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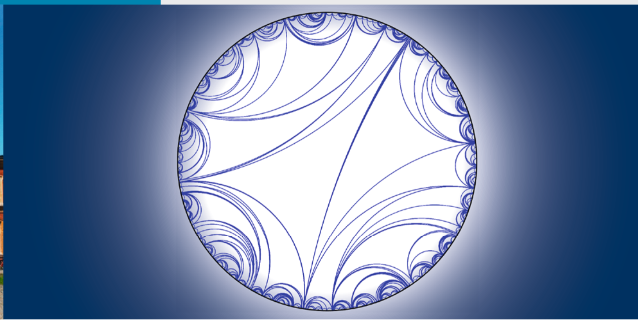

Sylvie Paycha (Ed.) | Pierre J. Clavier (Co-Ed.)

Integral Fourier Operators

Proceedings of a Summer School, Ouagadougou 14–25 September 2015

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Foreword

This volume provides an introduction to Fourier Integral Operators (FIO) for a readership of Master and PhD students as well as any interested layperson. Considering the wide spectrum of their applications and the richness of the mathematical tools they involve, FIOs lie the cross-road of many a field. The lectures to follow provide the reader with the necessary background, whether analytic or geometric, to get acquainted with FIOs, complemented by more advanced material presenting various aspects of active research in that area.

These contributions are based on lectures delivered at a school on Fourier Integral Operators held in Ouagadougou, Burkina Faso, 14–26 September 2015. It took place in the midst of the turmoil following the coup by General Diendéré, on Wednesday September 16th 2015, a date that probably none of the participants of the school will ever forget. As described in an article[1] *Recollections of a singular school* published in the European Mathematical Society Newsletter, this very singular school turned into a form of resistance to Diendéré's diktat. In spite of the very difficult conditions in which the school was held, the speakers were throughout the school, ready to deliver their knowledge the participants shared very eagerly. Keeping busy with mathematics turned out to be a very efficient way to dispel the worries. I admire the courage of the speakers, organisers and participants and I would like to express my gratitude to them. In particular, let me thank Marie-Fançoise Ouedraogo, Bernard Bonzi, Stanislas Ouaro, Hamidou Touré and Cyril Lévy, who actively contributed in setting up this school. Let me also thank Catherine Ducourtioux and Pierre Clavier who diligently and efficiently assisted me in preparing these proceedings.

[1] I am very grateful to the European Mathematical Society for allowing us to include it in these proceedings. We refer the reader to EMS NL № 98, pp. 45–47, December 2015 for the original article.

The volume encompasses 5 chapters, three in English and two in French which we chose to leave in the orginal language, since French was the language used during the school.

- ◇ *Lagrangian submanifolds* by Michèle Audin
- ◇ *Préambule aux opérateurs Fourier intégraux: les opérateurs pseudodifférentiels* by Catherine Ducourtioux and Marie-Françoise Ouédraogo
- ◇ *An introduction to the concepts of microlocal analysis* by René Schulz
- ◇ *Fourier multipliers in Hilbert spaces* by Julio Delgado and Michael Ruzhansky
- ◇ *Transformation de Bargmann et analyse microlocale analytique* by Gilles Lebeau

The first chapter by Michèle Audin provides relevant concepts of symplectic geometry, then Catherine Ducourtioux and Marie-Françoise Ouédraogo introduce the langauge of pseudodifferential operators, which are very special examples of the FIOs discussed in René Schulz's lecture. The latter two chapters present more advanced material on two very elegant topics, Fourier multipliers discussed by Julio Delgado and Michael Ruzhansky and the Bargmann transform discussed by Gilles Lebeau.

Let me thank the authors of these beautiful lectures most warmly.

Let us introduce the main protagonists of this volume, namely FIOs, which were initially introduced to host solution operators to differential equations such as the Cauchy problem for a hyperbolic operator. They therefore play, in the theory of hyperbolic equations, the role that pseudodifferential operators play in the theory of elliptic equations. As such, FIOs include parametrices of strictly hyperbolic equations. FIOs actually form a large class of transformations, for instance the Fourier transform, pseudodifferential operators, and diffeomorphisms can be viewed as FIOs. Using L. Hörmander's $S(M,g)$ calculus, J.M. Bony could show that the evolution unitary groups associated to a large class of Schrödinger equations are also Fourier integral operators (in a generalised sense).

Some history on FIOs

The precise origin of the concept of FIO is difficult to trace back; V.P. Maslov[2], G. Eskin[3] (1967), P. Lax (1957) and D. Ludwig[4] (1960) definitely contributed in an essential way to the birth of the theory. The early stages consisted in constructing parametrices in the Cauchy problem for hyperbolic equations and FIOs have indeed since then been widely employed to represent solutions to Cauchy problems, in the framework of both pure and applied mathematics. Thanks to the systematic development of the calculus of FIOs by L. Hörmander in the 70's (see the last volume of *The Analysis of Linear Partial Differential Operators*), they have become a central tool in the theory of partial differential equations (PDEs). J. Sjöstrand and A. Melin then developed (1974) a theory of Fourier integral operators with complex phase functions. The book of J. Duistermaat published in the 90's (but circulating as a Courant Institute preprint since the 70's) later provided a more geometric perspective on FIOs.

FIOs at the cross-roads of various areas of mathematics

FIOs lie at the cross-roads of various areas of mathematics beyond analysis and PDEs – to which they are usually attached to – including geometry, topology and operator algebra. In particular, they require the use of symplectic geometric tools, due to the role played by Lagrangian submanifolds in the theory. The various approaches that were developed to provide a consistent framework for FIOs, reflect the different types of techniques they involve. J. Duistermaat's geometric approach gave a new outlook on L. Hörmander's original analytic presentation of FIOs. In spite of its analytic nature, L. Hörmander's approach had an operator algebraic flair to it, which was later further explored by R. Melrose and M. Joshi. Other authors such as G. Eskin, adopt a more pragmatic PDE perspective on FIOs,

[2] In the introduction to his book *Introduction to Pseudodifferential and Fourier Integral Operators* published in 1980, F. Trèves writes *I kept my allegiance to the established term integral Fourier operator, although I am willing to agree that this term is not particularly good, and calling such operators Maslov operators would possibly do more justice*.

[3] In his historical account *25 years of Fourier Integral Operators* in *Mathematics past and present. Fourier integral operators,* edited by J. Brüning and V. Guillemin, V. Guillemin puts Maslov and Eskin on an equal footing in their contribution to the origin of F.I.Os.

[4] In his contribution *Applications of Fourier integral operators* to the proceedings of the *International Congress of Mathematics in Vancouver* in 1974, Duistermaat mentions P. Lax and D. Ludwig as initiators of the concept.

A recent revival in pure mathematics

Interestingly, FIOs have recently gained a new interest in pure mathematics, for example through the work of V. Mathai and R. Melrose on the geometry of pseudo-differential algebra bundles; there the projective invertible Fourier integral operators form the automorphism group of the filtered algebra of pseudodifferential operators. On the other hand, recently, D. Perrot develops a local index theory, in the sense of non-commutative geometry, for operators associated to non-proper and non-isometric actions of Lie groupoids on smooth submersions. The non-properness or non-isometry of the action brings in FIOs which are seen here as generalisations of the pseudodifferential operators that one usually expects in the context of local index theory. While T. Hartung and S. Scott generalised the Kontsevich-Vishik trace to FIOs, J.-M. Lescure and S. Vassout defined FIOs on groupoids, with applications to singular manifolds in view.

A broad range of applications

FIOs further offer a parallel to Hamiltonian mechanics for they can also be viewed as a quantization of a canonical transformation and are to linear partial differential equations what canonical transformations are to Hamiltonian mechanics. They have many applications in mathematical physics where they have become an important tool in the theory of partial differential equations. In particular, they can be used to (approximately) solve linear partial differential equations, or to transform such equations into a more convenient form.

FIOs lie at the core of many other applications beyond partial differential equations and related problems in physics. Indeed, their applications reach out to seismology, medical imaging and geometrical optics. For example, the mathematical theory of seismic reflection surveys is based on the framework of linearized inverse scattering problems. In this framework, the forward modeling operator is a Fourier integral operator which maps singularities of the subsurface into singularities of the wavefield recorded at the surface. The adjoint of this Fourier integral operator then allows to form seismic images from seismic data. Moreover, the solution operator to typical Cauchy problems that appear in exploration seismology can be approximated by the composition of global Fourier integral operators with complex phases.

To conclude this introduction, I would like to thank the Volkswagen Foundation for their generous support which made possible the school these proceedings are based on, as well as the University of Ouagadougou, Burkina Faso and the University of Potsdam. My very warm thanks to Steffanie Rahn who put much effort into the logistical organisation of the school. And last but not least, I would like to address my gratitude to Bernard Helffer for the precious scientific advice he provided at the early stages of the organisation of the school.

Sylvie Paycha
Berlin, 30th May 2017

Recollection of a Singular School

Some 40 participants are seated in the large seminar room; they have come from 12 different African countries[1] to Ouagadougou, the capital of the landlocked country of Burkina Faso[2] to learn about Fourier integral operators and their many concrete applications inside and outside the realm of mathematics. The school[3], funded by the Volkswagen Foundation, started three days ago and, for the last talk of the day, I am about to explain to them how to use the inverse Fourier transform to build pseudo-differential operators from rational functions, when my colleague Bernard Bonzi, co-organiser of the school, steps into the room calling me to the door.

I follow him along the corridor, where three other colleagues and co-organisers, Marie-Francoise Ouedraogo, Stanislas Ouaro and Hamidou Touré, are waiting with anxious looks; something serious has happened, they tell me. Is it that serious that I cannot have another 10 minutes to conclude my presentation? My colleagues seem reluctant to let me finish but finally nod approvingly, insisting that I should conclude hastily. I still have no idea why there is this sudden tension and I am suspecting a problem with the premises we are using, a three-storey, medium size, cream-coloured building on the outskirts of Ouagadougou, some 20 minute bus drive from the two hotels most of us are staying in and not too far from the campus of the University of Ouagadougou (co-organising institution of the event). Some 10 participants are staying at the guest house of the university, which is near the conference hall but far from the two hotels in the city centre where most of the participants are housed. The local organising committee has rented the conference hall

[1]Benin, Cameroon, Chad, Congo, Democratic Republic of Congo (RDC), Ethiopia, Ivory Coast, Mali, Morocco, Niger, Nigeria and Senegal.

[2]The “Land of people of integrity” (“Le pays des hommes intègres”), formerly Upper Volta.

[3]Summer school on Fourier integral operators and applications, Ouagadougou, 14–25 September 2015.

as well as a nearby room where we gather for lunch and coffee breaks from the private University of Saint Thomas d'Aquin.

Back in front of the audience patiently waiting for me, I mumble a few words explaining, with a touch of irony, that we should conclude rapidly before we are required to leave the room. I feel I am legitimately and dutifully finishing the explanations I had started. The students who had got actively involved in the discussion about the correspondence between symbols and operators via the Fourier transform seem somewhat disappointed when we stop soon after the interruption. My colleagues, who still look very preoccupied, take me away into a room, which adds to the mystery of their sudden interruption, and answer my questioning look: "There seems to have been a putsch; it still is not confirmed but considering the potential danger of the situation, we should all get back to the hotels immediately." By the time I have digested the news and further questioned my colleagues, we realise that the participants have already left for the hotels with the hotel van. In the car that is driving us back, I insist that we should go to the hotels to inform the participants and speakers of the situation. My colleagues hesitantly comply as I argue that informing the participants of the situation is the best thing to do at that stage, after which they leave for their respective homes on the other side of the city. Back in our hotel after the sudden interruption of the talks, we gather with other guests staying at the hotel around the television set in the reception hall to hear that "the event" was indeed a putsch by General Gilbert Diendéré.

This was how the school was suddenly interrupted on Wednesday 16 September 2015, a date that probably none of the participants of the school will ever forget.

Diendéré had served for three decades as former President Blaise Compaoré's Chief of Staff. This coup was supported by the presidential guard known as the RSP, the 1,200 strong Regiment of Presidential Security, of which Diendéré was also seen as the figurehead commander despite having retired from the force in 2014. The RSP arrested both President Michel Kafando of the transition government and Prime Minister Isaac Zida. The air and terrestrial borders were closed for some days[4] and a curfew from 7 p.m. to 6 a.m. (lasting some 10 days) was declared[5].

Little had I anticipated such a dramatic event during the opening ceremony[6], which

[4]The airport was briefly reopened later to let the presidential delegation of Macky Sall (President of Senegal) and Thomas Boni Yayi (President of Benin) land to start the negotiations with General Diendéré.

[5]The curfew restrictions were to be lessened a week later to 11 p.m. to 5 a.m.

[6]The ceremony had been postponed to the third day of the meeting, due to my late arrival.

had taken place that very morning in the presence of the President of the University of Ouagadougou.

I had expressed my hopes that this school might help overcome the obstacles that set walls between our respective nations and continents. A putsch was not among the obstacles I had envisaged but now, the school seemed to be doomed to end three days after it had started.

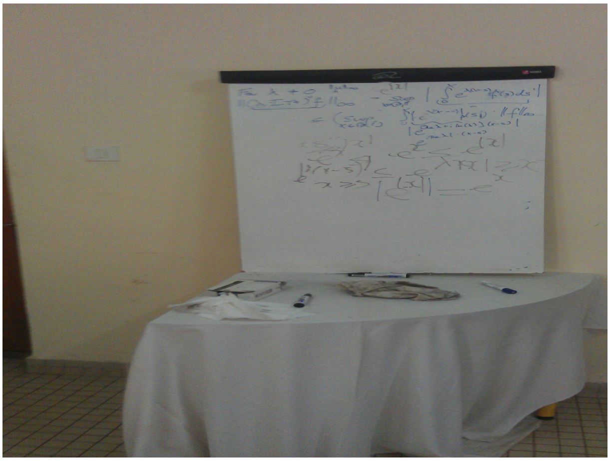
Place de la Nation (formerly Place de la Révolution).

The "Revolution square" (Place de la Révolution[7]) where protests started that very Wednesday evening – nearly a year after the October 2014 protests that had forced the former President Blaise Compaoré to resign following some 27 years in power – was to separate the local organisers and participants in their homes from us foreign organisers, speakers and participants in our hotels on the other side of the city. What was named "the event" for a while was to set up an imaginary yet tangible wall between us. Apart from a very brief visit of a couple of local organisers to one of the hotels three days after the "event", only some eight days later (a day after the Tabaski celebrations[8], seriously hampered by the lack of food in the city) when the city seemed to start getting back to normal life did the local organisers dare to venture back to the hotels. How frustrating and disappointing for them when they had put so much energy into organising the school!

[7]Nickname for the "Square of the Nation" (place de la nation), the main square of Ouagadougou after the 2014 protests initiated by the "Balai Citoyen" (see footnote below).

[8]Eid al-Adha, "Feast of the Sacrifice", a Muslim celebration honouring the willingness of Abraham to sacrifice his son, as an act of submission to God's command.

Little had they anticipated “the event”, especially as, on the request of the Volkswagen Foundation, they had asked the President of Ouagadougou University to confirm that organising such a school in Ouagadougou would be safe.



The white board on which the talks were given.

Gradually measuring the importance of what was happening, and the potential danger of violent confrontations between the population and the RSP, as co-organiser of the school, I began to worry about what to do in such circumstances. What were we to do on the morning after the putsch, and the days to follow? One option was to declare the school over but then what would the participants and speakers now stuck in their hotels do all day; would they not start panicking? An alternative was to try to adapt the organisation of the school to the circumstances, an a priori risky solution considering the instability of the situation. Indeed, the Balai Citoyen (the Civic Broom[9]) had grown in determination and efficiency, so a violent reaction could be expected from the Burkinabé people, who were surely not going to accept Diendéré’s diktat. Going back to the conference hall in the outskirts of Ouagadougou was therefore impossible due to potential riots and shooting on the streets. But going from the hotel, where most of the speakers were staying, to the nearby hotel where most of the participants were lodged (a 10 minute walk) seemed feasible. And this is indeed what we did.

The hotel manager kindly lent us a small seminar room, which seemed unused. We

[9]A civic organisation, founded by two musicians in 2013, which played a central role in the protest movement that forced Blaise Compaoré to resign in October 2014.

found a tiny, narrow whiteboard to lean against the wall, a board we had to hold up straight on the table with one hand while writing with the other. In that small room and on that board were held some 40 lectures (four a day over 10 days), thanks to the speakers[10] who gave talks in such difficult conditions. We could only count on those who were fit enough to give a talk – for many fell ill over several days due to the preventative medication against Malaria – and who were ready to take the risk of walking ten minutes from their hotel to the new, improvised conference room in the other hotel nearby. Participants searched their pockets for a few marker pens to give us and, when we ran out of pens on the fourth day, the hotel sent out an employee on a difficult mission (considering the circumstances) to drive through Ouagadougou, where most shops had remained closed since the coup, in search of a stationers who might sell markers. Thus, we could go on covering the white board from top to bottom with semi-groups, distributions, wave front sets, Fourier transforms, pseudo-differential operators, characteristics, singular supports, Lagrangian submanifolds, Bergmann transforms, Fourier integral operators, fundamental groups and Maslov indices. With the "event", singularities, which were the central theme of the school, had become a characteristic of this very singular school.

Phones would ring during the talks and participants (and even the speakers) would leave the room for a moment to reassure a relative worried by the news of the coup and return to the seminar room with a gloomy look, after having heard that their flight back had been cancelled because of the coup, or sometimes a happy face, having been informed of the new departure time of their plane. The lecturers' moods, looks and tones of voice varied from day to day according to their state of health and the latest news they had received but their faces and voices would invariably look and sound happier and more enthusiastic as their lecture evolved. I had not suspected how far a mathematics talk can pull the speaker and the audience away from the distressing reality around them, an observation shared by both participants and speakers. During the talks, a short silence would follow what we thought might be the sound of guns or any other suspicious sound but no one dared make a comment, so uncertain was the situation. In the midst of this overwhelming tension, one would hear the participants making jokes about the situation, such as the rather repetitive lunch menu: due to the acute food shortage caused by the putsch, the hotel manager had arranged for a relative living in the outskirts of the city to

[10]Michèle Audin (Strasbourg), Viet Ngyuen Dang (Lyon), Julio Delgado (London), Catherine Ducourtioux (Corte), Massimiliano Esposito (London), Matthias Krüger (Göttingen), Gilles Lebeau (Nice), Cyril Lévy (Albi/Toulouse), Michael Ruzhansky (London) and René Schulz (Hannover).

Hardly no one on the otherwise lively streets of Ouagadougou (left) and participants on their way to the improvised conference venue (right).

provide him with chickens, which were served for lunch every single day!

Four days after the coup, the negotiations between General Diendéré and the presidential delegation had concluded in favour of an amnesty for General Diendéré and the eligibility of the former CDP (Congrès pour la démocratie et le progrès, Blaise Compaoré's party) members. This was clearly a threat to peace; how could the population accept such a deal? Violent protests were to be expected. On the Monday following the coup, the regular army marched into Ouagadougou, publicly announcing its intention to disarm the RSP while avoiding any fighting. The night before the announced military manoeuvre, rumours had spread that the army would march into the city overnight. I packed my backpack with what I considered important belongings in case I had to suddenly flee from the fighting during the night and woke up at dawn, worried by some voices I could hear outside my bedroom window which led to a large terrace roof. A glimpse through the window reassured me; it was only the Radio France International two man team broadcasting the morning news from the terrace. No military confrontation could have taken place since the army had not yet reached the city. People gathered at sunset cheering

The market closed because of the putsch.

on the highway as they waited for the anticipated entry of Burkina Faso's regular army, who vowed to disarm the RSP. That morning[11] the streets, which had remained silent and empty since the coup, seemed to come back to life. In the afternoon, Diendéré publicly gave rather contradictory and inadequate apologies, asking the people of Burkinabé to forget about the putsch but claiming full responsibility for it and promising to restore civilian government.

But, by the evening, the situation radically changed; we heard that Michel Kafando had asked for protection from the French Embassy. A veil of silence covered the city again. Macky Sall, President of Senegal, who had come over the weekend with Thomas Boni Yayi, President of Benin, to negotiate with Diendéré, had failed to find a resolution to the crisis in spite of his political weight and diplomatic experience. Following an extraordinary summit meeting of ECOWAS (Economic Community of West African States), another delegation of presidents[12] arrived a couple of days later to calm down the situation. This time their intervention had an effect; a week after the coup,[13] an agreement was passed and a peace deal was presented to the Mogho Nabaa, King of Burkina Faso's leading Mossi tribe. Michel Kafando, who had been under house arrest for some days after his first detention, was now free and announcing his return to power.

[11] On Monday 21 September.
[12] A delegation comprising the Presidents of Ghana and Benin, as well as the Vice-President of Nigeria.
[13] On Tuesday 22 September.

The school went on running in the midst of the turmoil, a form of resistance to Diendéré's diktat. The applications to climate change and seismology we had planned for the second week were never discussed during the school. The flights of the speakers[14] who were due to arrive at the end of the first week had been cancelled and the airport remained closed until the middle of the second week. Yet, the participants were eager and happy to learn about the fundaments of FIOs and indeed learned a lot of abstract material during the talks and informal discussions with the speakers. A couple of participants from Benin had spent several days on a coach to reach Ouagadougou, having had to wait on the coach for the border to reopen, and were all the more determined to make the most out of the school. One could perceive the anxiety of some of the participants and most of the speakers but all agreed that, under the circumstances, it was best to go on with the talks. Keeping busy with mathematics, claimed many participants, was a very efficient way to dispel the worries, and various speakers asked to give more talks to keep their minds occupied preparing them. The particular circumstances the school was now held in were actually more propitious to informal interactions between the speakers and the participants than the more formal setup the school might have allowed for had the "event" not happened. I am very grateful to all the speakers and participants and admire their courage.

Despite questions raised as to the sincerity of Diendéré's public apologies, eight days after the putsch and one day after the Tabaski celebrations, the tension one had felt on the streets of Ouagadougou melted down and the sun dared to venture back. The preceding days had not been too hot, with sudden wind blasts and strong rain showers, as is to be expected during the rainy season. The city of Ouagadougou was now glowing with the pride of victory over the usurpers. With this coup, we (participants, speakers and organisers of the school) had unexpectedly borne witness to the complex, painful and still ongoing emancipation of the Burkinabé people from 27 years of dictatorial leadership and its ramifications.

Sylvie Paycha
December 2015

[14] Nicolas Burq (Orsay), David Dos Santos Ferreira (Nancy) and Jérôme Le Rousseau (Orléans).

Contents

Chapter 1

Lagrangian submanifolds

*Michèle Audin**

Abstract. *This is an introduction to Lagrangian and special Lagrangian submanifolds. I give the basic definitions and the most classical examples of the theory of Lagrangian and special Lagrangian submanifolds and immersions in $\mathbb{C}^n$. I then come to the global situation of symplectic and Calabi-Yau manifolds. I describe the local structure of the moduli space of Lagrangian submanifolds (Weinstein theorem) and that of special Lagrangian submanifolds (McLean theorem).*

Dans cette introduction aux sous-variétés lagrangiennes et lagrangiennes spéciales, je donne pour commencer les définitions et les exemples les plus classiques de la théorie des immersions lagrangiennes et lagrangiennes spéciales dans $\mathbb{C}^n$. Je passe ensuite à la situation plus globale des variétés symplectiques et de Calabi-Yau. Je décris localement l'espace de modules des sous-variétés lagrangiennes (un théorème de Weinstein) et celui des sous-variétés lagrangiennes spéciales (un théorème de McLean).

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This contribution corresponds to pages 1-83 of the book "Symplectic geometry of integrable Hamiltonian systems", by M. Audin, A. Cannas da Silva, E. Lerman, Advanced courses in Mathematics CRM Barcelona, Birkhauser, 2003. We thank the author for authorising its publication in this volume.

1 Introduction

This text is an introduction to Lagrangian and special Lagrangian submanifolds. Special Lagrangian submanifolds were invented twenty years ago by Harvey and Lawson [19]. They have become very fashionable recently, after a series of papers by Hitchin [20], [21] and Donaldson [12].

My aim here is mainly to present as many examples as possible. I have taken some time to explain why we know so many Lagrangian and so few special Lagrangian submanifolds and immersions. There are mainly two reasons:

- ◇ To be Lagrangian is, eventually, a *linear* property. On the other hand, the property to be special Lagrangian is, in dimension 3 and more, non linear.
- ◇ The moduli space of Lagrangian submanifolds that are close to a given one is an infinite dimensional manifold, while the corresponding moduli space of special Lagrangian submanifolds is finite dimensional.

This will be apparent in the number and nature of the examples I describe in these notes.

To prepare these lectures, in addition to the papers mentioned above, I have used standard textbooks on manifolds and vector fields as [23], on symplectic geometry as [4], [7], [25], [10] and on complex manifolds and Hodge theory as [8], [16].

I have used standard notation but, although this text pretends to be written in English, I have kept a preference for (transparent) French standards, for instance $\mathscr{P}^n(\mathbb{K})$ for the projective space of dimension n over the field $\mathbb{K}$ and ${}^tA$ for the transpose of a matrix A.

I thank Étienne Mann, Édith Socié, Thomas Vogel and Jean-Yves Welschinger for their comments and their help during the preparation of these notes. Special thanks to Mihai Damian, Alicia Jurado and Sébastien Racanière.

. . . .

2 Lagrangian and special Lagrangian immersions in $\mathbb{C}^n$

In this chapter, I define Lagrangian and special Lagrangian immersions in $\mathbb{C}^n$. To begin with, I explain that $\mathbb{C}^n$ is the standard *real* vector space endowed with a non degenerate alternated bilinear form (§2.1) and use this "symplectic structure" to define Lagrangian subspaces and immersions (§§2.2, 2.3 and 2.4). Later, I use the complex structure as well, to define *special* Lagrangian immersions (§2.5).

2.1 Symplectic form on $\mathbb{C}^n$, symplectic vector spaces

2.1.1 Symplectic vector spaces

Consider the vector space $\mathbb{C}^n$ with the Hermitian form

$$\langle Z, Z' \rangle = \sum_{j=1}^{n} \overline{Z_j} Z'_j$$

(note that it is anti-linear in the first entry and linear in the second). Decompose it in real and imaginary parts:

$$\langle Z, Z' \rangle = (Z, Z') - i\omega(Z, Z').$$

The real part is the standard inner product (Euclidean structure) of $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$,

$$(Z, Z') = \sum_{j=1}^{n} (X_j X'_j + Y_j Y'_j) = X \cdot X' + Y \cdot Y',$$

a symmetric non degenerate (real) bilinear form. The imaginary part defines a (real) bilinear form

$$\omega = \sum_{j=1}^{n} (X'_j Y_j - X_j Y'_j) = X' \cdot Y - X \cdot Y'$$

that is *alternated*, this meaning that $\omega(Z, Z) = 0$ for all Z. Equivalently, ω is skew-symmetric, that is,

$$\omega(Z', Z) = -\omega(Z, Z').$$

To write these formulas, I have decomposed the complex vectors of $\mathbb{C}^n$ as

$$Z = X + iY, \qquad X, Y \in \mathbb{R}^n$$

and I have used the inner product $X \cdot Y$ of $\mathbb{R}^n$. The form ω is non degenerate too, as

$$\big(\omega(X, Y) = 0 \text{ for all } Y\big) \Rightarrow X = 0.$$

More generally, on a real vector space E, a *symplectic form* is a non degenerate alternated bilinear form. A vector space endowed with a symplectic form is said to be a *symplectic* vector space.

2.1.2 Symplectic bases

Fix a complex unitary basis $(e_1,\dots,e_n)$ of $\mathbb{C}^n$. Put $f_j=-ie_j$, so that

$$(e_1,\dots,e_n,f_1,\dots,f_n)$$

is a basis of the *real* vector space $\mathbb{C}^n$. Compute ω on the vectors of this basis:

$$\omega(e_i,e_j)=\Im\langle e_i,e_j\rangle=\Im\delta_{i,j}=0,$$

also

$$\omega(f_i,f_j)=\Im\langle ie_i,ie_j\rangle=\Im\langle e_i,e_j\rangle=0$$

and eventually

$$\omega(e_i,f_j)=\Im\langle e_i,-ie_j\rangle=\Re\langle e_i,e_j\rangle=\delta_{i,j}.$$

Inspired by these properties, we say that a basis $(e_1,\dots,e_n,f_1,\dots,f_n)$ of a symplectic vector space is a *symplectic basis* if

$$\omega(e_i,f_j)=\delta_{i,j}\quad\text{and}\quad\omega(e_i,e_j)=\omega(f_i,f_j)=0\quad\text{for all } i \text{ and } j.$$

There are symplectic bases in all symplectic spaces, thanks to the following proposition.

Proposition 1.1 Let ω be a symplectic form on a finite dimensional vector space E. There exists a basis $(e_1,\dots,e_n,f_1,\dots,f_n)$ of E such that $\omega(e_i,f_j)=\delta_{i,j}$ and $\omega(e_i,e_j)=\omega(f_i,f_j)=0$.

Proof. As ω is non degenerate, it is not identically zero so that one can find two vectors e_1 and f_1 such that $\omega(e_1,f_1)=1$. One then checks that the restriction of ω to the orthogonal complement (with respect to ω) of the plane $\langle e_1,f_1\rangle$ is non degenerate. One eventually concludes by induction on the dimension—once noticed that an alternated bilinear form on a 1-dimensional vector space is zero. □

In particular, the dimension of E is an even number and this is the only invariant of the isomorphism type of (E,ω). If E has dimension $2n$, then E with its symplectic form is isomorphic to $\mathbb{C}^n$ with the form ω. This result can be called a “linear Darboux theorem”, in reference with the forthcoming (Darboux) theorem 1.77.

More generally, an alternated bilinear form has a *rank*, that is the dimension of the largest subspace on which it is non degenerate, and is an even number.

Matrices. In a symplectic basis, the matrix of the symplectic form is

$$J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

Notice that the matrix J satisfies

$$J^2 = -\mathrm{Id}\,.$$

As the matrix of an endomorphism, this is a *complex structure*. In the symplectic basis of $\mathbb{C}^n$ associated with the canonical (complex) basis $(e_1,\dots,e_n)$, J is nothing other that the matrix of multiplication by i.

2.1.3 The symplectic form as a differential form

One can write ω as a differential form

$$\omega = \sum_{j=1}^{n} \mathrm{d}y_j \wedge \mathrm{d}x_j.$$

This is an *exact* differential form (the differential of a degree 1-form):

$$\omega = \mathrm{d}\left(\sum_{j=1}^{n} y_j \mathrm{d}x_j\right) = \mathrm{d}(Y\cdot X).$$

The form $\lambda = Y\cdot \mathrm{d}X$ is called *Liouville form* (see §3.1 below).

2.1.4 The symplectic group

This is the group of isometries of ω. A transformation g of $\mathbb{C}^n$ is *symplectic* if it satisfies

$$\omega(gZ,gZ') = \omega(Z,Z') \text{ for all } Z,Z' \in \mathbb{C}^n.$$

Call $\mathrm{SP}(2n)$ the symplectic group of the space $\mathbb{C}^n$ of dimension $2n$. Consider all the groups $\mathrm{O}(2n)$, $\mathrm{GL}(n;\mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SP}(2n)$ as subgroups of $\mathrm{GL}(2n;\mathbb{R})$.

Proposition 1.2 The following equalities hold

$$\mathrm{SP}(2n)\cap\mathrm{O}(2n)=\mathrm{SP}(2n)\cap\mathrm{GL}(n;\mathbb{C})=\mathrm{O}(2n)\cap\mathrm{GL}(n;\mathbb{C})=\mathrm{U}(n).$$

Proof. Let us characterize our subgroups of $\mathrm{GL}(2n;\mathbb{R})$:

1) $g\in\mathrm{GL}(n;\mathbb{C})$ if and only if g is $\mathbb{C}$-linear, that is, if and only if

$$g(iZ)=ig(Z)\text{ for all }Z.$$

For a matrix A, this is to say that $AJ=JA$.

2) $g\in\mathrm{SP}(2n)$ if and only if g preserves ω, that is, if and only if $\omega(gZ,gZ')=\omega(Z,Z')$ for all Z and Z'. For a matrix A, this is

$${}^tAJA=J.$$

3) $g\in\mathrm{O}(2n)$ if and only if $(gZ,gZ')=(Z,Z')$. For a matrix A, this is ${}^tAA=\mathrm{Id}$.

One then checks that two of these conditions imply the third:

◇ 2) and 3) imply that

$$\langle gZ,gZ'\rangle=\langle Z,Z'\rangle$$

thus that $g\in\mathrm{U}(n)\subset\mathrm{GL}(n;\mathbb{C})$.

◇ 3) and 1) imply that

$$\omega(gZ,gZ')=\omega(gZ,-ig(iZ'))=(gZ,g(iZ'))=(Z,iZ')=\omega(Z,Z')$$

thus that $g\in\mathrm{SP}(2n)$.

◇ in the same way 1) and 2) imply 3).

In matrix terms, the intersection $\mathrm{SP}(2n)\cap\mathrm{O}(2n)$ is the set of matrices

$$\begin{pmatrix} U & -V \\ V & U \end{pmatrix} \in \mathrm{GL}(n;\mathbb{C}) \subset \mathrm{GL}(2n;\mathbb{R})$$

such that

$$\begin{cases} {}^tUV = {}^tVU \\ {}^tUU + {}^tVV = \mathrm{Id}. \end{cases}$$

This is exactly the condition that $U + iV$ be a unitary matrix. □

2.1.5 Orthogonality, isotropy

Write $F^\perp$ for the Euclidean orthogonal of the real subspace F of $\mathbb{C}^n$ and F° for its symplectic (that is, with respect to ω) orthogonal. As ω is non degenerate, one has

$$(F^\circ)^\circ = F \qquad \text{and} \qquad \dim F + \dim F^\circ = 2n = \dim_{\mathbb{R}} \mathbb{C}^n.$$

Notice however that a subspace and its orthogonal may have a non trivial intersection. The restriction of the non degenerate form ω to a subspace is not always a non degenerate form, in contradiction with what happens in the Euclidean case (which is due to the positivity of the inner product). In other words, all the subspaces of a symplectic space do not have the same behaviour with respect to the symplectic form. See Exercises 1.54 and 1.55.

One says that a subspace F is *isotropic* if $F \subset F^\circ$, *co-isotropic* if $F \supset F^\circ$. For instance, a (real) line is always isotropic, as it lies in its orthogonal which is a (real, co-isotropic) hyperplane. Notice that F is isotropic if and only if F° is co-isotropic. Notice also that the dimension of an isotropic subspace is at most equal to n, half the dimension of $\mathbb{C}^n$.

2.2 Lagrangian subspaces

2.2.1 Definition of Lagrangian subspaces

The isotropic subspaces of maximal dimension n are *Lagrangian*. For instance, $\mathbb{R}^n \subset \mathbb{C}^n$ is a Lagrangian subspace. More generally, a subspace generated by "one half" of a symplectic basis is Lagrangian. Conversely, if F is an isotropic subspace of dimension $k \leq n$, it is possible to complete any basis $(e_1, \ldots, e_k)$ of F in a symplectic basis and thus to obtain Lagrangian subspaces containing F.

Let us use now the complex multiplication in $\mathbb{C}^n$ to state:

Lemma 1.3 A real subspace P of $\mathbb{C}^n$ is Lagrangian if and only if $P^\perp = iP$.

Proof. This is a straightforward computation:

$$\omega(Z,Z') = 0 \Leftrightarrow \Im\langle Z,Z'\rangle = 0 \Leftrightarrow \Re\langle Z,iZ'\rangle = 0 \Leftrightarrow (Z,iZ') = 0. \qquad \square$$

Lemma 1.4 Let P be a Lagrangian subspace of $\mathbb{C}^n$ and let $(x_1,\dots,x_n)$ be an orthonormal basis of this real subspace. Then $(x_1,\dots,x_n)$ is a complex *unitary* basis of $\mathbb{C}^n$. Conversely, if $(x_1,\dots,x_n)$ is a unitary basis of $\mathbb{C}^n$, the real subspace it spans is Lagrangian.

Proof. If $(x_1,\dots,x_n)$ is an orthonormal basis of the Lagrangian P, the previous lemma says that the basis $(x_1,\dots,x_n,ix_1,\dots,ix_n)$ is an orthonormal basis of the real space $\mathbb{C}^n$, thus that $(x_1,\dots,x_n)$ is a complex basis of $\mathbb{C}^n$. Moreover, one has

$$\langle x_i,x_j\rangle = (x_i,x_j) - i\omega(x_i,x_j) = \delta_{i,j} - 0,$$

thus this is a unitary basis. The converse is even more obvious. $\square$

2.2.2 The symplectic reduction

This is a simple but useful operation, essentially contained in the next lemma.

Lemma 1.5 Let P be a Lagrangian subspace and F be a co-isotropic subspace of $\mathbb{C}^n$, such that

$$P + F = \mathbb{C}^n.$$

Then the restriction of the projection

$$P \cap F \subset F \rightarrow F/F^\circ$$

is injective, the space F/F° is symplectic and the image of $P \cap F$ is a Lagrangian subspace.

Proof. The symplectic form of $\mathbb{C}^n$ clearly induces a non degenerate form on F/F°, as F° is the kernel of the restriction of ω to F. The kernel of the composition$P \cap F \subset F \rightarrow F/F^\circ$ is

$$P \cap F \cap F^\circ = P \cap F^\circ \qquad\qquad F \text{ being co-isotropic, } F \supset F^\circ$$

$$
\begin{aligned}
&= (P^\circ + F)^\circ && \text{since } (A+B)^\circ = A^\circ \cap B^\circ,\\
&= (P + F)^\circ && \text{as } P \text{ is Lagrangian, } P = P^\circ\\
&= (\mathbb{C}^n)^\circ && \text{because } P + F = \mathbb{C}^n\\
&= 0 && \text{as } \omega \text{ is non degenerate.}
\end{aligned}
$$

The map is thus injective. $P \cap F$ is isotropic and has dimension

$$\dim P \cap F = \dim P + \dim F - \dim(P+F) = \dim F - n,$$

half the dimension of the symplectic space F/F°, that is

$$\dim F/F^\circ = \dim F - (2n - \dim F) = 2(\dim F - n). \qquad \square$$

See more generally Exercise 1.57.

2.3 The Lagrangian Grassmannian

We consider now the set Λ_n of all Lagrangian subspaces of $\mathbb{C}^n$.

2.3.1 The Grassmannian Λ_n as a homogeneous space

Look again at lemma 1.4. If P_1 and P_2 are two Lagrangian subspaces of $\mathbb{C}^n$, choose an orthonormal basis for each. We thus have two unitary bases of $\mathbb{C}^n$. There exists a unitary transformation (an element of the unitary group $\mathrm{U}(n)$) that maps the basis of P_1 on that of P_2, and thus *a fortiori* the Lagrangian P_1 on the Lagrangian P_2.

In other words, the group $\mathrm{U}(n)$ *acts transitively* on the set of Lagrangian subspaces of $\mathbb{C}^n$. The stabilizer of the Lagrangian $\mathbb{R}^n$ is the group $\mathrm{O}(n)$ of orthonormal basis changes in $\mathbb{R}^n$. We have defined this way a bijection

$$\mathrm{U}(n)/\mathrm{O}(n) \to \Lambda_n$$

with the help of which we identify the two sets. Notice that this provides Λ_n with a topology, namely that of $\mathrm{U}(n)/\mathrm{O}(n)$, the quotient topology of the topology of the matrix group $\mathrm{U}(n)$.

Example 1.6 As all lines are isotropic, the space Λ_1 is the space of real lines in $\mathbb{C}=\mathbb{R}^2$, namely the projective space $\mathscr{P}^1(\mathbb{R})$. The unitary group $\mathrm{U}(1)$ is a circle and the orthogonal group $\mathrm{O}(1)$ is the group with two elements $\{\pm 1\}$.

As the unitary group $\mathrm{U}(n)$ is compact (being closed and bounded in the space of matrices) and path-connected (exercise), the space Λ_n is a compact path-connected topological space.

2.3.2 The manifold Λ_n

Let us firstly describe a neighbourhood of $P \in \Lambda_n$ in Λ_n. Put

$$U_P = \{Q \in \Lambda_n \mid Q \cap (iP) = 0\}.$$

This is an open subset: using a unitary matrix, one can assume that $P = \mathbb{R}^n$, but then $U_{\mathbb{R}^n}$ is the image in Λ_n of the (saturated) open subset of $\mathrm{U}(n)$ consisting of all the unitary bases the real parts of whose vectors form a basis of $\mathbb{R}^n$. This is, clearly, a neighbourhood of P.

Lemma 1.7 The open set U_P is homeomorphic to the real vector space of all symmetric endomorphisms of P.

Proof. The subspaces Q that intersect iP only at 0 are the graphs of the linear maps $\varphi : P \to iP$. It is more convenient to call $i\varphi$ the linear map, so that φ is a linear map from P to itself. Write now that Q is Lagrangian, namely that $\forall x, y \in P, \quad \omega\big(x + i\varphi(x), y + i\varphi(y)\big) = 0$. We have

$$\begin{aligned}\omega\big(x+i\varphi(x), y+i\varphi(y)\big) &= -\Im\langle x+i\varphi(x), y+i\varphi(y)\rangle \\ &= \omega(x,y) + \omega\big(\varphi(x),\varphi(y)\big) + \big(\varphi(x),y\big) - \big(x,\varphi(y)\big) \\ &= \big(\varphi(x),y\big) - \big(x,\varphi(y)\big)\end{aligned}$$

P being Lagrangian. The subspace Q is Lagrangian if and only if the last expression vanishes for all x and y in P, namely if and only if φ is symmetric[1]. We have thus defined a bijection that maps 0 to P

$$\mathrm{sym}(P) \to U_P, \qquad \varphi \mapsto \text{graph of } i\varphi$$

[1] See also Exercise 1.56.

and is clearly a homeomorphism. $\square$

Remark 1.8 Consider for instance the "vertical" Lagrangian $i\mathbb{R}^n \subset \mathbb{C}^n$. We see that Λ_n is a disjoint union

$$\Lambda_n = \Lambda_n^0 \cup \Sigma_n$$

where Σ_n is the set of all Lagrangians that are not transversal to $i\mathbb{R}^n$ and Λ_n^0 is identified with the space of $n \times n$ real symmetric matrices.

We intend to prove now that the open sets U_P define the structure of a manifold on Λ_n. Notice firstly that any n-dimensional subspace Q of $\mathbb{R}^n \times \mathbb{R}^n$ may be represented by a rank n matrix

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}, \text{ with } 2n \text{ lines and } n \text{ columns,}$$

the column vectors of which form a basis of Q. Two matrices Z and Z' describe the same subspace if and only if there exists an $n \times n$ invertible matrix $g \in \mathrm{GL}(n;\mathbb{R})$, such that $Zg = Z'$.

Lemma 1.9 The subspace Q is Lagrangian if and only if the two matrices X and Y are such that

$${}^tXY = {}^tYX.$$

Proof. Let u, $u' \in \mathbb{R}^n$ and z, z' the corresponding vectors in Q:

$$z = \begin{pmatrix} X \\ Y \end{pmatrix} u, \qquad z' = \begin{pmatrix} X \\ Y \end{pmatrix} u'.$$

Note that Xu, Yu, Xu' and Yu' are vectors of $\mathbb{R}^n$. We compute:

$$\begin{aligned}
\omega(z,z') &= \omega((Xu,Yu),(Xu',Yu')) \\
&= (Xu)\cdot(Yu') - (Yu)\cdot(Xu') && \text{(inner product in } \mathbb{R}^n) \\
&= {}^tu{}^tXYu' - {}^tu{}^tYXu' && \text{(since } U\cdot V = {}^tUV) \\
&= {}^tu\left({}^tXY - {}^tYX\right)u'.
\end{aligned}$$

$\square$

Remark 1.10 If Q is the graph of a linear map $\mathbb{R}^n \to i\mathbb{R}^n$, it can be represented by a matrix $Z = \begin{pmatrix} \mathrm{Id} \\ A \end{pmatrix}$. The relation in lemma 1.9 simply expresses the fact that the matrix A

is symmetric.

Consider more generally a subset J of $\{1,\dots,n\}$ and the Lagrangian subspace P_J of $\mathbb{R}^n \times \mathbb{R}^n$ spanned by $\left\{(e_j)_{j\in J},(ie_j)_{j\notin J}\right\}$. Denote U_{P_J} by U_J (for simplicity). Any element of U_J is described by a unique matrix Z such that, if we extract from Z the matrix containing the lines j (for $j \in J$) and $j+n$ (for $j \notin J$), we get the identity matrix. The 2^n open sets U_J clearly cover Λ_n. Moreover, as we have said it, each of them can be identified with the subspace $\mathrm{sym}(n;\mathbb{R})$ of $n \times n$ symmetric matrices. The U_J's, with their identification with $\mathrm{sym}(n;\mathbb{R})$ are coordinate charts. Changes of coordinate are given by

$$\begin{array}{ccccc} \mathrm{sym}(n;\mathbb{R}) & \xrightarrow{\varphi_J^{-1}} & U_J \cap U_{J'} & \xrightarrow{\varphi_{J'}} & \mathrm{sym}(n;\mathbb{R}) \\ A & \longrightarrow & Z_J(A) = Z_{J'}(B) & \longrightarrow & B \end{array}$$

where $Z_J(A)$ is the matrix obtained from $\begin{pmatrix} \mathrm{Id} \\ A \end{pmatrix}$ by mapping the n first lines on the lines j (for $j \in J$) and $j+n$ (for $j \notin J$). The matrix $Z_{J'}(B)$ is obtained by multiplying $Z_J(A)$ by the inverse matrix of the (invertible!) matrix of the lines corresponding to J' in $Z_J(A)$. The coordinate change $A \mapsto B$ is clearly smooth (it is actually rational, thus analytic).

Proposition 1.11 The Grassmannian Λ_n is a compact and connected manifold of dimension $\dfrac{n(n+1)}{2}$.

2.3.3 The tautological vector bundle

Consider the space

$$E_n = \{(P,x) \in \Lambda_n \times \mathbb{C}^n \mid x \in P\}.$$

Together with its projection on Λ_n, this is a rank-n vector bundle over Λ_n. The fiber of E_n at $P \in \Lambda_n$ is the Lagrangian subspace P itself, a reason why this bundle is qualified as "tautological".

The property expressed in Lemma 1.3, namely $P^\perp = iP$ is translated, in terms of the bundle E_n, by the fact that $E_n \otimes_{\mathbb{R}} \mathbb{C}$, the complexified bundle, is trivial (has a canonical trivialization). The (global) trivialization is the isomorphism of complex vector bundles

$$\begin{aligned} E_n \otimes_{\mathbb{R}} \mathbb{C} &\to \Lambda_n \times \mathbb{C}^n \\ \big(P, x \otimes (a+ib)\big) &\mapsto \big(P, (a+ib)x\big). \end{aligned}$$

2.3.4 The tangent bundle to Λ_n

The canonical identification of the open subset U_P with the space of symmetric endomorphisms of P allows to identify the tangent bundle of Λ_n with the bundle $\mathrm{sym}(E_n)$. It is also possible to describe this bundle from the tangent bundle of $\mathrm{U}(n)$. The group $\mathrm{U}(n)$ is described as a submanifold of the space of all complex matrices by the equation ${}^t\bar{A}A = \mathrm{Id}$, so that we have

$$T_A\,\mathrm{U}(n) = \left\{X \in \mathrm{GL}(n;\mathbb{C}) \mid {}^t\bar{A}X + {}^t\bar{X}A = 0\right\}.$$

Call $\mathrm{G_u}(n)$ the vector space $T_{\mathrm{Id}}\,\mathrm{U}(n)$ of skew-Hermitian matrices. There is an isomorphism

$$\begin{aligned} T_A\,\mathrm{U}(n) &\to \mathrm{G_u}(n) \\ X &\mapsto {}^t\bar{A}X \end{aligned}$$

identifying the tangent bundle $T\,\mathrm{U}(n)$ with the trivial bundle $\mathrm{U}(n) \times \mathrm{G_u}(n)$—as any Lie group, $\mathrm{U}(n)$ is parallelizable. Consider the Lagrangian $\mathbb{R}^n$, image in Λ_n of the identity matrix Id. One can write

$$T_{\mathbb{R}^n}\big(\mathrm{U}(n)/\mathrm{O}(n)\big) = \mathrm{G_u}(n)/\mathrm{G_o}(n),$$

this is the quotient of the vector space of anti-Hermitian matrices by that of skew-symmetric real matrices. We thus identify

$$T_{\mathbb{R}^n}\Lambda_n = i\,\mathrm{sym}(n;\mathbb{R}),$$

as the real part of a skew-Hermitian matrix is skew-symmetric and its imaginary part is symmetric.

Let P be any Lagrangian subspace. Choose a unitary matrix A such that $P = A\cdot\mathbb{R}^n$. As we have identified the quotient $\mathrm{G_u}(n)/\mathrm{G_o}(n)$ with the subspace $i\,\mathrm{sym}(n;\mathbb{R})$ of $\mathrm{G_u}(n)$, we identify the quotient $T_{[A]}\Lambda_n$ with a subspace of $T_A\,\mathrm{U}(n)$:

$$\begin{tikzcd}
\mathrm{isym}(n;\mathbb{R}) \arrow[r, "X \to A.X"] \arrow[d] & T_{[A]}\Lambda_n \\
G_u(n) \arrow[r] & T_A U(n) \arrow[u]
\end{tikzcd}$$

We derive an isomorphism

$$\begin{aligned} i\,\mathrm{sym}(n;\mathbb{R}) &\rightarrow T_P\Lambda_n \\ X &\mapsto A\cdot X. \end{aligned}$$

Remark 1.12 This isomorphism depends on the choice of A, this is why it does not follow that Λ_n is parallelizable (it is actually not, as soon as $n \geq 2$).

2.3.5 The case of oriented Lagrangian subspaces

One can also consider the space $\widetilde{\Lambda}_n$ of *oriented* Lagrangian subspaces. Replacing "orthonormal basis" by "positive orthonormal basis" in what preceeds, we get an identification of $\widetilde{\Lambda}_n$ with ${}^{\mathrm{U}(n)}/_{\mathrm{SO}(n)}$.

2.3.6 The determinant and the Maslov class

The "determinant" mapping

$$\det : \mathrm{U}(n) \rightarrow S^1$$

descends to the quotient by $\mathrm{SO}(n)$ and, in the same way, its square

$$\det{}^2 : \mathrm{U}(n) \rightarrow S^1$$

to the quotient by $\mathrm{O}(n)$. This allows to compute the fundamental groups of Λ_n and $\widetilde{\Lambda}_n$.

Proposition 1.13 The fundamental group of Λ_n (resp. $\widetilde{\Lambda}_n$) is isomorphic to $\mathbb{Z}$. The covering $\widetilde{\Lambda}_n \rightarrow \Lambda_n$ shows $\pi_1(\widetilde{\Lambda}_n)$ as an index-2 subgroup in $\pi_1(\Lambda_n)$.

Proof. Recall first that the group $\mathrm{SU}(n)$ is simply connected. This can be proved by induction on n: $\mathrm{SU}(1)$ is a point and $\mathrm{SU}(n+1)$ acts transitively on the unit sphere S^{2n+1} of $\mathbb{C}^{n+1}$ with stabilizer $\mathrm{SU}(n)$, so that the exact sequence

$$\pi_1 \mathrm{SU}(n) \to \pi_1 \mathrm{SU}(n+1) \to \pi_1 S^{2n+1}$$

gives the result. As the determinant mapping

$$\det : \mathrm{U}(n) \to S^1$$

is a fibration with fiber $\mathrm{SU}(n)$, it induces an isomorphism

$$\det_\star : \pi_1 \mathrm{U}(n) \to \pi_1(S^1).$$

The fiber of the determinant mapping $\widetilde{\Lambda}_n \to S^1$ is $\mathrm{SU}(n)/\mathrm{SO}(n)$, which is simply connected, thus

$$\det_\star : \pi_1 \widetilde{\Lambda}_n \to \pi_1 S^1$$

is an isomorphism. What is left to prove is a consequence of the fact that the diagram is commutative. □

$$\begin{tikzcd}
\tilde{\Lambda}_n \arrow[r] \arrow[d, "\det"'] & \Lambda_n \arrow[d, "\det^2"] \\
S^1 \arrow[r, "z \to z^2"] & S^1
\end{tikzcd}$$

"The" generator of $\pi_1 \Lambda_n$ is called the *Maslov class*. One also calls "Maslov class" the cohomology class $\mu \in H^1(\Lambda_n; \mathbb{Z})$ that it defines by duality. Using the notation of Remark 1.8, it can be shown that μ is the dual class to the integral homology class represented by Σ_n (see [1], [13]).

2.4 Lagrangian submanifolds in $\mathbb{C}^n$

We are going now to globalize the notion of Lagrangian subspace, considering submanifolds of $\mathbb{C}^n$ whose tangent space at any point is Lagrangian. We will not really need actual submanifolds, but maps

$$f : V \to \mathbb{C}^n$$

from an n-dimensional manifold to $\mathbb{C}^n$, the tangent mapping of which

$$T_x f : T_x V \to \mathbb{C}^n$$

is an injection for any point x of V, with image a Lagrangian subspace. It is then said that f is a *Lagrangian immersion*.

For instance, any immersion of a curve (real manifold of dimension 1) in $\mathbb{C}$ is a Lagrangian immersion. Any product of Lagrangian immersions is a Lagrangian immersion (into the product target space), we thus obtain Lagrangian immersions of tori (products of circles). Our next aim is to describe examples of Lagrangian submanifolds and immersions in $\mathbb{C}^n$ and to give a necessary (and sufficient) condition for a given manifold to have a Lagrangian immersion into $\mathbb{C}^n$.

2.4.1 Lagrangian submanifolds described by functions

We consider firstly graphs.

Proposition 1.14 The graph of a map $F : \mathbb{R}^n \to (i)\mathbb{R}^n$ is a Lagrangian submanifold if and only if F is the gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$.

Proof. The tangent space to the graph at the point $\big(x, F(x)\big)$ is the graph of $(\mathrm{d}F)_x$, the differential of F at the point x. This graph is a Lagrangian subspace if and only if $(\mathrm{d}F)_x$ is a symmetric endomorphism (see the proof of Lemma 1.7). The matrix $\partial F_i/\partial x_j$ is symmetric for all x if and only if the differential form $\sum F_i \mathrm{d}x_i$ over $\mathbb{R}^n$ is closed or, equivalently, exact:

$$F_i = \frac{\partial f}{\partial x_i}, \quad \text{namely } F = \nabla f. \qquad \square$$

See, more generally, Proposition 1.70.

The Lagrangian submanifolds obtained as graphs have a very specific property: the projection of the Lagrangian submanifold on $\mathbb{R}^n$ is a diffeomorphism. We would like to consider more general Lagrangian immersions, for instance immersions of compact manifolds. Here is a way to construct Lagrangian immersions using the reduction process of §2.2.2. We start from a Lagrangian submanifold[2] $L \subset \mathbb{C}^{n+k}$. We want to construct a Lagrangian immersion into $\mathbb{C}^n$. To write $\mathbb{C}^n$ as F/F°, we choose the co-isotropic subspace $F = \mathbb{C}^n \oplus \mathbb{R}^k$, the orthogonal of which is $F^\circ = 0 \oplus \mathbb{R}^k$. We suppose that the submanifold L is "transversal to F" in the sense that, for all x,

$$T_xL + F = \mathbb{C}^{n+k}.$$

The Lagrangian subspace T_xL thus satisfies the assumption of the reduction lemma (Lemma 1.5). Hence the composition

$$T_xL \cap F \subset F \to F/F^\circ = \mathbb{C}^n$$

is the injection of a Lagrangian subspace.

Consider now the intersection V of the submanifold L with F. With the transversality assumption we have made on L, V is an n-dimensional submanifold of F (a consequence of the inverse function theorem) whose tangent space T_xV is the intersection of T_xL with F. Thus, the reduction lemma asserts, at the level of each tangent space, that, for all x in $V = L \cap F$, we have the injection of a Lagrangian subspace

$$T_xV \to \mathbb{C}^n.$$

In other words, the composition

$$V = L \cap F \subset F \to F/F^\circ = \mathbb{C}^n$$

is a Lagrangian immersion.

Remark 1.15 Even if one starts from a Lagrangian submanifold, what we get in general is only an immersion.

[2] Or a Lagrangian immersion.

Generating functions. We generalize the "graph" construction, using the reduction process as explained. Let us start with a nice and useful example.

Example 1.16 (The Whitney immersion) Consider the unit sphere in $\mathbb{R}^{n+1}$

$$S^n = \left\{(x,a) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\|^2 + a^2 = 1\right\}$$

and the map

$$f : S^n \to \mathbb{C}^n, \qquad (x,a) \mapsto (1+2ia)x.$$

The tangent space to the sphere is

$$T_{(x,a)}S^n = \{(\xi,\alpha) \in \mathbb{R}^n \times \mathbb{R} \mid x\cdot\xi + a\alpha = 0\}$$

and the tangent mapping to f is

$$T_{(x,a)}f : T_{(x,a)}S^n \to \mathbb{C}^n, \qquad (\xi,\alpha) \mapsto \xi + 2i(a\xi + \alpha x).$$

The map $T_{(x,a)}f$ is injective for all $(x,a) \in S^n$: if $T_{(x,a)}f(\xi,\alpha) = 0$, then $\xi = 0$ and $\alpha x = 0$; if $x = 0$, we have $a = \pm 1$ and the equality $x\cdot\xi + a\alpha = 0$ gives $\alpha = 0$. Thus we have $\xi = 0$ and $\alpha = 0$, so that f is an immersion. Moreover, we have

$$\begin{aligned}\omega\big(\xi + 2i(a\xi + \alpha x), \xi' + 2i(a\xi' + \alpha' x)\big) &= 2\big(\xi\cdot(a\xi' + \alpha' x) - \xi'\cdot(a\xi + \alpha x)\big)\\ &= 2(\alpha'\xi\cdot x - \alpha\xi'\cdot x) = 0\end{aligned}$$

so that the image of $T_{(x,a)}f$ is an isotropic subspace of dimension n, a Lagrangian subspace. In conclusion, the map f is a Lagrangian immersion. It has a unique double point (North and South poles of the sphere are mapped to 0). In dimension 1, this is a "figure eight". Below (in §2.4.2) we will draw pictures in dimensions 1 and 2.

Obviously, the Whitney sphere is not the graph of a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let us show that it can nevertheless be described from the graph of a map defined on a larger space. We start from a function

$$f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}.$$

As we have seen it above, the graph of ∇f is a Lagrangian subspace of $\mathbb{C}^{n+k}$. We reduce $\mathbb{C}^{n+k}$ as in §2.2.2 using the co-isotropic subspace $F = \mathbb{C}^n \oplus \mathbb{R}^k$. Here we intersect the graph of ∇f with F, namely we consider

$$V = \left\{ (x,a) \in \mathbb{R}^n \times \mathbb{R}^k \mid \frac{\partial f}{\partial a_1} = \ldots = \frac{\partial f}{\partial a_k} = 0 \right\} \subset \mathbb{R}^n \times \mathbb{R}^k.$$

The transversality assumption above is equivalent to the assumption that V is a submanifold of $\mathbb{R}^n \times \mathbb{R}^k$, in other words that the map

$$\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k, \qquad (x,a) \mapsto \left(\frac{\partial f}{\partial a_1}, \ldots, \frac{\partial f}{\partial a_k} \right)$$

is a submersion along V. In terms of partial derivatives, this is to say that the matrix

$$\left(\left(\frac{\partial^2 f}{\partial a_i \partial a_j} \right)_{\substack{1 \le i \le k \\ 1 \le j \le k}} \left(\frac{\partial^2 f}{\partial x_i \partial a_j} \right)_{\substack{1 \le i \le n \\ 1 \le j \le k}} \right)$$

has maximal rank k. In terms of tangent subspaces, this is to say that the Lagrangian subspaces that are tangent to the graph of ∇f are transversal to the co-isotropic subspace F. The reduction lemma 1.5 says that the map

$$V \to \mathbb{R}^n \times \mathbb{R}^n = \mathbb{C}^n, \qquad (x,a) \mapsto \left(x, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

is a Lagrangian immersion.

Example 1.17 (The Whitney immersion, again) With $k = 1$ and $f(x,a) = a\|x\|^2 + \dfrac{a^3}{3} - a$, we get

$$\frac{\partial f}{\partial a} = \|x\|^2 + a^2 - 1 = 0,$$

an equation which describes the sphere $S^n \subset \mathbb{R}^n \times \mathbb{R}$, and $\partial f/\partial x = 2ax$ gives the Whitney map.

Example 1.18 (An unfolding) Unfoldings are deeply related with Lagrangian submanifolds (see [2]). I will not explain here the general theory but rather show an example. Let $P \in \mathbb{R}[X]$ be a degree-$(n+1)$ polynomial

$$P(X) = X^{n+1} + x_1 X^{n-1} + \ldots + x_{n-1} X$$

where $x_1, \ldots, x_{n-1} \in \mathbb{R}$. These coefficients are going to vary, this is the reason why they are named as variables. Call P_x the polynomial corresponding to $x = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ and consider the map

$$f : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}, \qquad (x_1, \ldots, x_{n-1}, a) \mapsto P_x(a)$$

to which we apply the previous techniques. The manifold V is

$$\begin{aligned} V &= \left\{ (x,a) \in \mathbb{R}^{n-1} \times \mathbb{R} | \ \frac{\partial f}{\partial a}(x_1, \ldots, x_{n-1}, a) = 0 \right\} \\ &= \left\{ (x,a) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid P_x'(a) = 0 \right\}, \end{aligned}$$

this is the set of critical points of P_x (zeroes of its derivative P_x') when x varies. The condition that V actually be a submanifold is that the matrix of partial derivatives

$$\left(\left(\frac{\partial^2 f}{\partial a^2} \right), \left(\frac{\partial^2 f}{\partial x_i \partial a} \right)_{1 \le i \le n-1} \right)$$

has rank 1. But

$$\frac{\partial f}{\partial a} = P_x'(a) = (n+1)a^n + (n-1)x_1 a^{n-2} + \ldots + x_{n-1}$$

so that $\partial^2 f / \partial x_{n-1} \partial a$ is identically 1. Thus V is indeed a submanifold. The Lagrangian immersion is

$$V \to T^\star \mathbb{R}^{n-1}, \qquad (x,a) \mapsto \left(x, \frac{\partial P_x}{\partial x_1}(a), \ldots, \frac{\partial P_x}{\partial x_{n-1}}(a) \right).$$

For instance, starting from the family

$$P_x(X) = X^4 + x_1 X^2 + x_2 X,$$

we get

$$V = \left\{ (x_1, x_2, a) \in \mathbb{R}^3 \mid 4a^3 + 2x_1 a + x_2 = 0 \right\}$$

and the Lagrangian immersion from V into $\mathbb{R}^2 \times \mathbb{R}^2$ is the map

$$(x_1, x_2, a) \to (x_1, x_2, a^2, a).$$

Figure 1.1 shows V with its projection on the plane $\mathbb{R}^2$ of coefficients (x_1, x_2). The cusp curve is the discriminant of the family of degree-3 polynomials, the set of points x such that P'_x has a multiple root. It is obtained here as the set of critical values of the projection $V \to \mathbb{R}^2$. Over such a point x in the space of coefficients are the (one or three) roots of the polynomial P'_x.

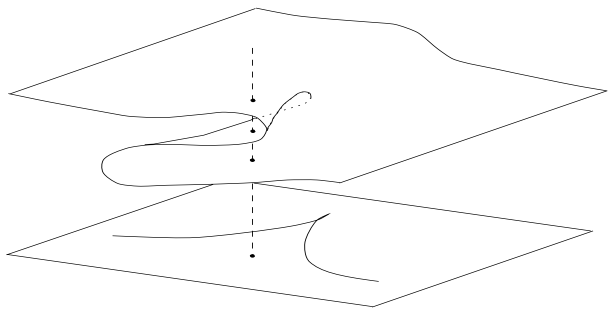


Figure 1.1: The discriminant of degree-3 polynomials.

2.4.2 Wave fronts

Exact Lagrangian immersions. If $f : V \to \mathbb{C}^n$ is a Lagrangian immersion, the 2-form $f^\star\omega$ is zero, so that $\mathrm{d}(f^\star\lambda) = 0$ and $f^\star\lambda$ is a closed 1-form on V. If, for some reason, for instance because $H^1_{\mathrm{DR}}(V) = 0$, this form is exact, there exists a function

$$F : V \to \mathbb{R}$$

such that $f^\star\lambda = dF$. The immersion f is qualified as *exact Lagrangian immersion*. The mapping

$$F \times f : L \to \mathbb{C}^n \times \mathbb{R}$$

has the property[3]

$$(f \times F)^\star \left(\mathrm{d}z - \sum_{j=1}^{n} y_j \mathrm{d}x_j \right) = 0.$$

[3] The manifold $\mathbb{C}^n \times \mathbb{R}$ is a "contact manifold" and $F \times f$ is a "Legendrian immersion" lifting the Lagrangian immersion F.

Wave fronts. Instead of looking at the Lagrangian immersion f, consider the projection

$$\begin{array}{cccccc} L & \xrightarrow{f\times F} & \mathbb{C}^n\times\mathbb{R} & \longrightarrow & \mathbb{R}^n\times\mathbb{R} \\ & & (X+iY,z) & \longrightarrow & (X,z). \end{array}$$

We will assume here that, at a general point of the Lagrangian, the tangent space is transversal to the subspace of coordinates Y. The image of the Lagrangian immersion is then a hypersurface of $\mathbb{R}^n\times\mathbb{R}$. This hypersurface is the *wave front*. Of course, it will in general be singular. Precisely, at a point of L where L is not a graph over $\mathbb{R}^n$, the projection $X+iY\mapsto X$ is singular. However, as $(f\times F)^\star\left(\mathrm{d}z-\sum_{j=1}^n y_j\mathrm{d}x_j\right)=0$, at every point of the wave front, there is a tangent hyperplane, the hyperplane

$$z=\sum_{j=1}^n Y_jx_j \qquad \text{in the space } \mathbb{R}^n\times\mathbb{R} \text{ of } (x,z) \text{ coordinates}$$

at the point image of (X,Y,z). Notice that, as the coefficient of z in this equation is non zero, the hyperplane is always transversal to the z-axis. Conversely, if a singular hypersurface of $\mathbb{R}^n\times\mathbb{R}$ has at every point a tangent hyperplane that is transversal to the z-axis, this hyperplane has a unique equation of the form $z=\sum Y_jx_j$ and it is possible to reconstruct a (maybe singular) Lagrangian submanifold from the "slopes" Y_j.

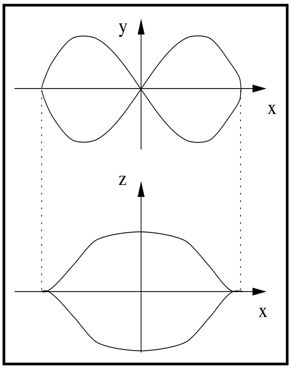


Figure 1.2: The eye.

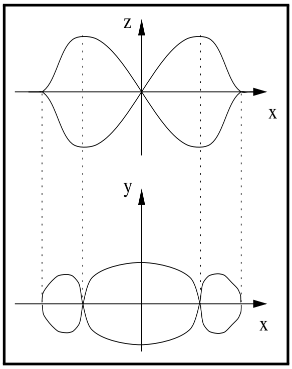


Figure 1.3: The crossbow.

We begin with an example of dimension 1, that of the Whitney immersion again. Notice that this is indeed an exact Lagrangian immersion: the restriction of the Liouville form $y\mathrm{d}x$ to the curve is exact because $\int y\mathrm{d}x=0$ (the "algebraic" area surrounded by the curve is zero). A primitive of $y\,\mathrm{d}x$ is easily found. The curve is parametrized by $t \mapsto (\cos t, \sin 2t)$ and

$$y\,\mathrm{d}x = -2\sin^2 t \cos t = -\frac{2}{3}\mathrm{d}\left(\sin^3 t\right).$$

A map to the (x,z) space is thus

$$t \mapsto (x,z) = \left(\cos t, -\frac{2}{3}\sin^3 t\right).$$

This is depicted on Figure 1.2, in an old-fashion "descriptive geometry" mood. It can be seen that the singular points of the (x,z) curve correspond to the tangents to the (x,y) curve that are vertical, and that the double point of the latter corresponds to the two tangents to the "eye" at points with the same x coordinate that are parallel.

Figure 1.3 represents an example in which we start from the wave front to reconstruct the Lagrangian. One can see from the shape of the wave front that the Lagrangian curve has two double points and two "vertical" tangents.

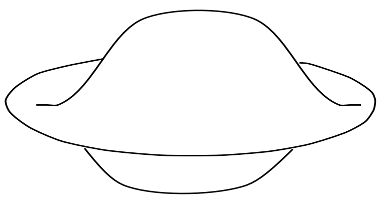


Figure 1.4: The flying saucer.

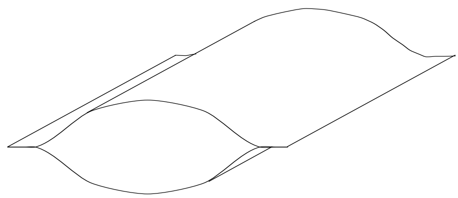


Figure 1.5: A cylinder.

One might wonder what use it can be to replace an immersed curve by a singular one. Notice that, in higher dimensions, the wave front is a hypersurface in $\mathbb{R}^n \times \mathbb{R}$ and it replaces a submanifold of the same dimension n in $\mathbb{R}^n \times \mathbb{R}^n$. Even for $n = 2$, this is very useful as this allows to represent exact Lagrangian surfaces of $\mathbb{R}^4$ by (singular) surfaces in a dimension-3 space. Here are some beautiful examples. Rotate the eye (Figure 1.2) about the z-axis to get the flying saucer depicted on Figure 1.4. The corresponding Lagrangian surface in $\mathbb{R}^2 \times \mathbb{R}^2$ is a Lagrangian immersion of the dimension-2 sphere in $\mathbb{C}^2$ with a double point. In Exercise, one checks that this is, indeed, the Whitney immersion—eventually drawn in dimension 2!

Figure 1.5 represents a cylinder constructed on the eye, namely a Lagrangian

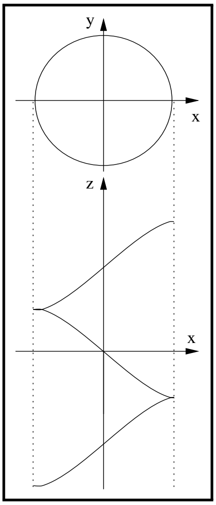


Figure 1.6: Wave front of the circle.

immersion of a cylinder, product of a figure eight with an interval, with two whole lines of singular points.

Singularities. As mentioned above, wave fronts are singular hypersurfaces. In dimension 1, they give rise to cusps, in dimension 2 to lines of cusps, and this can get more complicated, see Exercise 1.63.

Wave fronts of non exact Lagrangian immersions. Wave fronts are so nice that it is a pity not to have them for all Lagrangian immersions. In dimension 1, the problem is to represent by wave fronts curves that do not surround a zero area. Consider for instance the standard (round) circle in $\mathbb{C}$. As $\int y\,\mathrm{d}x \neq 0$, it seems that nothing can be done. Look, however, at the parametrization

$$t \mapsto (\cos t, \sin t).$$

It gives

$$y\mathrm{d}x = -\sin^2 t\,\mathrm{d}t = \mathrm{d}\left(\frac{\sin 2t}{4} - \frac{t}{2} + C\right).$$

Nothing forbids us to represent the Lagrangian (non exact) immersion of the circle by a piece of the (non closed) wave front[4] parametrized by

$$t \mapsto \left(\cos t, \frac{\sin 2t}{4} - \frac{t}{2} + C\right)$$

and depicted on Figure 1.6[5].

If we rotate the (unbounded) wave front of Figure 1.6 around a line parallel to the z-axis that does not intersect the wave front, we get the wave front of a Lagrangian torus, the one depicted on Figure 1.7. One can then use the cylinder represented on Figure 1.5 to perform connected sums of wave fronts. This way, Figure 1.8 represents (the wave front of) a genus-2 Lagrangian surface. In the same way, one constructs Lagrangian immersions of all orientable surfaces in $\mathbb{C}^2$. These figures are copied from Givental's paper [14], that contains many other examples.

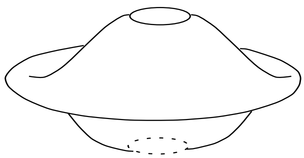


Figure 1.7: A torus.

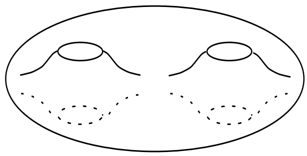


Figure 1.8: A genus-2 surface.

Remark 1.19 Except for the torus, all the surfaces depicted here have double points, that show up in the wave fronts as points having the same projection on the horizontal plane and parallel tangent planes. It is rather easy to prove that the torus is the only orientable surface that can be *embedded* as a Lagrangian submanifold in $\mathbb{C}^2$. As for non orientable surfaces, they can be embedded as Lagrangian surfaces when (and only when) their Euler characteristic is divisible by 4, with the exception of the Klein bottle, for which it is still unknown whether it can or cannot be embedded. See the pictures in [14].

Exact Lagrangian embeddings. Notice that all the examples of exact Lagrangian immersions we have given have double points. This is obviously necessary in dimension 2 ($n = 1$), due to Jordan theorem: an embedded curve cannot surround a zero

[4] This is a place where one can really appreciate the difference between closed and exact 1-forms.

[5] Notice that wave fronts are defined only up to a "vertical" translation, the actual constant C used in Figure 1.6 is $(\pi + 1)/4$.

area. This is also true in higher dimensions, due to a (hard) theorem of Gromov [18]: there is no exact Lagrangian submanifold in $\mathbb{C}^n$.

2.4.3 Other examples

Here are a few other examples.

Grassmannians. Consider the map

$$\mathrm{U}(n) \to \mathrm{sym}(n;\mathbb{C}), \qquad A \mapsto {}^t AA$$

from the group $\mathrm{U}(n)$ to the complex vector space of symmetric matrices.

Proposition 1.20 The map $A \mapsto {}^t AA$ defines a Lagrangian immersion

$$\Phi : \Lambda_n \to \mathrm{sym}(n;\mathbb{C}).$$

Proof. As ${}^t AA = \mathrm{Id}$ when $A \in \mathrm{O}(n)$, the map Φ is well defined. Call $[A]$ the class of a unitary matrix A in Λ_n. We have seen in §2.3.4 that the tangent space to Λ_n at the point $[A]$ can be identified with

$$T_{[A]}\Lambda_n = \{AH \mid H \in i\,\mathrm{sym}(n;\mathbb{R})\}.$$

It is mapped into $\mathrm{sym}(n;\mathbb{C})$ par $T_{[A]}\Phi$ as follows

$$AH \mapsto {}^t A \left(AH^t\bar{A} + \bar{A}^t H^t A\right) A.$$

The matrix $AH^t\bar{A} + \bar{A}^t H^t A$ has the form $K - \bar{K}$ for $K = AH^t\bar{A} = AHA^{-1}$ in $\mathrm{G_u}(n)$ and this describes all the matrices in the vector space $i\,\mathrm{sym}(n;\mathbb{R})$ when H varies in $i\,\mathrm{sym}(n;\mathbb{R})$. The image of the tangent mapping $T_{[A]}\Phi$ is, thus, the subspace $\rho(A) \cdot i\,\mathrm{sym}(n;\mathbb{R})$ where

$$\rho : \mathrm{U}(n) \to \mathrm{U}\left(\frac{n(n+1)}{2}\right), \qquad A \mapsto (B \mapsto {}^t ABA)$$

is the representation of $\mathrm{U}(n)$ operating on complex symmetric matrices. This image is, indeed, a Lagrangian subspace, being the image of the real part of the complex vector space $i\,\mathrm{sym}(n;\mathbb{C})$ by a unitary matrix. $\square$

Tori, integrable systems. Integrable systems (mechanical systems with many conserved quantities) yield many Lagrangian tori. See §2.6.3. We use here a few standard symplectic notions: Hamiltonian vector fields, Poisson bracket, commuting functions. See if necessary Appendix 2.6.3. Recall for instance that an integrable system on $\mathbb{C}^n = \mathbb{R}^{2n}$ is a map $f : \mathbb{R}^{2n} \to \mathbb{R}^n$ whose components $f_1, \ldots, f_n$ are functionally independent commuting functions.

This defines a local $\mathbb{R}^n$-action on $\mathbb{R}^{2n}$, which is locally free at the regular points (the points at which the derivatives of the functions f_i are actually independent). Call $X_1, \ldots, X_n$ the Hamiltonian vector fields associated with the functions f_i. These vector fields commute:

$$[X_i, X_j] = X_{\{f_i, f_j\}} = 0.$$

The $\mathbb{R}^n$-action is given by integration:

$$t \cdot x = \varphi_n^{t_n} \circ \varphi_{n-1}^{t_{n-1}} \circ \cdots \circ \varphi_1^{t_1}(x)$$

where φ_i denotes the flow of X_i and $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ is close to 0 (in order that $\varphi_i^{t_i}$ be defined). This local action is indeed locally free on the open set of regular points because the vector fields X_i give independent tangent vectors at these points.

Assume moreover that the vector fields X_i are complete, namely that the flows $\varphi_i^{t_i}$ are defined for all values of t. We then have a locally free action of $\mathbb{R}^n$ on the whole set of regular points. The vector fields X_i being tangent to the common level sets of the f_i's, this action preserve the level sets. The connected components of the regular level sets of f are thus homogeneous spaces, quotients of $\mathbb{R}^n$ by discrete subgroups. The discrete subgroups of $\mathbb{R}^n$ are the lattices $\mathbb{Z}^k$ in the linear subspaces of dimension k. The connected components of the regular level sets are thus diffeomorphic to $\mathbb{R}^{n-k} \times \mathbb{T}^k$ for some k such that $0 \leq k \leq n$. In particular, the compact connected components are tori $\mathbb{T}^n$ and these tori are Lagrangian[6], they are called the *Liouville tori*. The next proposition is the easiest part of the Arnold-Liouville theorem (see for instance [2, 6]).

Proposition 1.21 Compact connected components of the regular common level sets of an integrable system are Lagrangian tori.

There are many examples of integrable systems and thus of Lagrangian tori, coming

[6]Notice that on a compact connected component, the flows *are* complete.

from mechanical systems (spinning top, pendulum,…)[7]. The most classical example is that of the standard action of the torus

$$T^n = \left\{(t_1,\dots,t_n) \in \mathbf{C}^n \mid |t_j| = 1, \quad i = 1,\dots,n\right\}$$

on $\mathbf{C}^n$ by

$$(t_1,\dots,t_n)\cdot(z_1,\dots,z_n) = (t_1 z_1,\dots,t_n z_n),$$

the orbits of which are the common level sets of the functions

$$g_1 = \frac{1}{2}|z_1|^2,\dots,g_n = \frac{1}{2}|z_n|^2,$$

tori $S^1 \times \cdots \times S^1$ indeed, for the regular values of the g_i's (namely every g_i non zero). We will come back to these examples in §2.5.5.

Normal bundles. Let now $f : V \to \mathbf{R}^n$ be any immersion of a k-dimensional manifold into $\mathbf{R}^n$. Consider the total space of its normal bundle

$$Nf = \left\{(x,v) \in V \times \mathbf{R}^n \mid x \in V, \quad v \in \left(T_x f(T_x V)\right)^{\perp}\right\}.$$

It is naturally mapped into $\mathbf{R}^n \times \mathbf{R}^n$ by

$$\widetilde{f} : Nf \to \mathbf{R}^n \times \mathbf{R}^n, (x,v) \mapsto \left(f(x),v\right).$$

The manifold Nf has dimension $k+n-k = n$, and $\widetilde{f}$ is clearly an immersion. Moreover, it is Lagrangian. More precisely, we have:

Lemma 1.22 For λ the Liouville form on $\mathbf{R}^n \times \mathbf{R}^n$, one has $\widetilde{f}^\star \lambda = 0$.

Proof. Consider a vector $X \in T_{(x,v)}Nf$. Use the commutative diagram to compute

$$\begin{aligned}\left(\widetilde{f}^\star\lambda\right)_{(x,v)}(X) &= \lambda_{(f(x),v)}\left(T_{(x,v)}\widetilde{f}(X)\right)\\ &= v\cdot\left(T_{(f(x),v)}\pi \circ T_{(x,v)}\widetilde{f}(X)\right)\\ &= v\cdot\left(T_x f \circ T_{(x,v)}\pi(X)\right)\end{aligned}$$

[7] See for instance [6].

$$
\begin{tikzcd}
Nf \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & \mathbb{R}^n \times \mathbb{R}^n \arrow[d, "\pi"] \\
V \arrow[r, "f"] & \mathbb{R}^n
\end{tikzcd}
$$

$$= 0 \qquad \text{since } v \text{ is orthogonal to } T_x f(T_x V). \qquad \square$$

This method allows to construct many (non compact) examples and can be generalized by replacing $\mathbb{R}^n \times \mathbb{R}^n = T^\star \mathbb{R}^n$ by the cotangent bundle $T^\star M$ of a manifold and $V \to \mathbb{R}^n$ by an immersion into M. See §3.2.1.

2.4.4 The Gauss map

Let $f : V \to \mathbb{C}^n$ be a Lagrangian immersion. Its tangent space at any point is a Lagrangian subspace of $\mathbb{C}^n$. One can globalize the data consisting of all these tangent spaces to define the "Gauss map"

$$V \to \gamma(f)\Lambda_n, \qquad x \mapsto T_x f(T_x V).$$

By definition of the tautological bundle (§2.3.3), one has

$$\gamma(f)^\star E_n = TV.$$

In particular, the tangent bundle to V must have the same properties as E_n.

Proposition 1.23 For a manifold to have a Lagrangian immersion into $\mathbb{C}^n$, it is necessary that the complexification of its tangent bundle be trivializable.

The converse is true, but less easy to prove. This is an application of Gromov's h-principle [17], see also [24].

Example 1.24 1) Spheres. We have seen examples of Lagrangian immersions of spheres in $\mathbb{C}^n$ (in §2.4.1). One deduces that $TS^n \otimes_{\mathbb{R}} \mathbb{C}$ is a trivial complex bundle. Notice however that it is not true that the tangent bundle TS^n itself is trivial (except for $n = 0$, 1, 3 and 7).

2) Surfaces. All orientable surfaces and half the non orientable surfaces have Lagrangian immersions in $\mathbb{C}^2$ (as we have seen it in §2.4.2). This is not the case, neither for the real projective plane nor for the connected sums of an odd number of copies of this plane.

3) Normal bundles. This is a case where the tangent bundle itself is trivial (before complexification):

$$T_{(x,v)}(Nf) = \{(\xi, U) \mid \xi \in T_xV, U \perp T_xf(T_xV)\} = T_xV \oplus N_xf$$

which is canonically isomorphic to the ambient space $\mathbb{R}^n$.

4) Grassmannians. The Gauss map φ of the Lagrangian immersion Φ

$$\varphi : \Lambda_n \to \Lambda_{\frac{n(n+1)}{2}}$$

is of course such that

$$\varphi^\star E_{\frac{n(n+1)}{2}} = \mathrm{sym}(E_n).$$

The Maslov class. Every Lagrangian immersion has a Maslov class: use the Gauss map

$$\varphi(f) : V \to \Lambda_n$$

to pull back $\mu \in H^1(\Lambda_n; \mathbb{Z})$ to a class

$$\mu(f) \in H^1(V; \mathbb{Z}).$$

One can also, with the notation of Remark 1.8, define $\mu(f)$ as the cohomology class dual to $\gamma(f)^{-1}(\Sigma_n)$, see [25] for example.

2.5 Special Lagrangian submanifolds in $\mathbb{C}^n$

Lagrangian submanifolds are submanifolds of $\mathbb{C}^n$ whose tangent space at each point is a Lagrangian subspace. They have a Gauss map into the Grassmannian Λ_n, namely into $\mathrm{U}(n)/\mathrm{O}(n)$. We look now at the submanifolds whose Gauss map takes values in $\Lambda_n^{\mathrm{s}} = \mathrm{SU}(n)/\mathrm{SO}(n)$. These are the special Lagrangian submanifolds, invented by Harvey and Lawson [19].

2.5.1 Special Lagrangian subspaces

An oriented subspace P of $\mathbb{C}^n$ is said *special Lagrangian* if it has a positive orthonormal basis that is a *special unitary* basis of $\mathbb{C}^n$.

For instance, if $n = 1$, as $\mathbb{C}$ has a unique special unitary basis (the group $\mathrm{SU}(1)$ is the trivial group), there is only one special Lagrangian subspace in $\mathbb{C}$, the line $\mathbb{R} \subset \mathbb{C}$... this will not be a very interesting notion in dimension 1. Fortunately, for $n \geq 2$, this is more funny. Identify the space $\mathbb{C}^2$ with the skew-field $\mathscr{K}$ of quaternions:

$$\begin{aligned} Z = (z_1, z_2) &= X + iY \\ &= (x_1 + iy_1, x_2 + iy_2) \\ &= (x_1 + iy_1) + j(x_2 + iy_2) \\ &= (x_1 + jx_2) + i(y_1 - jy_2). \end{aligned}$$

The 2×2 matrices that are in $\mathrm{SU}(2)$ are the matrices of the form

$$\begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \qquad \text{with } |z_1|^2 + |z_2|^2 = 1.$$

Thus the special Lagrangian planes are those who have an orthonormal basis (Z, Z') with Z and Z' of the form

$$\begin{cases} Z = (x_1 + iy_1) + j(x_2 + iy_2) \\ Z' = (-x_2 + iy_2) + j(x_1 - iy_1). \end{cases}$$

Notice that

$$Z' = [(x_1 + iy_1) + j(x_2 + iy_2)]\, j = Zj.$$

Thus a basis (Z, Z') of $\mathbb{C}^2$ is special unitary if and only if $Z' = Zj$. Now put on the space $\mathscr{K}$ the structure of complex vector space defined by the multiplication by j to get:

Proposition 1.25 The special Lagrangian subspaces of $\mathbb{C}^2$ are the complex lines with respect to the complex structure defined by the multiplication by j. The Grassmannian L^{s} is a complex projective line.

Remark 1.26 Notice also that $L^{\mathrm{s}} = \mathrm{SU}(2)/\mathrm{SO}(2) = S^3/S^1$, this is indeed a dimension-2 sphere.

To distinguish the special Lagrangian subspaces among all the Lagrangian subspaces or the special unitary matrices among all the unitary matrices, one uses the (complex) determinant. To globalize the notion of special Lagrangian subspace and define special Lagrangian submanifolds, it will be practical (and natural) to describe the linear objects by differential forms. The form corresponding to the complex determinant is

$$\Omega = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n.$$

Expressing the definition of the determinant, namely

$$(Ae_1) \wedge \cdots \wedge (Ae_n) = (\det A) e_1 \wedge \cdots \wedge e_n,$$

we see that, for $A \in \mathrm{GL}(n;\mathbb{C})$, we have indeed

$$A^\star \Omega = (\det A)\Omega.$$

Hence

$$\det A = 1 \iff A^\star \Omega = \Omega.$$

In order to work with real subspaces, we need an additional notation: call α and β the two degree n real forms:

$$\alpha = \Re(\Omega), \qquad \beta = \Im(\Omega).$$

For instance, in dimension 1, $\Omega = \mathrm{d}z$, $\alpha = \mathrm{d}x$ and $\beta = \mathrm{d}y$. In dimension 2,

$$\begin{aligned}\Omega = \mathrm{d}z_1 \wedge \mathrm{d}z_2 &= (\mathrm{d}x_1 + i\mathrm{d}y_1) \wedge (\mathrm{d}x_2 + i\mathrm{d}y_2)\\ &= \mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}y_1 \wedge \mathrm{d}y_2 + i(\mathrm{d}y_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_1 \wedge \mathrm{d}y_2),\end{aligned}$$

that is

$$\begin{cases}\alpha = \mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}y_1 \wedge \mathrm{d}y_2\\ \beta = \mathrm{d}y_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_1 \wedge \mathrm{d}y_2.\end{cases}$$

Proposition 1.27 Let P be an oriented (real) vector subspace of dimension n in $\mathbb{C}^n$. The number $\Omega(x_1 \wedge \cdots \wedge x_n)$ depends only on P and not on the positive orthonormal basis $(x_1, \ldots, x_n)$ of P used to express it.

Proof. Consider the $2n$ vectors $(x_1, \ldots, x_n, ix_1, \ldots, ix_n)$ and the linear mapping $A : \mathbb{C}^n \to$

$\mathbb{C}^n$ defined by the images of the vectors of the canonical basis:

$$A(e_j) = x_j, \qquad A(ie_j) = ix_j$$

(so that A is complex linear). Then

$$\Omega(x_1 \wedge \cdots \wedge x_n) = \det_{\mathbb{C}} A.$$

If $(gx_1, \ldots, gx_n)$ is a positive orthonormal basis of P (that is, if $g \in \mathrm{SO}(n)$), one gets

$$\begin{aligned} \Omega(gx_1 \wedge \cdots \wedge gx_n) = \det_{\mathbb{C}}(gA) = \det_{\mathbb{C}} g \det_{\mathbb{C}} A &= \det_{\mathbb{R}} g \det_{\mathbb{C}} A \\ &= \det_{\mathbb{C}} A = \Omega(x_1 \wedge \cdots \wedge x_n) \end{aligned}$$

(since $g \in \mathrm{SO}(n) \subset \mathrm{GL}(n;\mathbb{C})$). □

We will thus denote $\Omega(P)$ the number $\Omega(x_1 \wedge \cdots \wedge x_n)$. Similarly, denote $\alpha(P)$ and $\beta(P)$ its real and imaginary parts.

Remark 1.28 Notice that $\Omega(P)$ is non zero if and only if the $2n$ vectors

$$(x_1, \ldots, x_n, ix_1, \ldots, ix_n)$$

form a basis of $\mathbb{C}^n$ over $\mathbb{R}$, that is, if and only if $P \cap iP = \{0\}$ or P does not contain any complex line. These subspaces are said *totally real*. This is in particular the case for Lagrangian subspaces.

Proposition 1.29 Let P be a real subspace of $\mathbb{C}^n$. For P to have an orientation for which it is a special Lagrangian subspace, it is necessary and sufficient that P be Lagrangian and that $\beta(P) = 0$.

Proof. Let P be a Lagrangian subspace. Choose $(x_1, \ldots, x_n)$, an orthonormal basis which is the image of the canonical basis of $\mathbb{C}^n$ by a unitary matrix A. Thus

$$\Omega(P) = \det_{\mathbb{C}} A \in S^1.$$

For P to have a positive basis that is special unitary, it is necessary and sufficient that

$\det_{\mathbb{C}} A$ be equal to ± 1, that is, that

$$\Im(\Omega(P)) = 0. \qquad \square$$

Here is a last elementary remark on linear subspaces:

Proposition 1.30 Let $Q \subset \mathbb{C}^n$ be an oriented isotropic linear subspace of dimension $n-1$. There exists a unique special Lagrangian subspace that contains Q.

Proof. Choose a positive orthonormal basis $(x_1, \dots, x_{n-1})$ of Q. In the complex line that is the orthogonal, with respect to the Hermitian form, of the complex subspace spanned by the x_i's, there is a unique vector x_n such that $(x_1, \dots, x_{n-1}, x_n)$ is a special unitary basis of $\mathbb{C}^n$. $\square$

2.5.2 Special Lagrangian submanifolds

A Lagrangian immersion

$$f : V \to \mathbb{C}^n$$

of an oriented manifold into $\mathbb{C}^n$ is *special* if $T_x f(T_x V)$ is a special Lagrangian subspace for every x. The Gauss map then takes values in $\Lambda_n^s \subset \widetilde{\Lambda}_n$.

Example 1.31 1) In dimension 1, the tangent space must be the unique special Lagrangian $\mathbb{R} \subset \mathbb{C}$ for all x. If V is connected, f must thus be the immersion of an open subset of $\mathbb{R}$ by $t \mapsto t + ia$. We already have noticed that this dimension will not be very exciting.

2) In dimension 2, $T_x f(T_x V)$ must be a j-complex line for all x, f is thus the immersion of a j-complex curve into $\mathbb{C}^2$. This gives quite a lot of examples.

Remark 1.32 The Maslov class of a *special* Lagrangian immersion into $\mathbb{C}^n$ is zero. Of course, as the examples above show it, there are many more Lagrangian immersions with zero Maslov class than there are special Lagrangian immersions.

In terms of forms, to say that the immersion

$$f : V \to \mathbb{C}^n$$

is special Lagrangian amounts to require that it satisfies

◇ firstly $f^\star\omega = 0$ (it is Lagrangian)

◇ secondly $f^\star\beta = 0$ (it is special).

Proposition 1.33 If f is a special Lagrangian immersion, $f^\star\Omega$ is a volume form on V.

Proof. The complex form Ω has type $(n,0)$ and defines an n-form $f^\star\Omega$ on V, which is real since its imaginary part vanishes on V. Let x be a point in V and let $(X_1,\dots,X_n)$ be a basis of T_xV. One has

$$\left(f^\star\Omega\right)_x(X_1,\dots,X_n) = \Omega_{f(x)}\left(T_xf(X_1),\dots,T_xf(X_n)\right) \neq 0$$

because of Remark 1.28 and since V is Lagrangian. Thus $f^\star\Omega$ never vanishes. □

In dimensions 1 and 2, the special Lagrangian submanifolds are non compact (in dimension 2, the Liouville theorem forbids complex curves in $\mathbb{C}^2$ to be compact). This is actually always the case, a straightforward application of Proposition 1.33:

Corollary 1.34 There is no special Lagrangian immersion from a compact manifold into $\mathbb{C}^n$.

Proof. If $f : V \to \mathbb{C}^n$ is a special Lagrangian immersion, $f^\star\Omega$ is a volume form on V. But Ω is an exact complex form:

$$\Omega = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n = \mathrm{d}(z_1\mathrm{d}z_2 \wedge \cdots \wedge \mathrm{d}z_n).$$

Decompose $z_1\mathrm{d}z_2 \wedge \cdots \wedge \mathrm{d}z_n$ into its real and imaginary parts to get $\alpha = \mathrm{d}\Re(z_1\mathrm{d}z_2 \wedge \cdots \wedge \mathrm{d}z_n) = \mathrm{d}\eta$ and eventually $f^\star\Omega = f^\star\alpha = \mathrm{d}(f^\star\eta)$. The manifold V thus has an exact volume form, and this prevents it of being compact. □

Let us give now examples of special Lagrangian submanifolds in $\mathbb{C}^n$, starting from the examples of Lagrangians constructed in section 2.4.

2.5.3 Graphs of forms

The condition "to be Lagrangian" is, to some extent, a linear condition, as it can be seen, for instance, when looking for 1-forms whose graphs are Lagrangian submanifolds: they are the graphs of closed forms—and the equation $\mathrm{d}\alpha = 0$ is *linear*.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. We now require the graph of ∇f, a Lagrangian submanifold, to be a special Lagrangian submanifold. The $n = 1$ case is not interesting. For $n = 2$, the Lagrangian immersion associated with the function f is

$$F : (x,y) \mapsto \left(x, y, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

and the form β is

$$\beta = \mathrm{d}y_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_1 \wedge \mathrm{d}y_2.$$

Then

$$\begin{aligned} F^\star \beta &= \mathrm{d}\left(\frac{\partial f}{\partial x}\right) \wedge \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}\left(\frac{\partial f}{\partial y}\right) \\ &= \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

We thus have:

Proposition 1.35 Let U be an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ a function of class $\mathscr{C}^2$. The graph of ∇f is a special Lagrangian submanifold of $\mathbb{C}^2$ if and only if f is a harmonic function.

Here again, the condition is linear. Starting from dimension 3, this is no more the case. The function f must satisfy a complicated non linear partial differential equation, expressed in Proposition 1.36 below. Let us begin by a notation. Denote by $\mathrm{Hess}(f)$ the Hessian matrix of f, namely the matrix

$$\mathrm{Hess}(f)_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

and by $\sigma_k\big(\mathrm{Hess}(f)\big)$ the k-th elementary symmetric functions of its eigenvalues. More generally, for an $n \times n$ real matrix A, write

$$\det(A - X\,\mathrm{Id}) = \sum_{k=0}^{n} (-1)^k \sigma_k(A) X^{n-k}.$$

For example, $\sigma_1\big(\mathrm{Hess}(f)\big)$ is the trace of the Hessian matrix, the Laplacian Δf of f.

Proposition 1.36 Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a function of class $\mathscr{C}^2$.

The graph of ∇f is a special Lagrangian submanifold of $\mathbb{C}^n$ if and only if f satisfies the partial differential equation

$$\sum_{k\geq 0}(-1)^k\sigma_{2k+1}\big(\mathrm{Hess}(f)\big)=0.$$

Example 1.37 For $n=1$, the differential equation is $f''(t)=0$ or $f'(t)$ constant, and this is precisely the differential equation of the special Lagrangian submanifolds. For $n=2$ again, only σ_1 appears in the (linear) relation, which expresses the fact that the function f must be harmonic. For $n=3$, the relation is

$$\sigma_1\big(\mathrm{Hess}(f)\big)=\sigma_3\big(\mathrm{Hess}(f)\big)$$

or

$$\Delta f=\det\big(\mathrm{Hess}(f)\big).$$

Proof of Proposition 1.36. The tangent space to the graph of ∇f at the point $(x,\nabla f_x)$ is the image of the plane $\mathbb{R}^n$ under the linear map $\mathrm{Id}+i(\mathrm{d}^2 f)_x$. This is a special Lagrangian subspace if and only if

$$\Im\left(\det_{\mathbb{C}}\big(\mathrm{Id}+i(\mathrm{d}^2 f)_x\big)\right)=0.$$

We still must check that, for any real symmetric matrix A, one has

$$\Im\left(\det_{\mathbb{C}}(\mathrm{Id}+iA)\right)=\sum_{k\geq 0}(-1)^k\sigma_{2k+1}(A).$$

Since A is real symmetric, it is diagonalizable in an orthonormal basis. It is clear that the two sides of the relation to be proved are invariant under conjugation by matrices in $\mathrm{O}(n)$. One may thus assume that the matrix A is the diagonal $(\lambda_1,\dots,\lambda_n)$. The left hand side is then $\Im\left(\prod_j(1+i\lambda_j)\right)$ and it clearly coincides with the right hand side. $\square$

2.5.4 Normal bundles of surfaces

Let $f:V\to\mathbb{R}^n$ be an immersion of a dimension-k manifold into $\mathbb{R}^n$. We know (see §2.4.3) that its normal bundle has a natural Lagrangian immersion into $\mathbb{R}^n\times\mathbb{R}^n$. Look now for the conditions under which this is a special Lagrangian immersion.

For the sake of simplicity, suppose here that $k=2$ and $n=3$ (case of surfaces in $\mathbb{R}^3$). There is a more general discussion in [19].

Fix a point x_0 in V, a unit normal vector field $n = n(x)$ on a neighbourhood of x_0 and an orthonormal basis (e_1, e_2) of $T_{x_0}V$. Assume that this basis is orthogonal with respect to the second fundamental form, that is, to the symmetric bilinear form defined on $T_{x_0}V$ by

$$II(X,Y) = -\big(T_{x_0}n(X), Y\big).$$

We have

$$T_{x_0}n(e_1) = -\lambda_1 e_1, \qquad T_{x_0}n(e_2) = -\lambda_2 e_2$$

where λ_1 and λ_2 are the two "principal curvatures" of V at x_0.

Consider now the tangent space to Nf at (x_0, v) where $v = \mu n(x_0) \in N_{x_0}f = \mathbb{R} \cdot n(x_0)$. The immersion of Nf into $\mathbb{R}^3 \times \mathbb{R}^3$ is

$$(x, \mu) \mapsto \big(\mu n(x), f(x)\big)$$

(notice that, this time, the immersion f appears in the second copy of $\mathbb{R}^n$, that of purely imaginary vectors). The tangent mapping is

$$\begin{aligned} P_0 = T_{(x_0,\mu)}(Nf) = T_{x_0}V \oplus N_{x_0}f &\to \mathbb{R}^3 \times \mathbb{R}^3 \\ (\xi, \eta) &\mapsto \big(\eta n(x_0) + \mu T_{x_0}n(\xi), T_{x_0}f(\xi)\big). \end{aligned}$$

The images of the basis vectors are

$$\begin{aligned} e_1 &\mapsto (-\mu\lambda_1 e_1, e_1) \\ e_2 &\mapsto (-\mu\lambda_2 e_2, e_2) \\ n &\mapsto (n, 0). \end{aligned}$$

Thus

$$\begin{aligned} \Omega(P_0) &= (\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3)\Big(\big((i - \mu\lambda_1)e_1\big) \wedge \big((i - \mu\lambda_2)e_2\big) \wedge n\Big) \\ &= (i - \mu\lambda_1)(i - \mu\lambda_2), \end{aligned}$$

so that P_0 is a special Lagrangian if and only if $\mu(\lambda_1 + \lambda_2) = 0$. This is to say that the trace of $T_{x_0}n$ is zero. In other words, we have shown:

Proposition 1.38 The immersion of the normal bundle of

$$f : V \to \mathbb{R}^3$$

into $\mathbb{C}^3$ is a special Lagrangian immersion if and only if f is a minimal immersion.

For more information on minimal surfaces, see, for example, the excellent surveys in [29] and the references quoted there.

Remark 1.39 It is true that we have already mentioned Riemannian metrics in these notes, but up to now, they have had only an auxiliary role. The result presented here is a genuine Riemannian one.

2.5.5 Starting from integrable systems

Being compact, Lagrangian tori obtained as "Liouville tori" cannot be special Lagrangian submanifolds in $\mathbb{C}^n$. One can try to replace them by special Lagrangian submanifolds with the help of the remark included in Proposition 1.30: the idea is to consider a (necessarily isotropic) subtorus in a Liouville torus T^n and to add a direction to construct another Lagrangian submanifold, which will be special.

Here is an example, coming from [19], of such a construction. Start from an orbit L of the standard action of T^n on $\mathbb{C}^n$ (see §2.4.3), namely a common level set of the functions

$$g_1(z) = \frac{1}{2}|z_1|^2, \dots, g_n(z) = \frac{1}{2}|z_n|^2,$$

say $g_i = a_i$, none of the a_i's being zero, so that L is a Lagrangian torus. Choose a subtorus of T^n: $T^{n-1} = \{(t_1, \dots, t_n) \in T^n \mid t_1 \cdots t_n = 1\}$. Let V be an orbit of this subtorus, an isotropic torus of dimension $n-1$. Consider the Hamiltonian vector fields $Y_1, \dots, Y_n$ associated to the functions g_i:

$$\begin{cases} Y_1(z_1, \dots, z_n) = (iz_1, 0, \dots, 0) \\ \qquad\qquad\qquad \vdots \\ Y_n(z_1, \dots, z_n) = (0, \dots, 0, iz_n). \end{cases}$$

Let $z = (z_1, \dots, z_n)$ be a point of V. The tangent space to L at z is spanned by the values of the Y_i's, the tangent space to V is the hyperplane consisting of the vectors $\sum \lambda_i Y_i$ satisfying

$\sum \lambda_i = 0$. It is spanned by the values at z of the vector fields

$$X_1 = Y_1 - Y_n, \dots, X_{n-1} = Y_{n-1} - Y_n,$$

that are the Hamiltonian vector fields of the functions

$$f_1 = g_1 - g_n, \dots, f_{n-1} = g_{n-1} - g_n.$$

We are looking now for an n-th function f such that the subspace spanned by the vectors $X_1, \dots, X_{n-1}$ and X_f is a special Lagrangian at each point where the vectors are independent. The subspace $F = \langle X_1, \dots, X_{n-1} \rangle$ is isotropic and has dimension $n-1$. Its orthogonal is

$$F^\circ = \langle X_1, \dots, X_{n-1}, Z, iZ \rangle$$

for any vector Z such that $\langle X_1, \dots, X_{n-1}, Z \rangle$ is Lagrangian. One can use for Z any linear combination of the Y_j's. We look for X_f of the form

$$X_f = \sum \lambda_j Y_j = (\lambda_1 i z_1, \dots, \lambda_n i z_n)$$

so that the determinant

$$\begin{vmatrix} iz_1 & & 0 & \lambda_1 i z_1 \\ 0 & \ddots & 0 & \vdots \\ \vdots & & iz_{n-1} & \vdots \\ -iz_n & \dots & -iz_n & \lambda_n i z_n \end{vmatrix}$$

is real. Subtracting the linear combination $\lambda_1 X_1 + \ldots + \lambda_{n-1} X_{n-1}$ from the last vector, this vector becomes $(\lambda_1 + \ldots + \lambda_n) Y_n$, so that the determinant is $i^n (\lambda_1 + \ldots + \lambda_n) z_1 \cdots z_n$. We are thus looking for functions f and λ_i's such that

$$X_f(z_1, \dots, z_n) = (\lambda_1 i z_1, \dots, \lambda_n i z_n) \quad \text{and} \quad i^n (\lambda_1 + \ldots + \lambda_n) z_1 \cdots z_n \quad \text{is real.}$$

For any index j, we must have:

$$2 \frac{\partial f}{\partial z_j} = \bar{\lambda}_j \bar{z}_j, \quad 2 \frac{\partial f}{\partial \bar{z}_j} = \lambda_j z_j, \quad \text{et} \quad i^n (\lambda_1 + \cdots + \lambda_n) z_1 \cdots z_n \in \mathbb{R}.$$

The functions

$$f(z_1,\dots,z_n) = z_1\cdots z_n + \overline{z_1\cdots z_n}, \qquad \lambda_j = 2\frac{\overline{z_1\cdots z_n}}{|z_j|^2}$$

give a solution when $i^n \in \mathbb{R}$, namely when n is even. When n is odd, we rather take

$$f(z_1,\dots,z_n) = \frac{1}{i}(z_1\cdots z_n - \overline{z_1\cdots z_n}), \qquad \lambda_j = 2\frac{\overline{z_1\cdots z_n}}{|z_j|^2}.$$

Proposition 1.40 The functions $f_1,\dots,f_n$ defined by

$$f_1(z_1,\dots,z_n) = \frac{1}{2}(|z_1|^2 - |z_n|^2), \dots, f_{n-1}(z_1,\dots,z_n) = \frac{1}{2}(|z_{n-1}|^2 - |z_n|^2), \quad \text{and}$$

$$f_n(z_1,\dots,z_n) = \begin{cases} \Re(z_1\cdots z_n) & \text{if } n \text{ is even} \\ \Im(z_1\cdots z_n) & \text{if } n \text{ is odd} \end{cases}$$

form an integrable system on $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$, all the regular common level sets of which are special Lagrangian cylinders $T^{n-1} \times \mathbb{R}$.

Proof. The only thing that is left to prove is that the regular levels are “cylinders” $T^{n-1} \times \mathbb{R}$. As we are dealing with an integrable system, we know that the levels are endowed with an $\mathbb{R}^n$-action. Here the $n-1$ first vector fields are periodic and in particular complete; the last one is complete too, because the level is a closed submanifold of $\mathbb{C}^n$. The action is thus an action of $T^{n-1} \times \mathbb{R}$ and this is a free action, as the level, being special Lagrangian, cannot be compact. $\square$

Exercise 1.67 describes essentially the same construction.

2.5.6 Special Lagrangian submanifolds invariant under SO(n)

The next and sporadic examples also come from [19]. Start from a curve Γ in

$$\mathbb{C} = \mathbb{C} \times \{0\} \subset \mathbb{C} \times \mathbb{C}^{n-1} = \mathbb{C}^n$$

and “rotate” it with the help of the diagonal SO(n)-action, namely

$$g \cdot (X + iY) = g \cdot X + ig \cdot Y \qquad \text{for } g \in \mathrm{SO}(n) \text{ and } X, Y \in \mathbb{R}^n.$$

We get a submanifold of $\mathbb{C}^n$:

$$V = \{(x+iy)u \mid x+iy \in \Gamma, u \in \mathbb{R}^n, u = g(e_1) \text{ for some } g \in \mathrm{SO}(n)\}$$

(notice that u describes a sphere $S^{n-1} \subset \mathbb{R}^n$). The tangent space to V at $(x+iy)u$ is spanned by the vectors $(x+iy)U$ with $U \in T_u S^{n-1}$ and the $(\xi + i\eta)u$ with $\xi + i\eta$ tangent to Γ at $x+iy$. The submanifold V is always Lagrangian, as is easily checked:

$$\begin{aligned}
\omega\big((x+iy)U, (x+iy)U'\big) &= xy(U \cdot U' - U' \cdot U) = 0,\\
\omega\big((x+iy)U, (\xi + i\eta)u\big) &= (x\eta - y\xi)U \cdot u = 0.
\end{aligned}$$

It is special Lagrangian if and only if, denoting $(U_1, \dots, U_{n-1})$ a basis of $T_u S^{n-1}$,

$$\det\nolimits_{\mathbb{C}}\big((x+iy)U_1, \dots, (x+iy)U_{n-1}, (\xi + i\eta)u\big) \in \mathbb{R}.$$

But this determinant is equal to $(x+iy)^{n-1}(\xi + i\eta)\det_{\mathbb{C}}(U_1, \dots, U_{n-1}, u)$, or to $(x+iy)^{n-1}(\xi + i\eta)\det_{\mathbb{R}}(U_1, \dots, U_{n-1}, u)$ since these vectors are in $\mathbb{R}^n \subset \mathbb{C}^n$. The condition is thus that

$$(x+iy)^{n-1}(\xi + i\eta) \in \mathbb{R} \quad \text{for any tangent vector } \xi + i\eta \text{ to } \Gamma.$$

We get eventually:

Proposition 1.41 The Lagrangian submanifold of $\mathbb{C}^n$

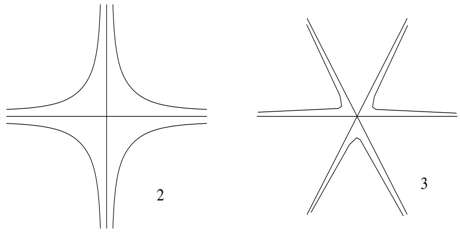
$$V = \left\{(x+iy)u \mid (x+iy) \in \Gamma, u \in S^{n-1} \subset \mathbb{R}^n\right\}$$

is special Lagrangian if and only if, on Γ, the function $\Im\left((x+iy)^n\right)$ is constant.

Remark 1.42 This method gives essentially one special Lagrangian submanifold in any dimension, which is not much!

Remark 1.43 Any connected component of Γ is diffeomorphic to $\mathbb{R}$, the special Lagrangian submanifolds obtained are (unions of) copies of $S^{n-1} \times \mathbb{R}$.

To draw a picture of the special Lagrangian submanifold, one draws first the curve Γ (in the (x,y) plane), then its wave front (in the (x,z) plane). One then notices that the



Liouville form $\lambda = Y \cdot \mathrm{d}X$ is, on V:

$$\begin{aligned}\lambda = Y \cdot \mathrm{d}X &= (yu) \cdot \mathrm{d}(xu)\\ &= (yu) \cdot \big((\mathrm{d}x)u + x\mathrm{d}u\big)\\ &= y\mathrm{d}x\end{aligned}$$

(since $u\mathrm{d}u = \frac{1}{2}\|u\|^2 = 0$) so that the wave front of V is

$$\big\{(xu, z) \in \mathbb{R}^n \times \mathbb{R} \mid (x, z) \text{ is a point of the wave front of } \Gamma\big\}.$$

For example, for $n = 2$, the curve Γ is a hyperbola $xy =$ constant, its wave front is the curve $z = \log x$ and the wave front of the special Lagrangian submanifold is the surface of revolution obtained by rotating the graph of the logarithm function about the z-axis (Figure 1.9).

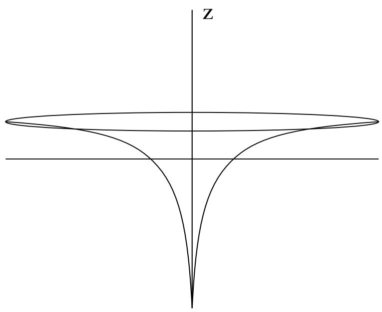


Figure 1.9: Rotating the graph of the logarithm function.

2.6 Appendices

2.6.1 The topology of the symplectic group

Proposition 1.44 The manifold $\mathrm{SP}(2n)$ is diffeomorphic to the Cartesian product of the group $\mathrm{U}(n)$ with a convex open cone of a vector space of dimension $n(n+1)$.

Corollary 1.45 The symplectic group $\mathrm{SP}(2n)$ is path connected. The injection of $\mathrm{U}(n)$ in $\mathrm{SP}(2n)$ induces an isomorphism

$$\mathbb{Z} = \pi_1 \mathrm{U}(n) \to \pi_1 \mathrm{SP}(2n).$$

Proof of Proposition 1.44. Let $A \in \mathrm{SP}(2n)$. As any invertible transformation of $\mathbb{R}^{2n}$, A can be written in a unique way as a product

$$A = S \cdot \Omega$$

where S is the positive definite symmetric matrix $S = \sqrt{A^t A}$ and Ω is the orthogonal matrix $\Omega = S^{-1}A$. As A is symplectic, the matrix S is also symplectic: ${}^t A$ and $A^t A$ are symplectic, the matrix $A^t A$ is symmetric, positive definite, thus it is diagonalizable in an orthonormal basis and S is the matrix that, in this basis, is the diagonal of the square roots of the eigenvalues of $A^t A$, so that S is indeed symplectic as is $A^t A$. One deduces that

$$\Omega = S^{-1}A \in \mathrm{SP}(2n) \cap \mathrm{O}(2n) = \mathrm{U}(n)$$

and thus that Ω is a unitary matrix. We have thus obtained a bijection

$$\mathrm{SP}(2n) \to \mathrm{U}(n) \times \mathscr{S}, \qquad A \mapsto \left((\sqrt{A^t A})^{-1}A, \sqrt{A^t A}\right)$$

where $\mathscr{S}$ denotes the set of positive definite symmetric matrices that are symplectic. We still have to prove that this space is an open convex cone in a vector space of dimension $n(n+1)$. Write the matrices as block matrices in a symplectic basis. Let $S \in \mathscr{S}$, we have

$$S = \begin{pmatrix} A & B \\ {}^t B & C \end{pmatrix} \qquad \text{with } A \text{ and } C \text{ positive definite symmetric and } {}^t SJS = J.$$

The last condition, that expresses the fact that S is symplectic, is equivalent to

$$BA \text{ is symmetric and } C = A^{-1}(\mathrm{Id} + B^2).$$

The mapping

$$\mathscr{S} \to \mathrm{sym}(n;\mathbb{R}) \times \mathrm{sym}^+(n;\mathbb{R}), \qquad S \mapsto (BA, A)$$

is the desired diffeomorphism. The open set $\mathrm{sym}^+(n;\mathbb{R})$ of all positive definite symmetric real matrices is obviously an open convex cone in the vector space $\mathrm{sym}(n;\mathbb{R})$ of all symmetric matrices, the product is an open convex cone of the product space, that has dimension $2\dfrac{n(n+1)}{2}$. □

Proof of Corollary 1.45. The convex cone $\mathrm{sym}(n;\mathbb{R}) \times \mathrm{sym}^+(n;\mathbb{R})$ is contractible. □

Remark 1.46 There is another beautiful proof of this type of contractibility results, due to Sévennec, in [5].

2.6.2 Complex structures

If E is a vector space endowed with a symplectic from ω, it is said that an endomorphism J of E is a complex structure *calibrated* by ω if $J^2 = -\mathrm{Id}$ (J is a complex structure),

$$\omega(Jv, Jw) = \omega(v, w)$$

(J is symplectic) and

$$g(v, w) = \omega(v, Jw)$$

is an inner product (namely positive definite) on E.

2.6.3 Hamiltonian vector fields, integrable systems

In this appendix, denote for simplicity $\mathbb{C}^n = \mathbb{R}^{2n}$ by W. It can be replaced by any symplectic manifold W (see §3.1).

Hamiltonian vector fields. To any function $H : W \to \mathbb{R}$, the symplectic form allows to associate a vector field, a kind of gradient, the *Hamiltonian vector field* X_H (sometimes called the "symplectic gradient" H). This is the vector field defined by the relation

$$\omega_x\big(Y, X_H(x)\big) = \big(\mathrm{d}H\big)_x(Y) \quad \text{for all } Y \in T_xW,$$

or by

$$\iota_{X_H}\omega = -\mathrm{d}H.$$

In coordinates, one has

$$X_H(x_1,\dots,x_n,y_1,\dots,y_n) = \left(\frac{\partial H}{\partial y_1},\dots,\frac{\partial H}{\partial y_n},-\frac{\partial H}{\partial x_1},\dots,-\frac{\partial H}{\partial x_n}\right).$$

Notice that the vector field X_H vanishes at x if and only if x is a critical point of the function H:

$$X_H(x) = 0 \Longleftrightarrow (dH)_x = 0.$$

In particular, the singularities (or zeroes) of a Hamiltonian vector field are the critical points of a function.

Notice also that the function H is constant along the trajectories, or integral curves, of the vector field X_H: as ω_x is skew symmetric, we have $(dH)(X_H) = 0$ or $X_H \cdot H = 0$.

The Poisson bracket. Assume now that f and g are two functions on W. Define their "Poisson bracket" $\{f,g\}$ by the formula

$$\{f,g\} = X_f \cdot g = \mathrm{d}g(X_f).$$

In coordinates, one has

$$\{f,g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial y_i}\frac{\partial f}{\partial x_i}\right).$$

Notice that

$$X_f \cdot g = \mathrm{d}g(X_f) = \omega(X_f,X_g) = -\omega(X_g,X_f) = -\mathrm{d}f(X_g) = -X_g \cdot f,$$

so that $\{f,g\} = -\{g,f\}$. This shows that the Poisson bracket is skew-symmetric in f and g. By definition, this is also a derivation (in both entries); in other words, the Poisson bracket satisfies the Leibniz identity

$$\{f,gh\} = \{f,g\}h + g\{f,h\}.$$

Using the general relation $\mathscr{L}_X\iota_Y - \iota_Y\mathscr{L}_X = \iota_{[X,Y]}$ and Cartan formula $\mathscr{L}_X = \mathrm{d}\iota_X + \iota_X\mathrm{d}$, we get

$$\begin{aligned}\iota_{[X_f,X_g]}\omega &= \mathscr{L}_{X_f}\iota_{X_g}\omega - \iota_{X_g}\mathscr{L}_{X_f}\omega\\ &= \mathrm{d}\iota_{X_f}\ iota_{X_g}\omega + \iota_{X_f}\mathrm{d}\iota_{X_g}\omega - \iota_{X_g}\mathrm{d}\iota_{X_f}\omega - \iota_{X_g}\iota_{X_f}\mathrm{d}\omega\\ &= \mathrm{d}\iota_{X_f}\iota_{X_g}\omega = \mathrm{d}\big(\omega(X_g,X_f)\big) = -\mathrm{d}\{f,g\},\end{aligned}$$

in other words

$$[X_f,X_g] = X_{\{f,g\}}.$$

We also have

$$[X_f,X_g]\cdot h = \big\{\{f,g\},h\big\}.$$

From this, we deduce that the Poisson bracket satisfies the Jacobi identity

$$\big\{f,\{g,h\}\big\} + \big\{g,\{h,f\}\big\} + \big\{h,\{f,g\}\big\} = 0$$

and thus defines a Lie algebra structure on $\mathscr{C}^\infty(W)$, the mapping

$$\mathscr{C}^\infty(W)\to\mathscr{X}(W), \qquad f\mapsto X_f$$

being a morphism of Lie algebras from $\mathscr{C}^\infty(W)$ (with the Poisson bracket) into the Lie algebra of vector fields (with the Lie bracket of vector fields).

Proof of the Jacobi identity. Apply the definition of the bracket of vector fields:

$$[X_f,X_g]\cdot h = X_f\cdot(X_g\cdot h) - X_g\cdot(X_f\cdot h),$$

and the equality above to get

$$\begin{aligned}\big\{\{f,g\},h\big\} &= [X_f,X_g]\cdot h = X_f\cdot(X_g\cdot h) - X_g\cdot(X_f\cdot h)\\ &= X_f\cdot\{g,h\} - X_g\cdot\{f,h\} = \big\{f,\{g,h\}\big\} - \big\{g,\{f,h\}\big\}.\end{aligned}$$

This, taking into account the skew-symmetry of the Poisson bracket, is equivalent to the Jacobi identity. $\square$

Integrable systems. As any vector field, the Hamiltonian vector field X_H defines a differential system on W, that is,

$$\dot{x}(t) = X_H\big(x(t)\big),$$

the *Hamiltonian system* associated with H. The function H is constant along the trajectories of this system, in other words

$$X_H \cdot H = 0 \quad \text{or} \quad \mathrm{d}H(X_H) = 0.$$

It is said that H is a first integral of the system. More generally, a function $f : W \to \mathbb{R}$ that is constant along the integral curves of a vector field X is called a *first integral* of X. In the case of a Hamiltonian vector field X_H, the equality $X_H \cdot f = 0$ is equivalent to $\{f, H\} = 0$, we say that the functions f and H *commute*.

It is said that a Hamiltonian system is integrable if it has "as many commuting first integrals as possible". Let us explain this:

- ◇ Let $f_1, \dots, f_k$ be commuting first integrals of the system X_H, so that $\left\{f_i, f_j\right\} = 0$ for all i and j. Each one is constant on the trajectories of the Hamiltonian system associated to each other one.
- ◇ The expression "as many as possible": at any point x of W, the subspace of T_xW spanned by the Hamiltonian vector fields of the functions f_i is isotropic:

  $$\omega(X_{f_i}, X_{f_j}) = \pm\left\{f_i, f_j\right\} = 0.$$

  Its dimension is thus at most $n = \frac{1}{2}\dim W$. It is required that, at least for x in an open dense subset of W, this subspace has maximal dimension n.
- ◇ Notice that the vectors X_{f_i} are independent at x if and only if the linear forms $(\mathrm{d}f_i)_x$ are independent.

Definition 1.47 The function H or the Hamiltonian vector field X_H on W is qualified as *integrable* if it has n independent commuting first integrals.

Example 1.48 Every function depending only of the coordinates y_i,

$$H = H(y_1, \dots, y_n)$$

is integrable: the functions y_i are independent commuting first integrals. Every Hamiltonian system on a $\mathbb{C}$ is integrable. Similarly, a Hamiltonian system on $\mathbb{C}^2$ is integrable if and only if it has a "second first integral".

Exercises

Exercise 1.49 Let V be a real vector space and $V^\star$ be its dual. Check that the form ω defined on $V \oplus V^\star$ by

$$\omega\big((v,\alpha),(w,\beta)\big) = \alpha(w) - \beta(v)$$

is a symplectic form.

Exercise 1.50 (Relative linear Darboux theorem) Let F be a vector subspace of a symplectic vector space E. Assume that the restriction of the symplectic form to F has rank $2r$. Show that there exists a symplectic basis $(e_1,\dots,e_n,f_1,\dots,f_n)$ of E such that $(e_1,\dots,e_r,e_{r+1},\dots,e_{r+k},f_1,\dots,f_r)$ is a basis of F (k is the integer defined by $2r+k = \dim F$).

Exercise 1.51 Show that the symplectic group of $\mathbb{C}$ is isomorphic with the special linear group $\mathrm{SL}(2;\mathbb{R})$.

Exercise 1.52 Prove directly that the symplectic group $\mathrm{SP}(2)$ is diffeomorphic to the product of a circle by an open disk.

Exercise 1.53 Let $A \in \mathrm{SP}(2n)$. Check that the matrices ${}^tA$ and A^{-1} are similar[8]. Show that λ is an eigenvalue of A if and only if λ^{-1} is also an eigenvalue, and that both occur with the same multiplicity.

Exercise 1.54 Check that a non zero vector of a symplectic space can be mapped to any other non zero vector by a symplectic transformation (in other words, the symplectic groups acts transitively on the set of non zero vectors).

Show that, for $n > 1$, the symplectic group does not act transitively on the set of real 2-dimensional subspaces of $\mathbb{C}^n$.

Exercise 1.55 Let $n > 1$ be an integer. Let P be a real plane (dimension-2 subspace) in $\mathbb{C}^n$. Show that P is either isotropic or symplectic. What are the orbits of the action of the symplectic group on the set of planes in $\mathbb{C}^n$?

[8]Thus A and A^{-1} are similar too.

Exercise 1.56 Let V be a vector space and $V^\star$ be its dual. Endow $V \oplus V^\star$ with the symplectic form defined in Exercise 1.49. Let $A : V \to V^\star$ be a linear map. Prove that the graph of A is a Lagrangian subspace if and only if the bilinear form defined by A on V is symmetric.

Exercise 1.57 Let E be a vector space endowed with a symplectic form ω and let F be (any) subspace of E. Prove that ω induces a symplectic structure on the quotient $F/F \cap F^\circ$.

Exercise 1.58 Let E be an even dimensional vector space and let ω, ω' be two symplectic forms on E. Prove that the symplectic groups $\mathrm{SP}(E,\omega)$ and $\mathrm{SP}(E,\omega')$ are conjugated subgroups of $\mathrm{GL}(E)$.

Let $\Omega(E)$ be the space of all symplectic forms on the vector space E. Prove that the linear group of E acts on this space by

$$\big(g \cdot \omega\big)(X,Y) = \omega(gX, gY).$$

Deduce that $\Omega(E)$ is in one-to-one correspondence[9] with the homogeneous space ${}^{\mathrm{GL}(E)}/_{\mathrm{SP}(E)}$, where $\mathrm{SP}(E)$ is the symplectic group $\mathrm{SP}(E,\omega_0)$ for a given form ω_0 on E.

Exercise 1.59 Prove that, on any symplectic vector space, there are complex structures. Prove that a complex structure is an isometry and that it is skew-symmetric for the inner product it defines.

Exercise 1.60 Let V be a real vector space. Using an inner product on V, construct a complex structure calibrated by the standard symplectic form on $V \oplus V^\star$ and such that

$$\big(J(v), w\big) = v \cdot w \quad \text{for all} \quad v, w \in V.$$

Exercise 1.61 Assume that the wave front

$$]-\alpha,\alpha[\to \mathbb{R}^2, \qquad t \mapsto \big(x(t), z(t)\big)$$

has an ordinary cusp for $t = 0$ with a tangent line transversal to the z-axis. Prove that this is the wave front of a Lagrangian *immersion* of $]-\alpha,\alpha[$ into $\mathbb{R}^2$.

[9] This is actually a homeomorphism.

Exercise 1.62 Prove that the wave front of the Whitney immersion $S^n \to \mathbb{C}^n$ is the hypersurface in $\mathbb{R}^{n+1}$ image of the sphere S^n by

$$(x,a) \mapsto \left(x, a\|x\|^2 + \frac{a^3}{3} - a\right)$$

(using the notation of Example 1.16). Find the singular points of this wave front and draw it in the cases $n = 1$ (this is the eye, Figure 1.2) and $n = 2$ (this is the flying saucer, Figure 1.4).

Exercise 1.63 (The swallow tail) Determine... and draw the wave front of the Lagrangian immersion described in §1.18 and on Figure 1.1.

Exercise 1.64 Prove that the Maslov class of the standard (Lagrangian) embedding of the circle is ± 2. What is that of the Whitney immersion? Of the immersion defined by the crossbow[10]?

Exercise 1.65 (Lagrangian cobordisms [3]) The space $\mathbb{C}^n$ is endowed with its Liouville form λ and its symplectic form $d\lambda$. It is said that a Lagrangian immersion $f : L \to \mathbb{C}^n$ is "cobordant to zero" if there exists an *oriented* manifold V of dimension $n+1$, with boundary, whose boundary is L, and a Lagrangian immersion

$$\widetilde{f} : V \to \mathbb{C}^{n+1}$$

transversal to the co-isotropic subspace $F = \mathbb{C}^n \oplus i\mathbb{R} \subset \mathbb{C}^{n+1}$, such that $\widetilde{f}^{-1}(F \cap V) = \partial V = L$ and such that the composition

$$L \to \widetilde{f}|_L F \to F/F^\circ = \mathbb{C}^n$$

is the immersion f.

1) Prove that the Whitney immersion $S^n \to \mathbb{C}^n$ (§1.16) is cobordant to zero.

2) Assume that $f : S^1 \to \mathbb{C}$ is cobordant to zero. What can be said of $\int_{S^1} f^\star \lambda$? Prove that, if a Lagrangian immersion $S^1 \to \mathbb{C}$ is cobordant to zero, it is exact.

[10]Hint: orient the circle and notice that the unit tangent vector to the Whitney immersion does not take all the values in the circle. For the crossbow, notice that this immersion of the circle into $\mathbb{C}$ may be deformed, among immersion, into the standard embedding.

3) Consider an exact Lagrangian immersion

$$f : S^1 \to \mathbb{C}$$

and its wave front in $\mathbb{R}^2$. Assume the singularities of the wave front are ordinary cusps. The tangent line to the front at any point is transversal to the z-axis. The circle S^1 is oriented. Count the cusps of type (a) with a $+$ sign, those of type (b) with a $-$ sign (Figure 1.10) and get a number $N(f) \in \mathbb{Z}$. What is the value of $N(f)$ for the Whitney immersion? For the crossbow (Figure 1.3)?

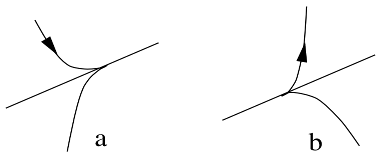


Figure 1.10: Two types of cusp.

4) The Lagrangian immersion $f : S^1 \to \mathbb{C}$ has a Gauss map $\gamma(f)$, taking its values in the Grassmannian $\widetilde{\Lambda}_1$ of oriented Lagrangians in $\mathbb{C}$, that is a circle S^1. Call σ the closed 1-form "dθ" on this circle. Prove that[11]

$$N(f) = \frac{1}{2\pi} \int_{S^1} \gamma(f)^\star \sigma.$$

5) Consider the mapping $j : \widetilde{\Lambda}_1 \to \widetilde{\Lambda}_2, \qquad P \mapsto P \oplus \mathbb{R} \subset \mathbb{C} \oplus \mathbb{R} \subset \mathbb{C}^2$. It can be shown (this is an additional question, use §2.3.6) that

$$j^\star : H^1(\widetilde{\Lambda}_2) \to H^1(\widetilde{\Lambda}_1)$$

is an isomorphism. Prove that if $f : S^1 \to \mathbb{C}$ is cobordant to zero, then $N(f) = 0$. Does there exist a Lagrangian immersion of a disk into $\mathbb{C}^2$ whose boundary is the crossbow?

Exercise 1.66 (From (x,y) to $(z,\bar{z})$) Writing

$$\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y, \qquad \mathrm{d}\bar{z} = \mathrm{d}x - i\mathrm{d}y$$

[11] This is to say that $N(f)$ is the Maslov class of the immersion f.

one gets a couple of relations between the expressions of the vector fields in coordinates (x,y) or $(z,\bar{z})$. Prove for instance that

$$X_f = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial y_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial y_j} \right) = \frac{i}{2} \sum_{j=1}^{n} \left(\frac{\partial f}{\partial z_j} \frac{\partial}{\partial \bar{z}_j} - \frac{\partial f}{\partial \bar{z}_j} \frac{\partial}{\partial z_j} \right).$$

Exercise 1.67 Consider the vector field X given on $\mathbb{C}^2$ by

$$X(z_1, z_2) = (i\alpha_1 z_1, i\alpha_2 z_2)$$

(α_1 and α_2 being two real parameters).

1) Check that

$$X(z_1, z_2) = \alpha_1 \left(iz_1 \frac{\partial}{\partial z_1} - i\bar{z}_1 \frac{\partial}{\partial \bar{z}_1} \right) + \alpha_2 \left(iz_2 \frac{\partial}{\partial z_2} - i\bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right)$$

and show that the form $\imath_X \Omega$ is holomorphic.

2) Show that X preserves ω and find a function H such that $X = X_H$.

3) Under which condition does the vector field X preserve Ω? Assume now that this condition holds. Find two functions g and h from $\mathbb{C}^2$ to $\mathbb{R}$ such that

$$\imath_X \Omega = \mathrm{d}g + i\mathrm{d}h.$$

Consider $H^{-1}(a) \cap h^{-1}(b)$. Show that, if a is a regular value of H, this is a special Lagrangian submanifold.

4) Describe the special Lagrangian submanifolds $H^{-1}(a) \cap h^{-1}(b)$ as complex j-curves, that is, by equations.

5) Check that they are diffeomorphic to $S^1 \times \mathbb{R}$. Hint: they are conics.

. . . .

3 Lagrangian and special Lagrangian submanifolds in symplectic and Calabi-Yau manifolds

3.1 Symplectic manifolds

In order to deform a Lagrangian submanifold in $\mathbb{C}^n$, we must understand how a tubular neighbourhood looks like. We prove here that a Lagrangian submanifold has a neighbourhood which is diffeomorphic to a neighbourhood of the zero section in its cotangent bundle. To be precise and explicit, we need to define a symplectic structure on the cotangent bundles and more generally to say what a symplectic structure on a manifold is.

A *symplectic manifold* is a manifold W endowed with a non degenerate 2-form ω, namely, a non degenerate alternated bilinear form ω_x on each tangent space T_xW, which is required to be *closed*, ($\mathrm{d}\omega = 0$). Notice that a symplectic manifold is even dimensional.

Example 1.68 1) The first example is of course $\mathbb{C}^n$ with the symplectic form we have used so far, considered as a differential form:

$$\omega = \sum_{j=1}^{n} \mathrm{d}y_j \wedge \mathrm{d}x_j$$

(where $(x_1 + iy_1, \ldots, x_n + iy_n)$ stands for the complex coordinates in $\mathbb{C}^n$). One also has:

$$\omega_z(Z, Z') = \omega(Z, Z') = \sum_{j=1}^{n} (X'_j Y_j - X_j Y'_j) = X' \cdot Y - X \cdot Y'.$$

And this is an exact, hence closed, form

$$\omega = \mathrm{d}\left(\sum_{j=1}^{n} y_j \mathrm{d}x_j\right).$$

2) The next example is that of cotangent bundles. Think that $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$, then that $\mathbb{R}^n \times \mathbb{R}^n = T^\star\mathbb{R}^n$ and simply replace $\mathbb{R}^n$ by any manifold V. On $W = T^\star V$, there is a *canonical* 1-form, the *Liouville form* λ, defined by the "compact" formula:

$$\lambda_{(x,\alpha)}(X) = \alpha\left(T_{(x,\alpha)}\pi(X)\right)$$

...in which x denotes a point of V, α an element of $T^\star_x V$ (namely a linear form on the tangent space T_xV), and π the projection $T^\star V \to V$ of the cotangent bundle. If $(x_1,\dots,x_n)$ are local coordinates on V and $(y_1,\dots,y_n)$ the cotangent coordinates, then

$$\lambda = \sum_{j=1}^{n} y_j \mathrm{d}x_j.$$

The 2-form $\mathrm{d}\lambda$ is both closed (!) and non degenerate.

3) Surfaces. On a surface W, any 2-form is closed. Moreover, in dimension 2, to say that a 2-form is non degenerate means that it nowhere vanishes, in other words that it is a volume form: all the orientable surfaces may be considered as symplectic manifolds.

4) The sphere. Consider, in particular, the unit sphere S^2 in $\mathbb{R}^3$, whose tangent space at a point v is the plane orthogonal to the unit vector v. Put

$$\omega_v(X,Y) = v \cdot (X \wedge Y) = \det(v,X,Y).$$

This is a non degenerate 2-form and thus a symplectic form.

5) The projective space $\mathscr{P}^n(\mathbb{C})$ is a symplectic manifold. The nicest thing to do is to define its symplectic form starting from that of $\mathbb{C}^{n+1}$ and using the symplectic reduction process. To define $\mathscr{P}^n(\mathbb{C})$, we factor out the unit sphere S^{2n+1} of $\mathbb{C}^{n+1}$ by the S^1-action (multiplication of coordinates):

$$t \cdot (z_1,\dots,z_{n+1}) = (tz_1,\dots,tz_{n+1})$$

At each point x of the sphere S^{2n+1}, the tangent space is the Euclidean orthogonal of x and the kernel of the restriction of the symplectic from is the line generated by ix. This line is also the tangent space to the circle through x on the sphere.

The symplectic form of $\mathbb{C}^{n+1}$ defines a non degenerate alternated bilinear form ω on $\mathscr{P}^n(\mathbb{C})$. Its pull-back on the sphere is closed, so that ω is closed. It is actually a (the standard) Kähler form on $\mathscr{P}^n(\mathbb{C})$.

6) *Complex* submanifolds of the projective space are symplectic. The compatibility of ω with the complex structure gives that $\omega(X,iX) > 0$ for any vector X that is tangent to the submanifold, so that ω is indeed non degenerate on this submanifold.

7) More generally, all Kähler manifolds are symplectic. We will come back to this remark.

Notice that, on cotangent bundles, as on $\mathbb{C}^n$, the symplectic form is *exact*. This cannot be the case on a compact symplectic manifold.

Proposition 1.69 On a compact manifold, there exists no 2-form that is both non degenerate and exact.

Proof. Let ω be a non degenerate 2-form on the $2n$-dimensional manifold W. To say that ω is non degenerate is to say that $\omega^{\wedge n}$ is a *volume form*. But then, if $\omega = \mathrm{d}\alpha$,

$$\omega^{\wedge n} = \mathrm{d}\big(\alpha \wedge \omega^{\wedge(n-1)}\big)$$

is also exact, thus W cannot be compact. □

Hamiltonian vector fields X_H for functions $H : W \to \mathbb{R}$ are defined exactly as in Appendix 2.6.3 and so is the Poisson bracket of two functions on W. Exercise 1.102 explains why it is required that a symplectic form be closed.

3.2 Lagrangian submanifolds and immersions

An immersion $f : L \to W$ into a symplectic manifold is *Lagrangian* if $f^\star\omega = 0$ and $\dim W = 2\dim L$.

3.2.1 In cotangent bundles

All what was done in $\mathbb{C}^n$ in §2.4.1 works as well in a cotangent bundle.

Graphs. Proposition 1.14 generalizes as:

Proposition 1.70 Let $\alpha : L \to T^\star L$ be a section of a cotangent bundle. Its image is a Lagrangian submanifold if and only if the 1-form α is closed.

Proof. The most elegant thing to do is to state first a property of the Liouville form (which explains why it is called the "canonical" 1-form): for any form α, one has

$$\alpha^\star\lambda = \alpha.$$

In this equality, α is considered as a section of the cotangent bundle in the left hand side and as a form in the right hand side. One has indeed:

$$\begin{aligned}
\left(\alpha^\star\lambda\right)_x(Y) &= \lambda_{(x,\alpha_x)}\left(T_x\alpha(Y)\right) && \text{by definition of } \alpha^\star \\
&= \alpha_x\left(T_{(x,\alpha_x)}\pi\circ T_x\alpha(Y)\right) && \text{by definition of } \lambda \\
&= \alpha_x(Y) && \text{because } \alpha \text{ is a section.}
\end{aligned}$$

Eventually, $\alpha^\star\omega = 0$ if and only if $\mathrm{d}(\alpha^\star\lambda) = 0$, thus the graph of α is a Lagrangian submanifold if and only if α is closed. □

Remark 1.71 In particular, the zero section of $L \subset T^\star L$ is a Lagrangian submanifold. What we plan to do next is to show that $L \subset T^\star L$ is a model for *all* Lagrangian embeddings of L into a symplectic manifold (Theorem 1.78).

Generating functions. A function

$$F : M \times \mathbb{R}^k \to \mathbb{R}$$

allows to construct a Lagrangian submanifold (the graph of $\mathrm{d}F$) into $T^\star M \times \mathbb{C}^k$ and then, by reduction, a Lagrangian immersion into $T^\star M$.

Wave fronts. Exact Lagrangian immersions into $T^\star M$ define wave fronts in $M \times \mathbb{R}$ and conversely.

Conormal bundles. Let $f : V \to M$ be any immersion. The *conormal* bundle is the subbundle of the pull back bundle

$$f^\star T^\star M = \left\{(x,\varphi) \mid x \in V, \varphi \in T^\star_{f(x)}M\right\} \to V$$

defined by

$$N^\star f = \left\{(x,\phi) \in f^\star T^\star M \mid \phi|_{T_x f(T_x V)} = 0\right\} = \left\{(x,\phi) \in f^\star T^\star M \mid \phi \circ T_x f = 0\right\}.$$

$N^\star f$ is mapped into $T^\star M$ by $F : (x,\phi) \mapsto \left(f(x),\phi\right)$. This is an immersion, since $T_{(x,\phi)}F(\xi,\psi) = \left(T_x f(\xi),\psi\right)$. It is Lagrangian, as we have $F^\star\lambda = 0$. Indeed, calling π

the two projections $T^\star M \to M$ et $N^\star f \to V$, we get

$$\begin{aligned}\left(F^\star\lambda\right)_{(x,\phi)}(X) &= \lambda_{(f(x),\phi)}\left(T_{(x,\phi)}F(X)\right)\\ &= \phi\left(T_{(f(x),\phi)}\pi \circ T_{(x,\phi)}F(X)\right)\\ &= \phi\left(T_x f\left(T_{(x,\phi)}\pi(X)\right)\right)\\ &= 0\end{aligned}$$

as ϕ vanishes on the vectors that are tangent to V.

One should check that the proof given for the normal bundle in §2.4.3 for the case where $M = \mathbb{R}^n$ is identical to the one given here, orthogonality there being an *ersatz* of duality here.

3.3 Tubular neighbourhoods of Lagrangian submanifolds

Let us now present a method, invented by Moser [28], which allows to describe a symplectic manifold in the neighbourhood of a point (they are all the same) or a neighbourhood of a Lagrangian submanifold in a symplectic manifold.

3.3.1 Moser's method

The next "lemma" contains all these results.

Lemma 1.72 Let W be a $2n$-dimensional manifold and let $Q \subset W$ be a *compact* submanifold. Assume that ω_0 and ω_1 are two closed 2-forms on W such that, at any point x of Q, ω_0 and ω_1 are equal and non degenerate on T_xW. Then there exists open neighbourhoods $\mathscr{V}_0$ and $\mathscr{V}_1$ of Q and a diffeomorphism

$$\psi : \mathscr{V}_0 \to \mathscr{V}_1$$

such that $\psi|_Q = \mathrm{Id}_Q$ and $\psi^\star\omega_1 = \omega_0$.

Remark 1.73 It is not easy to create a diffeomorphism "ex nihilo". The remarkable idea of Moser is to construct *a whole path* of diffeomorphisms starting from the identity and ending at some diffeomorphism which has the desired property.

Let us write the proof of Moser lemma when $W = \mathbb{C}^n$ and explain then what should be done to get it in the general case (essentially to replace the Euclidean structure by a

Riemannian metric). Consider the normal bundle to Q in $\mathbb{C}^n$,

$$NQ = \{(x,v) \in Q \times \mathbb{C}^n \mid v \perp T_x Q\}$$

and the open subset

$$\mathscr{U}_\varepsilon = \{(x,v) \in N_Q \mid \|v\| < \varepsilon\}.$$

Notice firstly that:

Lemma 1.74 Let Q be a compact submanifold of the Euclidean space $\mathbb{R}^m$. For ε small enough, the map

$$E : NQ \to \mathbb{R}^m, \qquad (x,v) \mapsto x+v$$

is a diffeomorphism from $\mathscr{U}_\varepsilon$ onto its image.

Proof. In a neighbourhood of a point x_0 of Q, we describe Q by local coordinates $u = (u_1, \dots, u_k)$, namely by a mapping $x : U \to \mathbb{R}^m$ where U is open in $\mathbb{R}^k$ and $x(0) = x_0$. One can choose vector fields $\big(v_1(u), \dots, v_{m-k}(u)\big)$ of $\mathbb{R}^m$ on U, that form, for all u, an orthonormal basis of the normal space of Q at $x(u)$. So we have local coordinates $(u_1, \dots, u_k, t_1, \dots, t_{m-k})$ on NQ in which the mapping E is $E(u,t) = x(u) + \sum_{i=1}^{m-k} t_i v_i(u)$. The partial derivatives are

$$\begin{cases} \dfrac{\partial E}{\partial u_i} = \dfrac{\partial x}{\partial u_i} + \displaystyle\sum_j t_j \dfrac{\partial v_j}{\partial u_j} \\ \dfrac{\partial E}{\partial t_k} = v_k. \end{cases}$$

The matrix of partial derivatives is invertible for $t = 0$, thus it is invertible also for $\|t\|$ small enough[12]. We conclude globally using the compactness of Q. □

Call $\mathscr{V}_0$ the image of a suitable $\mathscr{U}_\varepsilon$. This is a neighbourhood of Q in $\mathbb{C}^n$. We next prove:

Lemma 1.75 On $\mathscr{V}_0$, the 2-form $\tau = \omega_1 - \omega_0$ is exact.

First proof. The vector bundle NQ retracts on its zero section. The inclusion $j : Q \to \mathscr{V}_0$ thus induces an isomorphism $j^\star : H^2_{\mathrm{DR}}(\mathscr{V}_0) \to H^2_{\mathrm{DR}}(Q)$. As $j^\star[\omega_1] = j^\star[\omega_0]$, the cohomology classes of ω_1 and ω_0 are equal in $H^2_{\mathrm{DR}}(\mathscr{V}_0)$, which means that their difference is an exact form. □

[12] It is interesting to see "how far" we can go. This leads to the notion of *focal point*, see for example [27].

Second proof. We explicitly construct a 1-form σ that is a primitive of τ. Consider the dilation of factor t in the fibers

$$\varphi_t : \mathscr{V}_0 \to \mathscr{V}_0, \quad x+v \mapsto x+tv, \qquad t \in [0,1].$$

This is a diffeomorphism (onto its image) for $t > 0$ and we have $\varphi_0(\mathscr{V}_0) = Q$, $\varphi_1 = \mathrm{Id}_{\mathscr{V}_0}$ and $\varphi_t|_Q = \mathrm{Id}_Q$. The form $\tau = \omega_1 - \omega_0$ is a 2-form on $\mathbb{C}^n$. Consider its restriction to $\mathscr{V}_0$. It is identically zero along Q by assumption. We have

$$\varphi_0^\star \tau = 0, \qquad \varphi_1^\star \tau = \tau.$$

Consider now the (time depending) radial vector field X_t (tangent to the dilation) on $\mathscr{V}_0$. This is the vector field defined by

$$X_t(y) = \left(\frac{\mathrm{d}}{\mathrm{d}s}\varphi_s\right)\left(\varphi_t^{-1}(y)\right)\Big|_{s=t}.$$

It is defined only for $t > 0$, in the same way that φ_t is a diffeomorphism only for $t > 0$. In a very concrete way, the vector field is $X_t(x+v) = \frac{1}{t}v$. For all t, consider also the 1-form σ^t defined by

$$\sigma^t_{x+v}(Y) = \tau_{x+tv}\big(v, T_{x+v}\big(\varphi_t\big)(Y)\big).$$

Notice that, if y is in Q, one has $\varphi_t(y) = y$ and $\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(y) = 0$ thus σ^t is zero along Q. For $t > 0$, one has

$$\begin{aligned}
\big(\varphi_t^\star \iota_{X_t}\tau\big)_{x+v}(Y) &= \big(\iota_{X_t}\tau\big)_{x+tv}\big(X_t(x+tv), T_{x+v}(\varphi_t)(Y)\big)\\
&= \tau_{x+tv}\big(v, T_{x+v}(\varphi_t)(Y)\big)\\
&= \sigma^t_{x+v}(Y).
\end{aligned}$$

Hence, for $t > 0$,

$$\sigma^t = \varphi_t^\star \iota_{X_t}\tau$$

and consequently

$$\mathrm{d}\sigma^t = \mathrm{d}\left(\varphi_t^\star \iota_{X_t}\tau\right) = \varphi_t^\star\left(d\iota_{X_t}\tau + \iota_{X_t}d\tau\right)$$

$$= \varphi_t^\star (\mathscr{L}_{X_t}\tau) = \frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t^\star \tau).$$

Eventually, we get

$$\mathrm{d}\sigma^t = \frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t^\star \tau)$$

for $t > 0$ and thus also for all $t \in [0,1]$. Now

$$\tau = \tau - 0 = \varphi_1^\star \tau - \varphi_0^\star \tau = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t^\star \tau)\,\mathrm{d}t = \int_0^1 (\mathrm{d}\sigma^t)\mathrm{d}t = \mathrm{d}\sigma$$

writing $\sigma = \int_0^1 \sigma^t \mathrm{d}t$. We has thus proved that, in a neighbourhood of Q, $\omega_1 - \omega_0 = \mathrm{d}\sigma$ is an exact form (with σ identically zero on Q). □

Proof of Lemma 1.72. To finish the proof of Lemma 1.72, we use the actual method of Moser. We consider the path of symplectic forms

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0) = \omega_0 + t\mathrm{d}\sigma.$$

For $t = 0$, this is the non degenerate form ω_0. Also, along Q, this is the very same form ω_0. Restricting again $\mathscr{V}_0$ if necessary (using compactness again) one can assume that ω_t is non degenerate on $\mathscr{V}_0$ for all $t \in [0,1]$. Let Y_t be the vector field defined by

$$\iota_{Y_t}\omega_t = -\sigma$$

(the existence and uniqueness of Y_t are consequences of the fact that ω_t is non degenerate). Let ψ_t be its flow:

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = Y_t \circ \psi_t.$$

We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^\star \omega_t) &= \psi_t^\star \left(\frac{\mathrm{d}}{\mathrm{d}t}\omega_t + \mathscr{L}_{Y_t}\omega_t\right) \\
&= \psi_t^\star (\mathrm{d}(\sigma) + \mathrm{d}\iota_{Y_t}\omega_t) = \mathrm{d}(\psi_t^\star(\sigma + \iota_{Y_t}\omega_t)) \\
&= 0
\end{aligned}$$

by definition of Y_t. Hence $\psi_t^\star \omega_t = \psi_0^\star \omega_0 = \omega_0$ and eventually

$$\psi_1^\star \omega_1 = \omega_0. \qquad \square$$

Remark 1.76 In a general symplectic manifold W, the proof is identical to the one given here; what we need is the notion of a normal bundle, that is, of orthogonality in TW, and a way to replace the mapping $(x,v) \mapsto x+v$. One uses a Riemannian metric on W and its exponential mapping: the point $\exp_v(x)$ that replaces $x+v$ is the point reached at time 1 by a geodesic[13] starting from x (at time 0) with tangent vector v.

The most direct application of Lemma 1.72 is the Darboux theorem. This is the case where Q is a point x_0, ω_1 is a symplectic form on W and ω_0 is the symplectic form induced on $T_{x_0}W$.

Theorem 1.77 (Darboux theorem) Let x be a point of a manifold W endowed with a symplectic form ω. There exists local coordinates

$$(x_1, \dots, x_n, y_1, \dots, y_n)$$

centered at x in which $\omega = \sum \mathrm{d}y_i \wedge \mathrm{d}x_i$.

Proof. The form induced by ω_1 on $T_{x_0}W$ defines, using a diffeomorphism from a neighbourhood of 0 in $T_{x_0}W$ onto a neighbourhood of x_0 in W, a symplectic form ω_0 on a neighbourhood of x_0. Lemma 1.72 gives a diffeomorphism ψ from a neighbourhood of x_0 into itself, that fixes x_0 and satisfies $\psi^\star \omega_1 = \omega_0$. By definition of ω_0, there exists local coordinates centered at x_0 in which it can be written $\sum \mathrm{d}y_i \wedge \mathrm{d}x_i$. $\square$

3.3.2 Tubular neighbourhoods

The next application is a theorem of Weinstein that describes the tubular neighbourhoods of Lagrangian submanifolds.

Theorem 1.78 (Weinstein [33]) Let (W, ω) be a symplectic manifold and let $L \subset W$ be a compact Lagrangian submanifold. There exists a neighbourhood $\mathscr{N}_0$ of the zero section in $T^\star L$, a neighbourhood $\mathscr{V}_0$ of L in W and a diffeomorphism $\varphi : \mathscr{N}_0 \to \mathscr{V}_0$ such that

$$\varphi^\star \omega = -\mathrm{d}\lambda \quad \text{and} \quad \varphi\Big|_L = \mathrm{Id}.$$

[13] To extend the geodesics, we also need an assumption on the completeness of the metric, or on the manifold W.

Proof. Let us check that we can apply Lemma 1.72. The submanifold Q is the Lagrangian submanifold L and the form ω_0 is the restriction of ω. The form ω_1 is the symplectic form of $T^\star L$. We are going to compare them in $T^\star L$. As in the previous proof, let us assume firstly that $W = \mathbb{C}^n$. Let φ be the composed mapping

$$T^\star L \to N_L \to \mathbb{C}^n, \qquad (x,\alpha) \mapsto (x, Jv_\alpha) \mapsto x + Jv_\alpha$$

where

- $\alpha \mapsto v_\alpha$ is the isomorphism between cotangent and tangent given by the Euclidean structure of $\mathbb{C}^n$ restricted to L:
$$\alpha(u) = (u, v_\alpha),$$
- J is multiplication by i. Recall (see Lemma 1.3) that L is Lagrangian if and only if $TL^\perp = JTL$.

Call $\mathcal{N}_0$ a neighbourhood of the zero section in $T^\star L$, mapped onto a suitable $\mathcal{U}_\varepsilon$, so that $\varphi : \mathcal{N}_0 \to \mathbb{C}^n$ is a diffeomorphism onto its image. We want to compare, in $\mathcal{N}_0 \subset T^\star L$, the zero section $L \subset \mathcal{N}_0$ endowed with the two forms $\omega_1 = -\mathrm{d}\lambda$ and $\omega_0 = \varphi^\star \omega$. To apply Lemma 1.72, we have to check that they coincide along the zero section.

Let $(x,0) \in L \subset \mathcal{N}_0$. We have

$$T_{(x,0)}\mathcal{N}_0 = T_{(x,0)}\left(T^\star L\right) = T_x L \oplus T_x^\star L.$$

Recall that there is an exact sequence

$$0 \to \ker T_{(x,\alpha)}\pi \to T_{(x,\alpha)}\left(T^\star L\right) \to T_{(x,\alpha)}\pi T_x L \to 0$$

which splits along the zero section s by

$$T_x L \to T_x s T_{(x,0)}\left(T^\star L\right),$$

and that the kernel $\ker T_{(x,\alpha)}\pi$ is canonically identified with $T_x^\star L$. Compute then $\varphi^\star \omega$

along the zero section. For v, $w \in T_xL$ and α, $\beta \in T_x^\star L$, we have

$$\begin{aligned}
\left(\varphi^\star\omega\right)_{(x,0)}\big((v,\alpha),(w,\beta)\big) &= \omega_{\varphi(x,0)}\left(v+Jv_\alpha, w+Jv_\beta\right)\\
&= \omega_x\left(v+Jv_\alpha, w+Jv_\beta\right)\\
&= (v,v_\beta)-(w,v_\alpha)\\
&= \beta(v)-\alpha(w).
\end{aligned}$$

But we have seen in Exercise 1.100 that

$$\left(\mathrm{d}\lambda\right)_{(x,0)}\big((v,\alpha),(w,\beta)\big) = \left(\sum \mathrm{d}y_j\wedge \mathrm{d}x_j\right)\big((v,\alpha),(w,\beta)\big) = \alpha(w)-\beta(v).$$

The forms $\varphi^\star\omega$ and $-\mathrm{d}\lambda$ coincide along the zero section, so that we can apply the lemma. □

In the general situation where W is a symplectic manifold, we need a Riemannian metric and an analogue of J. We use an "almost complex structure" J calibrated by ω, namely an endomorphism J of the tangent bundle TW such that $J^2 = -\operatorname{Id}$ and

$$(X,Y)\mapsto\omega(X,JY)$$

is a Riemannian metric. Such structures exist and form a contractible set. See for instance [5], [25].

3.3.3 "Moduli space" of Lagrangian submanifolds

We consider now, for a given manifold L, the *space* of Lagrangian immersions

$$f: L\to W.$$

We call it a "space" because this set is actually a topological space, a fact which allows to consider immersions that are "close" to a given immersion. We use the Whitney $\mathscr{C}^1$-topology.

The $\mathscr{C}^1$-topology. Let V and W be two manifolds. The $\mathscr{C}^1$-topology is a topology on the space of $\mathscr{C}^1$-maps from V to W. Consider the vector bundle $\mathscr{L}(TV,TW)$ over $V\times W$, the fiber at (x,y) of which is the vector space $\mathscr{L}(T_xV,T_yW)$. The total space is usually called $J^1(V,W)$ rather than $\mathscr{L}(TV,TW)$. Every map $f \in \mathscr{C}^1(V,W)$ defines a mapping

$$j^1 f : V \to J^1(V,W), \qquad x \mapsto \bigl(x, f(x), T_x f\bigr).$$

If U is an open subset of $J^1(V,W)$, denote

$$\mathscr{V}(U) = \left\{ f \in \mathscr{C}^1(V,W) \mid j^1 f \in U \right\}.$$

The $\mathscr{C}^1$-topology is the topology for which the $\mathscr{V}(U)$ are a basis. It is said that a map f is "$\mathscr{C}^1$-close" to f_0 if it is close to f_0 for the $\mathscr{C}^1$-topology.

Diffeomorphism group. The group of diffeomorphisms of L acts on this space by $\varphi \cdot f = f \circ \varphi^{-1}$. We want to consider Lagrangian immersions only up to this action: we do not want to take into account the way the manifold L is "parametrized".

Moduli space. We consider the space of Lagrangian $\mathscr{C}^1$-immersions from L to W up to the action of the diffeomorphism group. The quotient space is called the "moduli space" of Lagrangian immersions from L to W and denoted $\mathscr{L}(L)$. The next theorem describes the Lagrangian immersions that are close to a fixed Lagrangian embedding of L into W.

Theorem 1.79 Let L be a compact and connected manifold. A neighbourhood of a Lagrangian embedding

$$L \to W$$

in the space $\mathscr{L}(L)$ can be identified with a neighbourhood of 0 in the vector space of closed 1-forms of class $\mathscr{C}^1$ on L.

Proof. Let $f_0 : L \to W$ be a Lagrangian embedding and $f : L \to W$ be a Lagrangian immersion close to f_0. In particular, f is close to f_0 for the "$\mathscr{C}^0$-topology[14]", we can consider that everything lies in a neighbourhood of L. Thanks to the tubular neighbourhood theorem (here Theorem 1.78) we can assume that everything takes place in a neighbourhood of the zero section in $T^\star L$. The map f is $\mathscr{C}^1$-close to the inclusion of the zero section $L \to T^\star L$ (this is what f_0 has become when we have identified the neighbourhood of $f_0(L)$ in W with a neighbourhood of the zero section in $T^\star L$). Thus the composition of f with

[14] The $\mathscr{C}^0$-topology, defined similarly to the $\mathscr{C}^1$-topology, is simply the compact open topology.

the projection of the cotangent is a $\mathscr{C}^1$-mapping $L \to L$, close to the identity. Recall the next lemma, which is a consequence of the inverse function theorem.

Lemma 1.80 Let L be a compact and connected manifold. Let f be a $\mathscr{C}^1$-map $L \to L$ that is $\mathscr{C}^1$-close to the identity. Then f is a diffeomorphism.

According to this lemma, that will be proved below, the composition is a diffeomorphism g of L. Composing with g^{-1}, we get an embedding

$$\alpha : L \to T^\star L$$

which is still $\mathscr{C}^1$-close to the zero section... but now the composition

$$L \to T^\star L \to L$$

is the identity. Thus α is a section, that is, a 1-form on L, and α is closed because the embedding is Lagrangian. Conversely, all the closed 1-forms that are close to the zero section define Lagrangian embeddings close to f_0. $\square$

Remark 1.81 One should have noticed that the section $L \to T^\star L$ defined by a 1-form is a $\mathscr{C}^1$-mapping if and only if the form is a $\mathscr{C}^1$-form. The $\mathscr{C}^1$-topology thus defines the structure of a topological vector space on the space of 1-forms. In §3.6 below, we will need a Banach space structure.

Remark 1.82 The vector space we have obtained is infinite dimensional. It can be considered as a neighbourhood of f_0 in the "manifold" of deformations of f_0, or as its tangent space at f_0.

Proof of Lemma 1.80. Let $f_0 : L \to L$ be a diffeomorphism (for example the identity map) and f be close to f_0. Let $\varepsilon > 0$ and $U \subset L$ be such that $f \in \mathscr{V}(\varepsilon, U, f_0)$. In particular

$$\forall x \in U, \quad \|T_x f - T_x f_0\| < \varepsilon.$$

Thus, if ε is small enough, $T_x f$ is invertible for every x in U. Cover L by open sets U such that

$$\exists \varepsilon_U \text{ with } f \in \mathscr{V}(\varepsilon_U, U, f_0).$$

Thanks to the compactness of L, there exists an $\varepsilon > 0$ that works for all U's. Hence $T_x f$ is invertible for every x in L and f is a local diffeomorphism $L \to L$. As L is connected and compact, f is a finite covering map. Being $\mathscr{C}^0$-close to the global diffeomorphism f_0, it must have the same degree, namely 1, and is thus a diffeomorphism. □

3.4 Calabi-Yau manifolds

We want now to describe, in a way analogous to what we have done in §3.3.2, the moduli space of special Lagrangian submanifolds. In order to apply Theorem 1.78 (special Lagrangian submanifolds are, firstly, Lagrangian submanifolds) we need a *compactness* assumption on the Lagrangian submanifold. Unfortunately, as we have seen it in §2.5.2, the special Lagrangian submanifolds of $\mathbb{C}^n$ are never compact. We thus need to consider more general manifolds, in which it is possible to define special Lagrangian submanifolds. These are the "Calabi-Yau" manifolds.

The point is to define a structure that globalizes the structures on $\mathbb{C}^n$ which have allowed us to speak of special Lagrangian submanifolds. Recall that, in addition to the $\mathbb{R}$-bilinear alternated form ω, we have used the form $\Omega = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$ of the complex determinant.

We will use here the best adapted definition of a Calabi-Yau manifold, the point is not to spend time on the Calabi-Yau manifold itself but rather on its special submanifolds. For more information on Calabi-Yau manifolds, see [32], [8] and the references inside.

3.4.1 Definition of the Calabi-Yau manifolds

Our manifolds should be complex and endowed with a symplectic form ω and a type-$(n,0)$ holomorphic form Ω that is nowhere zero (this is sometimes called a holomorphic volume form). Consider thus a manifold M, on which are given

- ◇ a complex structure J (multiplication by i),
- ◇ a closed non degenerate type $(1,1)$-form ω (the Kähler form)
- ◇ a Riemannian metric
$$g(X,Y) = \omega(X,iY),$$
- ◇ a Hermitian metric
$$h(X,Y) = g(X,Y) - i\omega(X,Y),$$

◇ a trivialization of the "canonical" bundle $\Lambda^n T^\star M$, namely a type-$(n,0)$ holomorphic form Ω which is nowhere zero.

We still need a relation between the forms ω and Ω. Notice that both forms $\omega^{\wedge n}$ and $\Omega\wedge\bar{\Omega}$ are of type (n,n) and both do not vanish on M, in particular, both are volume forms. We thus have

$$\Omega\wedge\bar{\Omega} = f\omega^{\wedge n}$$

for some function f on M. The additional compatibility condition is that f should be constant. Let us look at the case of $\mathbb{C}^n$. We have

$$\omega^n = \left(\sum_{j=1}^n \mathrm{d}y_j\wedge\mathrm{d}x_j\right)^{\wedge n} = n!(\mathrm{d}y_1\wedge\mathrm{d}x_1)\wedge\cdots\wedge(\mathrm{d}y_n\wedge\mathrm{d}x_n).$$

Writing $\mathrm{d}y = \frac{1}{2i}(\mathrm{d}z-\mathrm{d}\bar{z})$ and $\mathrm{d}x = \frac{1}{2}(\mathrm{d}z+\mathrm{d}\bar{z})$ and noticing that $\mathrm{d}y\wedge\mathrm{d}x = \frac{1}{4i}(\mathrm{d}z-\mathrm{d}\bar{z})\wedge(\mathrm{d}z+\mathrm{d}\bar{z}) = \frac{1}{2i}\mathrm{d}z\wedge\mathrm{d}\bar{z}$, we can also write

$$\omega^{\wedge n} = \frac{n!}{2^n i^n}\left(\mathrm{d}z_1\wedge\mathrm{d}\bar{z}_1\wedge\cdots\wedge\mathrm{d}z_n\wedge\mathrm{d}\bar{z}_n\right).$$

The computation of $\Omega\wedge\bar{\Omega}$ gives

$$\Omega\wedge\bar{\Omega} = (\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n)\wedge(\mathrm{d}\bar{z}_1\wedge\cdots\wedge\mathrm{d}\bar{z}_n).$$

We thus have

$$\omega^{\wedge n} = \frac{(-1)^{\frac{n(n-1)}{2}}n!}{2^n i^n}\Omega\wedge\bar{\Omega}.$$

We will use the same normalization formula to define a Calabi-Yau manifold in general.

Definition 1.83 A complex manifold M is said to be a *Calabi-Yau manifold* if it is Kähler, has a trivialized canonical bundle, and if the Kähler form ω and the type $(n,0)$ form Ω trivializing the bundle $\Lambda^n T^\star M$ are related by

$$\omega^{\wedge n} = \frac{(-1)^{\frac{n(n-1)}{2}}n!}{2^n i^n}\Omega\wedge\bar{\Omega}.$$

Remark 1.84 Recall that it is possible to express the fact that the form ω is Kähler by saying that the complex structure is "parallel" with respect to the Levi-Civitá connection associated with the metric it defines with ω. Similarly, it is possible to express the compatibility condition for Ω by saying that it is parallel with respect to the same connection.

Remark 1.85 In general, it is required that the Kähler metric be *complete*, in other words that it is possible to extend geodesics. This is equivalent to requiring that the manifold be complete (in the sense of metric spaces).

3.4.2 Yau's theorem

Consider a (complex algebraic) projective smooth manifold M of complex dimension n. Assume that all the $H^{p,0}(M)$ are zero for $1 \leq p \leq n-1$ and that the canonical bundle $\Lambda^n T^\star M = K_M$ is trivialized by a type $(n,0)$-form Ω. Notice that M is Kähler, call the Kähler form ω. Rescaling ω if necessary, we get

$$\int_M \omega^{\wedge n} = \frac{(-1)^{\frac{n(n-1)}{2}} n!}{2^n i^n} \int_M \Omega \wedge \bar{\Omega}.$$

A hard theorem of Yau [34] asserts that there exists a unique Kähler form $\widetilde{\omega}$ on M such that $[\widetilde{\omega}] = [\omega] \in H^2_{\mathrm{DR}}(M)$ and which, together with Ω, gives M the structure of a Calabi-Yau manifold.

3.4.3 Examples of Calabi-Yau manifolds

Of course $\mathbb{C}^n$ is a Calabi-Yau manifold.

Affine quadrics. We have defined in §3.1 a symplectic form on the unit sphere $S^2 \subset \mathbb{R}^3$ by the formula

$$\omega_x(X,X') = \det(x,X,X').$$

Similarly, the formula

$$\Omega_z(Z,Z') = \det_{\mathbb{C}}(z,Z,Z')$$

defines a "holomorphic symplectic" form of the complex quadric

$$Q = \left\{(z_1,z_2,z_3) \mid z_1^2 + z_2^2 + z_3^2 = 1\right\}.$$

In "differential" terms,

$$\Omega = z_1 \mathrm{d}z_2 \wedge \mathrm{d}z_3 + z_2 \mathrm{d}z_3 \wedge \mathrm{d}z_1 + z_3 \mathrm{d}z_1 \wedge \mathrm{d}z_2.$$

On the open subset of Q where $z_3 \neq 0$, z_1 et z_2 are coordinates and, using the relation

$$z_1 \mathrm{d}z_1 + z_2 \mathrm{d}z_2 + z_3 \mathrm{d}z_3 = 0,$$

we can write

$$\Omega = \frac{1}{z_3} \mathrm{d}z_1 \wedge \mathrm{d}z_2,$$

so that

$$\Omega \wedge \bar{\Omega} = \frac{1}{|z_3|^2} \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_1 \wedge \mathrm{d}\bar{z}_2.$$

Modifying the restriction ω_0 to Q of the standard Kähler form of $\mathbb{C}^3$, let us construct a Kähler form ω on Q such that

$$\omega \wedge \omega = \frac{1}{4} \Omega \wedge \bar{\Omega}.$$

Call h the restriction to Q of the function $|z|^2$. We look for ω of the form

$$\omega = \frac{i}{2} \partial \bar{\partial} (f \circ h)$$

for some function f. A straightforward computation (see also [30]) shows that $f(h) = \sqrt{h+1}$ works.

The quadric Q, equipped with Ω and ω is thus a Calabi-Yau manifold. Recall that Q is diffeomorphic to the tangent bundle TS^2 by

$$Q \to TS^2, \qquad X + iY \mapsto \left(\frac{X}{\sqrt{1 + \|Y\|^2}}, Y \right).$$

In this way, what we have got is the structure of a Calabi-Yau manifold on the tangent (or cotangent) bundle of the sphere S^2. It is possible (but a little more complicated) to do the same for the cotangent bundles of all the spheres S^n and more generally for those of all "rank-1 symmetric spaces" (see [30]).

Remark 1.86 Recall that we have identified $\mathbb{C}^2$ with the skew field $\mathscr{K}$ of quaternions (in §2.5.1). Similarly, the surface Q has the structure of a "quaternionic" or "hyperkähler" manifold.

Call I the complex structure defined on Q by that of $\mathbb{C}^3$ (this is the multiplication by i) and notice that the symmetric bilinear form that is an equation for Q is still non degenerate when restricted to $z^\perp = T_zQ$. Define an operator J_z on the tangent space T_zQ by the fact that $J_z(Z)$ is the unique vector in T_zQ that is orthogonal to Z for the complex bilinear form and such that

$$\det{}_{\mathbb{C}}\big(z, Z, J_z(Z)\big) = \|Z\|^2.$$

This is an almost complex structure since

$$\det{}_{\mathbb{C}}(z, J_zZ, -Z) = \|Z\|^2$$

thus $J_z^2 = -\mathrm{Id}$. This is an isometry since

$$\|JZ\|^2 = \det{}_{\mathbb{C}}\big(z, JZ, J^2(Z)\big) = \det{}_{\mathbb{C}}(z, JZ, -Z) = \|Z\|^2.$$

Moreover, J "anti-commutes" with I:

$$\begin{aligned} \det{}_{\mathbb{C}}(z, IZ, JIZ) &= \|IZ\|^2 = \|Z\|^2 && \text{on the one hand} \\ &= i \det{}_{\mathbb{C}}(z, Z, JIZ) && \text{by linearity.} \end{aligned}$$

We thus have

$$\det{}_{\mathbb{C}}(z, Z, JIZ) = -i\|Z\|^2 = -\det{}_{\mathbb{C}}(z, Z, IJZ)$$

so that $JI = -IJ$. Hence I, J and IJ form a quaternionic structure on Q. On Q, we thus have

- the Kähler form ω,
- the complex structure I defined by multiplication by i in $\mathbb{C}^3$,
- the associated Riemannian metric g, so that $\omega(X, IY) = g(X, Y)$,
- the "holomorphic symplectic form" Ω,
- the complex structure J defined in such a way that Ω be a J-kähler form, associated with the same metric g.

It is said that Q is hyperkähler. See Exercise 1.106 for a kind of converse statement.

Let us give now a few examples of compact Calabi-Yau manifolds.

Elliptic curves. The quotient M of $\mathbb{C}$ by a lattice Λ is an *elliptic curve*. The two forms

$$\omega = \frac{1}{2i} \mathrm{d}z \wedge \mathrm{d}\bar{z} \quad \text{and} \quad \Omega = \mathrm{d}z$$

give it the structure of a dimension-1 Calabi-Yau manifold. More generally, one can build the quotient of $\mathbb{C}^n$ by a lattice. Now is a good time for a remark: no other "explicit" example of compact Calabi-Yau manifold is known. In all the known examples, the existence of the Kähler metric with all the desired properties is obtained as a consequence of the Yau theorem (§3.4.2).

Hypersurfaces. Recall that complex elliptic curves can be considered as degree-3 curves in $\mathscr{P}^2(\mathbb{C})$, thanks to the Weierstrass $\wp$-function. They are thus the $n = 1$ case in the next theorem.

Theorem 1.87 A degree-d hypersurface in $\mathscr{P}^{n+1}(\mathbb{C})$ is a dimension-n Calabi-Yau manifold if and only if $d = n+2$.

Proof. The condition on the degree is necessary, as we show it now by the computation of the first Chern classes. We want that the bundle $\Lambda^n T^\star M$ be trivializable, we must thus have $c_1(T^\star M) = -c_1(TM) = 0$. Calling j the inclusion of M in $\mathscr{P}^{n+1}(\mathbb{C})$, we have

$$c_1(TM) + j^\star c_1\big(\mathscr{O}(d)\big) = j^\star c_1\big(TP^{n+1}(\mathbb{C})\big)$$

since the normal bundle of M in $\mathscr{P}^{n+1}(\mathbb{C})$ is $\mathscr{O}(d)$. Denoting by t the dual class to the hyperplane section in $H^2\big(\mathscr{P}^{n+1}(\mathbb{C})\big)$, we have

$$(n+2-d)j^\star t = 0$$

so that $d = n+2$.

Assume conversely that $d = n+2$. Let us construct explicitly a holomorphic n-form on M. Let F be a degree-$(n+2)$ homogeneous polynomial that describes the hypersurface M. Every point of M lies in an affine chart $Z_i \neq 0$ of $\mathscr{P}^{n+1}(\mathbb{C})$. In affine coordinates $z_k = Z_k/Z_i$, there is an index j such that

$$\frac{\partial}{\partial z_j} F(z_0, \dots, 1, \dots, z_{n+1}) \neq 0$$

since M is smooth. The formula

$$\Omega = (-1)^{i+j-1} \frac{\mathrm{d}z_0 \wedge \cdots \wedge \widehat{\mathrm{d}z_i} \wedge \cdots \wedge \widehat{\mathrm{d}z_j} \wedge \cdots \wedge \mathrm{d}z_{n+1}}{\frac{\partial F}{\partial z_j}(z_0, \ldots, 1, \ldots, z_{n+1})}$$

defines a homogeneous holomorphic n-form on M that is nowhere zero. This is a consequence of the theorem of Yau (§3.4.2) that there is, indeed, in the same cohomology class as the standard Kähler form ω, another Kähler form $\omega + i\partial\bar{\partial}\varphi$ giving a Calabi-Yau structure on M. □

Remark 1.88 The form Ω above is defined as "Poincaré residue[15]" starting from the $n+1$-form on $\mathscr{P}^{n+1}(\mathbb{C})$ with poles along M defined by

$$\sigma_i = (-1)^i \frac{\mathrm{d}z_0 \wedge \cdots \wedge \widehat{\mathrm{d}z_i} \wedge \cdots \wedge \mathrm{d}z_{n+1}}{F(z_0, \ldots, 1, \ldots, z_{n+1})}$$

in the affine chart $Z_i \neq 0$.

Remark 1.89 Calabi-Yau manifolds of dimension 2 are hyperkähler. The proof of this fact is the subject of Exercises 1.105 and 1.106.

3.4.4 Special Lagrangian submanifolds

An immersion $f : V \to M$ from a manifold of real dimension n into a Calabi-Yau manifold M of complex dimension n is said *special Lagrangian* if it satisfies $f^\star\omega = 0$ and $f^\star\beta = 0$. As in the case of $\mathbb{C}^n$, the form $f^\star\Omega = f^\star\alpha$ is then a volume form.

3.5 Special Lagrangians in real Calabi-Yau manifolds

3.5.1 Real manifolds

A complex analytic manifold is *real* if it is endowed with a "real structure", that is, with an anti-holomorphic involution S : an involution such that, for any holomorphic function f over an open subset U of M, $\overline{f \circ S}$ is a holomorphic function. For example, on the algebraic submanifolds of $\mathscr{P}^N(\mathbb{C})$ described by real polynomial equations, the complex conjugation is an anti-holomorphic involution. These manifolds are thus real manifolds. In particular, the projective space $\mathscr{P}^N(\mathbb{C})$ itself is a real manifold.

[15] See p. 147 of [16].

The real part, or set of real points of a real manifold is, by definition, the set of fixed points of S. For example, the real part of the real manifold $\mathscr{P}^N(\mathbb{C})$ is $\mathscr{P}^N(\mathbb{R})$. Notice that there exists respectable real manifolds that have no real point at all, as is, for example, the "Euclidean quadric"

$$\sum_{i=1}^{N+1} X_i^2 = 0$$

in $\mathscr{P}^N(\mathbb{C})$.

Proposition 1.90 The real part of a real manifold of complex dimension $2n$, if it is non empty, is a submanifold all connected components of which have dimension n.

Proof. The connected components of the set of fixed points of the action of a finite group (here the order-2 group generated by S) are always submanifolds. The tangent space at x to such a component is the subspace of fixed points of the $\mathbb{R}$-linear involution $\sigma = T_x S$.

The fact that S is a real structure implies that $\overline{f \circ \sigma}$ is a complex linear form for any complex linear form f on the tangent space at x. We have to check that the fixed subspace of σ has dimension n. To do this, we simply verify that the eigensubspaces associated with the eigenvalues 1 and -1 are isomorphic. Indeed, if $\sigma(X) = X$, then for any *complex* linear form f, we have

$$\overline{f \circ \sigma}(iX) = i\overline{f \circ \sigma}(X) = i\bar{f}(X) = \bar{f}(-iX).$$

For any complex linear form f, we thus have

$$f\big(\sigma(iX)\big) = f(-iX)$$

so that $\sigma(iX) = -iX$. Hence, there are "as many" eigenvectors for the eigenvalue -1 than there are for the eigenvalue 1. □

3.5.2 Real Calabi-Yau manifolds

A Calabi-Yau manifold is *real* if it is both a Calabi-Yau manifold and a real manifold, with a couple of compatibility conditions

$$S^\star \omega = -\omega \quad \text{and} \quad S^\star \Omega = \bar{\Omega}$$

(similarly to what happens in $\mathbb{C}^n$ with the complex conjugation and the two usual forms Ω and ω).

Example 1.91 $\diamond$ The affine quadric $\sum z_i^2 = 1$ of $\mathbb{C}^3$, endowed with the complex conjugation of coordinates is a real manifold. It is also clear that this is a real Calabi-Yau manifold. Its real part is simply the unit sphere $S^2 \subset \mathbb{R}^3$. If we consider Q as the tangent bundle to S^2, notice that the complex conjugation is the multiplication by -1 on the fibers and the real part is the zero section.

$\diamond$ A real hypersurface of degree $n+2$ in $\mathscr{P}^{n+1}(\mathbb{C})$ is a real Calabi-Yau manifold. This is checked by computing $S^\star\Omega$ and $S^\star(\omega + i\partial\bar{\partial}\varphi)$, for S the involution induced by the real structure (complex conjugation) of $\mathscr{P}^{n+1}(\mathbb{C})$ and Ω, ω as in the proof of Theorem 1.87.

3.5.3 Example: elliptic curves

Let us come back to the example of $\Gamma = \mathbb{C}/\Lambda$ where Λ is a lattice that we assume here to have the form

$$\Lambda = \{m + n\tau \mid m, n \in \mathbb{Z}\}$$

for some fixed τ such that $0 \leq \Re(\tau) < 1$ et $\Im(\tau) > 0$. To define a real structure on $\mathbb{C}/\Lambda$ from the complex conjugation in $\mathbb{C}$, it is necessary that Λ be invariant, that is, that

$$\bar{\tau} = m + n\tau$$

for some m, $n \in \mathbb{Z}$. Considering the real and imaginary parts of τ, it is seen that $m = 2\Re(\tau)$, thus $\Re(\tau) = \frac{1}{2}$ or 0.

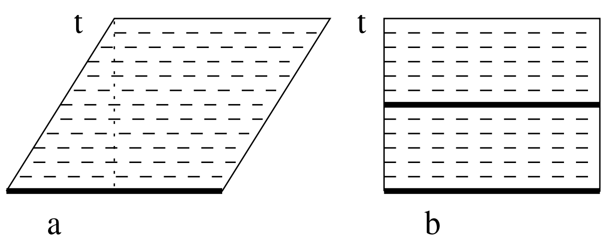


Figure 1.11: Real elliptic curves.

In the second case, the real part of Γ has two connected components, but in the first case, it has only one, as can be seen solving the equation

$$\bar{z} = z + m + n\tau$$

in both cases. These components are depicted in bold on Figure 1.11.

Notice (although this is a trivial remark) that the lines that are parallel to the x axis constitute a real foliation of $\mathbb{C}/\Lambda$ by circles (dimension-1 tori) that are special Lagrangian submanifolds of Γ, represented by dotted lines on Figure 1.11. The space of these special Lagrangian submanifolds is parametrized by the axis generated by τ or rather by its image in Γ, a circle.

We shall see more generally in §3.6 that the moduli space of special Lagrangian submanifolds in a Calabi-Yau manifold is, in the neighbourhood of a submanifold V, a manifold whose dimension is the first Betti number of V (here V is a circle and its first Betti number is 1).

3.5.4 Special Lagrangians in real Calabi-Yau manifolds

Assume now that M is a *real* Calabi-Yau manifold. We know that

$$S^\star\omega = -\omega \quad \text{and} \quad S^\star\Omega = \bar{\Omega}.$$

Assume now that the real part $M_\mathbb{R}$ is not empty. Call j the inclusion of $M_\mathbb{R}$ into M. We have $S \circ j = j$ and in particular

$$j^\star\omega = (S \circ j)^\star\omega = j^\star(S^\star\omega) = j^\star(-\omega) = -j^\star\omega$$

hence $j^\star\omega = 0$. Similarly

$$j^\star\Omega = (S \circ j)^\star\Omega = j^\star(S^\star\Omega) = j^\star\bar{\Omega} = \overline{j^\star\Omega}$$

thus $j^\star\beta = 0$. We have proved:

Proposition 1.92 Let M be a real Calabi-Yau manifold. The real part of M, if it is not empty, is a special Lagrangian submanifold of M. □

Let us describe now a few examples of this situation.

The affine quadric. The sphere S^2 is a special Lagrangian submanifold of the affine quadric $Q \in \mathbb{C}^3$. In other words, with the Calabi-Yau structure on TS^2 defined in §3.4.3 , the zero section is a special Lagrangian submanifold.

In the next examples, we consider a smooth hypersurface defined by a real homogeneous polynomial of degree $n+2$ in $\mathscr{P}^{n+1}(\mathbb{C})$, with its real Calabi-Yau structure.

Elliptic curves. The $n=1$ case, that of plane cubics, is isomorphic to the example of quotients of $\mathbb{C}$ by lattices (§3.5.3). The real part of a plane cubic has zero, one or two connected components (see Figure 1.11). All components are (topologically) circles. Cubics are foliated by special Lagrangian circles, drawn in dotted lines on Figure 1.11.

Degree-4 surfaces. Consider now real algebraic surfaces of degree 4 in $\mathscr{P}^3(\mathbb{C})$ (the real part of this subject has been investigated and explained in [22]). Here is an example from [9]. Consider the real polynomial $P(z_0,z_1,z_2,z_3) = z_0^4 + z_1^4 - z_2^4 - z_3^4$ that describes a smooth surface M with $M_{\mathbb{R}}$ non empty. This real part is

$$M_{\mathbb{R}} = \left\{(x_0,x_1,x_2,x_3)\in\mathbb{R}^4-\{0\} \mid x_0^4+x_1^4=x_2^4+x_3^4\right\}\big/(x\sim\lambda x).$$

Normalize the non zero vectors of $\mathbb{R}^4$ by the choice, in each real line through zero, of one of the two vectors such that

$$x_0^4+x_1^4+x_2^4+x_3^4=2.$$

Then $M_{\mathbb{R}}$ is the quotient

$$\left\{(x_0,x_1),(x_2,x_3)\in\mathbb{R}^2\times\mathbb{R}^2 \mid x_0^4+x_1^4=x_2^4+x_3^4=1\right\}\big/(u,v)\sim(-u,-v).$$

It is clear that the curve C described in $\mathbb{R}^2$ by the equation $x^4+y^4=1$ is diffeomorphic to a circle (radially). One further finds that

$$M_{\mathbb{R}} = (C\times C)\big/(u,v)\sim(-u-v)$$

is diffeomorphic to a torus. We have thus found a special Lagrangian torus in the Calabi-Yau surface M.

3.6 Moduli space of special Lagrangian submanifolds

We want now an analogue of Theorem 1.79, more precisely a description of a neighbourhood of a given special Lagrangian submanifold in the space of all special Lagrangian submanifolds.

Theorem 1.93 (McLean [26]) Let V be a compact manifold. The moduli space of the special Lagrangian embeddings of V in the Calabi-Yau manifold M is a manifold of finite dimension $b_1(V) = \dim H^1(V;\mathbb{R})$. Its tangent space at f_0 is isomorphic to the vector space of *harmonic* 1-forms on V.

Remark 1.94 There are two main differences between this statement and Theorem 1.79. The first one is that the moduli space here has finite dimension. The second one is that the condition "to be special Lagrangian" is no longer linear, so that this is indeed the tangent space that is identified to a space of differential forms.

Example 1.95 Let us come back to the example of the Calabi-Yau structure on TS^2 described in §3.4.3. We have said in §3.5.4 that the zero section is a special Lagrangian submanifold. As there are no non zero harmonic 1-forms on S^2, the theorem of McLean asserts that the zero section is "rigid", that is, it cannot be deformed. In the moduli space of special Lagrangian submanifolds, this is an isolated point.

Proof of Theorem 1.93. Using the tubular neighbourhood theorem (here in §3.3.2), replace M by a tubular neighbourhood of the submanifold V that is isomorphic to a neighbourhood of the zero section in the normal bundle of V (as are all tubular neighbourhoods) and to a neighbourhood of the zero section in the cotangent $T^\star V$. We will use the structures induced by those of M on this neighbourhood, keeping their names, for example $\Omega = \alpha + i\beta$.

We have said in §3.3.3 that the space of Lagrangian submanifolds can be identified with a neighbourhood of 0 in the space $Z^1(V)$ of the closed 1-forms on V. The special Lagrangian submanifolds are described, in this space, by the equation $F(\eta) = 0$ where

$$F : Z^1(V) \to \Omega^n(V)$$

is the mapping defined by $F(\eta) = \eta^\star\beta$. Although these spaces are infinite dimensional, the strategy of the proof is to show that F is submersive at 0. It is thus better to restrict, as much as possible, its target space. Notice first:

Lemma 1.96 The image of F is contained in the subspace $\mathrm{d}\Omega^{n-1}(V)$ of exact n-forms on V.

Proof of Lemma 1.96. If η is the zero form, the mapping $\eta : V \to T^\star V$ is the inclusion of the zero section, a special Lagrangian, thus $F(0) = 0$. Given a form η, it is possible to consider the path (segment) $(t\eta)_{t\in[0,1]}$ joining it to the zero form... and giving a homotopy from the section η to the zero section. The cohomology class of the closed form $(t\eta)^\star\beta$ does not depend on t, thus it is identically zero, and that means, indeed, that $\eta^\star\beta$ is an exact form. $\square$

We thus consider F as a mapping $F : Z^1(V) \to \mathrm{d}\Omega^{n-1}(V)$ and compute its differential at 0.

Lemma 1.97 The differential of F at 0 is the mapping

$$\left(\mathrm{d}F\right)_0(\eta) = -\mathrm{d}(\star\eta)$$

where $\star$ denotes the Hodge star operator[16] associated with the metric defined by the Calabi-Yau structure on the special Lagrangian V.

Proof of Lemma 1.97. To compute $\left(\mathrm{d}F\right)_0(\eta)$, one chooses a path of forms $\widetilde{\eta}_t$ whose tangent vector at 0 is the form η. Let η be a 1-form on V and X be the vector field that corresponds to it under the metric on V, that is, satisfies $g(X,\cdot) = \eta$. Let $Y = JX$ be the vector field normal to V. This is the vector corresponding to η under the isomorphism $NV \simeq T^\star V$. The vector field Y is only defined along V, we extend it (arbitrarily) in a vector field $\widetilde{Y}$ on the tubular neighbourhood under consideration.

Call $\widetilde{\varphi}_t$ the flow of $\widetilde{Y}$, so that $\widetilde{\varphi}_t$ is a diffeomorphism defined for t small enough. The restriction φ_t of $\widetilde{\varphi}_t$ to V is an embedding of V into NV (one pushes V using φ_t). For $t = 0$, this is the zero section. Hence for t small enough, this is still a section of NV. We have, for all x in V,

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x)\bigg|_{t=0} = Y(x).$$

Under the identification $NV \simeq T^\star V$, the section φ_t of NV corresponds to a section $\widetilde{\eta}_t$ of $T^\star V$ which is a path of forms, whose tangent vector at 0 is the form η.

[16] See Exercise 1.104.

Consider now the $(n,0)$-form Ω, still on our neighbourhood of the zero section in NV. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\varphi}_t^\star\Omega\Big|_{t=0} = \mathscr{L}_{\widetilde{Y}}\Omega = \mathrm{d}\iota_{\widetilde{Y}}\Omega$$

applying Cartan formula together with the fact that Ω is closed. For the embedding $\varphi_t : V \to NV$, we thus have

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^\star\Omega\Big|_{t=0} = \mathrm{d}\iota_{JX}\Omega = i\mathrm{d}(\iota_X\Omega)$$

hence Ω is $\mathbb{C}$-linear. We then have

$$\left(\mathrm{d}F\right)_0(\eta) = \frac{\mathrm{d}}{\mathrm{d}t}\eta_t^\star\beta\Big|_{t=0} = \Im\left(i\mathrm{d}(\iota_X\Omega)\right) = \Re\left(\mathrm{d}(\iota_X\Omega)\right) = \mathrm{d}(\iota_X\alpha).$$

We still have to convince ourselves that $\iota_X\alpha = \star\eta$. The $(n-1)$-form $\star\eta$ is the unique form satisfying

$$\psi\wedge(\star\eta) = g(\psi,\eta)\alpha$$

for any 1-form ψ. But, as the $(n+1)$-form $\psi\wedge\alpha$ is zero, its interior product by X is also zero and we have

$$\psi\wedge(\iota_X\alpha) = (\iota_X\psi)\alpha = g(\psi,\eta)\alpha$$

by definition of X and of the metric g on the space of 1-forms. □

To end the proof of the theorem, we need to determine what kind of implicit function theorem we use to go from “differential is surjective” to “inverse image is a submanifold”. The simplest here is to use the standard implicit function theorem for Banach spaces (see [11]). We need to endow the spaces of forms $Z^1(V)$ and $\Omega^n(V)$ with structures of Banach spaces. Let us precise the regularity of the forms we use. We consider forms of class $\mathscr{C}^{1,\varepsilon}$ in $Z^1(V)$ and of class $\mathscr{C}^{0,\varepsilon}$ in $\Omega^n(V)$. The Hölder norm used here on forms is deduced from the usual Hölder norm on functions: recall that $\mathscr{C}^{k,\varepsilon}(U)$ is the space of functions of class $\mathscr{C}^k$ on the open set U of $\mathbb{R}^n$ all the derivatives (of order $\leq k$) of which have a finite Hölder norm $\|u\|_\varepsilon$ (for $\varepsilon\in]0,1]$), with

$$\|u\|_\varepsilon = \sup_{x,y\in U}\frac{|u(x)-u(y)|}{\|x-y\|^\varepsilon} + \sup_{x\in U}|u(x)|.$$

The implicit function theorem gives the fact that $F^{-1}(0)$ is a submanifold in a neighbourhood of 0, whose tangent space at 0 is the kernel $\mathscr{H}^1(V)$ of $(\mathrm{d}F)_0$. It is important here that this kernel has finite dimension. The isomorphism between $H^1_{\mathrm{DR}}(V)$ and the space $\mathscr{H}^1(V)$ of harmonic 1-forms is the contents in degree 1 of the Hodge theorem, see [16]. □

Remark 1.98 The vector space $H^{n-1}_{\mathrm{DR}}(V)$ is isomorphic to the vector space dual to $H^1_{\mathrm{DR}}(V)$, so that $H^1_{\mathrm{DR}}(V) \oplus H^{n-1}_{\mathrm{DR}}(V)$ has a natural symplectic structure (see Exercise 1.49), here

$$\omega\big((\alpha,\eta),(\alpha',\eta')\big) = \int_V (\alpha\wedge\eta' - \alpha'\wedge\eta).$$

The space of harmonic 1-forms is a Lagrangian subspace, by

$$\mathscr{H}^1(V) \to H^1_{\mathrm{DR}}(V) \oplus H^{n-1}_{\mathrm{DR}}(V), \qquad \alpha \mapsto (\alpha, \star\alpha)$$

(this is the graph of the mapping $\star$, which is symmetric for the metric…see Exercise 1.56). If $j_0 : V \to W$ is a special Lagrangian submanifold, call $\mathscr{B}$ the moduli space in a neighbourhood of j_0. We thus have a Lagrangian subspace

$$T_{j_0}\mathscr{B} \to H^1_{\mathrm{DR}}(V) \oplus H^{n-1}_{\mathrm{DR}}(V)$$

and it is possible to "integrate" it in a Lagrangian embedding (see [20])

$$F : \mathscr{B} \to H^1_{\mathrm{DR}}(V) \oplus H^{n-1}_{\mathrm{DR}}(V).$$

See also [12] for a description of all these structures by symplectic reduction.

3.7 Towards mirror symmetry?

The "mirror conjecture" asserts the existence, for any Calabi-Yau manifold M, of another Calabi-Yau manifold $M^\star$ of the same dimension, related with M in the way we briefly describe now, sending the readers to [32] for missing detail.

Call $\mathscr{M}_M$ the space of isomorphism classes of

- ◇ a complex structure J_t deforming the complex structure J of M
- ◇ a "complexified Kähler class" on (M,J_t), namely a cohomology class of the form $\alpha + i\beta$, for some Kähler class (for J_t) $\alpha \in H^2_{\mathrm{DR}}(M)$ and some element $\beta \in H^2_{\mathrm{DR}}(M)/2\pi H^2(M;\mathbb{Z})$.

Notice that, locally, $\alpha + i\beta$ varies in an open subset of $H^2(M;\mathbb{C})$, so that the space $\mathscr{M}_M$ is, locally, a product. The manifold M and its "mirror" partner $M^\star$ should be related by an isomorphism of the moduli spaces

$$\mathscr{M}_M \to \mathscr{M}_{M^\star}$$

that exchanges the factors of this local decomposition as a product.

Using in an essential way the symplectic structure of the loop space of M and techniques that go far beyond the level of these notes, Givental has proved the conjecture in [15], following a series of previous papers, the references of which can be found in [15] and [32].

Special Lagrangian submanifolds have been a few years ago the central object of another approach to mirror symmetry, more speculative and having given so far very few results – but a very beautiful approach indeed, that I intend to describe very briefly here.

3.7.1 Fibrations in special Lagrangian submanifolds

We are no more interested in a single special Lagrangian submanifold but in a whole family. More precisely, we consider a compact Calabi-Yau manifold M and a differential mapping

$$p : M \to B$$

to a manifold B, whose general fibers are special Lagrangian submanifolds. The dimension of B, as that of the fibers of p, must be n. It is not required that p be everywhere regular. Some of the fibers may be singular. The other ones, who correspond to regular values of p, are said *general*.

We know (see §2.6.3 and [4]) that in any proper Lagrangian fibration, the general fibers are unions of tori, so this must be the case here. The first Betti number of a torus of dimension n is precisely n, so that it can be expected that B "looks like" the moduli space of special Lagrangian submanifolds.

So, let $b \in B$ be a regular value of p and let $V \subset p^{-1}(b)$ be a connected component of the fiber $p^{-1}(b)$. If $X \in T_bB$ is a tangent vector, there exists a unique vector field Y normal to V in M and such that, for all x in V,

$$T_x p(Y_x) = X.$$

To this field Y corresponds a harmonic 1-form η on V, as in the proof of the theorem of McLean (here Theorem 1.93). As B has dimension n, starting from n independent vectors $X_1,\dots,X_n$ in T_bB, one constructs n fields $Y_1,\dots,Y_n$, that are normal to V and linearly independent at each point of V. Dually, we thus have n harmonic 1-forms $\eta_1,\dots,\eta_n$ that form a basis of $\mathscr{H}^1(V)$ and are linearly independent at each point of V.

In order that such a fibration $p : M \to B$ exists in a neighbourhood of a special Lagrangian tors $V \subset M$, it is necessary that, for the metric induced by the Calabi-Yau structure on V, there exists a basis of $\mathscr{H}^1(V)$ consisting of forms that are independent at each point of V.

It is time to mention that (except in dimension 1) there is no known example having all the properties mentioned here.

- $\diamond$ Notice first that, abstractly, a basis of harmonic 1-forms that are independent at each point exists on the flat torus, the basis $\mathrm{d}x_1,\dots,\mathrm{d}x_n$ having this property. The metrics that are close enough to the flat metric thus have the same property.
- $\diamond$ We have seen in §3.5.3 that the situation of a Calabi-Yau manifold foliated by special Lagrangians submanifolds occurs in dimension 1.
- $\diamond$ In dimension 2, on a special Lagrangian torus, one always has a basis of harmonic 1-forms as expected. We have seen that a special Lagrangian submanifold in dimension 2 is simply a complex curve (for a different complex structure). Assuming the submanifold is a torus, it must be an elliptic curve and it has a nowhere vanishing holomorphic form. Actually, the real and imaginary part of this form are harmonic forms on V and they are independent at every point.

3.7.2 Mirror symmetry

The Strominger, Yau and Zaslow approach to mirror symmetry [31] is to associate, to a Calabi-Yau manifold M endowed with a fibration in special Lagrangian tori (assuming it exists), another Calabi-Yau manifold $M^\star$. The latter should be the "extended" moduli space of special Lagrangian submanifolds of M equipped with a flat unitary line bundle. Call, as above, $\mathscr{B}$ the moduli space of special Lagrangian submanifolds in the neighbourhood of V. Locally, the extended moduli space is $M^\star = \mathscr{B} \times H^1(V;\mathbb{R}/\mathbb{Z})$. Its tangent space at a point m is

$$T_mM^\star = H^1(V;\mathbb{R}) \oplus H^1(V;\mathbb{R}) \simeq H^1(V;\mathbb{R}) \otimes \mathbb{C}.$$

Thus, $M^\star$ has a natural almost complex structure, it is even Kähler:

Theorem 1.99 (Hitchin [20]) The complex structure on $M^\star$ is integrable, the metric of $H^1(V;\mathbb{R})$ defines a Kähler metric on $M^\star$.

We have seen (Remark 1.98 above) that $\mathscr{B}$ is a Lagrangian submanifold of $H^1_{\mathrm{DR}}(V) \oplus H^{n-1}_{\mathrm{DR}}(V)$, a symplectic vector space endowed by the metric of an almost complex structure (see Exercise 1.60). It can be shown (see [20]) that $M^\star$ is a Calabi-Yau manifold if $\mathscr{B}$ is... a special Lagrangian submanifold in this complex vector space. See [12], [21].

Exercises

Exercise 1.100 Check that the Liouville form λ of the cotangent $T^\star V$ satisfies

$$(\mathrm{d}\lambda)_{(x,0)}\big((v,\alpha),(w,\beta)\big) = \alpha(w) - \beta(v)$$

(see Exercise 1.49).

Exercise 1.101 Let $\varphi : L \to L$ be a diffeomorphism. Prove that the formula

$$\Phi(x,\alpha) = \big(\varphi(x), \big((\mathrm{d}\varphi)_x^{-1}\big)^\star \alpha\big)$$

defines a diffeomorphism of $T^\star L$ into itself. Determine $\Phi^\star\lambda$ and prove that Φ preserves the symplectic from.

Exercise 1.102 Let ω be a non degenerate 2-form on a manifold W. Define the Hamiltonian vector fields and Poisson brackets as above (this does not use the fact that ω is closed). Express

$$\big(\mathrm{d}\omega\big)_x(X,Y,Z)$$

when X, Y et Z are tangent vectors to W at x that are the values at x of the Hamiltonian vector fields of three functions f, g and h. Prove that ω is a closed form if and only if the Poisson bracket it defines satisfies the Jacobi identity.

Exercise 1.103 Assume X and Y are two "locally Hamiltonian" vector fields on a symplectic manifold, namely that $\iota_X\omega$ et $\iota_Y\omega$ are closed forms. Prove that thire Lie bracket $[X,Y]$ is a globally Hamiltonian vector field, namely that $\iota_{[X,Y]}\omega$ is an *exact* form.

Exercise 1.104 (The Hodge star operator) Let V be an n-dimensional oriented manifold endowed with a Riemannian metric g and let α be the Riemannian volume form. Check that the formula

$$g(u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_p) = \det\left(g(u_i, v_j)_{1 \le i,j \le p}\right)$$

defines an metric on $\Lambda^p T^\star V \ldots$ and that the map $\star : \Lambda^p T^\star V \to \Lambda^{n-p} T^\star V$ defined by $u \wedge (\star v) = g(u,v)\alpha$ for all $u \in \Lambda^p T^\star V$ defines, indeed, an operator, the *Hodge star operator*, which is an isometry. Check that

$$\star\star = (-1)^{p(n-1)} \operatorname{Id}_{\Lambda^p T^\star V}.$$

Exercise 1.105 (Multilinear algebra in $\mathbb{R}^4$) Consider the vector space $\mathbb{R}^4$, with its Euclidean structure $g(X,Y) = (X,Y)$ and canonical basis (e_1, e_2, e_3, e_4), and the vector space

$$\Lambda = \Lambda^2(\mathbb{R}^4)^\star$$

of alternated bilinear forms on $\mathbb{R}^4$.

1) What is the dimension of Λ? Check that Λ is isomorphic to the vector space of skew-symmetric endomorphisms of $\mathbb{R}^4$.

2) Endow Λ with the Euclidean structure $(\cdot,\cdot)$ induced by that of $\mathbb{R}^4$, namely such that the basis $(e_i^\star \wedge e_j^\star)/\sqrt{2}$ (for $1 \le i < j \le 4$) is orthonormal. Define the (Hodge) star operator $\star$ on Λ by the formula

$$(\star\alpha) \wedge \eta = (\alpha, \eta)\det \quad \text{for all } \eta \in \Lambda$$

(where det, the determinant, is the generator of $\Lambda^4(\mathbb{R}^4)^\star$ such that $\det(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 1$.

Check that $\star$ is an involution. Determine the $\star(e_i^\star \wedge e_j^\star)$ and the eigenspaces of $\star$.

3) Call Λ_+ the subspace of forms that are invariant by $\star$ (they are called "self-dual" forms). To any α in Λ_+, associate as in 1) a skew-symmetric endomorphism

$$J_\alpha : \mathbb{R}^4 \to \mathbb{R}^4.$$

Prove that $J_\alpha^2 = -\mathrm{Id}$ if and only if $(\alpha, \alpha) = 1$.

Exercise 1.106 (Calabi-Yau surfaces) Let M be a Calabi-Yau surface with Kähler form ω and holomorphic 2-form Ω.

1) There exists a local basis (φ_1, φ_2) of the vector space of holomorphic forms on M in which
$$\omega = \frac{1}{2i}\left(\phi_1 \wedge \bar{\phi}_1 + \phi_2 \wedge \bar{\phi}_2\right)$$
(see [16]). Prove that, on the open set where ϕ_1 and ϕ_2 are defined, one has
$$\Omega = \lambda \phi_1 \wedge \phi_2$$
for some *constant* λ.

2) Check that $\star\Omega = \bar{\Omega}$. Deduce that the real α and imaginary β parts of Ω are self-dual in the sense that $\star\alpha = \alpha$ and $\star\beta = \beta$.

3) Prove that the formula
$$\alpha(X, JY) = g(X, Y)$$
defines a skew-symmetric endomorphism J of the tangent bundle TM and that
$$JX \in \langle X, IX\rangle^\perp = (\mathbb{C}\cdot X)^\perp.$$

4) Prove that $J^2 = -\mathrm{Id}$, so that J is an almost[17] complex structure on M, and that J is an isometry for g.

5) Prove that M is endowed with a hyperkähler structure, namely with three isometries I, J and K that are almost complex structures and anti-commute and with three non degenerate 2-forms that are Kähler for the metric g and respectively for each of the complex structures I, J and K.

Exercise 1.107 Using the notation of Exercise 1.106, prove that the special Lagrangian submanifolds of the Calabi-Yau manifold M are the complex curves for the complex structure J.

[17] It is not very hard to prove that this is a *genuine* complex structure, namely that M is a complex manifold for some structure such that the multiplication by i is J.

Exercise 1.108 In this exercise, W denotes a complex analytic manifold[18] of complex dimension 2, endowed with the structure of a Calabi-Yau manifold, with the Kähler form ω, the holomorphic volume form Ω and the metric γ. Assume moreover that $H^1_{\mathrm{DR}}(W) = 0$.

Consider a vector field X on W, assume that it is not identically zero, and that it preserves ω and Ω, namely that it satisfies the relations $\mathscr{L}_X \omega = 0$ and $\mathscr{L}_X \Omega = 0$. Assume moreover that the 1-form $\iota_X \Omega$ is *holomorphic*.

The metric γ and the vector field X are assumed to be complete.

1) Prove that X is the Hamiltonian vector field of a function $H : W \to \mathbb{R}$.

2) Prove that $\iota_X \Omega$ is preserved by X and that there exists a holomorphic function $f : W \to \mathbb{C}$ such that $\iota_X \Omega = \mathrm{d}f$. Let $x \in W$ be a point such that $X_x \neq 0$. Prove that the kernel of $(\iota_X \Omega)_x$ is the complex line in $T_x W$ spanned by X_x.

3) Assume that L is a Lagrangian submanifold of W that is preserved by X (this means that $X_x \in T_x L$ for all $x \in L$). Check that the connected components of L are contained in the level sets $H^{-1}(a)$ of the Hamiltonian H.

4) Call g and h respectively the real and imaginary part of f. Assume now that L is a *special* Lagrangian submanifold. Prove that h is locally constant on L.

5) Let $a \in \mathbb{R}$ be a *regular* value of H and let $Q = H^{-1}(a)$ be the corresponding level set in W. Fix a point x in Q. Prove that the orthogonal of X_x for the metric γ in $T_x Q$ is a complex line D_x and that the complex linear form $(\iota_X \Omega)_x$ is non zero on D_x. Deduce that the two real linear forms $\mathrm{d}h(x)$ and $\mathrm{d}H(x)$ are independent. Prove that, for all $b \in \mathbb{R}$, $L = Q \cap h^{-1}(b)$ is a dimension-2 submanifold of W and that it is special Lagrangian.

6) Prove that $g\big|_L$ has no critical point.

7) Assume that the Hamiltonian vector field X is periodic. Prove that the connected components of L are diffeomorphic to $S^1 \times \mathbb{R}$.

[18] This exercise (slightly) generalizes the construction given in §2.5.5 and in particular that of Exercise 1.67.

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Bibliography

[1] V. I. Arnold, *A characteristic class entering in quantization conditions*, Funct. Anal. Appl. **1** (1965).

[2] — , *Mathematical methods of classical mechanics*, Springer, 1978.

[3] — , *Lagrange and Legendre cobordisms I and II*, Funct. Anal. Appl. **14** (1980), p. 167–177 and 252–260.

[4] V. I. Arnold & A. B. Givental, *Symplectic geometry*, in Dynamical systems IV, Encyclopaedia of Math. Sci., Springer, 1985, p. 1–138.

[5] M. Audin, *Symplectic and almost complex manifolds*, in [7], p. 41–74.

[6] — , *Les systèmes hamiltoniens et leur intégrabilité*, Cours Spécialisés, vol. 8, Société Mathématique de France & EDP Sciences, 2001.

[7] M. Audin & J. Lafontaine (éds.) – *Holomorphic curves in symplectic geometry*, Progress in Math., Birkhäuser, 1994.

[8] J. Bertin, J.-P. Demailly, L. Illusie & C. Peters, *Introduction à la théorie de Hodge*, Panoramas et Synthèses, 3, Société Mathématique de France, 1996.

[9] R. Bryant, *Some examples of special Lagrangian tori*, Adv. Theor. Math. Phys. **3** (1998), p. 83–90.

[10] A. Cannas da Silva, *Lectures on symplectic geometry*, Lecture Notes in Math., Springer, 2001.

[11] H. Cartan, *Calcul différentiel*, Méthodes, Hermann, Paris, 1967.

[12] S. Donaldson, *Moment maps and diffeomorphisms*, Asian Journal of Math. (2000), à paraître.

[13] D. B. Fuks, *Maslov-Arnold characteristic classes*, Soviet Math. Dokl. **9** (1968), p. 96–99.

[14] A. B. Givental, *Lagrangian imbeddings of surfaces and the open Whitney umbrella*, Funktsional. Anal. i Prilozhen. **20** (1986), no. 3, p. 35–41, 96.

[15] A. B. Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices **13** (1996), p. 613–663.

[16] P. A. Griffiths & J. Harris, *Principles of algebraic geometry*, Wiley, 1978.

[17] M. Gromov, *Partial differential relations*, Springer, Berlin, 1986.

[18] M. Gromov, *Pseudo-holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), p. 307–347.

[19] R. Harvey & H. B. Lawson, Jr., *Calibrated geometries*, Acta Math. **148** (1982), p. 47–157.

[20] N. J. Hitchin, *The moduli space of special Lagrangian submanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), no. 3-4, p. 503–515 (1998), Dedicated to Ennio De Giorgi.

[21] — , *Lectures on special Lagrangian submanifolds*, preprint (1999).

[22] V. M. Kharlamov, *On the classification of nonsingular surfaces of degree* 4 *in* $\mathbf{R}P^3$ *with respect to rigid isotopies*, Funktsional. Anal. i Prilozhen. **18** (1984), no. 1, p. 49–56.

[23] J. Lafontaine, *Introduction aux variétés différentielles*, Presses universitaires de Grenoble, 1996.

[24] J. A. Lees, *On the classification of Lagrange immersions*, *Duke Math. J.* **43** (1976), no. 2, p. 217–224.

[25] D. McDuff & D. Salamon, *Introduction to symplectic topology*, Clarendon Press, New York, 1995, Oxford Science Publications.

[26] R. C. McLean, *Deformations of calibrated submanifolds*, Comm. Anal. Geom. **6** (1998), no. 4, p. 705–747.

[27] J. Milnor, *Morse theory*, Princeton University Press, 1963.

[28] J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. **120** (1965), p. 286–294.

[29] H. Rosenberg & D. Hoffman, *Surfaces minimales et solutions de problèmes variationnels*, Société Mathématique de France, Paris, 1993.

[30] M. Stenzel, *Ricci-flat metrics on the complexification of a compact rank one symmetric space*, Manuscripta Math. **80** (1993), p. 151–163.

[31] A. Strominger, S. T. Yau & E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. **B 479** (1996), p. 243–259.

[32] C. Voisin, *Symétrie miroir*, Panoramas et Synthèses, 2, Société Mathématique de France, 1996, English translation: *Mirror symmetry*, SMF/AMS Texts and Monographs 1, 1999.

[33] A. Weinstein, *Lectures on symplectic manifolds*, CBMS Regional Conference Series in Mathematics, vol. 29, Amer. Math. Soc., 1977.

[34] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I*, Comm. Pure and Appl. Math. **31** (1978), p. 339–411.

Chapter 2

Préambule aux opérateurs Fourier intégraux: les opérateurs pseudo-différentiels

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Abstract. *Ces notes ont été rédigées sur la base d'un cours introductif donné à Ouagadougou en septembre 2015. Elles s'adressent à un lecteur qui souhaiterait s'initier à la notion d'opérateur pseudo-différentiel.*

1 Introduction

Les opérateurs pseudo-différentiels ont été introduits par Alberto Calderon en 1957. Ils contiennent les opérateurs différentiels et ils permettent d'inverser un opérateur différentiel elliptique à un opérateur régularisant près. Les opérateurs pseudo-différentiels

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sont donc un outil fondamental dans la théorie des équations aux dérivées partielles elliptiques. Pour traiter des équations hyperboliques on a besoin de les généraliser aux opérateurs Fourier intégraux (cf [3]). Dans ce cours introductif les opérateurs Fourier intégraux ne seront pas exposés. Par souci de simplicité, nous nous intéresserons à des opérateurs pseudo-différentiels agissant sur des fonctions de $\mathbb{R}^n$, n étant un entier naturel non nul, puis sur des espaces de distributions de $\mathbb{R}^n$. Les fonctions considérées seront toutes à valeurs complexes. En introduisant la notion de paramétrix, nous verrons comment inverser un opérateur différentiel elliptique. Nous présenterons le théorème de continuité des opérateurs pseudo-différentiels dans les espaces de Sobolev ce qui nous permettra d'aborder l'aspect qualitatif de la résolution des équations elliptiques. Ces résultats demeurent dans le cadre d'une variété compacte mais ce cadre ne sera pas envisagé ici. Les opérateurs pseudo-différentiels se situent dans le cadre plus large des opérateurs intégraux, pour ce point de vue non considéré ici, nous conseillons la lecture de [2].

Concernant la bibliographie, une introduction élémentaire aux opérateurs pseudo-différentiels se trouve dans les ouvrages de [7] et [11]. Une introduction plus complète et concise se trouve dans [1]. Pour des compléments au niveau de ce cours, on pourra consulter [5] ou [4] qui sont deux ouvrages accessibles. Pour aller plus loin, des références très complètes sont [6], [9], [10].

Afin de rentrer dans le vif du sujet, nous terminons cette introduction par des notations et un rappel.

On notera pour $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ et $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}, \qquad |\alpha| := \alpha_1 + \ldots + \alpha_n,$$
$$\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \qquad D_x^\alpha := (-i)^{|\alpha|} \partial_x^\alpha.$$

Si $\xi \in \mathbb{R}^n$, $x \cdot \xi := x_1\xi_1 + \cdots + x_n\xi_n$ est le produit scalaire canonique sur $\mathbb{R}^n$ et $\|\cdot\|$ la norme associée.

On introduit D_x^α car $D_x^\alpha(e^{ix\cdot\xi}) := \xi^\alpha e^{ix\cdot\xi}$.

On se servira souvent des majorations suivantes

$$|x^\alpha| = \prod_{i=1}^n |x_i|^{\alpha_i} \leq \|x\|^{|\alpha|} \leq \left(1 + \|x\|^2\right)^{|\alpha|/2} \leq \left(1 + \|x\|\right)^{|\alpha|}.$$

On dit que $\mu = (\mu_1, \dots, \mu_n) \leq \alpha = (\alpha_1, \dots, \alpha_n)$ si pour tout $i \in [[1,n]]$ on a $\mu_i \leq \alpha_i$ et le coefficient binomial généralisé C^μ_α est

$$C^\mu_\alpha = \frac{\mu!}{\alpha!(\mu-\alpha)!}$$

avec $\mu! = \prod_{i=1}^n \mu_i!$ et $\mu - \alpha = (\mu_1 - \alpha_1, \dots, \mu_n - \alpha_n)$.

On rappelle la formule de Leibniz de dérivation du produit de deux fonctions f et g de classe C^∞ sur $\mathbb{R}^n$: pour tout $\alpha \in \mathbb{N}^n$

$$\partial_x^\alpha (f \times g) = \sum_{\mu \leq \alpha} C^\mu_\alpha \partial_x^{\alpha-\mu} f \partial_x^\mu g.$$

Grâce à la transformée de Fourier, il est facile de voir les opérateurs pseudo-différentiels comme des généralisations naturelles des opérateurs différentiels.

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2 Transformée de Fourier sur $\mathscr{S}(\mathbb{R}^n)$

On note $C^\infty(\mathbb{R}^n)$ l'ensemble des fonctions de classe C^∞ sur $\mathbb{R}^n$ et $C_0^\infty(\mathbb{R}^n)$ l'ensemble des fonctions de classe C^∞ sur $\mathbb{R}^n$ à support compact.

$\mathscr{S}(\mathbb{R}^n)$ désigne l'espace de Schwartz des fonctions u de classe C^∞ sur $\mathbb{R}^n$ à décroissance rapide i. e.

$$\forall \alpha, \beta \in \mathbb{N}^n \ \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial_x^\beta u(x) \right| < +\infty.$$

On a clairement $C_0^\infty(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$ et toute fonction de $\mathscr{S}(\mathbb{R}^n)$ est intégrable sur $\mathbb{R}^n$.

De plus, comme $C_0^\infty(\mathbb{R}^n)$ est dense dans $L^p(\mathbb{R}^n)$ pour tout $1 \leq p < \infty$ et comme $C_0^\infty(\mathbb{R}^n)$ est contenu dans $\mathscr{S}(\mathbb{R}^n)$, $\mathscr{S}(\mathbb{R}^n)$ est dense dans $L^p(\mathbb{R}^n)$ pour tout $1 \leq p < \infty$. En particulier, $\mathscr{S}(\mathbb{R}^n)$ est dense dans $L^2(\mathbb{R}^n)$.

Exercice 2.1 a) Montrer que $u \in \mathscr{S}(\mathbb{R}^n)$ si et seulement si

$$\forall m \in \mathbb{N},\ \forall \beta \in \mathbb{N}^n,\ \big(1 + \|x\|\big)^m \left|\partial^\beta u(x)\right| \to 0$$

quand $\|x\| \to \infty$.

b) Vérifier que $\mathscr{S}(\mathbb{R}^n)$ est stable par la multiplication par des polynômes et par la dérivation.

L'espace $\mathscr{S}(\mathbb{R}^n)$ est muni de la topologie donnée par la suite croissante de semi-normes

$$(N_m)_{m\in\mathbb{N}} : \forall u \in \mathscr{S}(\mathbb{R}^n),\ N_m(u) = \sum_{|\alpha|,|\beta|\le m} \|u\|_{\alpha,\beta} \quad \text{où } \|u\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial_x^\beta u(x)\right|.$$

Dans toute la suite, lorsqu'on fera référence à une topologie sur $\mathscr{S}(\mathbb{R}^n)$, ce sera celle-ci. Un espace de Fréchet est un espace vectoriel topologique à base dénombrable de voisinages, localement convexe et complet. $\mathscr{S}(\mathbb{R}^n)$ est un espace de Fréchet.

En posant $\|u\|_{m,\mathscr{S}} = \sup_{x\in\mathbb{R}^n} \left(1+\|x\|\right)^m \sum_{|\beta|=0}^m \left|\partial_x^\beta u(x)\right|$, $\mathscr{S}(\mathbb{R}^n)$ est métrisable pour la métrique

$$d(u,v) = \sum_{m=1}^{+\infty} \frac{1}{2^m} \cdot \frac{\|u-v\|_{m,\mathscr{S}}}{1+\|u-v\|_{m,\mathscr{S}}}.$$

Proposition 2.2 $C_0^\infty(\mathbb{R}^n)$ est dense dans $\mathscr{S}(\mathbb{R}^n)$.

Preuve. Soit $\varphi \in \mathscr{S}(\mathbb{R}^n)$ et soit $\psi \in C_0^\infty(\mathbb{R}^n)$ telle que $\psi(x) = 1$ pour tout $\|x\| \le 1$. Pour tout $n \in \mathbb{N}^*$ on pose $\psi_n(x) = \varphi(x)\psi(x/n)$. Il est clair que pour tout $n \in \mathbb{N}^*$, $\psi_n \in C_0^\infty(\mathbb{R}^n)$ et $\psi_n(x) - \varphi(x) = 0$ pour tout $\|x\| \le n$ donc la suite (ψ_n) converge vers φ dans $\mathscr{S}(\mathbb{R}^n)$. $\square$

Une application linéaire L de $\mathscr{S}(\mathbb{R}^n)$ dans $\mathscr{S}(\mathbb{R}^n)$ est continue si et seulement si pour tout $m \in \mathbb{N}$ il existe $p \in \mathbb{N}$ et une constante $M_m > 0$ (pouvant dépendre de m) tels que pour tout u dans $\mathscr{S}(\mathbb{R}^n)$,

$$N_m(Lu) \le M_m N_p(u).$$

On remarquera que :

Lemme 2.3 Si pour tous multi-indices α, β dans $\mathbb{N}^n$, il existe deux multi-indices γ, δ dans $\mathbb{N}^n$ et une constante $M_{\alpha,\beta} > 0$ tels que pour tout $u \in \mathscr{S}(\mathbb{R}^n)$, $\|Lu\|_{\alpha,\beta} \le M_{\alpha,\beta}\|u\|_{\gamma,\delta}$ alors L est continue sur $\mathscr{S}(\mathbb{R}^n)$.

Exercice 2.4 Soit P une fonction polynomiale sur $\mathbb{R}^n$ et soit $\alpha \in \mathbb{N}^n$. Vérifier que les applications linéaires $u \to Pu$ et $u \to \partial^\alpha u$ sont continues de $\mathscr{S}$ dans $\mathscr{S}$.

On définit la transformée de Fourier d'une fonction u intégrable sur $\mathbb{R}^n$ par

$$\mathscr{F}u(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot\xi} u(x)\mathrm{d}x.$$

Nous rappelons ci-dessous les propriétés dont on aura besoin.

Proposition 2.5

1) Si $u \in \mathscr{S}(\mathbb{R}^n)$ alors $\mathscr{F}u \in S(\mathbb{R}^n)$ et l'application $u \to \mathscr{F}u$ est linéaire continue sur $\mathscr{S}(\mathbb{R}^n)$

2) Si $u \in \mathscr{S}(\mathbb{R}^n)$ alors pour tous α et β dans $\mathbb{N}^n$,

$$\mathscr{F}\big(D^\alpha u\big)(\xi) = \xi^\alpha \mathscr{F}u(\xi) \quad \text{et} \quad \big(D^\beta \mathscr{F}u\big)(\xi) = \mathscr{F}\big((-x)^\beta u\big)(\xi).$$

Théorème 2.6 La transformée de Fourier de $\mathscr{S}(\mathbb{R}^n)$ dans $\mathscr{S}(\mathbb{R}^n)$ admet un inverse défini par

$$\forall v \in \mathscr{S}(\mathbb{R}^n),\ \mathscr{F}^{-1}v(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} v(\xi)\mathrm{d}\xi$$

et $u \mapsto \mathscr{F}u$ est un isomorphisme topologique de $\mathscr{S}(\mathbb{R}^n)$.

Ces propriétés sont bien connues et leurs preuves ne présentent pas de difficulté. Sur la notion de transformée de Fourier on pourra consulter par exemple [7]. Pour le lecteur non familier avec la notion de transformée de Fourier, les exercices ci-dessous donnent des indications de preuve des résultats rappelés.

Exercice 2.7 Soit $u \in \mathscr{S}(\mathbb{R}^n)$ et $\alpha \in \mathbb{N}^n$. En intégrant par parties et en remarquant que la variation de la partie intégrée est nulle, vérifier que $\mathscr{F}\big(\partial_{x_1}u\big) = i\xi_1\mathscr{F}(u)$. En déduire que $\mathscr{F}\big(D^\alpha u\big)(\xi) = \xi^\alpha \mathscr{F}u(\xi)$.

Exercice 2.8 Soit $u \in \mathscr{S}(\mathbb{R}^n)$ et $\alpha, \beta \in \mathbb{N}^n$.

a) En utilisant le théorème de dérivation de Lebesgue sous le signe $\int$, vérifier que $\mathscr{F}u$ est de classe C^∞. On remarquera que $\partial_\xi^\beta \mathscr{F}u(\xi) = \mathscr{F}\big((-ix)^\beta u\big)(\xi)$.

b) En intégrant par parties vérifier que

$$\xi^\alpha \partial_\xi^\beta \mathscr{F}u(\xi) = (-1)^{|\alpha|}(-i)^{|\alpha|+|\beta|}\int_{\mathbb{R}^n} x^\beta \mathrm{e}^{-ix\cdot\xi}\partial_x^\alpha u(x)\mathrm{d}x.$$

En déduire que $\mathscr{F}u \in S(\mathbb{R}^n)$ et que l'application linéaire $u \mapsto \mathscr{F}u$ est continue sur $S(\mathbb{R}^n)$.

Exercice 2.9 Soit $u \in \mathscr{S}(\mathbb{R}^n)$ et soit g la fonction définie sur $\mathbb{R}^n$ par $g(x) = \mathrm{e}^{-\|x\|^2/2}$.

a) Vérifier que
$$\int_{\mathbb{R}^n} \mathscr{F}g(\xi)\mathrm{d}\xi = (2\pi)^n.$$
Pour tout $\varepsilon > 0$, on pose
$$u_\varepsilon(x) = \int_{\mathbb{R}^n} g(\varepsilon\xi)\mathscr{F}u(\xi)\mathrm{e}^{ix\cdot\xi}\mathrm{d}\xi.$$

b) En utilisant le théorème de convergence dominée de Lebesgue, montrer que
$$u_\varepsilon(x) \underset{\varepsilon\to 0}{\longrightarrow} \int_{\mathbb{R}^n} \mathscr{F}u(\xi)\mathrm{e}^{ix\cdot\xi}\mathrm{d}\xi.$$

c) En utilisant le théorème de Fubini, montrer que
$$u_\varepsilon(x) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} g(\varepsilon\xi)\mathrm{e}^{-i(y-x)\cdot\xi}\mathrm{d}\xi \right) u(y)\mathrm{d}y.$$

d) En déduire que
$$u_\varepsilon(x) = \int_{\mathbb{R}^n} \mathscr{F}g(t)u(x+\varepsilon t)\mathrm{d}t.$$

e) En utilisant de nouveau le théorème de convergence dominée de Lebesgue, montrer que
$$u_\varepsilon(x) \underset{\varepsilon\to 0}{\longrightarrow} (2\pi)^n u(x).$$

f) Déduire de b) et e) que
$$u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathscr{F}u(\xi)\mathrm{e}^{ix\cdot\xi}\mathrm{d}\xi.$$

. . . .

3 Des opérateurs différentiels aux opérateurs pseudo-différentiels

Définition 2.10 Une application linéaire A est dite opérateur différentiel à coefficients a_α dans $C^\infty(\mathbb{R}^n)$ s'il existe $m \in \mathbb{N}^*$ tel que

$$\forall u \in C^\infty(\mathbb{R}^n),\ A(u)(x) = \sum_{|\alpha| \leq m} a_\alpha(x)\big(D_x^\alpha u\big)(x)$$

où $\alpha = (\alpha_1, \ldots, \alpha_n)$ est un multi-indice d'entiers.

Dire que A est d'ordre m, c'est dire qu'il existe $\alpha \in \mathbb{N}^n$ tel que $|\alpha| = m$ et $a_\alpha \neq 0$.Par exemple, en dimension 2, l'opérateur différentiel $\partial_{x_1} + \partial_{x_1}\partial_{x_2}$ est d'ordre 2.

Le chemin des opérateurs différentiels aux opérateurs pseudo-différentiels passe par la notion de symbole que nous introduisons maintenant.

Définition 2.11 Soit A un opérateur différentiel à coefficients a_α. On appelle symbole de A la fonction $\sigma(A)$ définie sur $\mathbb{R}^n \times \mathbb{R}^n$ par

$$\sigma(A)(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha(x)\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n} = \sum_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha.$$

Ainsi le symbole d'un opérateur différentiel d'ordre m est un polynôme de degré m en les variables $\xi_1, \ldots, \xi_n$. Réciproquement, il est clair que la donnée d'un polynôme en les variables $\xi_1, \ldots, \xi_n$ à coefficients dans $C^\infty(\mathbb{R}^n)$ définit un unique opérateur différentiel.

Exemple 2.12 Le symbole de $\partial_{x_1} + \partial_{x_1}\partial_{x_2}$ est $-i\xi_1 - \xi_1\xi_2$. Le symbole de $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ est $-\xi_1^2 - \xi_2^2$.

La restriction aux fonctions u dans $\mathscr{S}(\mathbb{R}^n)$ fournit la représentation intégrale suivante d'un opérateur différentiel.

Théorème 2.13 Soit A un opérateur différentiel à coefficients a_α. Pour toute fonction u dans $\mathscr{S}(\mathbb{R}^n)$ on a

$$A\big(u\big)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi}\sigma\big(A\big)(x,\xi)\mathscr{F}u(\xi)\mathrm{d}\xi.$$

Preuve.

$$A\big(u\big)(x) = \sum_{|\alpha|\leq m} a_\alpha(x)\big(D_x^\alpha u\big)(x).$$

Par transformée de Fourier inverse,

$$A\big(u\big)(x) = \sum_{|\alpha|\leq m} a_\alpha(x)\mathscr{F}^{-1}\Big(\mathscr{F}\big(D_x^\alpha u\big)\Big)(x).$$

Comme la transformée de Fourier de D^α est la multiplication par ξ^α

$$\begin{aligned} A\big(u\big)(x) &= \sum_{|\alpha|\leq m} a_\alpha(x)\mathscr{F}^{-1}\big(\xi^\alpha \mathscr{F}u\big)(x) \\ &= \frac{1}{(2\pi)^n}\sum_{|\alpha|\leq m} a_\alpha(x)\int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi}\xi^\alpha \mathscr{F}u(\xi)\mathrm{d}\xi \\ &= \frac{1}{(2\pi^n)}\int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi}\left(\sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha\right)\mathscr{F}u(\xi)\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi}\sigma\big(A\big)(x,\xi)\mathscr{F}u(\xi)\mathrm{d}\xi. \end{aligned}$$ □

La dernière intégrale converge pour une classe de fonctions σ plus large que celle des polynômes en les variables $\xi_1,\ldots,\xi_n$ à coefficients dans $C^\infty(\mathbb{R}^n)$.

Définition 2.14 Soit $m \in \mathbb{R}$. Une application σ de classe C^∞ sur $\mathbb{R}^n \times \mathbb{R}^n$ à valeurs dans $\mathbb{C}$,

$$(x,\xi) \mapsto \sigma(x,\xi)$$

telle que pour tous multi-indices α et β, il existe une constante $C_{\alpha,\beta} > 0$ (pouvant dépendre de α et β) telle que

$$\forall x \in \mathbb{R}^n, \forall \xi \in \mathbb{R}^n,\ \big|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\big| \leq C_{\alpha,\beta}\big(1+\|\xi\|\big)^{m-|\beta|} \tag{2.1}$$

est dite symbole d'ordre m. Si σ vérifie la condition (2.1) pour tout $m \in \mathbb{R}$ on dit que σ est d'ordre $-\infty$.

On remarquera que l'ordre d'un symbole n'est pas défini de manière unique. Si σ est d'ordre $m \in \mathbb{R}$ alors σ est d'ordre m' pour tout nombre réel m' supérieur à m. Cette ambiguïté sur la notion d'ordre est due au fait qu'on autorise tout nombre réel à être un ordre. Si on se restreint à des ordres dans $\mathbb{Z}$ alors on peut lever l'ambiguïté en prenant le minimum dans $\mathbb{Z} \cup \{-\infty\}$ des m possibles.

Nous noterons $\mathrm{Symb}^m(\mathbb{R}^n)$ l'ensemble des symboles d'ordre m et $\mathrm{Symb}(\mathbb{R}^n) = \bigcup_{m\in\mathbb{R}} \mathrm{Symb}^m(\mathbb{R}^n)$ l'ensemble des symboles.

Exercice 2.15 Soit σ une application C^∞ sur $\mathbb{R}^n \times \mathbb{R}^n$.

a) Vérifier que s'il existe $R > 0$ tel que pour tout $(x,\xi) \in \mathbb{R}^n$: $\|\xi\| > R \Rightarrow \sigma(x,\xi) = 0$ alors σ est d'ordre $-\infty$.

b) Soit $u \in \mathscr{S}(\mathbb{R}^n)$. Vérifier que la fonction $(x,\xi) \mapsto u(\xi)$ sur $\mathbb{R}^n \times \mathbb{R}^n$ est un symbole d'ordre $-\infty$.

Exercice 2.16 Soit $a, b \in \mathrm{Symb}(\mathbb{R}^n)$, Si a est d'ordre m et si b est d'ordre m' montrer que la somme $a+b$ est un symbole d'ordre $\max(m,m')$.

Exercice 2.17 Soit $a, b \in \mathrm{Symb}(\mathbb{R}^n)$, Si a est d'ordre m et si b est d'ordre m' montrer que le produit ab est un symbole d'ordre $m+m'$. Si a ou b est d'ordre $-\infty$, montrer que ab est un symbole d'ordre $-\infty$.

Exercice 2.18 Soit $u \in \mathscr{S}(\mathbb{R}^n)$ et soit $\sigma \in \mathrm{Symb}(\mathbb{R}^n)$. Vérifier que pour tout $x \in \mathbb{R}^n$, la fonction $\xi \mapsto \sigma(x,\xi)u(\xi)$ appartient à $\mathscr{S}(\mathbb{R}^n)$.

Exercice 2.19 Soit σ un symbole d'ordre m et soit $\alpha \in \mathbb{N}^n$. Vérifier que

a) $\partial_x^\alpha \sigma$ est un symbole d'ordre m

b) $\partial_\xi^\alpha \sigma$ est un symbole d'ordre $m - |\alpha|$

c) $\xi^\alpha \sigma$ est un symbole d'ordre $m + |\alpha|$.

Exercice 2.20 Soit σ une application de classe C^∞ sur $\mathbb{R}^n \times \mathbb{R}^n$ à valeurs dans $\mathbb{C}$. Soit $m \in \mathbb{R}$. On suppose que

$$\forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n, \forall t > 0, \;\; \sigma(x,t\xi) = t^m \sigma(x,\xi)$$

i. e. σ est positivement homogène par rapport à la seconde variable. Soit $\varphi \in C^\infty(\mathbb{R}^n)$ telle que $\varphi(\xi) = 0$ pour $|\xi| \leq 1$ et $\varphi(\xi) = 1$ pour $|\xi| \geq 2$. Vérifier que $\varphi\sigma \in \mathrm{Symb}^m(\mathbb{R}^n)$.

Proposition 2.21 Soit A un opérateur différentiel d'ordre m sur $\mathbb{R}^n$. La fonction $\sigma(A)$ appartient à $\mathrm{Symb}^m(\mathbb{R}^n)$ si et seulement si tous les coefficients de A sont bornés ainsi que leurs dérivées de tous ordres.

Preuve. Supposons que tous les coefficients de A sont bornés ainsi que leurs dérivées de tous ordres. Soit α un multi-indice de longueur $|\alpha| = m$ et soit β un autre multi-indice. S'il existe $i \in [[1,n]]$ tel que $\beta_i > \alpha_i$ alors $\partial_\xi^\beta \xi^\alpha = 0$. Sinon on a

$$\partial_\xi^\beta \xi^\alpha = \frac{\alpha!}{(\alpha-\beta)!}\xi^{\alpha-\beta}$$

et comme dans ce cas $\alpha - \beta$ est un multi-indice, on a $\left|\xi^{\alpha-\beta}\right| \leq \left(1+\|\xi\|\right)^{m-|\beta|}$. La réciproque est claire en prenant $|\beta| = 0$ dans la formule (2.1). □

Exercice 2.22 Soit A un opérateur différentiel sur $\mathbb{R}^n$ tel que $\sigma(A) \in \mathrm{Symb}^m(\mathbb{R}^n)$. Vérifier que $A\big(\mathscr{S}(\mathbb{R}^n)\big) \subset \mathscr{S}(\mathbb{R}^n)$. On remarquera que $\mathscr{S}(\mathbb{R}^n)$ est stable par la multiplication par des fonctions de $C^\infty(\mathbb{R}^n)$ bornées sur $\mathbb{R}^n$ ainsi que toutes leurs dérivées.

Grâce au Théorème 2.13, on a vu que la restriction d'un opérateur différentiel à $\mathscr{S}(\mathbb{R}^n)$ s'écrit sous la forme d'une intégrale dans laquelle intervient le symbole de l'opérateur différentiel. En se restreignant donc à $\mathscr{S}(\mathbb{R}^n)$, on va voir que les opérateurs pseudo-différentiels généralisent les opérateurs différentiels.

Définition 2.23 On appelle opérateur pseudo-différentiel une application linéaire $Op(\sigma)$ défini à l'aide d'un symbole $\sigma \in \mathrm{Symb}(\mathbb{R}^n)$ sur $\mathscr{S}(\mathbb{R}^n)$ par

$$\forall u \in \mathscr{S}(\mathbb{R}^n),\ \ Op(\sigma)(u)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi}\sigma(x,\xi)\mathscr{F}u(\xi)\mathrm{d}\xi.$$

On a vu qu'on a une bijection entre les opérateurs différentiels et les symboles polynomiaux en $\xi_1,\ldots,\xi_n$. On peut se demander ce qu'il en est pour des opérateurs pseudo-différentiels. En utilisant la densité de $\mathscr{S}(\mathbb{R}^n)$ dans $L^2(\mathbb{R}^n)$ on peut démontrer (cf [1] dont l'exercice ci-dessous est extrait) que si σ_1 et σ_2 sont deux symboles dans $\mathrm{Symb}(\mathbb{R}^n)$ et si $Op(\sigma_1) = Op(\sigma_2)$ sur $\mathscr{S}(\mathbb{R}^n)$ alors $\sigma_1 = \sigma_2$.

Exercice 2.24 Soit $\sigma \in \mathrm{Symb}(\mathbb{R}^n)$ d'ordre m. Supposons que pour toute fonction $u \in \mathscr{S}$ et pour tout $x \in \mathbb{R}^n$,

$$Op(\sigma)(u)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi) \mathscr{F}u(\xi) d\xi = 0.$$

Pour x fixé, soit pour tout $\xi \in \mathbb{R}^n$

$$f(\xi) = \frac{\sigma(x,\xi)}{\left(1+\|\xi\|^2\right)^{m/2+n/4+1/2}}.$$

a) Vérifier que $f \in L^2(\mathbb{R}^n)$.

b) Montrer que f est orthogonal à $\mathscr{S}(\mathbb{R}^n)$ dans $L^2(\mathbb{R}^n)$.

c) En utilisant la densité de $\mathscr{S}(\mathbb{R}^n)$ dans $L^2(\mathbb{R}^n)$, conclure que $\sigma = 0$.

Définition 2.25 Si σ est d'ordre $m \in \mathbb{R} \cup \{-\infty\}$, on dit que $Op(\sigma)$ est d'ordre m.

Voici deux propriétés importantes des opérateurs pseudo-différentiels sur $\mathscr{S}(\mathbb{R}^n)$.

Proposition 2.26 Soit $\sigma \in \mathrm{Symb}(\mathbb{R}^n)$.

1) $Op(\sigma)$ envoie $\mathscr{S}(\mathbb{R}^n)$ dans $\mathscr{S}(\mathbb{R}^n)$.

2) $Op(\sigma)$ est linéaire continue sur $\mathscr{S}(\mathbb{R}^n)$.

Preuve. Soit $u \in \mathscr{S}(\mathbb{R}^n)$, $\alpha, \beta \in \mathbb{N}^n$.

1) Il suffit de montrer que

$$\sup_{x\in\mathbb{R}} \left| x^\alpha D_x^\beta \Big(Op(\sigma)(u) \Big)(x) \right| < +\infty.$$

Soit $x \in \mathbb{R}^n$. On a

$$\begin{aligned} D_x^\beta \Big(Op(\sigma)(u) \Big)(x) &= \frac{1}{(2\pi^n)} \int_{\mathbb{R}^n} D_x^\beta \Big(e^{ix\cdot\xi} \sigma(x,\xi) \Big) \mathscr{F}u(\xi) d\xi \\ &= \frac{1}{(2\pi^n)} \int_{\mathbb{R}^n} \sum_{\gamma\leq\beta} C_\beta^\gamma \xi^\gamma e^{ix.\xi} D_x^{\beta-\gamma} \sigma(x,\xi) \mathscr{F}u(\xi) d\xi \end{aligned}$$

donc

$$x^\alpha D_x^\beta \Big(Op(\sigma)(u) \Big)(x)$$

$$= \frac{1}{(2\pi^n)} \int_{\mathbb{R}^n} \sum_{\gamma \leq \beta} C_\beta^\gamma \xi^\gamma \left(D_\xi^\alpha e^{ix.\xi}\right) \left(D_x^{\beta-\gamma}\sigma\right)(x,\xi)\mathscr{F}u(\xi)d\xi.$$

On remarque que si $f \in \mathscr{S}(\mathbb{R}^n)$ alors, par intégration par parties,

$$\int_{\mathbb{R}^n} D_\xi^\alpha \left(e^{ix.\xi}\right) f(\xi)d\xi = (-1)^{|\alpha|} \int_{\mathbb{R}^n} e^{ix.\xi} D_\xi^\alpha f(\xi)d\xi.$$

On applique cette remarque à $f(\xi) = \xi^\gamma D_x^{\beta-\gamma}\sigma(x,\xi)\mathscr{F}u(\xi)$ qu'on sait être dans $\mathscr{S}(\mathbb{R}^n)$ car $\mathscr{F}u \in \mathscr{S}(\mathbb{R}^n)$ et $\xi^\gamma D_x^{\beta-\gamma}\sigma(x,\xi)$ est un symbole (cf. exercice 2.19). Il vient

$$\begin{aligned}
&x^\alpha D_x^\beta \Big(Op(\sigma)(u)\Big)(x) \\
&= (-1)^{|\alpha|} \frac{1}{(2\pi^n)} \int_{\mathbb{R}^n} \sum_{\gamma \leq \beta} C_\beta^\gamma e^{ix.\xi} D_\xi^\alpha \Big\{ \left(D_x^{\beta-\gamma}\sigma\right)(x,\xi)\xi^\gamma \mathscr{F}u(\xi)\Big\} d\xi \\
&= (-1)^{|\alpha|} \frac{1}{(2\pi^n)} \int_{\mathbb{R}^n} \sum_{\gamma \leq \beta} \sum_{\delta \leq \alpha} C_\beta^\gamma C_\alpha^\delta e^{ix.\xi} \left(D_\xi^{\alpha-\delta} D_x^{\beta-\gamma}\sigma(x,\xi)\right) D_\xi^\delta\left(\xi^\gamma \mathscr{F}u(\xi)\right)d\xi.
\end{aligned}$$

Comme σ est un symbole, en notant m son ordre, on en déduit qu'il existe une constante $C_{\alpha,\beta,\gamma,\delta}$ telle que

$$\forall x \in \mathbb{R}^n, \forall \xi \in \mathbb{R}^n, \ \left|D_\xi^{\alpha-\delta} D_x^{\beta-\gamma}\sigma(x,\xi)\right| \leq C_{\alpha,\beta,\gamma,\delta}\left(1+\|\xi\|\right)^{m-|\alpha|+|\delta|}.$$

Ainsi on obtient

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n} \left|x^\alpha D_x^\beta \Big(Op(\sigma)(u)\Big)(x)\right| \\
&\quad \leq \frac{1}{(2\pi^n)} \sum_{\gamma \leq \beta} \sum_{\delta \leq \alpha} C_{\alpha,\beta,\gamma,\delta} \int_{\mathbb{R}^n} \left(1+\|\xi\|\right)^{m-|\alpha|+|\delta|} \left|D_\xi^\delta\left(\xi^\gamma \mathscr{F}u(\xi)\right)\right| d\xi \\
&\quad \sup_{x\in\mathbb{R}^n} \left|x^\alpha D_x^\beta \Big(Op(\sigma)(u)\Big)(x)\right| \\
&\qquad \leq \frac{1}{(2\pi^n)} \sum_{\gamma \leq \beta} \sum_{\delta \leq \alpha} C_{\alpha,\beta,\gamma,\delta} \int_{\mathbb{R}^n} \left(1+\|\xi\|\right)^{m} \left|D_\xi^\delta\left(\xi^\gamma \mathscr{F}u(\xi)\right)\right| d\xi.
\end{aligned}$$

On sait que

- ◇ $\mathscr{F}u$ appartient à $\mathscr{S}(\mathbb{R}^n)$

$\diamond$ $\mathscr{S}(\mathbb{R}^n)$ est stable par la multiplication par des polynômes et par la dérivation donc la fonction $\xi \mapsto D^\delta_\xi\big(\xi^\gamma \mathscr{F}u(\xi)\big)$ est négligeable devant les fonctions $\xi \mapsto \big(1+\|\xi\|^2\big)^{-N}$ pour tout $N \in \mathbb{N}$. On choisit N tel que $m-2N<-n$ ce qui entraîne la convergence de l'intégrale

$$\int_{\mathbb{R}^n} \big(1+\|\xi\|\big)^m \big(1+\|\xi\|^2\big)^{-N} \mathrm{d}\xi.$$

2) Soit $N \in \mathbb{N}$ tel que $m-2N<-n$. On reprend l'inégalité obtenue ci-dessus. On écrit l'intégrale du membre de droite sous la forme

$$\int_{\mathbb{R}^n} \big(1+\|\xi\|\big)^m \big(1+\|\xi\|^2\big)^{-N} \ \big(1+\|\xi\|^2\big)^N \Big|D^\delta_\xi\big(\xi^\gamma \mathscr{F}u(\xi)\big)\Big| \mathrm{d}\xi.$$

Il suit de la continuité de $\mathscr{F}$ sur $\mathscr{S}(\mathbb{R}^n)$ que l'application linéaire $u \mapsto \big(1+\|\xi\|^2\big)^N D^\delta_\xi\big(\xi^\gamma \mathscr{F}u(\xi)\big)$ est continue sur $\mathscr{S}(\mathbb{R}^n)$ donc il existe une constante $M>0$ indépendante de u et deux multi-indices α' et β' tels que

$$\sup_{\xi\in\mathbb{R}^n} \Big[\big(1+\|\xi\|^2\big)^N \Big|D^\delta_\xi\big(\xi^\gamma \mathscr{F}u(\xi)\big)\Big|\Big] \le M\|u\|_{\alpha',\beta'}.$$

Comme on a choisi N de telle sorte que $\big(1+\|\xi\|\big)^m\big(1+\|\xi\|^2\big)^{-N}$ soit intégrable sur $\mathbb{R}^n$, on a l'existence d'une constante $M'>0$ indépendante de u telle que

$$\Big\|Op(\sigma)(u)\Big\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} \Big|x^\alpha D^\beta_x \Big(Op(\sigma)(u)\Big)(x)\Big| \le M'\|u\|_{\alpha',\beta'}$$

ce qui permet de conclure avec le lemme 2.3. $\square$

. . . .

4 Transformée de Fourier sur $\mathscr{S}'(\mathbb{R}^n)$

Par définition, $\mathscr{S}'(\mathbb{R}^n)$ est l'espace des formes linéaires continues sur $\mathscr{S}(\mathbb{R}^n)$. C'est le dual topologique de $\mathscr{S}(\mathbb{R}^n)$. Un élément de $\mathscr{S}'(\mathbb{R}^n)$ est appelé distribution tempérée. Si u est une forme sur linéaire sur $\mathscr{S}(\mathbb{R}^n)$,

$$u \in \mathscr{S}'(\mathbb{R}^n) \Longleftrightarrow \exists m \in \mathbb{N}, \exists C > 0 : \forall \varphi \in \mathscr{S}(\mathbb{R}^n),\ \left|\langle u, \varphi\rangle\right| \leq CN_m(\varphi),$$

et comme $\mathscr{S}(\mathbb{R}^n)$ est métrisable

$$u \in \mathscr{S}'(\mathbb{R}^n) \Longleftrightarrow \forall(\varphi_n) \text{ suite de } \mathscr{S}(\mathbb{R}^n) \text{ convergeant vers } 0,\ \left|\langle u, \varphi_n\rangle\right| \underset{n\to+\infty}{\longrightarrow} 0.$$

N.B. $(\varphi_n) \subset \mathscr{S}(\mathbb{R}^n)$ converge vers 0 si et seulement si

$$\forall m \in \mathbb{N},\ N_m(\varphi_n) \underset{n\to+\infty}{\longrightarrow} 0 \Longleftrightarrow \forall \alpha, \beta \in \mathbb{N}^n,\ \|\varphi_n\|_{\alpha,\beta} \underset{n\to+\infty}{\longrightarrow} 0.$$

Pour définir la transformée de Fourier sur $\mathscr{S}'(\mathbb{R}^n)$, nous allons utiliser la notion d'application transposée d'une application linéaire continue d'un espace vectoriel topologique dans un autre.

Soit E un espace vectoriel topologique. On appelle topologie faible sur E' la topologie la moins fine de E' rendant continues les applications $E' \to \mathbb{C}$, $u \mapsto u(x)$, $u \in E'$ $(x \in E)$. On rappelle que si E est un espace vectoriel topologique localement convexe séparé, la topologie faible sur E' est la topologie de la convergence simple.

Définition 2.27 Soient E et F deux espaces vectoriels topologiques et soient E' et F' leurs duaux topologiques respectifs munis de la topologie faible. Soit $f : E \to F$ une application linéaire continue. On appelle application transposée de f et on note f' l'application définie par

$$\forall u \in F',\ \ f'(u) = u \circ f.$$

L'application transposée f' de f est linéaire continue de F' dans E'.

Dans ce qui suit, on prend $E = F = \mathscr{S}(\mathbb{R}^n)$. $\mathscr{S}(\mathbb{R}^n)$ est un espace de Fréchet, donc $\mathscr{S}(\mathbb{R}^n)$ est un espace vectoriel topologique localement convexe séparé. On munit $\mathscr{S}'(\mathbb{R}^n)$ de la topologie de la convergence simple.

D'après la Proposition 2.5, $\mathscr{F}$ est linéaire continue de $\mathscr{S}(\mathbb{R}^n)$ dans $\mathscr{S}(\mathbb{R}^n)$. Soit $\mathscr{F}'$ l'application transposée de $\mathscr{F}$ i. e.

$$\forall u \in \mathscr{S}'(\mathbb{R}^n),\ \forall \varphi \in \mathscr{S}(\mathbb{R}^n),\ \ \langle \mathscr{F}'u, \varphi\rangle = \langle u, \mathscr{F}\varphi\rangle.$$

Pour la topologie de la convergence simple de $\mathscr{S}'(\mathbb{R}^n)$, $\mathscr{F}'$ est continue. On va voir que $\mathscr{F}'$ est l'unique prolongement continu de $\mathscr{F}$.

Si $u \in \mathscr{S}(\mathbb{R}^n)$, on peut voir u comme un élément de $\mathscr{S}'(\mathbb{R}^n)$ par l'identification

$$u \mapsto T_u : \forall v \in \mathscr{S}(\mathbb{R}^n),\ \langle T_u, v\rangle = \int_{\mathbb{R}^n} u(x)v(x)\mathrm{d}x.$$

En appliquant le théorème de Fubini, on a

$$\forall u, v \in \mathscr{S}(\mathbb{R}^n),\ \int_{\mathbb{R}^n} \mathscr{F}u(x)v(x)\mathrm{d}x = \int_{\mathbb{R}^n} u(x)\mathscr{F}v(x)\mathrm{d}x$$

i. e.

$$\forall u, v \in \mathscr{S}(\mathbb{R}^n),\ \langle \mathscr{F}u, v\rangle = \langle u, \mathscr{F}v\rangle.$$

Donc la transformée de Fourier de $\mathscr{S}'(\mathbb{R}^n)$ prolonge la transformée de Fourier de $\mathscr{S}(\mathbb{R}^n)$. L'unicité de ce prolongement sera donné par le corollaire 2.31 ci-dessous.

Définition 2.28 Etant donnés deux espaces topologiques E et F tels que E soit contenu dans F, on dit que E se plonge dans F si l'injection canonique $i : E \to F$, $x \mapsto x$, est continue et on écrira $E \hookrightarrow F$.

Proposition 2.29 $\mathscr{S}(\mathbb{R}^n) \hookrightarrow \mathscr{S}'(\mathbb{R}^n)$

Preuve. Pour tout $u \in \mathscr{S}(\mathbb{R}_n)$ on pose $T_u(v) = \int_{\mathbb{R}^n} u(x)v(x)\mathrm{d}x$. L'application linéaire $T : u \mapsto T_u$ est l'injection canonique de $\mathscr{S}(\mathbb{R}^n)$ dans $\mathscr{S}'(\mathbb{R}^n)$. Pour montrer que T est continue, comme $\mathscr{S}(\mathbb{R}^n)$ est métrisable, il suffit de montrer que si une suite (u_n) de $\mathscr{S}(\mathbb{R}^n)$ converge vers 0 dans $\mathscr{S}(\mathbb{R}^n)$ alors la suite $\big(T(u_n)\big)$ converge simplement vers 0 i. e. $\langle T_{u_n}, v\rangle$ converge vers 0 pour tout $v \in \mathscr{S}(\mathbb{R}^n)$. C'est clair par convergence uniforme de la suite (u_n) vers 0. □

Proposition 2.30 $C_0^\infty(\mathbb{R}^n)$ et $\mathscr{S}(\mathbb{R}^n)$ sont denses dans $\mathscr{S}'(\mathbb{R}^n)$.

Preuve. Cela résulte de la densité de $C_0^\infty(\mathbb{R}^n)$ dans $\mathscr{S}(\mathbb{R}^n)$ et de la proposition précédente. □

Corollaire 2.31 La transformée de Fourier de $\mathscr{S}'(\mathbb{R}^n)$ est l'unique prolongement continu de la transformée de Fourier de $\mathscr{S}(\mathbb{R}^n)$.

5 Extension des opérateurs pseudo-différentiels aux espaces de Sobolev

Dans ce paragraphe nous allons établir qu'un opérateur pseudo-différentiel se prolonge de façon unique en un opérateur borné d'un espace de Sobolev dans un autre. Nous commençons par rappeler ce que sont les espaces de Sobolev.

Pour tout $s \in \mathbb{R}$, on note

$$H^s(\mathbb{R}^n) = \Big\{ u \in \mathscr{S}'(\mathbb{R}^n) \ : \ \mathscr{F}u \text{ fonction de } \mathbb{R}^n \text{ dans } \mathbb{C} \\ \text{et } \int_{\mathbb{R}^n} \big(1+|\xi\|^2\big)^s \big|\mathscr{F}u(\xi)\big|^2 \mathrm{d}\xi < +\infty \Big\}.$$

Muni du produit hermitien

$$(u,v)_s = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \big(1+\|\xi\|^2\big)^s \mathscr{F}u(\xi)\overline{\mathscr{F}v(\xi)} \mathrm{d}\xi,$$

$H^s(\mathbb{R}^n)$ est un espace de Hilbert avec

$$\|u\|_s^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \big(1+\|\xi\|^2\big)^s \big|\mathscr{F}u(\xi)\big|^2 \mathrm{d}\xi.$$

On remarquera que si $s < t$ alors $H^t(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$.

Comme la transformée de Fourier $\mathscr{F}$ est continue sur $\mathscr{S}'(\mathbb{R}^n)$, il est clair que

$$\forall s \in \mathbb{R},\ \ \mathscr{S}(\mathbb{R}^n) \hookrightarrow H^s(\mathbb{R}^n).$$

Exercice 2.32 Soit $u \in H^s(\mathbb{R}^n)$ et soit $\varphi \in \mathscr{S}(\mathbb{R}^n)$.

1) Justifier que $\langle u, \varphi \rangle = \displaystyle\int_{\mathbb{R}^n} u(\xi)\varphi(\xi)\mathrm{d}\xi$.

2) En déduire que $\langle u, \varphi \rangle = \displaystyle\int_{\mathbb{R}^n} \mathscr{F}u(\xi)\mathscr{F}^{-1}\varphi(\xi)\mathrm{d}\xi$.

3) En appliquant l'inégalité de Cauchy-Schwarz et en remarquant que $\mathscr{F}^{-1}\varphi(\xi) = \mathscr{F}\big(\varphi(-\xi)\big)$, montrer que

$$\langle u, \varphi \rangle \leq (2\pi)^{2n}\|u\|_s\|\varphi\|_{-s}.$$

Cet exercice nous donne que $\forall s \in \mathbb{R},\ H^s(\mathbb{R}^n) \hookrightarrow \mathscr{S}'(\mathbb{R}^n)$.

On rappelle les propriétés suivantes des espaces de Sobolev :

◇ Lorsque $s \in \mathbb{N}$, $H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n)\ :\ \forall \alpha \in \mathbb{N}^n, |\alpha| \leq s,\ \partial^\alpha u \in L^2(\mathbb{R}^n)\}$. En particulier, $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.

◇ Pour tout $s \in \mathbb{R}$, le dual de $H^s(\mathbb{R}^n)$ s'identifie à $H^{-s}(\mathbb{R}^n)$.

◇ Pour tout $m \in \mathbb{N}$ et pour tout $s > m + n/2$, $H^s(\mathbb{R}^n)$ est contenu dans l'ensemble $C^m(\mathbb{R}^n)$ des fonctions de classe C^m sur $\mathbb{R}^n$.

Pour ces trois propriétés on renvoie à [4].

Le théorème suivant est essentiel.

Théorème 2.33 Pour tout $s \in \mathbb{R}$, $C_0^\infty(\mathbb{R}^n)$ et $\mathscr{S}(\mathbb{R}^n)$ sont denses dans $H^s(\mathbb{R}^n)$.

Preuve. Soit $s \in \mathbb{R}$. Dans la Proposition 2.2, on a vu que $C_0^\infty(\mathbb{R}^n)$ est dense dans $\mathscr{S}(\mathbb{R}^n)$ donc il suffit de montrer que $\mathscr{S}(\mathbb{R}^n)$ est dense dans $H^s(\mathbb{R}^n)$.

Par définition de $H^s(\mathbb{R}^n)$, l'application $\mathscr{I} : u \mapsto \left(1 + |\xi\|^2\right)^s \mathscr{F}u$ est une isométrie de $H^s(\mathbb{R}^n)$ dans $L^2(\mathbb{R}^n)$. L'espace $\mathscr{S}(\mathbb{R}^n)$ est dense dans $L^2(\mathbb{R}^n)$ donc $\mathscr{I}^{-1}\big(\mathscr{S}(\mathbb{R}^n)\big)$ est dense dans $H^s(\mathbb{R}^n)$. On peut vérifier que, pour tout $v \in \mathscr{S}(\mathbb{R}^n)$, l'application $\xi \in \mathbb{R}^n \mapsto \left(1 + |\xi\|^2\right)^s v(\xi)$ appartient à $\mathscr{S}(\mathbb{R}^n)$. On en déduit que $\mathscr{I}^{-1}\big(\mathscr{S}(\mathbb{R}^n)\big) = \mathscr{S}(\mathbb{R}^n)$ donc $\mathscr{S}(\mathbb{R}^n)$ est dense dans $H^s(\mathbb{R}^n)$. □

Exercice 2.34 Soit $s \in \mathbb{R}$. Soit A un opérateur différentiel d'ordre m à coefficients constants. Montrer qu'il existe une constante $C > 0$ telle que pour toute fonction $u \in C_0^\infty(\mathbb{R}^n)$,

$$\|Au\|_s \leq C\|u\|_{s+m}.$$

En déduire que A s'étend en un opérateur borné unique de $H^s(\mathbb{R}^n)$ dans $H^{s-m}(\mathbb{R}^n)$.

Ce résultat se généralise à un opérateur pseudo-différentiel. Pour en donner une démonstration nous aurons besoin de deux lemmes. Le premier est élémentaire et le deuxième est un résultat connu sur le produit de convolution.

Lemme 2.35 Pour tout $s \in \mathbb{R}$, pour tous $\eta, \xi \in \mathbb{R}^n$,

$$\left(1 + \|\eta\|^2\right)^{s/2} \left(1 + \|\xi\|^2\right)^{-s/2} \leq 2^{|s|/2} \left(1 + \|\eta - \xi\|^2\right)^{|s|/2}.$$

Preuve. Il suffit de remarquer que $1+\|\xi\|^2 \leq 2\big(1+\|\eta\|^2\big)\big(1+\|\xi-\eta\|^2\big)$. Si $s<0$ on utilise directement cette inégalité. Si $s>0$ on utilise l'inégalité obtenue en intervertissant ξ et η. □

Lemme 2.36 (Inégalité de Young) Soient f et g deux fonctions sur $\mathbb{R}^n$. Si f est de carré intégrable et si g est intégrable alors le produit de convolution $f*g$ est de carré intégrable et

$$\|f*g\|_{L^2} \leq \|f\|_{L^2}\|g\|_{L^1}.$$

Preuve.

$$\begin{aligned}\big(f*g\big)(x) &= \int f(x-y)g(y)\mathrm{d}y \\ \|f*g\|_{L^2}^2 &= \int \left|\int f(x-y)g(y)dy\right|^2 \mathrm{d}x.\end{aligned}$$

Par l'inégalité de Cauchy-Schwarz, on a

$$\begin{aligned}\left|\int f(x-y)g(y)\mathrm{d}y\right|^2 &\leq \left(\int |f(x-y)|\cdot|g(y)|\mathrm{d}y\right)^2 \\ &\leq \int |f(x-y)|^2\cdot|g(y)|\int |g(y)|\mathrm{d}y\end{aligned}$$

donc

$$\begin{aligned}\int\left|\int f(x-y)g(y)\mathrm{d}y\right|^2\mathrm{d}x &\leq \int |g(y)|\mathrm{d}y\int |g(y)|\left(\int |f(x-y)|^2\mathrm{d}x\right)\mathrm{d}y \\ &\leq \int |g(y)|\mathrm{d}y\int |g(y)|\left(\int |f(x)|^2\mathrm{d}x\right)\mathrm{d}y \\ &\leq \left(\int |g(y)|\mathrm{d}y\right)^2\int |f(x)|^2\mathrm{d}x\end{aligned}$$

i. e. $\|f*g\|_{L^2}^2 \leq \|g\|_{L^1}^2\cdot\|f\|_{L^2}^2$. □

Théorème 2.37 Soit $m\in\mathbb{R}$ et soit $\sigma\in\mathrm{Symb}^m(\mathbb{R}^n)$. Soit $s\in\mathbb{R}$.

1) Il existe une constante $C>0$ telle que

$$\forall u\in\mathscr{S}(\mathbb{R}^n)\ \ \left\|Op\big(\sigma\big)(u)\right\|_s \leq C\|u\|_{s+m}.$$

2) Il existe un prolongement borné unique de $Op(\sigma)$ à $H^s(\mathbb{R}^n)$ à valeurs dans $H^{s-m}(\mathbb{R}^n)$.

Preuve. Nous présentons une preuve lorsque σ est à support compact K en sa première variable.

1) Soit $u \in \mathscr{S}(\mathbb{R}^n)$. On a

$$\left\|Op(\sigma)(u)\right\|_s^2 = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(1+\|\eta\|^2\right)^s\left|\mathscr{F}\Big(Op(\sigma)(u)\Big)(\eta)\right|^2 \mathrm{d}\eta$$
$$\|u\|_{s+m}^2 = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(1+\|\xi\|^2\right)^{s+m}\left|\mathscr{F}u(\xi)\right|^2 \mathrm{d}\xi.$$

On remarque que

$$\begin{aligned}\mathscr{F}\Big(Op(\sigma)(u)\Big)(\eta) &= \int_{\mathbb{R}^n} \mathrm{e}^{-it\cdot\eta}\,Op(\sigma)(u)(t)\mathrm{d}t\\ &= \int_{\mathbb{R}^n}\mathscr{F}u(\xi)\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathrm{e}^{-it\cdot(\eta-\xi)}\sigma(t,\xi)\mathrm{d}t\right)\mathrm{d}\xi\end{aligned}$$

donc

$$\mathscr{F}\Big(Op(\sigma)(u)\Big)(\eta) = (2\pi)^{-n}\int_{\mathbb{R}^n}\mathscr{F}u(\xi)\mathscr{F}\big(\sigma(\cdot,\xi)\big)(\eta-\xi)\mathrm{d}\xi.$$

Soit N un entier naturel quelconque.

Dans la dernière intégrale on cherche à majorer $\left|\mathscr{F}\big(\sigma(\cdot,\xi)\big)(\eta-\xi)\right|$. Pour tout multi-indice α, on a

$$\eta^\alpha\mathscr{F}\big(\sigma(\cdot,\xi)\big)(\eta) = \int_K \mathrm{e}^{-i\eta\cdot t}D_t^\alpha\sigma(t,\xi)\mathrm{d}t$$

donc

$$\left|\eta^\alpha\mathscr{F}\big(\sigma(\cdot,\xi)\big)(\eta)\right| \le \int_K\left|D_t^\alpha\sigma(t,\xi)\right|\mathrm{d}t$$

et comme σ est un symbole d'ordre m il existe une constante $C_\alpha > 0$ telle que pour tout $t\in\mathbb{R}^n$ $\left|D_t^\alpha\sigma(t,\xi)\right| \le C_\alpha\left(1+\|\xi\|^2\right)^{m/2}$ donc il existe une constante $C_{\alpha,K}$ telle que pour tous $\eta,\xi\in\mathbb{R}^n$

$$\left|\eta^\alpha\mathscr{F}\big(\sigma(\cdot,\xi)\big)(\eta)\right| \le C_{\alpha,K}\left(1+\|\xi\|^2\right)^{m/2}.$$

Puisque $\left(1+\|\eta\|^2\right)^{N/2}$ est une combinaison linéaire de η^α avec $|\alpha|\le N$, il existe une constante $C_{N,K}$ telle que pour tous $\eta,\xi\in\mathbb{R}^n$

$$\left(1+\|\eta\|^2\right)^{N/2}\left|\mathscr{F}\left(\sigma(\,\cdot\,,\xi)\right)(\eta)\right|\le C_{N,K}\left(1+\|\xi\|^2\right)^{m/2}$$

d'où

$$\left|\mathscr{F}\left(\sigma(\,\cdot\,,\xi)\right)(\eta)\right|\le C_{N,K}\left(1+\|\xi\|^2\right)^{m/2}\left(1+\|\eta\|^2\right)^{-N/2}$$

et

$$\left|\mathscr{F}\left(\sigma(\,\cdot\,,\xi)\right)(\eta-\xi)\right|\le C_{N,K}\left(1+\|\xi\|^2\right)^{m/2}\left(1+\|\eta-\xi\|^2\right)^{-N/2}.$$

On aboutit ainsi à la majoration suivante

$$\begin{aligned}\left|\mathscr{F}\Big(Op(\sigma)(u)\Big)(\eta)\right|\le(2\pi)^{-n}C_{N,K}\int_{\mathbb{R}^n}\left|\mathscr{F}u(\xi)\right|\,\left(1+\|\eta-\xi\|^2\right)^{-N/2}&\\ \times\left(1+\|\xi\|^2\right)^{m/2}\mathrm{d}\xi.&\end{aligned}$$

Soit $s\in\mathbb{R}$. On note $C'_{N,K}=(2\pi)^{-n}C_{N,K}$. Afin d'utiliser l'inégalité de Young, on écrit

$$\begin{aligned}&\int_{\mathbb{R}^n}\left|\mathscr{F}u(\xi)\right|\,\left(1+\|\eta-\xi\|^2\right)^{-N/2}\,\left(1+\|\xi\|^2\right)^{m/2}\mathrm{d}\xi\\ &\qquad=\int_{\mathbb{R}^n}\left|\mathscr{F}u(\xi)\right|\,\left(1+\|\eta-\xi\|^2\right)^{(-N+|s|)/2}\,\left(1+\|\xi\|^2\right)^{(m+s)/2}\\ &\qquad\qquad\times\left(1+\|\eta-\xi\|^2\right)^{-|s|/2}\,\left(1+\|\xi\|^2\right)^{-s/2}\mathrm{d}\xi.\end{aligned}$$

D'après le Lemme 2.35, on a

$$\left(1+\|\eta-\xi\|^2\right)^{-|s|/2}\,\left(1+\|\xi\|^2\right)^{-s/2}\le 2^{|s|/2}\,\left(1+\|\eta\|^2\right)^{-s/2}$$

donc

$$\begin{aligned}&\left(1+\|\eta\|^2\right)^{s/2}\left|\mathscr{F}\Big(Op(\sigma)(u)\Big)(\eta)\right|\\ &\qquad\le 2^{|s|/2}C'_{N,K}\int_{\mathbb{R}^n}\left|\mathscr{F}u(\xi)\right|\,\left(1+\|\eta-\xi\|^2\right)^{(-N+|s|)/2}\,\left(1+\|\xi\|^2\right)^{(m+s)/2}\mathrm{d}\xi\end{aligned}$$

$$\text{donc:} \quad \int_{\mathbb{R}^n} \left(1+\|\eta\|^2\right)^s \left|\mathscr{F}\Big(Op(\sigma)(u)\Big)(\eta)\right|^2 \mathrm{d}\eta$$
$$\leq 2^{|s|} C'^2_{N,K} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\mathscr{F}u(\xi)| \left(1+\|\eta-\xi\|^2\right)^{(-N+|s|)/2} \left(1+\|\xi\|^2\right)^{(m+s)/2} \mathrm{d}\xi \right)^2 \mathrm{d}\eta.$$

obtenu en élevant au carré et en intégrant. Le premier membre de l'inégalité est $\left\|Op(\sigma)(u)\right\|_s^2$ au facteur multiplicatif près $(2\pi)^n$ et dans le deuxième membre on lit la norme L^2 au carré du produit de convolution $f * g$ avec

$$f(\xi) = \left|\mathscr{F}u(\xi)\right| \left(1+\|\xi\|^2\right)^{(m+s)/2} \quad \text{et} \quad g(\xi) = \left(1+\|\xi\|^2\right)^{(-N+|s|)/2}.$$

En appliquant l'inégalité de Young $\|f*g\|_{L^2}^2 \leq \|f\|_{L^2}^2 \cdot \|g\|_{L^1}^2$ on obtient

$$(2\pi)^n \left\|Op(\sigma)(u)\right\|_s^2 \leq 2^{|s|} C'^2_{N,K} \int_{\mathbb{R}^n} \left|\mathscr{F}u(\xi)\right|^2 \left(1+\|\xi\|^2\right)^{m+s} \mathrm{d}\xi$$
$$\times \int_{\mathbb{R}^n} \left(1+\|\xi\|^2\right)^{(-N+|s|)/2} \mathrm{d}\xi$$

d'où

$$\left\|Op(\sigma)(u)\right\|_s^2 \leq 2^{|s|} C'^2_{N,K} \|u\|_{s+m}^2 \int_{\mathbb{R}^n} \left(1+\|\xi\|^2\right)^{(-N+|s|)/2} \mathrm{d}\xi.$$

Comme N est un entier naturel quelconque, il suffit de le choisir strictement supérieur à $n+|s|$ pour assurer la convergence de l'intégrale et obtenir ainsi une constante C indépendante de u telle que $\left\|Op(\sigma)(u)\right\|_s \leq C\|u\|_{s+m}$.

2) On conclut par densité de $\mathscr{S}(\mathbb{R}^n)$ dans $H^{s+m}(\mathbb{R}^n)$. □

Dans le cas général, on ne peut pas conclure simplement en approchant σ par une suite de symboles à supports compacts. Dans la littérature, on commence par prouver le résultat pour σ d'ordre 0. Cela se fait en utilisant la notion d'adjoint d'un opérateur et des estimations de noyaux d'opérateurs intégraux. Puis on passe à un ordre quelconque par la composition d'opérateurs pseudo-différentiels. Pour un schéma de preuve nous renvoyons à [1], Proposition 5.2.

. . . .

6 Extension des opérateurs pseudo-différentiels aux distributions tempérées

Dans ce paragraphe, on étend les opérateurs pseudo-différentiels à $\mathscr{S}'(\mathbb{R}^n)$ en des opérateurs linéaires continus. On établira le lien avec l'extension aux espaces de Sobolev. Nous commençons par des rappels sur les distributions de $\mathbb{R}^n$.

Si on munit $C_0^\infty(\mathbb{R}^n)$ de la topologie induite par celle de $\mathscr{S}(\mathbb{R}^n)$, une forme linéaire u sur $C_0^\infty(\mathbb{R}^n)$ est continue si et seulement si

$$\exists m \in \mathbb{N}, \exists C > 0 \ : \ \forall \varphi \in C_0^\infty(\mathbb{R}^n), \ \left|\langle u, \varphi \rangle\right| \leq C N_m(\varphi).$$

Sans préciser la topologie de $C_0^\infty(\mathbb{R}^n)$ considérée (cf [8] pour une définition mais on n'en a pas besoin dans ce cours), ce n'est pas cette notion de continuité qu'on prend pour une forme linéaire sur $C_0^\infty(\mathbb{R}^n)$ mais la suivante:

Définition 2.38 On dit qu'une forme linéaire u sur $C_0^\infty(\mathbb{R}^n)$ est continue si pour tout compact K de $\mathbb{R}^n$, il existe un entier $m_K \in \mathbb{N}$ et une constante $C_K > 0$ tels que pour toute fonction $\varphi \in C_0^\infty(\mathbb{R}^n)$ de support contenu dans K

$$\left|\langle u, \varphi \rangle\right| \leq C_K \sum_{|\beta| \leq m_K} \sup_{x \in \mathbb{R}^n} \left|\partial_x^\beta \varphi(x)\right| \quad \text{i. e.} \quad \left|\langle u, \varphi \rangle\right| \leq C_K \sum_{|\beta| \leq m_K} \|\varphi\|_{0,\beta}.$$

Si m_K ne dépend pas de K, $m_K = m$ pour tout compact K de $\mathbb{R}^n$, on dit que la forme linéaire u est d'ordre m.

On note $\mathscr{D}'(\mathbb{R}^n)$ l'espace des formes linéaires continues sur $C_0^\infty(\mathbb{R}^n)$ et on dit que $\mathscr{D}'(\mathbb{R}^n)$ est l'espace des distributions sur $\mathbb{R}^n$.

On appelle support de $u \in \mathscr{D}'(\mathbb{R}^n)$ le complémentaire du plus grand ouvert V de $\mathbb{R}^n$ tel que $u\big|_V = 0$. La restriction de u à V est définie par

$$\forall \varphi \in C_0^\infty(V), \ \langle u_{|V}, \varphi \rangle = \langle u, \widetilde{\varphi} \rangle$$

où $\widetilde{\varphi} \in C_0^\infty(\mathbb{R}^n)$ est le prolongement de φ par 0 sur $\mathbb{R}^n \setminus V$.

On note $\mathscr{E}'(\mathbb{R}^n)$ le sous-espace de $\mathscr{D}'(\mathbb{R}^n)$ constitué des distributions à supports compacts. On rappelle que toute distribution de $\mathscr{E}'(\mathbb{R}^n)$ est d'ordre fini $m \in \mathbb{N}$.

Pour la topologie de $C^\infty(\mathbb{R}^n)$ donnée par la famille de semi-normes

$$N_{K,m}(\varphi) = \sup_{x\in K} \sum_{|\alpha|\le m} \left|\partial_x^\alpha \varphi(x)\right|,$$

K parcourant les compacts de $\mathbb{R}^n$ et m parcourant $\mathbb{N}$, $\mathscr{E}'(\mathbb{R}^n)$ s'identifie au dual de $C^\infty(\mathbb{R}^n)$.

Proposition 2.39

1) $\mathscr{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$.

2) Pour tout $u \in \mathscr{E}'(\mathbb{R}^n)$, il existe $s \in \mathbb{R}$ tel que $u \in H^s(\mathbb{R}^n)$.

3) $\bigcap_{s\in\mathbb{R}} H^s(R^n) \subset C^\infty(\mathbb{R}^n)$.

Preuve. 1) Montrons que $\mathscr{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$. On a l'inclusion $\mathscr{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$ donc une forme linéaire sur $C^\infty(\mathbb{R}^n)$ est une forme linéaire sur $\mathscr{S}(\mathbb{R}^n)$. Pour tout compact K de $\mathbb{R}^n$, pour tout $m \in \mathbb{N}$, pour toute fonction $\varphi \in \mathscr{S}(\mathbb{R}^n)$, on a clairement $N_{K,m}(\varphi) \le N_m(\varphi)$ donc $\mathscr{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$.

2) Soit $u \in \mathscr{E}'(\mathbb{R}^n)$ et K le support compact de u. Il existe $m \in \mathbb{N}$ et une constante $C > 0$ (m et C dépendant de u) tels que:

$$\forall \varphi \in C_0^\infty(\mathbb{R}^n),\ \left|\langle u,\varphi\rangle\right| \le C \sum_{|\alpha|\le m} \sup_{x\in\mathbb{R}^n} \left|\partial_x^\alpha \varphi(x)\right|.$$

Soit $s \in \mathbb{R}$ tel que $s > m + {}^{n}\!/_{2}$. Pour tout $\alpha \in \mathbb{N}^n$ tel que $|\alpha| \le m$, en appliquant $\mathscr{F}^{-1} \circ \mathscr{F}$ à $\partial_x^\alpha \varphi$ on a

$$\partial_x^\alpha \varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \xi^\alpha \mathscr{F}\varphi(\xi) \mathrm{e}^{ix\cdot\xi} \mathrm{d}\xi$$

donc

$$\left|\partial_x^\alpha \varphi(x)\right| \le \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left|\xi^\alpha\right| \left(1+\|\xi\|^2\right)^{-s/2} \left|\mathscr{F}\varphi(\xi)\right| \left(1+\|\xi\|^2\right)^{s/2} \mathrm{d}\xi$$

$$\le \frac{1}{(2\pi)^n} \left(\int_{\mathbb{R}^n} \left|\xi^\alpha\right|^2 \left(1+\|\xi\|^2\right)^{-s} \mathrm{d}\xi\right)^{1/2} \left(\int_{\mathbb{R}^n} \left|\mathscr{F}\varphi(\xi)\right|^2 \left(1+\|\xi\|^2\right)^{s} \mathrm{d}\xi\right)^{1/2}.$$

Comme $\left|\xi^\alpha\right|^2 \leq \left(1+\|\xi\|^2\right)^{|\alpha|}$ et comme $2|\alpha|-2s<-n$, l'intégrale $\int_{\mathbb{R}^n} \left|\xi^\alpha\right|^2 \left(1+\|\xi\|^2\right)^{-s} \mathrm{d}\xi$ converge donc il existe $C'>0$ tel que

$$\forall \varphi \in C_0^\infty(\mathbb{R}^n),\ \left|\langle u,\varphi\rangle\right| \leq CC'\|\varphi\|_s.$$

Par densité de $C_0^\infty(\mathbb{R}^n)$ dans $H^s(\mathbb{R}^n)$ on en déduit que u appartient au dual de $H^s(\mathbb{R}^n)$. Comme le dual de $H^s(\mathbb{R}^n)$ s'identifie à $H^{-s}(\mathbb{R}^n)$, u appartient à $H^{-s}(\mathbb{R}^n)$.

3) On a rappelé que pour tout $m \in \mathbb{N}$ et pour tout $s > m+n/2$, $H^s(\mathbb{R}^n) \subset C^m(\mathbb{R}^n)$ donc $\bigcap\limits_{s\in\mathbb{R}} H^s(R^n) \subset \bigcap\limits_{m\in\mathbb{N}} C^m(\mathbb{R}^n)$. □

Soit $\sigma \in \mathrm{Symb}(\mathbb{R}^n)$ et $A = Op(\sigma)$ l'opérateur pseudo-différentiel associé. On rappelle que A préserve $\mathscr{S}(\mathbb{R}^n)$ et que l'espace $\mathscr{S}(\mathbb{R}^n)$ agit sur lui même par la formule de dualité

$$\forall u,v \in \mathscr{S}(\mathbb{R}^n),\ \langle u,v\rangle = \int_{\mathbb{R}^n} u(x)v(x)\mathrm{d}x.$$

On cherche à définir un opérateur ${}^tA$ agissant sur $\mathscr{S}(\mathbb{R}^n)$ par

$$\forall u,v \in \mathscr{S}(\mathbb{R}^n),\ \langle Au,v\rangle = \langle u,{}^tAv\rangle.$$

Si ${}^tA$ existe, on appelle ${}^tA$ le transposé formel de A.

Exercice 2.40 Soit A un opérateur différentiel d'ordre m et de symbole $\sigma(A) \in \mathrm{Symb}^m(\mathbb{R}^n)$, $\sigma\big(A\big)(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$. Vérifier que le transposé formel de A est donné par

$$\forall v \in \mathscr{S}(\mathbb{R}^n),\ {}^tAv(x) = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D_x^\alpha\big(a_\alpha(x)v(x)\big).$$

En déduire que le symbole de ${}^tA$ est $\sigma\big({}^tA\big)(x,\xi) = \sum_{|\beta|\leq m} \frac{1}{\beta!}\partial_\xi^\beta D_x^\beta \sigma\big(A\big)(x,-\xi)$.

Nous admettrons le résultat suivant de démonstration délicate (cf [1]):

Théorème 2.41 Soit $\sigma \in \mathrm{Symb}(\mathbb{R}^n)$ et $A = Op(\sigma)$ l'opérateur pseudo-différentiel associé. Il existe un opérateur pseudo-différentiel ${}^tA$ tel que $\forall u,v \in \mathscr{S}(\mathbb{R}^n),\ \langle Au,v\rangle = \langle u,{}^tAv\rangle$. ${}^tA$ a même ordre que A et le symbole de ${}^tA$ est donné par

$$\sigma\big({}^tA\big)(x,\xi) = \sum_{|\beta|\leq N} \frac{1}{\beta!}\partial_\xi^\beta D_x^\beta \sigma\big(A\big)(x,-\xi) + r_N(x,\xi)$$

où pour tout $N \in \mathbb{N}$, $r_N \in \mathrm{Symb}^{m-N-1}(\mathbb{R}^n)$.

De même qu'on a défini la transformée de Fourier sur $\mathscr{S}'(\mathbb{R}^n)$ par transposition, on note $({}^tA)'$ l'opérateur linéaire continu de $\mathscr{S}'(\mathbb{R}^n)$ dans $\mathscr{S}'(\mathbb{R}^n)$ muni de la topologie de la convergence simple tel que

$$\forall u \in \mathscr{S}'(\mathbb{R}^n), \forall \varphi \in \mathscr{S}(\mathbb{R}^n),\ \big\langle ({}^tA)'u, \varphi \big\rangle = \langle u, {}^tA\varphi \rangle.$$

Proposition 2.42 L'opérateur linéaire continu $({}^tA)'$ prolonge à $\mathscr{S}'(\mathbb{R}^n)$ l'opérateur pseudo-différentiel A de $\mathscr{S}(\mathbb{R}^n)$.

Preuve. Soit $v \in \mathscr{S}(\mathbb{R}^n)$ et T_v la distribution tempérée associée:

$$\forall \varphi \in \mathscr{S}(\mathbb{R}^n),\ \langle T_v, \varphi \rangle = \int_{\mathbb{R}^n} v(x)\varphi(x)\mathrm{d}x.$$

Soit $\varphi \in \mathscr{S}(\mathbb{R}^n)$. On a

$$\big\langle ({}^tA)'T_v, \varphi \big\rangle = \langle T_v, {}^tA\varphi \rangle = \int_{\mathbb{R}^n} v(x)\big({}^tA\varphi\big)(x)\mathrm{d}x = \int_{\mathbb{R}^n} \big(Av\big)(x)\varphi(x)\mathrm{d}x$$

donc $({}^tA)'T_v = T_{Av}$ donc $({}^tA)'$ coïncide avec A sur $\mathscr{S}(\mathbb{R}^n)$. □

Proposition 2.43 L'opérateur $({}^tA)'$ est l'unique prolongement linéaire continu de A à $\mathscr{S}(\mathbb{R}^n)$.

Preuve. C'est immédiat par densité de $\mathscr{S}(\mathbb{R}^n)$ dans $\mathscr{S}'(\mathbb{R}^n)$. □

Soit $s \in \mathbb{R}$ et soit $\widetilde{A}^s$ l'unique prolongement borné de A allant de $H^s(\mathbb{R}^n)$ dans $H^{s-m}(\mathbb{R}^n)$. Comme $\mathscr{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ et comme $\mathscr{S}(\mathbb{R}^n)$ est dense dans $\mathscr{S}'(\mathbb{R}^n)$, $\mathscr{S}(\mathbb{R}^n)$ est dense dans $H^s(\mathbb{R}^n)$. Par ailleurs, $H^{s-m}(\mathbb{R}^n)$ se plonge dans $\mathscr{S}'(\mathbb{R}^n)$. Donc $\widetilde{A}^s$ s'étend en un unique opérateur continu $\widetilde{A}$ de $\mathscr{S}'(\mathbb{R}^n)$ dans $\mathscr{S}'(\mathbb{R}^n)$. Il est clair que $\widetilde{A}$ restreint à $\mathscr{S}(\mathbb{R}^n)$ coïncide avec A donc $\widetilde{A} = ({}^tA)'$.

Par la suite on notera plus simplement $\widetilde{A}$ l'unique prolongement linéaire continu d'un opérateur pseudo-différentiel A à $\mathscr{S}'(\mathbb{R}^n)$.

Lorsque A est différentiel, on obtient directement $\widetilde{A}$ en prolongeant la dérivation aux distributions tempérées par

$$\forall \alpha \in \mathbb{N}^n, \forall u \in \mathscr{S}'(\mathbb{R}^n), \forall \varphi \in \mathscr{S}(\mathbb{R}^n)\ \big\langle \partial^\alpha u, \varphi \big\rangle = (-1)^{|\alpha|}\big\langle u, \partial^\alpha \varphi \big\rangle.$$

Exercice 2.44 Soit $a \in C^\infty(\mathbb{R}^n)$ tel que pour tout $\alpha \in \mathbb{N}^n$, $\partial^\alpha a$ est bornée sur $\mathbb{R}^n$.

a) Montrer que la forme linéaire *au* définie par

$$\forall \varphi \in \mathscr{S}(\mathbb{R}^n),\ \ \langle au, \varphi \rangle = \langle u, a\varphi \rangle$$

est une distribution tempérée.

b) Vérifier que pour tout $\alpha \in \mathbb{N}^*$, la dérivation ∂^α étendue à $\mathscr{S}'(\mathbb{R}^n)$ est linéaire continue de $\mathscr{S}'(\mathbb{R}^n)$ dans $\mathscr{S}'(\mathbb{R}^n)$.

c) Vérifier que l'application linéaire $u \mapsto au$ de $\mathscr{S}'(\mathbb{R}^n)$ dans $\mathscr{S}'(\mathbb{R}^n)$ est continue.

d) Soit A un opérateur différentiel sur $\mathbb{R}^n$ de symbole $\sigma(A) \in \mathrm{Symb}(\mathbb{R}^n)$. Déduire des questions précédentes que A s'étend en un opérateur différentiel continu de $\mathscr{S}'(\mathbb{R}^n)$ dans $\mathscr{S}'(\mathbb{R}^n)$.

Proposition 2.45 Soit $\sigma \in \mathrm{Symb}(\mathbb{R}^n)$ d'ordre $-\infty$ et soit $A = Op(\sigma)$. On a

$$\widetilde{A}(\mathscr{E}'(\mathbb{R}^n)) \subset C^\infty(\mathbb{R}^n).$$

Preuve. On sait qu'il existe $s \in \mathbb{R}$ tel que $\mathscr{E}'(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$. Donc $\widetilde{A}(\mathscr{E}'(\mathbb{R}^n)) \subset \widetilde{A}(H^s(\mathbb{R}^n))$. Or $\widetilde{A}(H^s(\mathbb{R}^n)) = \widetilde{A}^s\big(H^s(\mathbb{R}^n)\big)$ et A est d'ordre $-\infty$ donc $\widetilde{A}^s\big(H^s(\mathbb{R}^n)\big) \subset \bigcap_{s\in\mathbb{R}} H^s(\mathbb{R}^n)$. Comme $\bigcap_{s\in\mathbb{R}} H^s(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$, on a le résultat. □

Remarque 2.46 C'est pour cela qu'on dit qu'un opérateur pseudo-différentiel de symbole d'ordre $-\infty$ est régularisant.

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7 Composition d'opérateurs pseudo-différentiels

Proposition 2.47 Soient deux opérateurs différentiels D_1 et D_2 de symboles respectifs σ_1 et σ_2. On a

$$\sigma\big(D_2 D_1\big)(x,\xi) = \sum_{\mu\in\mathbb{N}^n} \frac{\partial_\xi^\mu \sigma_2(x,\xi) D_x^\mu \sigma_1(x,\xi)}{\mu!},$$

la somme ne comportant qu'un nombre fini de termes.

Preuve. On utilise la formule de Leibniz

$$D_x^\alpha(f\times g)=\sum_{\mu\leq\alpha}C_\alpha^\mu D_x^{\alpha-\mu}fD_x^\mu g.$$

Soient $D_1=\sum_{|\alpha|\leq m_1}a_\alpha(x)D_x^\alpha$, $D_2=\sum_{|\beta|\leq m_2}b_\beta(x)D_x^\beta$. On a d'une part

$$\begin{aligned}D_2D_1&=\sum_{|\beta|\leq m_2}b_\beta(x)\sum_{|\alpha|\leq m_1}D_x^\beta\left(a_\alpha(x)D_x^\alpha\right)\\&=\sum_{|\beta|\leq m_2}b_\beta(x)\sum_{|\alpha|\leq m_1}\sum_{\mu\leq\beta}C_\beta^\mu D_x^\mu a_\alpha(x)D_x^{\alpha+\beta-\mu}\\&=\sum_{|\beta|\leq m_2}b_\beta(x)\sum_{|\alpha|\leq m_1}\sum_{\mu\leq\beta}C_\beta^\mu D_x^\mu a_\alpha(x)\xi^{\alpha+\beta-\mu},\end{aligned}$$

et d'autre part

$$\begin{aligned}\sum_{\mu\in\mathbb{N}^n}\partial_\xi^\mu\sigma_2(x,\xi)D_x^\mu\sigma_1(x,\xi)/\mu!&=\sum_{\mu\in\mathbb{N}^n}\partial_\xi^\mu\left(\sum_{|\beta|\leq m_2}b_\beta(x)\xi^\beta\right)D_x^\mu\left(\sum_{|\alpha|\leq m_1}a_\alpha(x)\xi^\alpha\right)\\&=\sum_{\mu\leq\beta}\sum_{|\beta|\leq m_2}b_\beta(x)\frac{\beta!}{(\beta-\mu)!}\xi^{\beta-\mu}\sum_{|\alpha|\leq m_1}\frac{\xi^\alpha D_x^\mu a_\alpha(x)}{\mu!}.\end{aligned}$$

Donc le résultat est clair. □

Soient A et B deux opérateurs pseudo-différentiels de symboles respectif a et b. Un calcul direct de $\sigma(BA)$ conduit à l'expression formelle

$$\sigma\big(BA\big)(x,\xi)=(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}b(x,\eta)\,a(y,\xi)\mathrm{e}^{i(x-y)(\eta-\xi)}\mathrm{d}\eta\mathrm{d}y.$$

Si m' l'ordre de B n'est pas strictement inférieur à $-n$ on n'est pas assuré de l'existence de l'intégrale double écrite ci-dessus. On peut lui donner un sens en utilisant la notion d'intégrale oscillante. Une intégrale oscillante est une distribution. En utilisant la formule de Taylor avec reste intégral, on montre que $\sigma(BA)$ est donné par le développement asymptotique suivant:

$$\sigma(BA) \sim \sum_{\mu \in \mathbb{N}^n} \frac{\partial_\xi^\mu b(x,\xi) D_x^\mu a(x,\xi)}{\mu!}$$

où $\mu! = \mu_1! \cdots \mu_n!$ et $\sim$ signifie que pour tout $N \in \mathbb{N}$

$$\sigma(BA) - \sum_{|\mu| \leq N} \frac{\partial_\xi^\mu b(x,\xi) D_x^\mu a(x,\xi)}{\mu!}$$

est un symbole d'ordre $m + m' - N - 1$ (ce résultat n'est pas facile à établir). Ici la terminologie "développement asymptotique" est à comprendre au sens de l'ordre de symbole lorsque $|\xi|$ tend vers $+\infty$. On remarquera que

$$\begin{array}{ll} ba & \text{est d'ordre } m+m', \\ \displaystyle\sum_{|\mu|=1} \partial_\xi^\mu b(x,\xi) D_x^\mu a(x,\xi) & \text{est d'ordre } m-1+m' \\ \vdots & \\ \displaystyle\sum_{|\mu|=N} \partial_\xi^\mu b(x,\xi) D_x^\mu a(x,\xi) & \text{est d'ordre } m-N+m'. \end{array}$$

Grâce aux développements asymptotiques de $\sigma(BA)$ et $\sigma(AB)$ on voit que si A est d'ordre m et si B est d'ordre m', alors BA et AB sont d'ordre $m+m'$. L'ordre du produit de composition de deux opérateurs pseudo-différentiels est la somme des ordres des opérateurs pseudo-différentiels. En particulier, si A et R sont deux opérateurs pseudo-différentiels et si R est d'ordre $-\infty$ alors AR et RA sont d'ordre $-\infty$. Nous terminons ce paragraphe en donnant la définition du calcul symbolique, autrement dit en posant une loi multiplicative sur les symboles héritée de la composition des opérateurs pseudo-différentiels.

Définition 2.48 Soient σ_1 et σ_2 deux symboles dans $\mathrm{Symb}(\mathbb{R}^n)$. On pose

$$\big(\sigma_2 \star \sigma_1\big)(x,\xi) = \sum_{\mu \in \mathbb{N}^n} \frac{\partial_\xi^\mu \sigma_2(x,\xi) D_x^\mu \sigma_1(x,\xi)}{\mu!}.$$

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8 Inversion approchée d'un opérateur différentiel elliptique

Dans cette section on considère un opérateur différentiel A d'ordre m sur $\mathbb{R}^n$ de symbole dans $\mathrm{Symb}(\mathbb{R}^n)$, $\sigma\big(A\big)(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$ où tous les coefficients a_α sont bornés ainsi que leurs dérivées de tous ordres.

Définition 2.49 Dans le symbole de A, la composante homogène de degré maximal s'appelle symbole principal de A qu'on notera $\sigma_P(A)$. Ainsi

$$\sigma_P\big(A\big)(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

Définition 2.50 On dit que A est elliptique si son symbole principal vérifie

$$\forall x \in \mathbb{R}^n,\ \forall \xi \in \mathbb{R}^n, \xi \neq 0, \quad \sigma_P\big(A\big)(x,\xi) \neq 0.$$

On suppose que A est elliptique.

Théorème 2.51 Pour tout $N \in \mathbb{N}^*$ il existe deux opérateurs pseudo-différentiels B_N et B'_N d'ordre $-m$ tels que

$$AB_N - \mathrm{Id} \quad \text{et} \quad B'_N A - \mathrm{Id}$$

soient des opérateurs pseudo-différentiels d'ordre $-N$.

Preuve. On pose $\sigma\big(A\big)(x,\xi) = a_m(x,\xi) + r_1(x,\xi)$ avec $a_m = \sigma_P(A)$.

Soit $\varphi \in C^\infty(\mathbb{R}^n)$ telle que $\varphi(\xi) = 0$ pour $|\xi| \leq 1$ et $\varphi(\xi) = 1$ pour $|\xi| \geq 2$. La fonction a_m^{-1} définie sur $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ n'est pas un symbole mais la fonction $\varphi(\xi)a_m^{-1}(x,\xi)$ prolongée par 0 pour $\xi = 0$ est un symbole d'ordre $-m$. On a

$$\begin{aligned}\sigma(A) \star \varphi(\xi)a_m^{-1}(x,\xi) = {} & \varphi(\xi)\ (a_m + r_1)\ a_m^{-1}(x,\xi) \\ & + \sum_{|\mu|\leq m, \mu\neq 0} \frac{\varphi(\xi)\partial_\xi^\mu \sigma\big(A\big)(x,\xi) D_x^\mu a_m^{-1}(x,\xi)}{\mu!}\end{aligned}$$

$$\text{donc}\quad \begin{aligned}\sigma(A) \star \varphi(\xi)a_m^{-1}(x,\xi) = {} & \varphi(\xi)\ \mathrm{Id}(x,\xi) + \varphi(\xi) r_1 a_m^{-1}(x,\xi) \\ & + \sum_{|\mu|\leq m, \mu\neq 0} \frac{\varphi(\xi)\partial_\xi^\mu \sigma\big(A\big)(x,\xi) D_x^\mu a_m^{-1}(x,\xi)}{\mu!}.\end{aligned}$$

Soit $\quad c_{-1}(x,\xi) = \varphi(\xi) r_1 a_m^{-1}(x,\xi) + \sum_{|\mu| \le m, \mu \ne 0} \dfrac{\varphi(\xi)\partial_\xi^\mu \sigma\big(A\big)(x,\xi) D_x^\mu a_m^{-1}(x,\xi)}{\mu!}.$

c_{-1} est un symbole d'ordre -1.

On pose $b_{-m} = \varphi a_m^{-1}$ et $b_{-m-1} = -a_m^{-1} c_{-1}$. L'ordre de b_{-m-1} est $-m-1$ et

$$\begin{aligned}\sigma(A) \star b_{-m-1}(x,\xi) &= \sigma(A) b_{-m-1}(x,\xi) + \sum_{|\mu| \le m, \mu \ne 0} \frac{\partial_\xi^\mu \sigma\big(A\big)(x,\xi) D_x^\mu b_{-m-1}(x,\xi)}{\mu!} \\ &= -c_{-1} - r_1 b_{-m-1} + \sum_{|\mu| \le m, \mu \ne 0} \frac{\partial_\xi^\mu \sigma\big(A\big)(x,\xi) D_x^\mu b_{-m-1}(x,\xi)}{\mu!}.\end{aligned}$$

Donc

$$\sigma(A) \star (b_{-m} + b_{-m-1}) = \varphi + c_{-2}$$

où

$$c_{-2}(x,\xi) = -r_1 b_{-m-1}(x,\xi) + \sum_{|\mu| \le m, \mu \ne 0} \frac{\partial_\xi^\mu \sigma\big(A\big)(x,\xi) D_x^\mu b_{-m-1}(x,\xi)}{\mu!}$$

est d'ordre -2. Par récurrence on définit une suite de symboles (b_{-m-N}) par: $\forall N \in \mathbb{N}^*$, $b_{-m-N} = -a_m^{-1} c_{-N}$ avec

$$c_{-N}(x,\xi) = -r_1 b_{-m-N+1}(x,\xi) + \sum_{|\mu| \le m, \mu \ne 0} \frac{\partial_\xi^\mu \sigma\big(A\big)(x,\xi) D_x^\mu b_{-m-N+1}(x,\xi)}{\mu!}.$$

On montre que, pour tout $N \in \mathbb{N}^*$, b_{-m-n} est d'ordre $-m-N$, c_{-N} est d'ordre $-N$ et $\sigma(A) \star (b_{-m} + \cdots + b_{-m-N+1}) = \varphi + c_{-N}$.

Comme $\mathrm{Id}(x,\xi) - \varphi(\xi)$ est un symbole d'ordre $-\infty$, l'opérateur de symbole $b_{-m} + \ldots + b_{-m-N+1}$ est un B_N.

De même pour l'existence de B'_N. □

Corollaire 2.52 Si A est un opérateur différentiel elliptique, il existe un opérateur pseudo-différentiel B et deux opérateurs pseudo-différentiels R_1 et R_2 d'ordre $-\infty$ tels que

$$AB = \mathrm{Id} + R_1, \quad \text{et} \quad BA = \mathrm{Id} + R_2.$$

Preuve. D'après le théorème précédent, on sait qu'il existe des opérateurs pseudo-différentiels B_1, B_2, R_1, R_2 tels que $B_1 A = \mathrm{Id} + R_1$ et $AB_2 = \mathrm{Id} + R_2$ avec R_1 et R_2 d'ordre

$-\infty$. La première égalité donne $B_1AB_2 = B_2 + R_1B_2$ et la deuxième égalité donne alors $B_1 + B_1R_2 = B_2 + R_1B_2$ donc $B_1 - B_2 = R_1B_2 - B_1R_2$ est d'ordre $-\infty$. □

Définition 2.53 L'opérateur B s'appelle une paramétrix de A.

C'est un inverse à droite et à gauche de A modulo les opérateurs d'ordre $-\infty$. On remarquera que le symbole de la paramétrix B appartient à une sous classe de la classe des symboles, celle des symboles classiques. On dit que σ d'ordre $m \in \mathbb{R}$ est un symbole classique si σ admet un développement asymptotique en composantes positivement homogènes par rapport à ξ:

$$\sigma \sim \sum_{j=0}^{+\infty} a_{m-j}$$

où pour tout $j \in \mathbb{N}$, $a_{m-j} \in \mathrm{Symb}^{m-j}(\mathbb{R}^n)$ et il existe $r > 0$ tel que tout $j \in \mathbb{N}$, pour tout $t > 0$, pour tout $\xi \in \mathbb{R}^n$ tel que $\|\xi\| \geq r$, $a_{m-j}(x, t\xi) = t^{m-j} a_{m-j}(x, \xi)$.

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9 Estimation a priori et régularité des solutions

Soit A un opérateur différentiel d'ordre $m \in \mathbb{N}^*$, de symbole dans $\mathrm{Symb}(\mathbb{R}^n)$ et elliptique.

Dans ce dernier paragraphe nous donnons un aperçu qualitatif de la résolution d'une équation aux dérivées partielles $Au = f$ en utilisant une paramétrix de A

Théorème 2.54 Pour tout $s \in \mathbb{R}$, il existe une constante $C > 0$ telle que pour tout $u \in C_0^\infty(\mathbb{R}^n)$,

$$\|u\|_s \leq C\big(\|Au\|_{s-m} + \|u\|_{s-1}\big).$$

Preuve. Soit $u \in C_0^\infty(\mathbb{R}^n)$. Comme A est différentiel, il est clair que $Au \in C_0^\infty(\mathbb{R}^n)$. Soit B une paramétrix de A: $BA = \mathrm{Id} + R$. On a $\|u\|_s \leq \|BAu\|_s + \|Ru\|_s$. Comme B est d'ordre $-m$ et R d'ordre -1, on conclut grâce au Théorème 2.37. □

Théorème 2.55 Soient $s, t \in \mathbb{R}$, $t < s + m$. Si $u \in H^t(\mathbb{R}^n)$ et si $\widetilde{A}u = f$ avec $f \in H^s(\mathbb{R}^n)$ alors $u \in H^{s+m}(\mathbb{R}^n)$.

Preuve. Soit B une paramétrix de A. On a $\widetilde{BA}u = u + \widetilde{R}u$, $\widetilde{BA} = \widetilde{B}\widetilde{A}$ et $\widetilde{A}u = f$ donc $u = \widetilde{B}f - \widetilde{R}u$. Or $f \in H^s(\mathbb{R}^n)$ et B est d'ordre $-m$ donc $\widetilde{B}f \in H^{s+m}(\mathbb{R}^n)$. L'opérateur R est d'ordre $-\infty$ donc, pour tout $k \in \mathbb{N}$, $\widetilde{R}u \in H^{t+k}(\mathbb{R}^n)$ donc, pour tout $k \in \mathbb{N}$, $u \in H^{t_k}(\mathbb{R}^n)$ avec $t_k = \min(s+m, t+k)$. Or il existe $k \in \mathbb{N}$ tel que $s + m < t + k$ donc $u \in H^{s+m}(\mathbb{R}^n)$. □

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Bibliographie

[1] S. Alinhac et P. Gérard, *Opérateurs pseudo-différentiels et théorème de Nash-Moser*, Savoirs actuels, InterEditions/Editions du CNRS, 1991.

[2] J. Dieudonné, *Eléments d'analyse*, tome 7, Gauther-Villars, 1978.

[3] J. J. Duistermaat, *Fourier Integral Operators*, Birkhäuser, 1996.

[4] G. Eskin, *Lectures on Linear Partial Differential Equations*, Graduate Studies in Mathematics, Volume 123, American Mathematical Society, 2011.

[5] Y. V. Egorov et B.-W. Schulze, *Pseudo-Differential Operators, Singularities, Applications*, Operator Theory, Advances and Applications, Volume 93, Birkhäuser, 1997.

[6] L. Hörmander, *The Analysis of Linear Partial Differential Operators I, III*, Springer-Verlag, 1985.

[7] P. Malliavin et H. Airault, *Intégration, analyse de Fourier, probabilités, analyse gaussienne*, Collection Maîtrise de mathématiques pures, MASSON, 1994.

[8] L. Schwartz, *Théories des distributions*, Hermann, 1966.

[9] M. E. Taylor, *Pseudodifferential Operators*, Princeton University Press, 1981.

[10] F. Trèves, *Introduction to pseudodifferential and Fourier Integral Operators*, The University Series in Mathematics, Plenum Press, New York, 1980.

[11] M. W. Wong, *An introduction to pseudo-differential operators*, World Scientific, 1999.

Chapter 3

An introduction to the concepts of microlocal analysis

René M. Schulz[*]

Abstract. *In this course, an elementary introduction to the concepts of microlocal analysis is given. Topics include distribution theory, pseudodifferential operators, wave front sets and oscillatory integrals.*

1 Introduction

Microlocal analysis is the field of study of a generalized function by means of localization in the spacial variables while taking the spectrum of this distribution – meaning its suitably localized Fourier transform – into account. Applied to the right classes of distributions – such as the kernels of partial differential operators – these methods often reveal a somewhat hidden geometry behind the analytical problems. The analysis of such distributions – and the problems in which they arise – may then be reformulated and often greatly simplified using methods of (symplectic) geometry.

There are many lecture notes and introductory reads available at the graduate level

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that are well-suited for a course on microlocal analysis – e. g. [8], [13], [14], [20], [24]. The present lecture notes essentially compile the topics treated during my introductory lectures for the Summer School 2015 in Ouagadougou, Burkina Faso.[1]

Their aim is to lay out an overview of microlocal techniques in a fast-paced, conceptual but rather informal style – suited for self-study and for readers with a minimal background in analysis who want to quickly learn the essential features of the theory.

The document is organized as follows. The first part of the notes are devoted to distribution theory and in particular Fourier analysis – the framework in which microlocal analysis is phrased.[2] A reader already familiar with these concepts might want to skip this section and may come back to them whenever they are referenced. The "microlocal" part of this document essentially begins with Section 2.3, where singularities of distributions are discussed.

In Section 3 we introduce pseudo-differential operators and discuss elements of their calculus from a conceptual point of view. In particular, we indicate the technique of pseudo-inversion, that is the parametrix construction.

Finally, Section 4 is devoted to oscillatory integrals, the prototype of Fourier Integral distributions, and their singularities.

In an appendix, some outlook perspectives and connections with other topics from microlocal analysis, which were treated in independent lectures, are given.

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2 Distribution theory

In the following, we will recall the notion of distributions as generalized functions and will quickly pass to the notion of their Fourier transform. We will do so on a rather informal level, skipping many key proofs. For further studies, excellent references for the theory of distributions are [6], [16], [19]. The original reference is [21]. A somewhat complete study of distributions geared towards microlocal analysis – on a very advanced level – may be found [10].

[1] See also [18] for a recollection of the extraordinary circumstances surrounding the school.

[2] Of course, there are many other ways to introduce microlocal concepts, among them hyperfunction theory, time-frequency analysis and many more.

Notation

In this document, we use multiindex notation. This means that for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we use the following notation:

$$|\alpha| := \sum_{j=1}^{d} |\alpha_j| \qquad \alpha! := \prod_{j=1}^{d} \alpha_j! \qquad x^\alpha := \prod_{j=1}^{d} x_j^{\alpha_j} \qquad \partial_x^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

It will often be useful to replace the usual partial derivative ∂_x^α by $D_x^\alpha := (-i)^{|\alpha|}\partial_x^\alpha$, where as usual $i = \sqrt{-1}$.

The basic function spaces we are going to use are continuous and smooth functions on $\mathbb{R}^d$, $\mathscr{C}(\mathbb{R}^d)$ and $\mathscr{C}^\infty(\mathbb{R}^d)$, as well as L^p-spaces. Recall that the L^p-norm $\|f\|_p$, of a locally integrable function f defined on $\mathbb{R}^d$ is given by

$$\|f\|_p := \sqrt[p]{\int_{\mathbb{R}^d} |f(x)|^p \, \mathrm{d}x} \quad \text{for } p \in [1,\infty); \qquad \|f\|_\infty := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |f(x)|.$$

The spaces $L^p(\mathbb{R}^d)$ then contain the functions[3] for which $\|f\|_p < \infty$. A special case is the space $L^2(\mathbb{R}^d)$, which is a Hilbert space for the inner product

$$(f,g) := \langle f, \overline{g} \rangle := \int_{\mathbb{R}^d} f(x)\overline{g(x)} \, \mathrm{d}x, \qquad f,g \in L^2(\mathbb{R}^d).$$

We call $\langle f, g \rangle$ the *real pairing* between functions.

2.1 Distributions as generalized functions

A formal approach to generalize differentiability

A first step in microlocal analysis is to generalize the notion of differential operator to a *pseudo-differential operator*. For that, we first have to wonder how to reformulate the concept of differentiablity and in order to do so, we consider some of the most general spaces in which it is possible to carry out differentiation.

A quite large space of functions to which partial differentiation in the usual sense – as the limit of the differential quotient – does not extend is that of $L^1(\mathbb{R}^d)$ functions. Suppose $g \in L^1(\mathbb{R}^d)$ with values in $\mathbb{C}$, we may then associate to g the application T_g,

[3] To be precise: L^p is formed by equivalence classes $[f]$ of measurable functions for which $\|f\|_p < \infty$, where the equivalence relation is given by $f \sim g$ if $f(x) = g(x)$ outside a set of measure zero.

which we want to define for some given function f – for now formally, since we don't specify a domain – as

$$f \mapsto T_g(f) := \langle g, f \rangle := \int_{\mathbb{R}^d} g(x) f(x) \, \mathrm{d}x. \tag{3.1}$$

The map T_g is a linear functional – it maps functions f to numbers and satisfies $T_g(f_1 + \lambda f_2) = T_g(f_1) + \lambda T_g(f_2)$.

Exercise 3.1 Prove that T_g is a linear functional on $L^\infty(\mathbb{R}^d)$ and that it satisfies an estimate of the form

$$|T_g(f)| \leq C \|f(x)\|_\infty,$$

meaning is continuous with respect to the norm $\|\cdot\|_\infty$.

Now suppose both g and f are differentiable (or absolutely continuous) and that they decay suitably fast as $|x| \to \infty$. Then we may integrate by parts and see that

$$T_{\partial_x^\alpha g}(f) = \int_{\mathbb{R}^d} g(x)(-\partial_x)^\alpha f(x) \, \mathrm{d}x. \tag{3.2}$$

Thus it makes sense to lift the notion of differentiability from functions to functionals by setting

$$\partial_x^\alpha T_g(f) = T_g\big((-\partial_x)^\alpha f\big). \tag{3.3}$$

Here, notice that since T_g is a functional, it is determined by its values on all functions f. In this way, we may define $\partial_x^\alpha T_g(f)$ even if g is **not** differentiable, by (3.3) – extending the notion of differentiability by passing to more general functionals.

Of course, so far, we argued on a formal level. We now examine what we have to adjust in order to obtain well-defined expressions. First, we see that if we want to differentiate at any level α, we need f to be smooth, i. e. $f \in \mathscr{C}^\infty(\mathbb{R}^d)$. Secondly, we should arrange for the boundary terms in the partial integration to vanish. For that we have to assume that $f \cdot g$ vanishes suitably fast at infinity. One way to ensure this is to ask for the support of either f or g to be compact, where we define the support of a continuous function f to be the closure of the points where f is unequal to zero:

$$\operatorname{supp}(f) := \overline{\{x \in \mathbb{R}^d \,|\, f(x) \neq 0\}}. \tag{3.4}$$

The previous considerations leads to the definition of three spaces of *test functions*. In the notation of Schwartz:

$$\mathscr{E}(\mathbb{R}^d) := \mathscr{C}^\infty(\mathbb{R}^d),$$
$$\mathscr{S}(\mathbb{R}^d) := \left\{ f \,|\, f \in \mathscr{C}^\infty(\mathbb{R}^d);\ \forall N \in \mathbb{N},\ \beta \in \mathbb{N}^d : \left\|(1+|x|)^N \partial_x^\beta f\right\|_{L^\infty(\mathbb{R}^d)} < \infty \right\},$$
$$\mathscr{D}(\mathbb{R}^d) := \mathscr{C}_c^\infty(\mathbb{R}^d) := \{ f \,|\, f \in \mathscr{C}^\infty(\mathbb{R}^d);\ \operatorname{supp}(f) \text{ is compact}\}.$$

We call these spaces the *spaces of test functions*, $\mathscr{S}(\mathbb{R}^d)$ is also called the *Schwartz space* and the *space of rapidly decaying functions*. Indeed, every element in $\mathscr{S}(\mathbb{R}^d)$ – and all its derivatives – has to decay faster than any polynomial grows.

Obviously we have

$$\mathscr{D}(\mathbb{R}^d) \subset \mathscr{S}(\mathbb{R}^d) \subset \mathscr{E}(\mathbb{R}^d). \tag{3.5}$$

We will now see in some examples that the spaces are non-trivial and not equal.

Example 3.2 The function $\exp(|x|^2)$ is an element of $\mathscr{E}(\mathbb{R}^d)$, but not of the other spaces. The function $\exp(-|x|^2)$ is an element of $\mathscr{S}(\mathbb{R}^d)$, but not of compact support. The function

$$\phi^0(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & |x| < 1 \\ 0 & \text{else} \end{cases}$$

is smooth and of compact support.

Exercise 3.3 Prove these facts. For the third, it is useful to first consider the function defined by $h(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & \text{else} \end{cases}$ and prove that it is smooth and that its derivatives are of the form $P(x^{-1})h(x)$, where $P(x)$ is a polynomial.

A fundamental tool in the study of distributions is that of localization. This is often carried out by means of a *cut-off function* or an *excision function*:

Definition 3.4 (Cut-offs and excision functions) A ϕ^{x_0} as above, i. e. a smooth function of compact support that is equal to one in a neighbourhood of $x_0 \in \mathbb{R}^d$ is called a *cut-off around* x_0.
A function of the form $\chi(x) = \left(1 - \phi^0(x)\right)$ is called an excision function.

Examples of such functions are given in Figure 3.40.

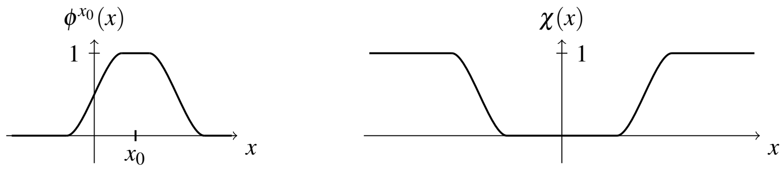

Figure 3.1: A cut-off and an excision function on $\mathbb{R}$.

We seek to define distributions as a generalization of (3.1), as the space of continuous linear functionals over the three test function spaces. For that, we first need to define a notion of convergence on these.

Topologies on test function spaces

We describe the topologies on the test function spaces by their notion of convergence. For any test function space $X \in \{\mathscr{D}, \mathscr{S}, \mathscr{E}\}$, we say that $f_n \xrightarrow{X} f$, meaning a sequence of elements $f_n \in X$ converges in the topology of X against $f \in X$, if $(f_n - f) \xrightarrow{\mathscr{D}} 0$. We now specify this convergence for the three test function spaces.

Let $f_n \in \mathscr{D}(\mathbb{R}^d)$, $n \in \mathbb{N}_0$, $f \in \mathscr{D}(\mathbb{R}^d)$. We say that $f_n \xrightarrow{\mathscr{D}} 0$ if there is a fixed compact set $K \subset \mathbb{R}^d$, such that for all f_n we have $\operatorname{supp}(f_n) \subset K$ and that for all $N \in \mathbb{N}$, $\|f_n\|_{N,K} := \sup_{x\in K} \sum_{|\alpha|\leq N} \|\partial^\alpha f\|$ converges to zero as $n \to \infty$. In words: f_n converges to zero if there is a bound for the supports of all test functions in the sequence and the functions – as well as all of their derivatives – converge uniformly to zero.

The topology on $\mathscr{S}(\mathbb{R}^d)$ is described by the expressions

$$\|f\|_{N,\alpha} := \|(1+|x|)^N \partial_x^\beta f\|_{L^\infty(\mathbb{R}^d)}$$

(for $N \in \mathbb{N},\ \beta \in \mathbb{N}^d$) that already occurred in the definition of the space. These form a family of *seminorms*, and $f_n \xrightarrow{\mathscr{S}} f$ if for all fixed $N \in \mathbb{N},\ \beta \in \mathbb{N}^d$ we have $\|f_n - f\|_{N,\alpha} \to 0$.

For $\mathscr{E}(\mathbb{R}^d)$ we say that $f_n \xrightarrow{\mathscr{E}} f$, if for all compact sets $K \subset \mathbb{R}^d$ and all $N \in \mathbb{N}$, $\|f_n\|_{N,K}$ converges to zero as $n \to \infty$.

Exercise 3.5 Prove that (3.5) is a topological inclusion, meaning that the inclusion maps are continuous. Prove furthermore, by use of a series of cut-offs with increasing support, that $\mathscr{D}(\mathbb{R}^d)$ is dense in $\mathscr{S}(\mathbb{R}^d)$ and in $\mathscr{E}(\mathbb{R}^d)$.

Spaces of distributions

We are now in the position to define the three distribution spaces we are interested in. That is we define them as the *dual spaces*

$$\begin{aligned}
\mathscr{D}'(\mathbb{R}^d):&\quad \text{The space of } \textit{distributions},\\
\mathscr{S}'(\mathbb{R}^d):&\quad \text{The space of } \textit{tempered distributions},\\
\mathscr{E}'(\mathbb{R}^d):&\quad \text{The space of } \textit{compactly supported distributions}.
\end{aligned}$$

That means u is in one of these spaces if it is a linear map from the associated test function space to the complex numbers which is continuous with respect to the topology on the test function space. Notation-wise we may write $u \in \mathscr{D}'(\mathbb{R}^d)$ as a map

$$u : \mathscr{D}(\mathbb{R}^d) \to \mathbb{C}; \qquad f \mapsto u(f) =: \langle u, f\rangle$$

For these dual spaces, we have the reverse inclusions

$$\mathscr{E}'(\mathbb{R}^d) \subset \mathscr{S}'(\mathbb{R}^d) \subset \mathscr{D}'(\mathbb{R}^d).$$

Since the previous definition is rather abstract, we will verify that our previous prototype of a distribution, T_g for a function $g \in L^1(\mathbb{R}^d)$, is contained in $\mathscr{D}'(\mathbb{R}^d)$:

Example 3.6 (L^1-regular distributions) Indeed, if $g \in L^1(\mathbb{R}^d)$, then for all $f \in \mathscr{D}(\mathbb{R}^d)$ we may estimate

$$|T_g(f)| = |\langle g, f\rangle| = \left|\int_{\mathbb{R}^d} gf \,\mathrm{d}x\right| = \left|\int_{\mathrm{supp}(f)} gf \,\mathrm{d}x\right| \le \underbrace{\int_{\mathrm{supp}(f)} |g| \,\mathrm{d}x}_{\le \|g\|_1} \|f\|_\infty$$

Thus we see that $|T_g(f)| \le C\|f\|_\infty$ for some $C > 0$, and consequently $T_g \in \mathscr{D}'(\mathbb{R}^d)$. Therefore we have $L^1(\mathbb{R}^d) \subset \mathscr{D}'(\mathbb{R}^d)$.

In fact, the above proof works for any $g \in L^1_{loc}(\mathbb{R}^d)$, that is for those g that are integrable over any compact subset of $\mathbb{R}^d$. We will now see that there are, however, distributions that are not of this kind.

Example 3.7 (The Dirac delta distribution) The *Dirac delta*, or Dirac mass, also called evaluation functional, is defined for any $x_0 \in \mathbb{R}^d$ via

$$\delta_{x_0}(f) := f(x_0).$$

Obviously, $|\delta_{x_0}(f)| \leq \sup_{x \in \{x_0\}} |f|$, and thus $\delta_{x_0} \in \mathscr{E}'(\mathbb{R}^d)$.

The object δ_{x_0} is not an L^1-function. Indeed, it has several seemingly strange properties. We first examine its support. Since we do not have the notion of point-wise evaluation for a distribution, we need to adapt the definition of support.

Definition 3.8 (The support of a distribution) For a distribution $u \in \mathscr{D}'(\mathbb{R}^d)$, we say $x_0 \in \operatorname{supp}(u)$ if for every open $U \subset \mathbb{R}^d$ with $x_0 \in U$ there exists some $f \in \mathscr{D}(\mathbb{R}^d)$ such that $\operatorname{supp}(f) \subset U$ and $u(f) \neq 0$.

For the interested reader

The reason why this definition coincides with the previous one for $u = T_g$ for some $g \in \mathscr{C}(\mathbb{R}^d)$ with that previously given is the following lemma:

Lemma 3.9 (Fundamental lemma of the calculus of variations) Let u be a continuous function on $\mathbb{R}^d$. Then

$$\int u(x)\phi(x)\,\mathrm{d}x = 0 \qquad \forall \phi \in \mathscr{D}(\mathbb{R}^d)$$

if and only if $u = 0$.

This is easily proved by contradiction, using a cut-off supported in a sufficiently small neighbourhood of a point where $u(x) \neq 0$.

Exercise 3.10 Prove the fundamental lemma of the calculus of variations. *Hint:* Use a cut-off supported in a sufficiently small neighbourhood of a point where $u(x) \neq 0$.

Exercise 3.11 Prove that for $h, g \in \mathscr{C}(\mathbb{R}^d)$ we have $T_g = T_h$ if and only if $h = g$.

Exercise 3.12 Prove that if $g \in \mathscr{C}(\mathbb{R}^d)$ then $T_g \in \mathscr{E}'(\mathbb{R}^d)$ if and only if $\operatorname{supp}(g)$ is compact.

By $\delta_{x_0}(f) = f(x_0)$, for any test function f, we may easily obtain $\operatorname{supp}(\delta_{x_0}) = \{x_0\}$. Hence δ_{x_0} cannot be an L^1-function – otherwise we would have $\delta \equiv 0$ almost everywhere and thus $\delta = 0$ as a functional.

Another observation about δ_{x_0} is that it is the derivative of a step function:

Exercise 3.13 Consider the distribution H on $\mathbb{R}$ defined as the L^1_{loc}-regular distribution associated to the "Heaviside function" given by the $H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$. Prove, using the fundamental theorem of calculus, that $\frac{\mathrm{d}}{\mathrm{d}x} H = \delta_0$.

Using the previous statement, we have $\delta_0 = \partial_x H(x)$. However, it is easy to see that $H(x) = \frac{1}{2}\partial_x(x + |x|)$. This means that while δ_0 is not a function itself, we may express δ_0 as the (second order) *distributional derivative of a continuous function*. This is, in fact, a general phenomenon:

Theorem 3.14 (The regularity theorem) Any $u \in \mathscr{D}'(\mathbb{R}^d)$ is locally a finite order derivative of a continuous function.

That means given $u \in \mathscr{D}'(\mathbb{R}^d)$ and any compact set K we may find some continuous function g such that for arbitrary $f \in \mathscr{D}(\mathbb{R}^d)$ with $\operatorname{supp}(f) \subset K$ we have

$$u(f) = \partial^\alpha T_g(f) = T_g\big((-1)^{|\alpha|}\partial^\alpha f\big). \tag{3.6}$$

It is furthermore possible to prove that any distribution may be obtained as the limit of functions:

Lemma 3.15 The set of T_g, $g \in \mathscr{D}(\mathbb{R}^d)$, is dense in $\mathscr{D}'(\mathbb{R}^d)$ and thus also in $\mathscr{S}'(\mathbb{R}^d)$ and $\mathscr{E}'(\mathbb{R}^d)$.

Exercise 3.16 Let $u \in \mathscr{D}(\mathbb{R}^d)$ with $u(0) = 1$. Write $u_n(x) := n^d u(nx)$. Then prove that $u_n \overset{\mathscr{D}'}{\to} \delta_0$ as $n \to \infty$.

Operations on distributions

Our initial motivation to study distributions was to extend the notion of differentiability to a wider class than $\mathscr{C}^\infty$ that contained all L^1-functions. This is achieved by the weak derivative

$$\partial_x^\alpha u(f) := u\big(v(-1)^{|\alpha|}v\partial_x^\alpha f v\big).$$

Since the distribution spaces are spaces of functionals, we need to check carefully for all other operations whether or not they extend – and in which sense – to distribution spaces. Of course, as dual spaces of some vector space, the distribution spaces are vector spaces, i. e. we may sum distributions and multiply them by a constant.

We now analyse the tensor product. Recall that for $f \in \mathscr{C}^\infty(\mathbb{R}^{d_1})$, $g \in \mathscr{C}^\infty(\mathbb{R}^{d_2})$ the map $(f \otimes g) \in \mathscr{C}^\infty(\mathbb{R}^{d_1+d_2})$ is given by

$$\big(f \otimes g\big)(x,y) = f(x)g(y), \qquad (x,y) \in (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}). \tag{3.7}$$

Lemma 3.17 If $u \in \mathscr{D}'(\mathbb{R}^{d_1})$, $v \in \mathscr{D}'(\mathbb{R}^{d_2})$ then there exists a distribution $u \otimes v \in \mathscr{D}'(\mathbb{R}^{d_1+d_2})$, uniquely determined by the property

$$u \otimes v(f \otimes g) = u(f)\, v(g), \qquad \text{for all } f \in \mathscr{C}^\infty(\mathbb{R}^{d_1}),\ g \in \mathscr{C}^\infty(\mathbb{R}^{d_2}). \tag{3.8}$$

The idea of the proof is to define $u \otimes v$ on functions of the form $f \otimes g$ (elementary tensors) and then approximate any function in $\mathscr{C}^\infty(\mathbb{R}^{d_1+d_2})$ by sums of elementary tensors.

Exercise 3.18 Prove that if u and v are $\mathscr{C}^\infty$, then $u \otimes v$ (defined by (3.7)) fulfils (3.8).

Next, one seeks to generalize the product of functions. Note that for any of the test function spaces, the pointwise product $(f,g) \to f \cdot g$ provides maps

$$\begin{gathered}
\mathscr{D}(\mathbb{R}^d) \times \mathscr{E}(\mathbb{R}^d) \to \mathscr{D}(\mathbb{R}^d) \\
\mathscr{S}(\mathbb{R}^d) \times \mathscr{C}^\infty_{\mathrm{pol}}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d) \\
\mathscr{E}(\mathbb{R}^d) \times \mathscr{E}(\mathbb{R}^d) \to \mathscr{E}(\mathbb{R}^d),
\end{gathered}$$

where $\mathscr{C}^\infty_{\mathrm{pol}}$ is the space of “polynomially bounded $\mathscr{C}^\infty$-functions”, i. e. every derivative of f is bounded by some polynomial. These may be extended to maps

$$\begin{gathered}
\mathscr{D}'(\mathbb{R}^d) \times \mathscr{E}(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d) \\
\mathscr{S}'(\mathbb{R}^d) \times \mathscr{C}^\infty_{\mathrm{pol}} \to \mathscr{S}'(\mathbb{R}^d) \\
\mathscr{E}'(\mathbb{R}^d) \times \mathscr{E}(\mathbb{R}^d) \to \mathscr{E}'(\mathbb{R}^d)
\end{gathered}$$

by defining for a distribution u and a function f the distribution fu by its action on test functions g – all in the corresponding spaces – via $fu(g) := u(fg)$.

The previous extensions of the product of functions are a bit unsatisfying: we want to treat distributions as generalized functions – meaning one would like to multiply freely. However, the product of two distributions – such as $(\delta_0)^2$ – is in general undefined.

For the interested reader

It is possible to define $\frac{1}{x}$ as a distribution in $\mathscr{D}'(\mathbb{R})$ by setting

$$Sf = \lim_{\varepsilon \to 0} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{f(x)}{x} \, \mathrm{d}x. \tag{3.9}$$

The distribution S is usually called the *principal value* of $\frac{1}{x}$. Then $(xS)(f) = S(xf) = \int f(x) \, \mathrm{d}x$. That means, as distributions, $xS = 1$. It is also easy to see that by $x\delta_0(f) = \delta_0(xf) = 0f(0) = 0$, that we have $x\delta_0 = 0$. This, however, leads to a contradiction if we calculate formally:

$$\delta_0 = (xS)\delta_0 \neq S(x\delta_0) = 0.$$

This shows that you have to be extremely careful when calculating with distributions as one would do with functions – a lot of seemingly harmless operations (exchanging integrals, products, ...) are simply not defined!

Exercise 3.19 Prove that (3.9) gives a well-defined distribution.

We will now see another operation on distributions that is extremely important in microlocal analysis: the Fourier transform.

2.2 The Fourier transform

The Fourier transform on functions

The Fourier transform is a transformation acting on functions and (tempered) distributions. It can be seen as a change of basis. Indeed, one may imagine a function $f \mapsto f(x)$ as an infinite vector, indexed by every point $x \in \mathbb{R}^d$. The idea is to write this in terms of the "function basis" given by the functions e_ξ, $\xi \in \mathbb{R}^d$ mapping $x \mapsto e_\xi(x) := \frac{1}{(2\pi)^{d/2}} e^{i(x\cdot\xi)}$.

Recall that for an orthonormal basis $e_1, \dots, e_n$ we may write $v \in \mathbb{R}^n$ via

$$v = \sum_{j=1}^{n} (v, e_j) e_j. \tag{3.10}$$

In the following, the "basis" will be given by the e_ξ. This motivates the following:

Definition 3.20 Let $f \in \mathscr{S}(\mathbb{R}^d)$. Then its Fourier transform, $\mathscr{F}(f)$ – sometimes also written $\widehat{f}$ – is given by the function

$$\mathscr{F}(f)(\xi) = (f, e_\xi) = \frac{1}{(2\pi)^{d/2}} \int e^{-ix\xi} f(x)\, \mathrm{d}x \tag{3.11}$$

Remark 3.21 There are different conventions in the literature regarding the pre-factor 2π.

Remark 3.22 The Fourier transform (in one dimension) may thus be seen as writing a function in terms of the "elementary frequency functions" $e_\xi(x) = \cos(\xi x) + i\sin(\xi x)$. Imagining f as an audio signal, the value of the Fourier transform of f at a given ξ is "the amount of frequency ξ" that f contains.

The interpretation non-withstanding, the usefulness of the Fourier transform is encoded in the following identities:

$$D_x^\alpha e_\xi = \xi^\alpha e_\xi, \qquad D_\xi^\beta e_\xi = x^\beta e_\xi. \tag{3.12}$$

This means that the e_ξ form a kind of eigenbasis for differentiation. This motivates why we are so interested in the Fourier transform: it "diagonalizes" differential operators with constant coefficients. This will be explored more thoroughly in Section 3.1. Indeed, if we integrate under the integral in (3.11) and perform partial integration, we obtain the identities

$$\mathscr{F}(D_x^\alpha u) = \xi^\alpha \mathscr{F}(u) \tag{3.13}$$

$$D_\xi^\beta \mathscr{F}(u) = \mathscr{F}(x^\beta u) \tag{3.14}$$

These identities, along with the estimate $|x| \leq \frac{1}{2}(1+|x|^2)$ may be used to prove:

Exercise 3.23 The Fourier transform is a continuous map $\mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$.

Exercise 3.24 Calculate the Fourier transform of $e^{-\frac{x^2}{2}}$. *Hint: Use completion of squares and* $\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\,\mathrm{d}x = \sqrt{\pi}$.

The Fourier transform turns out to be an isomorphism on $\mathscr{S}(\mathbb{R}^d)$, and its inverse is given in a form similar to (3.10):

Theorem 3.25 (Fourier inversion theorem) Let $f \in \mathscr{S}(\mathbb{R}^d)$. Then we have $f(x) = \mathscr{F}^{-1}\mathscr{F}f(x)$ where

$$\big(\mathscr{F}^{-1}f\big)(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\xi} f(\xi)\,\mathrm{d}\xi. \tag{3.15}$$

In particular this means $\mathscr{F}\big(\mathscr{F}f\big)(x) = f(-x)$.

Having assembled these facts about the Fourier transform, let us perform a simple calculation that hints at its applications: we want to solve the equation

$$(1-\Delta)u = f,$$

where $\Delta u = -\sum_{j=1}^{d} D_{x_j}^2 u$ is the Laplace operator and f, for now, is a function in $\mathscr{S}(\mathbb{R}^d)$. Using (3.13), it is easy to see that

$$\begin{aligned}(-\Delta+1)u = f \quad &\Leftrightarrow \quad \mathscr{F}(-\Delta+1)u = \mathscr{F}f \quad \Leftrightarrow \quad (|\xi|^2+1)\mathscr{F}u = \mathscr{F}f \\ &\Leftrightarrow \quad u = \mathscr{F}^{-1}(|\xi|^2+1)^{-1}\mathscr{F}f \\ &\qquad = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-y)\xi} \frac{f(y)}{1+|\xi|^2}\,\mathrm{d}y\,\mathrm{d}\xi.\end{aligned} \tag{3.16}$$

Therefore, we have found an inverse – that is a solution operator – to $(1-\Delta)$. These methods will be developed further during the course of this document. As a first step, we will now extend the Fourier transform in order to be able to treat more general right-hand sides, i. e., tempered distributions.

The Fourier transform on distributions

We now want to find a definition for the Fourier transform of distributions. For that, we want to start again by looking at regular distributions, since there both definitions should coincide. We therefore calculate for $f, g \in \mathscr{S}(\mathbb{R}^d)$

$$\langle \mathscr{F}g, f\rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-ix\xi} g(x)\,\mathrm{d}x f(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-ix\xi} f(\xi)\,\mathrm{d}\xi g(x)\,\mathrm{d}x = \langle g, \mathscr{F}f\rangle$$

and obtain

Proposition 3.26 (Plancherel's theorem) Let $f, g \in \mathscr{S}(\mathbb{R}^d)$. Then

$$T_{\mathscr{F}f}(g) = T_f(\mathscr{F}g). \tag{3.17}$$

Furthermore $\|f\|_{L^2(\mathbb{R}^d)} = \|\mathscr{F} f\|_{L^2(\mathbb{R}^d)}$.

Remark 3.27 As a consequence, by density of $\mathscr{S}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$, $\mathscr{F}$ extends to a unitary transformation on $L^2(\mathbb{R}^d)$.

The identity (3.17) motivates the following:

Theorem 3.28 (The Fourier transform of distributions) The Fourier transform and inverse Fourier transform may be extended to distributions $u \in \mathscr{S}'(\mathbb{R}^d)$ by setting for $f \in \mathscr{S}(\mathbb{R}^d)$

$$\big(\mathscr{F} u\big)(f) := u(\mathscr{F} f), \qquad \mathscr{F}^{-1} u(f) := u(\mathscr{F}^{-1} f). \tag{3.18}$$

These maps are continuous and inverse to each other.

Exercise 3.29 Prove that $\mathscr{F}\delta_{x_0} = (2\pi)^{-d/2} e^{-ix_0\xi}$. Note: for $x_0 = 0$ we obtain a constant function. *Hint:* Use Fourier's inversion theorem and recall that two distributions u_1 and u_2 are equal if and only if $u_1(f) = u_2(f)$ for arbitrary test functions.

The convolution product

By now, we have studied the Fourier transform to some extent and we have seen that it transforms multiplication by a monomial x^α into α-fold differentiation. It is thus a natural question what happens to general multiplication by a function. Let us first do a model calculation for $f, g \in \mathscr{S}(\mathbb{R}^d)$:

$$\begin{aligned}
\widehat{fg}(\xi) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\xi} f(x) g(x) \, \mathrm{d}x \\
&= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\xi} f(x) (\mathscr{F}^{-1}\hat{g})(x) \, \mathrm{d}x \\
&= \frac{1}{(2\pi)^{d}} \int_{\mathbb{R}^d} e^{-ix\xi} e^{ix\eta} f(x) \widehat{g}(\eta) \, \mathrm{d}x\mathrm{d}\eta \\
&= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \widehat{f}(\xi - \eta) \widehat{g}(\eta) \, \mathrm{d}\eta .
\end{aligned}$$

We have thus found the following

Theorem 3.30 (The convolution theorem) Let $f, g \in \mathscr{S}(\mathbb{R}^d)$. Then $\widehat{fg}(\xi) = \frac{1}{(2\pi)^{d/2}} \widehat{f} * \widehat{g}$, where the *convolution* of two functions $f, g \in \mathscr{S}(\mathbb{R}^d)$, $f * g \in \mathscr{S}(\mathbb{R}^d)$, is defined as

$$\big(f * g\big)(x) = \int_{\mathbb{R}^d} f(x-y) g(y) \, \mathrm{d}y = \int_{\mathbb{R}^d} f(y) g(x-y) \, \mathrm{d}y.$$

The convolution product is thus an alternate product on the Schwartz functions. Of course we now want to extend it to more general function and distribution spaces.

Definition 3.31 (Convolution of a function and a tempered distribution) Let $f \in \mathscr{S}(\mathbb{R}^d)$ and $u \in \mathscr{S}'(\mathbb{R}^d)$. Then $f * u$ is the distribution defined by

$$f * u := \frac{1}{(2\pi)^{d/2}} \mathscr{F}^{-1}(\widehat{f} \cdot \widehat{u}).$$

Since we have already defined the product of a function and a distribution, this definition is well-defined and generalizes the convolution on functions. Moreover, it immediately follows that for all $f \in \mathscr{S}(\mathbb{R}^d)$ we have

$$f * \delta_0 = f \tag{3.19}$$

meaning δ_0 is the identity with respect to the convolution product.

The convolution theorem however hints at another possible – Fourier independent – definition that turns out to be equivalent but also applicable to distribution spaces where it is not possible to apply the Fourier transform. In fact, one may define

$$\langle f * u, g \rangle = \langle u, f^{\vee} * g \rangle$$

where $f^{\vee}(x) = f(-x)$.

Remark 3.32 For a regular distribution, this is just a complicated way of writing

$$\langle u * f, g \rangle = \iint_{\mathbb{R}^d \times \mathbb{R}^d} u(y) f(x-y) g(x) \, \mathrm{d}x \mathrm{d}y.$$

The convolution may indeed be extended to wider classes of spaces. The most important generalization is that the convolution extends to a map

$$\mathscr{E}'(\mathbb{R}^d) \times \mathscr{D}'(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d),$$

for which we still have $\delta_0 * u = u$ for all $u \in \mathscr{D}'(\mathbb{R}^d)$.

One way in which the convolution is useful is that it can be used to regularize and approximate distributions. Indeed, we have

$$\partial_x^\alpha(u*v) = (\partial_x^\alpha u)*v = u*(\partial_x^\alpha v) \tag{3.20}$$

or, more generally,

Lemma 3.33 The convolution of a distribution u and a test function f is smooth, that means there exists some $g \in \mathscr{C}^\infty(\mathbb{R}^d)$ such that $u*f = T_g$.

Remark 3.34 In Exercise 3.16 we have discussed that δ_0 may be approximated by test functions. Together with the previous lemma, this may be used to prove Lemma 3.15, i. e., that every distribution may be approximated by smooth functions.

For the interested reader

An extremely useful estimate for convolutions is the following.

Theorem 3.35 (Young's inequality) Let $f,g \in \mathscr{S}(\mathbb{R}^d)$. Then

$$\|f*g\|_r \leq \|f\|_p \|g\|_q,$$

where $1 \leq p,q \leq \infty$ and $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$.

Consequently, for r,p,q given as above, the convolution extends to a map

$$L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \to L^r(\mathbb{R}^d).$$

For the interested reader

The following inclusion is obvious for test functions, from

$$f*g(x) = \int_{\mathbb{R}}^d f(x-y)g(y)\,\mathrm{d}y,$$

and carries over to distributions:

Proposition 3.36 For $u \in \mathscr{E}'(\mathbb{R}^d)$ and $v \in \mathscr{D}'(\mathbb{R}^d)$ we have

$$\operatorname{supp}(u*v) \subset \left\{x \in \mathbb{R}^d \,|\, x = x_1 + x_2 \text{ with } x_1 \in \operatorname{supp}(u),\ x_2 \in \operatorname{supp}(v)\right\}.$$

The Fourier transform and distributions of compact support

So far, we have defined the Fourier transform on $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$. Since $\mathscr{D}(\mathbb{R}^d) \subset \mathscr{S}(\mathbb{R}^d)$ and $\mathscr{E}'(\mathbb{R}^d) \subset \mathscr{S}'(\mathbb{R}^d)$ are subspaces, one may ask what their image under Fourier transform is. Phrased differently:

Is it possible to detect that a test function/distribution u is compactly supported by studying its Fourier transform $\mathscr{F}u$?

The answer to this question is yes, and is given by the theorem of Paley-Wiener-Schwartz. Let us do a model calculation to highlight the mechanism. Let $f \in \mathscr{D}(\mathbb{R})$, meaning f is smooth and compactly supported. We may suppose that $\operatorname{supp}(f) \subset [-R,R]$ for some $R > 0$. Then we may calculate

$$\mathscr{F}f(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\,\mathrm{d}x = \int_{-R}^{R} e^{-ix\xi} f(x)\,\mathrm{d}x.$$

Now, the latter integral is over a compact set and we may thus exchange it with any uniform limit. Now it is well-known $e^{-ix\xi}$ is actually an *analytic function* and may be extended to the complex plane by $e^{-ix\zeta}$ with $\zeta = \xi + i\eta \in \mathbb{C}$. Since we may exchange the integral and complex differentiation, $\mathscr{F}(\xi)$ is also complex differentiable and extends to an entire function $\mathscr{F}(\zeta)$. Furthermore, we may estimate

$$|\mathscr{F}f(\zeta)| = \left| \int_{\mathbb{R}} e^{-ix\xi} e^{x\eta} f(x)\,\mathrm{d}x \right| \leq \int_{-R}^{R} |e^{x\eta}||f(x)|\,\mathrm{d}x \leq e^{R|\eta|} \sup_{x\in[-R,R]} |f(x)|.$$

By inserting factors of $(1+|\xi|^2)^N(1+|\xi|^2)^{-N}$ and use of (3.13) we may observe for arbitrary $N \in \mathbb{N}$ that

$$\left|\mathscr{F}f(\zeta)\right| \leq Ce^{R|\eta|}(1+|\xi|^2)^{-N},$$

where C depends on R, f and N. This analysis cumulates in the following theorem:

Theorem 3.37 (Paley-Wiener-Schwartz)
For functions: Let $u \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(u) \subset \overline{B_R(0)} = \{x \in \mathbb{R}^d \,|\, |x| \leq R\}$. Then $\mathscr{F}u$ extends to an analytic function on all of $\mathbb{C}$ satisfying that for each $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$\mathscr{F}(u)(\zeta) \leq C_N(1+|\zeta|)^{-N} e^{R|\mathrm{Im}(\zeta)|}, \qquad \zeta \in \mathbb{C}^d. \tag{3.21}$$

Conversely, any entire function satisfying (3.21) is the Fourier transform of some $u \in \mathscr{C}_c^\infty(\mathbb{R}^d)$.

For distributions: Let $u \in \mathscr{E}'(\mathbb{R}^d)$ with $\operatorname{supp}(u) \subset \overline{B_R(0)}$. Then $\mathscr{F}u$ extends to an analytic function given by on all of $\mathbb{C}$

$$\zeta \mapsto u(e_\zeta).$$

Furthermore, $\mathscr{F}$ satisfies that there exists some $N \in \mathbb{N}$ and $C_N > 0$ such that

$$\mathscr{F}\big(u\big)(\zeta) \leq C(1+|\zeta|)^N e^{R|\operatorname{Im}(\zeta)|}, \qquad \zeta \in \mathbb{C}^d. \tag{3.22}$$

Conversely, any entire function satisfying (3.22) is the Fourier transform of some $u \in \mathscr{E}'(\mathbb{R}^d)$.

Exercise 3.38 Calculate the Fourier transform of the $L^1(\mathbb{R})$-regular distributions

$$\mathbb{1}_{[-R,R]}(x) = \begin{cases} 1 & |x| \leq R \\ 0 & x > R \end{cases}.$$

and $(x \cdot \mathbb{1}_{[-R,R]})$. Then use your findings to find the Fourier transform of $\frac{\sin^2(x)}{x^2}$.

2.3 Singularities of distributions

The Paley-Wiener-Schwartz theorem enabled us to determine whether a function is the Fourier transform of a compactly supported distribution or of a smooth function. This means we may distinguish the spaces $\mathscr{C}_c^\infty(\mathbb{R}^d)$ and $\mathscr{E}'(\mathbb{R}^d)$ by means of Fourier analysis. In the following, we will explore this further.

The singular support

Distributions are generalized functions – they are often needed when usual functions are not enough for calculations. Looking back at the calculation (3.16), we have by now solved the equation $(-\Delta+1)u = f$ in a *very* general set-up. In fact, the right-hand side may be *any tempered distribution*. In particular, it need not even be smooth – even discontinuous functions are allowed. However, u is then only a distributional (also called *weak*) solution. In fact, we have

$$f \in \mathscr{S}(\mathbb{R}^d) \quad \Leftrightarrow \quad u \in \mathscr{S}(\mathbb{R}^d)$$
$$f \in \mathscr{S}'(\mathbb{R}^d) \quad \Leftrightarrow \quad u \in \mathscr{S}'(\mathbb{R}^d)$$

We will now investigate a concept of that enables us to say that a distribution is "locally smooth". In later parts of this document we will then see how to relate (micro-)local smoothness of f to that of u.

Definition 3.39 An element $u \in \mathscr{D}'(\mathbb{R}^d)$ is called

- $\diamond$ (globally) $\mathscr{C}^\infty$-regular if $u \in \mathscr{C}^\infty(\mathbb{R}^d)$, i. e. there exists a $g \in \mathscr{C}^\infty(\mathbb{R}^d)$ such that $T_g = u$, meaning $\forall f \in \mathscr{D}(\mathbb{R}^d)$ we have $T_g(f) = u(f)$.
- $\diamond$ locally $\mathscr{C}^\infty$-regular at a given $x_0 \in \mathbb{R}^d$ if there exists some $\phi^{x_0} \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ such that $\phi^{x_0} = 1$ (i. e. a cut-off) in a neighbourhood of x_0 such that $\phi^{x_0} u \in \mathscr{C}^\infty(\mathbb{R}^d)$.

A distribution that is not $\mathscr{C}^\infty$-regular is called *singular*.

We see that these singularities are a localizable notion, see Figure 3.2. Indeed, we may attach a position to a singularity and we define the set of $\mathscr{C}^\infty$-singularities of a given distribution as its *singular support*:

$$\operatorname{singsupp}(u) = \left\{ x_0 \in \mathbb{R}^d \,\middle|\, u \text{ is } \textit{not } \mathscr{C}^\infty\text{-regular at } x_0. \right\}$$

We thus have $\operatorname{singsupp}(u) = \emptyset$ if and only if $u \in \mathscr{C}^\infty(\mathbb{R}^d)$.

We are now finally going to pass from the concept of *local smoothness* to that of *microlocal smoothness* of $u \in \mathscr{D}'(\mathbb{R}^d)$. For any cut-off ϕ^{x_0} we have $\phi^{x_0} u \in \mathscr{E}'(\mathbb{R}^d)$. By the Paley-Wiener theorem, Theorem 3.37, $\mathscr{F}(\phi^{x_0} u) \in \mathscr{C}^\infty(\mathbb{R}^d)$ and furthermore $\phi^{x_0} u \in \mathscr{C}^\infty(\mathbb{R}^d)$ if and only if for every $N \in \mathbb{N}$ there exists some $C_N > 0$ such that

$$|\mathscr{F}(\phi^{x_0} u)(\xi)| \leq C_N (1 + |\xi|)^{-N}. \tag{3.23}$$

Furthermore, the theorem guarantees that even if $u \notin \mathscr{C}^\infty(\mathbb{R}^d)$ we have for some $N \in \mathbb{N}$ that

$$\left|\mathscr{F}(\phi^{x_0} u)(\xi)\right| \leq C (1 + |\xi|)^N.$$

Observe that (3.23) is a *rapid decay condition*. We may summarize our previous observations as follows:

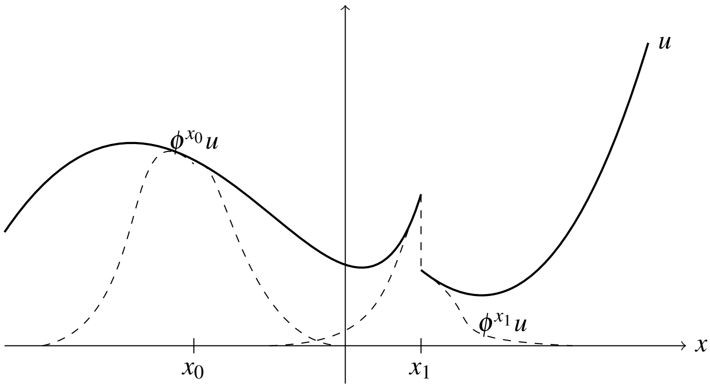

Figure 3.2: Localization of singularities by cut-offs (schematic).

A distribution u is singular at some x_0 if (and only if) for all cut-offs ϕ^{x_0} around x_0 we have that $\mathscr{F}(\phi^{x_0}u)$ fails to be everywhere rapidly decaying.

Now, again, we want to localize this concept of regularity. In this case, a singularity consists of a point $x_0 \in \mathbb{R}^d$ and a direction – this may be seen as a vector in $\mathbb{R}^d \setminus \{0\}$. The neighbourhood of a direction is an open *cone*.

Definition 3.40 A set $\Gamma \subset (\mathbb{R}^d \setminus \{0\})$ is a *cone* if for all $\lambda > 0$ we have $x \in \Gamma \Rightarrow \lambda x \in \Gamma$.

The following inequality is extremely useful when dealing with conic subsets: let $\Gamma_1,\ \Gamma_2 \subset (\mathbb{R}^d \setminus \{0\})$ two disjoint closed cones. Then there exists some $c > 0$ such that

$$|x-y| \geq c(|x|+|y|), \qquad x \in \Gamma_1,\ y \in \Gamma_2. \tag{3.24}$$

The *wave front set* – a finer concept than that of the singular support – is now precisely the collection of such singularities with the attached directions:

Definition 3.41 (The wave front set) An element $u \in \mathscr{D}'(\mathbb{R}^d)$ is called microlocally $\mathscr{C}^\infty$-regular at $(x_0, \xi_0) \in \mathbb{R}^d \times \big(\mathbb{R}^d \setminus \{0\}\big)$ if there exists

- $\diamond$ some $\phi^{x_0} \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ such that $\phi^{x_0} = 1$ in a neighbourhood of x_0
- $\diamond$ an open cone $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ with $\xi_0 \in \Gamma$

such that for any $N \in \mathbb{N}$ there exists some $C_N > 0$ such that

$$\left|\mathscr{F}(\phi^{x_0}u)(\xi)\right| \le C_N(1+|\xi|)^{-N} \qquad \forall \xi \in \Gamma. \tag{3.25}$$

Then $\mathrm{W}F(u) \subset \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ is defined as the collection of points where u is *not* microlocally $\mathscr{C}^\infty$-regular.

Proposition 3.42 (Properties of $\mathrm{W}F$**)** Let $u \in \mathscr{D}'(\mathbb{R}^d)$. Then

1) $\mathrm{W}F(u)$ is a closed subset of $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$

2) $\mathrm{W}F(u)$ is conic in the second variable, i. e.,

$$(x,\xi) \in \mathrm{W}F(u) \quad \Leftrightarrow \quad (x,\lambda\xi) \in \mathrm{W}F(u) \qquad \forall \lambda > 0$$

3) If $f \in \mathscr{C}^\infty(\mathbb{R}^d)$, then

$$\mathrm{W}F(fu) \subset \mathrm{W}F(u). \tag{3.26}$$

4) Let $\mathrm{pr}_1 : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ be the projection on the first set of variables $\mathrm{pr}_1(x,\xi) = x$. Then

$$\mathrm{pr}_1\big(\mathrm{W}F(u)\big) = \mathrm{singsupp}(u).$$

Exercise 3.43 Prove Property 3) as follows:

1. Pick for some fixed $\xi_0 \in \Gamma$ some (suitably small) conic neighbourhood $\Gamma_1 \subset \Gamma$, where $\mathscr{F}(\phi^{x_0}u)$ is rapidly decaying in Γ.
2. Write $\mathscr{F}(\phi^{x_0}fu) = (2\pi)^{-d/2}\mathscr{F}(f) * \mathscr{F}(\phi^{x_0}u)$
3. Now separate the convolution integral into the conic regions Γ and $\mathbb{R}^d \setminus \Gamma$.
4. Estimate the two parts separately using (3.24).

Exercise 3.44 Calculate $\mathrm{W}F(\mathbb{1} \otimes \delta_{x_0})$.

. . . .

3 Pseudo-differential operators

3.1 Partial differential operators and the Fourier transform

The starting point of our analysis was that we wanted to extend the notion of differentiability. We now want to harvest the benefits of our new language.

Assuming that the a_α, the coefficients, are smooth functions, we may write a partial differential operator (PDO) of order m on $\mathbb{R}^d$ in the form

$$P = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha \tag{3.27}$$

Special cases of interest are those where the a_α are polynomials or even constant. We first observe the following:

A differential operator P may be applied to any distribution $u \in \mathscr{D}'(\mathbb{R}^d)$. If P has polynomial coefficients and $u \in \mathscr{S}'(\mathbb{R}^d)$ then $Pu \in \mathscr{S}'(\mathbb{R}^d)$.

We may now rewrite the action of a such a P using the Fourier transform and (3.13). Let $u \in \mathscr{S}'(\mathbb{R}^d)$.

$$Pu = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha u = \sum_{|\alpha| \le m} a_\alpha(x) \mathscr{F}^{-1} \mathscr{F} D_x^\alpha u = \sum_{|\alpha| \le m} a_\alpha(x) \mathscr{F}^{-1} \xi^\alpha \mathscr{F} u. \tag{3.28}$$

Notice that the right-hand side in (3.28) does not contain any derivatives – just multiplication by functions in x and ξ and Fourier transforms.

Definition 3.45 To a PDO P we associate his *symbol* $p(x,\xi) \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ by

$$P = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha \quad \Longrightarrow \quad p(x,\xi) := \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha. \tag{3.29}$$

Then p is a polynomial with $\mathscr{C}^\infty$-coefficients. The *principal symbol* $p_m(x,\xi)$ of P is the collection of terms of order m in ξ, i. e.

$$p_m(x,\xi) := \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha.$$

We may consistently write $P = p(x, D_x)$, meaning we may obtain P back from its symbol by replacing ξ with D_x. This is an important fact: since the symbol is *uniquely associated to P*, we obtain an isomorphism

$$\text{PDOs with } \mathscr{C}^\infty\text{-coefficients} \longleftrightarrow \text{Polynomials with } \mathscr{C}^\infty\text{-coefficients}.$$

Phrased differently, all properties of a differential operator may be recovered from its symbol.

The advantage of studying the symbol is that it is a function – for whose analysis we have all the tools from analysis available. The difficulty lies in obtaining a "dictionary" how to relate properties of the operator to those of the symbol.

We first look at compositions of operators. By Leibniz' rule we may write:

$$\begin{aligned} p_1(x,D)\circ p_2(x,D) &= \left(\sum_{|\alpha|\le m_1} a_1^\alpha(x)D_x^\alpha\right)\circ\left(\sum_{|\beta|\le m_2} a_2^\beta(x)D_x^\beta\right) \\ &= \sum_{\substack{|\alpha|\le m_1\\|\beta|\le m_2}} a_1^\alpha(x)D_x^\alpha\left(a_2^\beta(x)D_x^\beta\right) \\ &= \sum_{\substack{|\alpha|\le m_1\\|\beta|\le m_2}}\sum_\gamma \frac{\alpha!}{\gamma!(\alpha-\gamma)!}a_1^\alpha(x)\big(D_x^\gamma a_2^\beta(x)\big)D_x^{\beta+\alpha-\gamma}. \end{aligned}$$

Note that we were able to sum over all γ since all terms for which $|\gamma|>m_1$ vanish. Taking symbols, we obtain that the symbol of $P_1\circ P_2$ may be written as

$$\begin{aligned} \sum_{\substack{|\alpha|\le m_1\\|\beta|\le m_2}}\sum_\gamma \frac{\alpha!}{\gamma!(\alpha-\gamma)!}a_1^\alpha(x)\big(D_x^\gamma a_2^\beta(x)\big)\xi^{\beta+\alpha-\gamma} &= \sum_\gamma\frac{1}{\gamma!}\sum_{\substack{|\alpha|\le m_1\\|\beta|\le m_2}} a_1^\alpha(x)\big(\partial_\xi^\gamma\xi^\alpha\big)\big(D_x^\gamma a_2^\beta(x)\big)\xi^\beta \\ &= \sum_\gamma\frac{(-i)^{|\gamma|}}{\gamma!}D_\xi^\gamma p_1(x,\xi)D_x^\gamma p_2(x,\xi). \end{aligned}$$

Lemma 3.46 (The symbol of a composition of PDOs) Let P_1 and P_2 two differential operators with symbols $p_1(x,\xi)$ and $p_2(x,\xi)$ of orders m_1 and m_2 respectively. Then $P_1\circ P_2$ is a PDO of order m_1+m_2 and in terms of symbols we have

$$P_1\circ P_2 = p_1(x,D)\circ p_2(x,D) = \big(p_1\#p_2\big)(x,D), \tag{3.30}$$

where

$$\big(p_1\#p_2\big)(x,\xi) := \sum_\gamma\frac{(-i)^{|\gamma|}}{\gamma!}D_\xi^\gamma p_1(x,\xi)D_x^\gamma p_2(x,\xi). \tag{3.31}$$

In particular, the principal symbol of $P_1\circ P_2$ is just the product $(p_1)_{m_1}(p_2)_{m_2}$.

Fundamental solutions

We are now interested in the local solvability of partial differential equations. Here, distribution theory is a potent tool, since it supplies us with the notion of fundamental solution.

Definition 3.47 Let P a partial differential operator. Then $u \in \mathscr{D}'(\mathbb{R}^d)$ is called a *fundamental solution* of P if $Pu = \delta_0$.

Exercise 3.48 Prove, using Green's first identity, that the $L^1(\mathbb{R}^3)$-regular distribution $\frac{1}{4\pi|x|}$ is a fundamental solution to $-\Delta$ in $\mathbb{R}^3$.

The reason why we are interested in fundamental solutions is the following: let u a fundamental solution to P, $v \in \mathscr{E}'(\mathbb{R}^d)$. Then

$$P(u*v) \overset{(3.20)}{=} (Pu)*v = \delta_0 * v \overset{(3.19)}{=} v.$$

Consequently, if we are able to find a fundamental solution, we can find solutions to the equation $Pu = v$ for arbitrary inhomogeneities $v \in \mathscr{E}'(\mathbb{R}^d)$. This means P is *locally solvable*.

Finding a fundamental solution is in general no easy task. Nevertheless, for operators with constant coefficients, it is fairly straight-forward: If P has constant coefficients, then its symbol $p(x,\xi) = p(\xi)$ is simply a polynomial in ξ and we may write $Pu = P(D)u = \mathscr{F}^{-1}P(\xi)\mathscr{F}u$. Consequently, if we look for a fundamental solution, we are led to consider

$$Pu = \mathscr{F}^{-1}P(\xi)\mathscr{F}u = \delta_0 \quad \Longrightarrow \quad P(\xi)\mathscr{F}u = (2\pi)^{-d/2}\mathbb{1}$$

Thus, we have the ansatz $u = (2\pi)^{-d/2}\mathscr{F}^{-1}T_{P(\xi)^{-1}}$ and this distribution defines a fundamental solution if it is well-defined – in general $P(\xi)$ could have zeroes and might not be invertible as a L^1_{loc}-function. It is nevertheless possible to make sense of $P(\xi)^{-1}$ as a distribution – we have already seen an example for this in (3.9) for the case of $\frac{1}{x}$. This regularization argument then yields:

Theorem 3.49 (Malgrange-Ehrenpreis) Every differential operator with constant coefficients has a fundamental solution.

The statement is in general false for differential operators with more general coefficients, even for polynomial coefficients.

We have seen that if P admits a fundamental solution u, then the operator given by $f \mapsto f*u$ provides a *right inverse* to P. We now want to study such operators. There is, however, a hurdle to overcome:

The inverse of a differential operator P of order $m \neq 0$, if it exists, is never a differential operator.

This is easily seen from Lemma 3.46. We want to study PDOs and their inverses[4] together as a class of operators. We are thus tasked with enlarging the algebra of differential operators. This will be achieved in the next section.

3.2 Pseudo-differential operators

What follows is a conceptual introduction to pseudo-differential operators with the intention to pseudo-invert differential operators, cf. also [4]. The starting point of pseudo-differential analysis is to replace the previous symbol $p(x,\xi)$ – a polynomial in ξ – with more general functions. We have already encountered the use of this in (3.16), where we have seen that $(1-\Delta)^{-1} = \mathscr{F}^{-1}q(\xi)\mathscr{F}$ for $q(\xi) = (1+|\xi|^2)^{-1}$. Thus we surely want to allow rational functions. Since we also want to employ our previous methods of localisation, for example by multiplying with excision functions in ξ. The following definition turns out to be suitable:

Definition 3.50 Let $a \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^s)$. Such an a is called a (global)[5] *symbol of order m*, $m \in \mathbb{R}$, if a satisfies for each $\alpha \in \mathbb{N}^d$, $\beta \in \mathbb{N}_s$ the estimates

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \tag{3.32}$$

for some $C_{\alpha,\beta} \geq 0$. We denote the space of symbols of fixed order $m \in \mathbb{R}$ by $S^m(\mathbb{R}^d \times \mathbb{R}^s)$.

Exercise 3.51 Prove that any function $a(x,\xi) = \chi\left(|\xi|\right)\frac{p(x,\xi)}{q(x,\xi)}$ where p and q are polynomials in ξ, bounded in x, and χ is an excision function vanishing in a neighbourhood of the zeroes of q, is a symbol of order $m = \deg(p) - \deg(q)$.

Exercise 3.52 Check that if a and b are symbols of order m_1 and m_2, then ab is a symbol of order $m_1 + m_2$. Furthermore, $\partial_x^\alpha a$ is a symbol of order $m_1 - |\alpha|$.

Definition 3.53 A pseudo-differential operator (ΨDO) is an operator that may be written in the form

$$u \mapsto a(x,D)u := \frac{1}{(2\pi)^d}\iint e^{i(x-y)\xi}a(x,\xi)u(y)\,\mathrm{d}y\mathrm{d}\xi \tag{3.33}$$

[4] More precisely, we want to study more general “approximate inverses” – since even simple differential operators are most of the time not invertible. Ex.: Already the simple operator ∂_x on $\mathbb{R}$ is not left-invertible, since the constant functions are in its kernel.

[5] Often, one rather looks at local symbols that only satisfy the inequalities on any given compact set in x. This is necessary when passing to pseudo-differential operators on general manifolds. For the purpose of this introduction, we limit ourselves to the global case.

for some symbol $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$.

As such, the expression defined in (3.33) is for $u \in \mathscr{S}(\mathbb{R}^d)$ well-defined. In fact, it may be written in the form

$$u \mapsto \mathscr{F}^{-1}_{\xi \to x} a(x,\xi)\big(\mathscr{F}u\big)(\xi).$$

We will now show that (3.33) defines a continuous map $\mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$. This may be validated by the following calculation:

$$\begin{aligned}
\left|x^\alpha \partial_x^\beta a(x,D)u\right| &= \left|\frac{1}{(2\pi)^{d/2}} \int x^\alpha \partial_x^\beta e^{ix\xi} a(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi\right| && (3.34)\\
&= \left|\frac{1}{(2\pi)^{d/2}} \int \sum_{\gamma\le\beta} c_{\gamma\beta}\, x^\alpha \xi^\gamma e^{ix\xi}\big(\partial_x^{\beta-\gamma} a\big)(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi\right| && (3.35)\\
&= \left|\frac{1}{(2\pi)^{d/2}} \int \sum_{\gamma\le\beta} c_{\gamma\beta}\, \xi^\gamma D_\xi^\alpha e^{ix\xi}\big(\partial_x^{\beta-\gamma} a\big)(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi\right| && (3.36)\\
&= \left|\frac{1}{(2\pi)^{d/2}} \int \sum_{\gamma\le\beta} c_{\gamma\beta}\, e^{ix\xi} D_\xi^\alpha\Big(\xi^\gamma\big(\partial_x^{\beta-\gamma} a\big)(x,\xi)\widehat{u}(\xi)\Big)\,\mathrm{d}\xi\right| && (3.37)\\
&\le C\int (1+|\xi|)^{-d-1}\sum_{\gamma\le\beta}\left|(1+|\xi|)^{d+1} D_\xi^\alpha\Big(\xi^\gamma\big(\partial_x^{\beta-\gamma} a\big)(x,\xi)\widehat{u}(\xi)\Big)\right|\,\mathrm{d}\xi. && (3.38)
\end{aligned}$$

Let's quickly assess what we have done: in (3.34) we have written out the definition, in (3.35) we used Leibniz' rule, in (3.36) we used $D_\xi e^{ix\xi} = xe^{ix\xi}$. Consequently, in (3.37), we have performed partial integration in ξ. Finally, in (3.38) we have inserted a factor of $(1+|\xi|)^{-d-1}$ and pulled the absolute modulus inside the integral. Summing up, we may estimate (using the fact that $(1+|\xi|)^{-d-1}$ is integrable), that

$$\left|x^\alpha \partial_x^\beta a(x,D)u(x)\right| \le C' \sup_{\xi\in\mathbb{R}^d}\sum_{\gamma\le\beta}\left|(1+|\xi|)^{d+1} D_\xi^\alpha\Big(\xi^\gamma\big(\partial_x^{\beta-\gamma} a\big)(x,\xi)\widehat{u}(\xi)\Big)\right|.$$

But now, since a fulfils the symbol estimates (3.32) and $\widehat{u} \in \mathscr{S}(\mathbb{R}^d)$, meaning we may use $\|(1+|x|)^N \partial_x^\beta f\|_{L^\infty(\mathbb{R}^d)} < \infty$, this proves

$$\sup_{x\in\mathbb{R}^d}\left|x^\alpha \partial_x^\beta a(x,D)u\right| < \infty,$$

meaning $a(x,D)u \in \mathscr{S}(\mathbb{R}^d)$ and $a(x,D) : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ is continuous.

It is possible, using the symbolic estimates and partial differentiation, to show the following:

Proposition 3.54 The adjoint of a ΨDO is again a ΨDO of the same order. As such it is possible to extend the action of a ΨDO to a continuous map $\mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ by setting for $u \in \mathscr{S}'(\mathbb{R}^d)$ and $f \in \mathscr{S}(\mathbb{R}^d)$:

$$\langle a(x,D)u, f\rangle = \langle u, {}^t a(x,D) f\rangle.$$

The symbol map

We had, for a differential operator, not only a map *Symbol* $\mapsto$ *Operator* but actually started from a map *Operators* $\mapsto$ *Symbol*. We want to now establish this for pseudo-differential operators.

Notice that Fourier's inversion theorem, Theorem 3.25, may be expressed as

$$f(x) = \delta_x(f) = \frac{1}{(2\pi)^d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y)\xi}\, f(y) \mathrm{d}y \mathrm{d}\xi. \tag{3.39}$$

Let again $e_\eta = \frac{1}{(2\pi)^{d/2}} e^{ix\eta}$ for fixed $\eta \in \mathbb{R}^d$. We may use (3.39) to compute

$$\begin{aligned} e_{-\eta} a(x,D) e_\eta &= e^{-ix\eta} \frac{1}{(2\pi)^{2d}} \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y)\xi} a(x,\xi) e^{iy\eta}\, \mathrm{d}y \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d} e^{-ix(\eta-\xi)} a(x,\xi) \underbrace{\int_{\mathbb{R}^d} e^{iy(\eta-\xi)}\, \mathrm{d}y}_{=(2\pi)^d \delta_\eta \text{ acting in } \xi}\, \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^d} a(x,\eta) \end{aligned}$$

We have achieved

Proposition 3.55 We may write a given ΨDO A as $A = a(x,D)$ where a is defined by $a(x,\xi) = (2\pi)^d e_{-\xi} a(x,D) e_\xi$.

Remark 3.56 Notice that the above calculation may be rewritten as

$$a(x,D) e_\eta = \frac{1}{(2\pi)^d} a(x,\eta) e_{-\eta}.$$

This means that $a(x,\eta)e_{-\eta}$ may be seen as the image of the "Fourier basis vector" under the action of $a(x,D)$.

This means we have extended the symbol isomorphism previously defined on PDOs to

$$\begin{array}{ccc} \Psi\text{DOs} & \longleftrightarrow & \text{Symbols} \\ \big\uparrow & & \big\uparrow \\ \text{PDOs} & \longleftrightarrow & \text{Polynomials w. } \mathscr{C}^\infty\text{-coeff.} \end{array} \tag{3.40}$$

This means we have finally found a bigger class of operators in which PDOs are contained and which are compatible with the symbolic structure. We will now verify that the class of ΨDOs is again an algebra, i. e., that any two ΨDOs may be composed.

Compositions of pseudo-differential operators

The composition theorem of pseudo-differential analysis is quite tricky to prove. We have for $f \in \mathscr{S}(\mathbb{R}^d)$

$$\big(a(x,D)\circ b(x,D)f\big)(x) = \frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^{4d}} e^{i(x-z)\eta}a(x,\eta)e^{i(z-y)\xi}b(z,\xi)f(y)\,\mathrm{d}y\mathrm{d}\xi\mathrm{d}z\mathrm{d}\eta$$

We want to write this again as a pseudo-differential operator, meaning as

$$c(x,D)f = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} e^{i(x-y)\xi}c(x,\xi)f(y)\,\mathrm{d}y\mathrm{d}\xi$$

and see that we thus have to set

$$\begin{aligned} c(x,\xi) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} e^{i(x-z)(\eta-\xi)}a(x,\eta)b(z,\xi)\,\mathrm{d}\eta\mathrm{d}z \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} e^{-iz\eta}a(x,\xi+\eta)b(z+x,\xi)\,\mathrm{d}\eta\mathrm{d}z \\ &=: \big(a\#b\big)(x,\xi). \end{aligned}$$

The latter expression is a so-called "twisted convolution" of a and b. It is now possible to prove that $c(x,\xi)$ is again a symbol, and furthermore that $c(x,\xi) = a(x,\xi)b(x,\xi) + r(x,\xi)$ where $r \in S^{m_1+m_2-1}(\mathbb{R}^d\times\mathbb{R}^d)$.

We have to skip the details of this lengthy computation and refer the reader to the literature. We however note an elementary fact: if $a \in S^{m_1}(\mathbb{R}^d \times \mathbb{R}^d)$ and $b \in S^{m_2}(\mathbb{R}^d \times \mathbb{R}^d)$, then $ab \in S^{m_1+m_2}(\mathbb{R}^d \times \mathbb{R}^d)$. Hence we summarize:

Theorem 3.57 (Compositions of pseudo-differential operators) Let $a \in S^{m_1}(\mathbb{R}^d \times \mathbb{R}^d)$, $b \in S^{m_2}(\mathbb{R}^d \times \mathbb{R}^d)$. Then $a(x,D) \circ b(x,D)$ is a pseudo-differential operator of order $m_1 + m_2$ with

$$a(x,D) \circ b(x,D) = \big(ab\big)(x,D) + r(x,D) \tag{3.41}$$

where $r \in S^{m_1+m_2-1}(\mathbb{R}^d \times \mathbb{R}^d)$.

Remark 3.58 It is in fact possible to prove an even stronger form of (3.41), resembling (3.31). In fact, if we denote the symbol of $a(x,D) \circ b(x,D)$ again by $a\#b$, then we have for any $N \in \mathbb{N}$

$$a\#b - \sum_{|\gamma| \le N} \frac{(-i)^{|\gamma|}}{\gamma!} D_\xi^\gamma a(x,\xi) D_x^\gamma b(x,\xi) \in S^{m_1+m_2-N}(\mathbb{R}^d \times \mathbb{R}^d).$$

This means the sum does not yield an exact expression for $a\#b$ as in the case for PDOs, it yields a "better and better approximation" in the sense of symbol orders.

Exercise 3.59 Prove the composition theorem for $d = 1$ when $a(x,D) = D_x$ and then for a general partial differential $a(x,D)$ on $\mathbb{R}^d$ by (repeated) use of (3.13) and (3.39).

3.3 The parametrix construction

We finally have established the existence of an algebra of operators containing the differential operators, compatible with the "filtration" provided by the order m. We will now harvest the benefits of the theory by "inverting" suitable partial differential operators – even those with variable coefficients.

Checking the product formula, we see that if $a(x,D)$ is a partial differential operator, then $a(x,D)b(x,D) = 1$ implies that $ab = 1 + r$ where r is of order -1 at most. That means that finding a suitable approximate inverse of an operator is connected with inverting its symbol. We make the following definition:

Definition 3.60 (Ellipticity) Let $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$. Then a is called *elliptic* if there exists a constant $R > 0$ and $c > 0$ such that

$$\big|a(x,\xi)\big| \ge c(1 + |\xi|)^m \qquad \forall |\xi| > R,\ x \in \mathbb{R}^d. \tag{3.42}$$

Likewise, a pseudo-differential operator A is called elliptic if its symbol is elliptic.

Example 3.61 In particular, $-\Delta$ is elliptic. More generally, any second order linear partial differential operator of the form

$$A = \sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}(x)\partial_{x_j}\partial_{x_k} + \sum_{j=1}^{d} b_j(x)\partial_{x_j} + c(x),$$

is elliptic if the coefficient matrix $A = (a_{jk})_{jk}$ satisfies $|\xi\cdot A\xi| \geq C|\xi|^2$ for all $\xi \neq 0$. Because of $\partial_{x_j}\partial_{x_k} = \partial_{x_k}\partial_{x_j}$, the matrix $\big(a_{jk}(x)\big)_{j,k}$ can be assumed symmetric. Then ellipticity at x is equivalent to all eigenvalues of $\big(a_{jk}(x)\big)_{j,k}$ being positive, which is the "standard" definition of ellipticity for quasilinear partial differential operators.

An elliptic operator may be "pseudo-inverted". Indeed, while we cannot hope for a true inverse (even $a(x,\xi)$ is in general not invertible), we may invert $a(x,\xi)$ out of bounded (in ξ) sets and thus "up to any order". Indeed, we will prove the following:

Proposition 3.62 (The symbolic parametrix construction) Let $a(x,\xi) \in S^m(\mathbb{R}^d\times\mathbb{R}^d)$ elliptic. Then there exists a sequence of symbols $b_j(x,\xi) \in S^{-m-j}(\mathbb{R}^d)$, $j \in \mathbb{N}$, such that for all $N \in \mathbb{N}$ we have

$$a\# \sum_{j=0}^{N} b_j = 1 + r_N \tag{3.43}$$

where $r_N \in S^{m-N-1}(\mathbb{R}^d\times\mathbb{R}^d)$.

Proof. Let $a(x,\xi)$ be elliptic. Then for some $C,R>0$, the estimate (3.42) is valid. Pick some excision function χ that vanishes for $|x| \leq R+1$. Then $a(x,\xi)$ is invertible on the support of χ and $b_0(x,\xi) := a(x,\xi)^{-1}\chi(x,\xi)$ defines a symbol $b_0 \in S^{-m}(\mathbb{R}^d\times\mathbb{R}^d)$. By Theorem 3.57 we have

$$a\#b = (ab_0) + r$$

with $r \in S^{-1}(\mathbb{R}^d\times\mathbb{R}^d)$. We may compute on the support of χ that

$$\big(ab_0\big)(x,\xi) = a(x,\xi)\chi(\xi)a^{-1}(x,\xi) = \chi(x,\xi) = 1-\phi^0(\xi)$$

where $\phi^0 \in \mathscr{C}_c^\infty(\mathbb{R}^d)$. Consequently

$$a\#b = 1 + r + \phi^0(\xi) := 1 + r_0$$

and we have inverted a “up to an error of order -1”.

We may now set $b_1 = -b_0 r_0$. Then

$$a\#b_1 = a\#b_0 + a\#b_1 = a\#b_0\#(1-r_0) = (1+r_0)\#(1-r_0) = 1 - r_0\#r_0.$$

Again by Theorem 3.57, $r_1 := r_0\#r_0 \in S^{-2}(\mathbb{R}^d \times \mathbb{R}^d)$, and we have inverted “up to an error of order -2”. Proceeding inductively, we may invert up to any error. The proof is complete. □

This construction is a nice feature, but it does not give us quite what we want: in fact, we would like to have an inversion “up to infinite order”. This is possible by a feature of pseudo-differential symbols called *asymptotic completeness*. In fact, we may not in general form $b = \sum_{j=1}^{\infty} b_j$, since the sum need not converge point-wise. However, using the symbolic estimates, it is possible to prove that for some excision function χ there exists a series of constants such that

$$q(x,\xi) := \sum_{j=1}^{\infty} \chi(\varepsilon\xi) b_j(x,\xi)$$

converges and yields a symbol $q \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$ that fulfils $a\#q = 1 + r$ where $r \in \bigcap_{m' \in \mathbb{R}} S^{m'}(\mathbb{R}^d \times \mathbb{R}^d)$. Such an r defines a *smoothing* operator, i. e. $r(x,D) : \mathscr{E}'(\mathbb{R}^d) \to \mathscr{C}^\infty(\mathbb{R}^d)$.

Exercise 3.63 Prove that $r \in \bigcap_{m' \in \mathbb{R}} S^{m'}(\mathbb{R}^d \times \mathbb{R}^d)$ yields a smoothing operator by showing that

$$k(x',x) = \int_{\mathbb{R}^d} e^{-i(x'-x)} r(x,\xi)\, \mathrm{d}\xi$$

is smooth. Since $r\big(x,D\big)u(x')$ may be written as $\langle u, k(x',\cdot)\rangle$, the smoothness of Au for $u \in \mathscr{E}'(\mathbb{R}^d)$ then follows.

Theorem 3.64 (Pseudodifferential parametrices) Let $A = a(x,D)$ with $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$ elliptic. Then there exists a pseudo-differential operator $Q = q(x,D)$ with $q \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$ elliptic such that

$$A \circ Q = 1 + R_r$$
$$Q \circ A = 1 + R_l$$

with R_r and R_l smoothing, i. e. $R_{r/l} : \mathscr{E}'(\mathbb{R}^d) \to \mathscr{C}^\infty(\mathbb{R}^d)$.

Exercise 3.65 Prove that the pseudo-differential operator given by the symbol $p(\xi) = \chi\big(|\xi|\big)|\xi|^{-2}$, for any excision function χ, gives a parametrix for $-\Delta$.

Remark 3.66 It should be mentioned that in the previous discussions, we made extensive use of the Fourier transform and we did not discuss the behaviour under changes of coordinates. It is, however, possible to generalize the concept of ΨDOs to manifolds and to identify the invariant parts of the theory. For these (quite elaborate) constructions, we refer the interested reader to [8], [13].

3.4 The wave front set revisited

To close off this section, let us note that the previous parametrix construction was purely on symbolic level – one might ask what happens when a symbol is not everywhere invertible, but only in the (conic) neighbourhood of some fixed (x_0, ξ_0). This leads to the notion of characteristic set.

Definition 3.67 (Characteristic set) A symbol $a \in S^m(\mathbb{R}^d \times \mathbb{R}^s)$ is called *non-characteristic* at $(x_0, \xi_0) \in \mathbb{R}^d \times \big(\mathbb{R}^s \setminus \{0\}\big)$ if there exists a constant R, an open conic neighbourhood $\Gamma \subset \big(\mathbb{R}^s \setminus \{0\}\big)$ of ξ_0 and some open neighbourhood $U \subset \mathbb{R}^d$ of x such that a satisfies the estimate (3.42) on $U \times \Gamma$.

The complement of all such (x_0, ξ_0) is the characteristic set of a, $\mathrm{char}(a)$.

Using the notion of characteristic set, the parametrix construction may be localized in the following sense: let $a \in S^m(\mathbb{R}^d \times \mathbb{R}^s)$. Then for every non-characteristic $(x_0, \xi_0) \in \mathbb{R}^d \times \big(\mathbb{R}^s \setminus \{0\}\big)$ there exists some conic neighbourhood $U \times \Gamma$ of (x_0, ξ_0) and a symbol $b \in S^{-m}(\mathbb{R}^d \times \mathbb{R}^s)$ such that $a\#b \equiv 1$ on $U \times \Gamma$.

One may ask why the concept of singularities is so essential in the theory of microlocal analysis. A (partial) reason for this is is the non-locality of ΨDOs: in general, we do not have for general $u \in \mathscr{E}'(\mathbb{R}^d)$ any inclusion relations between $\mathrm{supp}(Au)$ and $\mathrm{supp}(u)$. Indeed, by a theorem of Petree, we have $\mathrm{supp}(Au) \subset \mathrm{supp}(u)$ for all $u \in \mathscr{E}'(\mathbb{R}^d)$ if and only if A is already a *differential* operator. The singularities, however, do not share this non-locality. Indeed, the famous Hörmander-Sato-theorem states that $a \in S^m(\mathbb{R}^d \times \mathbb{R}^s)$ implies for $u \in \mathscr{E}'(\mathbb{R}^d)$ that

$$\mathrm{W}F\big(a(x,D)u\big) \subset \mathrm{W}F(u) \subset \mathrm{W}F\big(a(x,D)u\big) \cup \mathrm{char}(a).$$

This means: applying a pseudo-differential operator to u does not increase the wave front set. Furthermore, if we have an equation of the form $a(x,D)u = f$, we can determine the singularities of u from those of f and the characteristic set of a: without having to solve the equation for u, we can already get a bound for its singularities.

Example 3.68 For the wave equation $(\partial_t^2 - \Delta)u = f$ on $\mathbb{R}^{d+1}$ for some right-hand side f we easily compute that the symbol of the differential operator is given by $-\tau^2 + |\xi|^2$, denoting the covariables to (t,x) by (τ,ξ). Therefore, $\mathrm{char}(\partial_t^2 - \Delta)$ is given by

$$\mathbb{R}^{d+1} \times \Gamma = \mathbb{R}^{d+1} \times \Big\{(\tau,\xi) \in (\mathbb{R}^{d+1} \setminus \{0\}) \,\Big|\, \tau = \pm|\xi|\Big\}.$$

The set Γ is the so-called light-cone. The result tells us that singularities of u that are not singularities of f have something to do with the rays through points with $\pm|\xi| = \tau$. This is the starting point of the so-called study of propagation of singularities – the study of how singularities of solutions to evolution equations are transported under time evolution, a fascinating topic which exhibits the strength of the microlocal techniques, see e. g. [24] for a comprehensive introduction.

. . . .

4 Oscillatory integrals and their singularities

4.1 Motivation

The motivating example for the following paragraph is the following: consider the action of a pseudo-differential operator $a(x,D)$ on a test function f, viewed as a distribution. Then we may write

$$\langle a(x,D)f, g\rangle = (2\pi)^{-d} \int e^{-i(x-y)\xi} a(x,\xi) f(y) g(x) \mathrm{d}y \mathrm{d}\xi \mathrm{d}x = \langle I(a), f \otimes g\rangle \tag{3.44}$$

where we have set (as a formal expression)

$$I\big(a\big)(x,y) = (2\pi)^{-d} \int e^{-i(x-y)\xi} a(x,\xi) \mathrm{d}\xi. \tag{3.45}$$

This expression is called the (formal) *kernel* of $a(x,D)$. Of course, (3.45) does not in general converge absolutely as an L^1-integral: $a(x,\xi)$ does not even have to be bounded.

However, in (3.44) we only need it to be defined as a distribution acting on $f \otimes g$. This is precisely what we will establish in this section.

However, first we will enlarge the class of distributions of the form (3.45) to those of the form

$$I_\varphi\big(a\big)(x) = \int e^{i\varphi(x,\theta)} a(x,\theta)\mathrm{d}\theta. \tag{3.46}$$

where $a \in S^m(\mathbb{R}^d \times \mathbb{R}^s)$ and φ is a function on $\mathbb{R}^d \times \mathbb{R}^s$ that satisfies

- φ is a smooth, real-valued function on $\mathbb{R}^d \times \big(\mathbb{R}^s \setminus \{0\}\big)$,
- φ is 1-homogeneous in θ, meaning for $\lambda > 0$ we have $\varphi(x,\lambda\theta) = \lambda\varphi(x,\theta)$,
- φ is non-critical, meaning $\mathrm{d}\varphi \neq 0$ on all of $\mathbb{R}^d \times \big(\mathbb{R}^s \setminus \{0\}\big)$

We call a the *amplitude* and φ the *phase function* of the *oscillatory integral* $I_\varphi(a)$.

Let us first review the third criterion: we have for $(x,\theta) \in \big(\mathbb{R}^d \times \mathbb{R}^s \setminus \{0\}\big)$

$$\mathrm{d}\varphi(x,\theta) \neq 0 \quad \Longleftrightarrow \quad |\nabla_x\varphi|^2 + |\theta|^2|\nabla_\theta\varphi|^2 \neq 0$$

The right-hand side is homogeneous in θ, so it suffices to check this criterion for $|\theta| = 1$. The primary interest in these distributions stems from the fact that they appear as the kernels of Fourier Integral operators or as fundamental solutions to certain differential equations.

Example 3.69 Let $\tau : \mathbb{R}^d \to \mathbb{R}^d$ be a smooth diffeomorphism and consider the operator $f \mapsto \tau^* f = f \circ \chi$ for $f \in \mathscr{D}(\mathbb{R}^d)$. Using the Fourier transform, this may be written as $\tau^* f = \big(\mathscr{F}^{-1}\mathscr{F} f\big) \circ \chi$ and as such as

$$f \mapsto (2\pi)^{-d} \iint e^{i(\tau(x)-y)\xi} f(y)\, \mathrm{d}y\mathrm{d}\xi.$$

Example 3.70 The Cauchy problem to the 1-dimensional wave equation,

$$u_{tt} - c^2 u_{xx} = 0 \qquad \text{with} \quad u(0,x) = g(x) \quad \text{and} \quad u_t(0,x) = h(x)$$

is solved by D'Alembert's solution formula $u(x,t) = \frac{1}{2}\big[g(x-ct) + g(x+ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\,\mathrm{d}\xi$, which may be expressed as

$$u(x,t) = \frac{1}{2\pi}\iint e^{i\xi(x+ct)}e^{-i\xi y}\left(\frac{g(y)}{2}+\frac{h(y)}{2i\xi}\right)\mathrm{d}y\mathrm{d}\xi \\ + \frac{1}{2\pi}\iint e^{i\xi(x-ct)}e^{-i\xi y}\left(\frac{g(y)}{2}-\frac{h(y)}{2i\xi}\right)\mathrm{d}y\mathrm{d}\xi.$$

4.2 Oscillatory integrals as distributions

We now seek to define an oscillatory integral as a distribution. For $a \in S^m(\mathbb{R}^d \times \mathbb{R}^s)$ with $m < -s$, this is easily done by setting for $f \in \mathscr{D}(\mathbb{R}^d)$

$$\langle I(a), f\rangle := \int_{\mathbb{R}^d\times\mathbb{R}^s} e^{i\varphi(x,\theta)}a(x,\theta)f(x)\mathrm{d}\theta\mathrm{d}x \tag{3.47}$$

since in this case $I(a)$ is a locally integrable function in x, i. e. given by an L^1_{loc}-regular distribution

$$I_\varphi\big(a\big)(x) = \int_{\mathbb{R}^s} e^{i\varphi(x,\theta)}a(x,\theta)\mathrm{d}\theta$$

where the integral exists since $\big|a(x,\theta)\big| \leq C(1+|\theta|)^{-s-\varepsilon}$. Now, for general m, we recall that by Theorem 3.14, if $I(a) \in \mathscr{D}'$, we can realize it on the support of any given f as the derivative of a continuous function g, i. e. we have

$$\langle I_\varphi(a), f\rangle = \langle g, \partial^\alpha f\rangle$$

for some $\alpha \in \mathbb{N}^d$. We will now define oscillatory integrals as distributional derivatives. Assume, as a motivating example, that $s = d$ and $\varphi(x,\theta) = x\theta$, i. e. we are back in the case of a Fourier transform. Then we can make use of the identities (3.12) and calculate for $|x|+|\theta| > 0$ that

$$e^{ix\theta} = \frac{1}{|x|^2+|\theta|^2}\left(\sum_{j=1}^{d} -i\theta_j\partial_{x_j} + \sum_{k=1}^{d} -ix_j\partial_{\theta_k}\right)e^{ix\theta} =: Le^{ix\theta}. \tag{3.48}$$

Then, calculating on a formal level, we have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} e^{ix\theta}a(x,\theta)u(x)\mathrm{d}x\mathrm{d}\theta = \int L^N e^{ix\theta}a(x,\theta)f(x)\mathrm{d}x\mathrm{d}\theta \\ = \int e^{ix\theta}({}^tL)^N a(x,\theta)f(x)\mathrm{d}x\mathrm{d}\theta,$$

where ${}^tL$ is the formal transpose of L, a differential operator obtained from partial integration:

$$ {}^tLf = \left(\sum_{j=1}^{d} -i\theta_j \partial_{x_j} + \sum_{k=1}^{d} -ix_j \partial_{\theta_k} \right) \frac{f}{|x|^2 + |\theta|^2}. $$

This calculation is of course only valid, if all integrals are well-defined. For that, we'd have to introduce an excision function to "cut out" $\theta = 0$. Furthermore, a has to be integrable. Then, we notice one thing: if $a \in S^m(\mathbb{R}^d)$, then $({}^tL)^N a \in S^{m-N}(\mathbb{R}^d)$. This finally gives the following:

Lemma 3.71 If $a \in S^m(\mathbb{R}^d \times \mathbb{R}^s)$, $m < -s$, then for any excision function χ and $f \in \mathscr{D}'(\mathbb{R}^d)$ we may rewrite (3.47) as

$$ \langle I_\varphi(a), f \rangle = \big\langle I\big((1-\chi)a\big), f \big\rangle + \int e^{ix\theta} ({}^tL)^N \big[\chi(\theta) a(x,\theta) f(x)\big] \mathrm{d}x \mathrm{d}\theta \tag{3.49} $$

where for a general phase function we define L to be the first-order differential operator

$$ L = \frac{1}{|\nabla_x \varphi|^2 + |\nabla_\theta \varphi|^2} \left(\sum_{j=1}^{d} -i\nabla_{x_j} \varphi \partial_{x_j} + \sum_{r=1}^{s} -i\nabla_{\theta_r} \varphi \partial_{\theta_r} \right) $$

and ${}^tL$ by partial integration. Note that $Le^{i\varphi} = e^{i\varphi}$. Therein, $({}^tL)^N \big[\chi(\theta) a(x,\theta) f(x)\big] \in S^{m-|N|}(\mathbb{R}^d \times \mathbb{R}^s)$.

As a generalization to general symbols, we may remove the assumption on m by *regularizing I_φ*:

Definition 3.72 Let $a \in S^m(\mathbb{R}^d \times \mathbb{R}^s)$, φ be a phase function. Then the distribution $I_\varphi(a) \in \mathscr{D}'(\mathbb{R}^d)$, called the oscillatory integral with amplitude a and phase φ, is defined by

$$ \langle I_\varphi(a), f \rangle = \big\langle I\big((1-\chi)a\big), f \big\rangle + \int e^{ix\theta} ({}^tL)^N \big[\chi(\theta) a(x,\theta) f(x)\big] \mathrm{d}x \mathrm{d}\theta \tag{3.50} $$

for $f \in \mathscr{D}(\mathbb{R}^d)$, $N \in \mathbb{N}$ chosen large enough such that $m - N < -d$.

The following fact gives a way to actually calculate some oscillatory integrals explicitly:

Lemma 3.73 Let $a \in S^m(\mathbb{R}^d \times \mathbb{R}^s)$, φ be a phase function, $f \in \mathscr{D}(\mathbb{R}^d)$. Then $\langle I_\varphi(a), f\rangle$ may be calculated as

$$\langle I_\varphi(a), f\rangle = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i\varphi(x,\theta)} e^{-\varepsilon|\theta|^2} a(x,\theta) f(x) \mathrm{d}x \mathrm{d}\theta.$$

Exercise 3.74 Prove Lemma 3.73. For simplicity, you may assume that $\varphi(x,\theta)$ is of the form $x \cdot \theta$.

4.3 Singularities of oscillatory integrals

One feature of the previously treated oscillatory integrals is their clear singularity structure. Recall that, by Definition 3.41, $(x_0, \xi_0) \notin \mathrm{W}F(u)$ if there exists a cut-off ϕ^{x_0} around x_0 and a conic neighbourhood of ξ_0, such that

$$\left|\mathscr{F}\left(\phi^{x_0} u\right)(\xi)\right| \le C_N (1+|\xi|)^{-N} \qquad \forall \xi \in \Gamma. \tag{3.51}$$

For an oscillatory integral, $\mathscr{F}\left(\phi^{x_0} u\right)(\xi)$ takes the explicit form (with the θ-integration interpreted as a limit as above)

$$\mathscr{F}\left(\phi^{x_0} u\right)(\xi) = \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i[\varphi(x,\theta) - x\xi]} \varphi(x) a(x,\theta) u(x) \mathrm{d}x \mathrm{d}\theta. \tag{3.52}$$

We observe that (3.52) is again of “oscillatory form”. The function which plays the role of the phase function is now $\psi(x,\theta,\xi) = \varphi(x,\theta) - x\xi$. We seek to regularize the integral again by introducing a differential operator S which satisfies $Se^{i\psi} = S$. An operator which does the trick is

$$S = \frac{1}{|\xi - \nabla_x \varphi(x,\theta)|^2 + |\nabla_\theta \varphi(x,\theta)|^2} \left(\sum_{j=1}^{d} -i(\nabla_{x_j}\varphi - \xi_j)\partial_{x_j} + \sum_{r=1}^{s} -i\nabla_{\theta_r}\varphi \partial_{\theta_r} \right). \tag{3.53}$$

Analysing the structure of (3.53), we observe that if S is applied to some function in (x,ξ,η), the outcome will decay by one order lower in ξ. Therefore, this operator may be used to obtain an estimate like (3.51). For that, however, it needs to be well defined on the integrand, that is we need to have

$$\xi \neq \nabla_x \varphi(x,\theta), \qquad \nabla_\theta \varphi(x,\theta) \neq 0.$$

Filling out the proof one obtains:

Theorem 3.75 Let $a \in S^m(\mathbb{R}^d \times \mathbb{R}^s)$, φ be a phase function. Then the singularities of the distribution $I_\varphi(a)$ defined in Definition 3.72 satisfy

$$\begin{aligned} &\mathrm{W}F\big(I_\varphi(a)\big) \subset \Lambda_\varphi, \text{ where} \\ &\qquad \Lambda_\varphi := \Big\{(x,\xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \,\Big|\, \exists \theta \text{ s.t.} \nabla_x \varphi(x,\theta) = \xi \text{ and } \nabla_\theta \varphi(x,\theta) = 0\Big\}. \end{aligned}$$

With this result, this introductory write-up on microlocal analysis is finished. The reader should by now be quite familiar with the concepts and questions rised in microlocal analysis. Since these notes are intended as a "first read", let us provide some leads and references for further studies: some excellent starting references for distribution theory from a microlocal point of view are are [8], [13], [14], [20], [24] and there are many more. For those interested in the topic of FIOs, which is technically very challenging, the number of references is rather small, I recommend [1], [7], [5], [23], [9].

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5 Appendix: Outlook

In this section, we will briefly establish connections with further topics from microlocal analysis for which there were lectures held at the Summer school, among them

- ◇ Symplectic geometry,
- ◇ Fourier integral operators,
- ◇ Complex microlocal analysis by use of the Bargmann transform, see [12].

5.1 Symplectic geometry and oscillatory integrals

We briefly give an outline of the connection between the previous topics and the lecture on symplectic geometry. We notice that the set Λ_φ defined in Theorem 3.75 is only attached to φ and does not depend on a. We now observe that it is a geometric object. Indeed, Λ_φ is defined as the image of

$$C_\varphi = \big\{(x,\theta) \in \mathbb{R}^d \times \mathbb{R}^s \,\big|\, \nabla_\theta \varphi(x,\theta) = 0\big\}$$

under the map $(x,\theta) \mapsto (x, \nabla_x \varphi)$. Now C_φ is of the form of a null set. It is well known that a null set is a manifold if 0 is a *regular value* of the defining function, in this case $\nabla_\theta \varphi$. A phase function is called *non-degenerate*, if when $\nabla_\theta \varphi(x,\theta) = 0$, then $D_{(x,\theta)}(\partial_{\theta_r}\varphi)$ are linearly independent for all $r = 1,\ldots,s$. In this case, Λ_φ is an immersed submanifold of $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$.

Finally, a manifold $\Lambda \subset T^*\mathbb{R}^d$ is called *conic Lagrangian* if the canonical differential one-form $\alpha = \xi \mathrm{d}x$ vanishes when restricted to Λ. It is easy to check that this holds true on Λ_φ by θ-homogeneity of φ. A converse result also holds:

Proposition 3.76 A d-dimensional submanifold $\Lambda \subset T^*\mathbb{R}^d \setminus \{0\}$ is a conic Lagrangian manifold if and only if every $(x,\xi) \in \Lambda$ admits a conic neighbourhood Γ such that $\Lambda \cap \Gamma = \Lambda_\varphi$ for some locally defined non-degenerate phase function φ.

This means that to any (local) phase function φ, it is possible to attach a geometric object Λ_φ and vice versa. Let us state why this is important: If we change coordinates in an oscillatory integral, the phase function and symbol will in general change. However, the geometric content is preserved. This means that while oscillatory integrals with a fixed phase function do not form a “good class of geometric distributions”, the following do:

Definition 3.77 Let $\Lambda \subset T^*\mathbb{R}^d \setminus \{0\}$ be a conic Lagrangian manifold. We say that $u \in \mathscr{D}'(\mathbb{R}^d)$ is a *Lagrangian distribution* associated to Λ if it can be (micro-locally) written as $u = I_\varphi(a)$ where φ is a phase function that satisfies $\Lambda = \Lambda_\varphi$ in some small conic neighbourhood and a is a symbol.

This definition is a starting point for the geometric and global, that is invariant, study of oscillatory integrals.

5.2 Fourier Integral Operators

We have already noted, in (3.44), that a pseudo-differential operator can be identified with its so-called kernel. There is a deeper theorem behind this equivalence, the Schwartz kernel theorem.

Theorem 3.78 (Schwartz kernel theorem) There is a one-to-one correspondence between continuous, linear maps $A : \mathscr{D}(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d)$ and distributions $K_A \in \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$, called the kernel isomorphism. It is given by

$$\langle Au, v\rangle = \langle K_A,\ u \otimes v\rangle,$$

where u, $v \in \mathscr{D}(\mathbb{R}^d)$.

An important result from microlocal analysis describes how an operator "moves" singularities in terms of the singularities of its kernel:

Theorem 3.79 Let A be a continuous, linear map $A : \mathscr{D}(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d)$ with kernel $K_A \in \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ and $u \in \mathscr{E}'(\mathbb{R}^d)$. If $(y,\eta) \in \mathrm{W}F(u) \to ((x,y),(0,-\eta)) \notin \mathrm{W}F(K)$, then it is possible to extend the action of A to u, and Au satisfies

$$\begin{aligned}\mathrm{W}F(Au) \subset \big\{(x,\xi)\,\big|\,\exists y \text{ s.t. } (x,y,\xi,0) \in \mathrm{W}F(K_A)\big\}& \\ \cup \Big\{(x,y,\xi,\eta)\,|\,(y,\eta) \in \mathrm{W}F(u),\ \big((x,y),(\xi,-\eta)\big) \in \mathrm{W}F(K_A)\Big\}&. \qquad (3.54)\end{aligned}$$

Assuming that the first set is empty, we see that $\mathrm{W}F'(K_A) = \{(x,y,\xi,-\eta)\}$ can be viewed as a transformation

$$(\text{Singularities of } u) \quad (y,\eta) \mapsto (x,\xi) \quad (\text{Singularties of } Au).$$

Fourier integral operators are now a generalization of pseudo-differential operators where this transformation is (locally) a *canonical transformation* from symplectic geometry, or even more generally, their kernels are (microlocally) given by oscillatory integrals with phase functions for which Λ_φ is a *canonical relation*. They form a class of operators that also contains transformations as encountered in Example 3.69.

An introduction to the theory of Fourier integral operators goes well beyond the scope of this document and is subject to another course. Nevertheless, the methods exhibited in Section 4 highlight how their kernels may be treated from a distribution-theoretical point of view. The underlying geometry, as previously mentioned, is that of symplectic geometry, in particular the Hamilton-Jacoby theory underlying classical mechanics.

5.3 The wave front set in terms of the Bargmann transform

In this section, we want to briefly mention how the definition of the wave front set is related to its characterization in terms of the Bargmann (or "FBI") transform. Let $B_{x,\xi,\lambda} \in \mathscr{S}(\mathbb{R}^d)$ be the function

$$y \mapsto (2\pi\lambda^{3/2})^{-d/2} e^{-\frac{\lambda}{2}|y-x|^2} e^{-i\lambda y\xi} \qquad (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d,\ \lambda > 0.$$

Recall that the Bargmann transform (with parameter $\lambda > 0$) of $u \in \mathscr{S}'(\mathbb{R}^d)$ is given by $\mathscr{F}_\lambda\big(u\big)(x,\xi) = \langle u,\ B_{x,\xi,\lambda}\rangle$, that is for $u = T_g$ for $g \in \mathscr{S}(\mathbb{R}^d)$, it is given by

$$\mathscr{F}_\lambda\big(T_g\big)(x,\xi) = (2\pi\lambda^{3/2})^{-d/2}\int g(y)e^{-\frac{\lambda}{2}|y-x|^2}e^{-i\lambda y\xi}\,\mathrm{d}x.$$

In the lecture on microlocal analysis in the complex plane, [12], it was claimed that the wave front set of u may be characterized as follows:

Proposition 3.80 Let $u \in \mathscr{S}'(\mathbb{R}^d)$, $(x_0,\xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$. Then $(x_0,\xi_0) \notin \mathrm{W}F(u)$ if and only if there exists an open neighbourhood U of (x_0,ξ_0) such that for all $N \in \mathbb{N}$ there exists some $C_N > 0$ such that for all $\lambda > 1$

$$\Big|\mathscr{F}_\lambda\big(u\big)(x,\xi)\Big| \le C_N\lambda^{-N} \qquad (x,\xi) \in U. \tag{3.55}$$

A nice proof of this equivalence is given in [3], there in the context of Sobolev spaces. We will only motivate the method of proof. Comparing (3.55) to (3.25), we see that we may write (3.55) as

$$\mathscr{F}\big(\phi_\lambda^{x_0}u\big)(\lambda\xi) \le C_N\lambda^{-N} \qquad (x,\xi) \in U.$$

where $\phi_\lambda^{x_0}(y) = \lambda^{3d/2}e^{-\frac{\lambda}{2}|x_0-x|^2}$ and C_N is some constant. Now in the usual definition of $\mathrm{W}F$, we expected (3.25) to hold for all cut-offs ϕ^{x_0}. Here, we are given a Gaussian *centered* at x_0 and rescaled by λ. For increasing $\lambda > 0$, this is as good as checking (3.25) for all cut-offs. Indeed, the Gaussian "shrinks", see Figure 5.3, and becomes more and more centred around x_0. Using a partition of unity and an inversion formula for $\mathscr{F}_\lambda$, this train of thoughts may be used to prove Propostion 3.80. One use of it is immediately clear: it provides a method of computing $\mathrm{W}F(u)$ explicitly. For a cut-off ϕ^{x_0}, it is usually not possible to find explicit expressions for $\mathscr{F}(\phi^{x_0}u)$. The main reason for considering $\mathscr{F}_\lambda(u)$, however, is that these methods can be applied to study *analytic singularities* of u. For that, the usual definition in terms of cut-offs breaks down, since multiplication by cut-offs does not preserve analyticity.

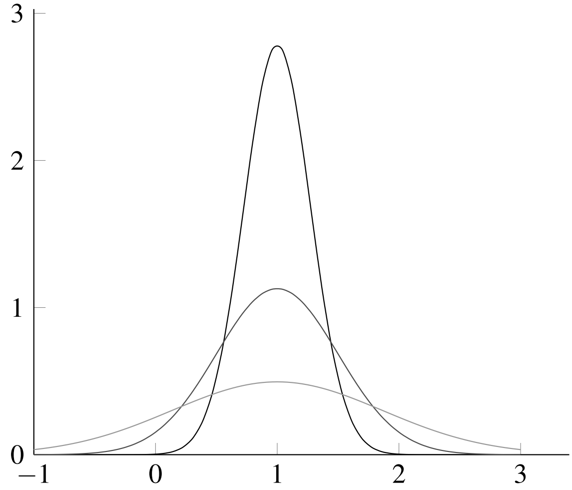

Figure 3.3: “Shrinking” Gaussian windows.

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Bibliography

[1] J. J. Duistermaat, V. W. Guillemin, L. Hörmander and J. Brüning, *Mathematics Past and Present: Fourier Integral Operators: Selected Classical Articles*, Springer, 1994.

[2] H. O. Cordes, *The Technique of Pseudodifferential Operators*, Cambridge Univ. Press, 1995.

[3] J. M. Delort, *FBI transformation: second microlocalization and semilinear caustics*, Vol. 1522, Springer-Verlag, Berlin, 1992.

[4] C. Ducourtioux and M.-F. Ouedraogo, *Préambule aux opérateurs Fourier intégraux: les opérateurs pseudo-différentiels, Inversion d'un opérateur différentiel elliptique*, in this volume.

[5] J. J.Duistermaat, *Fourier integral operators*, Springer, Berlin, 1996.

[6] J. J. Duistermaat and J. J. Kolk, *Distributions*, Birkhäuser, Basel, 2010.

[7] G. Eskin, *Lectures on Linear Partial Differential Equations*, American Mathematical Society, 2011.

[8] A. Grigis, and J. Sjöstrand, *Microlocal analysis for differential operators*, Cambridge University Press, 1994.

[9] L. Hörmander, *Fourier Integral operators I*, Acta Math. **127** 1 (1971), 79–183.

[10] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer, 1990.

[11] L. Hörmander, *The Analysis of Linear Partial Differential Operators III*, Springer, 1994.

[12] G. Lebeau, *Transformation de Bargmann et analyse microlocale analytique*, in this volume.

[13] M. S. Joshi, *Introduction to pseudo-differential operators*, lecture notes, available at http://arxiv.org/abs/math.AP/9906155.

[14] R. Melrose, *Lecture Notes for Graduate Analysis*, 2004, available online at http://www-math.mit.edu/∼ rbm/18.155-F04-notes/Lecture-notes.pdf.

[15] F. Nicola and R. Rodino, *Global Pseudo-differential Calculus on Euclidean Spaces*, Birkhäuser, Basel, 2010.

[16] M. Reed and B. Simon, *Methods of modern mathematical physics I*, Acad. Press, New York, 1972.

[17] M. Reed and B. Simon, *Methods of modern mathematical physics II*, Acad. Press, New York, 1975.

[18] S. Paycha, *Recollection of a Singular School*, Newsletter of the European Mathematical Society, Issue 98, December 2015.

[19] W. Rudin, *Functional analysis*, 2nd ed., International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1991.

[20] X. Saint Raymond, *Elementary Introduction to the Theory of Pseudodifferential Operators*, Studies in Advanced Mathematics (Book 3), CRC Press, 1991.

[21] L. Schwartz, *Théorie des distributions I–II*, Hermann, Paris, 1950–1951.

[22] M. Shubin, *Pseudodifferential operators and the spectral theory*. Springer Series in Soviet Mathematics. Springer, Berlin, 1987.

[23] F. Treves, *Introduction to pseudodifferential and Fourier integral operators*, Vol. 1-2. The University Series in Mathematics. Plenum Press, New York-London, 1980.

[24] J. Wunsch, *Microlocal Analysis and Evolution Equations*, lecture notes from 2008 CMI/ETH Summer school, 2008, available online at http://www.math.-northwestern.edu/∼jwunsch/micronotes.pdf.

Chapter 4

Fourier multipliers in Hilbert spaces

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Abstract. *This is a survey on a notion of invariant operators, or Fourier multipliers on Hilbert spaces. This concept is defined with respect to a fixed partition of the space into a direct sum of finite dimensional subspaces. In particular this notion can be applied to the important case of $L^2(M)$ where M is a compact manifold endowed with a positive measure. The partition in this case comes from the spectral properties of a a fixed elliptic operator E.*

1 Introduction

These notes are based on our paper [15] and have been prepared for the instructional volume associated to the *Summer School on Fourier Integral Operators held in Ouagadougou, Burkina Faso*, that the authors took part in from September 14th to September 26th, 2015.

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In this note we discuss invariant operators, or Fourier multipliers in a general Hilbert space $\mathscr{H}$. This notion is based on a partition of $\mathscr{H}$ into a direct sum of finite dimensional subspaces, so that a densely defined operator on $\mathscr{H}$ can be decomposed as acting in these subspaces. In the present exposition we follow our detailed description in [15], with which there are intersections and to which we also refer for further details.

There are two main examples of this construction: operators on $\mathscr{H} = L^2(M)$ for a compact manifold M as well as operators on $\mathscr{H} = L^2(G)$ for a compact Lie group G. The difference in approaches to these settings is in the choice of partitions of $\mathscr{H}$ into direct sums of subspaces: in the former case they are chosen as eigenspaces of a fixed elliptic pseudo-differential operator on M while in the latter case they are chosen as linear spans of matrix coefficients of inequivalent irreducible unitary representations of G.

Let M be a closed manifold (i. e. a compact smooth manifold without boundary) of dimension n endowed with a positive measure $\mathrm{d}x$. Given an elliptic positive pseudo-differential operator E of order ν on M, by considering an orthonormal basis consisting of eigenfunctions of E we can associate a discrete Fourier analysis to the operator E in the sense introduced by Seeley ([38], [40]).

These notions can be applied to the derivation of conditions characterising those invariant operators on $L^2(M)$ that belong to Schatten classes. Furthermore, sufficient conditions for the r-nuclearity on L^p-spaces can also be obtained as well as the corresponding trace formulas relating operator traces to expressions involving their symbols. More details on these applications can be found in Section 8 of [15].

A characteristic feature that appears is that no regularity is assumed neither on the symbol nor on the kernel. In the case of compact Lie groups, our results extend results on Schatten classes and on r-nuclear operators on L^p spaces that have been obtained in [13], [16]. This can be shown by relating the symbols introduced in this paper to matrix-valued symbols on compact Lie groups developed in [34], [33].

To formulate the notions more precisely, let $\mathscr{H}$ be a complex Hilbert space and let $T : \mathscr{H} \to \mathscr{H}$ be a linear compact operator. If we denote by $T^* : \mathscr{H} \to \mathscr{H}$ the adjoint of T, then the linear operator $(T^*T)^{\frac{1}{2}} : \mathscr{H} \to \mathscr{H}$ is positive and compact. Let $(\psi_k)_k$ be an orthonormal basis for $\mathscr{H}$ consisting of eigenvectors of $|T| = (T^*T)^{\frac{1}{2}}$, and let $s_k(T)$ be the eigenvalue corresponding to the eigenvector ψ_k, $k = 1, 2, \ldots$. The non-negative numbers $s_k(T)$, $k = 1, 2, \ldots$, are called the singular values of $T : \mathscr{H} \to \mathscr{H}$. If $0 < p < \infty$ and the sequence of singular values is p-summable, then T is said to belong to the Schatten class

$S_p(\mathscr{H})$, and it is well known that each $S_p(\mathscr{H})$ is an ideal in $\mathscr{L}(\mathscr{H})$. If $1 \le p < \infty$, a norm is associated to $S_p(\mathscr{H})$ by

$$\|T\|_{S_p} = \left(\sum_{k=1}^{\infty} \left(s_k(T)\right)^p \right)^{\frac{1}{p}}. \tag{4.1}$$

If $1 \le p < \infty$ the class $S_p(\mathscr{H})$ becomes a Banach space endowed by the norm $\|T\|_{S_p}$. If $p = \infty$ we define $S_\infty(\mathscr{H})$ as the class of bounded linear operators on $\mathscr{H}$, with $\|T\|_{S_\infty} := \|T\|_{\mathrm{op}}$, the operator norm. For the Schatten class S_2 we will sometimes write $\|T\|_{\mathrm{HS}}$ instead of $\|T\|_{S_2}$. In the case $0 < p < 1$ the quantity $\|T\|_{S_p}$ only defines a quasi-norm, and $S_p(\mathscr{H})$ is also complete. The space $S_1(\mathscr{H})$ is known as the *trace class* and an element of $S_2(\mathscr{H})$ is usually said to be a *Hilbert-Schmidt* operator. For the basic theory of Schatten classes we refer the reader to [19], [31], [42], [37].

It is well known that the class $S_2(L^2(M))$ is characterised by the square integrability of the corresponding integral kernels, however, kernel estimates of this type are not effective for classes $S_p(L^2(M))$ with $p < 2$. This is explained by Carleman's classical example [8] on the summability of Fourier coefficients of continuous functions (see [16] for a complete explanation of this fact). This obstruction explains the relevance of symbolic Schatten criteria and here we will clarify the advantage of the symbol approach with respect to this obstruction. With this approach, no regularity of the kernel needs to be assumed.

We introduce ℓ^p-style norms on the space of symbols Σ, yielding discrete spaces $\ell^p(\Sigma)$ for $0 < p \le \infty$, normed for $p \ge 1$. Denoting by σ_T the matrix symbol of an invariant operator T provided by Theorem 4.7, Schatten classes of invariant operators on $L^2(M)$ can be characterised in terms of symbols. Here, the condition that T is invariant will mean that T is strongly commuting with E (see Theorem 4.7).

On the level of the Fourier transform this means that $\widehat{Tf}(\ell) = \sigma(\ell)\widehat{f}(\ell)$ for a family of matrices $\sigma(\ell)$, i. e. T assumes the familiar form of a Fourier multiplier.

In Section 2 in Theorem 4.1 we discuss the abstract notion of symbol for operators densely defined in a general Hilbert space $\mathscr{H}$, and give several alternative formulations for invariant operators, or for Fourier multipliers, relative to a fixed partition of $\mathscr{H}$ into a direct sum of finite dimensional subspaces,

$$\mathscr{H} = \bigoplus_j H_j.$$

Consequently, in Theorem 4.3 we give the necessary and sufficient condition for the bounded extendability of an invariant operator to $\mathscr{L}(\mathscr{H})$ in terms of its symbol, and in Theorem 4.5 the necessary and sufficient condition for the operator to be in Schatten classes $S_r(\mathscr{H})$ for $0 < r < \infty$, as well as the trace formula for operators in the trace class $S_1(\mathscr{H})$ in terms of their symbols. As our subsequent analysis relies to a large extent on properties of elliptic pseudo-differential operators on M, in Sections 3 and 4 we specify this abstract analysis to the setting of operators densely defined on $L^2(M)$. The main difference is that we now adopt the Fourier analysis to a fixed elliptic positive pseudo-differential operator E on M, contrary to the case of an operator $E \in \mathscr{L}(\mathscr{H})$ in Theorem 4.2.

The notion of invariance depends on the choice of the spaces H_j. Thus, in the analysis of operators on M we take H_j's to be the eigenspaces of E. However, other choices are possible. For example, for $\mathscr{H} = L^2(G)$ for a compact Lie group G, choosing H_j's as linear spans of representation coefficients for inequivalent irreducible unitary representations of G, we make a link to the quantization of pseudo-differential operators on compact Lie groups as in [33]. These two partitions coincide when inequivalent representations of G produce distinct eigenvalues of the Laplacian; for example, this is the case for $G = \mathrm{SO}(3)$. However, the partitions are different when inequivalent representations produce equal eigenvalues, which is the case, for example, for $G = \mathrm{SO}(4)$. For the more explicit example on $\mathscr{H} = L^2(\mathbb{T}^n)$ on the torus see Remark 4.6. A similar choice could be made in other settings producing a discrete spectrum and finite dimensional eigenspaces, for example for operators in Shubin classes on $\mathbb{R}^n$, see Chodosh [9] for the case $n = 1$.

As an illustration we give an application to the spectral theory. The analogous concept to Schatten classes in the setting of Banach spaces is the notion of r-nuclearity introduced by Grothendieck [20]. It has applications to questions of the distribution of eigenvalues of operators in Banach spaces. In the setting of compact Lie groups these applications have been discussed in [16] and they include conclusions on the distribution or summability of eigenvalues of operators acting on L^p-spaces. Another application is the Grothendieck-Lidskii formula which is the formula for the trace of operators on $L^p(M)$. Once we have r-nuclearity, most of further arguments are then purely functional analytic, so they apply equally well in the present setting of closed manifolds.

The paper is organised as follows.

In Section 2 we discuss Fourier multipliers and their symbols in general Hilbert spaces. In Section 3 we associate a global Fourier analysis to an elliptic positive pseudo-differential operator E on a closed manifold M. In Section 4 we introduce the class of operators invariant relative to E as well as their matrix-valued symbols, and apply this to characterise invariant operators in Schatten classes in Section 5.

Throughout the paper, we denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Also $\delta_{j\ell}$ will denote the Kronecker delta, i. e. $\delta_{j\ell} = 1$ for $j = \ell$, and $\delta_{j\ell} = 0$ for $j \neq \ell$.

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2 Fourier multipliers in Hilbert spaces

In this section we present an abstract set up to describe what we will call invariant operators, or Fourier multipliers, acting on a general Hilbert space $\mathscr{H}$. We will give several characterisations of such operators and their symbols. Consequently, we will apply these notions to describe several properties of the operators, in particular, their boundedness on $\mathscr{H}$ as well as the Schatten properties.

We note that direct integrals (sums in our case) of Hilbert spaces have been investigated in a much greater generality, see e. g. Bruhat [6], Dixmier [12, Ch. 2, §2], [11, Appendix]. The setting required for our analysis is much simpler, so we prefer to adapt it specifically for consequent applications.

The main application of the constructions below will be in the setting when M is a compact manifold without boundary, $\mathscr{H} = L^2(M)$ and $\mathscr{H}^\infty = C^\infty(M)$, which will be described in detail in Section 3. However, several facts can be more clearly interpreted in the setting of abstract Hilbert spaces, which will be our set up in this section. With this particular example in mind, in the following theorem, we can think of $\{e_j^k\}$ being an orthonormal basis given by eigenfunctions of an elliptic operator on M, and d_j the corresponding multiplicities. However, we allow flexibility in grouping the eigenfunctions in order to be able to also cover the case of operators on compact Lie groups.

Theorem 4.1 Let $\mathscr{H}$ be a complex Hilbert space and let $\mathscr{H}^\infty \subset \mathscr{H}$ be a dense linear subspace of $\mathscr{H}$. Let $\{d_j\}_{j\in\mathbb{N}_0} \subset \mathbb{N}$ and let $\{e_j^k\}_{j\in\mathbb{N}_0, 1\leq k\leq d_j}$ be an orthonormal basis of

$\mathscr{H}$ such that $e_j^k \in \mathscr{H}^\infty$ for all j and k. Let $H_j := \mathrm{span}\{e_j^k\}_{k=1}^{d_j}$, and let $P_j : \mathscr{H} \to H_j$ be the orthogonal projection. For $f \in \mathscr{H}$, we denote

$$\widehat{f}(j,k) := (f, e_j^k)_{\mathscr{H}}$$

and let $\widehat{f}(j) \in \mathbb{C}^{d_j}$ denote the column of $\widehat{f}(j,k)$, $1 \le k \le d_j$. Let $T : \mathscr{H}^\infty \to \mathscr{H}$ be a linear operator. Then the following conditions are equivalent:

(A) For each $j \in \mathbb{N}_0$, we have $T(H_j) \subset H_j$.

(B) For each $\ell \in \mathbb{N}_0$ there exists a matrix $\sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}$ such that for all e_j^k

$$\widehat{Te_j^k}(\ell, m) = \sigma(\ell)_{mk}\delta_{j\ell}.$$

(C) If in addition, e_j^k are in the domain of T^* for all j and k, then for each $\ell \in \mathbb{N}_0$ there exists a matrix $\sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}$ such that

$$\widehat{Tf}(\ell) = \sigma(\ell)\widehat{f}(\ell) \tag{4.2}$$

for all $f \in \mathscr{H}^\infty$.

The matrices $\sigma(\ell)$ in (B) and (C) coincide. The equivalent properties (A) – (C) follow from the condition

(D) For each $j \in \mathbb{N}_0$, we have $TP_j = P_jT$ on $\mathscr{H}^\infty$.

If, in addition, T extends to a bounded operator $T \in \mathscr{L}(\mathscr{H})$ then (D) is equivalent to (A) – (C).

Under the assumptions of Theorem 4.1, we have the direct sum decomposition

$$\mathscr{H} = \bigoplus_{j=0}^{\infty} H_j, \qquad H_j = \mathrm{span}\{e_j^k\}_{k=1}^{d_j}, \tag{4.3}$$

and we have $d_j = \dim H_j$. The two applications that we will consider will be with $\mathscr{H} = L^2(M)$ for a compact manifold M with H_j being the eigenspaces of an elliptic pseudo-differential operator E, or with $\mathscr{H} = L^2(G)$ for a compact Lie group G with

$$H_j = \mathrm{span}\{\xi_{km}\}_{1 \le k,m \le d_\xi}$$

for a unitary irreducible representation $\xi \in [\xi_j] \in \widehat{G}$. The difference is that in the first case we will have that the eigenvalues of E corresponding to H_j's are all distinct, while in the second case the eigenvalues of the Laplacian on G for which H_j's are the eigenspaces, may coincide. In Remark 4.6 we give an example of this difference for operators on the torus $\mathbb{T}^n$.

In view of properties (A) and (C), respectively, an operator T satisfying any of the equivalent properties (A) – (C) in Theorem 4.1, will be called an *invariant operator*, or a *Fourier multiplier* relative to the decomposition $\{H_j\}_{j\in\mathbb{N}_0}$ in (4.3). If the collection $\{H_j\}_{j\in\mathbb{N}_0}$ is fixed once and for all, we can just say that T is *invariant* or a *Fourier multiplier*.

The family of matrices σ will be called the *matrix symbol* of T relative to the partition $\{H_j\}$ and to the basis $\{e_j^k\}$. It is an element of the space Σ defined by

$$\Sigma = \left\{\sigma : \mathbb{N}_0 \ni \ell \mapsto \sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}\right\}. \tag{4.4}$$

A criterion for the extendability of T to $\mathscr{L}(\mathscr{H})$ in terms of its symbol will be given in Theorem 4.3.

For $f \in \mathscr{H}$, in the notation of Theorem 4.1, by definition we have

$$f = \sum_{j=0}^{\infty}\sum_{k=1}^{d_j} \widehat{f}(j,k) e_j^k \tag{4.5}$$

with the convergence of the series in $\mathscr{H}$. Since $\{e_j^k\}_{j\geq 0}^{1\leq k\leq d_j}$ is a complete orthonormal system on $\mathscr{H}$, for all $f \in \mathscr{H}$ we have the Plancherel formula

$$\|f\|^2_{\mathscr{H}} = \sum_{j=0}^{\infty}\sum_{k=1}^{d_j} |(f, e_j^k)|^2 = \sum_{j=0}^{\infty}\sum_{k=1}^{d_j} |\widehat{f}(j,k)|^2 = \|\widehat{f}\|^2_{\ell^2(\mathbb{N}_0,\Sigma)}, \tag{4.6}$$

where we interpret $\widehat{f} \in \Sigma$ as an element of the space

$$\ell^2(\mathbb{N}_0,\Sigma) = \left\{ h : \mathbb{N}_0 \to \prod_d \mathbb{C}^d : h(j) \in \mathbb{C}^{d_j} \text{ and } \sum_{j=0}^{\infty}\sum_{k=1}^{d_j} |h(j,k)|^2 < \infty \right\}, \tag{4.7}$$

and where we have written $h(j,k) = h(j)_k$. In other words, $\ell^2(\mathbb{N}_{0,}\Sigma)$ is the space of all $h \in \Sigma$ such that

$$\sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |h(j,k)|^2 < \infty.$$

We endow $\ell^2(\mathbb{N}_0,\Sigma)$ with the norm

$$\|h\|_{\ell^2(\mathbb{N}_0,\Sigma)} := \left(\sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |h(j,k)|^2 \right)^{\frac{1}{2}}. \tag{4.8}$$

We note that the matrix symbol $\sigma(\ell)$ depends not only on the partition (4.3) but also on the choice of the orthonormal basis. Whenever necessary, we will indicate the dependance of σ on the orthonormal basis by writing $\left(\sigma, \{e_j^k\}_{j\geq 0}^{1\leq k\leq d_j}\right)$ and we also will refer to $\left(\sigma, \{e_j^k\}_{j\geq 0}^{1\leq k\leq d_j}\right)$ as the *symbol* of T. Throughout this section the orthonormal basis will be fixed and unless there is some risk of confusion, the symbols will be denoted simply by σ. In the invariant language, we have that the transpose of the symbol, $\sigma(j)^\top = T|_{H_j}$ is just the restriction of T to H_j, which is well defined in view of the property (A).

We will also sometimes write T_σ to indicate that T_σ is an operator corresponding to the symbol σ. It is clear from the definition that invariant operators are uniquely determined by their symbols. Indeed, if $T = 0$ we obtain $\sigma = 0$ for any choice of an orthonormal basis. Moreover, we note that by taking $j = \ell$ in (B) of Theorem 4.1, we obtain the formula for the symbol:

$$\sigma(j)_{mk} = \widehat{Te_j^k}(j,m), \tag{4.9}$$

for all $1 \leq k,m \leq d_j$. The Formula (4.9) furnishes an explicit formula for the symbol in terms of the operator and the orthonormal basis. The definition of Fourier coefficients tells us that for invariant operators we have

$$\sigma(j)_{mk} = (Te_j^k, e_j^m)_H. \tag{4.10}$$

In particular, for the identity operator $T = I$ we have $\sigma_I(j) = I_{d_j}$, where $I_{d_j} \in \mathbb{C}^{d_j\times d_j}$ is the identity matrix. Let us now indicate a formula relating symbols with respect to different orthonormal basis. If $\{e_\alpha\}$ and $\{f_\alpha\}$ are orthonormal bases of $\mathscr{H}$, we consider the unitary operator U determined by $U(e_\alpha) = f_\alpha$. Then we have

$$(Te_\alpha, e_\beta)_{\mathscr{H}} = (UTe_\alpha, Ue_\beta)_{\mathscr{H}} = (UTU^*Ue_\alpha, Ue_\beta)_{\mathscr{H}} = (UTU^*f_\alpha, f_\beta)_{\mathscr{H}}. \tag{4.11}$$

Thus, if $\big(\sigma_T, \{e_\alpha\}\big)$ denotes the symbol of T with respect to the orthonormal basis $\{e_\alpha\}$ and $\big(\sigma_{UTU^*}, \{f_\alpha\}\big)$ denotes the symbol of UTU^* with respect to the orthonormal basis $\{f_\alpha\}$ we have obtained the relation

$$\big(\sigma_T, \{e_\alpha\}\big) = \big(\sigma_{UTU^*}, \{f_\alpha\}\big). \tag{4.12}$$

Thus, the equivalence relation of basis $\{e_\alpha\} \sim \{f_\alpha\}$ given by a unitary operator U induces the equivalence relation on the set Σ of symbols given by (4.12). In view of this, we can also think of the symbol being independent of a choice of basis, as an element of the space $\Sigma/\sim$ with the equivalence relation given by (4.12).

We make another remark concerning part (C) of Theorem 4.1. We use the condition that e_j^k are in the domain $\mathrm{dom}(T^*)$ of T^* in showing the implication (B) $\Rightarrow$ (C). Since e_j^k's give a basis in $\mathscr{H}$, and are all contained in $\mathrm{dom}(T^*)$, it follows that $\mathrm{dom}(T^*)$ is dense in $\mathscr{H}$. In particular, by [32, Theorem VIII.1], T must be closable (in part (C)). These conditions are not restrictive for the further analysis since they are satisfied in several natural applications.

The principal application of the notions above will be as follows, except that in the sequel we sometimes need more general operators E unbounded on $\mathscr{H}$. In order to distinguish from this general case, in the following theorem we use the notation E_0.

Theorem 4.2 Continuing with the notation of Theorem 4.1, let $E_0 \in \mathscr{L}(\mathscr{H})$ be a linear continuous operator such that H_j are its eigenspaces:

$$E_0 e_j^k = \lambda_j e_j^k$$

for each $j \in \mathbb{N}_0$ and all $1 \le k \le d_j$. Then the equivalent conditions (A) – (C) imply the property

(E) For each $j \in \mathbb{N}_0$ and $1 \le k \le j$, we have $TE_0 e_j^k = E_0 T e_j^k$,

and if $\lambda_j \neq \lambda_\ell$ for $j \neq \ell$, then (E) is equivalent to properties (A) – (C).

Moreover, if T extends to a bounded operator $T \in \mathscr{L}(\mathscr{H})$ then equivalent properties (A) – (D) imply the condition

(F) $TE_0 = E_0 T$ on $\mathscr{H}$,

and if also $\lambda_j \neq \lambda_\ell$ for $j \neq \ell$, then (F) is equivalent to (A) – (E).

For an operator $T = F(E_0)$, when it is well-defined by the spectral calculus, we have

$$\sigma_{F(E_0)}(j) = F(\lambda_j) I_{d_j}. \tag{4.13}$$

In fact, this is also well-defined for a function F defined on λ_j, with finite values which are e. g. j-uniformly bounded (also for non self-adjoint E_o).

We have the following criterion for the extendability of a densely defined invariant operator $T : \mathscr{H}^\infty \to \mathscr{H}$ to $\mathscr{L}(\mathscr{H})$, which was an additional hypothesis for properties (D) and (F). In the statements below we fix a partition into H_j's as in (4.3) and the invariance refers to it.

Theorem 4.3 An invariant linear operator $T : \mathscr{H}^\infty \to \mathscr{H}$ extends to a bounded operator from $\mathscr{H}$ to $\mathscr{H}$ if and only if its symbol σ satisfies $\sup_{\ell \in \mathbb{N}_0} \|\sigma(\ell)\|_{\mathscr{L}(H_\ell)} < \infty$. Moreover, denoting this extension also by T, we have

$$\|T\|_{\mathscr{L}(\mathscr{H})} = \sup_{\ell \in \mathbb{N}_0} \|\sigma(\ell)\|_{\mathscr{L}(H_\ell)}. \tag{4.14}$$

We also record the formula for the symbol of the composition of two invariant operators:

Proposition 4.4 If $S, T : \mathscr{H}^\infty \to \mathscr{H}$ are invariant operators with respect to the same orthonormal partition, and such that the domain of $S \circ T$ contains $\mathscr{H}^\infty$, then $S \circ T : \mathscr{H}^\infty \to \mathscr{H}$ is also invariant with respect to the same partition. Moreover, if σ_S denotes the symbol of S and σ_T denotes the symbols of T with respect to the same orthonormal basis then

$$\sigma_{S \circ T} = \sigma_S \sigma_T, \tag{4.15}$$

i. e. $\sigma_{S \circ T}(j) = \sigma_S(j) \sigma_T(j)$ for all $j \in \mathbb{N}_0$.

We now show another application of the above notions to give a characterisation of Schatten classes of invariant operators in terms of their symbols.

Theorem 4.5 Let $0 < r < \infty$. An invariant operator $T \in \mathscr{L}(\mathscr{H})$ with symbol σ is in the Schatten class $S_r(\mathscr{H})$ if and only if

$$\sum_{\ell=0}^{\infty} \|\sigma(\ell)\|_{S_r(H_\ell)}^r < \infty.$$

Moreover

$$\|T\|_{S_r(\mathscr{H})} = \left(\sum_{\ell=0}^{\infty} \|\sigma(\ell)\|_{S_r(H_\ell)}^r \right)^{\frac{1}{r}}. \tag{4.16}$$

In particular, if T is in the trace class $S_1(\mathscr{H})$, then we have the trace formula

$$\operatorname{Tr}(T) = \sum_{\ell=0}^{\infty} \operatorname{Tr}\big(\sigma(\ell)\big). \tag{4.17}$$

Remark 4.6 We note that the membership in $\mathscr{L}(\mathscr{H})$ and in the Schatten classes $S_r(\mathscr{H})$ does not depend on the decomposition of $\mathscr{H}$ into subspaces H_j as in (4.3). However, the notion of invariance does depend on it. For example, let $\mathscr{H} = L^2(\mathbb{T}^n)$ for the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Choosing

$$H_j = \operatorname{span}\{e^{2\pi i j\cdot x}\}, \qquad j \in \mathbb{Z}^n,$$

we recover the classical notion of invariance on compact Lie groups and moreover, invariant operators with respect to $\{H_j\}_{j\in\mathbb{Z}^n}$ are the translation invariant operators on the torus $\mathbb{T}^n$. However, to recover the construction of Section 4 on manifolds, we take $\widetilde{H_\ell}$ to be the eigenspaces of the Laplacian E on $\mathbb{T}^n$, so that

$$\widetilde{H_\ell} = \bigoplus_{|j|^2=\ell} H_j = \operatorname{span}\left\{e^{2\pi i j\cdot x} :\ j \in \mathbb{Z}^n \text{ and } |j|^2 = \ell\right\}, \qquad \ell \in \mathbb{N}_0.$$

Then translation invariant operators on $\mathbb{T}^n$, i. e. operators invariant relative to the partition $\{H_j\}_{j\in\mathbb{Z}^n}$, are also invariant relative to the partition $\{\widetilde{H_\ell}\}_{\ell\in\mathbb{N}_0}$ (or relative to the Laplacian, in terminology of Section 4).

If we have information on the eigenvalues of E, like we do on the torus, we may sometimes also recover invariant operators relative to the partition $\{\widetilde{H_\ell}\}_{\ell\in\mathbb{N}_0}$ as linear combinations of translation invariant operators composed with phase shifts and complex conjugation.

. . . .

3 Fourier analysis associated to an elliptic operator

One of the main applications of the described setting is to study operators on compact manifolds, so we start this section by describing the discrete Fourier analysis associated to an elliptic positive pseudo-differential operator as an adaptation of the construction in Section 2. In order to fix the further notation we give some explicit expressions for notions of Section 2 in this setting.

Let M be a compact smooth manifold of dimension n without boundary, endowed with a fixed volume dx. We denote by $\Psi^{\nu}(M)$ the Hörmander class of pseudo-differential operators of order $\nu \in \mathbb{R}$, i. e. operators which, in every coordinate chart, are operators in Hörmander classes on $\mathbb{R}$ with symbols in $S^{\nu}_{1,0}$, see e. g. [41], [33]. For simplicity we may be using the class $\Psi^{\nu}_{cl}(M)$ of classical operators, i. e. operators with symbols having (in all local coordinates) an asymptotic expansion of the symbol in positively homogeneous components (see e. g. [18]). Furthermore, we denote by $\Psi^{\nu}_{+}(M)$ the class of positive definite operators in $\Psi^{\nu}_{cl}(M)$, and by $\Psi^{\nu}_{e}(M)$ the class of elliptic operators in $\Psi^{\nu}_{cl}(M)$. Finally,

$$\Psi^{\nu}_{+e}(M) := \Psi^{\nu}_{+}(M) \cap \Psi^{\nu}_{e}(M)$$

will denote the class of classical positive elliptic pseudo-differential operators of order ν. We note that complex powers of such operators are well-defined, see e. g. Seeley [39]. In fact, all pseudo-differential operators considered from now on will be classical, so we may omit explicitly mentioning it every time, but we note that we could equally work with general operators in $\Psi^{\nu}(M)$ since their powers have similar properties, see e. g. [45].

We now associate a discrete Fourier analysis to the operator $E \in \Psi^{\nu}_{+e}(M)$ inspired by those constructions considered by Seeley ([38], [40]), see also Greenfield and Wallach [21]. However, we adapt it to our purposes and in the sequel also indicate several auxiliary statements concerning the eigenvalues of E and their multiplicities, useful to us in the subsequent analysis. In general, the construction below is exactly the one appearing in Theorem 4.1 with a particular choice of a partition.

The eigenvalues of E (counted without multiplicities) form a sequence $\{\lambda_j\}$ which we order so that

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots . \tag{4.18}$$

For each eigenvalue λ_j, there is the corresponding finite dimensional eigenspace H_j of functions on M, which are smooth due to the ellipticity of E. We set

$$d_j := \dim H_j \quad \text{and} \quad H_0 := \ker E, \quad \lambda_0 := 0.$$

We also set $d_0 := \dim H_0$. Since the operator E is elliptic, it is Fredholm, hence also $d_0 < \infty$ (we can refer to [1], [23] for various properties of H_0 and d_0).

We fix an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of E:

$$\{e_j^k\}_{j\geq 0}^{1\leq k\leq d_j}, \tag{4.19}$$

where $\{e_j^k\}^{1\leq k\leq d_j}$ is an orthonormal basis of H_j. Let $P_j : L^2(M) \to H_j$ be the corresponding projection. We shall denote by $(\,\cdot\,,\cdot\,)$ the inner product of $L^2(M)$. We observe that we have

$$P_j f = \sum_{k=1}^{d_j} (f, e_j^k) e_j^k, \tag{4.20}$$

for $f \in L^2(M)$. The 'Fourier' series takes the form

$$f = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} (f, e_j^k) e_j^k, \tag{4.21}$$

for each $f \in L^2(M)$. The Fourier coefficients of $f \in L^2(M)$ with respect to the orthonormal basis $\{e_j^k\}$ will be denoted by

$$\big(\mathscr{F} f\big)(j,k) := \widehat{f}(j,k) := (f, e_j^k). \tag{4.22}$$

We will call the collection of $\widehat{f}(j,k)$ the *Fourier coefficients of f relative to E*, or simply the *Fourier coefficients of f*.

Since $\{e_j^k\}_{j\geq 0}^{1\leq k\leq d_j}$ forms a complete orthonormal system in $L^2(M)$, for all $f \in L^2(M)$ we have the Plancherel formula (4.6), namely,

$$\|f\|^2_{L^2(M)} = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |(f, e_j^k)|^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |\widehat{f}(j,k)|^2 = \|\widehat{f}\|^2_{\ell^2(\mathbb{N}_0,\Sigma)}, \tag{4.23}$$

where the space $\ell^2(\mathbb{N}_0,\Sigma)$ and its norm are as in (4.7) and (4.8).

We can think of $\mathscr{F}=\mathscr{F}_M$ as of the Fourier transform being an isometry from $L^2(M)$ into $\ell^2(\mathbb{N}_0,\Sigma)$. The inverse of this Fourier transform can be then expressed by

$$(\mathscr{F}^{-1}h)(x)=\sum_{j=0}^{\infty}\sum_{k=1}^{d_j}h(j,k)e_j^k(x). \tag{4.24}$$

If $f\in L^2(M)$, we also write

$$\widehat{f}(j)=\begin{pmatrix}\widehat{f}(j,1)\\ \vdots\\ \widehat{f}(j,d_j)\end{pmatrix}\in\mathbb{C}^{d_j}, \tag{4.25}$$

thus thinking of the Fourier transform always as a column vector. In particular, we think of

$$\widehat{e_j^k}(\ell)=\left(\widehat{e_j^k}(\ell,m)\right)_{m=1}^{d_\ell} \tag{4.26}$$

as of a column, and we notice that

$$\widehat{e_j^k}(\ell,m)=\delta_{j\ell}\delta_{km}. \tag{4.27}$$

Smooth functions on M can be characterised by

$$\begin{aligned} f\in C^\infty(M) \quad &\Longleftrightarrow \quad \forall N\,\exists C_N:\ |\widehat{f}(j,k)|\le C_N(1+\lambda_j)^{-N} \text{ for all } j,k\\ &\Longleftrightarrow \quad \forall N\,\exists C_N:\ |\widehat{f}(j)|\le C_N(1+\lambda_j)^{-N} \text{ for all } j, \end{aligned} \tag{4.28}$$

where $|\widehat{f}(j)|$ is the norm of the vector $\widehat{f}(j)\in\mathbb{C}^{d_j}$. The implication '$\Leftarrow$' here is immediate, while '$\Rightarrow$' follows from the Plancherel formula (4.6) and the fact that for $f\in C^\infty(M)$ we have $(I+E)^N f\in L^2(M)$ for any N.

For $u\in\mathscr{D}'(M)$, we denote its Fourier coefficient $\widehat{u}(j,k):=(\overline{e_j^k})$, and by duality, the space of distributions can be characterised by

$$f\in\mathscr{D}'(M) \quad \Longleftrightarrow \quad \exists M\,\exists C:\ |\widehat{u}(j,k)|\le C(1+\lambda_j)^M \text{ for all } j,k.$$

We will denote by $H^s(M)$ the usual Sobolev space over L^2 on M. This space can be defined in local coordinates or, by the fact that $E \in \Psi^{\nu}_{+e}(M)$ is positive and elliptic with $\nu > 0$, it can be characterised by

$$f \in H^s(M) \iff (I+E)^{s/\nu} f \in L^2(M) \tag{4.29}$$

$$\iff \{(1+\lambda_j)^{s/\nu} \widehat{f}(j)\}_j \in \ell^2(\mathbb{N}_0, \Sigma) \tag{4.30}$$

$$\iff \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} (1+\lambda_j)^{2s/\nu} |\widehat{f}(j,k)|^2 < \infty, \tag{4.31}$$

the last equivalence following from the Plancherel formula (4.6). For the characterisation of analytic functions (on compact manifolds M) we refer to Seeley [40].

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4 Invariant operators and symbols on compact manifolds

We now discuss an application of a notion of an invariant operator and of its symbol from Theorem 4.1 in the case of $\mathscr{H} = L^2(M)$ and $\mathscr{H}^\infty = C^\infty(M)$ and describe its basic properties. We will consider operators T densely defined on $L^2(M)$, and we will be making a natural assumption that their domain contains $C^\infty(M)$. We also note that while in Theorem 4.2 it was assumed that the operator E_0 is bounded on $\mathscr{H}$, this is no longer the case for the operator E here. Indeed, an elliptic pseudo-differential operator $E \in \Psi^{\nu}_{+e}(M)$ of order $\nu > 0$ is not bounded on $L^2(M)$.

Moreover, we do not want to assume that T extends to a bounded operator on $L^2(M)$ to obtain analogues of Properties (D) and (F) in Section 2, because this is too restrictive from the point of view of differential operators. Instead, we show that in the present setting it is enough to assume that T extends to a continuous operator on $\mathscr{D}'(M)$ to reach the same conclusions.

So, we combine the statement of Theorem 4.1 and the necessary modification of Theorem 4.2 to the setting of Section 3 as follows.

We also remark that Part (iv) of the following theorem provides a correct formulation for a missing assumption in [14, Theorem 3.1, (iv)].

Theorem 4.7 Let M be a closed manifold and let $T : C^\infty(M) \to L^2(M)$be a linear operator. Then the following conditions are equivalent:

(i) For each $j \in \mathbb{N}_0$, we have $T(H_j) \subset H_j$.

(ii) For each $j \in \mathbb{N}_0$ and $1 \leq k \leq j$, we have $TEe_j^k = ETe_j^k$.

(iii) For each $\ell \in \mathbb{N}_0$ there exists a matrix $\sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}$ such that for all e_j^k

$$\widehat{Te_j^k}(\ell, m) = \sigma(\ell)_{mk} \delta_{j\ell}. \tag{4.32}$$

(iv) If, in addition, the domain of T^* contains $C^\infty(M)$, then for each $\ell \in \mathbb{N}_0$ there exists a matrix $\sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}$ such that

$$\widehat{Tf}(\ell) = \sigma(\ell) \widehat{f}(\ell) \tag{4.33}$$

for all $f \in C^\infty(M)$.

The matrices $\sigma(\ell)$ in (iii) and (iv) coincide.

If T extends to a linear continuous operator $T : \mathscr{D}'(M) \to \mathscr{D}'(M)$ then the above properties are also equivalent to the following ones:

(v) For each $j \in \mathbb{N}_0$, we have $TP_j = P_j T$ on $C^\infty(M)$.

(vi) $TE = ET$ on $L^2(M)$.

If any of the equivalent conditions (i) – (iii) in Theorem 4.7 are satisfied, we say that the operator $T : C^\infty(M) \to L^2(M)$ is *invariant (or is a Fourier multiplier) relative to E*. We can also say that T is E-invariant or is an E-multiplier. This recovers the notion of invariant operators given by Theorem 4.1, with respect to the partitions H_j's in (4.3) which are fixed being the eigenspaces of E. When there is no risk of confusion we will just refer to such kind of operators as invariant operators or as Fourier multipliers. It is clear from (i) that the operator E itself or functions of E defined by the functional calculus are invariant relative to E.

We note that the boundedness of T on $L^2(M)$ needed for conditions (D) and (F) in Theorem 4.1 and in Theorem 4.2 is now replaced by the condition that T is continuous on $\mathscr{D}'(M)$ which explored the additional structure of $L^2(M)$ and allows application to differential operators.

We call σ in (iii) and (iv) the *matrix symbol of T* or simply the *symbol*. It is an element of the space $\Sigma = \Sigma_M$ defined by

$$\Sigma_M := \{\sigma : \mathbb{N}_0 \ni \ell \mapsto \sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}\}. \tag{4.34}$$

Since the expression for the symbol depends only on the basis e_j^k and not on the operator E itself, this notion coincides with the symbol defined in Theorem 4.1.

Let us comment on several conditions in Theorem 4.7 in this setting. Assumptions (v) and (vi) are stronger than those in (i) – (iv). On one hand, clearly (vi) contains (ii). On the other hand, it can be shown that Assumption (v) implies (i) without the additional hypothesis that T is continuous on $\mathscr{D}'(M)$.

In analogy to the strong commutativity in (v), *if T is continuous on $\mathscr{D}'(M)$*, so that all the Assumptions (i) – (vi) are equivalent, we may say that T is *strongly invariant relative to E* in this case.

The expressions in (vi) make sense as both sides are defined (and even continuous) on $\mathscr{D}'(M)$.

We also note that without additional assumptions, it is known from the general theory of densily defined operators on Hilbert spaces that Conditions (v) and (vi) are generally not equivalent, see e. g. Reed and Simon [32, Section VIII.5]. If T is a differential operator, the additional assumption of continuity on $\mathscr{D}'(M)$ for parts (v) and (vi) is satisfied. In [21, Section 1, Definition 1], Greenfield and Wallach called a differential operator D to be an E-invariant operator if $ED = DE$, which is our condition (vi). However, Theorem 4.7 describes more general operators as well as reformulates them in the form of Fourier multipliers that will be explored in the sequel.

There will be several useful classes of symbols, in particular the moderate growth class

$$\mathscr{S}'(\Sigma) := \{\sigma \in \Sigma : \exists N, C \text{ such that } \|\sigma(\ell)\|_{\mathrm{op}} \le C(1+\lambda_\ell)^N \ \forall \ell \in \mathbb{N}_0\}, \tag{4.35}$$

where

$$\|\sigma(\ell)\|_{\mathrm{op}} = \|\sigma(\ell)\|_{\mathscr{L}(H_\ell)}$$

denotes the matrix multiplication operator norm with respect to $\ell^2(\mathbb{C}^{d_\ell})$.

In the case when M is a compact Lie group and E is a Laplacian on G, left-invariant operators on G, i. e. operators commuting with the left action of G, are also invariant relative to E in the sense of Theorem 4.7. However, we need an adaptation of the above construction since the natural decomposition into H_j's in (4.3) may in general violate the Condition (4.18).

As in Section 2 since the notion of the symbol depends only on the basis, for the identity operator $T = I$ we have

$$\sigma_I(j) = I_{d_j},$$

where $I_{d_j} \in \mathbb{C}^{I_{d_j} \times I_{d_j}}$ is the identity matrix, and for an operator $T = F(E)$, when it is well-defined by the spectral calculus, we have

$$\sigma_{F(E)}(j) = F(\lambda_j) I_{d_j}. \tag{4.36}$$

We now discuss how invariant operators can be expressed in terms of their symbols.

Proposition 4.8 An invariant operator T_σ associated to the symbol σ can be written in the following way:

$$T_\sigma f(x) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \left(\sigma(\ell)\widehat{f}(\ell)\right)_m e_\ell^m(x) = \sum_{\ell=0}^{\infty} \left[\sigma(\ell)\widehat{f}(\ell)\right]^\top e_\ell(x), \tag{4.37}$$

where $\left[\sigma(\ell)\widehat{f}(\ell)\right]$ denotes the column-vector, and $\left[\sigma(\ell)\widehat{f}(\ell)\right]^\top e_\ell(x)$ denotes the multiplication (the scalar product) of the column-vector $\left[\sigma(\ell)\widehat{f}(\ell)\right]$ with the column-vector $e_\ell(x) = (e_\ell^1(x), \cdots, e_\ell^m(x))^\top$. In particular, we also have

$$\left(T_\sigma e_j^k\right)(x) = \sum_{m=1}^{d_j} \sigma(j)_{mk} e_j^m(x). \tag{4.38}$$

If $\sigma \in \mathscr{S}'(\Sigma)$ and $f \in C^\infty(M)$, the convergence in (4.37) is uniform.

Proof. Formula (4.37) follows from Part (iv) of Theorem 4.7, with uniform convergence for $f \in C^\infty(M)$ in view of (4.35). Then, using (4.37), (4.27), we can calculate

$$\begin{aligned}
\left(T_\sigma e_j^k\right)(x) &= \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \left(\sigma(\ell)\widehat{e_j^k}(\ell)\right)_m e_\ell^m(x) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \left(\sum_{i=1}^{d_\ell} \left(\sigma(\ell)\right)_{mi} \widehat{e_j^k}(\ell,i)\right) e_\ell^m(x) \\
&= \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \sum_{i=1}^{d_\ell} \left(\sigma(\ell)\right)_{mi} \delta_{j\ell} \delta_{ki} e_\ell^m(x) = \sum_{m=1}^{d_j} \left(\sigma(j)\right)_{mk} e_j^m(x),
\end{aligned}$$

yielding (4.38). □

Theorem 4.3 characterising invariant operators bounded on $L^2(M)$ now becomes

Theorem 4.9 An invariant linear operator $T : C^\infty(M) \to L^2(M)$ extends to a bounded operator from $L^2(M)$ to $L^2(M)$ if and only if its symbol σ satisfies

$$\sup_{\ell \in \mathbb{N}_0} \|\sigma(\ell)\|_{\mathrm{op}} < \infty,$$

where $\|\sigma(\ell)\|_{\mathrm{op}} = \|\sigma(\ell)\|_{\mathscr{L}(H_\ell)}$ is the matrix multiplication operator norm with respect to $H_\ell \simeq \ell^2(\mathbb{C}^{d_\ell})$. Moreover, we have

$$\|T\|_{\mathscr{L}\left(L^2(M)\right)} = \sup_{\ell \in \mathbb{N}_0} \|\sigma(\ell)\|_{\mathrm{op}}. \tag{4.39}$$

This can be extended to Sobolev spaces. We will use the multiplication property for Fourier multipliers which is a direct consequence of Proposition 4.4:

Proposition 4.10 If $S, T : C^\infty(M) \to L^2(M)$ are invariant operators with respect to E such that the domain of $S \circ T$ contains $C^\infty(M)$, then $S \circ T : C^\infty(M) \to L^2(M)$ is also invariant with respect to E. Moreover, if σ_S denotes the symbol of S and σ_T denotes the symbols of T with respect to the same orthonormal basis then

$$\sigma_{S\circ T} = \sigma_S \sigma_T, \tag{4.40}$$

i. e. $\sigma_{S\circ T}(j) = \sigma_S(j)\sigma_T(j)$ for all $j \in \mathbb{N}_0$.

Recalling Sobolev spaces $H^s(M)$ in (4.29) we have:

Corollary 4.11 Let an invariant linear operator $T : C^\infty(M) \to C^\infty(M)$ have symbol σ_T for which there exists $C > 0$ and $m \in \mathbb{R}$ such that

$$\|\sigma_T(\ell)\|_{\mathrm{op}} \le C(1+\lambda_\ell)^{\frac{m}{\nu}} \tag{4.41}$$

holds for all $\ell \in \mathbb{N}_0$. Then T extends to a bounded operator from $H^s(M)$ to $H^{s-m}(M)$ for every $s \in \mathbb{R}$.

Proof. We note that by (4.29) the condition that $T : H^s(M) \to H^{s-m}(M)$ is bounded is equivalent to the condition that the operator

$$S := (I+E)^{\frac{s-m}{\nu}} \circ T \circ (I+E)^{-\frac{s}{\nu}}$$

is bounded on $L^2(M)$. By Proposition 4.10 and the fact that the powers of E are pseudo-differential operators with diagonal symbols, see (4.36), we have

$$\sigma_S(\ell) = (1+\lambda_\ell)^{-\frac{m}{\nu}} \sigma_T(\ell).$$

But then $\|\sigma_S(\ell)\|_{\mathrm{op}} \leq C$ for all ℓ in view of the assumption on σ_T, so that the statement follows from Theorem 4.9. □

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5 Schatten classes of operators on compact manifolds

In this section we give an application of the constructions in the previous section to determine the membership of operators in Schatten classes and then apply it to a particular family of operators on $L^2(M)$.

As a consequence of Theorem 4.5, we can now characterise invariant operators in Schatten classes on compact manifolds. We note that this characterisation does not assume any regularity of the kernel nor of the symbol. Once we observe that the conditions for the membership in the Schatten classes depend only on the basis e_j^k and not on the operator E, we immediately obtain:

Theorem 4.12 Let $0 < r < \infty$. An invariant operator $T : L^2(M) \to L^2(M)$ is in $S_r\big(L^2(M)\big)$ if and only if $\sum\limits_{\ell=0}^{\infty} \|\sigma_T(\ell)\|_{S_r}^r < \infty$. Moreover

$$\|T\|^r_{S_r\left(L^2(M)\right)} = \sum_{\ell=0}^{\infty} \|\sigma_T(\ell)\|^r_{S_r}. \tag{4.42}$$

If an invariant operator $T : L^2(M) \to L^2(M)$ is in the trace class $S_1\big(L^2(M)\big)$, then

$$\mathrm{Tr}(T) = \sum_{\ell=0}^{\infty} \mathrm{Tr}\left(\sigma_T(\ell)\right). \tag{4.43}$$

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Bibliography

[1] M. F. Atiyah. *Global aspects of the theory of elliptic differential operators.* In Proc. Internat. Congr. Math. (Moscow, 1966), pages 57–64. Izdat. "Mir", Moscow, 1968.

[2] P. Boggiatto, E. Buzano, L. Rodino. *Global hypoellipticity and spectral theory*, volume 92 of *Mathematical Research*. Akademie Verlag, Berlin, 1996.

[3] E. Buzano, F. Nicola. *Pseudo-differential operators and Schatten-von Neumann classes.* In Advances in pseudo-differential operators, volume 155 of *Oper. Theory Adv. Appl.*, pages 117–130. Birkhäuser, Basel, 2004.

[4] E. Buzano, F. Nicola. *Complex powers of hypoelliptic pseudodifferential operators.* J. Funct. Anal., 245(2):353–378, 2007.

[5] A. Benedek, R. D. Panzone. *The spaces L^p, with mixed norms.* Duke. Math. J., 28:301–324, 1961.

[6] F. Bruhat. *Lectures on Lie groups and representations of locally compact groups*. Tata Institute of Fundamental Research, Bombay, 1968. Notes by S. Ramanan, Tata Institute of Fundamental Research Lectures on Mathematics, No. 14.

[7] E. Buzano, J. Toft. *Schatten-von Neumann properties in the Weyl calculus.* J. Funct. Anal., 259(12):3080–3114, 2010.

[8] T. Carleman. *Über die Fourierkoeffizienten einer stetigen Funktion.* Acta Math., 41(1):377–384, 1916. Aus einem Brief an Herrn A. Wiman.

[9] O. Chodosh. *Infinite matrix representations of isotropic pseudodifferential operators.* Methods Appl. Anal., 18(4):351–371, 2011.

[10] J. Delgado. *The trace of nuclear operators on $L^p(\mu)$ for σ-finite Borel measures on second countable spaces.* Integral Equations Operator Theory, 68(1):61–74, 2010.

[11] J. Dixmier. *C∗-algebras*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15.

[12] J. Dixmier. *Les algèbres d'opérateurs dans l'espace hilbertien (algèbres de von Neumann)*. Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Paris, 1996. Reprint of the second (1969) edition.

[13] J. Delgado and M. Ruzhansky. *Schatten classes and traces on compact Lie groups.* To appear in Math. Res. Lett.

[14] J. Delgado, M. Ruzhansky. *Kernel and symbol criteria for Schatten classes and r-nuclearity on compact manifolds.* C. R. Math. Acad. Sci. Paris, 352(10):779–784, 2014.

[15] J. Delgado, M. Ruzhansky. *Fourier multipliers, symbols and nuclearity on compact manifolds.* to appear in J. Anal. Math.

[16] J. Delgado, M. Ruzhansky. *L^p-nuclearity, traces, and Grothendieck-Lidskii formula on compact Lie groups.* J. Math. Pures Appl. (9), 102(1):153–172, 2014.

[17] J. Delgado, M. Ruzhansky. *Schatten classes on compact manifolds: kernel conditions.* J. Funct. Anal., 267(3):772–798, 2014.

[18] J. J. Duistermaat. *Fourier integral operators*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2011. Reprint of the 1996 edition [MR1362544], based on the original lecture notes published in 1973 [MR0451313].

[19] I. C. Gohberg, M. G. Kreĭn. *Introduction to the theory of linear nonselfadjoint operators*. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18. American Mathematical Society, Providence, R.I., 1969.

[20] A. Grothendieck. *Produits tensoriels topologiques et espaces nucléaires.* Mem. Amer. Math. Soc., 1955(16):140, 1955.

[21] S. J. Greenfield, N. R. Wallach. *Remarks on global hypoellipticity.* Trans. Amer. Math. Soc., 183:153–164, 1973.

[22] L. Hörmander. *The spectral function of an elliptic operator.* Acta Math., 121:193–218, 1968.

[23] L. Hörmander. *The Analysis of linear partial differential operators III*. Springer-Verlag, 1985.

[24] L. Hörmander. *The Analysis of linear partial differential operators IV*. Springer-Verlag, 1985.

[25] A. Hinrichs, A. Pietsch. *p-nuclear operators in the sense of Grothendieck.* Math. Nachr., 283(2):232–261, 2010.

[26] H. König. *Eigenvalues of p-nuclear operators.* In Proceedings of the International Conference on Operator Algebras, Ideals, and their Applications in Theoretical Physics (Leipzig, 1977), pages 106–113, Leipzig, 1978. Teubner.

[27] F. Nicola, L. Rodino. *Global pseudo-differential calculus on Euclidean spaces*, volume 4 of *Pseudo-Differential Operators. Theory and Applications*. Birkhäuser Verlag, Basel, 2010.

[28] R. Oloff. *p-normierte Operatorenideale.* Beiträge Anal., (4):105–108, 1972. Tagungsbericht zur Ersten Tagung der WK Analysis (1970).

[29] A. Pietsch. *Grothendieck's concept of a p-nuclear operator.* Integral Equations Operator Theory, 7(2):282–284, 1984.

[30] O. I. Reinov, Q. Laif. *Grothendieck-Lidskii theorem for subspaces of* $Lp-spaces$. Math. Nachr., (2–3):279–282, 2013.

[31] M. Reed, B. Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.

[32] M. Reed, B. Simon. *Methods of modern mathematical physics. I.* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, second edition, 1980. Functional analysis.

[33] M. Ruzhansky, V. Turunen. *Pseudo-differential operators and symmetries. Background analysis and advanced topics*, volume 2 of *Pseudo-Differential Operators. Theory and Applications*. Birkhäuser Verlag, Basel, 2010.

[34] M. Ruzhansky, V. Turunen. *Global quantization of pseudo-differential operators on compact Lie groups,* SU(2)*, 3-sphere, and homogeneous spaces.* Int. Math. Res. Not. IMRN, (11):2439–2496, 2013.

[35] M. Ruzhansky, N. Tokmagambetov. *Nonharmonic analysis of boundary value problems.* Int. Math. Res. Notices, (2016) 2016 (12), 3548–3615.

[36] B. Russo. *The Norm of the Lp-Fourier Transform on Unimodular Groups.* Trans. Amer. Math. Soc., 192(2):293–305, 1974.

[37] R. Schatten. *Norm ideals of completely continuous operators*. Second printing. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 27. Springer-Verlag, Berlin, 1970.

[38] R. T. Seeley. *Integro-differential operators on vector bundles.* Trans. Amer. Math. Soc., 117:167–204, 1965.

[39] R. T. Seeley. *Complex powers of an elliptic operator.* In *Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966)*, pages 288–307. Amer. Math. Soc., Providence, R.I., 1967.

[40] R. T. Seeley. *Eigenfunction expansions of analytic functions.* Proc. Amer. Math. Soc., 21:734–738, 1969.

[41] M. A. Shubin. *Pseudodifferential operators and spectral theory*. Springer-Verlag, Berlin, second edition, 2001. Translated from the 1978 Russian original by Stig I. Andersson.

[42] B. Simon. *Trace ideals and their applications*, volume 35 of Lond. Math. Soc. Lecture Note Series. Cambridge University Press, Cambridge, 1979.

[43] A. V. Sobolev. *On the Schatten–von Neumann properties of some pseudo-differential operators.* J. Funct. Anal., 266(9):5886–5911, 2014.

[44] E. M. Stein. *Topics in harmonic analysis related to the Littlewood-Paley theory.* Annals of Mathematics Studies, No. 63. Princeton University Press, Princeton, N.J., 1970.

[45] R. S. Strichartz. *A functional calculus for elliptic pseudo-differential operators.* Amer. J. Math., 94:711–722, 1972.

[46] C. Sogge, S. Zelditch. *Riemannian manifolds with maximal eigenfunction growth.* Duke Math. J., 114(3):387–437, 2002.

[47] J. Toft. *Schatten-von Neumann properties in the Weyl calculus, and calculus of metrics on symplectic vector spaces.* Ann. Global Anal. Geom., 30(2):169–209, 2006.

[48] J. Toft. *Schatten properties for pseudo-differential operators on modulation spaces.* In *Pseudo-differential operators*, volume 1949 of Lecture Notes in Math., pages 175–202. Springer, Berlin, 2008.

[49] J. Toth, S. Zelditch. *Riemannian manifolds with uniformly bounded eigenfunctions.* Duke Math. J., 111(1):97–132, 2002.

Chapter 5

Transformation de Bargmann et analyse microlocale analytique

*Gilles Lebeau**

Abstract. *Ce texte contient les notes d'un mini-cours donné à l'école "Opérateurs Intégraux de Fourier" à Ouagadougou du 14 au 26 septembre 2015. Il est consacré à une introduction à la théorie microlocale analytique.*

1 Introduction

Ce texte est une introduction élémentaire à la théorie microlocale analytique, en utilisant le formalisme des transformations de FBI introduit par J. Sjöstrand en 1982 dans [8].

C'est le mathématicien japonais M. Sato qui introduit à la fin des années 60 la notion de front d'onde analytique d'une distribution. Le mathématicien suédois L. Hörmander en donnera peu après une version C^∞.

On trouvera dans ce volume (le cours de R.-M. Shultz [7]) un exposé de la notion de front d'onde C^∞ de Hörmander. Rappelons brièvement de quoi il s'agit. Si f est une

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distribution sur $\mathbb{R}^d$ et x_0 un point de $\mathbb{R}^d$, on dit que x_0 n'est pas dans le support singulier C^∞ de f, et on note $x_0 \notin SupportSing(f)$, s'il existe un voisinage ouvert U de x_0 tel que $f|_U \in C^\infty(U)$. Le front d'onde C^∞ de Hörmander est une notion qui raffine la notion de support singulier d'une distribution: si f est une distribution sur $\mathbb{R}^d$ et (x_0, ξ_0) un point de $\mathbb{R}^d \times \mathbb{R}^d$ avec $\xi_0 \neq 0$, on dit que (x_0, ξ_0) n'est pas dans le front d'onde C^∞ de f, et on note $(x_0, \xi_0) \notin WF(f)$, si et seulement si il existe $\varphi \in C_0^\infty(\mathbb{R}^d)$, égal à 1 près de x_0, et $\alpha > 0$, tels que la transformée de Fourier $\widehat{\varphi f}(\xi)$ de φf vérifie:

$$\widehat{\varphi f}(\xi) \text{ est à décroissance rapide pour } |\xi| \to \infty \text{ dans le cône } \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < \alpha.$$

Comme il n'existe pas de fonction analytique non nulle φ appartenant à $C_0^\infty(\mathbb{R}^d)$, cette définition de Hörmander du front d'onde C^∞ ne peut pas s'adapter simplement au cadre analytique. La construction initiale de M. Sato utilise d'ailleurs des outils sophistiqués. Ce sont les physiciens J. Bros et D. Iagolnitzer qui dans [3], donneront la première définition analytiquement simple du front d'onde analytique. Leur construction sera reprise et généralisée par J. Sjöstrand en 1982 dans [8], et J. Sjöstrand donnera le nom de *Transformation de FBI*, acronyme pour Fourier-Bros-Iagolnitzer, aux transformations intégrales qui remplacent la transformation de Fourier pour la caractérisation du front d'onde analytique.

On étudiera dans ces notes de cours une transformation de FBI particulière, la transformation de Bargmann, qui permet des calculs exacts et explicites. L'étude de la transformation de Bargmann est effectuée dans le paragraphe 2. Ceci nous permet d'introduire dans le paragraphe 3 la définition du front d'onde analytique (ou spectre singulier) d' une distribution. Enfin, dans le paragraphe 4, nous introduisons les symboles analytiques, le calcul pseudodifférentiel dans les espaces de Sjöstrand H_φ, et nous donnons comme application la preuve d'un résultat fondamental: le théorème de régularité elliptique de Sato.

Nous utiliserons les notations suivantes, dont plusieurs communes à d'autres contributions à ce recueil.

Pour $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, on note $\alpha! = \alpha_1! \cdots \alpha_d!$, $|\alpha| = \alpha_1 + \ldots + \alpha_d$. Pour $z = (z_1, \dots, z_d) \in \mathbb{C}^d$, on note $z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$.

Pour U ouvert de $\mathbb{R}^d$, on note $C^\infty(U)$ l'espace des fonctions indéfiniment différentiables sur U, et $C^{\omega}(U)$ l'espace des fonctions analytiques sur U. On note $\mathscr{D}'(U)$ l'espace des distributions sur U.

On note $\mathscr{S}(\mathbb{R}^d)$ l'espace de Schwartz des fonctions C^∞ sur $\mathbb{R}^d$, à décroissance rapide avec toutes leurs dérivées. On note $\mathscr{S}'(\mathbb{R}^d)$ l'espace des distributions tempérées sur $\mathbb{R}^d$, et on note $\mathscr{E}'(\mathbb{R}^d)$ l'espace des distributions à support compact sur $\mathbb{R}^d$.

Rappelons qu'une fonction f définie sur un ouvert U de $\mathbb{C}^d \simeq \mathbb{R}^{2d}$, à valeur dans $\mathbb{C}$, mesurable et localement intégrable, ou plus généralement une distribution à valeurs complexes sur U, est holomorphe ssi on a au sens des distributions sur U:

$$\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) f = 0, \qquad \forall j \in \{1,\dots,d\}.$$

On a alors $f \in C^\infty(U,\mathbb{C})$. On note $\mathscr{O}(U)$ l'espace des fonctions holomorphes sur U.

Lorsque $U = \mathbb{C}^d$, les coefficients de la série de Taylor de f en $z=0$, $f_\alpha = \frac{1}{\alpha!}\partial^\alpha f(0)$, vérifient

$$\forall \varepsilon > 0,\ \exists A_\varepsilon,\ \text{tel que} \quad |f_\alpha| \le A_\varepsilon \varepsilon^{|\alpha|}$$

et f est somme de sa série de Taylor (convergence uniforme sur tout compact de $\mathbb{C}^d$),

$$f(z) = \sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha.$$

. . . .

2 La transformation de Bargmann

Définition 5.1 Soit $\varphi : \mathbb{C}^d \to \mathbb{R}$ une fonction continue et $\lambda \in]0,\infty[$. On note $\mathscr{H}_{\varphi,\lambda}$ l'espace de fonctions :

$$\mathscr{H}_{\varphi,\lambda} = \left\{ f : \mathbb{C}^d \to \mathbb{C},\ \text{f est holomorphe et } \int_{\mathbb{C}^d} |f(z)|^2 e^{-2\lambda\varphi(z)} \mathrm{d}x\mathrm{d}y \ < \infty \right\}.$$

Lemme 5.2 $\mathscr{H}_{\varphi,\lambda}$ est un espace de Hilbert pour le produit scalaire

$$(f|g)_{\mathscr{H}_{\varphi,\lambda}} = \int_{\mathbb{C}^d} f(z)\overline{g(z)} e^{-2\lambda\varphi(z)} \mathrm{d}x\mathrm{d}y.$$

L'espace $H = L^2(\mathbb{C}^d; e^{-2\lambda\varphi(z)}\mathrm{d}x\mathrm{d}y)$ est un espace de Hilbert et $\mathscr{H}_{\varphi,\lambda}$ est le sous espace de H: $\mathscr{H}_{\varphi,\lambda} = \{f \in H,\ \text{f est holomorphe}\}$. Ce sous espace est *fermé* dans H (donc est un espace de Hilbert), car si $f_n \in \mathscr{H}_{\varphi,\lambda}$ converge dans H vers $f \in H$, f_n converge vers f dans l'espace des distributions $\mathscr{D}'(\mathbb{C}^d)$, d'où $\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) f = \lim_{n\to\infty} \left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) f_n = 0$, la dérivation étant continue dans $\mathscr{D}'(\mathbb{C}^d)$. Donc f est holomorphe, donc $f \in \mathscr{H}_{\varphi,\lambda}$.

On fixe $\lambda \in]0,\infty[$. Pour $z \in \mathbb{C}^d$ et $x \in \mathbb{R}^d$, on note $(z-x)^2 = \sum_j (z_j - x_j)^2$ et on pose

$$g_\lambda(z,x) = e^{-\frac{\lambda(z-x)^2}{2}} .$$

Lemme 5.3 Pour tout $z \in \mathbb{C}^d$ on a $g_\lambda(z,\cdot) \in L^2(\mathbb{R}^d)$ et

$$\left\|g_\lambda(z,\cdot)\right\|_{L^2(\mathbb{R}^d)} = \left(\frac{\pi}{\lambda}\right)^{d/4} e^{\frac{\lambda|\Im(z)|^2}{2}} .$$

Rappelons l'expression des intégrales de fonctions gaussiennes:

$$\int_{\mathbb{R}^d} e^{-\lambda y^2}\mathrm{d}y = \left(\frac{\pi}{\lambda}\right)^{d/2} .$$

En posant $z = a + ib$, $a = \mathbb{R}(z)$, $b = \Im(z)$, on obtient:

$$\begin{aligned}\left\|g_\lambda(z,\cdot)\right\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \left|e^{-\frac{\lambda(z-x)^2}{2}}\right|^2 \mathrm{d}x = \int_{\mathbb{R}^d} \left|e^{-\lambda(z-x)^2}\right| \mathrm{d}x = \int_{\mathbb{R}^d} e^{-\lambda\mathbb{R}\left((z-x)^2\right)}\mathrm{d}x \\ &= \int_{\mathbb{R}^d} e^{-\lambda\left((a-x)^2-b^2\right)}\mathrm{d}x = e^{\lambda b^2}\int_{\mathbb{R}^d} e^{-\lambda y^2}\mathrm{d}y = e^{\lambda b^2}\left(\frac{\pi}{\lambda}\right)^{d/2},\end{aligned}$$

d'où le résultat.

Dans toute la suite, on note c_λ la constante

$$c_\lambda = \left(\frac{\lambda}{2\pi}\right)^{d/2}\left(\frac{\lambda}{\pi}\right)^{d/4} .$$

Définition 5.4 (Transformation de Bargmann) Pour $f \in L^2(\mathbb{R}^d)$ et $z \in \mathbb{C}^d$, on pose

$$T_\lambda(f)(z) = c_\lambda\left(g(z,\cdot)|\overline{f}\right)_{L^2(R^d)} = c_\lambda \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{\lambda(z-x)^2}{2}} f(x)\mathrm{d}x.$$

La propriété essentielle de la transformation de Bargmann T_λ est énoncée dans le théorème suivant. Ce résultat va résulter du fait que la transformation de Fourier est isométrique sur L^2.

Théorème 5.5 T_λ est une isométrie bijective de $L^2(\mathbb{R}^d)$ sur $\mathscr{H}_{\varphi,\lambda}$ avec $\varphi(z) = \frac{|\Im(z)|^2}{2}$.

Remarque 5.6 On munit $T^*\mathbb{C}^d = \{(z,\zeta) \in \mathbb{C}^{2d}\}$ de la structure de variété symplectique complexe définie par la 2-forme $\sum_j \mathrm{d}\zeta_j \wedge \mathrm{d}z_j$. La fonction $\Phi(z) = i\frac{(z-x)^2}{2}$ est fonction génératrice de la transformation canonique complexe χ de $T^*\mathbb{C}^d$ dans $T^*\mathbb{C}^d$

$$\left(x, \xi = -\frac{\partial \Phi}{\partial x}\right) \mapsto \left(z, \zeta = \frac{\partial \Phi}{\partial z}\right),$$
$$\chi(x,\xi) = \left(z = x - i\xi, \zeta = \xi\right).$$

On a en particulier, en notant $T^*\mathbb{R}^d = \{(x,\xi) \in \mathbb{R}^{2d}\} \subset T^*\mathbb{C}^d$

$$\chi(T^*\mathbb{R}^d) = \{(z,\zeta) \in T^*\mathbb{C}^d; \zeta = -Im(z)\}.$$

Donc $\chi(T^*\mathbb{R}^d)$ est paramétré par $z \in \mathbb{C}^d$. Ceci va permettre de ramener l'analyse de la fonction f *"microlocalement"* près de $(x_0,\xi_0) \in T^*\mathbb{R}^d$ à l'analyse *locale* de la fonction $T_\lambda(f)$ près de $z_0 = x_0 - i\xi_0 \in \mathbb{C}^d$ (on choisira le paramètre $\lambda \gg 1$ grand).

Avant de prouver le théorème 5.5, on vérifie que $T_\lambda(f)$ est bien défini pour toute distribution tempérée $f \in \mathscr{S}'(\mathbb{R}^d)$. Pour $g \in \mathscr{S}(\mathbb{R}^d)$ (l'espace de Schwartz des fonctions C^∞ à décroissance rapide avec toutes leurs dérivées), et $f \in \mathscr{S}'(\mathbb{R}^d)$, on note $\langle f, g\rangle$ la dualité $\mathscr{S}'(\mathbb{R}^d)$, $\mathscr{S}(\mathbb{R}^d)$. On a par définition de T_λ

$$T_\lambda(f)(z) = \left\langle f, c_\lambda \mathrm{e}^{-\frac{\lambda(z-x)^2}{2}} \right\rangle.$$

Proposition 5.7 L'application $f \mapsto T_\lambda(f)$ est continue de $\mathscr{S}'(\mathbb{R}^d)$ dans $\mathscr{O}(\mathbb{C}^d)$. De plus, on a:

$$\partial_{z_j} T_\lambda(f) = T_\lambda\left(\partial_{x_j} f\right), \qquad z_j T_\lambda(f) = T_\lambda\left(x_j f - \frac{1}{\lambda}\partial_{x_j} f\right). \tag{5.1}$$

La continuité résulte du fait que $z \mapsto \mathrm{e}^{-\frac{\lambda(z-x)^2}{2}}$ est holomorphe en $z \in \mathbb{C}^d$ à valeurs dans $\mathscr{S}(\mathbb{R}^d)$. Vérifions les formules (5.1). On a

$$\begin{aligned} \partial_{z_j} T_\lambda(f) - T_\lambda(\partial_{x_j} f) &= c_\lambda \left\langle f, \left(\partial_{x_j} + \partial_{z_j}\right) \mathrm{e}^{-\frac{\lambda(z-x)^2}{2}} \right\rangle = \langle f, 0 \rangle = 0, \\ z_j T_\lambda(f) - T_\lambda\left(x_j f - \frac{1}{\lambda}\partial_{x_j} f\right) &= c_\lambda \left\langle f, \left(z_j - x_j - \frac{1}{\lambda}\partial_{x_j}\right) \mathrm{e}^{-\frac{\lambda(z-x)^2}{2}} \right\rangle = \langle f, 0 \rangle = 0. \end{aligned} \tag{5.2}$$

Exemple 5.8 Voici quelques exemples:

1) $T_\lambda(\delta_a) = c_\lambda \mathrm{e}^{-\lambda \frac{(z-a)^2}{2}}, \quad a \in \mathbb{R}^d.$

2) $T_\lambda\left(\mathrm{e}^{ix\cdot\xi}\right) = \left(\frac{\lambda}{\pi}\right)^{d/4} \mathrm{e}^{iz\cdot\xi - \xi^2/2\lambda}, \quad \xi \in \mathbb{R}^d$. En effet, on a

$$\begin{aligned} T_\lambda\left(\mathrm{e}^{ix\cdot\xi}\right) = c_\lambda \int \mathrm{e}^{-\lambda\frac{(z-x)^2}{2} + ix\cdot\xi} \mathrm{d}x &= c_\lambda \mathrm{e}^{-\lambda\frac{z^2}{2}} \mathrm{e}^{\frac{\lambda}{2}\left(z + i\frac{\xi}{\lambda}\right)^2} \int \mathrm{e}^{-\frac{\lambda}{2}\left(x - z - i\frac{\xi}{\lambda}\right)^2} \mathrm{d}x \\ &= c_\lambda \left(\frac{2\pi}{\lambda}\right)^{d/2} \mathrm{e}^{iz\cdot\xi - \xi^2/2\lambda}. \end{aligned}$$

3) $T_\lambda\left(\mathrm{e}^{-\lambda\frac{x^2}{2}}\right) = \left(\frac{\lambda}{\pi}\right)^{d/4} 2^{-d/2} \mathrm{e}^{-\lambda\frac{z^2}{4}}$. En effet, on a

$$\begin{aligned} T_\lambda\left(\mathrm{e}^{-\lambda\frac{x^2}{2}}\right) = c_\lambda \int \mathrm{e}^{-\lambda\frac{(z-x)^2}{2} - \lambda\frac{x^2}{2}} \mathrm{d}x &= c_\lambda \mathrm{e}^{-\lambda\frac{z^2}{2}} \mathrm{e}^{\lambda\frac{z^2}{4}} \int \mathrm{e}^{-\lambda\left(x - \frac{z}{2}\right)^2} \mathrm{d}x \\ &= c_\lambda \left(\frac{\pi}{\lambda}\right)^{d/2} \mathrm{e}^{-\lambda\frac{z^2}{4}}. \end{aligned}$$

La formule d'inversion de Fourier fournit une expression de la transformée de Bargmann en terme de la transformée de Fourier, expression donnée dans le lemme suivant.

Lemme 5.9 Pour tout $f \in \mathscr{S}'(\mathbb{R}^d)$ on a

$$T_\lambda\big(f\big)(z) = (2\pi)^{-d}\left(\frac{\lambda}{\pi}\right)^{d/4}\left\langle \hat{f}, \mathrm{e}^{iz\cdot\xi-\frac{\xi^2}{2\lambda}}\right\rangle \tag{5.3}$$

où $\hat{f} \in \mathscr{S}'(\mathbb{R}^d)$ est la transformée de Fourier de f, et pour tout $a \in \mathbb{R}^d$

$$T_\lambda\big(f(x-a)\big)(z) = T_\lambda\big(f\big)(z-a). \tag{5.4}$$

La formule d'invariance par translation (5.4) est évidente. Par densité de $\mathscr{S}(\mathbb{R}^d)$ dans $\mathscr{S}'(\mathbb{R}^d)$, il suffit de vérifier la formule (5.3) pour $f \in \mathscr{S}(\mathbb{R}^d)$. Or par la formule d'inversion de Fourier, on a

$$f(x) = (2\pi)^{-d}\int e^{ix\cdot\xi}\hat{f}(\xi)\mathrm{d}\xi, \qquad \forall f \in \mathscr{S}(\mathbb{R}^d)$$

et il suffit alors d'utiliser la formule $T_\lambda\big(\mathrm{e}^{ix\cdot\xi}\big) = \left(\frac{\lambda}{\pi}\right)^{d/4}\mathrm{e}^{iz\cdot\xi-\frac{\xi^2}{2\lambda}}$.

Preuve du théorème 5.5. Pour $f \in L^2(\mathbb{R}^d)$ et avec $z = a+ib$, on a d'après la formule (5.3) du lemme 5.9

$$T_\lambda\big(f\big)(a+ib) = (2\pi)^{-d}\left(\frac{\lambda}{\pi}\right)^{d/4}\mathrm{e}^{\lambda\frac{b^2}{2}}\int \mathrm{e}^{ia\cdot\xi}\mathrm{e}^{-\lambda\frac{\left(b+\frac{\xi}{\lambda}\right)^2}{2}}\hat{f}(\xi)\,\mathrm{d}\xi.$$

Il en résulte $T_\lambda\big(f\big)(z)\mathrm{e}^{-\lambda\frac{|\Im(z)|^2}{2}} = H(a,b)$ avec

$$H(a,b) = (2\pi)^{-d}\left(\frac{\lambda}{\pi}\right)^{d/4}\int \mathrm{e}^{ia\cdot\xi}\mathrm{e}^{-\lambda\frac{\left(b+\frac{\xi}{\lambda}\right)^2}{2}}\hat{f}(\xi)\,\mathrm{d}\xi.$$

Par la formule d'inversion de Fourier en la variable $a \in \mathbb{R}^d$, on a donc

$$\hat{H}(\xi,b) = \left(\frac{\lambda}{\pi}\right)^{d/4}\mathrm{e}^{-\lambda\frac{\left(b+\frac{\xi}{\lambda}\right)^2}{2}}\hat{f}(\xi).$$

d'oú $$\iint|\hat{H}(\xi,b)|^2\mathrm{d}\xi\mathrm{d}b = \int\big|\hat{f}(\xi)\big|^2\left[\left(\frac{\lambda}{\pi}\right)^{d/2}\int \mathrm{e}^{-\lambda\left(b+\frac{\xi}{\lambda}\right)^2}\mathrm{d}b\right]\mathrm{d}\xi = \int\big|\hat{f}(\xi)\big|^2\mathrm{d}\xi.$$

On a donc par la formule de Plancherel $\displaystyle\int_{\mathbb{C}^d} \left| T_\lambda(f)(z) e^{-\lambda \frac{|\Im(z)|^2}{2}} \right|^2 \mathrm{d}a\mathrm{d}b = \int_{\mathbb{R}^d} |f(x)|^2 \mathrm{d}x$

ce qui signifie exactement

$$\|T_\lambda(f)\|_{\mathscr{H}_{\varphi,\lambda}} = \|f\|_{L^2(\mathbb{R}^d)}.$$

L'application T_λ de $L^2(\mathbb{R}^d)$ dans $\mathscr{H}_{\varphi,\lambda}$ est donc isométrique, en particulier est injective, et son image $\mathrm{Im}(T_\lambda) = T_\lambda(L^2(\mathbb{R}^d))$ est un sous espace *fermé* de $\mathscr{H}_{\varphi,\lambda}$. Pour finir la preuve du théorème 5.5, il reste à prouver que T_λ est surjective, c'est à dire

$$\mathrm{Im}(T_\lambda) = \mathscr{H}_{\varphi,\lambda}. \tag{5.5}$$

Pour montrer l'égalité (5.5), nous allons utiliser le lemme suivant.

Lemme 5.10 Soit $\varphi_0(z) = \frac{|z|^2}{4}$. Les monômes z^α, $\alpha \in \mathbb{N}^d$, forment une base orthogonale de $\mathscr{H}_{\varphi_0,\lambda}$.

Montrons alors l'égalité (5.5) On a $u(z) \in \mathscr{H}_{\varphi,\lambda}$ ssi $u(z)e^{\lambda \frac{z^2}{4}} \in \mathscr{H}_{\varphi_0,\lambda}$. D'après le lemme 5.10, les fonctions de la forme $p(z)e^{-\lambda \frac{z^2}{4}}$, où $p(z)$ est un polynôme, sont denses dans $\mathscr{H}_{\varphi,\lambda}$. Comme $\mathrm{Im}(T_\lambda)$ est fermé, il suffit donc de vérifier que pour tout polynôme $p(z)$, il existe $f_p \in L^2(\mathbb{R}^d)$ tel que $T_\lambda(f_p) = p(z)e^{-\lambda \frac{z^2}{4}}$. Posons

$$f_1(x) = 2^{d/2} \left(\frac{\pi}{\lambda}\right)^{d/4} e^{-\lambda \frac{x^2}{2}}.$$

On sait qu'on a (voir les exemples précédents) $T_\lambda(f_1) = e^{-\lambda \frac{z^2}{4}}$, et d'après les formules (5.1) de la proposition 5.7, on a

$$z^\alpha e^{-\lambda \frac{z^2}{4}} = z^\alpha T_\lambda(f_1) = T_\lambda\left(\left(x - \frac{1}{\lambda}\partial_x\right)^\alpha f_1 \right).$$

Comme on a $\left(x - \frac{1}{\lambda}\partial_x\right)^\alpha f_1 \in L^2(\mathbb{R}^d)$, ceci achève la preuve du théorème 5.5. □

Preuve du lemme 5.10. On vérifie facilement qu'on a $\left(z^\alpha | z^\beta\right)_{\mathscr{H}_{\varphi_0,\lambda}} = 0$ pour $\alpha \neq \beta$. Calculons $\|z^\alpha\|^2_{\mathscr{H}_{\varphi_0,\lambda}}$. On a avec $z = x + iy$

$$
\begin{aligned}
\|z^\alpha\|^2_{\mathscr{H}_{\varphi_0,\lambda}} &= \int_{\mathbb{C}^d} |z^\alpha|^2 \mathrm{e}^{-\lambda\frac{|z|^2}{2}} \mathrm{d}x\mathrm{d}y = \prod_j \int_{\mathbb{C}} |z_j^{\alpha_j}|^2 \mathrm{e}^{-\lambda\frac{|z_j|^2}{2}} \mathrm{d}x_j\mathrm{d}y_j \\
&= \prod_j \left(2\pi\int_0^\infty \rho^{2\alpha_j}\mathrm{e}^{-\lambda\frac{\rho^2}{2}}\rho\mathrm{d}\rho\right) = (2\pi)^d \prod_j \left(\int_0^\infty (2r)^{\alpha_j}\mathrm{e}^{-\lambda r}\mathrm{d}r\right) \\
&= (2\pi)^d \lambda^{-d-|\alpha|} 2^{|\alpha|}\alpha!
\end{aligned}
$$

Pour $f = \sum_\beta f_\beta z^\beta \in \mathscr{H}_{\varphi_0,\lambda}$, $\alpha \in \mathbb{N}^d$ et $R > 0$, posons

$$
\Theta(f,\alpha,R) = \int_{z\in\mathbb{C}^d,\,|z_j|<R} f(z)\,\overline{z}^\alpha\,\mathrm{e}^{-\lambda\frac{|z|^2}{2}}\,\mathrm{d}x\mathrm{d}y.
$$

On a, la série $\sum_\beta f_\beta z^\beta$ étant uniformément convergente sur tout compact de $\mathbb{C}^d$,

$$
\Theta(f,\alpha,R) = \sum_\beta f_\beta \int_{z\in\mathbb{C}^d,\,|z_j|<R} z^\beta\,\overline{z}^\alpha, \mathrm{e}^{-\lambda\frac{|z|^2}{2}}\,\mathrm{d}x\mathrm{d}y = f_\alpha \int_{z\in\mathbb{C}^d,\,|z_j|<R} \left|z^\alpha\right|^2 \mathrm{e}^{-\lambda\frac{|z|^2}{2}}\,\mathrm{d}x\mathrm{d}y.
$$

En faisant tendre R vers $+\infty$, on obtient donc $\left(f|z^\alpha\right)_{\mathscr{H}_{\varphi_0,\lambda}} = f_\alpha \left\|z^\alpha\right\|^2_{\mathscr{H}_{\varphi_0,\lambda}}$. Si f est orthogonale dans $\mathscr{H}_{\varphi_0,\lambda}$ à tous les monômes z^α, on a donc $f_\alpha = 0$ pour tout $\alpha \in \mathbb{N}^d$, donc $f = 0$, ce qui prouve le lemme 5.10. □

On remarquera que la preuve du lemme 5.10 fournit les formules suivantes:

$$
\begin{aligned}
\|f\|^2_{\mathscr{H}_{\varphi_0,\lambda}} &= \sum_\alpha |f_\alpha|^2 \left\|z^\alpha\right\|^2_{\mathscr{H}_{\varphi_0,\lambda}}, \qquad \forall f = \sum_\alpha f_\alpha z^\alpha \in \mathscr{H}_{\varphi_0,\lambda}, \\
\|z^\alpha\|_{\mathscr{H}_{\varphi_0,\lambda}} &= \left(\frac{2\pi}{\lambda}\right)^{d/2} (2/\lambda)^{|\alpha|/2} (\alpha!)^{1/2}.
\end{aligned}
$$

D'après le théorème 5.5, on a, en regardant T_λ comme opérateur de $L^2(\mathbb{R}^d)$ dans $L^2\left(\mathbb{C}^d, \mathrm{e}^{-\lambda|\Im(z)|^2}\mathrm{d}x\mathrm{d}y\right)$, et en notant T_λ^* l'adjoint de T_λ

$$
\begin{aligned}
T_\lambda^* T_\lambda(f) &= f \qquad && \forall f \in L^2(\mathbb{R}^d), \\
T_\lambda T_\lambda^*(g) &= \Pi_\lambda(g) \qquad && \forall g \in L^2\left(\mathbb{C}^d, \mathrm{e}^{-\lambda|\Im(z)|^2}\right),
\end{aligned}
$$

où Π_λ est le projecteur orthogonal de $L^2\left(\mathbb{C}^d, \mathrm{e}^{-\lambda|\Im(z)|^2}\right)$ sur $\mathscr{H}_{\varphi,\lambda}$. On appelle Π_λ le *projecteur de Bargmann*.

En utilisant la définition de l'adjoint $\big(T_\lambda(f)\big|g\big)_{L^2\left(\mathbb{C}^d,\mathrm{e}^{-\lambda|\Im(z)|^2}\right)} = \big(f\big|T_\lambda^*(g)\big)_{L^2(\mathbb{R}^d)}$, on obtient avec $z=a+ib$

$$T_\lambda^*\big(g\big)(x) = c_\lambda \int_{\mathbb{C}^d} g(z)\,\mathrm{e}^{-\lambda b^2-\lambda\frac{(\bar z-x)^2}{2}}\,\mathrm{d}a\mathrm{d}b, \tag{5.6}$$

ce qui implique pour tout $g\in L^2\left(\mathbb{C}^d,\mathrm{e}^{-\lambda|\Im(z)|^2}\right)$ et tout $w\in\mathbb{C}^d$

$$\Pi_\lambda\big(g\big)(w) = T_\lambda T_\lambda^*\big(g\big)(w) = \left(\frac{\lambda}{2\pi}\right)^d \int_{\mathbb{C}^d} \mathrm{e}^{-\lambda b^2-\lambda\frac{(\bar z-w)^2}{4}}\,g(z)\,\mathrm{d}a\mathrm{d}b.$$

Exercice 5.11 Vérifier directement à partir de la formule (5.6) que pour tout $g\in\mathscr{H}_{\varphi,\lambda}$ on a bien $\Pi_\lambda(g)=g$. On commencera par vérifier qu'on a $\Pi_\lambda(1)=1$.

Remarque 5.12 En utilisant la formule (5.6), on obtient pour tout $w\in\mathbb{C}^d$,

$$\mathrm{e}^{-\lambda\varphi(w)}\big|T_\lambda T_\lambda^*\big(g\big)(w)\big| \le \left(\frac{\lambda}{2\pi}\right)^d \int_{\mathbb{C}^d} \mathrm{e}^{-\lambda\frac{|z-w|^2}{4}}\left|g(z)\mathrm{e}^{-\lambda\varphi(z)}\right|\mathrm{d}a\mathrm{d}b.$$

En particulier, pour $\lambda\gg 1$ grand, le facteur $\mathrm{e}^{-\lambda\frac{|z-w|^2}{4}}$ montre que l'intégrale se localise près de $z=w$.

Comme on a $T_\lambda^*T_\lambda=\mathrm{Id}$, la formule (5.6) fournit une formule d'inversion pour la transformation de Bargmann T_λ. Cette formule d'inversion nécessite de connaitre $T_\lambda(f)(z)$ pour tout $z\in\mathbb{C}^d$.

Nous allons à présent donner dans le théorème 5.13 une autre formule d'inversion qui ne fait intervenir que les valeurs de $T_\lambda\big(f\big)(z)$ au voisinage du cylindre $\{z=a+ib, |b|=c\}$ où $c>0$ est une constante donnée. Cette formule d'inversion nous sera utile dans le paragraphe suivant consacré à l'étude du front d'onde.

On fixe $c>0$ et on note $\mathbb{S}_c=\{b\in\mathbb{R}^d, |b|=c\}$ la sphère de rayon c de $\mathbb{R}^d$. Pour $f\in\mathscr{E}'(\mathbb{R}^d)$, on note $M_\lambda\big(f\big)(z,b)$ la fonction de $(z,b)\in\mathbb{C}^d\times\mathbb{S}_c$, définie pour $\lambda>0$ par la formule

$$M_\lambda\big(f\big)(z,b) = (2\pi)^{-d}\lambda^{d-1}\frac{\mathrm{e}^{-\lambda\frac{c^2}{2}}}{2}c_\lambda\left(T_\lambda(f) - \frac{i}{\lambda c^2}\,b\cdot\partial_z T_\lambda(f)\right)(z-ib). \tag{5.7}$$

La fonction $M_\lambda\big(f\big)(z,b)$ ne dépend que de $T_\lambda(f)$ et de ses dérivées premières. Elle va nous permettre de construire la formule d'inversion du théorème 5.13.

La fonction $M_\lambda\big(f\big)(z,b)$ est holomorphe en $z \in \mathbb{C}^d$, et analytique en $b \in \mathbb{S}_c$. Soit $R > 0$ tel que le fermé $\operatorname{supp}(f)$ soit contenu dans la boule ouverte $\{|x| < R\}$. Il existe des constantes $C > 0$, $p \in \mathbb{N}$ telles qu'on ait pour tout $z \in \mathbb{C}^d$ et tout $\lambda \geq 0$

$$\big|T_\lambda(f)(z)\big| = c_\lambda \left|\left\langle f, \mathrm{e}^{-\frac{\lambda(z-x)^2}{2}}\right\rangle\right| \leq c_\lambda C \sup_{|x|\leq R, |\alpha|\leq p} \left|\partial_x^\alpha \mathrm{e}^{-\frac{\lambda(z-x)^2}{2}}\right|,$$

d'où en notant pour $u \in \mathbb{R}$, $u_+ = \sup(u,0)$, et avec une autre constante C,

$$\big|T_\lambda(f)(z)\big| \leq c_\lambda C\big(1+\lambda+|z|\big)^p \mathrm{e}^{\lambda\frac{\Im(z)^2}{2}} \mathrm{e}^{-\lambda\frac{\left((|\mathbb{R}(z)|-R)_+\right)^2}{2}}.$$

Comme on a $\frac{1}{\lambda}\partial_z T_\lambda(f) = -zT_\lambda(f) + T_\lambda(xf)$, il en résulte que pour tout β, il existe une constante C_β telle que pour tout $z \in \mathbb{C}^d$, tout $b \in \mathbb{S}_c$ et tout $\lambda > 0$, on ait

$$\left|\partial_b^\beta M_\lambda\big(f\big)(z,b)\right| \leq C_\beta \lambda^{d-1}\big(1+\lambda+|z|\big)^{p+1+|\beta|} \mathrm{e}^{-\lambda\left(\Im(z)\cdot b - \frac{\Im(z)^2}{2}\right)} \mathrm{e}^{-\lambda\frac{\left((|\mathbb{R}(z)|-R)_+\right)^2}{2}}. \tag{5.8}$$

Soit $\mathscr{D}$ l'ouvert de $\mathbb{C}^d \times \mathbb{S}_c$

$$\mathscr{D} = \left\{(z,b) \in \mathbb{C}^d \times \mathbb{S}_c, \Im(z)\cdot b > \frac{|\Im(z)|^2}{2}\right\}.$$

Il résulte de (5.8) que la fonction

$$M\big(f\big)(z,b) = \int_0^\infty M_\lambda\big(f\big)(z,b)\mathrm{d}\lambda \tag{5.9}$$

est bien définie pour $(z,b) \in \mathscr{D}$, et est holomorphe en z et C^∞ en b. Il résulte aussi de (5.8) que $M\big(f\big)(z,b)$ se prolonge en fonction holomorphe en z et C^∞ en b au voisinage de $\{|x| > R\} \times \mathbb{S}_c$. D'après (5.1), on a pour tout α

$$M\big(\partial_x^\alpha f\big)(z,b) = \partial_z^\alpha M\big(f\big)(z,b). \tag{5.10}$$

Théorème 5.13 1) Pour tout $f \in \mathscr{E}'(\mathbb{R}^d)$, et tout $b' \in \mathscr{S}_c$ vérifiant $b' \cdot b > 0$, la limite

$$K\big(f\big)(x,b) = \lim_{\varepsilon \to 0^+} M\big(f\big)(x + i\varepsilon b', b) \tag{5.11}$$

existe dans $\mathscr{D}'(\mathbb{R}^n \times \mathbb{S}_c)$, est indépendante de b', et appartient à $C^\infty(\mathbb{S}_c, \mathscr{D}'(\mathbb{R}^n))$.

2) Pour tout $f \in \mathscr{E}'(\mathbb{R}^d)$, on a

$$f(x) = \int_{|b|=c} K\big(f\big)(x,b)\,\mathrm{d}\sigma(b). \tag{5.12}$$

Preuve. Montrons 1). On doit vérifier que la limite (5.11) existe, est indépendante de $b' \in \mathscr{S}_c$ vérifiant $b' \cdot b > 0$ et appartient à $C^r\big(\mathbb{S}_c, \mathscr{D}'(\mathbb{R}^n)\big)$. Comme pour toute distribution f à support compact et tout entier k, il existe m et des fonctions g_α de classe C^k à support compact telles qu'on ait

$$f = \sum_{|\alpha| \leq m} \partial_x^\alpha g_\alpha,$$

on peut, d'après (5.10), supposer $f \in C_0^k$ avec k aussi grand qu'on veut. On aura alors en particulier $\big|\hat{f}(\xi)\big| \leq Cte\big(1 + |\xi|\big)^{-k}$. En choisissant k assez grand et en utilisant la formule (5.3) du lemme 5.9, qui fournit pour $|\Im(z)| \in [a,b]$, $0 < a < b$,

$$\lambda^{-d/4} \mathrm{e}^{-\lambda \frac{\Im(z)^2}{2}} \big|T_\lambda\big(f\big)(z)\big| \leq C \int \big(1 + |\xi|\big)^{-k} \mathrm{e}^{-\lambda \frac{\left(\frac{\xi}{\lambda} + \Im(z)\right)^2}{2}} \leq C_0 \lambda^d \left(1 + \frac{a\lambda}{2}\right)^{-k} + C_1 \mathrm{e}^{-\frac{a\lambda}{2}}$$

on obtient pour tout β, $|\beta| \leq r$,

$$\Big|\partial_b^\beta M_\lambda\big(f\big)(z,b)\Big| \leq C(1+\lambda)^{-2} \mathrm{e}^{-\lambda \frac{\Im(z) \cdot b - \Im(z)^2}{2}}.$$

Il en résulte que les fonctions $\partial_b^\beta M(f)(z,b)$ sont continues sur l'adhérence $\overline{\mathscr{D}}$ de $\mathscr{D}$; d'où le résultat.

Vérifions à présent le point 2). Comme $T_\lambda(f)$, donc aussi $M_\lambda(f)$, sont définis par des opérateurs de convolution, il suffit de vérifier que (5.12) est vrai pour $f = \delta_0$, où δ_0 est la mesure de Dirac à l'origine. Cela est conséquence de la proposition suivante, qui fournit une représentation intégrale de la mesure de Dirac à l'origine δ_0, où l'intégrale est convergente pour $x \neq 0$, ce qui n'est pas le cas de la formule d'inversion de Fourier $\delta_0 = (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{ix\xi} \mathrm{d}\xi$.

Proposition 5.14 Soit T la distribution sur $\mathbb{R}^d$ définie par l'intégrale oscillante

$$T(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\eta - x^2 \frac{|\eta|}{2c}} \left(1 + i\frac{x\eta}{2c|\eta|}\right) d\eta.$$

On a l'égalité

$$T = \delta_0.$$

Preuve. Soit $\varphi_\varepsilon(x) \in \mathscr{S}(\mathbb{R}^d)$ la fonction définie par

$$\varphi_\varepsilon(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\xi - \varepsilon\frac{\xi^2}{2}} d\xi.$$

On a $\lim_{\varepsilon\to 0, \mathscr{S}'(\mathbb{R}^d)} \varphi_\varepsilon(x) = \delta_0$. Comme la fonction $e^{ix\xi-\varepsilon\frac{\xi^2}{2}}$ est holomorphe en $\xi \in \mathbb{C}^d$, on peut, sans changer la valeur de l'intégrale qui définit $\varphi_\varepsilon(x)$, remplacer le contour d'intégration $\xi \in \mathbb{R}^d$ par le contour d'intégration $\Gamma_s : \xi = \eta + i\frac{sx|\eta|}{2c}$, $\eta \in \mathbb{R}^d$, où $s \in [0,1]$ est un paramètre. On a $\Gamma_0 = \mathbb{R}^d$, et sur le contour Γ_s la fonction $ix\xi - \varepsilon\frac{\xi^2}{2}$ vaut

$$ix\xi - \varepsilon\frac{\xi^2}{2} = ix\eta - \frac{sx^2|\eta|}{2c} - \frac{\varepsilon}{2}\left[\eta^2\left(1 - \frac{s^2x^2}{4c^2}\right) + i\frac{s|\eta|x\eta}{c}\right].$$

Pour que la déformation de contour de $\Gamma_0 = \mathbb{R}^d$ à Γ_1 soit licite, il faut assurer que l'intégrale en η reste absolument convergente, ce qui impose la condition $|x| < 2c$. On a donc obtenu la formule, pour tout $x \in \mathbb{R}^d$, $|x| < 2c$, en utilisant $d\xi = \left(1 + i\frac{sx\eta}{2c|\eta|}\right) d\eta$,

$$\begin{aligned}\varphi_\varepsilon(x) &= (2\pi)^{-d} \int_{\Gamma_1} e^{ix\xi - \varepsilon\frac{\xi^2}{2}} d\xi \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\eta - x^2\frac{|\eta|}{2c} - \frac{\varepsilon}{2}\left[\eta^2\left(1-\frac{x^2}{4c^2}\right) + i\frac{|\eta|x\eta}{c}\right]} \left(1 + i\frac{x\eta}{2c|\eta|}\right) d\eta.\end{aligned}$$

La distribution T est clairement analytique sur $\mathbb{R}^d \setminus \{0\}$, et dans l'ouvert $|x| < 2c$, on a $T = \lim_{\varepsilon\to 0} \varphi_\varepsilon$ au sens des distributions. Ceci prouve $T = \delta_0$, et achève la preuve de la proposition 5.14. $\square$

D'après la proposition 5.14, pour $\chi \in C_0^\infty(\mathbb{R})$ telle que $\chi(t) = 1$ pour $|t| \leq 1$, on a

$$\delta_0 = (2\pi)^{-d} \lim_{\mathscr{D}', R\to\infty} \int_{\mathbb{R}^d} \chi\left(\frac{|\eta|}{cR}\right) e^{ix\eta - x^2\frac{|\eta|}{2c}} \left(1 + i\frac{x\eta}{2c|\eta|}\right) d\eta.$$

En posant $\eta = \lambda b$, $|b| = c$, on a $\lambda = \frac{|\eta|}{c}$, $\mathrm{d}\eta = \lambda^{d-1}\mathrm{d}\lambda\mathrm{d}\sigma(b)$, on obtient

$$\delta_0 = (2\pi)^{-d} \lim_{\mathscr{D}', R\to\infty} \int_0^\infty \chi\left(\frac{\lambda}{R}\right) \lambda^{d-1} \left[\int_{|b|=c, y\in\mathbb{R}^d} \mathrm{e}^{-\lambda\frac{(x-ib)^2}{2}} \mathrm{e}^{-\lambda\frac{c^2}{2}} \left(1 + i\frac{x\cdot b}{2c^2}\right) \mathrm{d}\sigma(b)\right] \mathrm{d}\lambda.$$

La formule (5.12) avec $f = \delta_0$ en résulte en utilisant l'identité

$$2c_\lambda \int \mathrm{e}^{-\lambda\frac{(x-ib)^2}{2}} \left(1 + i\frac{xb}{2c^2}\right) = \left(T_\lambda(\delta_0) - i\frac{b}{\lambda c^2}\partial_z T_\lambda(\delta_0)\right)(x - ib).$$

Ceci achève la preuve du théorème 5.13. □

. . . .

3 Front d'onde

Le théorème suivant permet de caractériser la régularité locale d'une distribution f sur $\mathbb{R}^d$ en fonction du comportement de $T_\lambda(f)$ quand $\lambda \to +\infty$.

Soit $c > 0$ une constante. Pour tout $x_0 \in \mathbb{R}^d$, on note $K_{x_0,c}$ le compact de $\mathbb{C}^d$

$$K_{x_0,c} = \left\{z = x_0 - ib;\, b \in \mathbb{R}^d, |b| = c\right\}.$$

Théorème 5.15 Soit $c > 0$ fixé. Pour tout $f \in \mathscr{S}'(\mathbb{R}^d)$ et tout $x_0 \in \mathbb{R}^d$ les propriétés suivantes sont équivalentes:

1) Il existe un U voisinage de x_0 tel que $f|_U \in C^\infty(U)$ (resp. $f|_U \in C^\omega(U)$).

2) Il existe un V voisinage de $K_{x_0,c}$ tel que

$$N, \exists C_N \quad \text{tel que} \quad \sup_{z\in V} \left|T_\lambda\big(f\big)(z)\right| \mathrm{e}^{-\lambda\varphi(z)} \le C_N \lambda^{-N} \quad \forall\lambda \ge 1$$

$$(\text{resp. } \exists C > 0,\ \exists\varepsilon > 0 \quad \text{tel que} \quad \sup_{z\in V} \left|T_\lambda\big(f\big)(z)\right| \mathrm{e}^{-\lambda\varphi(z)} \le C\mathrm{e}^{-\varepsilon\lambda} \quad \forall\lambda \ge 1).$$

Preuve. Montrons déjà que 1) ⇒ 2). On commence par vérifier le cas C^∞. Soit $r > 0$ tel que $\{|x - x_0| \le r\} \subset U$ et $\psi \in C_0^\infty(U)$ tel que $\psi(x) = 1$ pour $|x - x_0| \le r$. On a $\psi f \in C_0^\infty(\mathbb{R}^d)$, $T_\lambda(f) = T_\lambda(\psi f) + T_\lambda\big((1-\psi)f\big)$ et on analyse séparément chacun des termes. On a

$$T_\lambda\big((1-\psi)f\big)(z) = \left\langle f, x \mapsto c_\lambda \big(1-\psi(x)\big)\mathrm{e}^{-\lambda\frac{(z-x)^2}{2}}\right\rangle$$

où $\langle\cdot,\cdot\rangle$ désigne la dualité entre $\mathscr{S}'(\mathbb{R}^d)$ et $\mathscr{S}(\mathbb{R}^d)$. Comme $f \in \mathscr{S}'(\mathbb{R}^d)$ est une distribution tempérée, il existe $C > 0$ et $p \in \mathbb{N}$ tels que

$$\left|\left\langle f, x \mapsto c_\lambda \big(1-\psi(x)\big)\mathrm{e}^{-\lambda\frac{(z-x)^2}{2}}\right\rangle\right| \leq CN_p\left(x \mapsto c_\lambda \big(1-\psi(x)\big)\mathrm{e}^{-\lambda\frac{(z-x)^2}{2}}\right)$$

où N_p est la norme

$$N_p(g) = \sum_{\substack{|\alpha|\leq p,\\ |\beta|\leq q}} \sup_{x\in\mathbb{R}^d} \left|x^\alpha \partial_x^\beta g(x)\right|.$$

Comme toutes les dérivées de $1-\psi$ sont des fonctions bornées sur $\mathbb{R}^d$ et à support dans $\{|x-x_0| \geq r\}$, il existe un polynôme $P(z,x,\lambda)$, en les variables z, x, λ tel qu'on ait

$$N_p\left(x \mapsto c_\lambda \big(1-\psi(x)\big)\mathrm{e}^{-\lambda\frac{(z-x)^2}{2}}\right) \leq c_\lambda \sup_{|x-x_0|\geq r}\left|P(z,x,\lambda)\mathrm{e}^{-\lambda\frac{\mathbb{R}(z-x)^2}{2}+\lambda\frac{\Im(z)^2}{2}}\right|$$

Pour tout $A > 0$, il existe donc une constante $C = C(A,r,P)$ telle qu'on ait

$$\sup_{\substack{|\mathbb{R}(z)-x_0|\leq\frac{r}{2},\\ |\Im(z)|\leq A}} \left|T_\lambda\big((1-\psi)f\big)(z)\right|\mathrm{e}^{-\lambda\frac{\Im(z)^2}{2}} \leq Cc_\lambda\lambda^{\deg(P)}\mathrm{e}^{-\lambda\frac{r^2}{4}}. \tag{5.13}$$

Ici, on a utilisé que pour tout entier k, il existe une constante C telle que

$$\sup_{t\geq r}\left(t^k\mathrm{e}^{-\lambda\frac{(t-\frac{r}{2})^2}{2}}\right) \leq Ce^{-\lambda\frac{r^2}{4}}, \qquad \forall\lambda\geq 1.$$

Pour estimer le terme $T_\lambda(\psi f)$, on utilise le lemme 5.9, formule (5.3), ce qui fournit, avec $z = a+ib$ $T_\lambda\big(\psi f\big)(z)\mathrm{e}^{-\lambda\frac{\Im(z)^2}{2}} = (2\pi)^{-d}\left(\frac{\lambda}{\pi}\right)^{d/4}\int \mathrm{e}^{ia\xi-\lambda\frac{\left(b+\frac{\xi}{\lambda}\right)^2}{2}}\widehat{\psi f}(\xi)\,\mathrm{d}\xi$. Posons $J_\lambda(z) = (2\pi)^d\left(\frac{\lambda}{\pi}\right)^{-d/4}\left|T_\lambda\big(\psi f\big)(z)\right|\mathrm{e}^{-\lambda\frac{\Im(z)^2}{2}}$. Pour tout $R > 0$ et $a \in \mathbb{R}^d$ on a

$$\left|J_\lambda(a+ib)\right| \leq \int_{|\lambda b+\xi|\leq\lambda R}\mathrm{e}^{-\lambda\frac{\left(b+\frac{\xi}{\lambda}\right)^2}{2}}\left|\widehat{\psi f}(\xi)\right|\mathrm{d}\xi + \int_{|\lambda b+\xi|>\lambda R}\mathrm{e}^{-\lambda\frac{\left(b+\frac{\xi}{\lambda}\right)^2}{2}}\left|\widehat{\psi f}(\xi)\right|\mathrm{d}\xi.$$

La deuxième intégrale est majorée par $\mathrm{e}^{-\lambda\frac{R^2}{2}}\|\widehat{\psi f}\|_{L^1}$. On choisit $R = c/4$, et on remarque qu'on a: $|b| \geq c/2$ et $|\lambda b+\xi| \leq \lambda R \Rightarrow |\xi| \geq \lambda\frac{c}{4}$. Comme $\widehat{\psi f} \in \mathscr{S}(\mathbb{R}^d)$ est à décroissance rapide, on obtient que pour tout N, il existe C_N tel que pour tout $\lambda \geq 1$ on a

$$\sup_{|b|\geq c/2}\int_{|\lambda b+\xi|\leq\lambda\frac{c}{4}} \mathrm{e}^{-\lambda\frac{\left(b+\frac{\xi}{\lambda}\right)^2}{2}}|\widehat{\psi f}(\xi)|\mathrm{d}\xi \leq \int_{|\xi|\geq\lambda\frac{c}{4}}|\widehat{\psi f}(\xi)|\mathrm{d}\xi \leq C_N\lambda^{-N}.$$

Ceci achève la preuve dans le cas C^∞. Dans le cas analytique, l'estimation (5.13) donne la décroissance exponentielle en λ du terme $\sup_{|\mathbb{R}(z)-x_0|\leq r/2,\,|\Im(z)|\leq A}\left|T_\lambda\big((1-\psi)f\big)(z)\right|$. Pour estimer le terme $T_\lambda(\psi f)$, on utilise le fait que par hypothèse, la fonction ψf est analytique dans la boule ouverte $|x-x_0| < r$. Il existe donc un $\delta \in]0,1]$ tel que ψf s'étende en fonction holomorphe dans le domaine complexe $\{w = x+iy,\ |x-x_0| < 3r/4,\ |y| < 2\delta\}$. Soit $\chi \in C_0^\infty\big(|x-x_0| < r/2\big)$, $0 \leq \chi \leq 1$, avec $\chi(x) = 1$ pour $|x-x_0| \leq r/4$. Dans l'intégrale de définition de $T_\lambda\big(\psi f\big)(a+ib)$, on peut, pour tout $s \in [0,\delta]$, remplacer le contour d'intégration réel $x \in \mathbb{R}^d$ par le contour $\Gamma_s : w = x+is\chi(x)b$, $x \in \mathbb{R}^d$. On a donc

$$T_\lambda\big(\psi f\big)(a+ib) = c_\lambda\int_{\Gamma_\delta}\mathrm{e}^{-\lambda\frac{(z-w)^2}{2}}\big(\psi f\big)(w)\,\mathrm{d}w.$$

Il en résulte

$$\left|T_\lambda\big(\psi f\big)(a+ib)\right| \leq c_\lambda\sup_{x\in\mathbb{R}^d}\left|\mathrm{e}^{-\lambda\frac{(a+ib-x-i\delta\chi(x)b)^2}{2}}\right|\cdot\left|\int_{\Gamma_s}\big(\psi f\big)(w)\,\mathrm{d}w\right|.$$

Il reste à observer que l'identité

$$-\mathbb{R}\left(\frac{\big(a+ib-x-i\delta\chi(x)b\big)^2}{2}\right) = b^2\frac{\big(1-\delta\chi(x)\big)^2}{2} - \frac{(a-x)^2}{2}$$

et $\chi(x) = 1$ pour $|x-x_0| \leq r/4$ implique

$$\sup_{\substack{|a-x_0|\leq r/8,\\ |b|\geq c/2}}\ \sup_{x\in\mathbb{R}^d}\left[-\mathbb{R}\left(\frac{\big(a+ib-x-i\delta\chi(x)b\big)^2}{2}\right)\right] \leq \frac{b^2}{2}-\varepsilon$$

avec $\varepsilon = \min\left(\frac{(r/8)^2}{2}, c^2\frac{2\delta-\delta^2}{8}\right) > 0$. On a donc obtenu une inégalité qui achève la preuve dans le cas analytique:

$$\sup_{\substack{|a-x_0|\leq r/8,\\ |b|\geq c/2}} \left|T_\lambda\left(\psi f\right)(a+ib)\right| \mathrm{e}^{-\lambda\frac{|b^2|}{2}} \leq c_\lambda \mathrm{e}^{-\varepsilon\lambda}\left|\int_{\Gamma_s}\left(\psi f\right)(w)\mathrm{d}w\right|.$$

Pour montrer l'implication 2) $\Rightarrow$ 1), on utilise le théorème d'inversion 5.13. On se limite ici à la vérification dans le cas analytique, le cas C^∞ étant similaire. L'hypothèse ii) et les inégalités de Cauchy pour estimer $\partial_z T_\lambda(f)$, impliquent qu'il existe $r > 0$, $C > 0$ tels que la fonction $M_\lambda\left(f\right)(z,b)$ définie par (5.7) vérifie pour tout $\lambda \geq 1$

$$\sup_{|z-x_0|\leq r, |b|=c} \left\|M_\lambda\left(f\right)(z,b)\right\| \leq C\mathrm{e}^{-\frac{\varepsilon\lambda}{2}}.$$

Il en résulte que la fonction $M\big(f\big)(z,b)$ définie par (5.9) est holomorphe en z dans la boule $|z-x_0| < r$, donc d'après le théorème 5.13, (5.12), f est analytique au voisinage de x_0. Ceci achève la preuve du théorème 5.15. □

D'après le théorème 5.15, la régularité C^∞ (resp. analytique) de f près de x_0 est équivalente à la décroissance rapide (resp. exponentielle) en $\lambda \to +\infty$ de la transformée de Bargmann $T_\lambda(f)$ dans l'espace L^2 à poids de fonctions holomorphes $\mathscr{H}_{\varphi,\lambda}$ près de $K_{x_0,c}$ (indépendemment du choix de la constante $c > 0$). Ceci permet de comprendre en quoi la définition suivante du front d'onde d'une distribution est un raffinement de la notion de régularité locale.

La définition du front d'onde d'une distribution est due à Sato dans le cas analytique, (voir [6]) et à Hörmander dans le cas C^∞ (voir [5]). Nous donnons ici la définition du frond d'onde en utilisant la transformation de Bargmann T_λ. Le fait que le front d'onde peut être caractérisé par transformation de Bargmann a été développé par Sjöstrand dans [8].

Soit Ω un ouvert de $\mathbb{R}^d$. A toute distribution $f \in \mathscr{D}'(\Omega)$, on associe un *fermé homogène* de l'espace cotangent $T^*(\Omega)\setminus\{0\}$, le *front d'onde de* f, qu'on note $WF(f)$ dans le cas C^∞ et $WF_a(f)$ dans le cas analytique.

Définition 5.16 Soit f une distribution sur Ω, et $(x_0,\xi_0)\in T^*(\Omega)$, avec $\xi_0\neq 0$. Pour W voisinage de $z_0 = x_0 - i\xi_0 \in \mathbb{C}^d$ et $\varphi \in C_0^\infty(\Omega)$, égale à 1 près de x_0, on pose:

$$\Theta_{W,\varphi}(\lambda) = \sup_{z\in W}\left| T_\lambda\left(\varphi f\right)(z)\mathrm{e}^{-\lambda\frac{|\Im(z)|^2}{2}}\right|.$$

1) On dit que (x_0,ξ_0) n'appartient pas au front d'onde C^∞ de f, et on note $(x_0,\xi_0)\notin WF(f)$ ssi il existe W et φ tels que $\Theta_{W,\varphi}(\lambda)$ est à décroissance rapide en $\lambda\to\infty$, i. e.
$$\forall N, \exists C_N, \forall \lambda\geq 1,\ \Theta_{W,\varphi}(\lambda)\leq C_N\lambda^{-N}.$$

2) On dit que (x_0,ξ_0) n'appartient pas au front d'onde analytique de f, et on note $(x_0,\xi_0)\notin WF_a(f)$ ssi il existe W et φ tels que $\Theta_{W,\varphi}(\lambda)$ est à décroissance exponentielle en $\lambda\to\infty$, i. e.
$$\exists C,\varepsilon>0, \forall\lambda\geq 1,\ \Theta(\lambda)\leq Ce^{-\varepsilon\lambda}.$$

On remarquera que d'après l'estimation (5.13), la définition précédente ne dépend pas du choix de la fonction de troncature $\varphi\in C_0^\infty(\Omega)$ égale à 1 près de x_0. Dans la suite, on se limitera à l'étude du front d'onde analytique. On renvoie aux cours [4] et [7] dans ce volume et bien sur aussi à l'ouvrage de L. Hörmander [5] pour l'étude du front d'onde C^∞. On notera toutefois qu'on a le lemme élémentaire suivant.

Lemme 5.17 Pour toute distribution f sur Ω, et tout $(x_0,\xi_0)\in T^*(\Omega)\setminus\{0\}$, les propriétés suivantes sont équivalentes:

1) $(x_0,\xi_0)\notin WF(f)$.

2) Il existe $g\in\mathscr{D}'(\Omega)$ avec $(x_0,\xi_0)\notin WF_a(g)$, et $h\in C^\infty(\Omega)$, telles que $f=g+h$.

Avec la définition 5.16 du front d'onde analytique $WF_a(f)$, il est évident que le front d'onde est fermé, et que pour $x_0\notin \operatorname{supp}(f)$, on a $(x_0,\xi_0)\notin WF_a(f)$ pour tout ξ_0.

Pour pouvoir affirmer que $WF_a(f)$ est un fermé homogène, c'est à dire fermé et invariant par l'action $s\cdot(x,\xi)=(x,s\xi)$ de $\mathbb{R}_+^*$ sur l'espace cotangent $T^*(\Omega)\setminus\{0\}$, il faut toutefois vérifier deux propriétés:

1) L'invariance par changement de variable analytique, c'est à dire: si Φ est un difféomorphisme d'un voisinage de x_0 sur un voisinage de y_0, avec $\Phi(x_0) = y_0$, on a pour toute distribution f définie au voisinage de y_0, et tout $\eta_0 \neq 0$

$$(y_0, \eta_0) \in WF_a(f) \quad \text{ssi} \quad (x_0, \xi_0) = \big(\Phi\big(x_0\big)(\eta_0)\big) \in WF_a(f\Phi).$$

2) Le front d'onde est homogène, i. e. on a pour tout $s > 0$

$$(x_0, \xi_0) \in WF_a(f) \quad \text{ssi} \quad (x_0, s\xi_0) \in WF_a(f).$$

Nous laisserons la vérification des points 1) et 2) en exercice, comme application du lemme suivant et de sa preuve, qui affirme que la formule (5.12)

$$f(x) = \int_{\mathscr{S}_c} K\big(f\big)(x,b)\mathrm{d}\sigma(b)$$

est une décomposition microlocale de la distribution f. Plus précisément, on a:

Lemme 5.18 Soit $f \in \mathscr{E}'(\mathbb{R}^d)$ une distribution à support compact. Pour tout $b \in \mathscr{S}_c$, on a

$$WF_a\big(K\big(f\big)(\,\cdot\,,b)\big) \subset \big\{(x,\xi), x \in \operatorname{supp}(f), \text{ et } \exists t > 0 \text{ tel que } \xi = tb\big\} \tag{5.14}$$

et pour tout fermé F de $\mathscr{S}_c$ on a

$$WF_a\left(\int_F K\big(f\big)(x,b)\mathrm{d}\sigma(b)\right) \subset \big\{(x,\xi), x \in \operatorname{supp}(f), \text{ et } \exists s > 0 \text{ tel que } s\xi \in F\big\}. \tag{5.15}$$

Preuve. Vérifions l'inclusion (5.14) (la vérification de (5.15) est analogue). D'après la preuve du théorème d'inversion 5.13, la distribution $K\big(f\big)(x,b)$ est C^∞ en $b \in \mathscr{S}_c$ et analytique en $x \in \mathbb{R}^d \setminus support(f)$. Il suffit donc de vérifier que $(x,\xi) \in WF\big(K\big(f\big)(\,\cdot\,,b)\big)$ implique qu'il existe $t > 0$ tel que $\xi = tb$.

On se donne $R > 0$ tel que $\operatorname{supp}(f) \subset \{|x| \leq R\}$ et $\psi \in C_0^\infty(|x| < 3R)$ avec $\psi = 1$ sur $|x| \leq 2R$, et on estime

$$T_\lambda\big(\psi K\big(f\big)(\,\cdot\,,b)\big)(z) = c_\lambda \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{\lambda(z-x)^2}{2}} \psi(x) K\big(f\big)(x,b)\mathrm{d}x.$$

$z = \alpha - i\xi$, $\xi \neq 0$ et supposons $\xi \notin \mathbb{R}_+^* b$. Il existe alors $b' \in \mathscr{S}_c$ tel qu'on ait $b' \cdot b > 0$ et $b' \cdot \xi < 0$. Soit $\theta \in C_0^\infty(|x| < 2R)$, $0 \leq \theta \leq 1$ et $\theta(x) = 1$ pour $|x| \leq 3R/2$. Comme $K(f)(x,b)$ est analytique en x pour $|x| > R$, on a pour $\varepsilon > 0$ petit

$$T_\lambda\big(\psi K(f)(\,\cdot\,,b)\big)(z) = c_\lambda \int_{\mathbb{R}^d} \mathrm{e}^{-\lambda \frac{(z-(x+i\varepsilon\theta(x)b'))^2}{2}} \psi\big(x+i\varepsilon\theta(x)b'\big) M\big(f\big)(x+i\varepsilon\theta(x)b',b)\,\mathrm{d}x.$$

On a

$$\sup_{|x|\leq 3R} \Big|\psi\big(x+i\varepsilon\theta(x)b'\big) M\big(f\big)(x+i\varepsilon\theta(x)b',b)\Big| < \infty$$

et on conclut en utilisant $\xi \cdot b' < 0$ et

$$\mathrm{e}^{\frac{-\lambda\xi^2}{2}} \left|\mathrm{e}^{\frac{-\lambda(z-(x+i\varepsilon\theta(x)b'))^2}{2}}\right| = \mathrm{e}^{\frac{-\lambda(\alpha-x)^2}{2}} \mathrm{e}^{\lambda \frac{\varepsilon\xi\cdot b'\theta(x)+\varepsilon^2|b'|^2\theta(x)^2}{2}}. \qquad \square$$

Exercice 5.19 En utilisant le lemme 5.18 et sa preuve, vérifier les points 1) et 2), i. e. le front d'onde WF_a est homogène et invariant par changement de variable analytique.

Nous terminons ce paragraphe par l'énoncé de deux théorèmes sur le comportement du front d'onde par intégration, et trace non caractéristique.

Théorème 5.20 Soit $f(x,y) \in \mathscr{E}'(\mathbb{R}^p \times \mathbb{R}^q)$ et $g(x) = \int f(x,y)dy \in \mathscr{E}'(\mathbb{R}^p)$. On a l'inclusion

$$WF_a(g) \subset \big\{(x,\xi),\, \exists y \in \mathbb{R}^q, (x,y,\xi,0) \in WF_a(f)\big\}. \tag{5.16}$$

Preuve. On note ici $c_d(\lambda) = \left(\frac{\lambda}{2\pi}\right)^{d/2}\left(\frac{\lambda}{\pi}\right)^{d/4}$. Pour $z \in \mathbb{C}^p$ et $w \in \mathbb{C}^q$, on pose

$$T_\lambda\big(g\big)(z) = c_p(\lambda)\left\langle g, \mathrm{e}^{-\lambda\frac{(z-x)^2}{2}}\right\rangle, \qquad \tilde{T}_\lambda\big(f\big)(z,w) = c_{p+q}(\lambda)\left\langle f, \mathrm{e}^{-\lambda\left(\frac{(z-x)^2}{2}+\frac{(w-y)^2}{2}\right)}\right\rangle.$$

Soit (x_0,ξ_0) tel que pour tout $y \in \mathbb{R}^q$, on ait $(x_0,y,\xi_0,0) \notin WF_a(f)$. On doit vérifier

$$(x_0,\xi_0) \notin WF_a(g).$$

Soit $R > 0$ tel qu'on ait $\mathrm{supp}(f) \subset \{|x|+|y| \leq R\}$. En utilisant le fait que le support en y de f est contenu dans la boule $|y| \leq R$, et que f est une distribution d'ordre fini, on obtient qu'il existe des constantes $A, B > 0$ tels qu'on ait pour tout $\lambda \geq 1$ et tout $y \in \mathbb{R}^q$ avec $|y| \geq 2R$

$$\left|\tilde{T}_\lambda\left(f\right)(z,y)\right| \leq A\lambda^B \mathrm{e}^{\lambda\frac{\Im(z)^2}{2}} \mathrm{e}^{-\lambda\frac{(|y|-2R)^2}{2}}. \tag{5.17}$$

Comme on suppose $(x_0, y, \xi_0, 0) \notin WF_a(f)$ pour tout $y \in \mathbb{R}^q$, en utilisant la compacité de la boule fermée $|y| \leq 3R$, on obtient qu'il existe un voisinage W de $z_0 = x_0 - i\xi_0$, et des constantes $A > 0$, $\varepsilon > 0$, telles qu'on ait pour tout $\lambda \geq 1$

$$\sup_{z\in W, |y|\leq 3R} \left|\tilde{T}_\lambda\left(f\right)(z,y)\mathrm{e}^{-\lambda\frac{\Im(z)^2}{2}}\right| \leq C\mathrm{e}^{-\varepsilon\lambda}. \tag{5.18}$$

Le théorème est alors une simple conséquence de (5.17), (5.18), et de la formule explicite suivante qui relie $T_\lambda(g)$ et $\tilde{T}_\lambda(f)$

$$c_p(\lambda)\int_{y\in\mathbb{R}^q} \tilde{T}_\lambda\left(f\right)(z,y)\mathrm{d}y = \left(\frac{2\pi}{\lambda}\right)^{q/2} c_{p+q}(\lambda) T_\lambda\left(g\right)(z). \qquad \square$$

Théorème 5.21 Soit $f(x,y) \in \mathscr{E}'(\mathbb{R}^p \times \mathbb{R}^q)$. Soit S la sous variété $S = \{y = 0\}$ et $T_S^* \subset T^*(\mathbb{R}^p \times \mathbb{R}^q)$ le fibré conormal à S. On suppose

$$WF_a(f) \cap T_S^* = \emptyset$$

Alors la trace de f sur S, $g(x) = f(x,0)$ est bien définie comme élément de $\mathscr{E}'(\mathbb{R}^p)$, et on a l'inclusion

$$WF_a(g) \subset \left\{(x,\xi) \in T^*(\mathbb{R}^p) \setminus \{0\}, \exists \eta \in \mathbb{R}^q, (x,0;\xi,\eta) \in WF_a(f)\right\}.$$

Exercice 5.22 Prouver le théorème 5.21. On utilisera le théorème d'inversion 5.13, et on commencera par vérifier que l'hypothèse $WF_a(f) \cap T_S^* = \emptyset$ implique qu'il existe $r > 0$ tel que la distribution à support compact f appartienne à l'espace $C^\infty\left(B_r, \mathscr{E}'(\mathbb{R}_x^p)\right)$ avec $B_r = \{y \in \mathbb{R}^q, |y| < r\}$, de sorte que la trace $g(x) = f(x,0)$ est bien définie.

Exercice 5.23 Cet exercice est une illustration du théorème 5.20. Soit $f = 1_{\{x^2+y^2<1\}}$ la fonction caractéristique du disque unité de $\mathbb{R}^2$.

1) Montrer que le support singulier de f est le cercle unité $\{(x,y), x^2 + y^2 = 1\}$, et

$$WF_a(f) = WF(f) = \left\{(x,y;\xi,\eta), x^2 + y^2 = 1, \exists s \neq 0, (\xi,\eta) = s(x,y)\right\}.$$

2) Vérifier qu'on a $g(x) = \int_{\mathbb{R}} f(x,y)\,\mathrm{d}y = 1_{|x|\leq 1} 2\sqrt{1-x^2}$, et en déduire $WF_a(g) = WF(g) = \{(x;\xi), x^2 = 1, \exists s \neq 0, \xi = sx\}$.

3) Comparer ce résultat avec la formule 5.16 du théorème 5.20.

. . . .

4 Calcul pseudodifférentiel analytique

4.1 Symboles analytiques

Les symboles analytiques ont étés introduits par L. Boutet de Monvel et P. Kree dans [2]. Dans cette section, nous suivons la présentation des symboles analytiques donnée par J. Sjöstrand dans [8].

Définition 5.24 Soit Ω un ouvert de $\mathbb{C}^{2d}$. Un symbole sur Ω est une *série formelle*

$$a(z,\zeta,\lambda) = \sum_{n\geq 0} (i\lambda)^{-n} a_n(z,\zeta)$$

où les fonctions $a_n(z,\zeta)$ sont holomorphes sur Ω. On dit que le symbole a est analytique si les a_n vérifient: pour tout compact $K \subset \Omega$, il existe des constantes $A_K > 0$, $B_K > 0$ telles qu'on ait

$$\sup_{(z,\zeta)\in K} |a_n(z,\zeta)| \leq A_K B_K^n n^n \tag{5.19}$$

Exemple 5.25 Les séries formelles vérifiant des estimations de type 5.19 interviennent naturellement dans l'étude des asymptotiques à l'infini. A titre d'exemple et d'exercice, le lecteur vérifiera, en utilisant la formule $\int_0^\infty \mathrm{e}^{-y} y^k \mathrm{d}y = k!$, que la fonction de $\lambda \in [0,\infty[$ définie par $f(\lambda) = \lambda \int_0^\infty \mathrm{e}^{-\lambda x} \frac{1}{1+x} \mathrm{d}x$ possède le développement asymptotique

$$f(\lambda) \simeq \sum_{n\geq 0} (-1)^n \lambda^{-n} n!, \qquad \lambda \to \infty$$

Remarque 5.26 La série de définition de $a(z,\zeta,\lambda)$ est une série formelle qui ne converge pas à priori. Toutefois, lorsqu'on suppose le symbole a analytique, les estimations (5.19) montrent que pour $C > B_K$ les fonctions $a_C(z,\zeta,\lambda) = \sum_{0\leq n\leq \lambda/C} (i\lambda)^{-n} a_n(z,\zeta)$ sont bien définies sur le compact K, et qu'on a pour $C' \geq C > B_K$

$$\sup_{(z,\zeta)\in K} \left| a_{C'}(z,\zeta) - a_{C'}(z,\zeta) \right| \leq A_K \frac{C}{C-B_K} \mathrm{e}^{-\varepsilon\lambda}, \qquad \varepsilon = \frac{1}{C'} \Lambda_n\left(\frac{C}{B_K}\right).$$

On munit l'espace des symboles sur Ω d'une structure d'algèbre en définissant le produit $a\sharp b$ de deux symboles a et b par la formule

$$a\sharp b(z,\zeta,\lambda) = \sum_{n\geq 0} (i\lambda)^{-n} \left[\sum_{|\alpha|+j+k=n} \frac{1}{\alpha!} \big(\partial_\zeta^\alpha a_j\big)(z,\zeta)\big(\partial_z^\alpha b_k\big)(z,\zeta) \right] \tag{5.20}$$

On a $a\sharp 1 = 1\sharp a = a$ pour tout symbole a. Il résulte de la formule de Leibniz que ce produit est associatif, i. e.

$$a\sharp(b\sharp c) = (a\sharp b)\sharp c.$$

Il résulte aussi des inégalités de Cauchy que lorsque a et b sont des symboles analytiques sur Ω, $a\sharp b$ est un symbole analytique sur Ω.

A un symbole a sur Ω, on associe une série formelle d'opérateurs différentiels à coefficients holomorphes sur Ω, $A = \mathrm{Op}(a)$ en posant

$$A = \mathrm{Op}(a) = \sum_n (i\lambda)^{-n} A_n(z,\zeta,\partial_z), \qquad A_n(z,\zeta,\partial_z) = \sum_{j+|\alpha|=n} \frac{1}{\alpha!} \partial_\zeta^\alpha a_j(z,\zeta)\partial_z^\alpha. \tag{5.21}$$

Les opérateurs différentiels A_n sont d'ordre $\leq n$, et on a

$$A_n(1)(z,\zeta) = a_n(z,\zeta). \tag{5.22}$$

La propriété essentielle de cette construction est qu'on a l'identité

$$\mathrm{Op}(a\sharp b) = \mathrm{Op}(a)\,\mathrm{Op}(b) \quad \text{i. e. pour tout n} \quad \mathrm{Op}(a\sharp b)_n = \sum_{j+k=n} \mathrm{Op}(a)_j\,\mathrm{Op}(b)_k.$$

où $\mathrm{Op}(a)_j\,\mathrm{Op}(b)_k$ est le composé des opérateurs différentiels $\mathrm{Op}(a)_j$ et $\mathrm{Op}(b)_k$.

On va maintenant construire des normes sur les symboles analytiques, permettant de contrôler le produit $a\sharp b$. On se donne pour $t \in [0,1]$ une famille croissante ω_t d'ouverts non vides de Ω telle que $\overline{\omega}_1 \subset \Omega$ et vérifiant pour un constante $\delta > 0$ et tout $0 \leq s \leq t \leq 1$

$$(w,\zeta) \in \omega_s \text{ et } |z-w| \leq \delta(t-s) \Rightarrow (z,\zeta) \in \omega_t. \tag{5.23}$$

Soit B_t l'espace de Banach des fonctions holomorphes et bornées sur ω_t muni de la norme $\|f\|_t = \sup_{(z,\zeta)\in\omega_t} |f(z,\zeta)|$. D'après les inégalités de Cauchy, et (5.23) l'opérateur ∂_z^α est borné de B_t dans B_s pour $s < t$, et on a

$$\|\partial_z^\alpha\|_{t,s} = \|\partial_z^\alpha; B_t \to B_s\| \leq \left(\frac{C_0|\alpha|}{\delta(t-s)}\right)^{|\alpha|} \tag{5.24}$$

où la constante C_0 ne dépend que de la dimension d.

Soit a un symbole, et $A = \mathrm{Op}(a)$. Comme A_n est un opérateur différentiel d'ordre $\leq n$ à coefficients holomorphes au voisinage de $\overline{\omega}_1$, il existe d'après (5.24) une constante M_n telle qu'on ait pour tout $0 \leq s < t \leq 1$

$$\|A_n\|_{t,s} = \|A_n; B_t \to B_s\| \leq M_n \left(\frac{n}{t-s}\right)^n$$

On note $C_0(A) = \|A_0\|_{1,1}$ et pour $n \geq 1$, $C_n(A)$ la meilleure constante telle qu'on ait pour tout $0 \leq s < t \leq 1$

$$\|A_n\|_{t,s} = \|A_n; B_t \to B_s\| \leq C_n(A) \left(\frac{n}{t-s}\right)^n$$

On pose pour $r \geq 0$

$$\|A\|_r = \sum_{n\geq 0} C_n(A) r^n \in [0, \infty].$$

Supposons à présent le symbole a analytique. Comme $\overline{\omega}_1 \subset \Omega$, d'après (5.19), il existe aussi des constantes $A > 0$, $B > 0$ telles que pour tout j on ait

$$\sup_{(z,\zeta)\in\omega_1} \left|\frac{1}{\alpha!}\partial_\zeta^\alpha a_j(z,\zeta)\right| \leq AB^j j^j.$$

Il résulte alors de (5.21) et (5.24) qu'on a, pour une constante M dépendante du symbole analytique a mais indépendante de $0 \leq s < t \leq 1$,

$$\|A_0\|_{t,t} \leq M \quad \text{et} \quad \|A_n\|_{t,s} = \|A_n; B_t \to B_s\| \leq M^{1+n} \left(\frac{n}{t-s}\right)^n. \tag{5.25}$$

D'après (5.25), pour tout symbole analytique a, il existe donc $r_a > 0$ tel qu'on ait $\|\mathrm{Op}(a)\|_r < \infty$ pour $r \leq r_a$.

Réciproquement, si a est un symbole tel que $m = \|A\|_r < \infty$ pour un $r > 0$, on a pour $s < 1$ en utilisant (5.22)

$$\sup_{(z,\zeta)\in B_s} |a_n(z,\zeta)| = \sup_{(z,\zeta)\in B_s} \left|A_n\big(1\big)(z,\zeta)\right| \leq m\left(\frac{n}{r(1-s)}\right)^n$$

et donc a est un symbole analytique sur l'ouvert ω_s pour tout $s<1$.

Proposition 5.27 Soient a,b des symboles analytiques sur Ω et $d = a\sharp b$. On suppose $\|\operatorname{Op}(a)\|_r < \infty$, $\|\operatorname{Op}(b)\|_r < \infty$. Alors on a $\|\operatorname{Op}(d)\|_r < \infty$ et

$$\|\operatorname{Op}(d)\|_r \leq \|\operatorname{Op}(a)\|_r \|\operatorname{Op}(b)\|_r \tag{5.26}$$

Preuve. On pose $A = \operatorname{Op}(a)$, $B = \operatorname{Op}(b)$, $D = \operatorname{Op}(d)$. Rappelons que la définition $a\sharp b$ du produit de deux symboles est exactement celle qui assure la propriété

$$D_n = \sum_{j+k=n} A_j B_k$$

où A_jB_k désigne le composé des opérateurs différentiels A_j et B_k. On a pour $r \in [s,t]$

$$\|A_jB_k\|_{t,s} \leq \|A_j\|_{r,s}\|B_k\|_{t,r} \leq C_j(A)C_k(B)\,(jr-s)^j\left(\frac{k}{t-r}\right)^k$$

En choisissant $r = s + \frac{j}{j+k}(t-s)$, donc $t-r = \frac{k}{j+k}(t-s)$, on obtient pour tout j,k tels que $j+k=n$ $\|A_jB_k\|_{t,s} \leq C_j(A)C_k(B)\left(\frac{n}{t-s}\right)^n$ donc $C_n(D) \leq \sum_{j+k=n} C_j(A)C_k(B)$ ce qui implique l'inégalité (5.26). □

Définition 5.28 Soit $a = \sum_{n\geq 0}(i\lambda)^{-n}a_n$ un symbole analytique sur Ω et K un compact de Ω. On dit que a est elliptique sur K s'il existe $c>0$ tel que

$$\min_{(z,\zeta)\in K} \left|a_0(z,\zeta)\right| \geq c > 0.$$

Le résultat suivant d'inversion des symboles elliptiques est du à L. Boutet de Monvel et P. Kree ([2]).

Théorème 5.29 Soit a un symbole analytique sur Ω. On suppose a elliptique sur un compact K de Ω. Il existe un ouvert U de $\mathbb{C}^d$, $K \subset U \subset \Omega$ et un unique symbole analytique b sur U tel qu'on ait l'identité, dans les symboles analytiques sur U

$$a\sharp b = b\sharp a = 1.$$

Preuve. On choisit l'ouvert U tel qu'on ait

$$\min_{(z,\zeta)\in K} \left|a_0(z,\zeta)\right| \geq \frac{c}{2} > 0. \tag{5.27}$$

On construit alors le symbole $b = \sum_{n\geq 0}(i\lambda)^{-n}b_n$ tel qu'on ait $a\sharp b = 1$ sur U. Ceci équivaut d'après (5.20) à choisir les $b_n(z,\zeta)$ tels qu'on ait

$$a_0(z,\zeta)b_0(z,\zeta) = 1, \qquad \forall (z,\zeta)\in U$$
$$a_0(z,\zeta)b_n(z,\zeta) = -\sum_{\substack{|\alpha|+j+k=n\\ k\neq n}} \frac{1}{\alpha!}\big(\partial_\zeta^\alpha a_j\big)(z,\zeta)\big(\partial_z^\alpha b_k\big)(z,\zeta), \qquad \forall (z,\zeta)\in U, \forall n\geq 1.$$

D'après (5.27), ce système détermine uniquement les fonctions holomorphes $b_n(z,\zeta)$ sur U. Montrons maintenant que $b = \sum_{n\geq 0}(i\lambda)^{-n}b_n$ est un symbole analytique sur U. Soit Q un compact de U. Il existe une famille croissante ω_t, $t\in[0,1]$ d'ouverts non vides de U telle que $Q\subset\omega_0\subset\overline{\omega}_1\subset\Omega$ et vérifiant pour une constante $\delta>0$ la propriété (5.23). On a $a\sharp b_0 = 1-d$ où $d = \sum_{n\geq 1}(i\lambda)^{-n}d_n$ est un symbole analytique sur U dont le terme principal, d_0, est nul. En particulier, comme $\|\operatorname{Op}(d)\|_r = \sum_{n\geq 1} C_n\big(\operatorname{Op}(d)\big)r^n$ est fini pour un $r>0$, on peut choisir r_0 tel que $\|\operatorname{Op}(d)\|_{r_0}\leq 1/2$.

Alors, d'après la proposition 5.27, la série formelle d'opérateurs en λ^{-1}, $S = \sum_{q\geq 0}\operatorname{Op}(d)^q$ est convergente pour la norme $\|\cdot\|_{r_0}$, et $\|S\|_{r_0}\leq 2$.

Par unicité de la série formelle d'opérateurs en λ^{-1}, on a $\operatorname{Op}(b) = \operatorname{Op}(b_0)\operatorname{Op}(S)$ donc toujours d'après la proposition 5.27, quitte à diminuer r_0, on obtient $\|\operatorname{Op}(b)\|_{r_0}<\infty$. Ceci prouve que b est un symbole analytique sur ω_0, donc au voisinage du compact Q pour tout compact Q de U, donc est un symbole analytique sur U. La vérification de $b\sharp a = 1$ est alors purement algébrique: par associativité, on a $b = b\sharp(a\sharp b) = (b\sharp a)\sharp b$. Comme b est aussi elliptique, on vient de voir qu'il existe un symbole analytique d tel que $b\sharp d = 1$, donc $1 = b\sharp d = \big((b\sharp a)\sharp b\big)\sharp d = (b\sharp a)\sharp(b\sharp d) = (b\sharp a)\sharp 1 = b\sharp a$. □

4.2 Opérateurs pseudodifférentiels analytiques

Soit ψ une fonction de classe C^2 définie sur $\mathbb{C}^d$ et à valeurs réelles.

Définition 5.30 Soit U un ouvert borné de $\mathbb{C}^d$. L'espace de Sjöstrand $H_\psi(U)$ est l'espace des fonctions $f(z,\lambda)$ définies sur $U \times [1,\infty[$, à valeurs dans $\mathbb{C}$, holomorphes en $z \in U$, et qui vérifient

$$\|f\|_{H_\psi(U)} := \sup_{z\in U, \lambda \geq 1} \left| f(z,\lambda) \mathrm{e}^{-\lambda \psi(z)} \right| < \infty.$$

Pour $z_0 \in \mathbb{C}^d$, on notera H_{ψ,z_0} l'espace des «germes modulo exponentiellement petit en λ» de fonctions de H_ψ. Un élément de H_{ψ,z_0} est donc la donnée d'un voisinage U de z_0 et d'une fonction $f(z,\lambda) \in H_\psi(U)$, avec la relation d'équivalence $f \simeq g$ suivante:

$$\begin{aligned} f \simeq g \quad \text{ssi} \quad & \exists \varepsilon > 0 \text{ et } W \text{ voisinage de } z_0 \text{ tels que} \\ & \sup_{z\in W, \lambda\geq 1} \left| \big(f-g\big)(z,\lambda) \mathrm{e}^{-\lambda[\psi(z)-\varepsilon]} \right| < \infty. \end{aligned}$$

A la fonction ψ, on associe une sous variété Λ_ψ de $T^*\mathbb{C}^d$ en posant

$$\Lambda_\psi = \left\{ \left(z, \zeta = \frac{2}{i} \partial_z \psi(z) \right); z \in \mathbb{C}^d \right\}.$$

Ici, ∂_z désigne la dérivée holomorphe, i. e., avec $z = x+iy$, on a $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$. En particulier, si on note $\mathrm{d}\psi$ la différentielle de la fonction ψ, qui est une forme $\mathbb{R}$-linéaire de $\mathbb{C}^d$ dans $\mathbb{R}$, on a pour tout $z \in \mathbb{C}^d$ et tout $u \in \mathbb{C}^d$

$$\mathrm{d}\psi\big(z\big)(u) = -\Im(\zeta u), \quad \text{pour} \quad \zeta = \frac{2}{i} \partial_z \psi(z). \tag{5.28}$$

On fixe $z_0 \in \mathbb{C}^d$ et on pose $\rho_0 = \big(z_0, \zeta_0 = \frac{2}{i} \partial_z \psi(z_0)\big) \in \mathbb{C}^d$. Soit Ω un voisinage de ρ_0 dans $T^*\mathbb{C}^d$. Nous allons associer à tout symbole analytique a sur Ω un opérateur $\mathrm{Op}_{\rho_0}(a)$ opérant sur H_{ψ,z_0}, et tel qu'on ait

$$\mathrm{Op}_{\rho_0}(a \sharp b) = \mathrm{Op}_{\rho_0}(a) \,\mathrm{Op}_{\rho_0}(b), \qquad \mathrm{Op}_{\rho_0}(1) = \mathrm{Id}.$$

Définition 5.31 Soit $a = \sum_{n\geq 0}(i\lambda)^{-n}a_n(z,\zeta)$ un symbole analytique sur Ω, et $C > 0$. On définit l'opérateur différentiel $\mathrm{Op}_{\rho_0,C}(a)$ par la formule

$$\mathrm{Op}_{\rho_0,C}\left(a\right)\left(f\right)(z,\lambda) = \sum_{n\leq \lambda/C} \mathrm{e}^{i\lambda(z-z_0)\zeta_0}(i\lambda)^{-n}A_n(z,\zeta_0,\partial_z)\left(\mathrm{e}^{-i\lambda(z-z_0)\zeta_0}f(z,\lambda)\right) \quad (5.29)$$

où l'opérateur différentiel A_n est défini par la formule (5.21).

On remarquera que dans (5.29), la somme sur n est finie, et que $\lambda \geq 1$ est un paramètre. Donc pour $f(z,\lambda)$ holomorphe près de z_0, $\mathrm{Op}_{\rho_0,C}\left(a\right)\left(f\right)(z,\lambda)$ est une fonction holomorphe bien définie au voisinage de z_0.

Pour $r > 0$, on note U_r le polydisque centré en z_0, $U_r = \left\{z \in \mathbb{C}^d, \sup_j |z_j - z_{0,j}| < r\right\}$. On pose $\tilde{\psi}(z) = \psi(z) + \Im\left((z-z_0)\zeta_0\right)$, de sorte qu'on a d'après (5.28) $d\tilde{\psi}(z_0) = 0$.

Pour $f(z,\lambda) \in H_\psi(U_r)$, on pose $\tilde{f}(z,\lambda) = \mathrm{e}^{-i\lambda(z-z_0)\zeta_0}f(z,\lambda)$. On a $\tilde{f}(z,\lambda) \in H_{\tilde{\psi}}(U_r)$, et $\|f\|_{H_\psi(U_r)} = \|\tilde{f}\|_{H_{\tilde{\psi}}(U_r)}$. On a donc

$$\mathrm{Op}_{\rho_0,C}\left(a\right)\left(f\right)(z,\lambda)\mathrm{e}^{-i\lambda(z-z_0)\zeta_0} = \sum_{n\leq \lambda/C} (i\lambda)^{-n}A_n(z,\zeta_0,\partial_z)\left(\tilde{f}(z,\lambda)\right) . \quad (5.30)$$

Comme a est un symbole analytique au voisinage de (z_0,ζ_0), il existe $r_0 > 0$ et des constantes $A > 0$, $B > 0$, $D > 0$ telles qu'on ait pour tout j et tout α

$$\sup_{z\in U_{r_0}} \left|\frac{1}{\alpha!}\partial_\zeta^\alpha a_j(z,\zeta_0)\right| \leq AB^j j^j D^{|\alpha|}. \quad (5.31)$$

Comme on a $\mathrm{d}\tilde{\psi}(z_0) = 0$, il existe $r_1 \in]0,r_0]$ tel qu'on ait

$$\sup_{z\in U_{r_1}} \left|\mathrm{d}\tilde{\psi}(z)\right| \leq \frac{1}{2dD}. \quad (5.32)$$

Soir $r \in]0,r_1]$ et $\tilde{f} \in H_{\tilde{\psi}}(U_r)$. Pour $r' \in]0,r[$ et $z \in U_{r'}$, on a par la formule de Cauchy, et pour tout $\rho \in]0,(r-r')[$

$$\partial_z^\alpha\left(\tilde{f}\right)(z) = \frac{\alpha!}{(2i\pi)^d}\int_{|z_j-w_j|=\rho} \frac{\tilde{f}(w)}{\prod_j(w_j-z_j)^{1+\alpha_j}}\,\mathrm{d}w. \quad (5.33)$$

Comme (5.32) implique $|\tilde{\psi}(z) - \tilde{\psi}(w)| \leq \frac{\rho}{2D}$ sur le support de l'intégrale (5.33) on obtient

$$\left\|\partial_z^\alpha(\tilde f)\right\|_{H_{\tilde\psi}(U'_r)} \leq \frac{\alpha!\mathrm{e}^{\lambda\frac{\rho}{2D}}}{\rho^{|\alpha|}}\|\tilde f\|_{H_{\tilde\psi}(U_r)}.$$

On peut choisir $\rho = \frac{2D|\alpha|}{\lambda}$ tant que α vérifie $|\alpha| < \lambda\frac{r-r'}{2D}$, et on obtient

$$\left\|\partial_z^\alpha(\tilde f)\right\|_{H_{\tilde\psi}(U'_r)} \leq \frac{\alpha!\mathrm{e}^{|\alpha|}\lambda^{|\alpha|}}{(2D)^\alpha|\alpha|^{|\alpha|}}\|\tilde f\|_{H_{\tilde\psi}(U_r)}, \qquad \forall\alpha \text{ vérifiant } |\alpha| < \lambda\frac{r-r'}{2D}. \tag{5.34}$$

En utilisant la définition (5.21) de A_n, (5.31) et (5.34), on obtient donc pour tout n vérifiant $n < \lambda\frac{r-r'}{2D}$

$$\left\|(i\lambda)^{-n}A_n\big(z,\zeta_0,\partial_z\big)(\tilde f)\right\|_{H_{\tilde\psi}(U'_r)} \leq A\sum_{j+k=n} p(k)\left(\frac{Bj}{\lambda}\right)^j 2^{-k}\|\tilde f\|_{H_{\tilde\psi}(U_r)}.$$

où $p(k)$ est un polynôme de k tel qu'on ait $\sum_{|\alpha|=k}\alpha! \leq p(k)\mathrm{e}^{-k}k^k$.

On choisit alors la constante C vérifiant

$$C \geq C_0(r-r',a) = \max\left(\frac{2D}{(r-r')}, 2B\right) \tag{5.35}$$

et on obtient pour tout $n \leq \lambda/C$, pour un autre polynôme $p(n)$

$$\left\|(i\lambda)^{-n}A_n\big(z,\zeta_0,\partial_z\big)(\tilde f)\right\|_{H_{\tilde\psi}(U'_r)} \leq Ap(n)2^{-n}\|\tilde f\|_{H_{\tilde\psi}(U_r)}. \tag{5.36}$$

Il en résulte que pour C vérifiant (5.35), l'opérateur $\mathrm{Op}_{\rho_0,C}(a)$ défini par (5.30) est borné de $H_\psi(U_r)$ dans $H_\psi(U'_r)$, et que si les constantes C_1, C_2 vérifient (5.35), l'opérateur $\mathrm{Op}_{\rho_0,C_1}(a) - \mathrm{Op}_{\rho_0,C_2}(a)$ de $H_\psi(U_r)$ dans $H_\psi(U'_r)$ est borné par $O(\mathrm{e}^{-\varepsilon\lambda})$ pour un $\varepsilon > 0$ dépendant de a et des constantes C_1, C_2.

Il résulte de ce qui précède que $\mathrm{Op}_{\rho_0,C}(a)$ définit un unique opérateur opérant dans H_{ψ,z_0}, l'espace des " germes modulo exponentiellement petit en λ" de fonctions de H_ψ. On note cet opérateur $\mathrm{Op}_{\rho_0}(a)$.

On a $\mathrm{Op}_{\rho_0}(1) = 1$ trivialement, et l'identité $\mathrm{Op}_{\rho_0}(a\sharp b) = \mathrm{Op}_{\rho_0}(a)\mathrm{Op}_{\rho_0}(b)$ résulte facilement des résultats du paragraphe 4.1 sur les symboles analytiques et de l'estimation (5.36).

4.3 Le théorème de régularité elliptique de Sato.

Dans ce paragraphe, on utilise la notation $D_x = \frac{1}{i}\partial_x$.

Soit $P(x,D_x)$ un opérateur différentiel d'ordre m à coefficients analytiques, défini sur un ouvert Ω de $\mathbb{R}^d$

$$P(x,D_x) = \sum_{|\alpha|\leq m} p_\alpha(x) D_x^\alpha, \qquad p_\alpha \in C^\omega(\Omega).$$

Le symbole principal de $P(x,D_x)$ est la fonction $p_m(x,\xi)$ définie sur le cotangent $T^*(\Omega)$ par

$$p_m(x,\xi) = \sum_{|\alpha|=m} p_\alpha(x)\xi^\alpha.$$

La variété caractéristique de P, $\mathrm{Car}(P)$, est le fermé homogène de $T^*\Omega\setminus\{0\}$ défini par

$$\mathrm{Car}(P) = \big\{(x,\xi)\in T^*\Omega\setminus\{0\},\, p_m(x,\xi)=0\big\}.$$

Le théorème suivant de régularité elliptique microlocale dans la catégorie analytique est du à M. Sato. Il a été ensuite démontré dans la catégorie C^∞ par L. Hörmander. On trouvera dans ce volume une preuve de la régularité microlocale C^∞ des opérateurs elliptiques (voir [4] théorème 8). C'est le résultat fondateur de l'analyse microlocale, puisqu'il relie explicitement le front d'onde des solutions d'un opérateur différentiel (objet d'analyse) et la variété caractéristique de l' opérateur (objet géométrique).

Théorème 5.32 Pour tout $f\in\mathscr{D}'(\Omega)$, on a l'inclusion

$$WF_a(f)\subset WF_a\big(P(f)\big)\cup \mathrm{Car}(P).$$

En particulier, si $P(f)=0$, on a $WF_a(f)\subset \mathrm{Car}(P)$.

Preuve. Nous nous limiterons ici à prouver le théorème dans le cas particulier où les coefficients $p_\alpha(x)$ sont des polynômes, laissant le cas général en exercice.

Soit $f\in\mathscr{D}'(\Omega)$ et $(x_0,\xi_0)\in T^*\Omega\setminus\{0\}$. Il s'agit de prouver l'implication

$$\big((x_0,\xi_0)\notin WF_a\big(P(f)\big) \text{ et } p_m(x_0,\xi_0)\neq 0\big) \Longrightarrow (x_0,\xi_0)\notin WF_a(f).$$

Supposons donc $(x_0,\xi_0)\notin WF_a\big(P(f)\big)$ et $p_m(x_0,\xi_0)\neq 0$. Soit $\psi\in C_0^\infty(\Omega)$, avec $\psi=1$ au voisinage de x_0. On a $P(\psi f)=\psi P(f)+[P,\psi](f)$. Comme x_0 n'appartient pas au support des distributions $[P,\psi](f)$ et $(1-\psi)P(f)$, on a $(x_0,\xi_0)\notin WF_a\big(P(\psi f)\big)$(voir la preuve du théorème 5.15). On a aussi $(x_0,\xi_0)\notin WF_a(f)$ ssi $(x_0,\xi_0)\notin WF_a(\psi f)$.On peut donc remplacer f par ψf, i. e. supposer que la distribution f est à support dans un compact K de Ω.

En notant k le degré maximal des polynômes $p_\alpha(x)$, on a

$$P(f)=\sum_{|\alpha|\le m}\sum_{|\beta|\le k}\frac{1}{\beta!}\partial_x^\beta p_\alpha(0)x^\beta D_x^\alpha(f)$$

D'après les formules de commutation (5.1) de la proposition 5.7, on a donc

$$\lambda^{-m}T_\lambda\big(P(f)\big)=\sum_{|\alpha|\le m}\sum_{|\beta|\le k}\frac{1}{\beta!}\,\partial_x^\beta p_\alpha(0)\left(z+\frac{1}{\lambda}\partial_z\right)^\beta\lambda^{-m}D_z^\alpha\,T_\lambda(f).$$

Posons $z_0=x_0-i\xi_0$, $\zeta_0=\xi_0$, et $\rho_0=(z_0,\zeta_0)$. Dans l'espace des germes H_{ψ,z_0}, avec $\psi(z)=\frac{\Im(z)^2}{2}$, on a les identités

$$\mathrm{Op}_{\rho_0}(z_j)f=z_jf,\qquad \mathrm{Op}_{\rho_0}(\zeta_j)f=\frac{1}{i\lambda}\partial_{z_j}f.$$

Il en résulte

$$\lambda^{-m}T_\lambda\big(P(f)\big)=\sum_{|\alpha|\le m}\sum_{|\beta|\le k}\frac{1}{\beta!}\partial_x^\beta p_\alpha(0)\,[\mathrm{Op}(z+i\zeta)]^\beta\,\lambda^{|\alpha|-m}\,\mathrm{Op}_{\rho_0}\big(\zeta^\alpha\big)\big(T_\lambda(f)\big).$$

Il est alors facile de vérifier qu'il existe un symbole analytique

$$a(z,\zeta,\lambda)=\sum_{0\le n\le N}(i\lambda)^{-n}a_n(z,\zeta),$$

au voisinage de $\rho_0=(z_0,\zeta_0)$ tel qu'on ait

$$\begin{gathered}a_0(z_0,\zeta_0)=\sum_{|\alpha|=m}\sum_{|\beta|\le k}\frac{1}{\beta!}\partial_x^\beta p_\alpha(0)(z_0+i\zeta_0)^\beta\zeta_0^\alpha=p_m(x_0,\xi_0)\\ \lambda^{-m}T_\lambda(Pf)=\mathrm{Op}_{\rho_0}(a)\,T_\lambda(f)\end{gathered}\tag{5.37}$$

D'après (5.37) et $(x_0,\xi_0) \notin WF_a\big(P(f)\big)$, on a $\mathrm{Op}_{\rho_0}(a)T_\lambda(f) = 0$ dans H_{ψ,z_0}. La condition $p_m(x_0,\xi_0) \neq 0$ équivaut à l'ellipticité de a en ρ_0. D'après le théorème 5.29, il existe un symbole analytique b au voisinage de ρ_0 tel que $b\sharp a = 1$. Il en résulte $T_\lambda(f) = 0$ dans H_{ψ,z_0} (donc $(x_0,\xi_0) \notin WF_a(f)$) en écrivant $T_\lambda(f) = \mathrm{Op}_{\rho_0}(1)T_\lambda(f)$ et en utilisant

$$\mathrm{Op}_{\rho_0}(1)T_\lambda(f) = \mathrm{Op}_{\rho_0}(b\sharp a)T_\lambda(f) = \mathrm{Op}_{\rho_0}(b)\big(\mathrm{Op}_{\rho_0}(a)T_\lambda(f)\big) = \mathrm{Op}_{\rho_0}\big(b\big)(0) = 0$$

Ceci achève la preuve du théorème 5.32. $\square$

Exercice 5.33 Démontrer le théorème de Sato avec des coefficients p_α analytiques sur Ω. On pourra se ramener à $x_0 = 0$ et prouver qu'il existe un symbole analytique au voisinage de (z_0,ζ_0) tel que la deuxième ligne de (5.37) reste vraie. Pour ce faire, on montrera que si f est analytique près de $x = 0$, il existe un symbole analytique $b = \sum(i\lambda)^{-n}b_n$, avec $b_0(z,\zeta) = f(z+i\zeta)$, et tel qu'on ait

$$\mathrm{Op}_{\rho_0}(b) = \sum_\beta \frac{1}{\beta!}\partial_x^\beta f(0)[\mathrm{Op}(z+i\zeta)]^\beta.$$

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Bibliographie

[1] V. Bargmann. *On a Hilbert space of analytic functions and an associated integral transform*. Communications on Pure and Applied Mathematics 14 (3), 1961, pp. 187–214.

[2] L. Boutet de Monvel, P. Kree. *Pseudodifferential operators and Gevrey classes*. Ann. Inst. Fourier 17, 1967, pp. 295–323.

[3] J. Bros, D. Iagolnitzer. *Support essentiel et structure analytique des distributions*. Séminaire Goulaouic-Lions-Schwart, 1975-1976, exposé 18.

[4] C. Ducourtioux, M.-F. Ouedraogo. *Préambule aux opérateurs Fourier intégraux: les opérateurs pseudo-différentiels*.
In this volume.

[5] L. Hörmander. *The Analysis of Linear Partial Differential Operators I,II,III, IV*. Springer Verlag, 1985, *Grundlehren der mathematischen Wissenschaften*, volume 275.

[6] M. Sato, T. Kawai, M. Kashiwara. *Microfunctions and pseudo differential equations*. Springer Lecture Notes in Mathematics, 287.

[7] R.-M. Schulz. *An introduction to the concept of microlocal analysis*.
In this volume.

[8] J. Sjöstrand. *Singularités analytiques microlocales.* Astérisque, vol. 95, publications de la SMF 1982, pp. 1–166.

This volume of contributions based on lectures delivered at a school on Fourier Integral Operators held in Ouagadougou, Burkina Faso, 14–26 September 2015, provides an introduction to Fourier Integral Operators (FIO) for a readership of Master and PhD students as well as any interested layperson. Considering the wide spectrum of their applications and the richness of the mathematical tools they involve, FIOs lie the cross-road of many a field. This volume offers the necessary background, whether analytic or geometric, to get acquainted with FIOs, complemented by more advanced material presenting various aspects of active research in that area.

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