and references therein. In all cases known to the author, where analyticity is not assumed, one instead assumes symmetry of the spacetime. In contrast to this, we do not assume neither analyticity nor symmetry of the spacetime.

The thesis is structured as follows. We start by introducing our notation and prove some basic results on some linear differential equations in Chapter 2. In Chapter 3, we recall the formulation of the Cauchy problem for the Einstein equation and formulate the corresponding Cauchy problem for the linearised Einstein equation. Chapter 4 contains our first main result of Part I, the well-posedness of the linearised Einstein equation in the sense described above. In Chapter 5, we discuss the linearised constraint equation by solving it in special cases and using the results of Chapter 4 to show that there are arbitrarily irregular solutions to the linearised Einstein equation. Chapter 6 is the main chapter of Part II, where we introduce the exact notions of Cauchy horizons and prove the well-posedness statement for the Goursat problem. We conclude the thesis in Chapter 7 by using the result of Chapter 6 to prove our result on non-linear wave equations.

2. Notation and mathematical preliminaries

In this thesis, differential operators will act on spaces of sections. Let us start by introducing these spaces and show how differential operators act on them. We then prove some basic results on certain linear differential equations. Most of the results presented here are well-known, however, some are only to find in a different setting. All manifolds, vector bundles and metrics will be smooth in this thesis, but the sections will have various regularity.

2.1. The function spaces

We will work with differential operators acting on sections of vector bundles, most commonly on various tensor fields.

2.1.1. Smooth sections of vector bundles and differential operators

Assume that (M, g) is a smooth manifold and let $E \to M$ be a real vector bundle over M. We denote the *space of smooth sections* in E by

$$C^\infty(M, E).$$

We will sometimes write $C^\infty(M)$ instead of $C^\infty(M, E)$ if it is clear from the context what vector bundle is meant. Let us recall how the standard Fréchet topology on $C^\infty(M, E)$ is defined. Given a connection ∇^E on E and the Levi-Civita connection ∇^{T^*M}, we get a connection ∇ on $(T^*M)^l \otimes E$ for any $l \in \mathbb{N}_0$. We denote the l:th covariant derivative of a section by

$$\nabla^l u := \nabla \dots \nabla u \in C^\infty\left(M, (T^*M)^l \otimes E\right).$$

Let us choose auxiliary norms $|\cdot|$ on $C^\infty(M, (T^*M)^l \otimes E)$. For any compact subset $K \subset M$ and $m \in \mathbb{N}_0$, define the semi-norm

$$\|u\|_{m,K,\nabla,|\cdot|} := \max_{l=0,\dots,m} \max_{x \in K} \left|\nabla^l u(x)\right|.$$

Define the *topology* on $C^\infty(M, E)$ using these semi-norms. Since K is compact, this gives the same topology independently of the choice of $\nabla, |\cdot|$. This turns $C^\infty(M, E)$ into a Fréchet space. For a compact set $K \subset M$, let

$$C^\infty_K(M, E)$$

be the smooth sections defined on all of M with support contained in K. It is clear that

$$C_K^\infty(M,E) \subset C^\infty(M,E)$$

is a closed subspace. Define the space of *smooth sections of compact support* in E by

$$C_c^\infty(M,E) := \bigcup_{\substack{K\subset M\\ \text{compact}}} C_K^\infty(M,E)$$

with the strict inductive limit topology. For the definition of the strict inductive limit topology and basic properties, see for example [31]. This implies the following notion of convergence of sequences (or nets): $\varphi_n \to \varphi \in C_c^\infty(M,E)$ if and only if there exists a compact $K \subset M$ such that $\operatorname{supp}(\varphi_n) \subset K$ for all $n \in \mathbb{N}_0$ and $\|\varphi_n - \varphi\|_{m,K} \to 0$ as $n \to \infty$ for all $m \in \mathbb{N}_0$.

Let W be a locally convex topological vector space and let V be a strict inductive limit of a sequence of Fréchet spaces (V_n), such that $V_n \hookrightarrow V_{n+1}$ has closed image. A linear map $F : V \to W$ is continuous, if and only if

$$F|_{V_n} : V_n \to W$$

is continuous, for all $n \in \mathbb{N}$. This applies in particular to $V = C_c^\infty(M,E)$, but also to several other cases that we introduce later.

Let us now define differential operators acting on smooth sections. Let $E, F \to M$ be real vector bundles.

Definition 2.1.1 (Linear differential operator)**.** We say that P is a *linear differential operator* from E to F of order $m \in \mathbb{N}_0$ if there are sections $A_j \in C^\infty(M, (TM)^j \otimes \operatorname{Hom}(E,F))$, for $j = 0, \dots, m$ such that

$$P = \sum_{j=0}^m A_j(\nabla^j).$$

We write $\operatorname{Diff}_m(E,F)$ for the vector space of linear differential operators from E to F of order m.

It is clear that if ∇ and $\hat{\nabla}$ are connections on $(T^*M)^j \otimes \operatorname{Hom}(E,F)$ for all $j = 0, \dots, m-1$ as constructed above, then if

$$P = \sum_{j=0}^m A_j(\nabla^j),$$

then there exists $\hat{A}_j \in C^\infty(M, (TM)^j \otimes \operatorname{Hom}(E,F))$, for $j = 0, \dots, m$ such that

$$P = \sum_{j=0}^m \hat{A}_j(\hat{\nabla}^j).$$

Therefore Definition 2.1.1 is meaningful.

Example 2.1.2. Any smooth vector field X acting as a differentiation ∂_X on functions is a linear differential operator of order 1 from the trivial line bundle to itself. The connection is the trivial connection and A_1 is the insertion of X.

Let $P \in \mathrm{Diff}_m(E, F)$. Then P defines continuous maps

$$\begin{aligned} P &: C^\infty(M, E) \to C^\infty(M, F), \\ P &: C^\infty_K(M, E) \to C^\infty_K(M, F), \\ P &: C^\infty_c(M, E) \to C^\infty_c(M, F), \end{aligned}$$

for each compact subset $K \subset M$. An important tool to classify linear differential operators is the following invariant.

Definition 2.1.3 (Principal symbol)**.** Let $P \in \mathrm{Diff}_m(E, F)$. The section $\sigma_P \in C^\infty(M, \otimes^m_{sym} T^*M \otimes \mathrm{Hom(E, F)})$ defined by

$$\sigma_P(\xi) := A_m(\xi, \ldots, \xi, \cdot)$$

is called the *principal symbol of* P .

One checks that σ_P is independent of the choice of connection.

2.1.2. Distributional sections of vector bundles

We now turn to distributional sections of a real vector bundle $E \to M$. Let $E^* \to M$ denote the vector bundle dual to E. We define the *space of distributions* $\mathcal{D}'(M, E)$ as the space of continuous linear functionals on $C^\infty_c(M, E^*)$, equipped with the weak*-topology. Thus convergence of sequences (or nets) is given by

$$T_n \to T \Leftrightarrow T_n[\varphi] \to T[\varphi]$$

for all $\varphi \in C^\infty_c(M, E^*)$. Recall that any locally integrable section f in E can be considered as a distribution by

$$\varphi \mapsto \int_M f(\varphi) d\mu_g,$$

where $d\mu_g$ is the volume density induced by a semi-riemannian metric g. For example if $f \in C^\infty_c(M, \mathbb{R})$ and we interpret f as a distribution, then $f[1] = \int_M f d\mu_g$. We define the support of a distribution T, denoted $\mathrm{supp}(T)$, by defining its complement $\mathrm{supp}(T)^c$ as the union of all open subsets $U \subset M$ such that $T[\varphi] = 0$ for all $\varphi \in C^\infty_c(M, E^*)$ with $\mathrm{supp}(\varphi) \subset U$. It follows that $\mathrm{supp}(T)^c$ is open, which means that $\mathrm{supp}(T)$ is closed. Let us show how linear differential operators act on distributional sections. For this, given a linear differential operator $P \in \mathrm{Diff}_m(E, F)$, define the *formally dual operator operator* $P' \in \mathrm{Diff}_m(F^*, E^*)$ as the unique differential operator such that

$$\int_M \psi(P'\varphi) d\mu_g = \int_M (P\psi)(\varphi) d\mu_g, \tag{2.1}$$

for all $\psi \in C_c^\infty(M,E)$ and $\varphi \in C_c^\infty(M,F^*)$. One can prove that the image of the embedding

$$C^\infty(M,E) \hookrightarrow \mathcal{D}'(M,E)$$

is dense. Now P can be extended to act on distributions by the formula

$$PT[\varphi] = T[P'(\varphi)].$$

This coincides with equation (2.1) when T can be identified with a compactly supported smooth section. P extends to a continuous map

$$P : \mathcal{D}'(M,E) \to \mathcal{D}'(M,F).$$

In case there are positive definite metrics $\langle \cdot,\cdot \rangle_E$ and $\langle \cdot,\cdot \rangle_F$ on E and F respectively and let $P \in \mathrm{Diff}_m(E,F)$, there is a simple way of computing the dual operator. We make the identifications $\sharp : E^* \to E$ and $\sharp : F^* \to F$ induced by the positive definite metrics. Define the formal adjoint operator $P^* \in \mathrm{Diff}_m(F,E)$ by the unique differential operator such that

$$\int_M \langle P^*\varphi, \psi \rangle_E = \int_M \langle \varphi, P\psi \rangle_F d\mu_g,$$

for all $\psi \in C_c^\infty(M,E)$ and $\varphi \in C_c^\infty(M,F)$. It follows that

$$P' \langle \cdot, \varphi \rangle_F = \langle \cdot, P^*\varphi \rangle_E$$

for all $\varphi \in C_c^\infty(M,F)$.

Example 2.1.4. As we will commonly work with distributional tensors, let us give some examples homomorphisms that we will use throughout. Homomorphisms are linear differential operators of zero order. Let g be a smooth semi-riemannian metric on M, extended to tensor fields.

- If $X \in \mathcal{D}'(M,TM)$ and $Y \in C^\infty(M,TM)$, then the distribution $g(X,Y)$ is given by

  $$g(X,Y)[\varphi] = X[\varphi g(\cdot, Y)].$$

  This is well-defined since $\varphi g(\cdot,Y) \in C_c^\infty(M,T^*M)$. Using this, we can project X to subvector bundles for example.

- Similarly, if $a \in \mathcal{D}'(M,(T^*M)^2)$ and $b \in C^\infty(M,(T^*M)^2)$, then the distribution

  $$g(a,b)[\varphi] := a[\varphi g(\cdot, b)].$$

  In particular, the trace of a with respect to g is defined and equals

  $$\mathrm{tr}_g(a) := g(g,a).$$

2.1.3. Sobolev spaces on Riemannian manifolds

Let us first define Sobolev spaces on closed (compact without boundary) Riemannian manifolds. Let E be a real vector bundle over a closed Riemannian manifold (M, g). We assume that E comes with a positive definite metric $\langle \cdot, \cdot \rangle_E$ and a metric connection ∇. For $u, v \in C^\infty(M, E)$, define the L^2-inner product by

$$\langle u, v \rangle_{L^2(M,E)} := \int_M \langle u, v \rangle_E d\mu_g,$$

where $d\mu_g$ is the smooth volume density induced by g. Let the space of *square integrable sections*, denoted by $L^2(M, E)$, be the completion of $C^\infty(M, E)$ with respect to the norm

$$\|u\|_{L^2(M,E)} := \sqrt{\langle u, u \rangle_{L^2(M,E)}}.$$

The linear differential operator

$$\nabla^*\nabla + 1 : C^\infty(M; E) \to C^\infty(M; E)$$

is elliptic, positive and essentially selfadjoint on $L^2(M; E)$. We extend $\nabla^*\nabla + 1$ to a selfadjoint operator on $L^2(M; E)$ and denote the extension with the same symbol. Using functional calculus, define for $k \in \mathbb{R}$

$$D^k := (\nabla^*\nabla + 1)^{k/2}.$$

Since $\nabla^*\nabla + 1$ is formally self-adjoint, $\langle D^k u, v \rangle_{L^2(ME)} = \langle u, D^k v \rangle_{L^2(ME)}$ for all $u, v \in C^\infty_c(M, E)$. We define the *Sobolev space* of sections in E of degree $k \in \mathbb{R}$, denoted by $H^k(M, E)$, to be the closure of $C^\infty(M, E)$ with respect to the norm

$$\|u\|_{H^k(M,E)} := \left\|D^k u\right\|_{L^2(M,E)}.$$

For $k \in \mathbb{N}_0$, the H^k-norm is for example equivalent to the norm

$$\sum_{i=0}^{k} \left\|\nabla^i u\right\|_{L^2(M,(T^*M)^i \otimes E)}.$$

Let us now drop the assumption that M is compact. Let $K \subset M$ be a compact subset and let K' be a closed manifold such that an open neighbourhood of K embeds isometrically into K'. Such an extension always exists, see for example [7, Section 1.6.2]. We can also extend the vector bundle $E|_K$ to K', let us for simplicity denote also the extension by E. Define the *Sobolev sections with support in* K, denoted by $H^k_K(M, E)$, to be the closure of $C^\infty_K(M, E)$ with respect to the norm

$$\|u\|_{H^k_K(M,E)} := \|\hat{u}\|_{H^k(K',E)},$$

where $\hat{u}$ is the image of u under the induced embedding $C^\infty_K(M,E) \to C^\infty(K',E)$. Let the space of *Sobolev sections of compact support* be

$$H^k_c(M,E) := \bigcup_{\substack{K\subset M\\ compact}} H^k_K(M,E)$$

with the strict inductive limit topology. Similar to the smooth compactly supported sections, we claim that we have the following notion of convergence of sequences (or nets): $u_n \to u \in H^k_c(M,E)$ if and only if there exists a compact $K \subset M$ such that $\mathrm{supp}(u_n) \subset K$ for all $n \in \mathbb{N}_0$ and $\|u_n - u\|_{H^k_K(M,E)} \to 0$ as $n \to \infty$. To see this, we prove the following lemma.

Lemma 2.1.5. *Assume that $V \subset H^k_c(M,E)$ is bounded. Then there is a compact subset $K \subset M$ such that $V \subset H^k_K(M,E)$. In particular, if $u_n \to u$ is a converging sequence (or net), then $u_n \to u$ in $H^k_K(M,E)$.*

The method of proof is standard.

Proof. Assume to reach a contradiction, that the statement is not true. Let $K_1 \subset K_2 \subset \ldots$ be a exhaustion by compact subsets of M. By assumption, for each $i \in \mathbb{N}$ there is an $f_i \in V$ such that $\mathrm{supp}(f_i) \not\subset K_i$. Hence there are test sections $\varphi_i \in C^\infty_c(M,E^*)$ such that $\mathrm{supp}(\varphi_i) \subset K_i^{\mathsf{c}}$ and $f_i[\varphi_i] \neq 0$. Consider the convex subset containing zero, given by

$$W := \left\{ f \in H^k_c(M,E) \mid |f[\varphi_i]| < \frac{|f_i[\varphi_i]|}{i},\ \forall i \right\} \subset H^k_c(M,E).$$

We claim that W is open. We have

$$W \cap H^k_{K_j}(M,E) = \bigcap_{i=1}^{j-1} \left\{ f \in H^k_{K_j}(M,E) \mid |f[\varphi_i]| < \frac{|f_i[\varphi_i]|}{i} \right\}.$$

Since $f \mapsto |f[\varphi_i]|$ is a continuous function on $H^k_{K_j}(\Sigma,E)$, this is a finite intersection of open sets and hence open. It follows that W is open. Note that for each $T > 0$, we have $f_i \notin T \cdot W$ if $i > T$. It follows that V is not bounded. □

Finally, define the space of *locally Sobolev sections* as

$$H^k_{loc}(M,E) := \{u \in \mathcal{D}'(M,E) \mid \varphi u \in H^k_c(M,E), \forall \varphi \in C^\infty_c(M,\mathbb{R})\}.$$

We define the topology on $H^k_{loc}(M,E)$ as follows. Let $K_1 \subset K_2 \subset \ldots$ be an exhaustion of M by compact subsets. Choose $\{\varphi_n\}_{n=1}^\infty \subset C^\infty_c(M,\mathbb{R})$ such that $\varphi_n|_{K_n} = 1$. We define semi-norms on $H^k_{loc}(M,E)$ by

$$\|u\|_{k,K_n} := \|\varphi_n u\|_{H^k_{K_n}(M,E)}.$$

Defining the topology of $H^k_{loc}(M,E)$ using these semi-norms makes $H^k_{loc}(M,E)$ a Fréchet space. We have the following continuous embeddings

$$H^k_K(M,E) \hookrightarrow H^k_c(M,E) \hookrightarrow H^k_{loc}(M,E) \hookrightarrow \mathcal{D}'(M,E),$$

for each compact $K \subset M$. Note that all four spaces are independent of the choice of riemannian metric g. We can extend any $P \in \mathrm{Diff}_m(E,F)$ to a differential operator on Sobolev sections by considering them as distributional sections. We get continuous maps

$$\begin{aligned} P &: H^k_{loc}(M,E) \to H^{k-m}_{loc}(M,F), \\ P &: H^k_K(M,E) \to H^{k-m}_K(M,F), \\ P &: H^k_c(M,E) \to H^{k-m}_c(M,F), \end{aligned}$$

for each $k \in \mathbb{R}$ and compact subset $K \subset M$. Sometimes it will be convenient to write

$$\begin{aligned} H^\infty_{loc}(M,E) &:= \bigcap_{k\in\mathbb{R}} H^k_{loc}(M,E) = C^\infty(M,E), \\ H^\infty_K(M,E) &:= \bigcap_{k\in\mathbb{R}} H^k_K(M,E) = C^\infty_K(M,E), \\ H^\infty_c(M,E) &:= \bigcap_{k\in\mathbb{R}} H^k_c(M,E) = C^\infty_c(M,E). \end{aligned}$$

Let us conclude the section with the following important lemma.

Lemma 2.1.6. *Let $k \in \mathbb{R} \cup \{\infty\}$ and let $P \in \mathrm{Diff}_m(E,F)$. Then the induced subspace topology on*

$$H^k_c(M,E) \cap \ker(P)$$

is the same as the inductive limit topology induced by the embeddings $H^k_K(M,E) \cap \ker(P) \hookrightarrow H^k_c(M,E) \cap \ker(P)$.

Proof. Let $u_n \to u$ be a net converging in $H^k_c(M,E) \cap \ker(P)$ with respect to the subspace topology. Then $u_n \to u$ in $H^k_c(M,E)$, which by Lemma 2.1.5 means that there is a compact subset $K \subset M$ such that $u_n \to u$ in $H^k_K(M,E)$. It follows that $u_n \to u$ in $H^k_K(M,E) \cap \ker(P)$ and hence in $H^k_c(M,E) \cap \ker(P)$, since the embedding is continuous.

For the other direction, assume that $U \subset H^k_c(M,E) \cap \ker(P)$ is an open convex neighbourhood of 0 in the subspace topology. Then there is an open convex neighbourhood $\hat{U} \subset H^k_c(M,E)$ of 0 such that $U = \hat{U} \cap \ker(P)$. By definition, $\hat{U} \cap H^k_K(M,E)$ is open, for all compact subsets $K \subset M$. It follows that $U \cap H^k_K(M,E) = \hat{U} \cap H^k_K(M,E) \cap \ker(P) \subset H^k_K(M,E) \cap \ker(P)$ is open for all compact subsets $K \subset M$. But this means that $U \subset H^k_c(M,E) \cap \ker(P)$ is an open convex neighbourhood of 0 in the strict inductive limit topology. This concludes the proof. $\square$

2.1.4. Finite energy spaces on globally hyperbolic spacetimes

Assume now that (M, g) is a smooth globally hyperbolic spacetime. By [8, Theorem 1.1] there is a Cauchy temporal function $t : M \to \mathbb{R}$, i.e. for all $\tau \in t(M)$, $\Sigma_\tau := t^{-1}(\tau)$ is a smooth spacelike Cauchy hypersurface and $\mathrm{grad}(t)$ is timelike and past directed. The metric can then be written as

$$g = -N^2 dt^2 + \tilde{g}_t,$$

where $N : M \to \mathbb{R}$ is a positive function and $\tilde{g}_\tau$ denotes a riemannian metric on Σ_τ, depending smoothly on $\tau \in t(M)$. For each $k \in \mathbb{R}$, we get a Fréchet vector bundle

$$(H^k_{loc}(\Sigma_\tau, E|_{\Sigma_\tau}))_{\tau \in t(M)}.$$

We denote the C^m-sections in this vector bundle by

$$C^m \left(t(M), H^k_{loc}(\Sigma_\cdot, E|_{\Sigma_\cdot})\right).$$

This is a Fréchet space. When solving wave equations, the solutions typically lie in the following spaces of sections of *finite energy of infinite order*:

$$CH^k_{loc}(M, E, t) := \bigcap_{j=0}^{\infty} C^j \left(t(M), H^{k-j}_{loc}(\Sigma_\cdot, E|_{\Sigma_\cdot})\right).$$

The spaces $CH^k_{loc}(M, E, t)$ carry a natural induced Fréchet topology. Note that we have the continuous embedding

$$CH^k_{loc}(M, E, t) \hookrightarrow H^{\lfloor k \rfloor}_{loc}(M, E),$$

where $\lfloor k \rfloor$ is the largest integer smaller than or equal to k. The finite energy sections can be considered as distributions defined by

$$u[\varphi] := \int_{t(M)} u(\tau) \left[(N\varphi)|_{\Sigma_\tau}\right] d\tau.$$

For any subset $A \subset M$, let $J^{-/+}(A)$ denote the causal past/future of A and denote their union $J(A)$. Similarly, denote by $I^{-/+}(A)$ the chronological past/future of A and their union by $I(A)$. A subset $A \subset M$ is called *spatially compact* if $A \subset J(K)$ for some compact subset $K \subset M$. For each spatially compact subset $A \subset M$, the space

$$CH^k_A(M, E, t) := \{f \in CH^k_{loc}(M, E, t) \mid \mathrm{supp}(f) \subset A\} \subset CH^k_{loc}(M, E, t)$$

is closed and therefore also a Fréchet space. We define the *finite energy sections of spatially compact support* by

$$CH^k_{sc}(M, E, t) := \bigcup_{\substack{A \\ \text{spatially compact}}} CH^k_A(M, E, t)$$

with the inductive limit topology . This can be done, using that if K_i is an exhaustion of a Cauchy hypersurface Σ, then $J(K_i)$ is an exhaustion by spatially compact set of M. Similar to before, the notion of convergence is given by the following lemma.

Lemma 2.1.7. *Assume that $V \subset CH^k_{sc}(M, E, t)$ is bounded. Then there is a compact subset $K \subset \Sigma$ such that $V \subset CH^k_{J(K)}(M, E, t)$. In particular, if $u_n \to u$ is a converging sequence (or net), then $u_n \to u$ in $CH^k_{J(K)}(M, E, t)$.*

The proof is analogous to the proof of Lemma 2.1.5. Any $P \in \mathrm{Diff}_m(E, F)$ extends to a continuous differential operator

$$P : CH^k_{loc}(M, E, t) \hookrightarrow CH^{k-m}_{loc}(M, E, t)$$
$$P : CH^k_A(M, E, t) \hookrightarrow CH^{k-m}_A(M, E, t)$$
$$P : CH^k_{sc}(M, E, t) \hookrightarrow CH^{k-m}_{sc}(M, E, t)$$

for any spatially compact set $A \subset M$. It will sometimes be convenient to write

$$CH^\infty_{loc}(M, E, t) := \bigcap_{k\in\mathbb{R}} CH^k_{loc}(M, E, t) = C^\infty(M, E),$$
$$CH^\infty_A(M, E, t) := \bigcap_{k\in\mathbb{R}} CH^k_A(M, E, t) = C^\infty_A(M, E),$$
$$CH^\infty_{sc}(M, E, t) := \bigcap_{k\in\mathbb{R}} CH^k_{sc}(M, E, t) = C^\infty_{sc}(M, E),$$

for any spatially compact $A \subset M$.

The following lemma is analogous to Lemma 2.1.6.

Lemma 2.1.8. *Let $k \in \mathbb{R} \cup \{\infty\}$ and let $P \in \mathrm{Diff}_m(E, F)$. Then the induced subspace topology on*

$$CH^k_{sc}(M, E) \cap \ker(P)$$

is the same as the inductive limit topology induced by the embeddings

$$CH^k_{J(K)}(M, E) \cap \ker(P) \hookrightarrow CH^k_{sc}(M, E) \cap \ker(P).$$

The proof is analogous to the proof of Lemma 2.1.6, but we use Lemma 2.1.7 instead of lemma 2.1.5.

2.2. Some linear differential equations on manifolds

In this section, we sum up results on various linear differential equations on manifolds that will be used in the thesis.

2.2.1. First order differential equations

Lemma 2.2.1. *Assume that M is a closed manifold. Let $X \in C^\infty(M, TM)$ be a vector field that is nowhere zero and $\alpha \in C^\infty(M)$ be a function that is nowhere zero. Then*

$$P : C^\infty(M) \to C^\infty(M)$$
$$u \mapsto \partial_X u + \alpha u$$

is an isomorphism of topological vector spaces.

Proof. Without loss of generality, we can assume that $\alpha = 1$, by substituting X with $\frac{1}{\alpha}X$.

First we show injectivity of P. Assume therefore that $Pu = 0$. Since M is compact, the maximum and minimum values of u are attained at, let us say, $x_{\max}$ and $x_{\min}$. At these points $\partial_X u(x_{\min}) = 0 = \partial_X u(x_{\max})$. Since $Pu = 0$, it follows that $u(x_{\min}) = 0 = u(x_{\max})$. Hence $u = 0$, which proves injectivity.

To prove surjectivity, let $f \in C^\infty(M)$ be given. For each point $p \in M$, let $c_p : \mathbb{R} \to M$ be the maximal integral curve of X such that $c_p(0) = p$. We claim that the function u defined for each point $p \in M$ by

$$u(p) := \int_{-\infty}^{0} e^s f \circ c_p(s) ds \tag{2.2}$$

is smooth and solves $Pu = f$. This is well-defined for each $p \in M$, since f is bounded. For a fixed $s \in \mathbb{R}$, it is a standard result that

$$\begin{aligned} M &\to M \\ p &\mapsto c_p(s) \end{aligned}$$

is a diffeomorphism. Hence $p \mapsto f \circ c_p(s)$ is smooth. By the Lebesgue dominated convergence theorem, it follows that $p \mapsto u(p)$ is smooth. Furthermore, we calculate that

$$\begin{aligned} \partial_X u|_p &= \frac{d}{dt}\Big|_{t=0} \int_{-\infty}^{0} e^s f \circ c_{c_p(t)}(s) ds \\ &= \frac{d}{dt}\Big|_{t=0} \int_{-\infty}^{0} e^s f \circ c_p(s+t) ds \\ &= \frac{d}{dt}\Big|_{t=0} e^{-t} \int_{-\infty}^{t} e^w f \circ c_p(w) dw \\ &= -u(p) + f \circ c_p(0) \\ &= -u(p) + f(p). \end{aligned}$$

Hence $Pu = f$ as claimed. □

Remark 2.2.2. The explicit formula (2.2) for the solution shows that if $f > 0$ (resp. $f < 0$), then $u > 0$ (resp. $u < 0$) as well.

2.2.2. Linear elliptic operators

Lemma 2.2.3 (Fredholm alternative on closed manifolds)**.** *Assume that M is a closed manifold, $k \in \mathbb{R}$ and*

$$P : H^{k+m}(M, E) \to H^k(M, F)$$

is a differential operator of order m with injective principal symbol. Then

$$H^k(M, F) = \operatorname{im}(P) \oplus \ker(P^*), \tag{2.3}$$

where P^* is the formal adjoint as an operator

$$P^* : H^k(M,F) \to H^{k-m}(M,E).$$

Extend or restrict P and P^* act on the spaces

$$\tilde{P} : H^{-k+m}(M,E) \to H^{-k}(M,F),$$
$$\tilde{P}^* : H^{-k}(M,F) \to H^{-k-m}(M,E).$$

Then $\operatorname{im}(\tilde{P})$ *is the annihilator of* $\ker(P^*)$ *and* $\ker(\tilde{P}^*)$ *is the annihilator of* $\operatorname{im}(P)$ *under the isomorphism $H^{-k}(M,E) \cong H^k(M,E)'$.*

In particular, if $k \geq 0$, the sum in (2.3) *is L^2-orthogonal. In case $k = \infty$, equation* (2.3) *holds true.*

Proof. See for example [9, Appendix I] for equation (2.3) when $k \geq 0$. For any $k \in \mathbb{R}$, the map

$$\begin{aligned} H^{-k}(M,E) &\to H^k(M,E)' \\ f &\mapsto (\varphi \mapsto \langle D^{-k} f, D^k \varphi \rangle_{L^2(M,E)}) \end{aligned}$$

is an isomorphism. Let $k \geq 0$. Equation (2.3) implies that

$$H^{-k}(M,E) \cong H^k(M,E)' = \operatorname{im}(P)' \oplus \ker(P^*)',$$

where $\operatorname{im}(P)'$ and $\ker(P^*)'$ can be identified with the annihilator of $\ker(P^*)$ and of $\operatorname{im}(P)$ respectively. If we denote the annihilator by Ann, we therefore have

$$H^{-k}(M,E) = \operatorname{Ann}(\operatorname{im}(P)) \oplus \operatorname{Ann}(\ker(P^*)). \tag{2.4}$$

We claim now that $\operatorname{Ann}(\operatorname{im}(P)) = \ker(\tilde{P}^*)$. For all $f \in H^{-k}(M,)$ and all $u \in H^k(M,)$, we have

$$\begin{aligned} \langle D^{-k} f, D^k P u \rangle &= \langle D^{-k} f, D^k P D^{-m-k} D^{m+k} u \rangle \\ &= \langle D^{-k-m} \tilde{P}^* f, D^{m+k} u \rangle. \end{aligned}$$

This shows that $\operatorname{Ann}(\operatorname{im}(P)) = \ker(\tilde{P}^*)$. Similarly, one shows that $\operatorname{Ann}(\operatorname{im}(\tilde{P})) = \ker(P^*)$. It follows that $\operatorname{im}(\tilde{P}) = \operatorname{Ann}(\operatorname{Ann}(\operatorname{im}(\tilde{P}))) = \operatorname{Ann}(\ker(P^*))$. Inserting this into equation (2.4) finishes the proof. □

One part of the previous lemma generalises to non-compact manifold.

Lemma 2.2.4. *Let M be a possibly non-compact manifold and let $K \subset M$ be a compact subset and let $k \in \mathbb{R} \cup \{\infty\}$. Assume that*

$$P : H^{k+m}_K(M,E) \to H^k_K(M,F)$$

is a differential operator of order m with injective principal symbol. Assume furthermore that P is injective. Then

$$\operatorname{im}(P) \subset H^k_K(M,F)$$

is closed and P is an isomorphism of Hilbert spaces onto its image.

Proof. By [7, Section 1.6.2.], we can embed an open neighbourhood U of K isometrically into a closed riemannian manifold $(K', \tilde{g}')$. Denote the embedding by $\iota : U \hookrightarrow K'$. Moreover, we can extend the vector bundles in a smooth way, let us for simplicity still denote them by E and F. For any section $f : M \to E$, define $\iota_* f : K' \to E$ such that $f|_K = (\iota_* f) \circ \iota|_K$, just by multiplying by a bump function which equals 1 on K and vanishes outside U. It follows that there is a differential operator with injective principal symbol

$$Q : H^{k+m}(K', E) \to H^k(K', F)$$

such that the following diagram commutes:

$$\begin{array}{ccc} H_K^{k+m}(M, E) & \xrightarrow{P} & H_K^k(M, F) \,. \\ \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} \\ H^{k+m}(K', E) & \xrightarrow{Q} & H^k(K', F) \end{array}$$

Choose a function $\lambda : K' \to \mathbb{R}$ such that $\lambda(x) > 0$ for all $x \in K' \backslash \iota(K)$ and $\lambda|_{\iota(K)} = 0$. We claim that

$$Q^* Q + \lambda : H^{k+m}(K', E) \to H^{k-m}(K', E)$$

is an isomorphism of Hilbert spaces (in the smooth case, $k = \infty$, we claim that this is an isomorphism of Fréchet spaces). By Lemma 2.2.3, it suffices to show that $\ker(Q^*Q + \lambda) = \{0\}$, since $Q^*Q + \lambda$ is formally self-adjoint. For any $a \in \ker(Q^*Q + \lambda)$ it follows that a is smooth and

$$\int_{K'} |Qa|^2 + \lambda \, |a|^2 \, dVol = 0.$$

Hence $\mathrm{supp}(a) \subset \iota(K)$ and $Qa = 0$. This implies that $b := \iota^* a$, extended to whole M by zero, solves $P(b) = 0$. Since $\mathrm{supp}(b) \subset K$ and P is injective, this implies that $b = 0$ and hence $a = 0$. We conclude the claim.

Assume now that $P(u_n) \to f$ in $H_K^k(M, F)$, with $u_n \in H_K^{k+m}(M, E)$. It follows that $Q(\iota_* u_n) \to \iota_* f$ in $H^k_{\iota(K)}(K', F)$ and $\iota_* u_n \in H^{k+m}_{\iota(K)}(K', E)$. Hence

$$(Q^*Q + \lambda)(\iota_* u_n) = Q^* Q(\iota_* u_n) \to Q^*(\iota_*(f))$$

in $H^{k-m}_{\iota(K)}(K', E)$. Therefore, there is a $v \in H^{k+m}(K', E)$ such that $\iota_* u_n \to v$ in $H^{k+m}(K', E)$. Since $\mathrm{supp}(\iota_* u_n) \subset \iota(K)$ and $\iota_* u_n \to v$ as distributions, the support of v cannot be larger than $\iota(K)$. Hence $v \in H^k_{\iota(K)}(K', E)$. Now define

$$u := \iota^* v \in H_K^{k+m}(U, E)$$

and extend it by zero to an element in $H_K^{k+m}(M, E)$. Note that $u_n \to u$ in $H_K^{k+m}(M, E)$. It follows that

$$P(u) = \lim_{n \to \infty} P(u_n) = f,$$

as claimed (in the case $k = \infty$, the last line is to be thought of as a limit of a net). □

Definition 2.2.5 (Laplace type operators)**.** A differential operator $P \in \mathrm{Diff}_2(E, E)$ is called a Laplace type operator if

$$\sigma_P(\xi) = -g(\xi, \xi)\mathrm{id},$$

for all $\xi \in T^*M$.

In local coordinates, this is equivalent to P having the form

$$P = -\sum_{i,j} g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + l.o.t.$$

We will need the following theorem, known as the *Strong unique continuation property.* We quote the statement from [5]. For a proof, see [4, Theorem on p. 235 and Remark 3 on p. 248].

Theorem 2.2.6 (Aronszajn's Unique Continuation Theorem)**.** *Let (M, g) be a connected riemannian manifold and let P be a Laplace type operator acting on sections of a vector bundle $E \to M$. Assume that $Pu = 0$ and that u vanishes at some point of infinite order, i.e. that all derivatives vanish at that point. Then $u = 0$.*

Corollary 2.2.7. *Let $k \in \mathbb{R} \cup \{\infty\}$. Assume that M is connected. Let $K \subset M$ be a compact subset such that $K \neq M$. Assume that*

$$P : H_K^{k+2}(M, E) \to H_K^k(M, E)$$

is a Laplace-type operator. Then

$$\mathrm{im}(P) \subset H_K^k(M, E)$$

is closed and P is an isomorphism of Hilbert spaces (Fréchet spaces if $k = \infty$) onto its image.

Proof. We only need to show that P is injective. Assume that $Pu = 0$. Since $u|_{M \setminus K} = 0$, Theorem 2.2.6 implies that $u = 0$. □

2.2.3. Linear wave equations

In the literature, there are various statements concerning the well-posedness of the Cauchy problem for linear wave equations with initial data of Sobolev regularity. The statement that is relevant for our purposes is not in the form we need it in the literature, so we give a proof here.

Definition 2.2.8 (Wave operator)**.** A differential operator $P \in \mathrm{Diff}_m(E, E)$ is called a wave operator if

$$\sigma_P(\xi) = -g(\xi, \xi)\mathrm{id},$$

for all $\xi \in T^*M$.

Wave operators are sometimes also called *normally hyperbolic operators*. Wave operators correspond to Laplace type operators, in the Lorentzian setting. Note that P is a wave operator if and only if in local coordinates it takes the form

$$P = -\sum_{i,j} g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + l.o.t.$$

Assume that (M, g) is a smooth globally hyperbolic spacetime with Cauchy hypersurface $\Sigma \subset M$ and $t : M \to \mathbb{R}$ a Cauchy temporal function such that $\Sigma = t^{-1}(t_0)$ for some $t_0 \in t(M)$. Let $I := t(M)$. Let $E \to M$ be a real vector bundle and let P be a wave operator acting on sections in E. Denote by ν the future pointing unit normal vector field on Σ. It follows that $\nu = -\frac{1}{N}\mathrm{grad}(t)|_\Sigma$. Let us use the notation

$$\nabla_t := \nabla_{\mathrm{grad}(t)}.$$

Theorem 2.2.9 (Existence and uniqueness of solution). *Let $k \in \mathbb{R} \cup \{\infty\}$ be given. For each $(u_0, u_1, f) \in H^k_{loc}(\Sigma, E|_\Sigma) \oplus H^{k-1}_{loc}(\Sigma, E|_\Sigma) \oplus CH^{k-1}_{loc}(M, E, t)$, there is a unique $u \in CH^k_{loc}(M, E, t)$ such that*

$$\begin{aligned} Pu &= f, \\ u|_\Sigma &= u_0, \\ \nabla_\nu u|_\Sigma &= u_1. \end{aligned}$$

Moreover, we have finite speed of propagation, i.e.

$$\mathrm{supp}(u) \subset J\left(\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup K\right),$$

for any subset $K \subset M$ such that $\mathrm{supp}(f) \subset J(K)$.

Corollary 2.2.10 (Continuous dependence on initial data). *Let $k \in \mathbb{R} \cup \{\infty\}$ be given. Then the map*

$$\begin{aligned} CH^k_{loc}(M, E, t) \cap \ker(P) &\to H^k_{loc}(\Sigma, E|_\Sigma) \oplus H^{k-1}_{loc}(\Sigma, E|_\Sigma) \\ u &\to (u|_\Sigma, \nabla_\nu u|_\Sigma) \end{aligned}$$

is an isomorphism between topological vector spaces. In particular, the inverse map is continuous.

Proof of the corollary. By the preceding theorem, this map is continuous and bijective between Fréchet spaces. The open mapping theorem for Fréchet spaces implies the statement. □

The proof of the theorem is a standard method, translating the result in ([7, Theorem 13] and [6, Theorem 3.2.11]), where spatially compact support was assumed, to the general case.

Proof of the theorem. Let us first prove uniqueness of solution. Assume that $u \in CH^k_{loc}(M, E, t)$ satisfies

$$\begin{aligned} Pu &= 0, \\ u|_\Sigma &= 0 \\ \nabla_\nu u|_\Sigma &= 0. \end{aligned}$$

Let $x \in M$, we want to prove that $u|_U = 0$ for some open set $U \ni x$ (distributions cannot be evaluated at points in general). Since M is globally hyperbolic, it follows that $\Sigma \cap J(x) \subset \Sigma$ is compact. Choose $\chi \in C^\infty_c(\Sigma)$ such that $\chi = 1$ on an open neighbourhood U of $\Sigma \cap J(x)$. Extend χ to $\hat{\chi} \in C^\infty(M)$ by the condition $\partial_{\mathrm{grad}(t)}\hat{\chi} = 0$ and $\hat{\chi}|_\Sigma = \chi$. It follows that $\hat{u} := \hat{\chi} u \in CH^k_{sc}(M, E, t)$ and satisfies

$$\begin{aligned} \hat{u}|_\Sigma &= (\hat{\chi} u)|_\Sigma = 0 \\ \nabla_\nu \hat{u}|_\Sigma &= -\frac{1}{N}\nabla_t(\hat{\chi} u)|_\Sigma = \hat{\chi}\nabla_\nu u|_\Sigma = 0. \end{aligned}$$

Moreover, we note that $P\hat{u} = P(\hat{\chi}u) = Qu + \chi P u = Qu$, where Q is a first order differential operator. Hence it follows that

$$P\hat{u} \in CH^{k-1}_{loc}(M, E, t)$$

and $\mathrm{supp}(P\hat{u})$ is spatially compact. By [7, Remark 16], we conclude that

$$\mathrm{supp}(\hat{u}) \subset J(K),$$

for each compact subset $K \subset \Sigma$ such that $\mathrm{supp}(P\hat{u}) \subset J(K)$. Choose

$$K := \mathrm{supp}(\chi) \backslash U.$$

Hence $K \cap J(x) = K \cap (\Sigma \cap J(x)) \subset K \cap U = \emptyset$. It follows that $x \notin J(K)$ and hence that $x \in \mathrm{supp}(u)^c$ as claimed. Since x was arbitrary, we conclude that $u = 0$.

Let us now show existence of solution. Let $(u_0, u_1, f) \in H^k_{loc}(\Sigma, E|_\Sigma) \oplus H^{k-1}_{loc}(\Sigma, E|_\Sigma) \oplus CH^{k-1}_{loc}(M, E, t)$ be given. Choose a locally finite partition of unity $\{\varphi_i\}_{i=1}^\infty$ of Σ, such that for each compact set $K \subset \Sigma$, $\mathrm{supp}(\varphi_i) \cap K \neq \emptyset$ for only finitely many i. Extend φ_i to $\hat{\varphi}_i \in C^\infty(M, \mathbb{R})$ by $\partial_{\mathrm{grad}(t)}\hat{\varphi}_i = 0$ and $\hat{\varphi}_i|_\Sigma = \varphi_i$. It follows that $\mathrm{supp}(\hat{\varphi}_i) \subset J(\mathrm{supp}(\varphi_i))$. By [7, Theorem 13], for every i, there is a $u^i \in C^0(t(M), H^k_{loc}(\Sigma_.)) \cap C^1(t(M), H^{k-1}_{loc}(\Sigma_.))$ such that

$$\begin{aligned} u^i|_\Sigma &:= \varphi_i \cdot u_0, \\ \nabla_\nu u^i|_\Sigma &:= \varphi_i \cdot u_1, \\ Pu^i &:= \hat{\varphi}_i f. \end{aligned}$$

By [7, Remark 16], it follows that $\mathrm{supp}(u^i) \subset J(\mathrm{supp}(\varphi_i))$. Since $J(x) \cap \Sigma$ is compact, we have $\mathrm{supp}(\varphi_i) \cap (J(x) \cap \Sigma) \neq \emptyset$ only for finitely many i. Hence $x \in \mathrm{supp}(u^i) \subset J(\mathrm{supp}(\varphi_i))$ for finitely many i. Hence the following sum makes sense:

$$u := \sum_{i=1}^{\infty} u^i,$$

as distributions. It follows that $u \in C^0(t(M), H^k_{loc}(\Sigma.)) \cap C^1(t(M), H^{k-1}_{loc}(\Sigma.))$ and

$$\begin{aligned} Pu &= \sum_{i=1}^{\infty} \hat{\varphi}_i f = f \\ u|_{\Sigma} &= \sum_{i=1}^{\infty} \varphi_i u_0 = u_0, \\ \nabla_\nu u|_{\Sigma} &= \sum_{i=1}^{\infty} \varphi_i u_1 = u_1. \end{aligned}$$

Moreover, by [7, Remark 16] we have

$$\begin{aligned} \mathrm{supp}(u) &= \bigcup_{i=1}^{\infty} \mathrm{supp}(u^i) \\ &\subset \bigcup_{i=1}^{\infty} J\left(\mathrm{supp}(\varphi_i u_0) \cup \mathrm{supp}(\varphi_i u_1) \cup K_i\right), \end{aligned}$$

where $K_i \subset \Sigma$ are compact subsets such that $\mathrm{supp}(\hat{\varphi}_i f) \subset J(K_i)$. Define $K = \cup_{i=1}^{\infty} K_i$ and conclude that

$$\mathrm{supp}(u) \subset J(\mathrm{supp}(\varphi_i u_0) \cup \mathrm{supp}(\varphi_i u_1) \cup K)$$

as claimed.

We know that $u \in C^0(t(M), H^k_{loc}(\Sigma.)) \cap C^1(t(M), H^{k-1}_{loc}(\Sigma.))$. What remains is to show that in fact $u \in CH^k_{loc}(M, E, t)$. We assumed that $f \in CH^{k-1}_{loc}(M, E, t)$. Then since $Pu = f$ and P is a wave operator, we can locally write

$$\nabla_t \nabla_t u = B(u) + f$$

where B is a differential operator, which differentiates at most once in time. Hence

$$\nabla_t \nabla_t u \in C^0(t(M), H^{k-2}_{loc}(\Sigma.)),$$

which means that $u \in C^2(I, H^{k-2}_{loc}(\Sigma.))$. This implies that

$$\nabla_t \nabla_t u = B(u) + f \in C^1(t(M), H^{k-3}_{loc}(\Sigma.)),$$

which means that $u \in C^3(t(M), H^{k-3}_{loc}(\Sigma.))$. Iterating this implies the assertion. □

Part I.

The Cauchy problem for the linearised Einstein equation

3. Linearising the Einstein equation

Let throughout the chapter (M, g) denote a Lorentz manifold. We let $\mathrm{ric} := \mathrm{ric}_g$ and $\mathrm{scal} := \mathrm{scal}_g$ denote the Ricci and scalar curvature of (M, g). Let us give our convention of the divergence of a $(m+1)$-tensor, for $m \geq 0$:

$$\nabla \cdot h(X_1, \ldots, X_m) := \mathrm{tr}_g(\nabla_{(\cdot)} h(\cdot, X_1, \ldots, X_m)),$$

for $X_1, \ldots, X_m \in TM$.

3.1. The Cauchy problem of the (non-linear) Einstein equation

In this section, we recall the classical results about the Cauchy problem of the Einstein equation. The main result is the existence of the maximal globally hyperbolic development.

Definition 3.1.1 (The Einstein (vacuum) equation)**.** A smooth Lorentzian manifold (M, g) of dimension at least 3 satisfying

$$\mathrm{ric} - \frac{1}{2}\mathrm{scal} g = 0, \tag{3.1}$$

or equivalently

$$\mathrm{ric} = 0,$$

is said to be a *solution to the Einstein (vacuum) equation.*

We start by recalling that if

$$\mathrm{ric}_g(\nu, \cdot) - \frac{1}{2}\mathrm{scal}_g g(\nu, \cdot) = 0 \tag{3.2}$$

on a spacelike hypersurface $\Sigma \subset M$, where ν is the unit normal vector field, then the induced first and second fundamental forms $(\tilde{g}, \tilde{k})$ satisfy

$$\mathrm{scal}_{\tilde{g}} + (\mathrm{tr}_{\tilde{g}}\tilde{k})^2 - \tilde{g}(\tilde{k}, \tilde{k}) = 0, \tag{3.3}$$

$$\tilde{\nabla} \cdot \tilde{k} - d(\mathrm{tr}_{\tilde{g}}\tilde{k}) = 0. \tag{3.4}$$

Reversely, given a manifold Σ, we can ask if a Riemannian metric $\tilde{g}$ and a symmetric 2-tensor $\tilde{k}$ satisfy equations (3.3) and (3.4). Equation 3.3 is called the *Hamiltonian constraint equation* and equation (3.4) is called the *momentum constraint.* We can now formulate the initial value problem (the Cauchy problem of relativity).

Definition 3.1.2 (The Cauchy problem of the Einstein (vacuum) equation)**.** Assume that $(\Sigma, \tilde{g}, \tilde{k})$ satisfies the constraint equations (3.3) and (3.4). A Lorentz manifold (M, g) is said to satisfy the *Cauchy problem of the Einstein (vacuum) equation* if (M, g) satisfies

$$\mathrm{ric}_g = 0$$

and there is an isometric embedding $\iota : \Sigma \to M$ such that $\tilde{k}$ is the second fundamental form of $\iota(\Sigma)$.

Let us state the famous result by Choquet-Bruhat [18], Choquet-Bruhat and Geroch [12], saying that given initial data, there is a maximal globally hyperbolic development that is unique up to isometry.

Theorem 3.1.3. *Assume that* $(\Sigma, \tilde{g}, \tilde{k})$ *satisfies the constraint equations* (3.3) *and* (3.4)*. Then there exists a globally hyperbolic manifold* (M, g) *and an isometric embedding* $\iota : \Sigma \to M$ *such that*

- $\mathrm{ric}_g = 0$*,*
- $\iota(\Sigma)$ *is a spacelike Cauchy hypersurface,*
- $\tilde{k}$ *is the second fundamental form of* $\iota(\Sigma)$*,*
- *if* (M', g') *together with an isometric embedding* $\iota' : \Sigma \to M'$ *satisfy the above three conditions, then there exists an isometric embedding* $\varphi : M' \to M$ *such that* $\varphi \circ \iota' = \iota$*.*

A solution as in the theorem is called a *maximal globally hyperbolic development* of $(\Sigma, \tilde{g}, \tilde{k})$. Note that the maximal globally hyperbolic development is only determined up to isometry.

3.2. The linearised Einstein equation

Assume in the rest of Part I that (M, g) is a globally hyperbolic spacetime satisfying the *Einstein equation*, i.e.

$$\mathrm{ric}_g = 0.$$

We do not require (M, g) to be maximal in the sense of Theorem 3.1.3. Let us now linearise the Einstein equation around the solution g. For this, we first define the Lichnerowicz operator:

$$\Box_L h := \nabla^* \nabla h - 2\mathring{R}h,$$

where

$$\nabla^* \nabla := -\mathrm{tr}_g(\nabla^2),$$
$$\mathring{R}h(X, Y) := \mathrm{tr}_g(h(R(\cdot, X)Y, \cdot)),$$

for all $X, Y \in TM$.

Lemma 3.2.1 (Linearising the Ricci curvature)**.** *A curve of smooth Lorentz metrics g_s such that $g_0 = g$ satisfies*

$$\frac{d}{ds}\Big|_{s=0} \mathrm{ric}_{g_s} = \frac{1}{2}\left(\Box_L h + \mathcal{L}_{(\nabla\cdot\overline{h})^\sharp} g\right),$$

where

$$h := \frac{d}{ds}\Big|_{s=0} g_s$$

and $\mathcal{L}$ is the Lie derivative, $\overline{h} := h - \frac{1}{2}\mathrm{tr}_g(h)g$ and $\sharp$ is the musical isomorphism ("raising an index").

Proof. This straight forward computation can be found in for example [9, Theorem 1.174]. □

We are going to use a definition of the Lie derivative that extends to distributions, namely $\mathcal{L}_V g(X,Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X)$. The lemma motivates the following definition. Let us denote the vector bundle of symmetric 2-tensors on M by

$$S^2M := \otimes^2_{sym} T^*M.$$

Definition 3.2.2 (The linearised Einstein equation)**.** We define the linearised Ricci curvature

$$D\mathrm{ric}(h) := \frac{1}{2}\left(\Box_L h + \mathcal{L}_{(\nabla\cdot\overline{h})^\sharp} g\right)$$

for any $h \in \mathcal{D}'(M, S^2M)$. We say that h satisfies the *linearised Einstein equation* if

$$D\mathrm{ric}(h) = 0.$$

Remark 3.2.3. Note that $\Box_L$ is a wave operator, but $D\mathrm{ric}$ is not (c.f. Section 2.2.3).

There are certain solutions of the linearised Einstein equation, called "gauge solutions", which are due to "infinitesimal isometries".

Lemma 3.2.4 (Gauge invariance of the linearised Einstein equation)**.** *For any vector field $V \in \mathcal{D}'(M, TM)$, we have*

$$D\mathrm{ric}(\mathcal{L}_V g) = 0.$$

Proof. Let us first restrict to smooth objects. There is a natural geometric argument inherited from the diffeomorphism invariance of the nonlinear Einstein equation. Let $\varphi_s : M \to M$ be a curve of diffeomorphisms such that $\varphi_0 = \mathrm{id}$ and such that $\frac{d}{ds}\big|_{s=0}\varphi_s = V$. Differentiating the equation

$$0 = \varphi_s^*\mathrm{ric}(g) = \mathrm{ric}(\varphi_s^* g)$$

gives

$$D\mathrm{ric}(\mathcal{L}_V g) = 0.$$

By density of smooth sections in distributional sections, the result extends to the general case. □

3.3. The linearised constraint equation

Assume throughout the rest of Part I that $\Sigma \subset M$ is a smooth spacelike Cauchy hypersurface with future pointing unit normal vector field ν. Let $(\tilde{g}, \tilde{k})$ denote the, by g, induced first and second fundamental forms. As mentioned in the Section 3.1, $(\tilde{g}, \tilde{k})$ will satisfy the *constraint equation*

$$\Phi(\tilde{g}, \tilde{k}) := \begin{pmatrix} \Phi_1(\tilde{g}, \tilde{k}) \\ \Phi_2(\tilde{g}, \tilde{k}) \end{pmatrix} = 0,$$

where

$$\begin{aligned} \Phi_1(\tilde{g}, \tilde{k}) &= \mathrm{scal}_{\tilde{g}} - \tilde{g}(\tilde{k}, \tilde{k}) + (\mathrm{tr}_{\tilde{g}}\tilde{k})^2, \\ \Phi_2(\tilde{g}, \tilde{k}) &= \tilde{\nabla} \cdot \tilde{k} - d(\mathrm{tr}_{\tilde{g}}\tilde{k}), \end{aligned}$$

and $\tilde{\nabla}$ is the Levi-Civita connection on Σ with respect to $\tilde{g}$. We linearise the constraint equation around $(\tilde{g}, \tilde{k})$, analogously to Lemma 3.2.1.

Definition 3.3.1 (The linearised constraint equations)**.** A pair of tensors $(\tilde{h}, \tilde{m}) \in \mathcal{D}'(\Sigma, S^2\Sigma) \times \mathcal{D}'(\Sigma, S^2\Sigma)$ is said to satisfy the *linearised constraint equation*, linearised around $(\tilde{g}, \tilde{k})$, if

$$D\Phi(\tilde{h}, \tilde{m}) := \begin{pmatrix} D\Phi_1(\tilde{h}, \tilde{m}) \\ D\Phi_2(\tilde{h}, \tilde{m}) \end{pmatrix} = 0,$$

in $\mathcal{D}'(\Sigma, \mathbb{R}) \times \mathcal{D}'(\Sigma, T^*\Sigma)$, where

$$\begin{aligned} D\Phi_1(\tilde{h}, \tilde{m}) :=& \tilde{\nabla} \cdot (\tilde{\nabla} \cdot \tilde{h} - d\mathrm{tr}_{\tilde{g}}\tilde{h}) - \tilde{g}(\tilde{\mathrm{ric}}_{\tilde{g}}, \tilde{h}) \\ &+ 2\tilde{g}(\tilde{k} \circ \tilde{k} - (\mathrm{tr}_{\tilde{g}}\tilde{k})\tilde{k}, \tilde{h}) - 2\tilde{g}(\tilde{k}, \tilde{m} - (\mathrm{tr}_{\tilde{g}}\tilde{m})\tilde{g}) \qquad (3.5) \\ D\Phi_2(\tilde{h}, \tilde{m})(X) :=& -\tilde{g}(\tilde{h}, \tilde{\nabla}_{(\cdot)}\tilde{k}(\cdot, X)) - \tilde{g}\left(\tilde{k}(\cdot, X), \tilde{\nabla} \cdot (\tilde{h} - \frac{1}{2}(\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g})\right) \\ &- \frac{1}{2}\tilde{g}(\tilde{k}, \tilde{\nabla}_X\tilde{h}) + d(\tilde{g}(\tilde{k}, \tilde{h}))(X) + \tilde{\nabla} \cdot (\tilde{m} - (\mathrm{tr}_{\tilde{g}}\tilde{m})\tilde{g})(X), \qquad (3.6) \end{aligned}$$

for any $X \in T\Sigma$, where $\tilde{k} \circ \tilde{k}(X, Y) := \tilde{g}(\tilde{k}(X, \cdot), \tilde{k}(Y, \cdot))$ for any $X, Y \in T\Sigma$.

Similarly to the non-linear case, we will be given initial data satisfying the linearised constraint equations and require a solution to induce these initial data as linearised first and second fundamental forms. Therefore, we need to linearise the following expressions

$$\begin{aligned} \tilde{g}(X, Y) :=& g(X, Y), \\ \tilde{k}(X, Y) :=& g(\nabla_X \nu, Y), \end{aligned}$$

analogously to Lemma 3.2.1. In order to make sense of the restriction of distributional tensors to Σ, we assume some regularity.

Definition 3.3.2 (Linearised first and second fundamental forms)**.** Given $h \in CH^k_{loc}(M, S^2M, t)$, we define $(\tilde{h}, \tilde{m}) \in H^k_{loc}(\Sigma, S^2\Sigma) \times H^{k-1}_{loc}(\Sigma, S^2\Sigma)$ as

$$\begin{aligned}
\tilde{h}(X, Y) &= h(X, Y),\\
\tilde{m}(X, Y) &= -\frac{1}{2}h(\nu, \nu)\tilde{k}(X, Y) - \frac{1}{2}\nabla_X h(\nu, Y) - \frac{1}{2}\nabla_Y h(\nu, X) + \frac{1}{2}\nabla_\nu h(X, Y),
\end{aligned}$$

for any $X, Y \in T\Sigma$. $\tilde{h}$ and $\tilde{m}$ are called the *linearised first and second fundamental forms* induced by h.

Analogously to the non-linear case, one shows that if $\tilde{h}$ and $\tilde{m}$ are the linearised first and second fundamental forms induced by h, then using that $\mathrm{ric} = 0$ we get

$$\mathrm{tr}_g(D\mathrm{ric}(h)) + 2D\mathrm{ric}(h)(\nu, \nu) = D\Phi_1(\tilde{h}, \tilde{m}), \tag{3.7}$$

$$D\mathrm{ric}(h)(\nu, \cdot) = D\Phi_2(\tilde{h}, \tilde{m}). \tag{3.8}$$

In particular, if $D\mathrm{ric}(h) = 0$, the induced initial data $(\tilde{h}, \tilde{m})$ must satisfy $D\Phi(\tilde{h}, \tilde{m}) = 0$. Let us now formulate the Cauchy problem of the linearised Einstein equation.

Definition 3.3.3 (The Cauchy problem)**.** Let $(\tilde{h}, \tilde{m}) \in H^k_{loc}(\Sigma, S^2\Sigma) \times H^{k-1}_{loc}(\Sigma, S^2\Sigma)$ satisfy $D\Phi(\tilde{h}, \tilde{m}) = 0$. If $h \in CH^k_{loc}(M, S^2M, t)$ satisfies

$$D\mathrm{ric}(h) = 0$$

and induces $(\tilde{h}, \tilde{m})$ as linearised first and second fundamental forms, then we call h a *solution to the Cauchy problem of the linearised Einstein equation* with initial data $(\tilde{h}, \tilde{m})$.

4. The Cauchy problem for the linearised Einstein equation

The main purpose of this chapter is to show well-posedness of the Cauchy problem for the linearised Einstein equations. Recall first the setting. (M, g) is a smooth globally hyperbolic vacuum spacetime with smooth spacelike Cauchy hypersurface $\Sigma \subset M$ and induced first and second fundamental forms $(\tilde{g}, \tilde{k})$. In particular, ric $= 0$ on M and $\Phi(\tilde{g}, \tilde{k}) = 0$ on Σ. $D\Phi$ denotes the linearisation of the constraint map around $(\tilde{g}, \tilde{k})$ and Dric denotes the linearisation of the Ricci curvature around g. We fix a Cauchy temporal function $t : M \to \mathbb{R}$ such that $\Sigma = t^{-1}(t_0)$ for some $t_0 \in t(M)$.

4.1. Existence of solution

We start by proving that given initial data satisfying the linearised constraint equation, there is a solution to the linearised Einstein equation. The crucial point in the proof is to translate the initial data to initial data for a wave equation. For the (non-linear) Einstein equation, this is usually done by choosing so called "wave coordinates". It is not obvious how to choose wave coordinates for the linearised equations, it is more natural to work with a reformulation of this method, using so called "gauge source functions". Our idea here was inspired by the use of "gauge source functions" presented in for example [29]. Recall the notation

$$\bar{h} := h - \frac{1}{2}\mathrm{tr}_g(h)g.$$

Lemma 4.1.1. *For $k \in \mathbb{R}$, let $(\tilde{h}, \tilde{m}) \in H^k_{loc}(\Sigma, S^2\Sigma) \times H^{k-1}_{loc}(\Sigma, S^2\Sigma)$. Assume that $h \in CH^k_{loc}(M, S^2M, t)$ satisfies*

$$\begin{aligned} h(X,Y) &= \tilde{h}(X,Y), & \nabla_\nu h(X,Y) &= 2\tilde{m}(X,Y) - (\tilde{h}\circ\tilde{k} + \tilde{k}\circ\tilde{h})(X,Y), \\ h(\nu,X) &= 0, & \nabla_\nu h(\nu,X) &= \tilde{\nabla}\cdot\left(\tilde{h} - \frac{1}{2}(\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g}\right)(X), \\ h(\nu,\nu) &= 0, & \nabla_\nu h(\nu,\nu) &= -2\mathrm{tr}_{\tilde{g}}\tilde{m}, \end{aligned}$$

for all $X, Y \in T\Sigma$, where $\tilde{h}\circ\tilde{k}(X,Y) := g(\tilde{h}(X,\cdot), \tilde{h}(Y,\cdot))$ for all $X, Y \in T\Sigma$. Then $\tilde{h}, \tilde{m}$ are the, by h, induced first and second linearised fundamental forms and

$$\nabla \cdot \overline{h}|_\Sigma = 0.$$

The proof is a simple computation. Let us now state the existence theorem.

Theorem 4.1.2 (Existence of solution)**.** *For $k \in \mathbb{R} \cup \{\infty\}$, assume that $(\tilde{h}, \tilde{m}) \in H^k_{loc}(\Sigma, S^2\Sigma) \times H^{k-1}_{loc}(\Sigma, S^2\Sigma)$ satisfies*

$$D\Phi(\tilde{h}, \tilde{m}) = 0.$$

Then there exists a unique

$$h \in CH^k_{loc}(M, S^2M, t),$$

inducing linearised first and second fundamental forms $(\tilde{h}, \tilde{m})$, such that $h|_\Sigma$ and $\nabla_\nu h|_\Sigma$ are as in Lemma 4.1.1 and

$$\begin{aligned} \Box_L h =& 0, \\ \nabla \cdot \overline{h} =& 0. \end{aligned}$$

In particular

$$D\mathrm{ric}(h) = 0.$$

Moreover

$$\mathrm{supp}(h) \subset J\left(\mathrm{supp}(\tilde{h}) \cup \mathrm{supp}(\tilde{m})\right). \tag{4.1}$$

From Section 2.1.4, we conclude that in fact $h \in H^{\lfloor k \rfloor}_{loc}(M)$.

Remark 4.1.3. The property (4.1) is called *finite speed of propagation.* If the initial data are compactly supported, the solution will have spatially compact support. Note however that (4.1) will not hold for all solutions with initial data $(\tilde{h}, \tilde{m})$. If for example $V \in C^\infty(M, TM)$ with support not intersecting Σ, then $h + \mathcal{L}_V g$ is going to be a solution with the same initial data. The support of $\mathcal{L}_V g$ needs not be contained in $J\left(\mathrm{supp}(\tilde{h}) \cup \mathrm{supp}(\tilde{m})\right)$.

Using Theorem 4.1.2, we get the following stability result.

Corollary 4.1.4 (Stable dependence on intial data)**.** *For $k \in \mathbb{R} \cup \{\infty\}$, assume that $(\tilde{h}_i, \tilde{m}_i)_{i\in\mathbb{N}} \in H^k_{loc}(\Sigma) \times H^{k-1}_{loc}(\Sigma)$ such that $D\Phi(\tilde{h}_i, \tilde{m}_i) = 0$ and*

$$(\tilde{h}_i, \tilde{m}_i) \to (\tilde{h}, \tilde{m}) \in H^k_{loc}(\Sigma) \times H^{k-1}_{loc}(\Sigma)$$

in $H^k_{loc}(\Sigma) \times H^{k-1}_{loc}(\Sigma)$. Then there exists a solution $h \in CH^k_{loc}(M, t)$ inducing initial data $(\tilde{h}, \tilde{m})$ and a sequence of solutions $h_i \in CH^k(M, t)$, inducing $(\tilde{h}_i, \tilde{m}_i)$ as initial data, such that

$$h_i \to h$$

in $CH^k_{loc}(M, t)$ and $\nabla \cdot \overline{h}_i = 0$.

Proof. Since $(\tilde{h}_i, \tilde{m}_i) \to (\tilde{h}, \tilde{m})$, the equations in Lemma 4.1.1 imply that $(h_i|_\Sigma, \nabla_\nu h_i|_\Sigma) \to (h|_\Sigma, \nabla_\nu h|_\Sigma)$. Since

$$\Box_L h = \Box_L h_i = 0,$$

we conclude by continuous dependence on initial data for linear wave equations, Corollary 2.2.10, that $h_i \to h$. □

It is important to note that given converging initial data, the previous corollary gives *one* sequence of converging solutions, inducing the correct initial data. Not every sequence of solutions, inducing the correct initial data, will converge. One could just add a gauge solution similar to Remark 4.1.3. This is the reason why the question of continuous dependence on initial data a priori does not make sense. This will be solved in Section 4.4, by considering *equivalence classes* of solutions. Let us now turn to the proof of the theorem.

Lemma 4.1.5. *If $h \in \mathcal{D}'(M, S^2M)$, then*

$$\nabla \cdot \overline{D\mathrm{ric}} = \nabla \cdot \left(D\mathrm{ric}(h) - \frac{1}{2}\mathrm{tr}_g(D\mathrm{ric}(h))g \right) = 0.$$

Proof. For any Lorentzian metric $\hat{g}$,

$$\hat{\nabla} \cdot \left(\mathrm{ric}_{\hat{g}} - \frac{1}{2}\mathrm{tr}_{\hat{g}}(\mathrm{ric}_{\hat{g}})\hat{g} \right) = 0,$$

where $\hat{\nabla}$ is the Levi-Civita connection with respect to $\hat{g}$. Linearising this equation around g, using $\mathrm{ric} = 0$, gives the equation for smooth h. Since the smooth sections are dense in the distributional sections, this proves the lemma. □

A calculation that will be very useful on many places in Part I is the following.

Lemma 4.1.6. *Assume that $(N, \hat{g})$ is a semi-Riemannian manifold with Levi-Civita connection $\hat{\nabla}$. Then*

$$\hat{\nabla} \cdot \overline{\mathcal{L}_V \hat{g}} = \hat{\nabla} \cdot \left(\mathcal{L}_V \hat{g} - \frac{1}{2}\mathrm{tr}_{\hat{g}}(\mathcal{L}_V \hat{g})\hat{g} \right) = -\hat{\nabla}^*\hat{\nabla}V^\flat + \mathrm{ric}_{\hat{g}}(V, \cdot). \tag{4.2}$$

Proof. Let $(e_1, \ldots, e_n)$ be a local orthonormal frame with respect to $\hat{g}$ and define $\epsilon_i := g(e_i, e_i) \in \{-1, 1\}$. We have

$$\begin{aligned} \hat{\nabla} \cdot \overline{\mathcal{L}_V \hat{g}}(X) &= \sum_{i=1}^n \epsilon_i \left(\hat{\nabla}_{e_i} \mathcal{L}_V \hat{g}(e_i, X) - \partial_X \hat{g}(\hat{\nabla}_{e_i} V, e_i) \right) \\ &= \sum_{i=1}^n \epsilon_i \left(\hat{g}(\hat{\nabla}^2_{e_i, e_i} V, X) + \hat{g}(\hat{\nabla}^2_{e_i, X} V, e_i) - \hat{g}(\hat{\nabla}^2_{X, e_i} V, e_i) \right) \\ &= -\hat{\nabla}^*\hat{\nabla}V^\flat(X) + \mathrm{ric}_{\hat{g}}(V, X). \end{aligned}$$

□

Proof of Theorem 4.1.2. Consider the Cauchy problem

$$\Box_L h = 0$$

with $h|_\Sigma$ and $\nabla_\nu h|_\Sigma$ defined as in Lemma 4.1.1, using $(\tilde{h}, \tilde{m})$. Note that $(h|_\Sigma, \nabla_\nu h|_\Sigma) \in H^k_{loc}(\Sigma, S^2M|_\Sigma) \times H^{k-1}_{loc}(\Sigma, S^2M|_\Sigma)$. Therefore, by Theorem 2.2.9 there is a unique solution

$h \in CH^k_{loc}(M, S^2M, t)$ to this Cauchy problem. Moreover, it follows that $\mathrm{supp}(h) \subset J(\mathrm{supp}(\tilde{h}) \cup \mathrm{supp}(\tilde{m}))$. We claim that $\nabla \cdot \overline{h} = 0$. Since $h \in CH^k_{loc}(M, t)$ it follows by Section 2.1.4 that $\nabla \cdot \overline{h} \in CH^{k-1}_{loc}(M, t)$. Lemma 4.1.5 implies that

$$\begin{aligned} 0 =& \nabla \cdot \left(D\mathrm{ric}(h) - \frac{1}{2}\mathrm{tr}_g(D\mathrm{ric}(h))g \right) \\ =& \frac{1}{2}\nabla \cdot \left(\mathcal{L}_{(\nabla\cdot\overline{h})^\sharp} g - \frac{1}{2}\mathrm{tr}_g \left(\mathcal{L}_{(\nabla\cdot\overline{h})^\sharp} g \right) g \right) \\ \overset{(4.2)}{=}& -\frac{1}{2}\nabla^*\nabla(\nabla \cdot \overline{h}), \end{aligned}$$

since ric $= 0$. Also from Lemma 4.1.1, we know that $\nabla \cdot \overline{h}|_\Sigma = 0$. Let us now use the assumption that $D\Phi(\tilde{h}, \tilde{m}) = 0$ to show that $\nabla_\nu(\nabla \cdot \overline{h})|_\Sigma = 0$. Since we know that $\Box_L h = 0$ and $\nabla \cdot \overline{h}|_\Sigma = 0$, equations (3.7) and(3.8) imply that

$$\begin{aligned} 0 &= D\Phi_1(\tilde{h}, \tilde{m}) \\ &= \mathrm{tr}_g(D\mathrm{ric}(h)) + 2D\mathrm{ric}(h)(\nu, \nu) \\ &= \frac{1}{2}\left(\mathrm{tr}_g(\mathcal{L}_{(\nabla\cdot\overline{h})^\sharp} g) + 2\mathcal{L}_{(\nabla\cdot\overline{h})^\sharp} g(\nu, \nu) \right) \\ &= \nabla_\nu(\nabla \cdot \overline{h})(\nu), \\ 0 &= D\Phi_2(\tilde{h}, \tilde{m})(X) \\ &= D\mathrm{ric}(h)(\nu, X) \\ &= \frac{1}{2}\nabla_\nu(\nabla \cdot \overline{h})(X), \end{aligned}$$

for each $X \in T\Sigma$. Altogether we have shown that $\nabla \cdot \overline{h} \in CH^{k-1}_{loc}(M, t)$ satisfies

$$\begin{aligned} \nabla^*\nabla(\nabla \cdot \overline{h}) &= 0, \\ \nabla \cdot \overline{h}|_\Sigma &= 0, \\ \nabla_\nu(\nabla \cdot \overline{h})|_\Sigma &= 0. \end{aligned}$$

Theorem 2.2.9 now implies that $\nabla \cdot \overline{h} = 0$. This finishes the proof. □

4.2. Uniqueness up to gauge

We continue by showing that the solution is unique up to addition of a gauge solution.

Theorem 4.2.1 (Uniqueness up to gauge). *Let $k \in \mathbb{R} \cup \{\infty\}$. Assume that $h \in CH^k_{loc}(M, S^2M, t)$ satisfies*

$$D\mathrm{ric}(h) = 0$$

and that the induced first and second linearised fundamental forms vanish. Then there exists a vector field $V \in CH^{k+1}_{loc}(M, TM, t)$ such that

$$h = \mathcal{L}_V g.$$

If $\operatorname{supp}(h) \subset J(K)$ *for some compact* $K \subset \Sigma$*, we can choose* V *such that* $\operatorname{supp}(V) \subset J(K)$*.*

We start by proving a technical lemma, that reminds of elliptic regularity theory. The difference is that we work with finite energy spaces and not Sobolev spaces.

Lemma 4.2.2. *Let* $V \in CH^k_{loc}(M, TM, t)$ *with* $\mathcal{L}_V g \in CH^k_{loc}(M, S^2M, t)$*. Then* $V \in CH^{k+1}_{loc}(M, TM, t)$*.*

Proof of Lemma 4.2.2. For each $\tau \in t(M)$, let $(\tilde{g}_\tau, \tilde{k}_\tau)$ be the induced first and second fundamental forms on Σ_τ. Let $V|_{\Sigma_\tau} =: V^\perp|_{\Sigma_\tau}\nu_\tau + V^\parallel|_{\Sigma_\tau}$ be the projection onto parallel and normal components with respect to Σ_τ, where ν_τ is the future pointing normal vector field along Σ_τ. Using this, we get a split $TM \cong \mathbb{R} \oplus T\Sigma.$. Note that

$$\mathcal{L}_{V^\parallel|_{\Sigma_\tau}} \tilde{g}_\tau(X, Y) = \mathcal{L}_V g|_{\Sigma_\tau}(X, Y) - 2V^\perp|_{\Sigma_\tau}\tilde{k}_\tau(X, Y) \in H^k_{loc}(\Sigma_\tau)$$

for all $X, Y \in T\Sigma_\tau$. Since

$$V^\parallel \mapsto \mathcal{L}_{V^\parallel}\tilde{g}_{(\cdot)} \in \operatorname{Diff}_1(T\Sigma., S^2\Sigma.)$$

is a differential operator of injective principal symbol, elliptic regularity theory implies that $V^\parallel \in C^0(t(M), H^{k+1}(\Sigma.))$. Let us generalise this by showing that $(\nabla^j_{t,\dots,t}V)^\parallel \in C^0(t(M), H^{k+1-j}_{loc}(\Sigma.))$ for all integers $j \geq 0$. We know already that $(\nabla^j_{t,\dots,t}V) \in C^0(t(M), H^{k-j}_{loc}(\Sigma.))$ for all integers $j \geq 0$. Moreover, we have

$$\begin{aligned}\mathcal{L}_{\nabla^j_{t,\dots,t}V} g(X, Y) &= g(\nabla_X \nabla^j_{t,\dots,t}V, Y) + g(\nabla_Y \nabla^j_{t,\dots,t}V, X)\\ &= (\nabla_t)^j \mathcal{L}_V g(X, Y) + P_j(V)(X, Y) \in CH^{k-j}_{loc}(M, t),\end{aligned}$$

since P_j is some differential operator of order j. By the argument above, it follows that $(\nabla^j_{t,\dots,t}V)^\parallel \in C^0(t(M), H^{k+1-j}_{loc}(\Sigma.))$ for all integer $j \geq 0$. Using this, we conclude that

$$\begin{aligned}&\partial_X((\nabla^j_{t,\dots,t}V)^\perp)\\ &\quad= g(\nabla_X \nabla^j_{t,\dots,t}V, \operatorname{grad}(t))\\ &\quad= \nabla^j_{t,\dots,t}\mathcal{L}_V(\operatorname{grad}(t), X) - g((\nabla^{j+1}_{t,\dots,t}V)^\parallel, X) + Q_j(V)(X) \in C^0_{loc}(t(M), H^{k-j}_{loc}(\Sigma.))\end{aligned}$$

for all $X \in T\Sigma.$, since $(\nabla^{j+1}_{t,\dots,t}V)^\parallel \in C^0_{loc}(t(M), H^{k-j}_{loc}(\Sigma.))$ and $Q_j(V)$ is some differential operator of order j. We conclude that

$$d((\nabla^j_{t,\dots,t}V)^\perp) \in C^0_{loc}(t(M), H^{k-j}_{loc}(\Sigma., T^*\Sigma.)).$$

Since $d \in \operatorname{Diff}_1(\Sigma. \times \mathbb{R}, T\Sigma.)$, mapping functions to one-forms on $\Sigma.$, has injective principal symbol on each leaf $\Sigma.$, we conclude that $(\nabla^j_{t,\dots,t}V)^\perp \in C^0(t(M), H^{k+1-j}_{loc}(\Sigma.))$ for all integer $j \geq 0$. We conclude that

$$\nabla^j_{t,\dots,t}V \in C^0(t(M), H^{k+1-j}_{loc}(\Sigma.))$$

for all integer $j \geq 0$, which is the same as $V \in CH^{k+1}_{loc}(M, TM, t)$. $\square$

The proof of the Theorem 4.2.1 is a generalisation of the proof of [16, Theorem 3.3] to solutions of low regularity.

Proof of Theorem 4.2.1. By Section 2.1.4, we know that $\nabla \cdot \overline{h} \in CH^{k-1}_{loc}(M, S^2M, t)$. By Theorem 2.2.9, we can define

$$V \in CH^k_{loc}(M, TM, t)$$

as the unique solution to

$$\begin{aligned} \nabla^*\nabla V =& - \nabla \cdot \overline{h}^\sharp, \\ V|_\Sigma =& 0, \\ \nabla_\nu V|_\Sigma =& \frac{1}{2} h(\nu, \nu)\nu + h(\nu, \cdot)^\sharp, \end{aligned} \tag{4.3}$$

where $\sharp : T^*M \to TM$ is the musical isomorphism with inverse $\flat : TM \to T^*M$. If $\mathrm{supp}(h) \subset J(K)$ for some subset $K \subset \Sigma$, then [7, Remark 16] implies that $\mathrm{supp}(V) \subset J(K)$. We have

$$\nabla \cdot \overline{\mathcal{L}_V g} = -\nabla^*\nabla V^\flat = \nabla \cdot \overline{h},$$

where $\overline{\mathcal{L}_V g} := \mathcal{L}_V g - \frac{1}{2}\mathrm{tr}_g (\mathcal{L}_V g)\, g$. Hence

$$\begin{aligned} 0 =& 2D\mathrm{ric}_g(h - \mathcal{L}_V g) \\ =& \Box_L(h - \mathcal{L}_V g) + \mathcal{L}_{\nabla\cdot(\overline{h} - \overline{\mathcal{L}_V g})^\sharp} g \\ =& \Box_L(h - \mathcal{L}_V g). \end{aligned}$$

Since $V \in CH^k_{loc}(M, TM, t)$, we know that $\mathcal{L}_V g \in CH^{k-1}_{loc}(M, S^2M, t)$, which implies that $h - \mathcal{L}_V g \in CH^{k-1}_{loc}(M, S^2M, t)$. Hence, if we knew that

$$(h - \mathcal{L}_V g)|_\Sigma = 0, \tag{4.4}$$

$$\nabla_\nu (h - \mathcal{L}_V g)|_\Sigma = 0, \tag{4.5}$$

then Theorem 2.2.9 would imply that $h - \mathcal{L}_V g = 0$ as asserted. We start by showing (4.4). Since $V|_\Sigma = 0$ and $\nabla_\nu V|_\Sigma = \frac{1}{2}h(\nu, \nu)\nu + h(\nu, \cdot)^\sharp$ and $\tilde{h} = 0$, we get for all $X, Y \in T\Sigma$,

$$\begin{aligned} h(X, Y) &= \tilde{h}(X, Y) \\ &= 0 \\ &= g(\nabla_X V, Y) + g(\nabla_Y V, X) \\ &= \mathcal{L}_V g(X, Y), \\ h(X, \nu) &= g(\nabla_\nu V, X) \\ &= g(\nabla_\nu V, X) + g(\nabla_X V, \nu) \\ &= \mathcal{L}_V g(\nu, X), \\ h(\nu, \nu) &= 2g(\nabla_\nu V, \nu) \\ &= \mathcal{L}_V g(\nu, \nu). \end{aligned}$$

We continue by showing (4.5). Since $\tilde{m} = 0$, we get (recall Definition 3.3.2)

$$\nabla_\nu h(X,Y) = h(\nu,\nu)\tilde{k}(X,Y) + \nabla_X h(\nu,Y) + \nabla_Y h(\nu,X).$$

Using $\tilde{h} = 0$, we get (recall Definition 3.3.2)

$$\begin{aligned}
\nabla_\nu \mathcal{L}_V g(X,Y) &= g(\nabla^2_{\nu,X}V,Y) + g(\nabla^2_{\nu,Y}V,X) \\
&= g(\nabla^2_{X,\nu}V,Y) + g(\nabla^2_{Y,\nu}V,X) + R(\nu,X,V,Y) + R(\nu,Y,V,X) \\
&= \partial_X g(\nabla_\nu V,Y) - g(\nabla_\nu V,\nabla_X Y) + \partial_Y g(\nabla_\nu V,X) - g(\nabla_\nu V,\nabla_Y X) \\
&= \partial_X h(\nu,Y) - h(\nu,\nabla_X Y) - \frac{1}{2}h(\nu,\nu)g(\nu,\nabla_X Y) \\
&\quad + \partial_Y h(\nu,X) - h(\nu,\nabla_Y X) - \frac{1}{2}h(\nu,\nu)g(\nu,\nabla_Y X) \\
&= \nabla_X h(\nu,Y) + \nabla_Y h(\nu,X) + h(\nu,\nu)\tilde{k}(X,Y) \\
&= \nabla_\nu h(X,Y),
\end{aligned}$$

since $\nabla_X\nu \in T\Sigma$ and therefore $\nabla_{\nabla_X\nu}V = 0$. What remains is to show that $\nabla_\nu(h - \mathcal{L}_V g)|_\Sigma(\nu,\cdot) = 0$. Recall that

$$\nabla \cdot \overline{\mathcal{L}_V g} = \nabla \cdot \overline{h},$$

which is equivalent to

$$\nabla \cdot \mathcal{L}_V g(W) - \frac{1}{2}\partial_W \mathrm{tr}_g(\mathcal{L}_V g) = \nabla \cdot h(W) - \frac{1}{2}\partial_W \mathrm{tr}_g(h), \tag{4.6}$$

for all $W \in TM$. Note that from what is shown above, we know that $\mathrm{tr}_g(\mathcal{L}_V g)|_\Sigma = \mathrm{tr}_g(h)|_\Sigma$. Therefore, for $X \in T\Sigma$, we have $\partial_X \mathrm{tr}_g(\mathcal{L}_V g) = \partial_X \mathrm{tr}_g(h)$, so

$$\nabla \cdot \mathcal{L}_V g(X) = \nabla \cdot h(X),$$

which simplifies to

$$\nabla_\nu \mathcal{L}_V g(X,\nu) = \nabla_\nu h(X,\nu).$$

Instead inserting ν into equation (4.6), gives

$$\begin{aligned}
0 =& \nabla \cdot \overline{\mathcal{L}_V g}(\nu) - \nabla \cdot \overline{h}(\nu) \\
=& \nabla \cdot (\mathcal{L}_V g - h)(\nu) - \frac{1}{2}\partial_\nu \left(\mathrm{tr}_g(\mathcal{L}_V g) - \mathrm{tr}_g(h)\right) \\
=& \nabla \cdot (\mathcal{L}_V g - h)(\nu) - \frac{1}{2}\mathrm{tr}_g(\nabla_\nu \left(\mathcal{L}_V g - h\right)) \\
=& - \nabla_\nu \left(\mathcal{L}_V g - h\right)(\nu,\nu) + \frac{1}{2}\nabla_\nu \left(\mathcal{L}_V g - h\right)(\nu,\nu) \\
=& - \frac{1}{2}\nabla_\nu \left(\mathcal{L}_V g - h\right)(\nu,\nu).
\end{aligned}$$

We conclude that

$$\nabla_\nu(h - \mathcal{L}_V g)(\nu,\nu) = 0.$$

This shows that $h = \mathcal{L}_V g$. Lemma 4.2.2 implies the regularity of V. □

4.3. Gauge producing initial data and gauge solutions

In this section, we consider the structure of the space of gauge solutions and gauge producing initial data. We consider from now on *compactly supported initial data* and *spatially compactly supported solutions*. The purpose is to show that the spaces

$$\text{Initial data on } \Sigma \big/ \text{Gauge producing i.d.} \quad \text{and} \quad \text{Global solutions on } M \big/ \text{Gauge solutions}$$

equipped with the quotient topology are topological vector spaces.

Definition 4.3.1. Define the *solutions* of smooth and finite energy regularity $k \in \mathbb{R}$ as

$$\mathcal{Sol}^{\infty}_{sc}(M) := C^{\infty}_{sc}(M, S^2M) \cap \ker(D\mathrm{ric}),$$
$$\mathcal{Sol}^{k}_{sc}(M,t) := CH^{k}_{sc}(M, S^2M, t) \cap \ker(D\mathrm{ric}),$$

with the induced topology.

Since Dric is a linear differential operator, it is continuous as an operator on distributions. Therefore, the solution spaces are closed subspaces and therefore topological vector spaces. Let us now define the subspace of gauge solutions.

Definition 4.3.2. Define the *gauge solutions* of smooth and finite energy regularity $k \in \mathbb{R}$ as

$$\mathcal{G}^{\infty}_{sc}(M) := \{\mathcal{L}_V g \mid V \in C^{\infty}_{sc}(M, TM)\} \subset \mathcal{Sol}^{\infty}_{sc}(M),$$
$$\mathcal{G}^{k}_{sc}(M,t) := \{\mathcal{L}_V g \mid V \in CH^{k+1}_{sc}(M, TM, t)\} \subset \mathcal{Sol}^{k}_{sc}(M,t),$$

with the induced topology.

We show later that the spaces of gauge solutions are closed subspaces of the solution spaces, which implies that the quotient spaces are topological vector spaces. Let us define space of solutions to the constraint equation.

Definition 4.3.3. Define the *initial data* of smooth and Sobolev regularity $k \in \mathbb{R}$ as

$$\mathcal{ID}^{\infty}_{c}(\Sigma) := \big(C^{\infty}_{c}(\Sigma, S^2\Sigma) \times C^{\infty}_{c}(\Sigma, S^2\Sigma)\big) \cap \ker(D\Phi),$$
$$\mathcal{ID}^{k,k-1}_{c}(\Sigma) := \big(H^{k}_{c}(\Sigma, S^2\Sigma) \times H^{k-1}_{c}(\Sigma, S^2\Sigma)\big) \cap \ker(D\Phi),$$

with the induced topology.

Let

$$\pi_\Sigma : \mathcal{Sol}^{\infty}_{sc}(M) \to \mathcal{ID}^{\infty}_{c}(\Sigma)$$
$$\pi_\Sigma : \mathcal{Sol}^{k}_{sc}(M,t) \to \mathcal{ID}^{k,k-1}_{c}(\Sigma)$$

be the map that assigns to a solution the induced initial data, i.e. linearised first and second fundamental forms. It follows from Definition 3.3.2 that π_Σ is continuous.

Definition 4.3.4. Define the *gauge producing initial data* of smooth and Sobolev regularity $k \in \mathbb{R}$ as

$$\mathcal{GP}_c^\infty(\Sigma) := \pi_\Sigma(\mathcal{G}_{sc}^\infty(M)) \subset \mathcal{ID}_c^\infty(\Sigma),$$
$$\mathcal{GP}_c^{k,k-1}(\Sigma) := \pi_\Sigma(\mathcal{G}_{sc}^k(M,t)) \subset \mathcal{ID}_c^{k,k-1}(\Sigma).$$

It will sometimes be necessary to consider only sections supported in a fixed compact set $K \subset \Sigma$ or $J(K) \subset M$, for example $\mathcal{ID}_K^\infty(\Sigma)$ or $\mathcal{Sol}_{J(K)}^\infty(M)$. The definitions in this case are analogous to Definitions 4.3.1, 4.3.2, 4.3.3 and 4.3.4.

Let us study the space of gauge producing initial data $\mathcal{GP}_c^\infty(\Sigma)$ and $\mathcal{GP}_c^{k,k-1}(\Sigma)$ in more detail. For $V \in CH_{sc}^{k+1}(M,TM,t)$, define $(N,\beta) \in H_c^{k+1}(\Sigma, \mathbb{R} \oplus T\Sigma)$ by projecting $V|_\Sigma$ to normal and tangential components, i.e. $V|_\Sigma =: N\nu + \beta$. Now define

$$\tilde{h}_{N,\beta} := \mathcal{L}_\beta \tilde{g} + 2\tilde{k}N, \tag{4.7}$$
$$\tilde{m}_{N,\beta} := \mathcal{L}_\beta \tilde{k} + \mathrm{Hess}(N) + \left(2\tilde{k} \circ \tilde{k} - \mathrm{ric}_{\tilde{g}} - (\mathrm{tr}_{\tilde{g}}\tilde{k})\tilde{k}\right) N. \tag{4.8}$$

We claim that $(\tilde{h}_{N,\beta}, \tilde{m}_{N,\beta}) = \pi_\Sigma(\mathcal{L}_V g)$. Indeed, for each $X, Y \in T\Sigma$, we have

$$\begin{aligned}
\tilde{h}_{N,\beta}(X,Y) &= \mathcal{L}_V g(X,Y) \\
&= g(\nabla_X(\beta + N\nu), Y) + g(\nabla_Y(\beta + N\nu), X) \\
&= \mathcal{L}_\beta \tilde{g}(X,Y) + 2N\tilde{k}(X,Y), \\
\tilde{m}_{N,\beta}(X,Y) &= -\frac{1}{2}\mathcal{L}_V g(\nu,\nu)\tilde{k}(X,Y) - \frac{1}{2}\nabla_X \mathcal{L}_V g(\nu, Y) \\
&\quad - \frac{1}{2}\nabla_Y \mathcal{L}_V g(\nu, X) + \frac{1}{2}\nabla_\nu \mathcal{L}_V g(X,Y) \\
&= -g(\nabla_\nu V, \nu)\tilde{k}(X,Y) - \frac{1}{2} g(\nabla^2_{X,Y} V + \nabla^2_{Y,X} V, \nu) \\
&\quad + \frac{1}{2} R(\nu, X, V, Y) + \frac{1}{2} R(\nu, Y, V, X).
\end{aligned}$$

Using that ric $= 0$ together with the Codazzi and Gauss equations, one calculates that the last line coincides with (4.8). In particular, $\mathcal{GP}_c^\infty(\Sigma)$ and $\mathcal{GP}_c^{k,k-1}(\Sigma)$ can be defined intrinsically on Σ by equations (4.7) and (4.8) and are therefore independent of the chosen temporal function t on M, as the notation implies. We have shown that

$$\mathcal{GP}_c^\infty(\Sigma) = \{(\tilde{h}_{N,\beta}, \tilde{m}_{N,\beta}) \text{ as in (4.7) and (4.8)} \mid (N,\beta) \in C_c^\infty(\Sigma, \mathbb{R} \oplus T\Sigma)\},$$
$$\mathcal{GP}_c^{k,k-1}(\Sigma) = \{(\tilde{h}_{N,\beta}, \tilde{m}_{N,\beta}) \text{ as in (4.7) and (4.8)} \mid (N,\beta) \in H_c^{k+1}(\Sigma, \mathbb{R} \oplus T\Sigma)\}.$$

We are now in shape to prove that the space of gauge producing initial data is a closed subspace of the space of initial data.

Lemma 4.3.5. *Let $k \in \mathbb{R}$. The spaces*

$$\mathcal{GP}_c^\infty(\Sigma) \subset \mathcal{ID}_c^\infty(\Sigma),$$
$$\mathcal{GP}_c^{k,k-1}(\Sigma) \subset \mathcal{ID}_c^{k,k-1}(\Sigma),$$

are closed subspaces. The statement still holds if we substitute c and sc with K and $J(K)$ respectively, for a fixed compact subset $K \subset \Sigma$.

Proof. Consider the linear differential operator given by

$$\begin{aligned} Q : H_c^{k+1}(\Sigma, \mathbb{R} \oplus T\Sigma) &\to H_c^k(\Sigma, S^2\Sigma) \times H_c^{k-1}(\Sigma, S^2\Sigma), \\ (N, \beta) &\mapsto (\tilde{h}_{N,\beta}, \tilde{m}_{N,\beta}). \end{aligned}$$

The lemma is proven if we can show that Q has closed image. First assume that Σ is compact. Define the operator

$$P := \begin{pmatrix} \sqrt{\tilde{\nabla}^*\tilde{\nabla} + 1} & 0 \\ 0 & 1 \end{pmatrix} \circ Q : H^{k+1}(\Sigma, \mathbb{R} \oplus T\Sigma) \to H^{k-1}(\Sigma, S^2\Sigma \oplus S^2\Sigma).$$

Note that

$$P^*P = Q^* \begin{pmatrix} \tilde{\nabla}^*\tilde{\nabla} + 1 & 0 \\ 0 & 1 \end{pmatrix} Q$$

is an elliptic differential operator of order 4. Lemma 2.2.3 implies that

$$H^{k-3}(\Sigma, \mathbb{R} \oplus T\Sigma) = \mathrm{im}(P^*P) \oplus \ker(P^*P).$$

We claim that

$$H^{k-1}(\Sigma, S^2\Sigma \oplus S^2\Sigma) = \mathrm{im}(P) \oplus \ker(P^*), \tag{4.9}$$

where $P^* : H^{k-1}(\Sigma, S^2\Sigma \oplus S^2\Sigma) \to H^{k-3}(\Sigma, \mathbb{R} \oplus T\Sigma)$. We do this by a standard method, found for example in [9, Appendix I]. We start with the case $k = 1$. Then $H^{k-1}(\Sigma) = L^2(\Sigma)$. Since $\overline{\mathrm{im}(P)}^\perp = \ker(P^*)$, we have

$$L^2(\Sigma) = \overline{\mathrm{im}(P)} \oplus \ker(P^*),$$

since $\ker(P^*)$ is closed. Hence it suffices to show that $\mathrm{im}(P) = \ker(P^*)^\perp$. Assume that $f \in L^2(\Sigma)$ such that $f \perp \ker(P^*)$. Since $P^*f \in H^{-2}(\Sigma) \cong H^2(\Sigma)'$ annihilates any element in $\ker(P) = \ker(P^*P)$, Lemma 2.2.3 implies that $P^*f \in \mathrm{im}(P^*P)$, i.e. there exists a $u \in H^2(\Sigma)$ such that $P^*Pu = P^*f$. This implies that $Pu - f \in \ker(P^*)$. Since $Pu - f \perp \ker(P^*)$, we conclude that $Pu = f$ and hence $f \in \mathrm{im}(P)$. We have thus proven (4.9) for $k = 1$. Let us now prove (4.9) for all $k \geq 1$. Given $v \in H^{k-1}(\Sigma)$, we know that there is a $u \in H^2(\Sigma)$ and an $w \in L^2(\Sigma)$ such that $P^*w = 0$ and $v = Pu + w$. It follows that $P^*Pu = Pv \in H^{k-3}(\Sigma)$, so elliptic regularity theory implies that $u \in H^{k+1}(\Sigma)$. This implies that $w = v - Pu \in H^{k-1}(\Sigma)$, so we conclude equation (4.9) for $k \geq 1$. By the argument given in the proof of Lemma 2.2.3, we conclude equation (4.9) for all $k \in \mathbb{R}$. Hence $\mathrm{im}(P)$ is closed for any $k \in \mathbb{R}$. Since

$$\begin{pmatrix} \sqrt{\tilde{\nabla}^*\tilde{\nabla} + 1} & 0 \\ 0 & 1 \end{pmatrix} : H^k(\Sigma, S^2\Sigma) \times H^{k-1}(\Sigma, S^2\Sigma) \to H^{k-1}(\Sigma, S^2\Sigma \oplus S^2\Sigma)$$

is an isomorphism, it follows that $\mathrm{im}(Q)$ is closed for any $k \in \mathbb{R}$. This proves the statement if Σ is closed.

Assume now instead that Σ is non-compact. We need to show that for each compact subset $K \subset \Sigma$, $\mathrm{im}(Q) \cap H^k_K(\Sigma) \times H^{k-1}_K(\Sigma) \subset H^k_K(\Sigma) \times H^{k-1}_K(\Sigma)$ is closed. For a fixed compact subset $K \subset \Sigma$, let us construct a set $L \subset \Sigma$, containing K, such that if $\mathrm{supp}(Q(N,\beta)) \subset L$ and $\mathrm{supp}(N,\beta)$ is compact, then $\mathrm{supp}(N,\beta) \subset L$. We construct L as follows. Since K is compact, ∂K is compact, which implies that $M \backslash \mathring{K}$ has a finite amount of connected components. Define L to be the union of K with all *compact* connected components of $M \backslash \mathring{K}$. It follows that L is compact, $K \subset L$ and that all components of $M \backslash \mathring{L}$ are non-compact. Let us show that L has the desired properties. One calculates that the differential operator P, defined by

$$\begin{aligned} P(N,\beta) &:= \begin{pmatrix} -\tilde{\nabla} \cdot (\cdot) + \frac{1}{2} d\mathrm{tr}(\cdot) & 0 \\ 0 & -\mathrm{tr}(\cdot) \end{pmatrix} Q(N,\beta) \\ &= \begin{pmatrix} \tilde{\nabla}^* \tilde{\nabla} \beta^\flat + l.o.t. \\ \tilde{\nabla}^* \tilde{\nabla} N + l.o.t. \end{pmatrix} \in H^{k-1}_K(\Sigma, T^*\Sigma \oplus \mathbb{R}) \end{aligned}$$

is a Laplace type operator. If $\mathrm{supp}(Q(N,\beta)) \subset L$, and $\mathrm{supp}(N,\beta)$ is compact, it follows that $\mathrm{supp}(P(N,\beta)) \subset L$ and that $(M \backslash \mathring{L}) \cap \mathrm{supp}(N,\beta) = \mathrm{supp}(N,\beta) \backslash (\mathrm{supp}(N,\beta) \cap \mathring{L})$ is compact. Since each component of $M \backslash \mathring{L}$ was non-compact, Theorem 2.2.6 now implies that $\mathrm{supp}(N,\beta) \subset L$ as claimed. Now if $Q(N_n,\beta_n) \to (\tilde{h}, \tilde{m})$ in $H^k_K(\Sigma) \times H^{k-1}_K(\Sigma)$, then $\mathrm{supp}(N_n,\beta_n) \subset L$ and

$$P(N_n,\beta_n) \to \begin{pmatrix} -\nabla \cdot (\tilde{h}) + \frac{1}{2} d\mathrm{tr}(\tilde{h}) \\ -\mathrm{tr}(\tilde{m}) \end{pmatrix}$$

in $H^{k-1}_K(\Sigma) \times H^{k-1}_K(\Sigma)$. By Corollary 2.2.7, we conclude that

$$P : H^{k+1}_L(\Sigma) \to H^{k-1}_L(\Sigma)$$

is an isomorphism onto its image and therefore there is a $(N,\beta) \in H^{k+1}_L(\Sigma)$ such that $(N_n,\beta_n) \to (N,\beta)$. We conclude that

$$(\tilde{h}, \tilde{m}) = \lim_{n \to \infty} Q(N_n,\beta_n) = Q(N,\beta),$$

which finishes the proof. □

For later use, we need the following technical observation.

Lemma 4.3.6. *For $k \in \mathbb{R}$,*

$$\begin{aligned} \mathcal{G}^\infty_{sc}(M) &= {\pi_\Sigma}^{-1}(\mathcal{GP}^\infty_c(\Sigma)), \\ \mathcal{G}^k_{sc}(M,t) &= {\pi_\Sigma}^{-1}(\mathcal{GP}^{k,k-1}_c(\Sigma)). \end{aligned}$$

In particular,

$$\begin{aligned} \mathcal{G}^\infty_{sc}(M) &\subset \mathcal{Sol}^\infty_{sc}(M), \\ \mathcal{G}^k_{sc}(M,t) &\subset \mathcal{Sol}^k_{sc}(M,t), \end{aligned}$$

are closed subspaces. The statement still holds if we substitute c with K and sc with J(K), for a fixed compact subset $K \subset \Sigma$.

Proof. Assume that $h \in \mathcal{Sol}^k_{sc}(M,t)$ and that

$$\pi_\Sigma(h) = \pi_\Sigma(\mathcal{L}_V g)$$

for some $\mathcal{L}_V g \in \mathcal{G}^k_{sc}(M,t)$. Then $D\mathrm{ric}(h - \mathcal{L}_V g) = 0$ and $\pi_\Sigma(h - \mathcal{L}_V g) = 0$. By Theorem 4.2.1, there is a $W \in CH^{k+1}_{sc}(M, TM, t)$, such that

$$h = \mathcal{L}_V g + \mathcal{L}_W g = \mathcal{L}_{V+W} g$$

which proves the statement. The smooth case is analogous. Since π_Σ is continuous, Lemma 4.3.5 implies the second statement. □

The next lemmas give a natural way to understand the topology of the quotient spaces.

Lemma 4.3.7. *Let $k \in \mathbb{R}$. The topological vector spaces*

$$\mathcal{ID}^\infty_c(\Sigma) \Big/ \mathcal{GP}^\infty_c(\Sigma),$$
$$\mathcal{ID}^{k,k-1}_c \Big/ \mathcal{GP}^{k,k-1}_c(\Sigma),$$

are the strict inductive limits of the topological vector spaces

$$\mathcal{ID}^\infty_K(\Sigma) \Big/ \mathcal{GP}^\infty_K(\Sigma),$$
$$\mathcal{ID}^{k,k-1}_K(\Sigma) \Big/ \mathcal{GP}^{k,k-1}_K(\Sigma),$$

for compact subsets $K \subset \Sigma$, with respect to the natural inclusions.

Proof. Let us simplify notation by writing $\mathcal{ID}_K := \mathcal{ID}^{k,k-1}_K(\Sigma)$, $\mathcal{ID}_c := \mathcal{ID}^{k,k-1}_K(\Sigma)$, $\mathcal{GP}_K := \mathcal{GP}^{k,k-1}_K(\Sigma)$ and $\mathcal{GP}_K := \mathcal{GP}^{k,k-1}_K(\Sigma)$. The proof follows a standard argument.

First note that the continuous maps $\mathcal{ID}_K \hookrightarrow \mathcal{ID}_c$ extend to continuous maps

$$\mathcal{ID}_K \Big/ \mathcal{GP}_K \hookrightarrow \mathcal{ID}_c \Big/ \mathcal{GP}_c,$$

for any compact $K \subset \Sigma$. It follows that all convex open neighbourhoods of 0 in $\mathcal{ID}_c \Big/ \mathcal{GP}_c$ are open convex neighbourhoods of 0 in $\mathcal{ID}_K \Big/ \mathcal{GP}_K$. What remains to show is the converse statement. For this, assume that

$$U + \mathcal{GP}_c \subset \mathcal{ID}_c \Big/ \mathcal{GP}_c$$

is a convex neighbourhood of 0 such that

$$(U + \mathcal{GP}_c) \cap \mathcal{ID}_K \Big/ \mathcal{GP}_K \subset \mathcal{ID}_K \Big/ \mathcal{GP}_K$$

is open for all compact subsets $K \subset \Sigma$. Note that

$$\begin{aligned}\big(U + \mathcal{GP}_c\big) \cap \mathcal{ID}_K \Big/ \mathcal{GP}_K \\ &= \big(U + \mathcal{GP}_c\big) \cap \mathcal{ID}_K + \mathcal{GP}_K \\ &= \big(U + \mathcal{GP}_c\big) \cap \mathcal{ID}_K,\end{aligned}$$

where we in the last line have used that $0 \in U$. We conclude that

$$\big(U + \mathcal{GP}_c\big) \cap \mathcal{ID}_K \subset \mathcal{ID}_K,$$

is open as a set, for all compact subsets $K \subset \Sigma$. By (a slight modification of) Lemma 2.1.6, $\mathcal{ID}_c^{k,k-1} = \ker(D\Phi) \cap (H_c^k \times H_c^{k-1})$ is the strict inductive limit of $\mathcal{ID}_K^{k,k-1} = \ker(D\Phi) \cap (H_K^k \times H_K^{k-1})$, it follows that

$$U + \mathcal{GP}_c \subset \mathcal{ID}_c$$

is open as a set. But this means that

$$U + \mathcal{GP}_c \subset \mathcal{ID}_c \Big/ \mathcal{GP}_c$$

is open as a set of equivalence classes. This finishes the proof. □

Lemma 4.3.8. *Let $k \in \mathbb{R}$. The topological vector spaces*

$$\begin{aligned}&\mathcal{Sol}_{sc}^{\infty}(M) \Big/ \mathcal{G}_{sc}^{\infty}(M),\\ &\mathcal{Sol}_{sc}^{k}(M,t) \Big/ \mathcal{G}_{sc}^{k}(M,t),\end{aligned}$$

are the strict inductive limits of the topological vector spaces

$$\begin{aligned}&\mathcal{Sol}_{J(K)}^{\infty}(M) \Big/ \mathcal{G}_{J(K)}^{\infty}(M),\\ &\mathcal{Sol}_{J(K)}^{k}(M,t) \Big/ \mathcal{G}_{J(K)}^{k}(M,t),\end{aligned}$$

for compact subsets $K \subset \Sigma$, with respect to the natural inclusions.

Proof. The proof is analogous to the proof of Lemma 4.3.7, using Lemma 2.1.8 instead of Lemma 2.1.6. □

4.4. Continuous dependence on initial data

Let us now state and prove the main result of this chapter, the well-posedness of the Cauchy problem of the linearised Einstein equation.

Recall Section 4.3 for the definitions of the function spaces below.

Theorem 4.4.1 (Wellposedness of the Cauchy problem)**.** *Let $k \in \mathbb{R}$. The linear solution maps*

$$\mathrm{Solve}^{\infty} : \mathcal{ID}^{\infty}_{c}(\Sigma) \Big/ \mathcal{GP}^{\infty}_{c}(\Sigma) \to \mathcal{Sol}^{\infty}_{sc}(M) \Big/ \mathcal{G}^{\infty}_{sc}(M)$$
$$\mathrm{Solve}^{k} : \mathcal{ID}^{k,k-1}_{c}(\Sigma) \Big/ \mathcal{GP}^{k,k-1}_{c}(\Sigma) \to \mathcal{Sol}^{k}_{sc}(M,t) \Big/ \mathcal{G}^{k}_{sc}(M,t)$$

are isomorphisms of topological vector spaces.

The theorem implies in particular global existence of solution, that the equivalence class of solutions is unique and that the solution depends continuously on equivalence classes of initial data. Since projection maps are continuous and surjective, we immediately get the following corollary.

Corollary 4.4.2 (Continuous dependence on initial data)**.** *Let $k \in \mathbb{R}$. The linear solution maps*

$$\widetilde{\mathrm{Solve}}^{\infty} : \mathcal{ID}^{\infty}_{c}(\Sigma) \to \mathcal{Sol}^{\infty}_{sc}(M) \Big/ \mathcal{G}^{\infty}_{sc}(M)$$
$$\widetilde{\mathrm{Solve}}^{\mathrm{k}} : \mathcal{ID}^{k,k-1}_{c}(\Sigma) \to \mathcal{Sol}^{k}_{sc}(M,t) \Big/ \mathcal{G}^{k}_{sc}(M,t)$$

are continuous and surjective.

Before proving the theorem, let us discuss some more remarks and corollaries.

Remark 4.4.3 (Distributional initial data)**.** Since any compactly supported distribution is of some real Sobolev regularity, any compactly supported distributional section lies in some $H^{k}_{c}(\Sigma)$. Hence Theorem 4.4.1 covers the case of any compactly supported distributional initial data.

A priori, the solution spaces depend on the time function. After quoting out the gauge solutions, this is not anymore the case.

Corollary 4.4.4 (Independence of the Cauchy temporal function)**.** *Let t and τ be Cauchy temporal functions on M. Then for every $k \in \mathbb{R}$ there is an isomorphism*

$$\mathcal{Sol}^{k}_{sc}(M,t) \Big/ \mathcal{G}^{k}_{sc}(M,t) \to \mathcal{Sol}^{k}_{sc}(M,\tau) \Big/ \mathcal{G}^{k}_{sc}(M,\tau)$$

which is the identity on smooth solutions.

Proof. The proof is analogous to the proof of [7, Corollary 18], using Theorem 4.4.1. □

As a final observation, let us note that if Σ is compact, we obtain a natural Hilbert space structure on the solution space.

Corollary 4.4.5 (Hilbert space structure on the phase space)**.** *Let $k \in \mathbb{R}$. In case Σ is compact, Theorem 4.4.1 implies that*

$$\mathcal{Sol}^k(M,t) \Big/ \mathcal{G}^k(M,t)$$

carries a Hilbert space structure, induced by Solve^k.

Proof. Since Σ is compact, the space $\mathcal{ID}^{k,k-1}(\Sigma) \Big/ \mathcal{GP}^{k,k-1}(\Sigma)$ gets a natural Hilbert space structure induced by the Sobolev norms. Theorem 4.4.1 and Corollary 4.4.4 imply therefore that we get an induced Hilbert space structure on $\mathcal{Sol}^k(M,t) \Big/ \mathcal{G}^k(M,t)$ which is independent of the choice of Cauchy hypersurface Σ. □

Let us turn to the proof of Theorem 4.4.1.

Lemma 4.4.6. *Let $k \in \mathbb{R}$ and fix a compact subset $K \subset \Sigma$. The linear maps*

$$\begin{aligned}
\mathrm{Solve}^\infty_K &: \mathcal{ID}^\infty_K(\Sigma) \Big/ \mathcal{GP}^\infty_K(\Sigma) \to \mathcal{Sol}^\infty_{J(K)}(M)) \Big/ \mathcal{G}^\infty_{J(K)}(M) \\
\mathrm{Solve}^k_K &: \mathcal{ID}^{k,k-1}_K(\Sigma) \Big/ \mathcal{GP}^{k,k-1}_K(\Sigma) \to \mathcal{Sol}^k_{J(K)}(M,t) \Big/ \mathcal{G}^k_{J(K)}(M,t)
\end{aligned}$$

are isomorphisms of topological vector spaces.

Proof. Lemma 4.3.5 and Lemma 4.3.6 imply that the quotient spaces are well defined *Fréchet spaces*. By Theorem 4.1.2, Theorem 4.2.1 and Lemma 4.3.6, the map Solve^k_K is a well defined linear bijection. We prove that it indeed is an isomorphism of topological vector spaces. Recall that the map that assigns to each solution its initial data

$$\pi_\Sigma : \mathcal{Sol}^k_{J(K)}(M,t) \to \mathcal{ID}^{k,k-1}_K(\Sigma).$$

is continuous. By definition of the quotient space topology, π_Σ induces a continuous map

$$\hat{\pi}_\Sigma : \mathcal{Sol}^k_{J(K)}(M,t) \Big/ \mathcal{G}^k_{J(K)}(M,t) \to \mathcal{ID}^{k,k-1}_K(\Sigma) \Big/ \mathcal{GP}^{k,k-1}_K(\Sigma)$$

between Fréchet spaces. Since $\hat{\pi}_\Sigma$ is the inverse of Solve^k_K, the open mapping theorem for Fréchet spaces implies the statement. The smooth case is analogous. □

Proof of Theorem 4.4.1. Again, Lemma 4.3.5 and Lemma 4.3.6 imply that the quotient spaces are well defined topological vector spaces. By Theorem 4.1.2, Theorem 4.2.1 and Lemma 4.3.6, the map Solve^k is a well defined linear bijection. Therefore it remains to prove that it is an isomorphism of topological vector spaces. By Lemma 4.4.6, the map

$$\begin{aligned}
\mathcal{ID}^{k,k-1}_K(\Sigma) \Big/ \mathcal{GP}^{k,k-1}_K(\Sigma) &\xrightarrow{\mathrm{Solve}^k_K} \mathcal{Sol}^k_{J(K)}(M,t) \Big/ \mathcal{G}^k_{J(K)}(M,t) \\
&\hookrightarrow \mathcal{Sol}^k_{sc}(M,t) \Big/ \mathcal{G}^k_{sc}(M,t)
\end{aligned}$$

is continuous, for every compact subset $K \subset \Sigma$. By Lemma 4.3.7, this implies that Solve^k is continuous. Similarly, by Lemma 4.4.6, the composed map

$$\mathcal{Sol}^k_{J(K)}(M,t) \Big/ \mathcal{G}^k_{J(K)}(M,t) \xrightarrow{\hat{\pi}_\Sigma} \mathcal{ID}^{k,k-1}_K(\Sigma) \Big/ \mathcal{GP}^{k,k-1}_K(\Sigma) \hookrightarrow \mathcal{ID}^{k,k-1}_c(\Sigma) \Big/ \mathcal{GP}^{k,k-1}_c(\Sigma)$$

is continuous, for every compact subset $K \subset \Sigma$. By Lemma 4.3.8, this implies that $\left(\mathrm{Solve}^k\right)^{-1}$ is continuous. The smooth case is analogous. □

5. The linearised constraint equations

As we have seen in the previous chapter, (equivalence classes of) solutions of the linearised constraint equations correspond to (equivalence classes of) solutions to the linearised Einstein equation. The goal of the following chapter is to understand the quotient

$$\mathcal{Sol}^k_{sc}(M,t) \Big/ \mathcal{G}^k_{sc}(M,t)\,,$$

at least in certain cases. By Theorem 4.4.1 it suffices to understand

$$\mathcal{ID}^{k,k-1}(\Sigma) \Big/ \mathcal{GP}^{k,k-1}(\Sigma)\,.$$

5.1. The case of vanishing second fundamental form

A closed Riemannian manifold $(\Sigma, \tilde{g})$ such that $\mathrm{scal}_{\tilde{g}} = 0$ is a solution of the (non-linear) constraint equations with $\tilde{k} = 0$. Assume in this section that Σ is closed and $\mathrm{scal}_{\tilde{g}} = 0$. Recall that for any $k \in \mathbb{R} \cup \{\infty\}$, $(\tilde{h}, \tilde{m}) \in \mathcal{ID}^{k,k-1}(\Sigma)$ if and only if

$$\begin{aligned}\tilde{\nabla} \cdot (\tilde{\nabla} \cdot \tilde{h} - d\mathrm{tr}_{\tilde{g}}\tilde{h}) - \tilde{g}(\mathrm{ric}_{\tilde{g}}, \tilde{h}) &= 0,\\ \tilde{\nabla} \cdot (\tilde{m} - (\mathrm{tr}_{\tilde{g}}\tilde{m})\tilde{g}) &= 0.\end{aligned}$$

The gauge producing initial data $\mathcal{GP}^{k,k-1}(\Sigma)$ are in this case given by the image of

$$\begin{aligned}P : H^{k+1}(\Sigma, T\Sigma \oplus \mathbb{R}) &\to H^k(\Sigma, S^2\Sigma) \times H^{k-1}(\Sigma, S^2\Sigma),\\ (\beta, N) &\mapsto (\mathcal{L}_\beta \tilde{g}, \mathrm{Hess}(N) - \mathrm{ric}_{\tilde{g}} N).\end{aligned}$$

The formal adjoint of P is given by

$$P^*(\tilde{h}, \tilde{m}) = (-2\tilde{\nabla} \cdot \tilde{h}, \tilde{\nabla} \cdot \tilde{\nabla} \cdot \tilde{m} - \tilde{g}(\mathrm{ric}_{\tilde{g}}, \tilde{m})).$$

Define now

$$\Gamma^{k,k-1}(\Sigma) := \mathcal{ID}^{k,k-1}(\Sigma) \cap \ker(P^*).$$

We first prove a special case of the classical Moncrief's splitting theorem, see [24], generalised to arbitrary regularity. Our argument is somewhat simpler, making use of the assumption $\tilde{k} = 0$.

Proposition 5.1.1. *Assume that Σ is closed. Let $k \in \mathbb{R} \cup \{\infty\}$. The map*

$$\begin{aligned}\Gamma^{k,k-1}(\Sigma) &\to \mathcal{ID}^{k,k-1}(\Sigma) \Big/ \mathcal{GP}^{k,k-1}(\Sigma),\\ (\tilde{h}, \tilde{m}) &\mapsto [(\tilde{h}, \tilde{m})],\end{aligned}$$

is an isomorphism of topological vector spaces.

Proof. Recall by Lemma 4.3.5, that we know that $\mathrm{im}(P) = \mathcal{GP}^{k,k-1}(\Sigma) \subset \mathcal{ID}^{k,k-1}(\Sigma)$ is closed. We claim that

$$H^k(\Sigma, S^2\Sigma) \oplus H^{k-1}(\Sigma, S^2\Sigma) = \mathrm{im}(P) \oplus \ker(P^*). \tag{5.1}$$

We first prove this when $k \leq 0$. Define

$$\begin{aligned}P_0 : H^1(\Sigma, T\Sigma) \times H^2(\Sigma, \mathbb{R}) &\to L^2(\Sigma, S^2\Sigma \oplus S^2\Sigma),\\ (\beta, N) &\mapsto P(\beta, N).\end{aligned}$$

It follows, when $k \leq 0$,

$$\begin{aligned}L^2(\Sigma, S^2\Sigma \oplus S^2\Sigma) &= \overline{\mathrm{im}(P_0)} \oplus \ker(P_0^*)\\ &\subset \mathrm{im}(P) \oplus \ker(P^*)\\ &\subset H^k(\Sigma, S^2\Sigma) \oplus H^{k-1}(\Sigma, S^2\Sigma).\end{aligned}$$

Since $\mathrm{im}(P) \oplus \ker(P^*) \subset H^k(\Sigma, S^2\Sigma) \oplus H^{k-1}(\Sigma, S^2\Sigma)$ is closed and $L^2(\Sigma, S^2\Sigma \oplus S^2\Sigma) \subset H^k(\Sigma, S^2\Sigma) \oplus H^{k-1}(\Sigma, S^2\Sigma)$ is dense, the claim follows when $k \leq 0$. Assume now that $k > 0$ and that $(\tilde{h}, \tilde{m}) \in H^k(\Sigma) \times H^{k-1}(\Sigma)$. Since we know equation (5.1) when $k = 0$, we conclude that there is $(N, \beta) \in H^1(\Sigma)$ and $(\tilde{h}_0, \tilde{m}_0) \in L^2(\Sigma) \times H^{-1}(\Sigma)$ such that $P^*(\tilde{h}_0, \tilde{m}_0) = 0$ and

$$(\tilde{h}, \tilde{m}) = P(N, \beta) + (\tilde{h}_0, \tilde{m}_0).$$

It follows that $P^*P(N, \beta) = P^*(\tilde{h}, \tilde{m}) \in H^{k-1}(\Sigma) \times H^{k-3}(\Sigma)$. Note that

$$\begin{pmatrix}\tilde{\nabla}^*\tilde{\nabla} & 0\\ 0 & 1\end{pmatrix} \circ P^*P : H^{k+1}(\Sigma) \to H^{k-3}(\Sigma)$$

is an elliptic differential operator. It follows that $(N, \beta) \in H^{k+1}(\Sigma)$ and hence $(\tilde{h}_0, \tilde{m}_0) = (\tilde{h}, \tilde{m}) - P(N, \beta) \in H^k(\Sigma) \times H^{k-1}(\Sigma)$. This proves the claim for $k > 0$. Since $\mathrm{im}(P) = \mathcal{GP}^{k,k-1}(\Sigma) \subset \mathcal{ID}^{k,k-1}(\Sigma)$, it follows now that

$$\begin{aligned}\mathcal{ID}^{k,k-1}(\Sigma) &= \mathcal{GP}^{k,k-1}(\Sigma) \oplus \left(\mathcal{ID}^{k,k-1}(\Sigma) \cap \ker(P^*)\right)\\ &= \mathcal{GP}^{k,k-1}(\Sigma) \oplus \Gamma^{k,k-1}(\Sigma)\end{aligned}$$

which concludes the proof. □

By combining the above equations, note that $(\tilde{h}, \tilde{m}) \in \Gamma^{k,k-1}(\Sigma)$ if and only if

$$\tilde{\nabla}^* \tilde{\nabla} \mathrm{tr}_{\tilde{g}} \tilde{h} = \tilde{g}(\mathrm{ric}_{\tilde{g}}, \tilde{h}), \tag{5.2}$$

$$\tilde{\nabla} \cdot \tilde{h} = 0. \tag{5.3}$$

$$\tilde{\nabla}^* \tilde{\nabla} \mathrm{tr}_{\tilde{g}} \tilde{m} = -\tilde{g}(\mathrm{ric}_{\tilde{g}}.\tilde{m}), \tag{5.4}$$

$$\tilde{\nabla} \cdot (\tilde{m} - (\mathrm{tr}_{\tilde{g}} \tilde{m})\tilde{g}) = 0. \tag{5.5}$$

In particular, the equations decouple. That is not the case for general $\tilde{k}$. Let

$$L\omega := \mathcal{L}_{\omega^\sharp} g - \frac{2}{\dim(\Sigma)} \tilde{\nabla} \cdot \omega$$

denote the conformal Killing operator on one-forms.

Theorem 5.1.2. *Let $k \in \mathbb{R} \cup \{\infty\}$. Assume that $(\Sigma, \tilde{g})$ is a closed Riemannian manifold of dimension $n \geq 2$ with $\mathrm{scal}_{\tilde{g}} = 0$ and let $\tilde{k} = 0$. Then for each $(\alpha, \beta) \in H^k(\Sigma, S^2\Sigma) \times H^{k-1}(\Sigma, S^2\Sigma)$, there is a unique decomposition*

$$\alpha = \tilde{h} + L\omega + C\mathrm{ric}_{\tilde{g}} + \phi\tilde{g},$$
$$\beta = \tilde{m} + L\eta + C'\mathrm{ric}_{\tilde{g}} + \psi\tilde{g},$$

where $(\tilde{h}, \tilde{m}) \in \Gamma^{k,k-1}(\Sigma)$, $(\omega, \eta) \in H^{k+1}(\Sigma, T^*\Sigma) \times H^k(\Sigma, T^*\Sigma)$, $(C, C') \in \mathbb{R}^2$ and $(\phi, \psi) \in H^k(\Sigma, \mathbb{R}) \times H^{k-1}(\Sigma, \mathbb{R})$ such that $\phi[1] = 0 = \psi[1]$, where 1 is the function with value 1.

Let $\Gamma_1^k(\Sigma)$ denote the solutions to (5.2) and (5.3) and let $\Gamma_2^{k-1}(\Sigma)$ denote the solutions to (5.4) and (5.5). We have

$$\Gamma^{k,k-1}(\Sigma) = \Gamma_1^k(\Sigma) \times \Gamma_2^{k-1}(\Sigma).$$

Remark 5.1.3. Note that Theorem 5.1.2 is equivalent to showing that

$$H^k(\Sigma, S^2\Sigma) = \Gamma_1^k(\Sigma) \oplus \mathrm{im}(L) \oplus \mathbb{R}\mathrm{ric}_{\tilde{g}} \oplus \widehat{H}^k(\Sigma, \mathbb{R})\tilde{g},$$
$$H^k(\Sigma, S^2\Sigma) = \Gamma_2^k(\Sigma) \oplus \mathrm{im}(L) \oplus \mathbb{R}\mathrm{ric}_{\tilde{g}} \oplus \widehat{H}^k(\Sigma, \mathbb{R})\tilde{g},$$

where $L : H^{k+1}(\Sigma, T^*\Sigma) \to H^k(\Sigma, S^2\Sigma)$ and $\widehat{H}^k(\Sigma, \mathbb{R}) := \{\phi \in H^k(\Sigma, \mathbb{R}) \mid \phi[1] = 0\}$.

Remark 5.1.4. Recall the classical L^2-spliting

$$H^k(\Sigma, S^2\Sigma) = \left(\ker(\tilde{\nabla}\cdot) \cap \ker(\mathrm{tr}_{\tilde{g}})\right) \oplus \mathrm{im}(L) \oplus H^k(\Sigma, \mathbb{R})\tilde{g}.$$

In case $\mathrm{ric}_{\tilde{g}} = 0$, we have

$$\Gamma_1^k(\Sigma) = \mathbb{R}\tilde{g} \oplus \left(\ker(\tilde{\nabla}\cdot) \cap \ker(\mathrm{tr}_{\tilde{g}})\right)$$
$$\Gamma_2^{k-1}(\Sigma) = \mathbb{R}\tilde{g} \oplus \left(\ker(\tilde{\nabla}\cdot) \cap \ker(\mathrm{tr}_{\tilde{g}})\right).$$

The space $\ker(\tilde{\nabla}\cdot) \cap \ker(\mathrm{tr}_{\tilde{g}})$ is called the space of *transverse traceless-tensors* or *TT-tensors*. Hence Theorem 5.1.2 can be seen as a generalisation of the above splitting.

Example 5.1.5. Let us give two examples of closed Riemannian manifolds with vanishing scalar curvature.

- For each $n \in \mathbb{N}$, the flat torus $T^n := \mathbb{R}^n/\mathbb{Z}^n$ is flat, in particular $\mathrm{scal}_{\tilde{g}} = 0$.
- For each $n \in \mathbb{N}$, there is a Berger metric on S^{4n-1} with vanishing scalar curvature. In case $n = 1$, the scalar flat Berger metric is given by $\frac{5}{2}{\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2$, where $\sigma_1, \sigma_2, \sigma_3$ are orthonormal left invariant one-forms on S^3. Note that this metric does not have vanishing Ricci curvature.

On these manifolds, Theorem 5.1.2 applies.

Let us turn to the proof of Theorem 5.1.2. Note that $\tilde{h} = \alpha - L\omega - C\mathrm{ric}_{\tilde{g}} - \phi\tilde{g} \in \Gamma_1^k(\Sigma)$ if and only if

$$\begin{aligned}\Delta\phi - \frac{1}{n}\tilde{g}(\mathrm{ric}_{\tilde{g}}, L\omega) &= -\frac{1}{n}\tilde{g}(\mathrm{ric}_{\tilde{g}}, \alpha) + \frac{1}{n}\Delta\mathrm{tr}_{\tilde{g}}\alpha + \frac{C}{n}\tilde{g}(\mathrm{ric}_{\tilde{g}}, \mathrm{ric}_{\tilde{g}}),\\ L^*L\omega - 2d\phi &= -2\tilde{\nabla}\cdot\alpha\end{aligned}$$

and $\tilde{m} = \beta - L\eta - C'\mathrm{ric}_{\tilde{g}} - \psi\tilde{g} \in \Gamma_2^{k-1}(\Sigma)$ if and only if

$$\begin{aligned}\Delta\psi + \frac{1}{n}\tilde{g}(\mathrm{ric}_{\tilde{g}}, L\eta) &= \frac{1}{n}\tilde{g}(\mathrm{ric}_{\tilde{g}}, \beta) + \frac{1}{n}\Delta\mathrm{tr}_{\tilde{g}}\beta - \frac{C'}{n}\tilde{g}(\mathrm{ric}_{\tilde{g}}, \mathrm{ric}_{\tilde{g}}),\\ L^*L\eta + 2(n-1)d\psi &= -2\tilde{\nabla}\cdot(\beta - (\mathrm{tr}_{\tilde{g}}\beta)\tilde{g}),\end{aligned}$$

using that $\tilde{\nabla}\cdot\mathrm{ric}_{\tilde{g}} = \frac{1}{2}d\mathrm{scal}_{\tilde{g}} = 0$ and $\mathrm{tr}_{\tilde{g}}\mathrm{ric}_{\tilde{g}} = \mathrm{scal}_{\tilde{g}} = 0$. The idea is to consider the right hand side as given and find (ϕ, ω) and (ψ, η) solving the equations. For this, we need the following lemma.

Lemma 5.1.6. *Assume that $(\Sigma, \tilde{g})$ is closed Riemannian manifold of dimension $n \geq 2$ such that $\mathrm{scal}_{\tilde{g}} = 0$. Let $a, b \in \mathbb{R}$ such that $0 < ab < 2$. For any $k \in \mathbb{R} \cup \{\infty\}$, consider the elliptic differential operator*

$$\begin{aligned}P &: H^{k+2}(\Sigma, \mathbb{R} \oplus T^*\Sigma) \to H^k(\Sigma, \mathbb{R} \oplus T^*\Sigma),\\ P(\phi, \omega) &:= \begin{pmatrix}\Delta\phi + a\tilde{g}(\mathrm{ric}_{\tilde{g}}, L\omega)) \\ L^*L\omega + bd\phi\end{pmatrix}.\end{aligned}$$

Then

$$\ker(P) = \ker(P^*) = \ker(d) \oplus \ker(\mathcal{L}),$$

i.e. constant functions and Killing one-forms.

In our case, we have first that $(a, b) = (-\frac{1}{n}, -2)$, which implies that $ab = \frac{2}{n}$ and secondly that $(a, b) = (\frac{1}{n}, 2(n-1))$, which implies that $ab = \frac{2(n-1)}{n}$. In both cases $0 < ab < 2$, for all $n \geq 2$, so the lemma applies.

We will use the following differential operators acting on one-forms ω and functions ϕ on Σ:

$$\begin{aligned}\delta\omega :=& -\nabla\cdot\omega,\\ \Delta\omega :=& (d\delta+\delta d)\omega,\\ \Delta\phi :=& (d\delta+\delta d)\phi = \delta d\phi.\end{aligned}$$

Proof of Lemma 5.1.6. Let us start by calculating L^*L:

$$\begin{aligned}L^*L\omega &= -2\tilde\nabla\cdot\left(\mathcal{L}\omega - \frac{1}{n}\mathrm{tr}_{\tilde g}(\mathcal{L}\omega)\tilde g\right)\\ &= -2\tilde\nabla\cdot\left(\overline{\mathcal{L}\omega} + \left(\frac{1}{2}-\frac{1}{n}\right)\mathrm{tr}_{\tilde g}(\mathcal{L}\omega)\tilde g\right)\\ &\overset{(4.2)}{=} 2\tilde\nabla^*\tilde\nabla\omega - 2\mathrm{ric}(\omega^\sharp,\cdot) + \left(2-\frac{4}{n}\right)d\delta\omega\\ &= 2\Delta\omega - 4\mathrm{ric}(\omega^\sharp,\cdot) + \left(2-\frac{4}{n}\right)d\delta\omega,\end{aligned}$$

where we used the Weitzenböck identity

$$\Delta\omega = \tilde\nabla^*\tilde\nabla\omega + \mathrm{ric}_{\tilde g}(\omega^\sharp,\cdot)$$

in the last step. We start by showing that $\ker(P) = \ker(d)\oplus\ker(\mathcal{L})$. For this, assume that

$$P(\phi,\omega) = 0.$$

It follows, using that $\tilde\nabla\cdot\mathrm{ric}_{\tilde g} = 0$, that

$$\begin{aligned}\delta(L^*L\omega) &= \left(4-\frac{4}{n}\right)\Delta\delta\omega + 2\tilde g(\mathrm{ric}_{\tilde g}, L\omega)\\ &= \left(4-\frac{4}{n}\right)\Delta\delta\omega - \frac{2}{a}\Delta\phi.\end{aligned}$$

On the other hand,

$$\delta(L^*L\omega) = -b\Delta\phi$$

which implies that

$$\Delta\left(\left(4-\frac{4}{n}\right)\delta\omega + \left(b-\frac{2}{a}\right)\phi\right) = 0.$$

Since Σ is closed,

$$\phi = \frac{4-\frac{4}{n}}{\frac{2}{a}-b}\delta\omega$$

where C is constant, which implies that

$$\begin{aligned}L^*L\omega &= -bd\phi \\ &= \frac{4-\frac{4}{n}}{1-\frac{2}{ab}} d\delta\omega.\end{aligned}$$

Since $0 < ab < 2$, it follows that

$$\|L\omega\|_{L^2}^2 = \frac{4-\frac{4}{n}}{1-\frac{2}{ab}} \|\delta\omega\|_{L^2}^2 \leq 0.$$

We conclude that

$$L\omega = 0, \quad \delta\omega = 0.$$

Hence ω is a Killing one-form. It follows that $d\phi = 0$ and hence $\phi = C$ as claimed.

We continue by calculating $\ker(P^*)$. We have

$$P^*(\phi, \omega) = \begin{pmatrix} \Delta\phi + b\delta\omega \\ L^*L\omega - 2a\mathrm{ric}(\mathrm{grad}\phi, \cdot) \end{pmatrix}.$$

From the calculation in the beginning of the proof, we get

$$-2a\mathrm{ric}_{\bar{g}}(\mathrm{grad}(\phi), \cdot) = -2a\left(1-\frac{1}{n}\right) d\Delta\phi + \frac{a}{2}L^*Ld\phi.$$

Using this, we conclude that

$$L^*L\left(\omega + \frac{a}{2}d\phi\right) = 2a\left(1-\frac{1}{n}\right) d\Delta\phi.$$

It follows that

$$\begin{aligned}\left\|L\left(\omega + \frac{a}{2}d\phi\right)\right\|^2 &= 2a\left(1-\frac{1}{n}\right)\langle d\Delta\phi, \omega + \frac{a}{2}d\phi\rangle \\ &= a^2\left(1-\frac{1}{n}\right)\|\Delta\phi\|^2 + 2a\left(1-\frac{1}{n}\right)\langle\Delta\phi, \delta\omega\rangle \\ &= \left(a^2 - \frac{2a}{b}\right)\left(1-\frac{1}{n}\right)\|\Delta\phi\|^2 \\ &= a^2\left(1-\frac{2}{ab}\right)\left(1-\frac{1}{n}\right)\|\Delta\phi\|^2 \\ &\leq 0,\end{aligned}$$

since $0 < ab < 2$. It follows that

$$L\left(\omega + \frac{a}{2}d\phi\right) = 0$$

and hence $\Delta\phi = 0$. Since Σ is closed, it follows that ϕ is constant and hence $L\omega = 0$. Since $b \neq 0$, it follows that $\delta\omega = 0$ and hence ω is a Killing one-form as claimed. □

Proof of Theorem 5.1.2. We first show that $\widehat{H}^k(\Sigma,\mathbb{R})\tilde{g} \oplus \mathrm{im}(L) \oplus \mathbb{R}\mathrm{ric}_{\tilde{g}}$ really is a direct sum. Since $\mathrm{scal}_{\tilde{g}} = 0$, we have for all $f \in \widehat{H}^k(\Sigma,\mathbb{R})$ that

$$f\tilde{g}[C\mathrm{ric}_{\tilde{g}}] = f[C\tilde{g}(\tilde{g},\mathrm{ric}_{\tilde{g}})] = f[0] = 0$$

and hence $\widehat{H}^k(\Sigma,\mathbb{R})\tilde{g} \cap \mathbb{R}\mathrm{ric}_{\tilde{g}} = \{0\}$. Since $\omega \mapsto L\omega$ has injective principal symbol, Lemma 2.2.3 implies that

$$H^k(\Sigma, S^2\Sigma) = \mathrm{im}(L) \oplus \ker(L^*).$$

Since $\mathrm{scal}_{\tilde{g}} = 0$, $L^*(\mathrm{ric}_{\tilde{g}}) = -2\tilde{\nabla} \cdot \mathrm{ric}_{\tilde{g}} = -d\mathrm{scal}_{\tilde{g}} = 0$ and hence $\mathbb{R}\mathrm{ric}_{\tilde{g}} \subset \ker(L^*)$ which implies that $\mathbb{R}\mathrm{ric}_{\tilde{g}} \cap \mathrm{im}(L) = \{0\}$. That $\widehat{H}^k(\Sigma,\mathbb{R})\tilde{g} \cap \mathrm{im}(L) = \{0\}$ is clear, since $\mathrm{tr}_g(L\omega) = 0$. This proves the first claim. Let us now prove that $\left(\widehat{H}^k(\Sigma,\mathbb{R})\tilde{g} \oplus \mathrm{im}(L) \oplus \mathbb{R}\mathrm{ric}_{\tilde{g}}\right) \cap \Gamma_1^k(\Sigma) = \{0\}$. For this, assume that

$$0 = \tilde{h} + \phi\tilde{g} + L\omega + C\mathrm{ric}_{\tilde{g}} \in \Gamma_1^k(\Sigma),$$

with $\phi \in \widehat{H}^k(\Sigma,\mathbb{R})$ and $\omega \in H^{k+1}(\Sigma, T^*\Sigma)$. We know that $\tilde{h} \in \Gamma_1^k(\Sigma)$ if and only if

$$P(\phi,\omega) = \begin{pmatrix} -\frac{C}{n}\tilde{g}(\mathrm{ric}_{\tilde{g}},\mathrm{ric}_{\tilde{g}}) \\ 0 \end{pmatrix},$$

with $(a,b) = (-\frac{1}{n}, -2)$. By Lemma 2.2.3 and Lemma 5.1.6, it follows that $C\tilde{g}(\mathrm{ric}_{\tilde{g}},\mathrm{ric}_{\tilde{g}})$ must be orthogonal to the constant functions, i.e. that

$$\int_\Sigma C\tilde{g}(\mathrm{ric}_{\tilde{g}},\mathrm{ric}_{\tilde{g}}) d\mu_{\tilde{g}} = 0.$$

Since $\tilde{g}(\mathrm{ric}_{\tilde{g}},\mathrm{ric}_{\tilde{g}}) \geq 0$, we conclude that either $\mathrm{ric}_{\tilde{g}} = 0$ or $C = 0$ which in both cases implies that $C\mathrm{ric}_{\tilde{g}} = 0$. Hence $(\phi,\omega) \in \ker(P)$, which by Lemma 5.1.6 implies that ϕ is constant and ω is a Killing one-form. Hence $L\omega = 0$ and since $0 = \phi[1] = \int_\Sigma \phi d\mu_{\tilde{g}}$, it follows that $\phi = 0$. This proves that $\left(\widehat{H}^k(\Sigma,\mathbb{R})\tilde{g} \oplus \mathrm{im}(L) \oplus \mathbb{R}\mathrm{ric}_{\tilde{g}}\right) \cap \Gamma_1^k(\Sigma) = \{0\}$. Similarly, one proves that $\left(\widehat{H}^k(\Sigma,\mathbb{R})\tilde{g} \oplus \mathrm{im}(L) \oplus \mathbb{R}\mathrm{ric}_{\tilde{g}}\right) \cap \Gamma_2^k(\Sigma) = \{0\}$.

It remains to show that

$$H^k(\Sigma, S^2\Sigma) \subseteq \widehat{H}^k(\Sigma,\mathbb{R})\tilde{g} \oplus \mathrm{im}(L) \oplus \mathbb{R}\mathrm{ric}_{\tilde{g}} \oplus \Gamma_1^k(\Sigma).$$

Given $\alpha \in H^k(\Sigma, S^2\Sigma)$ we want to find $\phi \in \widehat{H}^k(\Sigma,\mathbb{R})$ and $\omega \in H^{k+1}(\Sigma, T^*\Sigma)$ such that $\tilde{h} := \alpha - \phi\tilde{g} - L\omega - C\mathrm{ric} \in \Gamma_1^k(\Sigma)$. Note that $\tilde{h} \in \Gamma_1^k(\Sigma)$ if and only if

$$P(\phi,\omega) = \begin{pmatrix} -\frac{1}{n}\tilde{g}(\alpha,\mathrm{ric}_{\tilde{g}}) + \frac{1}{n}\Delta\mathrm{tr}_{\tilde{g}}\alpha + \frac{C}{n}\tilde{g}(\mathrm{ric}_{\tilde{g}},\mathrm{ric}_{\tilde{g}}) \\ -2\tilde{\nabla}\cdot\alpha \end{pmatrix}. \tag{5.6}$$

By Lemma 2.2.3 and Lemma 5.1.6 we find $(\phi,\omega) \in H^k(\Sigma, \mathbb{R} \oplus T^*\Sigma)$ if and only if we choose

$$C := \frac{\tilde{g}(\alpha,\mathrm{ric}_{\tilde{g}})[1]}{\int_\Sigma g(\mathrm{ric},\mathrm{ric}) d\mu_{\tilde{g}}},$$

when $\mathrm{ric}_{\tilde{g}} \neq 0$. If $\mathrm{ric}_{\tilde{g}} = 0$, it does not matter how we choose C, $C\mathrm{ric}_{\tilde{g}} = 0$ anyway. What remains is to show that $L\omega \in H^k(\Sigma, S^2\Sigma)$, up to now we only know that $L\omega \in H^{k-1}(\Sigma, S^2\Sigma)$. But from equation (5.6), we know that $L^*L\omega = 2d\varphi - 2\tilde{\nabla} \cdot \alpha \in H^{k-1}(\Sigma, T^*\Sigma)$. Elliptic regularity theory implies that in fact $\omega \in H^{k+1}(\Sigma, T^*\Sigma)$ which implies that $L\omega \in H^k(\Sigma, S^2\Sigma)$. The inclusion $H^k(\Sigma, S^2\Sigma) \subseteq \widehat{H}^k(\Sigma, \mathbb{R})\tilde{g} \oplus \mathrm{im}(L) \oplus \mathbb{R}\mathrm{ric}_{\tilde{g}} \oplus \Gamma^k_2(\Sigma)$ is proven analogously. □

5.2. Arbitrarily irregular solutions

The purpose of this section is to construct non-gauge solutions of the linearised Einstein equation that there are *arbitrarily irregular*. This shows that there is no analogue of "elliptic regularity theory" for the linearised Einstein equation. Rather, as one might expect, the regularity theory is more similar to that of linear wave equations. The example is given on generalised Kasner spacetimes and on Minkowski spacetime. The example works for both compact and non-compact spatial topology.

We construct the generalised Kasner spacetimes by first specifying its initial data and applying Theorem 3.1.3 to get the maximal globally hyperbolic development. Let Σ be any quotient of $\mathbb{R}^n$ with a discrete group. We define the initial data on $\mathbb{R}^n$ which induce initial data on Σ. Let $\tilde{g}$ be the standard metric and define $\tilde{k}$ by

$$\tilde{k}(\partial_i, \partial_j) = p_i \delta_{ij},$$

where $p_i \in \mathbb{R}$ are constants. Note that since $\mathrm{scal}_{\tilde{g}} = 0$, $\tilde{\nabla}\tilde{k} = 0$ and $\mathrm{tr}_{\tilde{g}}\tilde{k}$ is constant, $(\tilde{g}, \tilde{k})$ satisfies the (non-linear) vacuum constraint equations if and only if $(\mathrm{tr}_{\tilde{g}}\tilde{k})^2 = |\tilde{k}|^2$. This is equivalent to the condition

$$\sum_{i=1}^n p_i = \sqrt{\sum_{i=1}^n {p_i}^2} =: p \geq 0,$$

which we assume from now on. Since the initial data are translation invariant, they induce well-defined initial data on the quotient Σ. We conclude that $(\Sigma, \tilde{g}, \tilde{k})$ is an initial data set for the Einstein equation.

Theorem 5.2.1. *Assume that $\Sigma = \mathbb{R} \times (S^1)^{n-1}$ or $\Sigma = (S^1)^n$ and assume that $(\tilde{g}, \tilde{k})$ is as above. Let $k, k' \in \mathbb{R}$ such that $k \neq k'$. Then*

$$\mathcal{ID}_c^{k,k-1}(\Sigma) \Big/ \mathcal{GP}_{sc}^{k,k-1}(\Sigma) \neq \mathcal{ID}_c^{k',k'-1}(\Sigma) \Big/ \mathcal{GP}_{sc}^{k',k'-1}(\Sigma).$$

Let (M, g) be the maximal globally hyperbolic vacuum development of $(\Sigma, \tilde{g}, \tilde{k})$. If $p = 0$, the metric g is just the flat Minkowski metric. If $p \neq 0$, we call (M, g) a *generalised Kasner spacetime*. In the special case when $n = 3$ and $p = 1$, the metric g is given by

$$g = -dt^2 + \sum_{j=1}^3 t^{p_j}(dx^j)^2$$

on $M = (0,\infty) \times \Sigma$. In this case one can check that

$$|\mathrm{riem}_g|^2 = \frac{4}{t^4}\left(p_1^2p_2^2 + p_1^2p_3^2 + p_2^2p_3^2 + \sum_{j=1}^{3}(p_j^2 - p_j)^2\right),$$

so (M, g) is not flat in this case, unless $p_1 = 1$ and $p_2 = p_3 = 0$ or permutations thereof.

By applying Theorem 4.4.1, we obtain the following corollary.

Corollary 5.2.2. *Assume the same as in Theorem 4.4.1. Let $k, k' \in \mathbb{R}$ such that $k \neq k'$. Then*

$$\mathcal{Sol}^k_{sc}(M,t) \Big/ \mathcal{G}^k_{sc}(M,t) \neq \mathcal{Sol}^{k'}_{sc}(M,t) \Big/ \mathcal{G}^{k'}_{sc}(M,t).$$

In particular, there are flat and non-flat vacuum spatially compact and spatially non-compact spacetimes with arbitrarily irregular solutions to the linearised Einstein equation.

Proof of Theorem 5.2.1. We prove the theorem when $\Sigma = \mathbb{R} \times (S^1)^{n-1}$, the other case is proven exactly the same way. Let us assume that $k < k'$. Choose a distribution $\hat{f} \in H^k_c(\mathbb{R},\mathbb{R}) \backslash H^{k'}_c(\mathbb{R},\mathbb{R})$ such that there is no $a \in \mathcal{D}'(\mathbb{R},\mathbb{R})$ such that $a' = f$. This can always be achieved just by adding a positive compactly supported smooth function to $\hat{f}$ if necessary. Define $f \in H^k_c(\Sigma,\mathbb{R}) \backslash H^{k'}_c(\Sigma,\mathbb{R})$ by

$$f(x^1, x^2, \dots, x^n) := \hat{f}(x^1).$$

The formula does only make sense for functions, but it is clear how to generalise it to distributions. Define $(\tilde{h}, \tilde{m}) \in H^k_c(\Sigma, S^2\Sigma \oplus S^2\Sigma) \backslash H^{k'}_c(\Sigma, S^2\Sigma \oplus S^2\Sigma)$ by

$$\tilde{h} := f dx^1 \otimes dx^1,$$
$$\tilde{m} := p_1 f dx^1 \otimes dx^1.$$

It follows that $\tilde{\nabla}\cdot(\tilde{h} - (\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g}) = \partial_1 f dx^1 - df = 0$, since f only depends on the x^1-coordinate. Moreover,

$$2\tilde{g}(\tilde{k} \circ \tilde{k} - (\mathrm{tr}_{\tilde{g}}\tilde{k})\tilde{k}, \tilde{h}) - 2\tilde{g}(\tilde{k}, \tilde{m} - (\mathrm{tr}_{\tilde{g}}\tilde{m})\tilde{g}) = 2{p_1}^2 f - 2pp_1 f - (2{p_1}^2 f - 2pp_1 f) = 0.$$

Since $\mathrm{ric}_{\tilde{g}} = 0$, we have proven that $D\Phi_1(\tilde{h}, \tilde{m}) = 0$. Since $\tilde{\nabla}\tilde{k} = 0$ and $\tilde{\nabla}\cdot(\tilde{m} - (\mathrm{tr}_{\tilde{g}}\tilde{m})\tilde{g}) = p_1\tilde{\nabla}\cdot(\tilde{h} - (\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g}) = 0$, we only need to check that

$$-\tilde{g}\left(\tilde{k}(\cdot, X), \tilde{\nabla}\cdot\left(\tilde{h} - \frac{1}{2}(\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g}\right)\right) + \frac{1}{2}\tilde{g}(\tilde{k}, \tilde{\nabla}_X\tilde{h}) = 0$$

for all $X \in T\Sigma$ in order to prove that $D\Phi_2(\tilde{h}, \tilde{m}) = 0$. Similar to before, $\tilde{\nabla}\cdot\left(\tilde{h} - \frac{1}{2}(\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g}\right) = \frac{1}{2}df$ and $\tilde{\nabla}_X\tilde{h} = (\partial_X f)dx^1 \otimes dx^1 = (\partial_1 f)dx^1(X)dx^1 \otimes dx^1$. It follows that

$$-\tilde{g}\left(\tilde{k}(\cdot, X), \tilde{\nabla}\cdot\left(\tilde{h} - \frac{1}{2}(\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g}\right)\right) + \frac{1}{2}\tilde{g}(\tilde{k}, \tilde{\nabla}_X\tilde{h}) =$$
$$-\frac{p_1}{2}(\partial_1 f)(dx^1(X)) + \frac{p_1}{2}(\partial_1 f)(dx^1(X)) = 0.$$

Hence we have shown that $(\tilde{h}, \tilde{m}) \in \mathcal{ID}^{k,k-1}(\Sigma) \setminus \mathcal{ID}^{k',k'-1}(\Sigma)$. We need to make sure that $(\tilde{h}, \tilde{m})$ is not gauge producing initial data. Assume therefore that it is, i.e. that there is $(N, \beta) \in H_c^{k+1}(\Sigma, \mathbb{R} \oplus T\Sigma)$ such that

$$\begin{aligned} f dx^1 \otimes dx^1 &= \tilde{h}_{N,\beta}, \\ p_1 f dx^1 \otimes dx^1 &= \tilde{m}_{N,\beta}. \end{aligned}$$

Write $\beta = \sum_{i=1}^n \beta_i \partial_i$. If we insert $\partial_i \otimes \partial_i$ using (4.7) and (4.8), we see that

$$\begin{aligned} f dx^1(\partial_i)^2 &= \tilde{h}_{N,\beta}(\partial_i, \partial_i) = 2\partial_i \beta_i + 2p_i N, \\ p_1 f dx^1(\partial_i)^2 &= \tilde{m}_{N,\beta}(\partial_i, \partial_i) = 2p_i \partial_i \beta_i + \partial_i^2 N + N(2{p_i}^2 - p_i). \end{aligned}$$

Simplifying and summing over i implies that

$$\sum_{i=1}^n \partial_i^2 N - pN = 0.$$

In case Σ is non-compact, it follows that $N = 0$, since N is compactly supported. In case Σ is compact and $p > 0$ it follows that $N = 0$, since $\sum_i^n \partial_i^2$ is a non-positive operator. If $p = 0$, it follows that N is constant. But if $p = 0$, all $p_i = 0$ as well. We conclude that $Np_i = 0$ in any case. Inserting $Np_i = 0$ and N is constant into the above equations gives

$$\begin{aligned} 0 &= \partial_i \beta_i, \quad i \neq 1 \\ f &= 2\partial_1 \beta_1. \end{aligned}$$

Inserting $\partial_i \otimes \partial_j$ with $i \neq j$, we conclude that

$$0 = \tilde{h}_{N,\beta}(\partial_i, \partial_j) = \partial_i \beta_j + \partial_j \beta_i.$$

Differentiating this expression with respect to i and summing over i gives for $j \neq 1$:

$$\begin{aligned} 0 &= \sum_{i=1}^n \partial_i^2 \beta_j + \partial_j \partial_i \beta_i \\ &= \sum_{i=1}^n \partial_i^2 \beta_j + \frac{1}{2} \partial_j f \\ &= \sum_{i=1}^n \partial_i^2 \beta_j. \end{aligned}$$

Hence we conclude that $\beta_j = 0$ for $j \neq 1$ in the non-compact case and $\beta_j = C_j$ are constant for $j \neq 1$ in the compact case. It follows that

$$0 = \partial_j \beta_1$$

for $j \neq 0$. Altogether, this implies that β_1 only depends on the first variable. In other words, $\beta_1(x_1, \dots, x_n) = \beta_1(x_1, q_2, \dots, q_n)$, where $(q_2, \dots, q_n) \in (S^1)^{n-1}$ is some fixed point. Since

$$f = 2\partial_1 \beta_1,$$

this means that we have found the distribution $x_1 \to \beta_1(x_1, q_2, \dots, q_n)$, which has $\hat{f}$ as its derivative. This is a contradiction to the assumptions on $\hat{f}$. Hence $(\tilde{h}, \tilde{m})$ is not gauge producing initial data and $[(\tilde{h}, \tilde{m})]$ defines a non-trivial equivalence class. □

Part II.

Scalar wave equations with initial data on compact Cauchy horizons

6. Well-posedness for linear wave equations

We state and prove the well-posedness statement for linear wave equations with initial data on compact non-degenerate Cauchy horizons.

6.1. Cauchy horizons

Let $(\hat{M}, g)$ denote a spacetime throughout the chapter. We recall a basic definition in Lorentzian geometry. For an achronal subset $A \subset \hat{M}$, the *past/future domain of dependence of* A is given by

$$D^{-/+}(A) = \{p \in \hat{M} \mid \text{every future/past inextendible causal curve though } p \text{ meets } A\}.$$

The *domain of dependence of* A is denoted $D(A) := D^-(A) \cup D^+(A)$. Let us recall the basic structure of Cauchy horizons.

Definition 6.1.1. ([26, Def. 14.49]) Let $\hat{M}$ be a spacetime and let $\Sigma \subset \hat{M}$ be an achronal set. Its *past Cauchy horizon* and *future Cauchy horizon* are defined as

$$\begin{aligned} H_-(\Sigma) &:= \overline{D^-(\Sigma)} \backslash I^+(D^-(\Sigma)), \\ H_+(\Sigma) &:= \overline{D^+(\Sigma)} \backslash I^-(D^+(\Sigma)). \end{aligned}$$

We will only be interested in the case when $D(\Sigma)$ is an open globally hyperbolic submanifold of $\hat{M}$. We also want the Cauchy horizons to be disjoint from Σ. Let us therefore assume that $\Sigma \subset \hat{M}$ is a closed acausal topological hypersurface. Here we mean that Σ is "closed as a set", we do not necessarily demand Σ to be compact. By [26, Theorem 14.38, Lemma14.43], it follows that $M := D(\Sigma)$ is globally hyperbolic and Σ is a Cauchy hypersurface of M. Let $\mathcal{H}_- := H_-(\Sigma)$ and $\mathcal{H}_+ := H_+(\Sigma)$ denote the past and future (possibly empty) Cauchy horizons of Σ.

Lemma 6.1.2 ([26, Lemma 14.51, Proposition 14.53])**.** *Assume that* $\Sigma \subset \hat{M}$ *is a closed acausal topological hypersurface. Then*

$$\begin{aligned} \mathcal{H}_- &= I^-(\Sigma) \cap \partial D^-(\Sigma) = \overline{D^-(\Sigma)} \backslash D^-(\Sigma), \\ \mathcal{H}_+ &= I^+(\Sigma) \cap \partial D^+(\Sigma) = \overline{D^+(\Sigma)} \backslash D^+(\Sigma), \end{aligned}$$

and

$$\overline{D^{-/+}(\Sigma)} = \{p \in \hat{M} \mid \textit{every future/past inextendible timelike curve though } p \textit{ meets } \Sigma\}.$$

In particular, $\partial M = \mathcal{H}_- \sqcup \mathcal{H}_+$*. Moreover,* $\mathcal{H}_-$ *and* $\mathcal{H}_+$ *are closed, achronal topological hypersurfaces.*

Lemma 6.1.3. *Assume that* $\Sigma \subset \hat{M}$ *is a closed acausal topological hypersurface and that* $\mathcal{H}_- \subset \hat{M}$ *is a (non-empty) smooth manifold. Then*

$$M \sqcup \mathcal{H}_-$$

is a smooth manifold with boundary. Moreover, $\mathcal{H}_-$ *is a lightlike hypersurface. The same result holds with* $\mathcal{H}_-$ *replaced by* $\mathcal{H}_+$*.*

Proof. We start by proving the first statement. Since M is already a manifold, we only need to check the property on the boundary of $M \sqcup \mathcal{H}_-$. By the previous lemma, $\mathcal{H}_-$ is the boundary of $M \sqcup \mathcal{H}_-$ in $\hat{M}$ (as a topological space). Let $x \in \mathcal{H}_-$. Since $\mathcal{H}_- \subset \hat{M}$ is a smooth hypersurface and $\mathcal{H}_+ \subset \hat{M}$ is closed, there is a connected open subset $W \subset \hat{M}$, such that $x \in W \cap (\hat{M} \backslash \mathcal{H}_+)$ and such that $W \backslash \mathcal{H}_- = U_1 \sqcup U_2$, where U_1 and U_2 are two disjoint open subsets of $\hat{M}$. It follows that $\partial M \cap W = \mathcal{H}_- \cap W$. We claim that one of U_1 and U_2 has to be contained in M and the other one contained in $\hat{M} \backslash M$. Since $x \in \partial M$, $W \cap M \neq \emptyset$ and $W \cap (\hat{M} \backslash M) \neq \emptyset$. Let therefore $y_1 \in W \cap M$ and $y_2 \in W \cap (\hat{M} \backslash M)$. Assume, to reach a contradiction, that $y_1, y_2 \in U_1$. Since U_1 is connected, we can choose a path γ in U_1 between y_1 and y_2. It follows that some point on γ must lie in $\partial M \cap W = \mathcal{H}_- \cap W$, which is a contradiction, since $\mathcal{H}_- \cap U_1 = \emptyset$. Similarly, not both y_1 and y_2 can lie in U_2. This proves the first statement.

The second statement follows by [22, Proposition 1.13, Proposition 1.16]. □

Recall that any smooth lightlike hypersurface N of a Lorentzian manifold has an induced nowhere vanishing (unique up to scaling) lightlike vector field V such that $\nabla_V V = \kappa V$ for $\kappa \in C^\infty(N)$.

Definition 6.1.4. Assume that $N \subset \hat{M}$ is a smooth lightlike hypersurface. We say that N is *non-degenerate* if there exists an induced non-vanishing lightlike vector field V such that $\nabla_V V = \kappa V$, for $\kappa \in C^\infty(N)$ with $\kappa \neq 0$. We say that N is *degenerate* if it is not non-degenerate.

Lemma 6.1.5. *Let* N *be a compact smooth lightlike hypersurface of* $\hat{M}$*. Then the following are equivalent*

1. *N is non-degenerate.*

2. *There is a nowhere vanishing lightlike vector field V such that $\nabla_V V = V$.*

Proof. That $2 \Rightarrow 1$ is trivial. Let us show that $1 \Rightarrow 2$. Let us make the ansatz $\tilde{V} := fV$. Then $\nabla_{\tilde{V}} \tilde{V} = \tilde{V}$ if and only if $\partial_V f + \kappa f = 1$. By Lemma 2.2.1 and Remark 2.2.2 there is a solution $f \neq 0$, since $\kappa \neq 0$. □

Remark 6.1.6. Let N be a compact smooth lightlike hypersurface of $\hat{M}$. We claim that if $\kappa = 0$ for some lightlike vector field V, then N is degenerate. Indeed, assume that there was a non-vanishing function f on N such that $1 = \partial_V f + \kappa f = \partial_V f$. The function f would grow linearly along the integral curve of V. Since N is compact, the integral curve of V exists for all times and hence f would not be bounded. This is a contradiction, so the claim is proven.

Definition 6.1.7. We call a smooth submanifold $N \subset \hat{M}$ *totally geodesic* if all geodesics in $\hat{M}$, starting tangent to N, stay in N.

Remark 6.1.8. Let N be a smooth lightlike hypersurface of $\hat{M}$. By [21, Theorem 30], N is totally geodesic if and only if

$$g(\nabla_X V, Y) = 0,$$

for some non-vanishing lightlike vector field V tangent to N and all $X, Y \in TN$.

The following recent result is of central importance as motivation for our assumptions on the Cauchy horizon.

Theorem 6.1.9 ([22, Corollary 1.43], [23, Theorem 18])**.** *Assume that $\hat{M}$ satisfies the null energy condition, i.e.* $\mathrm{ric}_g(V, V) \geq 0$ *for all lightlike vectors $V \in T\hat{M}$. Let $\Sigma \subset \hat{M}$ be an closed acausal topological hypersurface and let $\mathcal{H}$ be its past or future Cauchy horizon. Assume that $\mathcal{H}$ is compact. Then $\mathcal{H}$ is a smooth and totally geodesic null hypersurface.*

Remark 6.1.10. In [22, Corollary 1.43], it is assumed that $\Sigma \subset \hat{M}$ is acausal and that $\mathrm{edge}(\Sigma) = \emptyset$. This is by [26, Corollary 14.63] equivalent to assuming that Σ is a closed acausal topological hypersurface.

Let $\mathcal{H}$ be a future or past smooth Cauchy horizon. Then for any $p \in \mathcal{H}$, $T_p\mathcal{H} \subset T_p\hat{M}$ is a linear subspace of codimension 1. $T_p\hat{M} \backslash T_p\mathcal{H} =: T_p\hat{M}_p^+ \sqcup T_p\hat{M}_p^-$ has two disjoint components, where $T_p\hat{M}_p^+$ contains future directed timelike vectors and $T_p\hat{M}_p^-$ contains past directed timelike vectors. We will need the following definition.

Definition 6.1.11. Let $p \in \mathcal{H}$. We say that a vector $w \in T_p\hat{M}$ is *outward pointing* if $w \in T\hat{M}_p^-$ and $\mathcal{H}$ is a past Cauchy horizon or if $w \in T\hat{M}_p^+$ and $\mathcal{H}$ is a future Cauchy horizon.

6.2. The statement

We use the following convention for the d'Alembert operator, acting on scalar valued functions:

$$\Box := \nabla^*\nabla = -\sum_{i=0}^{n} \partial_{e_i}\partial_{e_i} - \partial_{\nabla_{e_i} e_i}$$

in some g-orthonormal frame $(e_0, \ldots, e_n)$, with $\epsilon_i := g(e_i, e_i) \in \{-1, 1\}$. A general wave operator acting on scalar valued functions can be written on the following form:

$$\Box + \partial_W + \alpha$$

where $W \in C^\infty(\hat{M}, T\hat{M})$ and $\alpha \in C^\infty(\hat{M})$.

Definition 6.2.1 (Admissible wave operator)**.** Let $\mathcal{H}$ be a smooth past or future Cauchy horizon. We call a wave operator admissible with respect to $\mathcal{H}$ if $W|_\mathcal{H}$ is *nowhere outward pointing.*

For example, $\Box + \alpha$ is always admissible, for any $\alpha \in C^\infty(\hat{M})$. Our main result of this chapter is the following theorem.

Theorem 6.2.2. *Let $\hat{M}$ be a spacetime and let $\Sigma \subset \hat{M}$ be a closed acausal topological hypersurface. Let $\mathcal{H}$ be the past or future Cauchy horizon of Σ. Assume that $\mathcal{H}$ is a non-empty, smooth, compact, totally geodesic and non-degenerate hypersurface of $\hat{M}$. Let P be an admissible wave operator with respect to $\mathcal{H}$. Then $M := D(\Sigma)$ is globally hyperbolic, $M \sqcup \mathcal{H}$ is a smooth manifold with boundary and for every $u_0 \in C^\infty(\mathcal{H})$ and $f \in C^\infty(M \sqcup \mathcal{H})$ there is a unique $u \in C^\infty(M \sqcup \mathcal{H})$ such that*

$$Pu = f,$$
$$u|_\mathcal{H} = u_0.$$

Moreover, the solution u depends continuously on the data (u_0, f).

Remark 6.2.3. By Theorem 6.1.9, our assumptions on the Cauchy horizon are fulfilled if $\hat{M}$ satisfies the null energy condition and if $\mathcal{H}$ is non-empty, compact and non-degenerate. The null energy condition is satisfied by many matter models in general relativity, for example by vacuum models.

Let us give a simple example where Theorem 6.2.2 applies.

Example 6.2.4 (The Misner spacetime)**.** We define the Misner spacetimes by

$$\hat{M}_\pm := (\mathbb{R} \times S^1, \pm 2dtdy + tdy^2). \tag{6.1}$$

Both these spacetimes are (inequivalent) extensions of the spacetime

$$(\mathbb{R}_+ \times S^1, -\frac{1}{t}dt^2 + tdx^2).$$

The spacetime $\hat{M}_+$ is obtained by defining $y := x - \log(t)$ and $\hat{M}_-$ is obtained by defining $y := x + \log(t)$, as can easily be checked. If we choose $\Sigma := \{1\} \times S^1$ in $\hat{M}_\pm$, the past Cauchy horizon is given by $\mathcal{H}_- = \{0\} \times S^1$ and the future Cauchy horizon is empty. It is clear that $\mathcal{H}_-$ is totally geodesic and we claim that $\mathcal{H}_-$ is non-degenerate. Choosing $V := \partial_y$, one calculates that

$$
\begin{aligned}
g(\nabla_V V, \partial_t) &= g(\nabla_{\partial_y}(\partial_y), \partial_t) = -g(\partial_y, \nabla_{\partial_y}\partial_t) = -\frac{1}{2}\partial_t g(\partial_y, \partial_y) = -\frac{1}{2}, \\
g(\nabla_V V, \partial_y) &= g(\nabla_{\partial_y}(\partial_y), \partial_y) = \frac{1}{2}\partial_y g(\partial_y, \partial_y) = 0,
\end{aligned}
$$

which implies that $\nabla_V V = \mp\frac{1}{2}V$ on $\hat{M}_\pm$. This shows that $\mathcal{H}_-$ is non-degenerate. Therefore Theorem 6.2.2 applies.

Remark 6.2.5 (Neither existence nor uniqueness holds for non-admissible wave operators)**.** The d'Alembert operator on the Misner spacetime $\hat{M}_+$ is

$$\Box = \partial_t(t\partial_t - 2\partial_y).$$

We give examples of wave operators that are not admissible in the sense of Definition 6.2.1, such that uniqueness respectively existence of solution does not hold. We conclude that the assumption in Theorem 6.2.2, that the wave operator is admissible, is indeed necessary.

- First consider the non-admissible operator $P := \Box - \partial_t$ on Misner spacetime. Note that $u(t, x) = Ct$ solves $Pu = 0$ for all $C \in \mathbb{R}$ and $u|_{\mathcal{H}_-} = 0$. We conclude that *uniqueness does not hold for all wave operators.*

- Now consider the non-admissible operator $P := \Box - \partial_t + 1$ on Misner spacetime. Assume that $Pu = 0$. Then $Pu|_{\mathcal{H}} = -2\partial_y\partial_t u|_{\mathcal{H}} + u_0 = 0$. But if we integrate this equation over S^1, we conclude that $\int_{S^1} u_0(y)dy = 0$, which is a strong restriction on the initial data. The conclusion is that *existence of solution does not hold for all wave operators.*

Remark 6.2.6 (Not all solutions extend to the Cauchy horizon)**.** Note that $u = \ln(t)$ satisfies $\Box u = 0$ on the Misner spacetimes $\hat{M}_\pm$. However, u does not extend continuously to the horizon $\mathcal{H}_- = \{0\} \times S^1$. We conclude that *there are solutions to $\Box u = 0$, defined on M that "blow up" at $\mathcal{H}_-$.*

Example 6.2.7 (The Taub-NUT-spacetime)**.** An important example of a *vacuum* spacetime containing two compact Cauchy horizons is the Taub-NUT spacetime. The Taub-NUT spacetimes is given by $\mathbb{R} \times S^3$ with the metrics

$$\pm 4l dt\sigma_1 + 4l^2 U(t){\sigma_1}^2 + (t^2 + l^2)({\sigma_2}^2 + {\sigma_3}^2),$$

where

$$U(t) := \frac{(t_+ - t)(t - t_-)}{t^2 + l^2}$$

where $t_\pm := m \pm \sqrt{m^2 + l^2}$, where $m \in \mathbb{R}$, $l > 0$ and $\sigma_1, \sigma_2, \sigma_3$ are orthonormal left invariant one-forms on S^3. These spacetimes are (inequivalent) extensions of the Taub-region, given by the manifold $(\mathbb{R}_+ \times S^3)$ with the metric

$$-\frac{1}{U(t)}dt^2 + 4l^2U(t){\sigma_1}^2 + (t^2 + l^2)({\sigma_2}^2 + {\sigma_3}^2).$$

Note that $\Sigma_\tau := \{\tau\} \times S^3$ for $\tau \in (t_-, t_+)$ are acausal hypersurfaces that are closed as sets. The past and future Cauchy horizons are given by

$$\begin{aligned}\mathcal{H}_- &:= \{t_-\} \times S^3,\\ \mathcal{H}_+ &:= \{t_+\} \times S^3,\end{aligned}$$

which are clearly compact. Similarly to the Misner spacetimes, one calculates that both the future horizon at t_+ and the past horizon at t_- are non-degenerate. By Remark 6.2.3, the horizons are totally geodesic, since the Taub-NUT spacetime is vacuum. Hence Theorem 6.2.2 applies to both $\mathcal{H}_-$ and $\mathcal{H}_+$.

Example 6.2.8 (The generalised Misner spacetime)**.** Let us investigate further the properties of a compact Cauchy horizon being totally geodesic and non-degenerate. For this, let (Σ, σ) be a Riemannian closed manifold and assume that $W \in C^\infty(\Sigma, T\Sigma)$ is a *non-vanishing* vector field such that $\sigma(W, W) = 1$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth function. Consider the manifold $\mathbb{R} \times \Sigma$ and the Lorentzian metric defined by

$$\begin{aligned}&g(\partial_t, \partial_t) = 0,\\ &g(\partial_t, X) = \sigma(X, W),\\ &g(X, Y) = \sigma(X, Y) + (\varphi(t) - 1)\sigma(X, W)\sigma(Y, W),\end{aligned}$$

for $X, Y \in T\Sigma$. Choose a local orthonormal frame $(e_1 = W, e_2, \dots, e_n)$ of $T\Sigma$ with respect to σ. The metric g takes the form

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & \varphi & 0 \\ 0 & 0 & \delta_{ij} \end{pmatrix},$$

in the basis $(\partial_t, W, e_2, \dots, e_n)$. For each $t \in \mathbb{R}$, the sign of $\varphi(t)$ decides whether the hypersurface $\{t\} \times \Sigma$ is timelike, lightlike or spacelike. If $\varphi(t) < 0$, it is timelike, if $\varphi(t) = 0$, it is lightlike and if $\varphi(t) > 0$, it is spacelike. Hence if $\varphi(t) \leq 0$, there is a closed causal curve in $\{t\} \times \Sigma$. If we instead assume that there are $t_- < t_+$ such that $\varphi(t) > 0$ for all $t \in (t_-, t_+)$, then $M := (t_-, t_+) \times \Sigma$ is globally hyperbolic. If $\varphi(t_-) = 0 = \varphi(t_+)$, then $\mathcal{H}_- := \{t_-\} \times \Sigma$ and $\mathcal{H}_+ := \{t_+\} \times \Sigma$ are the past and future Cauchy horizon of $\{t\} \times \Sigma$ for any $t \in (t_-, t_+)$. Let us check when for example $\mathcal{H}_-$ is non-degenerate. Choose the nowhere vanishing lightlike vector field $V := W|_{\mathcal{H}_-}$. For all $X \in T\mathcal{H}_-$, we have

$$\begin{aligned} g(\nabla_V V, X) &= \partial_V g(V, X) - g(V, \nabla_V X) \\ &= -g(V, [V, X]) - g(V, \nabla_X V) \\ &= -\frac{1}{2}\partial_X g(V, V) \\ &= 0. \end{aligned}$$

Moreover, using $[\partial_t, W] = 0$, we have

$$\begin{aligned} g(\nabla_V V, \partial_t) &= \partial_V g(V, \partial_t) - g(V, \nabla_V \partial_t) \\ &= -g(V, \nabla_{\partial_t} W) \\ &= -\frac{1}{2}\partial_t g(W, W) \\ &= -\frac{1}{2}\partial_t \varphi(t_-). \end{aligned}$$

Hence $\nabla_V V = -\frac{1}{2}\partial_t \varphi(t_-) V$. It follows by Remark 6.1.6 that $\mathcal{H}_-$ *is non-degenerate if and only if* $\partial_t \varphi(t_-) \neq 0$. Let us now calculate under what condition $\mathcal{H}_-$ is totally geodesic. The null second fundamental form of $\mathcal{H}_-$ with respect to V is

$$g(\nabla_X V, Y) = \frac{1}{2}\mathcal{L}_V g(X, Y)$$

for all $X, Y \in T\mathcal{H}_-$. By the above calculations, $g(\nabla_X V, Y) = 0$ if $X = V$ or $Y = V$. Hence, let us assume that $X, Y \perp V$ with respect to σ. We get

$$\begin{aligned} \mathcal{L}_V g(X, Y) &= \mathcal{L}_V \sigma(X, Y) - \mathcal{L}_V(\sigma(\cdot, V)\sigma(\cdot, V))(X, Y) \\ &= \mathcal{L}_V \sigma(X, Y) - \partial_V(\sigma(X, V)\sigma(Y, V)) - \sigma([V, X], V)\sigma(Y, V) \\ &\quad - \sigma(X, V)\sigma([V, Y], V) \\ &= \mathcal{L}_V \sigma(X, Y), \end{aligned}$$

since $\sigma(X, V) = 0 = \sigma(Y, V)$. It follows that $\mathcal{H}_-$ *is totally geodesic if and only if* V *is a Killing field with respect to* σ.

Let us use this construction to provide examples with interesting features. First we show that Theorem 6.2.2 would be false if we did not assume that the horizon is non-degenerate. In fact, not even the equation $\Box u = 0$ is solvable for arbitrary initial data.

Remark 6.2.9 (Non-degeneracy is necessary)**.** Let $(\mathbb{R} \times \Sigma, g)$ be the spacetime from Example 6.2.8, with φ chosen such that there is a $t_- \in \mathbb{R}$ with $\varphi(t_-) = 0$ such that $\partial_t \varphi(t_-) = 0$ and a $t_+ \in \mathbb{R} \cup \{\infty\}$ such that $\varphi(t) > 0$ for all $t \in (t_-, t_+)$. We choose $\Sigma := S^1 \times N$, with metric $d\theta^2 + \tilde{g}$, where $(N, \tilde{g})$ is a closed Riemannian manifold with $\dim(N) \geq 1$ and let $W = \partial_\theta$, where θ is the coordinate on S^1. It follows that $V = W|_{\mathcal{H}_-}$ is a Killing field. By the calculations in Example 6.2.8, we conclude that $\mathcal{H}_- := \{t_-\} \times \Sigma$ is a compact *degenerate* and totally geodesic past Cauchy horizon of the globally hyperbolic part $M := (t_-, t_+) \times \Sigma$. The metric is given by

$$g = 2dtd\theta + \varphi(t)d\theta^2 + \tilde{g}$$

and the d'Alembert operator by

$$\Box = \partial_t(\varphi(t)\partial_t - 2\partial_\theta) + \Delta_{\tilde{g}}.$$

Assume now that $\Box u = 0$ for some $u \in C^\infty(M \sqcup \mathcal{H}_-)$. Define $u_0 := u|_{\mathcal{H}_-}$Then

$$0 = \Box u|_{\mathcal{H}_-} = -2\partial_\theta \partial_t u|_{\mathcal{H}_-} + \Delta_{\tilde{g}} u_0.$$

Integrating this equation over S^1, i.e. over the θ-coordinate gives

$$\Delta_{\tilde{g}} \int_{S^1} u_0(\theta, \cdot)d\theta = 0.$$

It follows that the function

$$\begin{aligned} N &\to \mathbb{R} \\ p &\mapsto \int_{S^1} u_0(\theta, p)d\theta \end{aligned}$$

is constant. Since $\dim(N) \geq 1$,this is a strong restriction on the initial data u_0. We conclude that *Theorem 6.2.2 would be false if we dropped the assumption that the horizon is non-degenerate.*

Remark 6.2.10 (Domain of dependence of the Cauchy horizon)**.** Consider again Example 6.2.8. Assume that $\varphi(t_0) = 0$, i.e. that $\{t_0\} \times \Sigma$ is a compact lightlike hypersurface. The vector field $-\frac{\varphi}{2}\partial_t + W$ will be lightlike and its integral curves will "spiral" around $\{t_0\} \times \Sigma$ for t close to t_0 but never intersect $\{t_0\} \times \Sigma$, unless the integral curve starts in $\{t_0\} \times \Sigma$. This implies that the domain of dependence of $\{t_0\} \times \Sigma$ is nothing but the hypersurface itself. This applies in particular to the case when $\{t_0\} \times \Sigma$ is a Cauchy horizon.

Let us use the construction in Example 6.2.8 to construct an example where Theorem 2.2.9 applies and the integral curves of the non-vanishing lightlike vector field on the horizon do not close. This is interesting because the techniques of [25] and [19], where certain wave equations are solved for initial data on compact Cauchy horizons, rely on the fact that the integral curves are closed.

Example 6.2.11 (A compact Cauchy horizon with non-closed generators)**.** For any $n \in \mathbb{N}$, let $T^n := (S^1)^n$ be the flat n-dimensional torus. Choose the vector field W in Example 6.2.8 to be parallel with "irrational angle", i.e. so that the integral curves of W do not close. Let $(\mathbb{R} \times \Sigma, g)$ be the spacetime from Example 6.2.8, with φ chosen such that there is a $t_- \in \mathbb{R}$ with $\varphi(t_-) = 0$ and $\partial_t \varphi(t_-) > 0$. Let $t_+ \in \mathbb{R} \cup \{\infty\}$ be such that $\varphi(t) > 0$ for all $t \in (t_-, t_+)$. By the calculations in Example 6.2.8, we conclude that $\mathcal{H}_- := \{t_-\} \times \Sigma$ is a past non-degenerate compact Cauchy horizon of the globally hyperbolic region $M := (t_-, t_+) \times \Sigma$.

The induced lightlike vector field $V := W|_{\mathcal{H}_-}$ has non-closed integral curves, also called non-closed generators. Since V is parallel, it is a Killing field and by the discussion in Example 6.2.8 it follows that $\mathcal{H}_-$ is totally geodesic. Hence we have an example of a spacetime where *the Cauchy horizon has non-closed generators and Theorem 6.2.2 applies.*

Remark 6.2.12 (No finite speed of propagation)**.** Consider Example 6.2.11. We can choose V so that any integral curve of V densely fills the Cauchy horizon. This implies that for any point $p \in \mathcal{H}_-$, $M \subset J_+(p)$. Usually, when dealing with wave equations, one can localise the problem to a coordinate patch, because of finite speed of propagation (c.f. Theorem 2.2.9). As we see by this example, *proving Theorem 6.2.2 is a non-local problem* and cannot be studied locally on one coordinate patch at the time.

In Theorem 6.2.2, we show that the solution exists and is unique on the globally hyperbolic region. Let us give an example that shows that the solution fails to be unique on the non-globally hyperbolic part.

Remark 6.2.13 (Non-uniqueness in the non-globally hyperbolic part)**.** Again, let $(\mathbb{R} \times \Sigma, g)$ be the spacetime from Example 6.2.8, with φ chosen such that for some interval $(t_0, t_1) \subset \mathbb{R}$, we have $\varphi|_{(t_0,t_1)} = 0$. Assume also that $\varphi(t_-) = 0$ and $\partial_t \varphi(t_-) > 0$ for some $t_- > t_1$. It follows that for some $t_+ \in \mathbb{R}$, $(t_-, t_+) \times \Sigma$ is globally hyperbolic with smooth and non-degenerate past Cauchy horizon $\mathcal{H}_- = \{t_-\} \times \Sigma$. Choose $\Sigma = S^1 \times N$, with metric $d\theta^2 + \tilde{g}$, where $(N, \tilde{g})$ is a closed Riemannian manifold. Since ∂_θ is a Killing vector field, it follows that $\mathcal{H}_-$ is totally geodesic. The spacetime $(t_0, t_1) \times \Sigma$ is not globally hyperbolic. In fact, the integral curves of $W := \partial_\theta$ will be closed lightlike curves such that every point in $(t_0, t_1) \times \Sigma$ lies on one of these curves. The d'Alembert operator takes the form

$$\Box = -2\partial_t \partial_\theta + \Delta_{\tilde{g}}.$$

We see that function of the form $u(t, x) = a(t)$, where $a \in C_c^\infty(t_0, t_1)$ solves $\Box u = 0$. This shows that *the solution, if it exists, does not need to be unique in the non-globally hyperbolic region, even if the past Cauchy horizon at $\mathcal{H}_-$ is compact, non-degenerate and totally geodesic.*

6.3. The null time function

The first step towards the proof of Theorem 6.2.2 is to foliate a small future neighbourhood of the, say, past Cauchy horizon and express the metric in terms of the obtained "null time function". We assume from now on that $\mathcal{H}$ is a *past* Cauchy horizon, which is non-empty, compact, smooth, totally geodesic and non-degenerate. The case when $\mathcal{H}$ is a future Cauchy horizon is obtained by a time reversal. Recall by Lemma 6.1.5 that we can choose a lightlike non-vanishing vector field V on $\mathcal{H}$ such that $\nabla_V V = V$. By assumption,

$$g(\nabla_X V, Y) = 0$$

for all $X, Y \in T\mathcal{H}$. Hence for all $X \in T\mathcal{H}$,

$$\nabla_X V = \omega(X)V$$

where ω is a smooth one-form on $\mathcal{H}$. Moreover, $\omega(V)V = \nabla_V V = V$ and hence $\omega(V) = 1$. It follows that

$$T\mathcal{H} = \mathbb{R}V \oplus \ker(\omega).$$

Hence defining $E := \ker(\omega)$ gives the splitting

$$T\mathcal{H} = \mathbb{R}V \oplus E.$$

Proposition 6.3.1 (The null time function)**.** *There is an open neighbourhood $U \subset M \sqcup \mathcal{H}$, containing $\mathcal{H}$ and a smooth function $t : U \to \mathbb{R}$ such that $(U, g|_U)$ is isometric to*

$$[0, \epsilon) \times \mathcal{H},$$

where t is the coordinate on $[0, \epsilon)$ and the metric takes the form

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -\psi & 0 \\ 0 & 0 & \tilde{g} \end{pmatrix} \tag{6.2}$$

with respect to the splitting $T([0, \epsilon) \times \mathcal{H}) = \mathbb{R}\partial_t \oplus \mathbb{R}\mathrm{grad}(t) \oplus E$. Here $\psi \in C^\infty([0, \epsilon) \times \mathcal{H})$ is such that $\partial_t\psi(t, \cdot), \psi(t, \cdot) > 0$ for all $t \in (0, \epsilon)$ and $\partial_t\psi(0, \cdot) = 2$ and $\psi(0, \cdot) = 0$. The induced (time-dependent) metric $\tilde{g}$ on the vector bundle E is positive definite. Moreover, ∂_t is a lightlike geodesic vector field and $\Sigma_t := \{t\} \times \mathcal{H}$ are Cauchy hypersurfaces for $t \in (0, \epsilon)$.

Proof. Note that $T\hat{M}|_{\mathcal{H}} = E \oplus E^\perp$, since E is a Riemannian subbundle of $T\mathcal{H}$ and hence of $T\hat{M}|_{\mathcal{H}}$. Since $E^\perp$ is a Lorentzian subbundle of rank 2, there is a unique future pointing lightlike vector field $L \in E^\perp$ such that $g(L, V) = -1$. Consider now the map

$$\begin{aligned} f_s : \mathcal{H} &\to \hat{M}, \\ x &\mapsto \exp_x(L_x s), \end{aligned}$$

for those $s \in \mathbb{R}$ where this map is defined. By compactness of $\mathcal{H}$, there is an $\epsilon > 0$ such that f_s is defined $s \in [0, \epsilon)$ and such that the map

$$\begin{aligned} F : [0, \epsilon) \times \mathcal{H} &\to \hat{M}, \\ (s, x) &\mapsto f_s(x), \end{aligned}$$

is a diffeomorphism (of manifolds with boundary) onto its image. Note that for each $x \in \mathcal{H}$, $s \mapsto f_s(x)$ is a future pointing lightlike curve. By Lemma 6.1.3, $M \sqcup \mathcal{H}$ is a manifold with boundary, so $f_s(x) \in M$ for small positive s or small negative s. If $f_s(x) \in M$ for small negative s, it would follow by continuity, that there exists a past directed timelike curve γ, such that $\gamma(0) = x$ and $\gamma(s') \in M$ for all s' small. Since M is globally hyperbolic, we can extend γ to a past directed timelike curve reaching Σ. In other words, $x \in I_+(\Sigma)$. On the other hand, by [26, Proposition 53 (1)], $x \in \mathcal{H} \subset I_-(\Sigma)$. This contradicts achronality of Σ, which shows that $f_s(x) \in M$ for small positive s. It follows by compactness of $\mathcal{H}$ and Lemma 6.1.3 that, after shrinking ϵ if necessary, $F((0,\epsilon) \times \mathcal{H}) \subset M$.

Denote the coordinate on $[0,\epsilon)$ by t. By construction we have $\nabla_{\partial_t}\partial_t = 0$, i.e. ∂_t is a geodesic vector field. Let now $(e_i)_{i=2}^n$ be a local orthonormal basis of the Riemannian subbundle $E \subset T\mathcal{H}$. Extend the $(e_i)_{i=2}^n$ to $T([0,\epsilon) \times \mathcal{H})$ by $(0, e_i)$, which we still call e_i. It follows that $[e_i, \partial_t] = 0$. Moreover, it follows that $\partial_t g(\partial_t, e_i) = g(\nabla_{\partial_t}\partial_t, e_i) + g(\partial_t, \nabla_{\partial_t} e_i) = g(\partial_t, \nabla_{e_i}\partial_t) = \frac{1}{2}\partial_{e_i} g(\partial_t, \partial_t) = 0$. Since $g(\partial_t, e_i)|_{\{0\}\times\mathcal{H}} = g(L, e_i) = 0$, it follows that $g(\partial_t, e_i) = 0$ everywhere. Since $\mathrm{grad}(t)|_{\{0\}\times\mathcal{H}} = -V$, we note that ∂_t and $\mathrm{grad}(t)$ are linearly independent on $[0,\epsilon) \times \mathcal{H}$, making ϵ even smaller if necessary. Moreover, $g(\mathrm{grad}(t), e_i) = dt(e_i) = \partial_{e_i} t = 0$ and $g(\mathrm{grad}(t), \partial_t) = 1$ everywhere. Hence we have shown up to now that

$$\begin{aligned} g(\partial_t, \partial_t) &= 0, \\ g(\partial_t, \mathrm{grad}(t)) &= 1, \\ g(\partial_t, e_i) &= 0, \\ g(\mathrm{grad}(t), e_i) &= 0. \end{aligned}$$

for all $i = 2, \ldots, n$. We define $\psi := -g(\mathrm{grad}(t), \mathrm{grad}(t))$, which completes the form of the metric stated in (6.2). Since $\mathrm{grad}(t)|_{\{0\}\times\mathcal{H}} = -V$ it follows that $\psi(0,\cdot) = g(-V,-V) = 0$. In order to calculate $\partial_t\psi(0,\cdot)$, first extend the vector field V to $T([0,\epsilon) \times \mathcal{H})$ by $(0, V)$, still denoting it V. We have $[V, \partial_t] = 0$. It follows that $\partial_t\psi(0,\cdot) = -2g(\nabla_{\partial_t}\mathrm{grad}(t), -V)|_{\{0\}\times\mathcal{H}} = -2\partial_t g(\mathrm{grad}(t), -V)|_{\{0\}\times\mathcal{H}} + 2g(-V, -\nabla_{\partial_t} V)|_{\{0\}\times\mathcal{H}} = 2g(V, \nabla_V \partial_t)|_{\{0\}\times\mathcal{H}} = -2g(\nabla_V V, \partial_t)|_{\{0\}\times\mathcal{H}} = 2$. Shrinking ϵ if necessary, we can make sure that $\partial_t\psi(t,\cdot) > 0$ for all $t \in [0,\epsilon)$ and $\psi(t,\cdot) > 0$ for all $t \in (0,\epsilon)$.

Choosing ϵ smaller if needed, we can make sure that $\mathrm{grad}(t)$ is timelike on $(0,\epsilon) \times \mathcal{H}$, which implies that hypersurfaces $\{t\}\times\mathcal{H}$ are compact spacelike hypersurfaces in the globally hyperbolic Lorentz manifold M, for all $t \in (0,\epsilon)$. By [11, Theorem 1] it follows (the statement is given in $n = 3$, but the proof goes through in any dimension) that the level sets $\{t\} \times \mathcal{H}$ are Cauchy hypersurfaces, for all $t \in (0,\epsilon)$. □

The idea is to first solve the wave equation locally, on the manifold $[0,\epsilon) \times \mathcal{H}$ for some small $\epsilon > 0$, with the above form of the metric. For this, it will be useful to use the basis $(\partial_t, \mathrm{grad}(t), e_2, \ldots, e_n)$. Let us define the "$E$-Laplace operator"

$$\bar{\Delta} u := -\sum_{i,j\geq 2} g^{ij} \left(\partial_{e_i}\partial_{e_j} u - \partial_{\nabla_{e_i} e_j} u\right).$$

for $u \in C^\infty([0,\epsilon) \times \mathcal{H})$. Since E is a subbundle of $T([0,\epsilon) \times \mathcal{H})$, this is well-defined (independent of the choice of basis). Let us also introduce some notation. Often, we will consider vector fields $X \in C^\infty([0,\epsilon) \times \mathcal{H}, T([0,\epsilon) \times \mathcal{H}))$ that are tangent to the Cauchy horizon and to the Cauchy hypersurfaces everywhere , i.e. $X \in T(\{t\} \times \mathcal{H})$ for all $t \in [0, \epsilon)$. To simplify notation, let us just write

$$X \in T(\{t\} \times \mathcal{H}),$$

for all $t \in [0, \epsilon)$ and also mean that X is a smooth vector field, unless stated otherwise.

Lemma 6.3.2. *The d'Alembert operator is given by*

$$\Box = -\partial_t(2\partial_{\mathrm{grad}(t)} + \psi\partial_t) + \partial_Y + \bar{\Delta},$$

with $Y \in T(\{t\} \times \mathcal{H})$, for all $t \in [0, \epsilon)$.

Proof. By Proposition 6.3.1, the metric is given by

$$g_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\psi & 0 \\ 0 & 0 & g_{ij} \end{pmatrix} \Rightarrow g^{\alpha\beta} = \begin{pmatrix} \psi & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & g^{ij} \end{pmatrix},$$

for $i, j \geq 2$, in the basis $(\partial_t, \mathrm{grad}(t), e_2, \ldots, e_n)$. This implies that

$$\begin{aligned} \Box u &= -\mathrm{tr}_g(\nabla^2)u \\ &= -2\left(\partial_t\partial_{\mathrm{grad}(t)}u - \partial_{\nabla_{\partial_t}\mathrm{grad}(t)}u\right) - \psi{\partial_t}^2 u - \sum_{i,j\geq 2} g^{ij}\nabla^2_{e_i,e_j}u. \end{aligned}$$

Since $g(\nabla_{\partial_t}\mathrm{grad}(t), \partial_t) = 0$ and $g(\nabla_{\partial_t}\mathrm{grad}(t), \mathrm{grad}(t)) = -\frac{1}{2}\partial_t\psi$, we have

$$\nabla_{\partial_t}\mathrm{grad}(t) = -\frac{1}{2}(\partial_t\psi)\partial_t + \sum_{i,j\geq 2} g(\nabla_{e_i}\mathrm{grad}(t), \partial_t)g^{ij}e_j.$$

Since $e_j \in T(\{t\} \times \mathcal{H})$, we have proven that

$$\Box u = -\partial_t(2\partial_{\mathrm{grad}(t)}u + \psi\partial_t u) + \partial_Y u + \bar{\Delta}u,$$

where $Y := 2\sum_{i,j\geq 2} g(\nabla_{e_i}\mathrm{grad}(t), \partial_t)g^{ij}e_j \in T(\{t\} \times \mathcal{H})$ for all t. □

6.4. A Riemannian metric on the Cauchy horizon

To do analysis in a neighbourhood of the Cauchy horizon, it will be convenient to define a Riemannian metric on $\mathcal{H}$, which corresponds to $\{0\} \times \mathcal{H}$ under the isometry of the previous section. We define it as follows:

$$\sigma(X, Y) := g(X, Y) + g(X, \partial_t)g(Y, \partial_t),$$

for all $X, Y \in T\mathcal{H}$. Denote the induced Levi-Civita connection on $T\mathcal{H}$ by ∇^σ. Note that a g-orthonormal basis $(e_i)_{i=2}^n$ of E together with V will be an orthonormal basis with respect to σ on $\mathcal{H}$. Using the classical Koszul formula, one calculates that

$$\nabla^\sigma_{e_i} e_i = \sum_{j=2}^n g(\nabla_{e_i} e_i, e_j) e_j \tag{6.3}$$

for $i = 2, \dots, n$. Since $\sigma(\nabla^\sigma_V V, V) = 0$, we have

$$\begin{aligned} \mathrm{div}_\sigma(V) &= -\sum_{i=2}^n g(\nabla^\sigma_{e_i} e_i, V) \\ &= -\sum_{i,j=2}^n g(\nabla_{e_i} e_i, e_j) g(e_j, V) \\ &= 0. \end{aligned} \tag{6.4}$$

Let us choose the metric $dt^2 + \sigma$ on $[0, \epsilon) \times \mathcal{H}$ and consider the induced Levi-Civita connection $\nabla^{dt^2+\sigma}$. Since for all $X, Y \in T(\{t\} \times \mathcal{H})$, $\nabla^{dt^2+\sigma}_X Y = (0, \nabla^\sigma_X Y) \in T(\{t\} \times \mathcal{H})$ we are going to suppress notation and write ∇^σ for $\nabla^{dt^2+\sigma}$.

Let us define the "gradient in E-direction"

$$\bar{\nabla} u := \sum_{i,j=2}^n g^{ij} (\partial_{e_i} u) e_j$$

for $u \in C^\infty([0, \epsilon) \times \mathcal{H})$. The following lemma is of central importance for the energy estimates.

Lemma 6.4.1. *There exist a vector field $X \in T(\{t\} \times \mathcal{H})$ for all $t \in [0, \epsilon)$ and $\varphi \in C^\infty([0, \epsilon) \times \mathcal{H})$ with $\varphi(0, \cdot) = 0$ such that for all $u, v \in C^\infty((0, \epsilon) \times \mathcal{H})$*

$$\int_{\mathcal{H}} v \bar{\Delta} u d\mu_\sigma = \int_{\mathcal{H}} g(\bar{\nabla} v, \bar{\nabla} u) d\mu_\sigma + \int_{\mathcal{H}} v(\partial_X + \varphi \partial_t) u d\mu_\sigma. \tag{6.5}$$

Proof. Choose a local orthonormal frame $(e_1 = V, e_2, \dots, e_n)$ with $e_2, \dots, e_n \in E$. We calculate

$$\begin{aligned} \mathrm{div}_\sigma((\bar{\nabla} u) v) &= \sum_{k=1}^n \sigma(\nabla^\sigma_{e_k}((\bar{\nabla} u) v), e_k) \\ &= \sum_{k=1}^n \sigma(\bar{\nabla} u, e_k) \partial_{e_k}(v) + v \sum_{k=1}^n \sigma(\nabla^\sigma_{e_k}(\bar{\nabla} u), e_k) \\ &= \sum_{i,j=2}^n g^{ij} (\partial_{e_i} u)(\partial_{e_j} v) + v \sum_{i,j=2}^n \partial_{e_j}(g^{ij})(\partial_{e_i} u) + g^{ij} \partial_{e_i} \partial_{e_j} u + g^{ij} (\partial_{e_i} u) \mathrm{div}_\sigma(e_j) \\ &= g(\bar{\nabla} u, \bar{\nabla} v) - v \bar{\Delta} u + v \sum_{i,j=2}^n \partial_{e_j}(g^{ij})(\partial_{e_i} u) + g^{ij} \partial_{\nabla_{e_i} e_j} u + g^{ij} (\partial_{e_i} u) \mathrm{div}_\sigma(e_j). \end{aligned}$$

It follows that the last sum is independent of the choice of orthonormal frame. The only term that does not in general differentiate u in $\{t\}\times\mathcal{H}$-direction is the term $\sum_{i,j\geq 2} g^{ij}\nabla_{e_i}e_j$. However, at $t=0$, we have

$$\sum_{i,j\geq 2} g(g^{ij}\nabla_{e_i}e_j|_{\{0\}\times\mathcal{H}}, V) = -\sum_{i,j\geq 2} g^{ij} g(e_j, \nabla_{e_i}V)|_{\{0\}\times\mathcal{H}} = 0,$$

since $\{0\}\times\mathcal{H}$ is totally geodesic. This implies that $g^{ij}\nabla_{e_i}e_j|_{\{0\}\times\mathcal{H}} \in T(\{0\}\times\mathcal{H})$. Using Stoke's theorem on $(\mathcal{H},\sigma)$, implies the statement. □

6.5. The L^2-energy estimate and uniqueness of solution

Let us now start by deriving the first energy estimate. From this estimate we will be able to conclude uniqueness of solution to admissible wave operators. For two functions $u, v \in C^\infty([0,\epsilon)\times\mathcal{H})$ define the L^2-inner product

$$\langle u, v\rangle := \int_{\mathcal{H}} (uv) d\mu_\sigma,$$

and the L^2-norm by

$$\|u\|^2 := \langle u, u\rangle.$$

Recall that the metric g on vector bundle $E \to [0,\epsilon)\times\mathcal{H}$, seen as a subbundle of $T([0,\epsilon)\times\mathcal{H})$, is positive definite. This means that we can define the L^2-inner product, defined for two smooth vector fields $X, Y \in C^\infty([0,\epsilon)\times\mathcal{H}, E)$ by

$$\langle X, Y\rangle := \int_{\mathcal{H}} g(X,Y) d\mu_\sigma$$

and the L^2-norm

$$\|X\|^2 := \langle X, X\rangle.$$

We define the L^2-energy by

$$E^0(u,t) := \left\|2\partial_{\mathrm{grad}(t)}u + \psi\partial_t u\right\|^2 + \left\|\sqrt{\psi}\partial_t u\right\|^2 + \left\|\bar{\nabla}u\right\|^2 + \left\|\sqrt{\psi}\bar{\nabla}u\right\|^2 + \|u\|^2.$$

Remark 6.5.1. Note that $Z := \mathrm{grad}(t) + \psi\partial_t \in T(\{t\}\times\mathcal{H})$, since $g(Z, \mathrm{grad}(t)) = 0$ and $\mathrm{grad}(t)$ is normal to the hypersurfaces $\{t\}\times\mathcal{H}$. It will sometimes be useful to write $\mathrm{grad}(t) = Z - \psi\partial_t$. Since $Z|_{\{0\}\times\mathcal{H}} = \mathrm{grad}(t)|_{\{0\}\times\mathcal{H}} = -V$, it follows that $T(\{t\}\times\mathcal{H}) = \mathbb{R}Z \oplus E|_{\{t\}\times\mathcal{H}}$ for all $t\in[0,\epsilon)$, where we shrink ϵ further if necessary. The first term in the energy can thus be rewritten, using

$$2\partial_{\mathrm{grad}(t)}u + \psi\partial_t u = 2\partial_Z u - \psi\partial_t u.$$

Before proving the first energy estimate, let us observe the following.

Lemma 6.5.2. *For every smooth vector field $X \in T(\{t\} \times \mathcal{H})$ and all $\alpha \in \mathbb{R}$, there is a constant $C > 0$ such that*

$$\begin{aligned}
\|\psi^\alpha\|_\infty &\leq Ct^\alpha, \\
\|\partial_X(\psi^\alpha)\|_\infty &\leq Ct^{\alpha+1}, \\
\langle \partial_X u, \partial_t u \rangle &\leq \frac{C}{\sqrt{t}} E^0(u,t), \\
\langle \partial_X u, 2\partial_Z u - \psi \partial_t u \rangle &\leq C E^0(u,t), \\
\|\partial_X u\|^2 &\leq C E^0(u,t),
\end{aligned}$$

for all $u \in C^\infty([0,\epsilon) \times \mathcal{H})$.

Proof. The first inequality follows by observing that

$$\frac{\|\psi^\alpha\|_\infty}{t^\alpha} = \left\| \left(\frac{\psi}{t} \right)^\alpha \right\|_\infty \to \|\partial_t \psi(0,\cdot)^\alpha\|_\infty = 2^\alpha,$$

as $t \to 0$, using that $\psi(0,\cdot) = 0$. For the second inequality note that

$$\frac{\|\partial_X(\psi^\alpha)\|_\infty}{t^\alpha} = \left\| \partial_X \left(\frac{\psi}{t} \right)^\alpha \right\|_\infty \to \|\partial_X(\partial_t \psi(0,\cdot)^\alpha)\|_\infty = 0,$$

as $t \to 0$. Since $T(\{t\} \times \mathcal{H}) = \mathbb{R}Z \oplus E|_{\{t\}\times\mathcal{H}}$ for all t, there is a vector field $e \in C^\infty([0,\epsilon) \times \mathcal{H}, E)$ and a function $a \in C^\infty([0,\epsilon) \times \mathcal{H})$ such that

$$X = aZ + e.$$

Therefore

$$\begin{aligned}
\langle \partial_X u, \partial_t u \rangle &= \langle a\partial_Z u, \partial_t u \rangle + \langle \partial_e u, \partial_t u \rangle \\
&= \langle a(\partial_Z - \psi\partial_t)u, \partial_t u \rangle + \langle a\psi\partial_t u, \partial_t u \rangle + \langle g(e, \bar{\nabla} u), \partial_t u \rangle \\
&\leq \int_{\mathcal{H}} \frac{a}{\sqrt{\psi}} ((\partial_Z - \psi\partial_t)u)(\sqrt{\psi}\partial_t u) d\mu_\sigma + \int_{\mathcal{H}} a(\sqrt{\psi}\partial_t u)^2 d\mu_\sigma \\
&\quad + C \int_{\mathcal{H}} \frac{1}{\sqrt{\psi}} \left| \bar{\nabla} u \right| (\sqrt{\psi}\partial_t u) d\mu_\sigma \\
&\leq C \left\| \frac{1}{\sqrt{\psi}} \right\|_\infty E^0(u,t) + CE^0(u,t) \\
&\leq \frac{C}{\sqrt{t}} E^0(u,t).
\end{aligned}$$

where we in the last line have used the first inequality. The fourth inequality is shown analogously to the third. For the last inequality, writing $X = aZ + e$, we have

$$\begin{aligned}
\|\partial_X u\|^2 &\leq C \left(\|\partial_Z u\|^2 + \|\partial_e u\|^2 \right) \\
&\leq C \left(\|\partial_Z u\|^2 + \left\| g(e, \bar{\nabla} u) \right\|^2 \right) \\
&\leq CE^0(u,t).
\end{aligned}$$

□

Proposition 6.5.3 (The L^2-energy estimate)**.** *Let P be an admissible wave operator with respect to $\mathcal{H}$. There is a constant $C > 0$ such that for all $t_1 \geq t_0 \geq 0$, we have*

$$\sqrt{E^0(u,t_1)} \leq C\sqrt{E^0(u,t_0)} + C\int_{t_0}^{t_1} \frac{\|Pu\|}{\sqrt{s}} ds.$$

for all $u \in C^2([t_0,t_1]\times\mathcal{H})$.

Corollary 6.5.4 (Local uniqueness of solution)**.** *Let P be an admissible wave operator with respect to $\mathcal{H}$. Assume that $u \in C^2([0,\epsilon)\times\mathcal{H})$ such that*

$$\begin{aligned} Pu &= 0, \\ u|_{\{0\}\times\mathcal{H}} &= 0. \end{aligned}$$

Then $u = 0$.

Proof of the corollary. We have

$$E^0(u,0) = 0.$$

By the energy estimate, it follows that $E^0(u,t) = 0$ for all $t > 0$. Hence we conclude that $\|u(t,\cdot)\| = 0$ for all $t > 0$. It follows that $u = 0$. ∎

For the energy estimate, recall equation (6.4), saying that

$$\operatorname{div}_\sigma(V)|_{\{0\}\times\mathcal{H}} = 0.$$

Proof of Proposition 6.5.3. Using $\operatorname{grad}(t) = Z - \psi\partial_t$, the d'Alembert operator takes the form

$$\Box = -\partial_t(2\partial_Z - \psi\partial_t) + \partial_Y + \bar{\Delta}. \tag{6.6}$$

Let us now write C for the constants in the estimates. The value of C can change from line to line. Its exact value is not important. We first look at the derivative of the second term in the energy, we have

$$\begin{aligned} \frac{d}{dt}\left(\left\|\sqrt{\psi}\partial_t u\right\|^2\right) &= 2\langle\partial_t\left(\sqrt{\psi}\partial_t u\right), \sqrt{\psi}\partial_t u\rangle \\ &= 2\langle\partial_t\left(\psi\partial_t u\right), \partial_t u\rangle - \int_{\mathcal{H}}(\partial_t\psi)(\partial_t u)^2 d\mu_\sigma \\ &\leq 2\langle\partial_t(\psi\partial_t u), \partial_t u\rangle \\ &= 2\langle\Box u, \partial_t u\rangle + 2\langle 2\partial_t\partial_Z u - \partial_Y u - \bar{\Delta}u, \partial_t u\rangle \\ &= 2\langle\Box u, \partial_t u\rangle + 2\int_{\mathcal{H}}\partial_Z(\partial_t u)^2 d\mu_\sigma \\ &\quad + 2\langle 2\partial_{[\partial_t,Z]}u - \partial_Y u, \partial_t u\rangle - 2\langle\bar{\Delta}u, \partial_t u\rangle \\ &\leq 2\langle\Box u, \partial_t u\rangle - 2\int_{\mathcal{H}}\operatorname{div}_\sigma(Z)(\partial_t u)^2 d\mu_\sigma \\ &\quad + \frac{C}{\sqrt{t}}E(u,t) - 2\langle\bar{\Delta}u, \partial_t u\rangle, \end{aligned}$$

where we in the last step have used Lemma 6.5.2 and that $[\partial_t, Z] \in T(\{t\} \times \mathcal{H})$, since $Z \in T(\{t\}\times\mathcal{H})$. Moreover, since $Z|_{\{0\}\times\mathcal{H}} = \mathrm{grad}(t)|_{\{0\}\times\mathcal{H}} = -V$, we have $\mathrm{div}(Z)|_{\{0\}\times\mathcal{H}} = 0$. Since $\mathrm{div}(Z)$ is smooth up to $t = 0$ and $\mathrm{div}(Z) \to 0$ as $t \to 0$, it follows that $\mathrm{div}(Z) \leq C\psi$ and hence

$$\int_{\mathcal{H}} \mathrm{div}(Z)(\partial_t u)^2 d\mu \leq C \left\| \sqrt{\psi}\partial_t u \right\|^2 \leq CE(u,t).$$

What remains is to estimate $-2\langle \bar{\Delta} u, \partial_t u\rangle$. For this, we need the following calculation:

$$\begin{aligned} 2g(\bar{\nabla} u, \bar{\nabla}\partial_t u) &= 2\sum_{i,k\geq 2} g^{ik}(\partial_{e_i} u)(\partial_{e_k}\partial_t u) \\ &= \partial_t g(\bar{\nabla} u, \bar{\nabla} u) - \sum_{i,k\geq 2}(\partial_t g^{ik})(\partial_{e_i} u)(\partial_{e_k} u) \\ &= \partial_t g(\bar{\nabla} u, \bar{\nabla} u) + (\partial_t g)(\bar{\nabla} u, \bar{\nabla} u), \end{aligned} \tag{6.7}$$

since $[\partial_t, e_i] = 0$. Using this, together with Lemma 6.4.1, we see that

$$\begin{aligned} -2\langle \bar{\Delta} u, \partial_t u\rangle &= 2\langle \bar{\nabla} u, \bar{\nabla}\partial_t u\rangle + 2\langle \partial_X u, \partial_t u\rangle + 2\langle \varphi\partial_t u, \partial_t u\rangle \\ &\leq -2\int_{\mathcal{H}} g(\bar{\nabla} u, \bar{\nabla}\partial_t u) d\mu_\sigma + \frac{C}{\sqrt{t}} E^0(u,t) \\ &= -\partial_t(\left\|\bar{\nabla} u\right\|^2) - \int_{\mathcal{H}} (\partial_t g)(\bar{\nabla} u, \bar{\nabla} u) d\mu_\sigma + \frac{C}{\sqrt{t}} E^0(u,t) \end{aligned}$$

where we used that φ is smooth up to $t = 0$ and $\varphi(0, \cdot) = 0$ and therefore $\varphi(t, \cdot) \leq C\psi(t, \cdot)$. Now, since g restricted to the vector bundle E is Riemannian and $\mathcal{H}$ is compact, we get the estimate

$$\begin{aligned} -(\partial_t g)(\bar{\nabla} u, \bar{\nabla} u) &\leq Cg(\bar{\nabla} u, \bar{\nabla} u) \\ &\leq CE^0(u,t). \end{aligned}$$

Altogether, we have so far proven that

$$\frac{d}{dt}\left(\left\|\sqrt{\psi}\partial_t u\right\|^2 + \left\|\bar{\nabla} u\right\|^2\right) \leq \frac{C}{\sqrt{t}} E^0(u,t) + 2\langle \Box u, \partial_t u\rangle.$$

Next we look at the time derivative of the first term in the energy.

$$\begin{aligned} &\frac{d}{dt}\left\|2\partial_Z u - \psi\partial_t u\right\|^2 \\ &\quad= 2\langle \partial_t(2\partial_Z u - \psi\partial_t u), 2\partial_Z u - \psi\partial_t u\rangle \\ &\quad= 2\langle -\Box u + \partial_Y u + \bar{\Delta} u, 2\partial_Z u - \psi\partial_t u\rangle \\ &\quad= -2\langle \Box u, 2\partial_Z u - \psi\partial_t u\rangle + 2\langle \partial_Y u, 2\partial_Z u - \psi\partial_t u\rangle + 2\langle \bar{\Delta} u, 2\partial_Z u - \psi\partial_t u\rangle \\ &\quad\leq -2\langle \Box u, 2\partial_Z u - \psi\partial_t u\rangle + CE^0(u,t) + 2\langle \bar{\Delta} u, 2\partial_Z u - \psi\partial_t u\rangle, \end{aligned}$$

where we used Lemma 6.5.2. Hence it remains to estimate $2\langle \bar{\Delta} u, 2\partial_Z u - \psi \partial_t u\rangle$. By Lemma 6.4.1, we have

$$
\begin{aligned}
&2\langle \bar{\Delta} u, 2\partial_Z u - \psi\partial_t u\rangle \\
&\quad = 2\langle \bar{\nabla} u, \bar{\nabla}(2\partial_Z u - \psi\partial_t) u\rangle + 2\langle \partial_X u, 2\partial_Z u - \psi\partial_t u\rangle + 2\langle \varphi\partial_t u, 2\partial_Z - \psi\partial_t u\rangle \\
&\quad \leq 4\langle \bar{\nabla} u, \bar{\nabla}\partial_Z u\rangle - 2\langle \bar{\nabla} u, (\bar{\nabla}\psi)\partial_t u\rangle - 2\langle \bar{\nabla} u, \psi\bar{\nabla}\partial_t u\rangle + CE^0(u,t) \\
&\quad \leq 4\langle \bar{\nabla} u, \bar{\nabla}\partial_Z u\rangle + CE^0(u,t)\left\| \frac{\bar{\nabla}\psi}{\sqrt{\psi}} \right\|_\infty - 2\langle \bar{\nabla} u, \psi\bar{\nabla}\partial_t u\rangle + CE^0(u,t).
\end{aligned}
$$

We estimate the first term as

$$
\begin{aligned}
4\langle \bar{\nabla} u, \bar{\nabla}\partial_Z u\rangle &= 4\int_{\mathcal{H}} g(\bar{\nabla} u, \bar{\nabla}\partial_Z u) d\mu_\sigma \\
&= 2\int_{\mathcal{H}} \partial_Z g(\bar{\nabla} u, \bar{\nabla} u) d\mu_\sigma + 4\int_{\mathcal{H}} g(\nabla_{\bar{\nabla} u} Z, \bar{\nabla} u) d\mu_\sigma \\
&\leq -2\int_{\mathcal{H}} \mathrm{div}_\sigma(Z) g(\bar{\nabla} u, \bar{\nabla} u) + C\left\|\bar{\nabla} u\right\|^2 \\
&\leq CE^0(u,t),
\end{aligned} \tag{6.8}
$$

where we used that

$$
\begin{aligned}
\partial_Z g(\bar{\nabla} u, \bar{\nabla} u) &= 2\partial_Z g(\bar{\nabla} u, \bar{\nabla} u) - \partial_Z g(\bar{\nabla} u, \bar{\nabla} u) \\
&= 2\partial_Z \partial_{\bar{\nabla} u} u - 2g(\nabla_Z \bar{\nabla} u, \bar{\nabla} u) \\
&= 2\partial_{\bar{\nabla} u}\partial_Z u + 2\partial_{[Z,\bar{\nabla} u]} u - 2g(\nabla_{\bar{\nabla} u} Z, \bar{\nabla} u) - 2g([Z, \bar{\nabla} u], \bar{\nabla} u) \\
&= 2g(\bar{\nabla}\partial_Z u, \bar{\nabla} u) - 2g(\nabla_{\bar{\nabla} u} Z, \bar{\nabla} u).
\end{aligned}
$$

We estimate the second term as

$$
\begin{aligned}
-2\langle \bar{\nabla} u, \psi\bar{\nabla}\partial_t u\rangle &= -2\int_{\mathcal{H}} \psi g(\bar{\nabla} u, \bar{\nabla}\partial_t u) d\mu_\sigma \\
&= -\partial_t\left(\left\|\sqrt{\psi}\bar{\nabla} u\right\|^2\right) + \int_{\mathcal{H}} (\partial_t\psi) g(\bar{\nabla} u, \bar{\nabla} u) d\mu_\sigma \\
&\quad - \int_{\mathcal{H}} \psi(\partial_t g)(\bar{\nabla} u, \bar{\nabla} u) d\mu_\sigma \\
&\leq -\partial_t\left(\left\|\sqrt{\psi}\bar{\nabla} u\right\|^2\right) + CE^0(u,t),
\end{aligned}
$$

where we have used (6.7). Hence, we have proven that

$$
\frac{d}{dt}\left(\|2\partial_Z u - \psi\partial_t u\|^2 + \left\|\sqrt{\psi}\bar{\nabla} u\right\|^2\right) \leq \frac{C}{\sqrt{t}} E^0(u,t) - 2\langle \Box u, 2\partial_Z u - \psi\partial_t u\rangle.
$$

The last term in the energy is estimated as

$$\begin{aligned}\frac{d}{dt}\left(\|u\|^2\right) &= 2\langle\partial_t u, u\rangle \\ &\leq 2\|\partial_t u\|\|u\| \\ &\leq \frac{C}{\sqrt{t}}\left\|\sqrt{\psi}\partial_t u\right\|\|u\| \\ &\leq \frac{C}{\sqrt{t}}E^0(u,t).\end{aligned}$$

Altogether, we have shown that

$$\frac{d}{dt}E^0(u,t) \leq \frac{C}{\sqrt{t}}E^0(u,t) + 2\langle\Box u, \partial_t u - 2\partial_Z u + \psi\partial_t u\rangle.$$

Recall that $Pu = \Box u + \partial_W u + \alpha u$, where $W|_{\mathcal{H}}$ is nowhere outward pointing. This implies that

$$W = a_1\partial_t + a_2\partial_t + X,$$

where $a_1, a_2 \in C^\infty([0,\epsilon)\times\mathcal{H})$ such that $a_1(0,\cdot) = 0$, $a_2 \geq 0$ and $X \in T(\{t\}\times\mathcal{H})$. Hence $a_1 \leq C\psi$. Let us insert this above. We get

$$\begin{aligned}\frac{d}{dt}E^0(u,t) &\leq \frac{C}{\sqrt{t}}E^0(u,t) + 2\langle Pu - \partial_W u - \alpha u, \partial_t u - 2\partial_Z u + \psi\partial_t u\rangle \\ &\leq \frac{C}{\sqrt{t}}E^0(u,t) + \frac{C}{\sqrt{t}}\sqrt{E^0(u,t)}\,\|Pu\| - 2\langle a_1\partial_t u, \partial_t u\rangle - 2\langle a_2\partial_t u, \partial_t u\rangle \\ &\quad - \langle\partial_X u, \partial_t u\rangle + \langle(a_1+a_2)\partial_t u, 2\partial_Z u - \psi\partial_t u\rangle + \langle\partial_X u, 2\partial_Z u - \psi\partial_t u\rangle \\ &\quad - 2\langle\alpha u, \partial_t u\rangle + 2\langle\alpha u, 2\partial_Z - \psi\partial_t u\rangle \\ &\leq \frac{C}{\sqrt{t}}E^0(u,t) + \frac{C}{\sqrt{t}}\sqrt{E^0(u,t)}\,\|Pu\| + C\left\|\sqrt{\psi}\partial_t u\right\|^2 - 2\int_{\mathcal{H}} a_2(\partial_t u)^2 d\mu_\sigma \\ &\leq \frac{C}{\sqrt{t}}E^0(u,t) + \frac{C}{\sqrt{t}}\sqrt{E^0(u,t)}\,\|Pu\|\end{aligned}$$

since $a_2 \geq 0$. This is equivalent to

$$\frac{d}{dt}\sqrt{E^0(u,t)} \leq \frac{C}{\sqrt{t}}\sqrt{E^0(u,t)} + \frac{C}{\sqrt{t}}\|Pu\|.$$

We conclude that

$$\frac{d}{dt}\left(\sqrt{E^0(u,t)}e^{-2C\sqrt{t}}\right) \leq Ce^{-2C\sqrt{t}}\frac{\|Pu\|}{\sqrt{t}}$$

and hence

$$\sqrt{E^0(u,t_1)} \leq e^{2C(\sqrt{t_1}-\sqrt{t_0})}\sqrt{E^0(u,t_0)} + Ce^{2C\sqrt{t_1}}\int_{t_0}^{t_1} e^{-2C\sqrt{s}}\frac{\|Pu\|}{\sqrt{s}}ds.$$

Since $0 \leq t_0 \leq t_1 < \epsilon$, we can estimate the exponential functions by new constants and obtain the statement of the proposition. □

6.6. The higher energy estimates

For the existence result, we will prove slightly different energy estimates compared to the one for the uniqueness result. We will be able to control arbitrary Sobolev norms, however to the cost of less control on the growth as $t \to 0$. This estimate will hold for *all* wave operators. That the wave operator is admissible is used later in the existence proof. Recall that the Sobolev norms with respect to σ are defined as follows. Let Δ denote the Laplace operator on $\mathcal{H}$ with respect to σ, i.e.

$$\Delta := -\mathrm{tr}_\sigma((\nabla^\sigma)^2).$$

Since σ is independent of time, we have $[\partial_t, \Delta] = 0$ on functions. Since we are only interested in smooth solutions here, it is convenient to restrict ourselves to even Sobolev degrees $2m$, since $(1+\Delta)^m$ is a differential operator. For an integer m, recall the Sobolev inner products

$$\langle u, v\rangle_{2m} := \langle (1+\Delta)^m u, (1+\Delta)^m v\rangle,$$

with induced Sobolev norms

$$\|u\|_{2m}^2 := \langle u, u\rangle_{2m}.$$

Recall that (see for example [1, Theorem 5.23]) if $2m > \frac{\dim(\mathcal{H})}{2}$, there is a constant $C_m > 0$ such that

$$\|uv\|_{2m} \leq C_m \|u\|_{2m} \|v\|_{2m}, \tag{6.9}$$

for all $u, v \in \mathcal{H}$. Let us therefore from now on assume that $2m > \frac{\dim(\mathcal{H})}{2}$. We define the $2m$-energy as follows:

$$\begin{aligned} E^{2m}(u,t) := {} & \left\|2\partial_{\mathrm{grad}(t)} u + \psi \partial_t u\right\|_{2m}^2 + \left\|\sqrt{\psi}\partial_t u\right\|_{2m}^2 + \left\|\bar{\nabla}(1+\Delta)^m u\right\|^2 \\ & + \left\|\sqrt{\psi}\bar{\nabla}(1+\Delta)^m u\right\|^2 + \|u\|_{2m}^2 . \end{aligned}$$

Before we derive the energy estimates, let us prove two useful lemmas.

Lemma 6.6.1. *For each $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ and smooth vector field $X \in T(\{t\} \times \mathcal{H})$, there is a constant $C = C(\alpha, m)$ such that*

$$\begin{aligned} \|\psi^\alpha\|_{2m} &\leq Ct^\alpha, \\ \|\partial_X(\psi^\alpha)\|_{2m} &\leq Ct^{\alpha+1}, \\ \|[(1+\Delta)^m, \psi^\alpha]u\| &\leq Ct^{\alpha+1}\|u\|_{2m-1}, \\ \|[(1+\Delta)^m, \partial_t\psi]u\| &\leq Ct\|u\|_{2m-1}, \end{aligned}$$

for all $u \in C^\infty([0,\epsilon) \times \mathcal{H})$.

Proof. To obtain the first inequality, note that

$$\frac{\|\psi^\alpha\|_{2m}}{t^\alpha} = \left\|\left(\frac{\psi}{t}\right)^\alpha\right\|_{2m} \to \|(\partial_t \psi|_{t=0})^\alpha\|_{2m} = \|2^\alpha\|_{2m}.$$

as $t \to 0$. In particular there is a constant C such that

$$\frac{\|\psi^\alpha\|_{2m}}{t^\alpha} \leq C$$

for all $t \in (0, \epsilon)$ as claimed. For the second inequality, note that

$$\frac{\|\partial_X(\psi^\alpha)\|_{2m}}{t^\alpha} = \left\| \partial_X \left(\frac{\psi}{t} \right)^\alpha \right\|_{2m} \to \|\partial_X(2^\alpha)\|_{2m} = 0.$$

The third inequality follows since $[(1+\Delta)^m, \psi^\alpha]$ is a differential operator of order $2m-1$ and

$$\frac{\|[(1+\Delta)^m, \psi^\alpha]u\|}{t^\alpha} = \left\| \left[(1+\Delta)^m, \left(\frac{\psi}{t} \right)^\alpha \right] u \right\| \to \|[(1+\Delta)^m, 2^\alpha]u\| = 0.$$

The last inequality follows by evaluating the left hand side at $t = 0$, i.e.

$$\|[(1+\Delta)^m, \partial_t\psi]u\| \, |_{t=0} = \|[(1+\Delta)^m, 2]u\| = 0.$$

□

Lemma 6.6.2. *For each $m \in \mathbb{N}$ such that $2m > \frac{\dim \mathcal{H}}{2}$, we have*

$$\|u\|^2_{2m+1} \leq CE^{2m}(u,t)$$

and

$$\|\partial_t u\|^2_{2m} \leq \frac{C}{t} E^{2m}(u,t)$$

for $t \in (0, \epsilon)$.

This means that the energy can control the value of the function at $t = 0$ but not the first time derivative. This is what one expects, since we only specify the value and not the first derivative at the Cauchy horizon.

Proof. We will use that $\mathrm{grad}(t) = -\psi\partial_t + Z$, where $Z|_\mathcal{H} = -V$ and $Z \in T(\{t\} \times \mathcal{H})$. By Remark 6.5.1, we know that $T(\{t\} \times \mathcal{H}) = \mathbb{R}Z \oplus E|_t$ for all t and hence the norm $\|\partial_Z u\|_{2m} + \|\bar{\nabla} u\|_{2m} + \|u\|_{2m}$ is equivalent to $\|u\|_{2m+1}$, since $\{t\} \times \mathcal{H}$ is compact. We have

$$\begin{aligned}
\|u\|^2_{2m+1} &\leq C(\|\partial_Z u\|^2_{2m} + \|\bar{\nabla} u\|^2_{2m} + \|u\|^2_{2m}) \\
&\leq C \|(2\partial_Z - \psi\partial_t)u\|^2_{2m} + C\|\psi\partial_t u\|^2_{2m} + CE^{2m}(u,t) \\
&\leq C \left\| \sqrt{\psi}\partial_t u \right\|^2_{2m} \left\| \sqrt{\psi} \right\|^2_{2m} + CE^{2m}(u,t) \\
&\leq CE^{2m}(u,t).
\end{aligned}$$

The second statement follows by

$$\begin{aligned}
\|\partial_t u\|^2_{2m} &\leq C \left\| \frac{1}{\sqrt{\psi}} \right\|^2_{2m} \left\| \sqrt{\psi}\partial_t u \right\|^2_{2m} \\
&\leq \frac{C}{t} E^{2m}(u,t).
\end{aligned}$$

□

Let us now prove the energy estimate for the higher energies. Note that it does not require the wave operator to be admissible in the sense of Definition 6.2.1.

Proposition 6.6.3 (The higher energy estimates)**.** *Let P be any wave operator. Given an integer m such that $2m > \frac{\dim(\mathcal{H})}{2}$, there exists a constant $C > 1$ such that for all $t_1 \geq t_0 > 0$, we have*

$$\frac{1}{t_1{}^C}\sqrt{E^{2m}(u,t_1)} \leq \frac{1}{t_0{}^C}\sqrt{E^{2m}(u,t_0)} + \int_{t_0}^{t_1} \frac{C}{t^{C+1/2}} \left\|Pu\right\|_{2m} dt$$

for all $u \in C^\infty([t_0,t_1] \times \mathcal{H})$.

Proof. We will throughout the proof use Lemma 6.6.1 and Lemma 6.6.2 without explicitly mentioning it. Again, let us write $\operatorname{grad}(t) = -\psi\partial_t + Z$, where $Z \in T(\{t\} \times \mathcal{H})$ for all t. The d'Alembert operator takes the form

$$\Box = \partial_t(\psi\partial_t - 2\partial_Z) + \partial_Y + \bar{\Delta}.$$

Similar to the the proof of Proposition 6.5.3, let us start with the second term in the energy, we have

$$\begin{aligned}
\frac{d}{dt}\left(\left\|\sqrt{\psi}\partial_t u\right\|_{2m}^2\right) &= 2\langle\partial_t\left(\sqrt{\psi}\partial_t u\right), \sqrt{\psi}\partial_t u\rangle_{2m} \\
&= -\langle\frac{\partial_t\psi}{\psi}\sqrt{\psi}\partial_t u, \sqrt{\psi}\partial_t u\rangle_{2m} + 2\langle\frac{1}{\sqrt{\psi}}\partial_t(\psi\partial_t u), \sqrt{\psi}\partial_t u\rangle_{2m} \\
&= -\langle\frac{\partial_t\psi}{\psi}\sqrt{\psi}\partial_t u, \sqrt{\psi}\partial_t u\rangle_{2m} + 2\langle\frac{1}{\sqrt{\psi}}\Box u, \sqrt{\psi}\partial_t u\rangle_{2m} \\
&\quad + 4\langle\frac{1}{\sqrt{\psi}}\partial_Z\partial_t u, \sqrt{\psi}\partial_t u\rangle_{2m} + 2\langle\frac{1}{\sqrt{\psi}}(2\partial_{[\partial_t,Z]}u - \partial_Y u), \sqrt{\psi}\partial_t u\rangle_{2m} \\
&\quad - 2\langle\frac{1}{\sqrt{\psi}}\bar{\Delta}u, \sqrt{\psi}\partial_t u\rangle_{2m}.
\end{aligned} \tag{6.10}$$

Using that $\frac{\partial_t\psi}{\psi} > 0$, the first term in equation (6.10) is estimated as

$$\begin{aligned}
-\langle\frac{\partial_t\psi}{\psi}\sqrt{\psi}\partial_t u, \sqrt{\psi}\partial_t u\rangle_{2m} &= -\langle(1+\Delta)^m(\frac{\partial_t\psi}{\psi}\sqrt{\psi}\partial_t u), (1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle \\
&= -\langle[(1+\Delta)^m, \frac{1}{\psi}](\partial_t\psi\sqrt{\psi}\partial_t u), (1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle \\
&\quad - \langle\frac{1}{\psi}[(1+\Delta)^m, \partial_t\psi]\sqrt{\psi}\partial_t u, (1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle \\
&\quad - \int_{\mathcal{H}} \frac{\partial_t\psi}{\psi}\left((1+\Delta)^m(\sqrt{\psi}\partial_t u)\right)^2 d\mu_\sigma \\
&\leq C\left\|[(1+\Delta)^m, \frac{1}{\psi}](\partial_t\psi\sqrt{\psi}\partial_t u)\right\| \left\|\sqrt{\psi}\partial_t u\right\|_{2m} \\
&\quad + C\left\|\frac{1}{\psi}\right\|_\infty \left\|[(1+\Delta)^m, \partial_t\psi]\sqrt{\psi}\partial_t u\right\| \left\|\sqrt{\psi}\partial_t u\right\|_{2m} \\
&\leq CE^{2m}(u,t).
\end{aligned} \tag{6.11}$$

We leave the second term in equation (6.10) for the moment and estimate the third term in equation (6.10), using Lemma 6.5.2, as

$$\begin{aligned}
&4\langle \frac{1}{\sqrt{\psi}}\partial_Z\partial_t u, \sqrt{\psi}\partial_t u\rangle_{2m} \\
&= 4\langle \frac{1}{\sqrt{\psi}}\partial_Z(\frac{1}{\sqrt{\psi}})\sqrt{\psi}\partial_t u, \sqrt{\psi}\partial_t u\rangle_{2m} + 4\langle \frac{1}{\psi}\partial_Z(\sqrt{\psi}\partial_t u), \sqrt{\psi}\partial_t u\rangle_{2m} \\
&= 2\langle \partial_Z(\frac{1}{\psi})\sqrt{\psi}\partial_t u, \sqrt{\psi}\partial_t u\rangle_{2m} \\
&\quad + 4\langle (1+\Delta)^m(\frac{1}{\psi}\partial_Z(\sqrt{\psi}\partial_t u)), (1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle \\
&\leq C\left\|\partial_Z(\frac{1}{\psi})\right\|_{2m}\left\|\sqrt{\psi}\partial_t u\right\|_{2m}^2 \\
&\quad + 4\langle [(1+\Delta)^m, \frac{1}{\psi}]\partial_Z(\sqrt{\psi}\partial_t u), (1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle \\
&\quad + 4\langle \frac{1}{\sqrt{\psi}}(1+\Delta)^m\partial_Z(\sqrt{\psi}\partial_t u), \frac{1}{\sqrt{\psi}}(1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle \\
&\leq CE^{2m}(u,t) + \left\|\partial_Z(\sqrt{\psi}\partial_t u)\right\|_{2m-1}\left\|\sqrt{\psi}\partial_t u\right\|_{2m} \\
&\quad + 4\langle \frac{1}{\sqrt{\psi}}[(1+\Delta)^m, \partial_Z](\sqrt{\psi}\partial_t u), \frac{1}{\sqrt{\psi}}(1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle \\
&\quad + 2\int_{\mathcal{H}} \frac{1}{\psi}\partial_Z((1+\Delta)^m(\sqrt{\psi}\partial_t u))^2 d\mu_\sigma \\
&\leq CE^{2m}(u,t) + 4\langle \frac{1}{\sqrt{\psi}}[(1+\Delta)^m, \partial_Z](\sqrt{\psi}\partial_t u)), \frac{1}{\sqrt{\psi}}(1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle \\
&\quad - 2\int_{\mathcal{H}} \partial_Z(\frac{1}{\psi})((1+\Delta)^m(\sqrt{\psi}\partial_t u))^2 d\mu_\sigma \\
&\quad - 2\int_{\mathcal{H}} \frac{\mathrm{div}_\sigma(Z)}{\psi}((1+\Delta)^m(\sqrt{\psi}\partial_t u))^2 d\mu_\sigma \\
&\leq CE^{2m}(u,t) + 4\langle \frac{1}{\sqrt{\psi}}[(1+\Delta)^m, \partial_Z](\sqrt{\psi}\partial_t u), \frac{1}{\sqrt{\psi}}(1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle \\
&\quad + 2\left\|\partial_Z(\frac{1}{\psi})\right\|_\infty\left\|\sqrt{\psi}\partial_t u\right\|_{2m}^2 + 2\left\|\frac{\mathrm{div}_\sigma(Z)}{\psi}\right\|_\infty\left\|\sqrt{\psi}\partial_t u\right\|_{2m}^2 \\
&\leq CE^{2m}(u,t) + 4\langle \frac{1}{\sqrt{\psi}}[(1+\Delta)^m, \partial_Z](\sqrt{\psi}\partial_t u), \frac{1}{\sqrt{\psi}}(1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle, \qquad (6.12)
\end{aligned}$$

since $\mathrm{div}_\sigma(Z) \to -\mathrm{div}_\sigma(V) = 0$ as $t \to 0$. To estimate the remaining term in equation (6.12), note first that $[(1+\Delta)^m, \partial_Z]$ is a differential operator of order $2m$. We have the following estimate

$$4\langle\frac{1}{\sqrt{\psi}}[(1+\Delta)^m,\partial_Z](\sqrt{\psi}\partial_t u),\frac{1}{\sqrt{\psi}}(1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle \leq C\left\|\frac{1}{\psi}\right\|_\infty\left\|\sqrt{\psi}\partial_t u\right\|_{2m}^2 \leq \frac{C}{t}E^{2m}(u,t). \tag{6.13}$$

Note that this is the only term that has to be estimated by $\frac{C}{t}E^{2m}(u,t)$ instead of $\frac{C}{\sqrt{t}}E^{2m}(u,t)$. The fourth term in equation (6.10) is estimated as

$$2\langle\frac{1}{\sqrt{\psi}}(2\partial_{[\partial_t,Z]}u-\partial_Y u),\sqrt{\psi}\partial_t u\rangle_{2m} \leq C\left\|\frac{1}{\sqrt{\psi}}\right\|_{2m}\|u\|_{2m+1}\left\|\sqrt{\psi}\partial_t u\right\|_{2m} \leq \frac{C}{\sqrt{t}}E^{2m}(u,t) \tag{6.14}$$

since $[\partial_t,Z]-Y\in T(\{t\}\times\mathcal{H})$ for all t. The fifth term in equation (6.10) is estimated as

$$\begin{aligned}
&-2\langle\frac{1}{\sqrt{\psi}}\bar{\Delta}u,\sqrt{\psi}\partial_t u\rangle_{2m}\\
&=-2\langle(1+\Delta)^m(\frac{1}{\sqrt{\psi}}\bar{\Delta}u),(1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle\\
&=-2\langle[(1+\Delta)^m,\frac{1}{\sqrt{\psi}}]\bar{\Delta}u,(1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle\\
&\quad-2\langle\frac{1}{\sqrt{\psi}}[(1+\Delta)^m,\bar{\Delta}]u,(1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle\\
&\quad-2\langle\bar{\Delta}(1+\Delta)^m u,\frac{1}{\sqrt{\psi}}(1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle\\
&\overset{(6.5)}{\leq}\frac{C}{\sqrt{t}}\|u\|_{2m+1}\left\|\sqrt{\psi}\partial_t u\right\|_{2m}-2\langle\bar{\nabla}(1+\Delta)^m u,\bar{\nabla}(\frac{1}{\sqrt{\psi}}(1+\Delta)^m(\sqrt{\psi}\partial_t u))\rangle\\
&\quad-2\langle(\partial_X+\varphi\partial_t)(1+\Delta)^m u,\frac{1}{\sqrt{\psi}}(1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle\\
&\leq\frac{C}{\sqrt{t}}E^{2m}(u,t)-2\langle\bar{\nabla}(1+\Delta)^m u,\bar{\nabla}(\frac{1}{\sqrt{\psi}}[(1+\Delta)^m,\sqrt{\psi}]\partial_t u)\rangle\\
&\quad-2\langle\bar{\nabla}(1+\Delta)^m u,\bar{\nabla}(1+\Delta)^m\partial_t u\rangle\\
&\quad+C\left\|\frac{1}{\sqrt{\psi}}\right\|_\infty\|u\|_{2m+1}\left\|\sqrt{\psi}\partial_t u\right\|_{2m}+C\left\|\frac{\varphi}{\sqrt{\psi}}\right\|_\infty\|\partial_t u\|_{2m}\left\|\sqrt{\psi}\partial_t u\right\|_{2m}\\
&\leq\frac{C}{\sqrt{t}}E^{2m}(u,t)+C\|u\|_{2m+1}\left\|\left(\bar{\nabla}\frac{1}{\sqrt{\psi}}\right)[(1+\Delta)^m,\sqrt{\psi}]\partial_t u\right\|\\
&\quad+C\|u\|_{2m+1}\left\|\frac{1}{\sqrt{\psi}}\bar{\nabla}[(1+\Delta)^m,\sqrt{\psi}]\partial_t u)\right\|\\
&\quad-2\langle\bar{\nabla}(1+\Delta)^m u,\bar{\nabla}(1+\Delta)^m\partial_t u\rangle\\
&\leq\frac{C}{\sqrt{t}}E^{2m}(u,t)-2\langle\bar{\nabla}(1+\Delta)^m u,\bar{\nabla}(1+\Delta)^m\partial_t u\rangle.
\end{aligned}\tag{6.15}$$

The remaining term in equation (6.15) is estimated as

$$
\begin{aligned}
-2\langle\bar\nabla(1+\Delta)^m u, \bar\nabla(1+\Delta)^m\partial_t u\rangle &= -2\langle\bar\nabla(1+\Delta)^m u, \bar\nabla\partial_t(1+\Delta)^m u\rangle \\
&\overset{(6.7)}{=} -\frac{d}{dt}\left(\left\|\bar\nabla(1+\Delta)^m u\right\|^2\right) \\
&\quad - \int_{\mathcal{H}}(\partial_t g)(\bar\nabla(1+\Delta)^m u, \bar\nabla(1+\Delta)^m u)d\mu_\sigma \\
&\le -\frac{d}{dt}\left(\left\|\bar\nabla(1+\Delta)^m u\right\|^2\right) + CE^{2m}(u,t). \qquad (6.16)
\end{aligned}
$$

Combine the estimates (6.10 - 6.16) to get

$$
\frac{d}{dt}\left(\left\|\bar\nabla(1+\Delta)^m u\right\|^2 + \left\|\sqrt{\psi}\partial_t u\right\|_{2m}^2\right) \le \frac{C}{t}E^{2m}(u,t) + 2\langle\frac{1}{\sqrt{\psi}}\Box u, \sqrt{\psi}\partial_t u\rangle_{2m}.
$$

We continue with the first term of the energy.

$$
\begin{aligned}
\frac{d}{dt}\left\|2\partial_Z u - \psi\partial_t u\right\|_{2m}^2 &= 2\langle\partial_t(2\partial_Z u - \psi\partial_t u), 2\partial_Z u - \psi\partial_t u\rangle_{2m} \\
&= 2\langle-\Box u + \partial_Y u + \bar\Delta u, 2\partial_Z u - \psi\partial_t u\rangle_{2m} \\
&\le -2\langle\Box u, 2\partial_Z u - \psi\partial_t u\rangle_{2m} + C\left\|u\right\|_{2m+1}\left\|2\partial_Z u - \psi\partial_t u\right\|_{2m} \\
&\quad + 2\langle\bar\Delta u, 2\partial_Z u - \psi\partial_t u\rangle_{2m} \\
&\le -2\langle\Box u, 2\partial_Z u - \psi\partial_t u\rangle_{2m} + CE^{2m}(u,t) \\
&\quad + 2\langle\bar\Delta u, 2\partial_Z u - \psi\partial_t u\rangle_{2m}. \qquad (6.17)
\end{aligned}
$$

The last term in equation (6.17) is estimated as follows:

$$
\begin{aligned}
2\langle\bar\Delta u, 2\partial_Z u - \psi\partial_t u\rangle_{2m} &= 2\langle[(1+\Delta)^m, \bar\Delta]u, (1+\Delta)^m(2\partial_Z u - \psi\partial_t u)\rangle \\
&\quad + 2\langle\bar\Delta(1+\Delta)^m u, (1+\Delta)^m(2\partial_Z u - \psi\partial_t u)\rangle \\
&\le C\left\|u\right\|_{2m+1}\left\|2\partial_Z u - \psi\partial_t u\right\|_{2m} \\
&\quad + 2\langle\bar\nabla(1+\Delta)^m u, \bar\nabla(1+\Delta)^m(2\partial_Z u - \psi\partial_t u)\rangle \\
&\quad + 2\langle(\partial_X + \varphi\partial_t)(1+\Delta)^m u, (1+\Delta)^m(2\partial_Z u - \psi\partial_t u)\rangle \\
&\le CE^{2m}(u,t) + 4\langle\bar\nabla(1+\Delta)^m u, \bar\nabla(1+\Delta)^m\partial_Z u\rangle \\
&\quad - 2\langle\bar\nabla(1+\Delta)^m u, \bar\nabla(1+\Delta)^m(\psi\partial_t u)\rangle \\
&\quad + C\left\|u\right\|_{2m+1}\left\|2\partial_Z u - \psi\partial_t u\right\|_{2m} \\
&\quad + C\left\|\varphi\right\|_\infty\left\|\partial_t u\right\|_{2m}\left\|2\partial_Z u - \psi\partial_t u\right\|_{2m} \\
&\le 2CE^{2m}(u,t) + 4\langle\bar\nabla(1+\Delta)^m u, \bar\nabla(1+\Delta)^m\partial_Z u\rangle \\
&\quad - 2\langle\bar\nabla(1+\Delta)^m u, \bar\nabla(1+\Delta)^m(\psi\partial_t u)\rangle \qquad (6.18)
\end{aligned}
$$

The first term in equation (6.18) is estimated analogously to (6.8) as

$$
\begin{aligned}
4\langle\bar\nabla(1+\Delta)^m u, \bar\nabla(1+\Delta)^m\partial_Z u\rangle &= 4\langle\bar\nabla(1+\Delta)^m u, \bar\nabla[(1+\Delta)^m, \partial_Z]u\rangle \\
&\quad + 4\langle\bar\nabla(1+\Delta)^m u, \bar\nabla\partial_Z(1+\Delta)^m u\rangle \\
&\le C\left\|u\right\|_{2m+1}^2 + CE^{2m}(u,t) \\
&\le CE^{2m}(u,t). \qquad (6.19)
\end{aligned}
$$

The last term in equation (6.18) is estimated analogously to (6.7) as

$$\begin{aligned}
&-2\langle\bar{\nabla}(1+\Delta)^m u, \bar{\nabla}(1+\Delta)^m(\psi\partial_t u)\rangle \\
&\quad\leq -2\langle\bar{\nabla}(1+\Delta)^m u, [\bar{\nabla}(1+\Delta)^m, \psi]\partial_t u)\rangle \\
&\qquad - 2\langle\bar{\nabla}(1+\Delta)^m u, \psi\bar{\nabla}(1+\Delta)^m\partial_t u\rangle \\
&\quad\leq C\left\|u\right\|_{2m+1}\left\|\partial_t u\right\|_{2m} \\
&\qquad - 2\int_{\mathcal{H}}\psi g(\bar{\nabla}(1+\Delta)^m u, \bar{\nabla}\partial_t(1+\Delta)^m u)d\mu_\sigma \\
&\quad\leq C\left\|\frac{1}{\sqrt{\psi}}\right\|_{2m} E^{2m}(u,t) \\
&\qquad - \frac{d}{dt}\int_{\mathcal{H}}\psi g(\bar{\nabla}(1+\Delta)^m u, \bar{\nabla}(1+\Delta)^m u)d\mu_\sigma \\
&\qquad + \int_{\mathcal{H}}(\partial_t\psi)g(\bar{\nabla}(1+\Delta)^m u, \bar{\nabla}(1+\Delta)^m u)d\mu_\sigma \\
&\qquad - \int_{\mathcal{H}}\psi(\partial_t g)(\bar{\nabla}(1+\Delta)^m u)(\bar{\nabla}(1+\Delta)^m u)d\mu_\sigma \\
&\quad\leq \frac{C}{\sqrt{t}}E^{2m}(u,t) - \frac{d}{dt}\left\|\sqrt{\psi}\bar{\nabla}(1+\Delta)^m u\right\|^2.
\end{aligned} \tag{6.20}$$

Combining estimates (6.17 - 6.20) gives

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|2\partial_Z u - \psi\partial_t u\right\|_{2m}^2 + \left\|\sqrt{\psi}\bar{\nabla}(1+\Delta)^m u\right\|^2\right) \\
&\quad\leq \frac{C}{\sqrt{t}}E^{2m}(u,t) - 2\langle\Box u, 2\partial_Z u - \psi\partial_t u\rangle_{2m}.
\end{aligned}$$

The last term in the energy is estimated as

$$\begin{aligned}
\frac{d}{dt}\left(\left\|u\right\|_{2m}^2\right) &= 2\langle\partial_t u, u\rangle_{2m} \\
&\leq 2\left\|\partial_t u\right\|_{2m}\left\|u\right\|_{2m} \\
&\leq \frac{C}{\sqrt{t}}\left\|\sqrt{\psi}\partial_t u\right\|_{2m}\left\|u\right\|_{2m} \\
&\leq \frac{C}{\sqrt{t}}E^{2m}(u,t).
\end{aligned}$$

Altogether, we have proven that

$$\begin{aligned}
\frac{d}{dt}E^{2m}(u,t) \leq \frac{C}{t}E^{2m}(u,t) &+ 2\langle\frac{1}{\sqrt{\psi}}\Box u, \sqrt{\psi}\partial_t u\rangle_{2m} \\
&- 2\langle\Box u, 2\partial_Z u - \psi\partial_t u\rangle_{2m}.
\end{aligned} \tag{6.21}$$

A general wave operator can be written as $P = \Box + \partial_W + \alpha$, where $W \in C^\infty([0,\epsilon) \times \mathcal{H})$ and $\alpha \in C^\infty([0,\epsilon) \times \mathcal{H})$. Let us write $W = a\partial_t + X$, with $X \in T(\{0\} \times \mathcal{H})$ for all t. Inserting $\Box u = Pu - a\partial_t u - \partial_X - \alpha u$ into equation (6.21) gives

$$\begin{aligned}
&\frac{d}{dt}E^{2m}(u,t) \\
&\quad \leq \frac{C}{t}E^{2m}(u,t) + 2\langle Pu - a\partial_t u - \partial_X u - \alpha u, \partial_t u - 2\partial_Z u + \psi\partial_t u\rangle_{2m} \\
&\quad \leq \frac{C}{t}E^{2m}(u,t) + \frac{C}{\sqrt{t}}\sqrt{E^{2m}(u,t)}\left\|Pu\right\|_{2m} + C\left\|\partial_t u\right\|_{2m}^2 \\
&\qquad + \left\|\partial_t u\right\|_{2m}\left\|2\partial_Z - \psi\partial_t u\right\|_{2m} + C\left\|u\right\|_{2m+1}\left(\left\|\partial_t u\right\|_{2m} + \left\|2\partial_Z u - \psi\partial_t u\right\|_{2m}\right) \\
&\quad \leq \frac{C}{t}E^{2m}(u,t) + \frac{C}{\sqrt{t}}\sqrt{E^{2m}(u,t)}\left\|Pu\right\|_{2m}.
\end{aligned}$$

Thus we have proven that

$$\frac{d}{dt}\sqrt{E^{2m}(u,t)} \leq \frac{C}{t}\sqrt{E^{2m}(u,t)} + \frac{C}{\sqrt{t}}\left\|Pu\right\|_{2m},$$

which implies that

$$\frac{d}{dt}\left(\frac{\sqrt{E^{2m}(u,t)}}{t^C}\right) \leq C\frac{\left\|Pu\right\|_{2m}}{t^{C+\frac{1}{2}}},$$

for some constant $C > 0$. Integrating this between t_0 and t_1 gives the result. $\Box$

6.7. The asymptotic solution

The first step towards the existence proof is to show that an asymptotic solution exists, if the wave operator is admissible in the sense of Definition 6.2.1.

Proposition 6.7.1 (The asymptotic solution)**.** *Let P be an admissible wave operator with respect to $\mathcal{H}$ and let $u_0 \in C^\infty(\mathcal{H})$ and $f \in C^\infty([0,\epsilon) \times \mathcal{H})$ be given. There are functions $(u_j)_{j=0}^\infty \subset C^\infty(\mathcal{H})$ such that for each $N \in \mathbb{N}$, the function*

$$w^N(t,x) := \sum_{j=0}^{N} \frac{u_j(x)}{j!}t^j,$$

satisfies

$$\begin{aligned}
(\partial_t)^n \left(Pw^N - f\right)|_{t=0} &= 0, \\
w^N|_{t=0} &= u_0,
\end{aligned}$$

for all $n \leq N - 2$.

Proof. We start by writing

$$P = \psi\partial_t\partial_t + L_1\partial_t + L_2,$$

where L_1, L_2 are differential operators only differentiating in $\{t\}\times\mathcal{H}$-direction, with smooth coefficients up to $t = 0$. We compute the u_j's inductively, with u_0 given. The equation $(\partial_t)^n(Pw^N - f)|_{t=0} = 0$ is equivalent to

$$\sum_{j=0}^{n}\binom{n}{j}\big(((\partial_t)^j\psi)|_{t=0}u_{n-j+2} + ((\partial_t)^j L_1)|_{t=0}u_{n-j+1} + ((\partial_t)^j L_2)|_{t=0}u_{n-j}\big) = (\partial_t)^n f|_{t=0}. \tag{6.22}$$

Since $\psi(0,\cdot) = 0$ and $\partial_t\psi(0,\cdot) = 2$, we can rewrite it as

$$2nu_{n+1} + L_1|_{t=0}u_{n+1} = Q_n(u_0,\dots,u_n), \tag{6.23}$$

where Q_n is some differential operator, only differentiating in $\{t\}\times\mathcal{H}$-direction. Since P is an admissible wave operator, there is a *non-negative* function $\beta \in C^\infty(\mathcal{H})$ such that $P|_{t=0} - \Box|_{t=0} - \beta\partial_t$ is a differential operator of first order, only differentiating in $T(\{0\}\times\mathcal{H})$-direction. Hence by Lemma 6.3.2, using that $\operatorname{grad}(t) = -\psi\partial_t + Z$, with $Z|_{\mathcal{H}} = -V$, it follows that

$$\begin{aligned} L_1|_{t=0} &= 2\partial_V + \partial_t\psi(0,\cdot) + \beta \\ &= 2\partial_V + 2 + \beta. \end{aligned}$$

It follows that equation (6.23) becomes

$$\partial_V u_{n+1} + ((n+1) + \frac{1}{2}\beta)u_{n+1} = \frac{1}{2}Q_n(u_0,\dots,u_n),$$

which can be solved uniquely by Lemma 2.2.1, since $(n+1) + \frac{1}{2}\beta > 0$.

□

6.8. Construction of the local solution

Let $u_0 \in C^\infty(\mathcal{H})$ and $f \in C^\infty([0,\epsilon)\times\mathcal{H})$ be given. The purpose of this section is to show that if N is large enough, we can use w^N to construct a solution u to

$$\begin{aligned} Pu &= f, \\ u|_{t=0} &= u_0, \end{aligned}$$

where P is an admissible wave operator. For each $\tau \in (0,\epsilon)$, define $v_\tau^N \in C^\infty((0,\epsilon)\times\mathcal{H})$ by

$$\begin{aligned} Pv_\tau^N &= f, \\ v_\tau^N(\tau,\cdot) &= w^N(\tau,\cdot), \\ \partial_t v_\tau^N(\tau,\cdot) &= \partial_t w^N(\tau,\cdot). \end{aligned}$$

Similarly to above, note that $w_\tau \in C^\infty([0,\epsilon) \times \mathcal{H})$ defined by

$$w_\tau^N(t,x) := \sum_{j=0}^{N} \frac{(\partial_t)^j v_\tau^N(\tau,\cdot)}{j!}(t-\tau)^j.$$

satisfies

$$(\partial_t)^n (P w_\tau^N - f)|_{t=\tau} = 0,$$

for all $n \leq N-2$. Our first lemma is to show that w_τ^N approximates w^N as $\tau \to 0$.

Lemma 6.8.1. *As $\tau \to 0$, we have*

$$w_\tau^N \to w^N$$

in $C^\infty([0,\epsilon) \times \mathcal{H})$.

Proof. Define $u_j^\tau := (\partial_t)^j v_\tau^N(\tau,\cdot)$ for $0 \leq j \leq N$. Note that we would be done if we could show that

$$u_j^\tau \to u_j$$

in $C^\infty(\mathcal{H})$ as $\tau \to 0$. We claim first that u_j^τ are smooth in τ up to $\tau = 0$. Since

$$\begin{aligned} u_0^\tau &= v_\tau^N(\tau,\cdot) = w^N(\tau,\cdot), \\ u_1^\tau &= \partial_t v_\tau^N(\tau,\cdot) = \partial_t w^N(\tau,\cdot), \end{aligned}$$

it is clear that u_0^τ and u_1^τ are smooth in τ up to $\tau = 0$. Moreover, we have

$$\begin{aligned} \left(\frac{d}{d\tau}\right)^{k_1}\Big|_{\tau=0} u_0^\tau &= u_{k_1}, \\ \left(\frac{d}{d\tau}\right)^{k_2}\Big|_{\tau=0} u_1^\tau &= u_{k_2+1}, \end{aligned}$$

for all $k_1 \leq N$ and $k_2 \leq N-1$. Assume that for each $k \leq n+1$ that $u_k^\tau \to u_k$ as $\tau \to 0$, u_k^τ is smooth in τ up to $\tau = 0$ and that $\left(\frac{d}{d\tau}\right)^j\Big|_{\tau=0} u_k^\tau = u_{k+j}$ for all $j \leq N-k$. We know that this is true when $k = 0, 1$, so the induction step is what remains. Let us write as in the proof of Proposition 6.7.1

$$P = \psi \partial_t \partial_t v + L_1 \partial_t v + L_2 v,$$

where L_1, L_2 are differential operators only differentiating in $\{t\} \times \mathcal{H}$-direction, with smooth coefficients up to $t = 0$. The equation $(\partial_t)^n (Pv - f) = 0$ becomes

$$\sum_{j=0}^{n} \binom{n}{j} \left(((\partial_t)^j \psi)(\partial_t)^{n-j+2} v + ((\partial_t)^j L_1)(\partial_t)^{n-j+1} v + ((\partial_t)^j L_2)(\partial_t)^{n-j} v \right) = (\partial_t)^n f.$$

Evaluating this equation at $t = 0$ with $v = w^N$, gives the defining equation for the u_j's, namely

$$\sum_{j=0}^{n} \binom{n}{j} \left(((\partial_t)^j \psi)|_{t=0} u_{n-j+2} + ((\partial_t)^j L_1)|_{t=0} u_{n-j+1} + ((\partial_t)^j L_2)|_{t=0} u_{n-j} \right) = (\partial_t)^n f|_{t=0}. \tag{6.24}$$

Evaluating the same equation, but at $t = \tau$ and $v = w^N_\tau$, gives the defining equations for u^τ_j, namely,

$$\sum_{j=0}^{n} \binom{n}{j} \left(((\partial_t)^j \psi)|_{t=\tau} u^\tau_{n-j+2} + ((\partial_t)^j L_1)|_{t=\tau} u^\tau_{n-j+1} + ((\partial_t)^j L_2)|_{t=\tau} u^\tau_{n-j} \right) = (\partial_t)^n f|_{t=\tau}. \tag{6.25}$$

Subtracting equation (6.24) from equation (6.25) implies that

$$\begin{aligned} u^\tau_{n+2} = & -\frac{\tau}{\psi|_{t=\tau}} \frac{\sum_{j=1}^n \binom{n}{j} \left(-((\partial_t)^j \psi)|_{t=\tau} u^\tau_{n-j+2} + ((\partial_t)^j \psi)|_{t=0} u_{n-j+2} \right)}{\tau} \\ & - \frac{\tau}{\psi|_{t=\tau}} \frac{\sum_{j=0}^n \binom{n}{j} \left(((\partial_t)^j L_1)|_{t=\tau} u^\tau_{n-j+1} - ((\partial_t)^j L_1)|_{t=0} u_{n-j+1} \right)}{\tau} \\ & - \frac{\tau}{\psi|_{t=\tau}} \frac{\sum_{j=0}^n \binom{n}{j} \left(((\partial_t)^j L_2)|_{t=\tau} u^\tau_{n-j} - ((\partial_t)^j L_2)|_{t=0} u_{n-j} \right)}{\tau} \\ & + \frac{\tau}{\psi|_{t=\tau}} \frac{(\partial_t)^n f|_{t=\tau} - (\partial_t)^n f|_{t=0}}{\tau}. \end{aligned}$$

If follows that $u^\tau_{n+2} = \frac{h(\tau) - h(0)}{\tau}$ for a smooth function h, which implies that u^τ_{n+2} is smooth up to $\tau = 0$. Hence we can differentiate equation (6.25) with respect to τ, at $\tau = 0$ and use what we have assumed to obtain

$$\begin{aligned} (2(n+1) + L_1|_{\tau=0}) \lim_{\tau \to 0} u^\tau_{n+2} = & -\sum_{j=2}^{n+1} \binom{n+1}{j} \left(((\partial_t)^j \psi)|_{t=0} u_{n-j+3} \right) \\ & - \sum_{j=1}^{n+1} \binom{n+1}{j} \left(((\partial_t)^j L_1)|_{t=0} u_{n-j+2} \right) \\ & - \sum_{j=0}^{n+1} \binom{n+1}{j} \left(((\partial_t)^j L_2)|_{t=0} u_{n-j+1} \right). \end{aligned}$$

Comparing with equation (6.24) with n replaced by $n + 1$, we conclude that

$$(2(n+1) + L_1|_{\tau=0}) \lim_{\tau \to 0} u^\tau_{n+2} = (2(n+1) + L_1|_{t=0}) u_{n+2}.$$

We recall from the proof of Proposition 6.7.1 that

$$L_1|_{t=0} = 2\partial_V + 2 + \beta,$$

for some function $\beta \in C^\infty(\mathcal{H})$ with $\beta \geq 0$. Hence it follows that

$$(2\partial_V + 2(n+2) + \beta)(\lim_{\tau \to 0} u^\tau_{n+2} - u_{n+2}) = 0.$$

Lemma 2.2.1 now implies that $\lim_{\tau\to 0} u^\tau_{n+2} = u_{n+2}$. More general, differentiating equation (6.25) $k+1$ times and evaluating at $\tau = 0$ and comparing with equation (6.24) (with $n+k+1$ replacing n) gives recursively that

$$\left(\frac{d}{d\tau}\right)^k \Big|_{\tau=0} u^\tau_{n+2} = u_{n+2+k}$$

for all $k \leq N - (n+2)$ as claimed. By induction, we have shown that

$$\lim_{\tau\to 0} u^\tau_n = u_n,$$

for all $n \leq N$, which proves the lemma.

□

Lemma 6.8.2. *Fix a $T \in (0, \epsilon)$. For each integer m such that $2m \geq \frac{\dim(\mathcal{H})}{2}$, there is a constant $C_m > 0$ such that*

$$\left\|v^N_\tau(T,\cdot)\right\|_{2m+1} + \left\|\partial_t v^N_\tau(T,\cdot)\right\|_{2m} < C_m$$

for all $\tau > 0$.

Proof. The idea is to use the higher order energy estimates, Proposition 6.6.3, with $t_1 = T$ and $t_0 = \tau$. Note namely that at τ, we have

$$E^{2m}(v^N_\tau - w^N_\tau, \tau) = 0,$$

since $v^N_\tau(\tau,\cdot) = w^N_\tau(\tau,\cdot)$ and $\partial_t v^N_\tau(\tau,\cdot) = \partial_t w^N_\tau(\tau,\cdot)$. By construction, $Pv^N_\tau = f$ and hence by Lemma 6.6.2 and Proposition 6.6.3, it follows that

$$\begin{aligned}\left\|(v^N_\tau - w^N_\tau)(T,\cdot)\right\|_{2m+1} + \left\|(\partial_t v^N_\tau - \partial_t w^N_\tau)(T,\cdot)\right\|_{2m} &\leq \frac{C}{\sqrt{T}}\sqrt{E^{2m}(v^N_\tau - w^N_\tau, T)} \\ &\leq CT^{C-\frac{1}{2}} \int_\tau^T \frac{1}{t^{C+1/2}} \left\|f - Pw^N_\tau\right\|_{2m} dt.\end{aligned}$$

Now, the previous lemma implies in particular that $Pw^N_\tau(t,\cdot) \to Pw^N(t,\cdot)$ in $H^{2m}(\{t\}\times\mathcal{H})$ uniformly in $t \in [0,T]$ and hence

$$\begin{aligned}&\limsup_{\tau\to 0}\left(\left\|(v^N_\tau - w^N_\tau)(T,\cdot)\right\|_{2m+1} + \left\|(\partial_t v^N_\tau - \partial_t w^N_\tau)(T,\cdot)\right\|_{2m}\right) \\ &\quad\leq CT^{C-\frac{1}{2}} \int_0^T \frac{1}{t^{C+\frac{1}{2}}} \left\|f - Pw^N\right\|_{2m} dt \\ &\quad\leq CT^C \sqrt{\int_0^T \frac{1}{t^{2C+1}} \left\|f - Pw^N\right\|^2_{2m} dt}.\end{aligned}$$

Note that

$$\left(\frac{d}{dt}\right)^n \left\|f - Pw^N\right\|_{2m}^2 = \sum_{j=0}^{n} \binom{n}{j} \langle (\partial_t)^j (f - Pw^N), (\partial_t)^{n-j}(f - Pw^N)\rangle_{2m}.$$

By Proposition 6.7.1 we conclude that

$$\left(\frac{d}{dt}\right)^j \left\|f - Pw^N\right\|_{2m}^2 |_{t=0} = 0$$

for every $j \leq N-2$. Hence there is a constant $C' > 0$ such that

$$\left\|f - Pw^N\right\|_{2m}^2 \leq C' t^{N-2} \tag{6.26}$$

for small t. If we choose $N > 2C + 3$, the integral is bounded. We have hence shown that

$$\left\|(v_\tau^N - w_\tau^N)(T, \cdot)\right\|_{2m+1} + \left\|(\partial_t v_\tau^N - \partial_t w_\tau^N)(T, \cdot)\right\|_{2m} \leq C_m,$$

where C_m is independent of τ. By the previous lemma, $(w_\tau^N, \partial_t w_\tau^N)(T, \cdot)$ is uniformly bounded in $H^{2m+1} \times H^{2m}(\mathcal{H})$. Hence we conclude the statement. □

Proposition 6.8.3 (The local existence)**.** *For each $u_0 \in C^\infty(\mathcal{H})$ and $f \in C^\infty([0, \epsilon) \times \mathcal{H})$, there is a $u \in C^\infty([0, \epsilon) \times \mathcal{H})$ such that*

$$\begin{aligned} Pu &= f, \\ u|_{t=0} &= u_0. \end{aligned}$$

Proof. The previous lemma and the Rellich Lemma implies that for each k large enough, there is a sequence $v_{\tau_j}^N$, such that $\tau_j \to 0$ and

$$(v_{\tau_j}^N, \partial_t v_{\tau_j}^N)(T, \cdot) \to (a_0^k, a_1^k) \in H^k(\mathcal{H}) \times H^{k-1}(\mathcal{H}),$$

as $j \to \infty$. By a diagonal sequence argument, we find a sequence τ_i such that

$$(v_{\tau_i}^N, \partial_t v_{\tau_i}^N)(T, \cdot) \to (a_0, a_1) \in C^\infty(\mathcal{H}) \times C^\infty(\mathcal{H})$$

as $i \to \infty$ for some $(a_0, a_1) \in C^\infty(\mathcal{H}) \times C^\infty(\mathcal{H})$. Define now $v^N \in C^\infty((0, \epsilon) \times \mathcal{H})$ by

$$\begin{aligned} Pv^N &= f, \\ v^N(T, \cdot) &= a_0, \\ \partial_t v^N(T, \cdot) &= a_1. \end{aligned}$$

By continuous dependence on initial data for the Cauchy problem (see e.g. [6, Theorem 3.2.12]), it follows that $v_{\tau_j}^N \to v^N$ in $C^\infty((0, \epsilon) \times \mathcal{H})$. Hence we also get the estimate for each integer m such that $2m > \frac{\dim \mathcal{H}}{2}$:

$$
\begin{aligned}
\big\|(v^N - w^N)(t,\cdot)\big\|_{2m+1} &+ \big\|(\partial_t v^N - \partial_t w^N)(t,\cdot)\big\|_{2m} \\
&= \lim_{\tau\to 0} \Big(\big\|(v^N_\tau - w^N_\tau)(t,\cdot)\big\|_{2m+1} + \big\|(\partial_t v^N_\tau - \partial_t w^N_\tau)(t,\cdot)\big\|_{2m} \Big) \\
&\le \lim_{\tau\to 0} \frac{C}{\sqrt{t}} \sqrt{E^{2m}(v^N_\tau - w^N_\tau, t)} \\
&\le \lim_{\tau\to 0} C t^C \sqrt{\int_\tau^t \frac{1}{s^{2C+1}} \|f - P w^N_\tau\|^2_{2m}\, ds} \\
&= C t^C \sqrt{\int_0^t \frac{1}{s^{2C+1}} \|f - P w^N\|^2_{2m}\, ds}.
\end{aligned}
$$

for $t \in [0, \epsilon)$, where C is independent of N. We claim that v^N is C^2-extendible to $t = 0$ if N is chosen large enough. Let $2m > \frac{\dim(\mathcal{H})}{2} + 1$. We choose $N > 2C + 4$, so that the above inequality, equation (6.26) and the Sobolev embedding theorem implies that

$$
\begin{aligned}
\big\|(v^N - w^N)(t,\cdot)\big\|_{C^2} &+ \big\|(\partial_t v^N - \partial_t w^N)(t,\cdot)\big\|_{C^1} \\
&\le C(\big\|(v^N - w^N)(t,\cdot)\big\|_{2m+1} + \big\|(\partial_t v^N - \partial_t w^N)(t,\cdot)\big\|_{2m}) \\
&\le C t^2.
\end{aligned}
$$

Since w^N is smooth up to $t = 0$, we conclude that v^N and $\partial_t v^N$ are in C^2 respectively C^1 up to $t = 0$. Now using that $P v^N = f$ and $P w^N = f + o(t^{N-2})$, we conclude that

$$
\begin{aligned}
\big\|(\partial_t)^2 v^N(t,\cdot) - (\partial_t)^2 w^N(t,\cdot)\big\|_{C^0} &\le \left\| \frac{1}{\psi} \left(L_1 \partial_t v^N(t,\cdot) - L_1 \partial_t w^N(t,\cdot) \right) \right\|_{C^0} \\
&\quad + \left\| \frac{1}{\psi} \left(L_2 v^N(t,\cdot) - L_2 w^N(t,\cdot) \right) \right\|_{C^0} + o(t^{N-2}) \\
&\le \frac{1}{t} \big\|\partial_t v^N(t,\cdot) - \partial_t w^N(t,\cdot)\big\|_{C^1} \\
&\quad + \frac{1}{t} \big\|v^N(t,\cdot) - w^N(t,\cdot)\big\|_{C^2} + o(t^{N-2}) \\
&\le C t.
\end{aligned}
$$

This shows that also the second time derivative is C^0 up to $t = 0$. Hence v^N is C^2 up to $t = 0$, for all $N > 2C+2$. Corollary 6.5.4 implies that all v^N are equal for $N > 2C+2$. Hence define $u := v^N$ for an $N > 2C+2$. Iterating the above arguments, using $(\partial_t)^n (P v^N - f) = 0$ and $(\partial_t)^n (P w^N - f) = o(t^{N-2-n})$ for all $n \le N - 2$ and increasing N whenever necessary, shows that u is smoothly extendible to $t = 0$. Since

$$
u|_{t=0} = w^N(0,\cdot) = u_0,
$$

this finishes the proof of the proposition. □

6.9. Finishing the proof

We finish the proof by going from the local existence and uniqueness result to global existence and uniqueness.

Finishing the proof of Theorem 6.2.2. Let us start by proving the global uniqueness. Assume that

$$\begin{aligned} Pu &= 0, \\ u|_{\mathcal{H}} &= 0. \end{aligned}$$

Using the isometry given in Proposition 6.3.1 and applying Corollary 6.5.4, we conclude that there is a Cauchy surface $\Sigma \in M$ such that

$$u|_{\Sigma} = 0.$$

By [6, Theorem 3.2.11], it follows that $u = 0$ everywhere on M, since M is globally hyperbolic. This proves the uniqueness statement. Let us now turn to the global existence statement. Using Proposition 6.3.1 and Proposition 6.8.3, we know that there is an open subset $U \subset M \sqcup \mathcal{H}$, containing $\mathcal{H}$ and a Cauchy hypersurface Σ of M and a function $\tilde{u} \in C^\infty(U)$ such that $P\tilde{u} = f$ on U and $\tilde{u}|_{\mathcal{H}} = u_0$. By Theorem [6, Thereom 3.2.11], we can now solve the Cauchy problem

$$\begin{aligned} P\hat{u} &= f \\ \hat{u}|_{\Sigma} &= \tilde{u}|_{\Sigma} \\ \partial_t \hat{u}|_{\Sigma} &= \partial_t \tilde{u}|_{\Sigma} \end{aligned}$$

on M, since M is a globally hyperbolic spacetime and $\Sigma \subset M$ is a Cauchy hypersurface. Pasting $\hat{u}$ and $\tilde{u}$ together gives the globally defined solution u. We have shown that the continuous map

$$\begin{aligned} C^\infty(M \sqcup \mathcal{H}) &\to C^\infty(\mathcal{H}) \times C^\infty(M \sqcup \mathcal{H}) \\ u &\mapsto (u|_{\mathcal{H}}, Pu) \end{aligned}$$

bijective. The open mapping theorem for Fréchet spaces now implies that the inverse is continuous as well, i.e. that the solution depends continuously on u_0 and f. □

7. Local well-posedness for non-linear wave equations

The purpose of the following chapter is to show that if the Ricci curvature vanishes at the Cauchy horizon, we have local existence and uniqueness of solutions to certain non-linear wave equations, given initial data on the horizon. Since the Misner spacetime and the Taub-NUT spacetime have vanishing Ricci curvature, the results of the following chapter applies. The idea for the proof is that under the assumption that the Ricci curvature vanishes at the boundary, we are able to improve the energy estimates enough for a standard "Picard iteration" to go through. To show that the solution really is smooth up to the boundary, we need to prove existence of an asymptotic solution, just as in the proof of Theorem 6.2.2. We will consider equations of the form

$$\Box u = f(u)$$

where $f \in C^\infty(\mathbb{R})$. The Cauchy horizon will be assumed to be compact, smooth, totally geodesic and non-degenerate, as in the previous chapter.

Example 7.0.1. The non-linear wave equation

$$\Box u = \sin(u)$$

is known as the *sine-Gordon equation*. Another commonly studied non-linear wave equation is

$$\Box u = u^k$$

for some $k \in \mathbb{N}$. Theorem 7.1.1 applies to both these equations.

7.1. The statement

Recall from the previous chapter, that when a smooth, compact and totally geodesic Cauchy horizon $\mathcal{H}$ is non-degenerate, then there is a lightlike vector field $V \in C^\infty(\mathcal{H}, T\mathcal{H})$ such that

$$\nabla_V V = V.$$

Theorem 7.1.1. *Let $\hat{M}$ be a spacetime and let $\Sigma \subset \hat{M}$ be a closed acausal topological hypersurface. Assume that the past or future Cauchy horizon $\mathcal{H}$ of Σ is a non-empty, smooth, compact, totally geodesic and non-degenerate hypersurface of $\hat{M}$. Assume that*

$\mathrm{ric}(X,V) = 0$ *for all* $X \in T\mathcal{H}$, *where* V *is the smooth lightlike vector field on* $\mathcal{H}$ *such that* $\nabla_V V = V$. *Let* $f \in C^\infty(\mathbb{R})$ *and* $u_0 \in C^\infty(\mathcal{H})$ *be given. Then* $M := D(\Sigma)$ *is globally hyperbolic,* $M \sqcup \mathcal{H}$ *is a smooth manifold with boundary and there is an open set* $U \subset M \sqcup \mathcal{H}$ *such that* $\mathcal{H} \subset U$ *and a unique* $u \in C^\infty(U)$ *such that*

$$\begin{aligned} \Box u &= f(u), \\ u|_{\mathcal{H}} &= u_0. \end{aligned}$$

Recall Remark 6.2.3, saying that if $\hat{M}$ is Ricci flat (vaccum) and $\mathcal{H}$ is non-empty, compact and non-degenerate, all the other assumptions will be fulfilled immediately.

Example 7.1.2. The Misner spacetime and Taub-NUT spacetime are Ricci-flat, see Example 6.2.7. The Cauchy horizon in the Misner spacetime and both Cauchy horizons in the Taub-NUT spacetime are non-degenerate, so Theorem 7.1.1 applies.

7.2. The energy estimate

By Proposition 6.3.1, there is an open neighbourhood of $\mathcal{H}$ which is isometric to

$$[0,\epsilon) \times \mathcal{H},$$

with the metric given by (6.2). We will choose the open set $U \subset M \sqcup \mathcal{H}$ in Theorem 7.1.1 on the form $[0,T) \times \mathcal{H}$. The following lemma is the key observation in order to improve the energy estimate.

Lemma 7.2.1. *If* $\mathrm{ric}_g(X,V) = 0$ *for all* $X \in T(\{0\} \times \mathcal{H})$, *then* $\mathcal{L}_V \sigma = 0$.

Proof. The crucial point is to see that $\nabla_V e \in E$ for all local sections e in $E|_{\{0\}\times\mathcal{H}}$. Let $e_2, \ldots, e_n \in E|_{\{0\}\times\mathcal{H}}$, together with V, be a local orthonormal frame of $T\mathcal{H}$ with respect to σ. By Proposition 6.3.1, the metric is given by

$$g_{\alpha\beta}|_{\{0\}\times\mathcal{H}} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \delta_{ij} \end{pmatrix} = g^{\alpha\beta}|_{\{0\}\times\mathcal{H}},$$

for $i,j \geq 2$, in the basis $(\partial_t, V, e_2, \ldots, e_n)$. By assumption, for all $X \in \{0\} \times \mathcal{H}$ we have

$$\begin{aligned} 0 &= \mathrm{ric}(e_i, V) \\ &= -R(V, e_i, V, \partial_t) + \sum_{j=2}^n R(e_j, e_i, V, e_j) \\ &= g(\nabla_{e_i}\nabla_V V, \partial_t) - g(\nabla_V \nabla_{e_i} V, \partial_t) - g(\nabla_{[e_i,V]} V, \partial_t) \\ &\quad + \sum_{j=2}^n \big(g(\nabla_{e_j}\nabla_{e_i} V, e_j) - g(\nabla_{e_i}\nabla_{e_j} V, e_j) - g(\nabla_{[e_j,e_i]} V, e_j) \big) \\ &= g(\nabla_{e_i} V, \partial_t) + g(\nabla_{\nabla_V e_i} V, \partial_t) \\ &= g(\nabla_{\nabla_V e_i} V, \partial_t), \end{aligned}$$

where we have used that $\nabla_V V = V$ and $\nabla_{e_i} V = 0$ and that $\{0\} \times \mathcal{H}$ is totally geodesic. Again since $\{0\} \times \mathcal{H}$ is totally geodesic, we have $\nabla_V e_i \in T(\{0\} \times \mathcal{H})$ and hence

$$g(\nabla_{\nabla_V e_i} V, X) = 0$$

for all $X \in T(\{0\} \times \mathcal{H})$. Altogether we conclude that $\nabla_{\nabla_V e_i} V = 0$ and hence $\nabla_V e_i \in E$. Now we compute $\mathcal{L}_V \sigma$. Since $\{0\} \times \mathcal{H}$ is totally geodesic, it is clear that $\mathcal{L}_V g(X, Y) = 0$ for all $X, Y \in T(\{0\} \times \mathcal{H})$. We have

$$\begin{aligned} \mathcal{L}_V \sigma(X, Y) &= \mathcal{L}_V g(X, Y) + \partial_V (g(X, \partial_t) g(Y, \partial_t)) \\ &\quad - g([V, X], \partial_t) g(Y, \partial_t) - g(X, \partial_t) g([V, Y], \partial_t) \\ &= \partial_V (g(X, \partial_t) g(Y, \partial_t)) - g([V, X], \partial_t) g(Y, \partial_t) - g(X, \partial_t) g([V, Y], \partial_t). \end{aligned}$$

Clearly $\mathcal{L}_V \sigma(V, V) = 0$ since $g(V, \partial_t) = -1$. Moreover, $\mathcal{L}_V \sigma(e_i, e_j) = 0$, since $g(e_i, \partial_t) = 0$ for all i. Now, from what we showed above, we know that $[V, e_i] \in E$, which means that $g([V, e_i], \partial_t) = 0$. This implies that also $\mathcal{L}_V \sigma(e_i, V) = 0$, which finishes the proof. □

Proposition 7.2.2. *If* $\mathrm{ric}_g(X, V) = 0$ *for all* $X \in T(\{0\} \times \mathcal{H})$ *and* m *is an integer such that* $2m > \frac{\dim(\mathcal{H})}{2}$*, then there is a constant* $C' > 0$ *such that for all* $t_1 \geq t_0 \geq 0$*, we have*

$$\sqrt{E^{2m}(u, t_1)} \leq C' \sqrt{E^{2m}(u, t_0)} + C' \int_{t_0}^{t_1} \frac{\|\Box u\|_{2m}}{\sqrt{s}} ds.$$

for all $u \in C^2([t_0, t_1] \times \mathcal{H})$*.*

Proof. Since we have stronger assumptions than those in Proposition 6.6.3, we conclude that the estimate (6.21) holds true. However, we claim that if $\mathrm{ric}_g(X, V) = 0$ for all $X \in T(\{0\} \times \mathcal{H})$, we in fact have

$$\begin{aligned} \frac{d}{dt} E^{2m}(u, t) \leq \frac{C}{\sqrt{t}} E^{2m}(u, t) + 2 \langle \frac{1}{\sqrt{\psi}} \Box u, \sqrt{\psi} \partial_t u \rangle_{2m} \\ - 2 \langle \Box u, 2\partial_Z u - \psi \partial_t u \rangle_{2m}. \end{aligned} \tag{7.1}$$

The difference is that the first factor $\frac{1}{t}$ has been substituted by $\frac{1}{\sqrt{t}}$. In the proof of Proposition 6.6.3, the loss of control on the growth was due to the estimate

$$4 \langle \frac{1}{\sqrt{\psi}} [(1 + \Delta)^m, \partial_Z](\sqrt{\psi} \partial_t u)), \frac{1}{\sqrt{\psi}} (1 + \Delta)^m (\sqrt{\psi} \partial_t u) \rangle \leq \frac{C}{t} E^{2m}(u, t),$$

see equation (6.13). Since we have now assumed that $\mathrm{ric}_g(X, V) = 0$ for all $X \in T(\{0\} \times \mathcal{H})$, we know from Lemma 7.2.1 that V is a Killing vector field with respect to σ. Since Killing vector fields commutes with the Laplace operator, it follows that

$[(1+\Delta)^m, \partial_Z]|_{\{0\}\times\mathcal{H}} = [(1+\Delta)^m, -\partial_V] = 0$. This implies that

$$
\begin{aligned}
4\langle \frac{1}{\sqrt{\psi}}[(1+\Delta)^m, \partial_Z](\sqrt{\psi}\partial_t u)), &\frac{1}{\sqrt{\psi}}(1+\Delta)^m(\sqrt{\psi}\partial_t u)\rangle \\
&\leq C \left\| \frac{1}{\psi} \right\|_\infty \left\| [(1+\Delta)^m, \partial_Z](\sqrt{\psi}\partial_t u) \right\| \left\| \sqrt{\psi}\partial_t u \right\|_{2m} \\
&\leq Ct \left\| \frac{1}{\psi} \right\|_\infty \left\| \sqrt{\psi}\partial_t u \right\|_{2m} \left\| \sqrt{\psi}\partial_t u \right\|_{2m} \\
&\leq CE^{2m}(u,t),
\end{aligned}
$$

which proves equation (7.1). We conclude that

$$
\frac{d}{dt}E^{2m}(u,t) \leq \frac{C}{\sqrt{t}}E^{2m}(u,t) + \frac{C}{\sqrt{t}}\sqrt{E^{2m}(u,t)}\, \|\Box u\|_{2m}.
$$

Integrating gives the desired statement. □

7.3. The proof of local well-posedness

Similarly to when we proved existence of solutions for the linear wave equation, we will need an *asymptotic solution* for the non-linear wave equation.

Proposition 7.3.1 (The asymptotic solution)**.** *Assume the same as in Theorem 7.1.1. Let $f \in C^\infty(\mathbb{R}, \mathbb{R})$ and $u_0 \in C^\infty(\mathcal{H})$ be given. Then there are functions $(u_j)_{j\in\mathbb{N}} \subset C^\infty(\mathcal{H})$ such that*

$$
w^N(x,t) := \sum_{j=0}^{N} \frac{u_j(x)t^j}{j!}
$$

satisfies

$$
\begin{aligned}
(\partial_t)^n(\Box w^N - f(w^N))|_{t=0} &= 0, \\
w^N|_{t=0} &= u_0,
\end{aligned}
$$

for all $n \leq N-2$.

Proof. The proof is a straightforward modification of the proof of Lemma 6.7.1, using that $(d_t)^n(\Box w^N - f(w^N))|_{t=0} = 0$ is equivalent to

$$
\partial_V u_{n+1} + (n+1)u_{n+1} = \frac{1}{2}(\partial_t)^n f(w^N)|_{t=0},
$$

where the right hand side only depends on $u_0 = w^N|_{t=0}, \ldots, u_n = (\partial_t)^n w^N|_{t=0}$. □

Let us prove Theorem 7.1.1.

Proof of Theorem 7.1.1. By Proposition 6.3.1, there is an open neighbourhood of $\mathcal{H}$ which is isometric to

$$[0, \epsilon) \times \mathcal{H},$$

with the metric described in Proposition 6.3.1. We will choose our neighbourhood U such that it corresponds under the isometry to an open set of the form $[0, T) \times \mathcal{H}$ for some $T \leq \epsilon$, yet to be chosen. Let us start with the existence part. The idea is to iteratively solve linear wave equations and show that the solutions converge to a smooth solution to the non-linear equation on $[0, T) \times \mathcal{H}$. Define $v_0 \in C^\infty([0, \epsilon) \times \mathcal{H})$ as the unique solution (given by Theorem 6.2.2) to the Goursat problem

$$\begin{aligned} \Box v_0 &= 0, \\ v_0|_{\{0\}\times\mathcal{H}} &= u_0. \end{aligned}$$

for all $(t, x) \in [0, \epsilon) \times \mathcal{H}$ and define $v_n \in C^\infty([0, \epsilon) \times \mathcal{H})$, for $n \in \mathbb{N}$, to be the solution of the Goursat problem

$$\begin{aligned} \Box v_{n+1} &= f(v_n), \\ v_{n+1}|_{\{0\}\times\mathcal{H}} &= u_0. \end{aligned}$$

By Theorem 6.2.2, there is a unique solution to the above Goursat problem. Now fix an $m \in \mathbb{N}$ such that $2m > \frac{\dim(\mathcal{H})}{2}$. Any constant used in the rest of the proof will be positive only depends on ϵ, m, u_0 and f but not on t and n. Let C and C' denote the constants from Lemma 6.6.2 and Proposition 7.2.2 respectively and let $C_1, C_2, \ldots, C_8$ denote the rest of the constants. The constants in this proof will not change from line to line.

Step 1: Show that there exists a $T > 0$ such that (v_n) is a Cauchy sequence in $C^0([0, T], H^{2m}(\mathcal{H}))$. For this, we start by showing that (v_n) is bounded in $C^0([0, T], H^{2m}(\mathcal{H}))$. For $n = 0$, the energy estimate implies that

$$\begin{aligned} \|v_0(t, \cdot)\|_{2m} &\leq \sqrt{E^{2m}(v_0, t)} \\ &\leq C' \sqrt{E^{2m}(v_0, 0)} \\ &=: C_1 \end{aligned}$$

since $\Box v_0 = 0$. Since the initial data is the same for all v_n, it follows that $C_1 = C' \sqrt{E^{2m}(v_n, 0)}$ for all n. We claim that there is a $T > 0$ such that $\|v_n(t, \cdot)\|_{2m} \leq C_1 + 1$ for all $n \in \mathbb{N}_0$ and $t \in [0, T]$. Before we move to the induction step, let us note the following. By the Sobolev embedding theorem, there is a constant C_2 such that for all $t \in [0, \epsilon)$ and $v \in C^\infty([0, \epsilon) \times \mathcal{H})$ with $\|v(t, \cdot)\|_{2m} \leq C_1 + 1$, we have

$$\begin{aligned} \|v(t, \cdot)\|_\infty &\leq C_2 \|v(t, \cdot)\|_{2m} \\ &\leq C_2(C_1 + 1). \end{aligned}$$

Hence there is a constant C_3 such that if $\|v(t, \cdot)\|_{2m} \leq D_m + 1$, then

$$\begin{aligned} \|f(v)(t, \cdot)\|_{2m} &\leq C_3 \sum_{j=0}^{2m} \left\|f^{(j)}(v)(t, \cdot)\right\|_\infty \|v(t, \cdot)\|_{2m}^j \\ &\leq 2mC_3 \|f\|_{C^{2m}([-C_2(C_1+1), C_2(C_1+1)])} (C_1 + 1)^{2m} \end{aligned} \tag{7.2}$$

for all $t \in [0, \epsilon)$. Since f is smooth and the interval $[-C_2(C_1+1), C_2(C_1+1)]$ is compact, we can control the C^{2m}-norm of f on this interval by a constant only depending on m. We conclude that there is a constant C_4 such that

$$\|f(v)(t,\cdot)\|_{2m} \le C_4,$$

for all $v \in C^\infty([0,\epsilon) \times \mathcal{H})$ with $\|v(t,\cdot)\|_{2m} \le C_1 + 1$. Let us now choose

$$T := \min\left(\frac{1}{(2C'C_4+1)^2}, \epsilon\right).$$

Note that T is independent of n. For the induction step, assume that $\|v_n(t,\cdot)\|_{2m} \le C_1+1$. We claim that $\|v_{n+1}\|_{2m} \le C_1 + 1$. The energy estimate implies that

$$\begin{aligned}
\|v_{n+1}\|_{2m} &\le \sqrt{E^{2m}(v_{n+1}, t)} \\
&\le C_1 + C' \int_0^T \frac{\|f(v_n)(s,\cdot)\|_{2m}}{\sqrt{s}} ds \\
&\le C_1 + C'C_4 \int_0^T \frac{1}{\sqrt{s}} ds \\
&= C_1 + 2C'C_4\sqrt{T} \\
&\le C_1 + 1
\end{aligned}$$

as claimed. We move on to show that (v_n) is a Cauchy sequence in $C^0([0,T], H^{2m}(\mathcal{H}))$. For this, define $A_n(t) := \|v_{n+1}(t,\cdot) - v_n(t,\cdot)\|_{2m}$. We know that

$$\begin{aligned}
\Box(v_n - v_{n-1}) &= f(v_n) - f(v_{n-1}), \\
(v_n - v_{n-1})|_{\{0\}\times\mathcal{H}} &= 0.
\end{aligned}$$

Applying the energy estimate in Proposition 7.2.2, we conclude that

$$\begin{aligned}
A_n(t) \le& \sqrt{E^{2m}(v_{n+1} - v_n, t)} \\
&\le C' \int_0^t \frac{\|f(v_n)(s,\cdot) - f(v_{n-1})(s,\cdot)\|_{2m}}{\sqrt{s}} ds.
\end{aligned}$$

Note that

$$\begin{aligned}
f(v_n) - f(v_{n-1}) &= \int_0^1 \frac{d}{d\tau} f(\tau v_n + (1-\tau)v_{n-1}) d\tau, \\
&= (v_n - v_{n-1}) \int_0^1 f'(\tau v_n + (1-\tau)v_{n-1}) d\tau,
\end{aligned}$$

which implies that there is a constant C_5 such that

$$\begin{aligned}
\|f(v_n) - f(v_{n-1})\|_{2m} &= \left\|(v_n - v_{n-1}) \int_0^1 f'(\tau v_n + (1-\tau)v_{n-1}) d\tau\right\|_{2m} \\
&\le C_5 \|(v_n - v_{n-1})\|_{2m} \int_0^1 \|f'(\tau v_n + (1-\tau)v_{n-1})\|_{2m} d\tau. \qquad (7.3)
\end{aligned}$$

Since

$$\begin{aligned}\|\tau v_n(t,\cdot)+(1-\tau)v_{n-1}(t,\cdot)\|_{2m} &\leq \tau\,\|v_n(t,\cdot)\|_{2m}+(1-\tau)\,\|v_{n-1}(t,\cdot)\|_{2m}\\ &\leq C_1+1\end{aligned}$$

by assumption and since f' is smooth, we estimate as in equation (7.2) and conclude that there is a constant C_6, such that

$$\int_0^1 \|f'(\tau v_n+(1-\tau)v_{n-1})\|_{2m}\,d\tau \leq C_6,$$

for some constant C_6. Altogether, we have

$$A_n(t)\leq C'C_5C_6\int_0^t \frac{A_{n-1}(s)}{\sqrt{s}}ds.$$

Defining $C_7 := C'C_5C_6$, we get by iteration that

$$\begin{aligned}A_n(t) &\leq C_7\int_0^t \frac{A_{n-1}(s)}{\sqrt{s}}ds\\ &\leq (C_7)^n\int_0^t \frac{1}{\sqrt{s_{n-1}}}\int_0^{s_{n-1}}\frac{1}{\sqrt{s_{n-2}}}\cdots\int_0^{s_1}\frac{A_0(s_0)}{\sqrt{s_0}}ds_0\dots ds_{n-1}\\ &\leq (C_7)^n\max_{s\in[0,T]}(A_0(s))\int_0^t \frac{1}{\sqrt{s_{n-1}}}\int_0^{s_{n-1}}\frac{1}{\sqrt{s_{n-2}}}\cdots\int_0^{s_1}\frac{1}{\sqrt{s_0}}ds_0\dots ds_{n-1},\end{aligned}$$

for all $t\in[0,T]$. The term $\max_{s\in[0,T]}(A_0(s))=\max_{s\in[0,T]}(\|v_1-v_0\|_{2m}(s))$ is independent of n. The multiple integrals can also be easily calculated using the formula

$$\int_0^t s^{\alpha-\frac{1}{2}}ds=\frac{t^{\alpha+\frac{1}{2}}}{\alpha+\frac{1}{2}},$$

for all $\alpha\in\mathbb{R}$, we get

$$\int_0^t \frac{1}{\sqrt{s_{n-1}}}\int_0^{s_{n-1}}\frac{1}{\sqrt{s_{n-2}}}\cdots\int_0^{s_1}\frac{1}{\sqrt{s_0}}ds_0\dots ds_{n-1}=\frac{1}{\frac{1}{2}}\frac{1}{\frac{2}{2}}\frac{1}{\frac{3}{2}}\cdots\frac{1}{\frac{n}{2}}t^{\frac{n}{2}}=\frac{(2\sqrt{t})^n}{n!}.$$

Hence there is a new constant C_8 such that

$$A_n(t)\leq C_8\frac{(2C_7\sqrt{t})^n}{n!},$$

for all $t\in[0,T]$. Now we can estimate $\|v_{n+k}-v_n\|_{2m}$ by

$$\begin{aligned}\|v_{n+k}(t,\cdot)-v_n(t,\cdot)\|_{2m} &\leq \sum_{j=0}^{k-1}A_{n+j}(t)\\ &\leq C_8\sum_{j=0}^{k-1}\frac{(2C_7\sqrt{T})^{n+j}}{(n+j)!},\end{aligned}$$

for all $t \in [0, T]$. This shows that (v_n) is a Cauchy sequence in $C^0([0,T], H^{2m}(\mathcal{H}))$. Hence there is a limit $u \in C^0([0,T], H^{2m}(\mathcal{H}))$ such that

$$v_n \to u$$

in $C^0([0,T], H^{2m}(\mathcal{H}))$, for any integer m such that $2m > \frac{\dim(\mathcal{H})}{2}$.

Step 2: We show that $u \in C^\infty((0,T) \times \mathcal{H})$. We start by showing that the limit function is in $C^1((0,T), H^{2m}(\mathcal{H}))$. For this, fix a small $\delta > 0$. Define $B_n(t) := \|\partial_t v_{n+1} - \partial_t v_n(t, \cdot)\|_{2m}$. By Lemma 6.6.2 and the energy estimate Proposition 7.2.2, we conclude that for all $t \in (\delta, T]$,

$$\begin{aligned} B_n(t) &\leq \frac{C}{\delta}\sqrt{E^{2m}(v_{n+1} - v_n, t)} \\ &\leq \frac{CC'}{\delta}\int_0^t \frac{\|f(v_n)(s,\cdot) - f(v_{n-1})(s,\cdot)\|_{2m}}{\sqrt{s}} ds \\ &\leq \frac{CC'}{\delta} C_7 \int_0^t \frac{A_{n-1}(s)}{\sqrt{s}} ds \\ &\leq \frac{CC'C_8}{\delta}\frac{(2C_7\sqrt{t})^n}{n!}, \end{aligned}$$

similar to above. We conclude that $(\partial_t v_n)$ is a Cauchy sequence in $C^0((\delta,T), H^{2m}(\mathcal{H}))$ for any $\delta > 0$ and integer m such that $2m > \frac{\dim(\mathcal{H})}{2}$. It follows that $(\partial_t v_n)$ is a Cauchy sequence in $C^0((0,T), H^{2m}(\mathcal{H}))$. Since $\Box v_n = f(v_{n-1})$, using Lemma 6.3.2, we concludes that

$$\psi(\partial_t)^2 v_n = Qv_n + f(v_{n-1}),$$

where Q is a linear differential operator of second order, differentiating at most once in ∂_t-direction. Since for all $t \in (0, \epsilon)$, $\psi(t, \cdot) > 0$, we conclude that $v_n \to u \in C^2((0,T), H^{2m-1}(\mathcal{H}))$ for any integer m such that $2m > \frac{\dim(\mathcal{H})}{2}$. Iterating this, making use of the fact that m can be chosen arbitrarily large, we conclude that $u \in C^\infty((\delta,T) \times \mathcal{H})$ as claimed. Since now $\delta > 0$ can be chosen arbitrarily small, we conclude that $u \in C^\infty((0,T) \times \mathcal{H})$.

Step 3: We show that $u \in C^\infty([0,T) \times \mathcal{H})$. By Proposistion 7.3.1 there is an asymptotic solution w^N such that

$$(\partial_t)^j(\Box w^N - f(w^N))|_{t=0} = 0$$

for all $j \leq N - 2$. By Proposition 7.2.2 and the Hölder inequality, we get

$$\begin{aligned} &\sqrt{E^{2m}(v_n - w^N, t)} \\ &\leq C' \int_0^t \frac{\left\|f(v_{n-1}) - \Box w^N\right\|_{2m}(s)}{\sqrt{s}} ds \\ &\leq C' \int_0^t \frac{\left\|f(v_{n-1}) - f(w^N)\right\|_{2m}(s)}{\sqrt{s}} ds + C' \int_0^t \frac{\left\|f(w^N) - \Box w^N\right\|_{2m}(s)}{\sqrt{s}} ds \\ &\leq C' \int_0^t \frac{\left\|f(v_{n-1}) - f(w^N)\right\|_{2m}(s)}{\sqrt{s}} ds + 2C'T^{\frac{1}{4}} \sqrt{\int_0^t \frac{\|f(w^N) - \Box w^N\|^2_{2m}(s)}{\sqrt{s}} ds}, \end{aligned}$$

for all $t \in [0,T]$. The second term is estimated using that

$$\begin{aligned}
&\left(\frac{d}{dt}\right)^j \left\|f(w^N) - \Box w^N\right\|_{2m}^2 |_{t=0} \\
&\quad = \sum_{i=0}^{j} \binom{j}{i} \langle (\partial_t)^i (f(w^N) - \Box w^N), (\partial_t)^{j-i}(f(w^N) - \Box w^N)\rangle_{2m}|_{t=0} \\
&\quad = 0,
\end{aligned}$$

for all $j \leq N-2$. It follows that

$$\sqrt{\int_0^t \frac{\left\|f(w^N) - \Box w^N\right\|_{2m}^2 (s)}{\sqrt{s}} ds} \leq D_1(N) t^{\frac{N}{2}-\frac{1}{4}},$$

where $D_1(N)$ is some constant depending on N. Since w^N is a smooth function and hence bounded on $[0,T] \times \mathcal{H}$ we can estimate $\left\|f(v_{n-1}) - f(w^N)\right\|_{2m}$ as in equation (7.3) to conclude that there is a constant $D_2(N)$ such that

$$\begin{aligned}
\left\|f(v_{n-1}) - f(w^N)\right\|_{2m}(t) &\leq D_2(N) \left\|v_{n-1} - w^N\right\|_{2m}(t) \\
&\leq D_2(N) \left\|v_n - w^N\right\|_{2m}(t) + D_2(N) A_{n-1}(t).
\end{aligned}$$

Hence, for all $t \in [0,T)$, we have

$$\begin{aligned}
\left\|v_n - w^N\right\|_{2m}(t) &\leq \sqrt{E^{2m}(v_n - w^N, t)} \\
&\leq C' \int_0^t \frac{\left\|f(v_{n-1}) - f(w^N)\right\|_{2m}(s)}{\sqrt{s}} ds \\
&\quad + 2C' T^{\frac{1}{4}} \sqrt{\int_0^t \frac{\left\|f(w^N) - \Box w^N\right\|_{2m}^2 (s)}{\sqrt{s}} ds} \\
&\leq C' D_2(N) \int_0^t \frac{\left\|v_n - w^N\right\|_{2m}(s)}{\sqrt{s}} ds + C' D_2(N) \int_0^t \frac{A_{n-1}(s)}{\sqrt{s}} ds \\
&\quad + 2C' T^{\frac{1}{4}} D_1(N) t^{\frac{N}{2}-\frac{1}{4}}, \\
&\leq C' D_2(N) \int_0^t \frac{\left\|v_n - w^N\right\|_{2m}(s)}{\sqrt{s}} ds + D_3(N) t^{\frac{N}{2}-\frac{1}{4}} + D_4(N) \frac{(2C_7\sqrt{t})^n}{n!},
\end{aligned}$$

where $D_3(N)$ and $D_4(N)$ are constants depending on N. Letting $n \to \infty$, using that $v_n \to u$ in $C^0([0,T), H^{2m}(\mathcal{H}))$ and that $\frac{1}{\sqrt{s}}$ is integrable on $[0,T)$, we conclude that

$$\left\|u - w^N\right\|_{2m}(t) \leq C' D_2(N) \int_0^t \frac{\left\|u - w^N\right\|_{2m}(s)}{\sqrt{s}} ds + D_3(N) t^{\frac{N}{2}-\frac{1}{4}}$$

for all $t \in [0,T)$. Now Grönwall's lemma implies that

$$\left\|u - w^N\right\|_{2m}(t) \leq D_5(N) t^{\frac{N}{2}-\frac{1}{4}},$$

where $D_5(N)$ is a constant depending on N, since $\frac{1}{\sqrt{s}}$ is integrable on $[0,T)$. Hence increasing N gives a better bound, as one might expect. It follows that for all $t \in (0,T)$, we get the estimate

$$\begin{aligned}
&\left\|\partial_t u - \partial_t w^N\right\|_{2m}(t) \\
&\quad \leq \frac{1}{\sqrt{t}}\sqrt{E^{2m}(u-w^N,t)} \\
&\quad \leq \frac{C'}{\sqrt{t}} D_2(N) \int_0^t \frac{\left\|u-w^N\right\|_{2m}(s)}{\sqrt{s}} ds + D_3(N) t^{\frac{N}{2}-\frac{3}{4}} \\
&\leq D_6(N) t^{\frac{N}{2}-\frac{3}{4}},
\end{aligned}$$

for some constant $D_6(N)$, depending on N. Choosing N large and letting $t \to 0$ shows that $\partial_t u$ is extendible continuously in $C^\infty(\mathcal{H})$ to $t=0$. Writing

$$\Box u = \psi(\partial_t)^2 u + L_1 \partial_t u + L_2 u = f(u),$$

and differentiating it with respect to t shows that we can control any time derivative, by increasing N if necessary, up to arbitrary Sobolev degree. This is done similar to the proof of Proposition 6.8.3. Hence all time derivatives of u are continuously extendible to $t=0$ as claimed.

Step 4: We show that the solution is unique. Assume for this that

$$\begin{aligned}
\Box u &= f(u), \\
\Box v &= f(v), \\
(u-v)|_{t=0} &= 0,
\end{aligned}$$

on $[0,T) \times \mathcal{H}$. The idea is to show that $u-v$ satisfies a homogeneous wave equation with trivial initial data and must hence vanish by Theorem 6.2.2. For this, define for any $p \in [0,T) \times \mathcal{H}$,

$$\alpha(p) := \int_0^1 f'(\tau u(p) + (1-\tau)v(p)) d\tau.$$

It follows that

$$\begin{aligned}
\Box(u-v) &= f(u) - f(v) \\
&= \int_0^1 f'(\tau u + (1-\tau)v) d\tau (u-v) \\
&= \alpha(u-v).
\end{aligned}$$

Hence

$$\begin{aligned}
(\Box - \alpha)(u-v) &= 0, \\
(u-v)|_{t=0} &= 0,
\end{aligned}$$

and we conclude that $u-v=0$ on $[0,T) \times \mathcal{H}$, since $\Box - \alpha$ is an admissible wave operator. This shows that the solution is unique. $\Box$

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