A Dynamic Life Cycle Model for Germany with Unemployment Uncertainty

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Chapter 1

Introduction

> *"There are at least two reasons why it is important to try to explain the intertemporal consumption and saving choices of individuals: first, it is intrinsically interesting; and second, it is useful as a means of understanding, and potentially forecasting, movements in aggregate consumption, and thus contributing to understanding and/or forecasts of aggregate fluctuations. The latter motivations need no further justification, given the priority which policy makers attach to trying to prevent fluctuations in economic activity. The former motivation – intrinsic interest – is often less stressed by economists, but it is hard to see why: it is surely worthwhile for humankind to improve its understanding of human behaviour, and economists, along with other social scientist, have much to contribute here."* Pemberton (2003), p.1 [142]

Dynamic stochastic life cycle models are the modern tools for simulating the economic behavior of individuals. These models are used to explain the consumption and saving behavior of different social groups, they are employed to see how consumers react to uncertainty and to policy measures that counteract uncertainty, such as security. And they help us to understand how the expected development of income changes the saving behavior of individuals.

Although life cycle models have so far failed to explain some empirical findings, they are still considered the best tool for describing the saving and consumption behavior of individuals (Attanasio (1999), p.744 [9], Browning (2001), p.1 [23], (Ludvigson (2001), p.644 [114]).

The life cycle model built in this thesis is based on already existing life cycle models, but extends these models in different ways. The goal of the present work is threefold: first, to build a life cycle model in which an agent's probability of becoming unemployed depends on his educational background and current employment status; second, to include a CRRA-CES-Stone-Geary function in order to explore how this function helps to better explain the consumption and saving behavior of individuals; and, third, to apply the life cycle model to Germany. In order to fulfill the latter objective, I built a life cycle model, henceforth referred to as "German life cycle model", and calibrated it to German data.

The agents of the German life cycle model are divided into three groups according to educational background and income. The first group has no educational qualifications, the second group has undergone vocational training, and the third group has a university degree or a degree in applied sciences. These groups not only earn different incomes but, more importantly, they also have different prospects on the employment market. The unemployment rate of the highly qualified is comparatively low, while people with no educational qualifications have a high probability of becoming or remaining unemployed.

An important aspect of unemployment is its state-dependency. Becoming unemployed is less likely for people who have been employed before than for people who start from a state of unemployment. People know this and consider it in their saving and consumption decisions. In the German life cycle model, this feature is implemented as a Markov chain.

The agents are single individuals who intertemporally optimize utility for their working and retirement years. In contrast to many similar models there is a security net: Agents in this model receive unemployment benefits (Hartz IV) when they are out of work. This influences the way people save significantly since they do not face destitution even when they lose their jobs.

Most utility functions in the life cycle literature are simple Constant Relative Risk Aversion functions (CRRA) with a single good. The utility function in this model includes three goods (durables, nondurables, necessities) and money (implemented through a Constant Elasticity of Substitution function (CES)), and contains subsistence levels, which guarantees that people will first spend their money on essential goods like food and housing before spending on anything else. The utility function represents the process of inheritance by applying a bequest function at the end of the agent's life.

The intertemporal optimization of the single agents is implemented using dynamic programming, and the evolution of an individual agent is given by the resulting policy function (defined for every period and each employment status). Representative evolutions of single agents are obtained by averaging over a large number of runs.

To be able to approximate the saving and consumption behavior of Germans, benchmark data from recent years has been collected and used to parameterize the model. Parameters, including the conditional probabilities of unemployment, the average wage for every group, the average pension, the average working years, the unemployment benefit, and the share parameters for the single goods, are based on data derived from German statistics.

1.1 Structure of the thesis

In the second chapter the origins of stochastic, dynamic life cycle models are traced, and the most important modern life cycle models are introduced. The differences between the latter and the German life cycle model are then explained.

The third chapter deals with the subject of saving and answers the following questions: What are commonly assumed saving motives and what role do they play in modelling the saving behavior of individuals? What are internationally valid stylized facts of saving, and what are typical saving patterns in Germany?

In the fourth chapter, the utility function is introduced and the different parts of the CRRA-CES-Stone-Geary utility function are explained and composed. The composite utility function is then compared to the way other authors have introduced complex utility functions.

The fifth chapter describes the mathematical model, shows how the optimization problem is transformed in order to be solved by dynamic programming, and describes the computational program.

The sixth chapter discusses parameter choices for the model. It explains the parameters used in the utility function and defines the German data which are used as initial conditions and as parameters of the dynamic constraints.

In the seventh chapter, the results of the simulations are presented. First, the influence different features of the model have on the results are presented graphically. Then, it is demonstrated how variations in the parameters influence the findings. Afterwards, the model outcomes for the three different agents are shown. In the second half of Chapter 7, the simulation results are compared with the stylized facts and the empirical German saving data. Finally, the results of the model are put into the context of the existing literature.

The conclusion rounds off the thesis and devises ideas for further development.

The appendix contains the sensitivity analysis for the German life cycle model, as well as the documented code for the program and the proofs that it is possible to aggregate the CRRA-CES-Stone-Geary utility function and that the entire optimization problem can only have one optimal path.

Chapter 2

The development of life cycle models

2.1 Historical origins

The idea of investigating the consumption and saving behavior of households in order to learn something about the economy was pioneered by John Maynard Keynes (Keynes (1936) [97]). According to him, people increase their consumption when their income increases, with the growth of consumption being slower than the growth of income:

> *"The fundamental psychological law, upon which we are entitled to depend with great confidence both a priori and from our knowledge of human nature and from the detailed facts of experience, is that men are disposed, as a rule and on the average, to increase their consumption, as their income increases, but not by as much as the increase in their income."* (Keynes (1936), p.96 [97])

Empirical investigations of US time series carried out after Keynes's pioneering idea revealed a constant ratio of aggregate consumption and aggregate income (Kuznets (1946) [108], (1952) [109], Reid (1956) [146]). Approached this finding independently, Friedman and Modigliani developed two explanatory theories, both based on expected income: the permanent income hypothesis (Friedman (1957) [70]) and the life cycle hypothesis (Modigliani and Brumberg (1980) [127]).

According to Friedman's permanent income hypothesis (PIH), expected income is divided into permanent income, which is anticipated by the agent, and transitory income. The main implication of his hypothesis is that temporary changes in the transitory income do not change the consumption behavior of the agents, only their saving behavior. Only if the income change is permanent will the agent also change his consumption behavior.

The agents in Modigliani's life cycle model maximize utility over their lifetime and have expectations about their future income. According to the life cycle hypothesis (LCH), an agent receives a constant income in his working life, which drops to zero once he retires. Since the consumer's goal is to have a continuously high level of consumption throughout his life, he is going to save part of his income until he retires and will then dissave the accumulated money until he dies. In Modigliani's model the agent knows the exact date of his death and has no intention of leaving anything; he will have spent all his wealth at the end of his life.

2.1.1 The introduction of stochastic dynamic programming

The next step on the way to modern life cycle models[1] was taken by Hall (1978) [82], who inserted uncertainty about future income into a life cycle model with finite horizon and discovered how to use an Euler equation to solve the problem. For solving his model, Hall used a quadratic utility function and came up with the same results as Modigliani did in his model without uncertainty[2].

The result of Hall's model was of course unsatisfying since, despite the introduction of uncertainty, the agents did not react to the risks they were faced with. Using the CRRA utility function[3] would have been a better choice for this model, since this function, in the case of uncertainty, always yields lower consumption and more savings than the certainty equivalent model with the quadratic utility function[4]. Unfortunately, introducing the CRRA function in the framework of Hall's model was not possible since the multi-period case

[1] The term life cycle model refers in this context not exclusively to Modigliani's model, but to models in which agents are forward-looking and solve intertemporal consumption problems, which may have different features such as uncertainty and bequest.

[2] Hall's model has become known as the certainty equivalent model.

[3] For a more detailed description of the CRRA function see section: 4.1.2.

[4] This new, additional saving is called precautionary saving.

with the improved utility function could not be solved with only a single Euler equation. In this case several Euler equations would have been needed[5], but using several Euler equations made optimizing the intertemporal life cycle problem with uncertainty as difficult as it had been before.

A way to solve multi-period problems under uncertainty with the improved utility was finally found by combining the method of dynamic programming with sufficient computer power. Even though dynamic programming had already been invented by Bellmann in 1952 [17], the computer power necessary to solve such problems did not become available until the 1980s. Zeldes (1989) [184] was the first to make use of this new combination, solving a finite life cycle problem with uncertainty and a CRRA function using backward induction for 15 periods.

Since then many applications of stochastic dynamic programming have been developed, the most influential examples being the models of Deaton (1991) [53], Hubbard, Skinner and Zeldes (1994) [91] and Carroll (1992) [32], (1997) [34][41]. The next section introduces these models in detail.

These three models, as well as more recent extensions of them, are all solved numerically since an analytical solution is not possible (Carroll (1997), p.6 [34], Hubbard et al. (1994), p.65 [91]). The life cycle models could still be solved without a computer as long as the modelers had no uncertainty or were using the certainty equivalent model (Hubbard et al. (1994), p.63 [91], Carroll (1997), p.6 [34]), but since the introduction of the CRRA utility or even more complicated functions, such as the one introduced in the German life cycle model, no way has been found to solve these models. All three papers described below use stochastic dynamic programming in connection with Euler equations to solve their models.

2.2 The standard models in life cycle theory in comparison to the German life cycle model

2.2.1 The Deaton Model

The first of the three important life cycle models was developed by Angus Deaton (Deaton (1991) [53]) and published in his paper "Saving and Liquid-

[5]In fact, one always needs one Euler equation less than the number of periods.

ity Constraints". His motive for developing a new life cycle model was to improve the reproduction of empirical facts. He especially wanted to improve the simulation of the saving behavior of individuals, which in the standard life cycle model is predicted to dissociate from income, whereas US saving data imply that consumption is positively related to predictions of income and that consumption tracks income closely[6].

Deaton's agents face income uncertainty (but no other form of uncertainty) and are unable to borrow money. Deaton ran his model with different kinds of income uncertainties and discovered that the most important factor influencing his agents' saving behavior was what they believed about the stochastic process generating their income.

At first, Deaton tested the model with incomes which were stationary, independently and identically distributed over time. This is a case which might be true for farmers in poor countries who are prone to weather risks, but for most people this kind of uncertainty does not play a role. The cautious agent who is faced with this kind of income shock tries to smoothen his income by accumulating assets, which here play the role of buffer stock. The more cautious the agent is, the more wealth he will accumulate.

In the second case, Deaton implemented positive serial correlation in the income process. This means that a good draw of income indicates that more good draws are to be expected, so that income can be expected to be higher in the future. The converse is true of bad draws. This kind of uncertainty influences the liquidity constrained consumers in such a way that they reduce income smoothing and do not use their assets as buffer stock, but instead consume their incomes.

In a third variant of modeling uncertainty, Deaton gives the agents incomes with stationary growth, where the growth rates mimic aggregate data and are positively serially correlated. This does not reproduce empirical saving data either, since the agents will save when their incomes shrink and consume when their incomes rise.

Deaton then realized that by using only aggregate income data, one cannot reproduce aggregate behavior; therefore he used income processes deduced from microeconomic data. In order to do so, he introduced two kinds of income (as Friedman had suggested before), permanent and transitory in-

[6]For sources on these empirical findings see Deaton (1991) [53].

come. A small part of the transitory income has aggregate fluctuations in income growth which are identical for all agents, whereas the other part of the transitory income is independent from the other agents. Using this income process, Deaton managed to reproduce many of the stylized facts in the actual data.

The agent in Deaton's model solves the typical optimization problem[7]:

$$\max E_t \sum_{t=1}^{T} \beta^{s-t} u(C_t) \tag{2.1}$$

where E_t is the expected value at time t, β is the discount factor, C_t is the consumption in period t and u is the per period utility function. Deaton requires the utility function to be increasing, strictly concave and differentiable. An example of such a function would be the CRRA utility function.

The evolution of assets is:

$$A_{t+1} = (1 + R)(A_t + Y_t - C_t) \tag{2.2}$$

where A_t are the assets, Y_t is the income and R is the interest rate. The state variable in this problem is "cash on hand" X_t[8], the money that can be spent in every period:

$$X_t = A_t + Y_t \tag{2.3}$$

[7]Deaton, as well as Carroll later on, solves the model for the infinite and the finite case. The results of these cases are extremely similar; thus, in the following, only the finite case is introduced.

[8]Deaton invents the term "cash on hand" and uses it here the first time. It is later adopted by other life cycle modelers, and is now used as a standard term.

where x_t evolves according to:

$$x_{t+1} = (1 + R)(X_t - C_t) + Y_{t+1} \tag{2.4}$$

Deaton, like Carroll later on, realizes that the common assumption of life cycle modelers, i. e. that the rate of time preference d equals the interest rate r, is not very realistic[9]. Deaton therefore stresses the fact that the agents in his model prefer to have some money now rather than keeping it for the future ($d > r$), and he shows that the widespread assumption of $d = r$, as well as the less common assumption of $d < r$, will lead to unfavorable or even wrong results (Deaton (1991), p.1125 [53]). The model with the impatient and prudent agent, who likes consuming today and only saves because he knows of his uncertain future, exhibits a buffer-stock behavior which is considered realistic by Deaton and which also plays a very important role in Carroll's model.

2.2.2 The Carroll Model

Christopher Carroll's model (Carroll (1992) [32], (1997) [34][41]) is similar to Deaton's in many respects; the most important differences lie in the sophisticated way in which Carroll introduces uncertainty and in the fact that his agents do not face liquidity constraints. Carroll's motivation for developing his life cycle model was grounded in the standard life cycle models' inability to describe the typical saving behavior of consumers in a satisfactory manner.

Carroll especially wanted to resolve the discrepancy between the empirical fact that people have mainly precautionary reasons for saving (meaning they save to buffer themselves against unforeseen events) and the answers of the standard life cycle models, where explain the saving behavior of their agents is explained by their intention to provide for retirement.

Carroll solves his model with a finite and with an infinite time horizon. In both versions his agents save according to the wealth-to-permanent income ratio[10], but for the finite horizon model the agents show buffer-stock behavior

[9] Where $\beta = 1/(1 + d)$.

[10] The wealth-to-permanent income ratio or the buffer-stock goal is a saving target the agent aspires to and which is not influenced by temporary shocks.

only up to the age of 45 - 50 years. After this point the agents in the finite model switch to the traditional life cycle motives and save for retirement.

The consumer solves the usual intertemporal optimization problem for a finite horizon:

$$\max E_t \sum_{t=1}^{T} \beta^{s-t} u(C_t) \tag{2.5}$$

with the usual CRRA function with the coefficient of risk aversion ρ:

$$u(C_t) = C_t^{(1-\rho)}/(1-\rho) \tag{2.6}$$

subject to the constraints:

$$W_{t+1} = R[W_t + Y_t - C_t] \tag{2.7}$$

$$Y_t = P_t V_t \tag{2.8}$$

where W_t is the stock of net wealth, R is the interest rate and Y_t is the current labor income, which is divided into the permanent part P_t and the transitory part V_t, both being subject to uncertainty.

The permanent part of the income is defined as follows:

$$P_t = G_t P_{t-1} n_t \tag{2.9}$$

$$\ln n \approx N(-\sigma^2_{\ln n}/2, \sigma^2_{\ln n}) \tag{2.10}$$

where G_t is the rate with which the permanent part of the income grows and P_t is subject to the shocks n_t which are lognormally distributed.

The transitory part V_t can be disturbed in two ways: with probability q this part of the income is zero (caused, for example, by unemployment) and with probability $1-q$ the transitory income is lognormally distributed:

$$V_t = \begin{cases} 0 \text{ with probability } q \\ Z \text{ with probability } 1-q \end{cases} \tag{2.11}$$

$$\ln Z \approx N(-\sigma^2_{\ln Z}/2, \sigma^2_{\ln Z}) \tag{2.12}$$

The optimal consumption depends on the total resources X_t[11]:

$$X_t = W_t + Y_t \tag{2.13}$$

The evolution of the total resources is given by:

$$X_{t+1} = R[X_t - C_t] + Y_{t+1} \tag{2.14}$$

At the heart of the solution of Caroll's model lies the Euler equation. Here, all variables are divided by the current level of permanent income, so as to reduce the state variables to two normalized state variables: consumption and total resources[12]:

[11] In more recent publications X_t is called "cash on hand", e.g. Carroll (2006) [39], after Deaton (1991, 1992).

[12] Carroll (2004) [38] shows that renormalizing the equation does not change the solution.

$$1 = R\beta E_{t-1}\left[\{c_t[R[x_{t-1} - c_{t-1}]/GN_t + V_t]\, GN_t/c_{t-1}\}^{-\rho}\right] \qquad (2.15)$$

The finite case is solved recursively, using the Euler equations for the per period optimization since the agents consume all their money in the last period[13].

In order to find a solution for the infinite case, the consumption rule has to converge. Carroll (2004) [38] shows that the consumption rule converges if the following condition holds:

$$\rho^{-1}(r - \delta) + (\rho/2)\sigma^2_{\ln N} < g - \sigma^2_{\ln N}/2 \qquad (2.16)$$

where $\ln[R] \approx r$, $\ln[\beta] \approx -\delta$ and $\ln[G] \approx g$.

Pemberton (2003) [142] calls this condition the impatience condition. If there is no uncertainty about the future (and the condition is $\sigma^{-1}(r - d) < g$), the consumers, having an equal present value of lifetime consumption and of lifetime income[14], will consume by borrowing against future consumption[15]. In fact, if there is no income uncertainty and $\sigma^2_{\ln N} = 0$, the agents in Carroll's model behave like the agents in a Modigliani life cycle model.

As already mentioned, the consumer in this model is mostly impatient (as in Deaton's model), with the rate of time preference being larger than the interest rate ($d > r$). The introduced uncertainty is offset by the caution of the consumers. Agents who exhibit both of these attributes, being impatient and being prudent, develop a wealth-to-permanent income ratio to which they aspire. If their savings are below that ratio, the agents behave prudently and will save more; if the savings lie above it, the agents behave impatiently and consume more. In this way the agents develop a buffer-stock behavior, where wealth acts as a buffer against income shocks. As the consumers' wealth goes

[13]For an explanation of this recursive technique, see Chapter 5.
[14]That is of course only possible because the consumers do not have a bequest function.
[15]The agents are able to do this, since there is no liquidity constraint.

toward infinity, the income uncertainty no longer influences consumption.

Even if the uncertainty is very small, the consumers will never borrow against future income since there is always a positive probability that the income will become zero, and no social net and no insurance guarantees the agents a minimum income. Saving is therefore the only possibility for the agents to protect themselves from total destitution. This differs from Deaton's model, where the agents have a consumption floor and are not allowed to borrow.

Carroll considers his model a very close substitute to Deaton's model, since despite the modeling differences (uncertainty and liquidity constraints), the emerging consumer behavior is very similar (Carroll (1991), p.8 [34]). Both Deaton and Carroll can explain an important stylized fact with their models which is not covered by the most important alternative life cycle models: the consumption/income parallel for the representative agent. When an agent has reached his wealth-to-permanent income ratio, a change in income will just change the consumption and not the amount of money saved.

Pemberton (2003) [142], who has looked into the development of dynamic stochastic life cycle models, finds some shortcomings in Carroll's framework. First, there are no imperfect markets in the model; and, second, there is only one type of non risk saving assets. The third simplification concerns the absence of any insurance possibility or social net. The agents in Carroll's model will always save, at least little, since they are always faced with the small possibility of total destitution. Pemberton argues that there is no evidence to support this assumption, since the agents in this model, if given a social net (and no risk of ending up in destitution), would start to borrow huge amounts of money.

In Pemberton's opinion, the first two simplifications do not lead to a distortion of empirical facts; he does, however, consider the lack of a social system a major flaw. Pemberton also has a suggestion of how to rescue the model: he proposes introducing a utility function with nation-specific subsistence levels (since social security nets differ in each country). This is exactly the way the German life cycle model is constructed; the utility function used has a subsistence level which is specific to Germany.

2.2.3 The Hubbard et al. model

Hubbard et al. (1994) [91] developed a model which can be considered an extended version of Deaton's and Carroll's models. The authors introduce several sources of uncertainty, including labor income, remaining length of life, and medical expenses. They also incorporate social security, which acts as a subsistence level, since the agents will always have enough money to consume a predefined minimum of goods. Finally, the agents are faced with a borrowing constraint, which is accomplished by specifying that assets are not allowed to be negative. Hubbard et al. have built the model for three groups of agents with different educational backgrounds and income: These are, no high school degree, high school degree and college.

Only the income uncertainty has a significant influence on the outcome of the model. Hubbard et al. split the income into a transitory and a permanent part, just as in Carroll's model, and use a similar way of empirically estimating the labor income risks. The variance in income shocks is small compared to the other life cycle models described here since the authors implement social insurance. Each educational group has special parameters estimating its equation of income uncertainty. Different from Carroll's model is the rate of income growth, which is lower in the model of Hubbard et al.

The agent in this model solves the following optimization:

$$\max E_t \sum_{t=1}^{T} \beta^{s-t} D_s u(C_s) \tag{2.17}$$

which is identical to the problem of Carroll's and Deaton's agents, except for the random date of death.

The utility function is again a CRRA function, and the problem is subject to the following transition constraint:

$$A_t = A_{t-1}(1+r) + W_t + TR_t - C_t - M_t \tag{2.18}$$

with

$$TR_t = TR(W_t, M_t, A_{t-1}(1+r)) \tag{2.19}$$

The transition equation (2.18) describes the accumulated assets. A_t are the financial assets, R is the interest rate, W_t is the income, M_t are the necessary medical expenses and TR_t are the government transfers. Government transfers depend on income, medical expenses, and assets and are defined as follows:

$$TR_t = \max\{[(\bar{C} + M_t) - (A_{t-1}(1-r) + W_t)], 0\} \tag{2.20}$$

where $\bar{C}$ is the consumption floor, i. e.the minimum level of consumption guaranteed. This definition is an approximation of government transfer programs in the US.

Restricting the assets to positive values makes sure that the agents cannot borrow:

$$A_t \geq 0 \tag{2.21}$$

The “cash on hand” X_t the agent can use for consumption is defined as:

$$X_t = A_{t-1}(1+r) + W_t - M_t + TR_t \tag{2.22}$$

The most important difference from Deaton as well as Carroll is the relation of time preference and interest rate. Whereas Deaton and Carroll use a time preference rate which is always larger than the interest rate, the agents of Hubbard et al. are more patient and the authors define $d = r$.

This model, like the other dynamic stochastic life cycle models, cannot be solved analytically and is therefore solved numerically. The technique is basically the same as the one used by Deaton and Carroll. The authors use the

Euler equation for the per period optimization and solve recursively from the last period to find the optimal consumption rule.

The behavior of the agents depends on their educational status and hence on their income. For agents with a high school degree, the existence of the consumption floor has almost no influence on their saving behavior, while the households from the lowest educational group have almost no incentive to save since they know there is a social net which ensures minimum consumption for them.

The model explains a range of empirical facts better than previous life cycle models. The wealth-income ratios of the agents are a better approximation of reality than in alternative models. The fraction of households which consume their current income is better matched. Further, the model shows that the design of government expenditure can have an important impact on aggregate saving behavior whereas in other life cycle models government expenditure does not even play a role.

Hubbard et al. also have investigated the observation that poor people often display a different saving behavior from the rest of the population, a result which cannot be reproduced by standard life cycle models (Hubbard et al. (1995) [92]). The authors claim that this heterogeneity of wealth accumulation can be explained by the existence of a social net: people with little income but a ground floor of consumption tend not to save. By dividing agents into different groups with different educational backgrounds and giving them social insurance, this behavior can be replicated.

In Pemberton's overview paper, the model of Hubbard et al. takes a prominent place, as does Carroll's model. Pemberton approves of the introduction of a social net but has some doubts about the differences of saving behavior between the agents with different educational backgrounds. In the model of Hubbard et al. the differences in saving behavior are only based on the introduction of a social net (for rich people the existence of such a net has no influence while poor people rely on the net and save less in its presence). But Pemberton shows that the saving behavior of rich and poor people also differs in the absence of a social net and therefore demands a better explanation.

2.2.4 The German life cycle model

The work done in this thesis has been inspired by the models described above. There are both similarities to features in these models and differences in places where a better approximation of reality seemed possible. The first difference is of a topographic nature: Deaton, Hubbard et al. and Carroll have crafted their models so as to replicate the saving behavior of the US and have therefore used North American data, whereas the German life cycle model uses German data and tries to better understand German consumer behavior.

The only source of uncertainty in this model is the income, as is the case in the models of Deaton and Carroll. The main difference, however, lies in the way in which the uncertainty is implemented. These models implement uncertainty as a random process in which agents are faced with stationary probabilities during every period; in the German life cycle model, however, the probability of being unemployed in the succeeding period depends on the current employment status of the agents[16].

The solution methods used in the German life cycle model and in the models described above are very similar: all of them have been solved using dynamic programming. Instead of the Euler equations employed by the above authors to find the optimal solutions per period, however, the German life cycle model uses numerical optimization algorithms[17]. An analytic solution was not feasible due to the complexity of the utility function and the Markov process[18].

One aspect which makes the present model more realistic than the models described above the utility function[19] employed. While Deaton, Hubbard et al. and Carroll use a simple CRRA function, the utility function implemented here has several special features. Instead of one kind of good, the German life cycle model has three different kinds of goods (necessities, durables and nondurables), includes money and subsistence levels[20].

Of the three models described above, only Hubbard et al. have introduced a social net. The absence of a social net in a life cycle model is, in the opinion

[16] In fact, none of the papers above use unemployment directly as a source of uncertainty; they use income distributions. A later paper by Carroll et al. uses employment explicitly as a source of uncertainty, but with stationary probability (Carroll et al. (1999) [40]).

[17] The optimization routine used is fmincon by Matlab.

[18] The Markov processes were implemented in such a way that it was not possible to derive an Euler equation.

[19] For a detailed description of the utility function see Chapter 4.

[20] A Stone-Geary function has also been proposed by Pemberton (2003), p.20 [142] to improve standard life cycle models.

of the author of this thesis, a major shortcoming since all industrialized countries have implemented some sort of social insurance and since the existence of a security net influences saving behavior[21]. Even the lowest unemployment insurance in the OECD amounts to 22% of the former wage (UK), followed by the US and Australia which provide 26%. Many European countries have unemployment insurance which equals 60% of the former wage (Nickel et al. (2001) [132]). For this reason, the implementation of a social net modeled to mirror the German safety net is one of the most important and distinguishing features of the German life cycle model.

In the German life cycle model, the rate of time preference is larger than the interest rate and the consumers are therefore impatient and prudent (similar to the models of Deaton and Carroll). In contrast to Carroll, but similar to Deaton and Hubbard et al., the agents in the German life cycle model are liquidity constrained. Like Hubbard et al. and one of Carroll's papers (1997a) [34], the German life cycle model is applied to three groups of agents which are differentiated by their education and their income.

As in the models described above, no uncertainty is implemented in the stock market; whatever the agents decide to save, they will keep and be able to collect the interest. It seems justified to neglect the variations in the stock market since they do not greatly influence the behavior of most people in Germany, and only a small percentage of the population actually invests in risky stocks on a large scale.

In contrast to the models above, the agents in the German life cycle model bequeath their assets, which means that they are not obliged to use up their savings before the end of their lives but leave money to their heirs.

[21] For a more thorough examination of how the introduction of a social net changes the outcome of buffer-stock models see: Michaelides (2003) [126].

Chapter 3

Patterns of Saving

In this chapter, three aspects of saving behavior are depicted. We will first examine why people save, then present stylized facts about saving behavior and finally illustrate empirical findings about saving in Germany. The stylized facts and empirical findings are important because they enable us to judge wether the results of the German life cycle model are consistent with the real world. The empirical findings presented in this chapter will be compared to the simulation results in Sections 7.2.2 and 7.2.1.

3.1 Motives for saving

In his book "The General Theory of Employment, Interest and Money", Keynes compiled a comprehensive list of saving motives (Keynes (1936) [97])[1]:

> *1. The precautionary motive (saving against unexpected income failures);*
> *2. The life cycle motive (saving for retirement);*
> *3. The intertemporal substitution motive (saving so as to be able to enjoy the money deriving from interest rates);*
> *4. The improvement motive (saving so as to be able to gradually increase expenditure);*
> *5. The independence motive (saving so as to be independent to do certain things or to be independent from work);*

[1]This list has been reformulated into present-day language by Browning and Lusardi (1996) [24].

6. The enterprise motive (to have some money to either speculate or found a business);
7. The bequest motive (to be able to bequeath money);
8. The avarice motive (saving out of stinginess).

Adapting the list to present-day conditions, Browning and Lusardi (1996) [24] have added another saving motive:

> *9. The downpayment motive (saving in order to be able to accumulate deposits to buy houses, cars and other durables).*

Many authors have pursued the question of saving motives since Keynes[2], but only three of these motives have had enough influence on the actual saving behavior to be considered in the standard saving models. These are the precautionary motive, the life cycle motive and the bequest motive.

In many cases, it is not only the modeler who finds it difficult to distinguish between different saving motives but also the modeled subjects, the people themselves. Very often reasons for saving such as precaution, retirement or bequest are mingled in a person's head, and they would not easily be able to say what the exact saving motives are. For this reason, and in spite of the existing, extensive literature, which would seem to suggest otherwise, the discussion about saving motives often does not further our understanding of saving behavior and is often not at the core of the problem. This idea has been discussed by Dynan et al. (2002), p.274 [58], who state, in the context of a life cycle model incorporating uncertainty:

> *"Allowing for uncertainty resolves the controversy over the importance of life-cycle and bequest saving by showing that these motives for saving are overlapping and cannot generally be distinguished."*

Furthermore, not only is the benefit of investigating the spending motives of people highly doubtful; it also appears to be extremely difficult to learn anyhing about these motives. Browning and Lusardi (1996), p.1849 [24],

[2] See, for example, Gourinchas and Parker (2001) [80], Gourinchas and Parker (2002) [81], Dynan et al. (2002) [58].

who have done extensive research on the issue of saving motives draw the following conclusion about the explanatory power of household saving data:

> *"While these data provide a more or less accurate description of who saves and how this has changed over time, they are much less effective in helping us to answer the basic question of why people save."*

3.2 Stylized facts about saving

To make general remarks about the behavior of individuals is always difficult, this also applies to people's saving and spending behavior. Nevertheless, it is possible to recognize certain patterns for industrialized countries and more specifically for Germany. Some of these patterns will be introduced in the following section.

3.2.1 The influence of income uncertainty

The relation between income uncertainty and saving rates is central when building a stochastic life cycle model. Christopher Carroll, who developed one of the standard buffer-stock models (see Chapter 2), has also concerned himself with this question. In a paper in which he analyzed income uncertainty and saving rates he conducted simulations with his life cycle model using data from the US consumer and expenditure survey (CEX) and concluded:

> *"... consumption responds strongly to uncertainty in future income"* (Carroll (1994), p.142 [33]).

Carroll produced similar findings in a more recent paper in which he used US data from the Panel Study of Income Dynamics (PSID), ascertaining that people behave according to the buffer-stock model of saving. This means that young households under fifty hold more liquid assets when faced with higher income uncertainty while households over fifty are less influenced by this uncertainty and tend to save for retirement (Carroll 1997 [34]).

Nikolaus Bartzsch (2006) [14] analyzed German saving data after 2002, when saving rates were rising and the situation on the labor market was very tense. He could not find any long-term correlation between high unemployment rates (and therefore high income uncertainty) and high saving rates, but he showed that saving rates were higher for households with higher uncertainty which had not yet reached their buffer-stock goal[3].

Fuchs-Schündeln and Schündeln (2005) [71] also investigated the relation between income uncertainty and saving rates. To do so, they used German data from the time before reunification. While in West Germany it was possible to work in the civil service, which, compared to other employment sectors, provided a secure income (something still possible in reunified Germany), the income risk was more equally distributed in East Germany. Thus, it was possible in West Germany for very risk-averse people to choose a job according to their preference; once such a secure job was chosen, these people had hardly any need for precautionary saving. When the authors used a model in which the self selection of risk-averse people in West Germany was controlled for, however, the amount of the precautionary savings was similar in both parts of the country. From this, the authors concluded that individuals in general act according to the theory of precautionary savings. Not controlling for self selection on the other hand leads to underestimating the precautionary savings.

Although it is difficult to say specifically how people behave when facing income uncertainty, it seems apparent that uncertainty does influence people and that an increase in uncertainty tends to lead to an increase in precautionary saving.

3.2.2 The influence of income growth

In his paper published in 1994, Carroll also investigated the correlation between income growth and savings and found that:

> *"Current consumption appears strongly related to predictable current income, and unrelated to predictable lifetime changes in income"* (Carroll (1994), p.124 [33]).

[3]For an explanation of "buffer-stock goal" and "buffer-stock behavior" see Section 2.2.2.

These findings were confirmed in another paper, published with a colleague in the same year. The authors found that among young households, those households which expected a faster income growth saved more than households which expected a slower income growth (Carroll and Weil (1994) [43]).

A similar result was also reached by Paxson (1996) [141], who examined the correlation between aggregate saving and economic growth in four countries: the US, Britain, Taiwan and Thailand. She found that higher growth in income will increase the saving rate of young people more than the rate of older individuals.

The strong correlation between current income growth and saving is also confirmed by those cases in which a sudden income boost resulted in higher saving rates. A 26 percent rise in pensions in West Germany in 1957, for example, led to an increase in the average saving rate of these pensioners ranging from 1.5 to 7 percent. Similar results were found when unanticipated payments were made to US army veterans in 1950 and Israelis received reparation payments from Germany in 1957-58. In both cases, saving rates were much higher from these sudden income boosts than from permanent income[4].

Sahra Wagenknecht, who has carried out an extensive literature survey on stylized facts about saving, concludes that it can be considered a stylized fact that households with higher income growth display higher saving rates than households with a similar income but lower income growth (Wagenknecht (2007) [175]).

3.2.3 Relative and absolute income

Carroll (1994) [33] and Carroll and Summers (1998) [42] documented two important stylized facts about income. Over several years, aggregated consumption closely follows growth in income, which means that the saving rate develops in the same way as the income. The authors call this the consumption/income parallel. The second stylized fact concerns the individuals: for them, the relation is quite different since consumption at the microeconomic level does not follow current income. Carroll (1997) [34] sees the explanation for this consumption/income divergence in people's buffer-stock behavior. Consumption and saving do not follow income with its transitory shocks

[4]For a summary of these studies see Bodkin (1959) [20], Kreinin (1961) [105], Landsberger (1966) [110] and Mayer (1972) [120].

since money from positive shocks is saved to buffer against negative shocks.

This is also similar to the findings of Wagenknecht (2007) [175]. According to her, relative income has a stronger effect on saving than absolute income. This means that a person with an income which is subject to many shocks will save more than a person with the same average, but permanent, income.

3.3 Empirical saving in Germany

Collecting data about the saving behavior of people classified according to their different educational attainments and different ages is difficult because it is then necessary to interview individuals instead of merely collecting data at the macro level. In Germany, the Federal Statistical Office conducts the Income and Expenditure Survey (EVS), in which this kind of detailed and differentiated data is collected. This survey is carried out every five years. About 0.2 percent of German households, about 60 000 in number are interviewed about their spending and saving behavior. Households with a monthly net income of over 18 000 Euros are excluded[5], as well as persons who are institutionalized (for example old people in nursing homes). The results are then extrapolated for all 38.1 million German households.

Another source of data of this kind is a poll conducted by the Spiegel-Verlag in 2003 (Der Spiegel (2004) [164]). The "Spiegel" interviewed 10 100 members of the German population aged 14 or higher and asked them about different aspects of their lives: their jobs, their education, their political ideas, their cars, their leisure time, their eating habits, etc. Thus, they were able to extrapolate their results to all 64.72 million German inhabitants older than 13 in order to get a comprehensive picture of the German population. Among other things, the "Spiegel" also asked their interview partners about their saving behavior.

There are two main differences between the two surveys outlined above. First, the EVS questioned households (where a household is classified according to the person who contributes most to the household income), while the "Spiegel" collected data from individuals. Households and individuals cannot, of course, be compared directly, but some patterns of their behavior are very similar, for example the fact that young people and young households

[5] As there are not enough rich households willing to participate, these results would in the end not be statistically significant.

at the beginning of their working lives save less than persons at the height of their careers. Looking at both data sets can therefore be very helpful in ascertaining some stylized facts about the saving behavior of people in Germany.

The second difference is the way the polls were conducted. The “Spiegel” confronted its interviewees with multiple choice questions, in which they could choose from several saving bands, without having to disclose their individual saving rates. The ”Spiegel” results cannot, therefore be expressed in average savings. The EVS accompanied each household for one year; the households were interviewed about their consumption behavior and had to fill in questionnaires, reporting how much they spent and saved every month. For this reason, the EVS saving data is much more detailed.

3.3.1 The age-saving profile

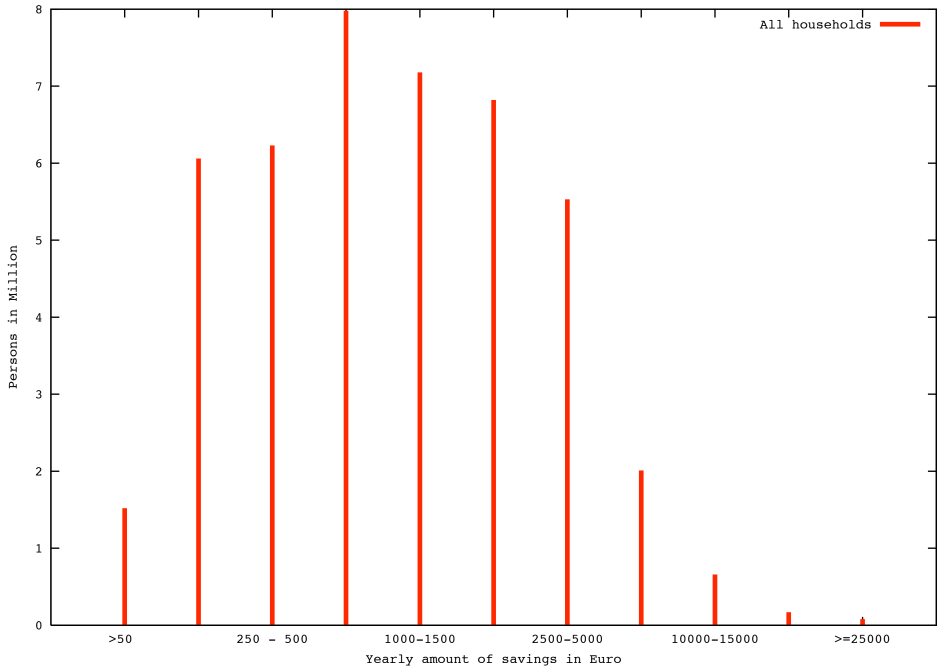

Figure 3.1: Yearly saving amounts (2003)

Source: Spiegel (2004)

The yearly savings of Germans are mostly between 50 and 5 000 Euros; 39.8

million of the over-14-year-olds save in that band. Only 1.52 million people save under 50 Euros and an insignificant number totalling a quarter of a million people save more than 15 000 Euros a year (see Figure 3.1).

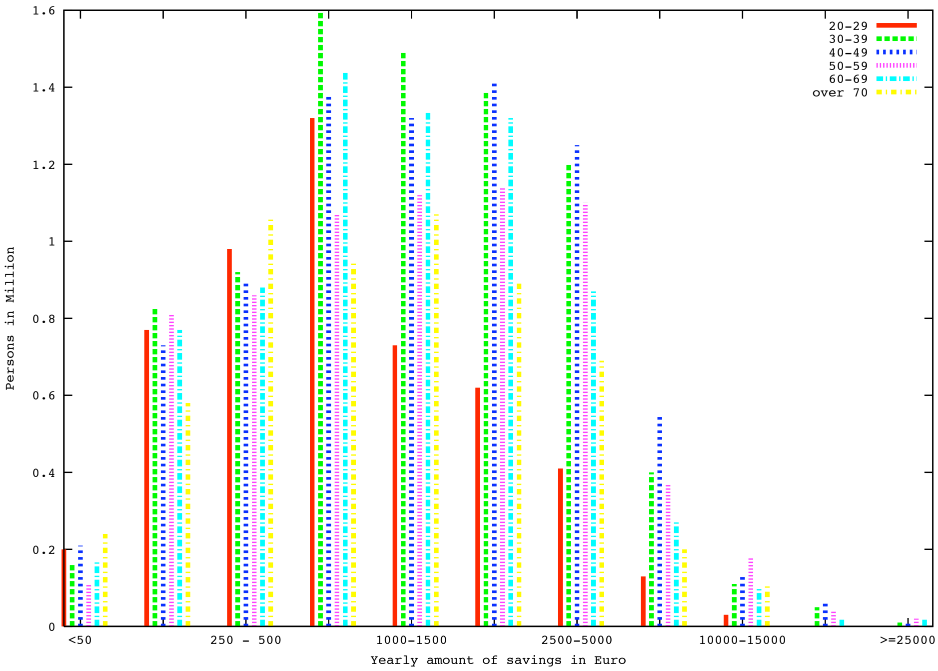

Figure 3.2: Saving amounts according to age (2003)

Source: Spiegel (2004)

Young people from 20 to 29 save the least, their savings being concentrated on the left of the graph. People from 30 to 39 save in a similar band to those from 40 to 49, but the saving hump in their profile is located a little more to the left. The 50- to 59-year-olds save relatively little in comparison while the 60- to 69-year-olds contribute strongly again. The elderly are still saving, but due to the fact that they are a comparatively small group their saving contribution is small for every saving band (see Figure 3.2).

Since the figures above do not disclose anything about the saving behavior of individuals in their age group compared to the other age groups, percentages of each age group are shown. The results are only partly clear. It is striking that young people (20 to 29 years) save mostly in the low band of savings of up to 1 000 Euros per year. The succeeding age groups behave similarly to each other, but small differences can be seen. In the saving profile of the age

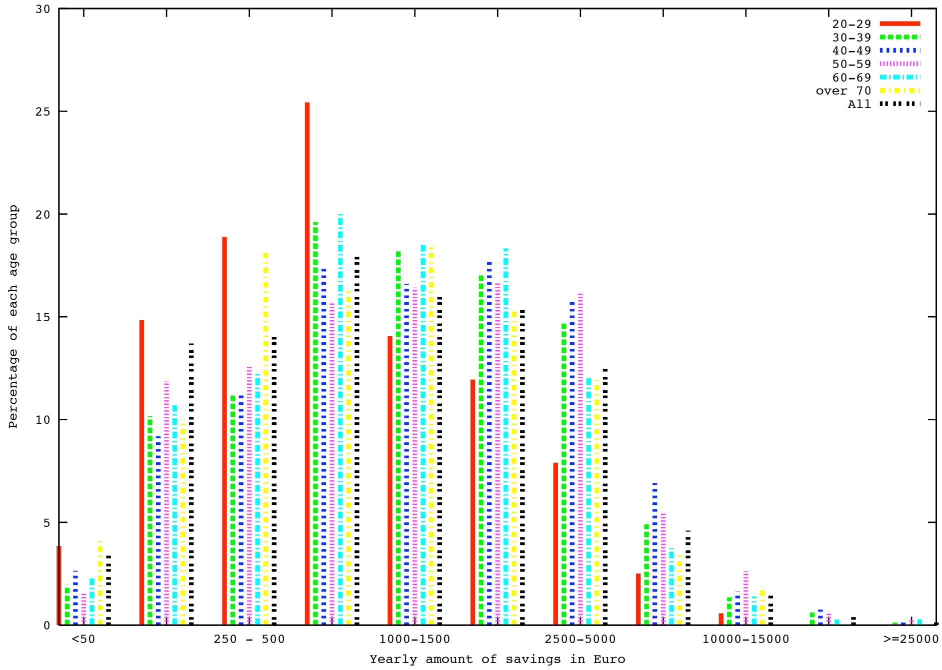

Figure 3.3: Yearly saving amounts according to age in percent of each group (2003)

Source: Spiegel (2004)

group 30 to 39 the hump is oriented more to the left (the range of the smaller saving amounts) compared to the 40- to 49- and 50- to 59-year-olds. The latter are also the ones who save most. From the age of 60 onward, people save in the smaller saving bands again (see Figure 3.3).

In the EVS data the saving pattern is clearer. Households save little when they are young, on average 71 Euros monthly and 852 Euro per year. They increase their savings until they reach their saving peak at an age of 35 to 55, with an average yearly saving rate of 5892 Euros for the 45- to 55-year-old. After that, the saving amounts decrease again until they reach their lowest point when the households are between 65 to 70 and then increase again as they grow older (see Figure 3.4).

The age group with the highest saving rate is the one prior to retirement age (in Germany, the average retirement age lies at around 60). The observation that households have the highest saving rates shortly before retirement also corresponds to various international studies presented in the book "Inter-

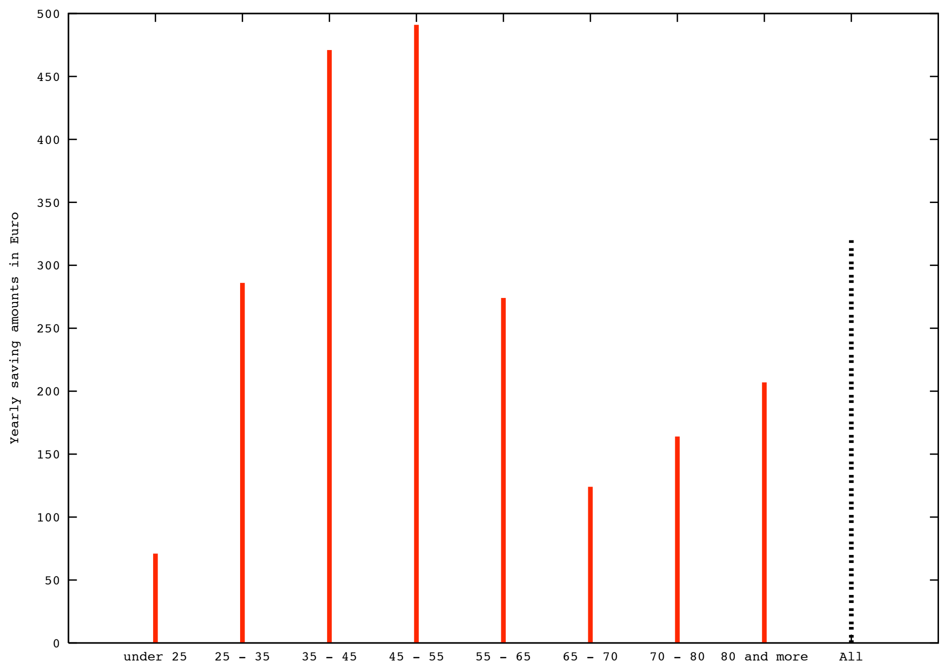

Figure 3.4: Monthly saving amounts of households according to age (2003)

Source: EVS 2003

national Comparisons of Household Saving", edited by James M. Poterba (1994) [143].

While the profile of saving rates looks similar to that of saving amounts, it is distinguished by the fact that the age-saving profile is flatter. The largest saving rate of 14.2 percent for the 35- to 45-year olds is three times larger than that of people under 25 years of age and those between 65 and 75, which is only 4.8 percent. On the other hand, the saving amount of the 45- to 55-year-olds, 5892 Euros, is seven times larger than the smallest amount of 852 Euros per year, the amount saved by the young and by the old at the beginning of their pension period. This means that the percentage households save from their income changes less than the absolute amount they save. This fact can be explained by taking into consideration that households at the height of their working lives not only earn the most, but also spend the most, for example by borrowing money to buy a house.

A further difference concerns the discrepancy between the two saving peaks. While in absolute saving amounts the 45- to 55-year-olds save most, in terms

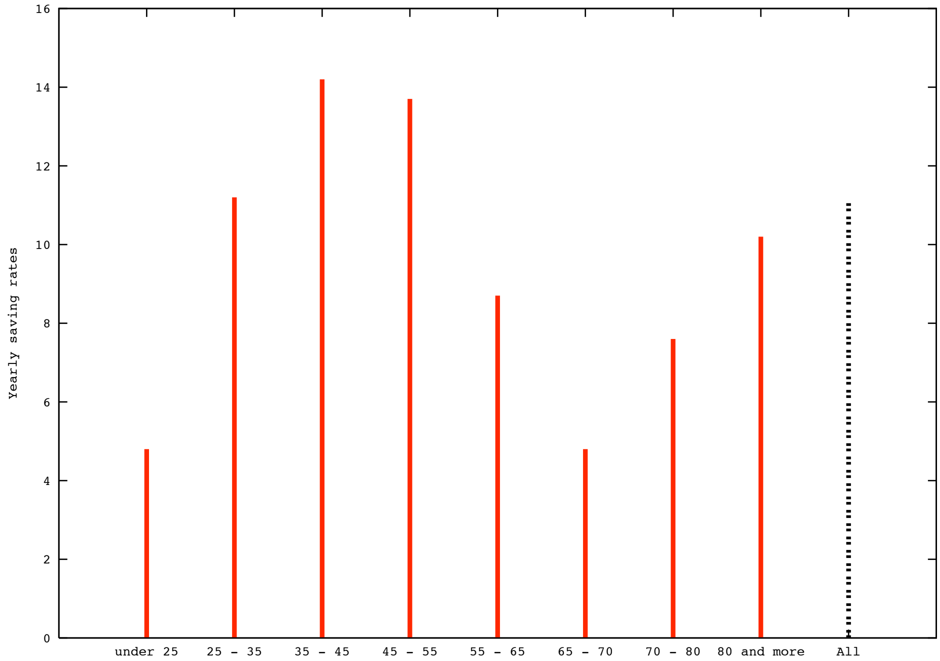

Figure 3.5: Monthly saving rates of households according to age (2003)

Source: EVS 2003

of saving rates the 35- to 45-year-olds are ahead. That means the 45- to-55-year-olds save more money but a smaller amount of their income (see Figure 3.5).

Comparing the EVS data with the “Spiegel” data is startling because of one contradictory fact, the way saving behavior changes over life spans. While in the EVS data saving increases again when people grow older and has its lowest point at the beginning of retirement age, in the “Spiegel” data very old people save less than the other age groups except for the very young.

This contradiction can be explained by the different ways in which the data was collected. As mentioned above, very rich households are not considered in the EVS; this fact explains the relatively low average saving rate. The reason that elderly start saving again in the EVS data but not in the ”Spiegel” data stems from the fact that elderly people in nursing homes are not included in the EVS data. Old people in such institutions usually spend a lot of money on staying there so that there is nothing left to save, while old people who stay at home are less mobile and consume less, and therefore have more money left to save. Finally, one must bear in mind the inherent difference in the two different data sets: one of them is about households and and the other about individuals.

But why does the saving profile of the German population look so different from Modigliani's life cycle model? Not only do people not dissave with age (which is a well-established observation); they actually start saving again when growing older. Börsch-Supan et al. (1992) [21], (2000) [22], who have investigated the "old-age saving puzzle", state the following reasons. The first reason lies in the way the data is collected; the EVS transforms the wealth people accumulate for retirement into a steady stream of income after retirement starts. This means, of course, that there is no dissaving to be seen in the saving graphs. The way the data is arranged explains why saving rates are not negative, but it still does still not explain why the saving rates increase with age. The second reason for the positive German saving rates during retirement lies in the generous German public pension system and the extensive coverage of health expenses by the mandatory health insurance, which make it unnecessary not only to use up savings during retirement (except for the elderly people who are institutionalized) but also to build up savings for retirement. Preparing for retirement is therefore not the major reason for saving, and the resulting age-saving profile will be flatter than in cases where people have to worry about their income when they reach old age. And, finally, old people are physically not fit enough to be able to engage in a lot of money consuming activities. As Börsch-Supan points out, this very special German age-saving profile might change when the pension system changes and people have the feeling that their pensions are no longer secure.

3.3.2 The education-saving profile

The largest group of people working in Germany have undergone vocational training[6] and are therefore also the people with the highest saving amounts in every saving band. This group comprises approximately 28.5 million people while the other groups of savers together only comprise 18 million people (see Figure 3.6).

The behavior of the members of this group can again be seen more clearly in the graph showing the percentage of the people each group comprises. As

[6] In German: Ausbildung.

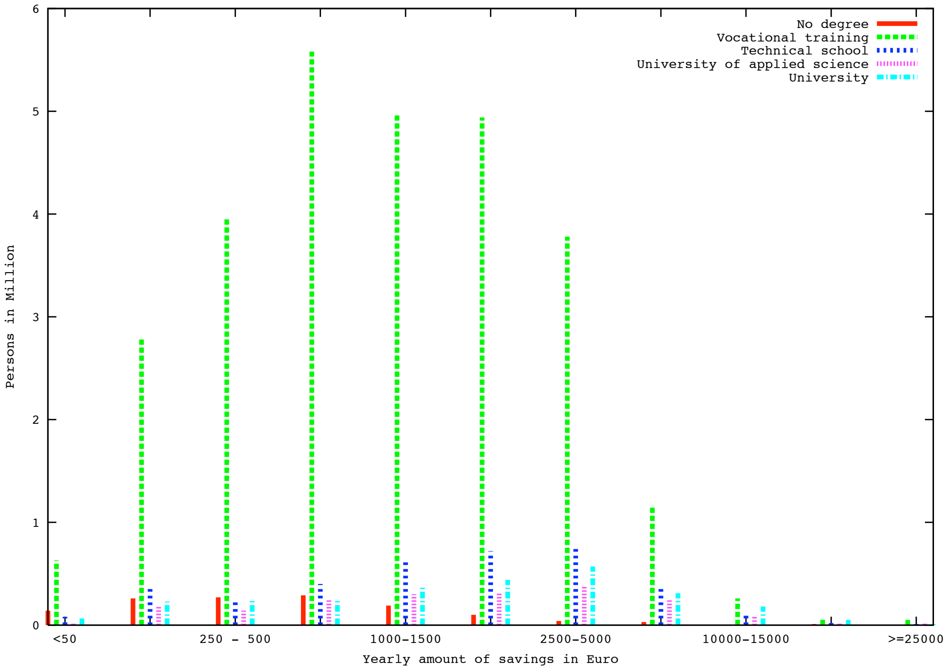

Figure 3.6: Yearly saving amounts according to educational degree (2003)

Source: Spiegel (2004)

expected, persons with no qualifications save the least, mostly in the range of between less than 50 to 1 500 Euros per year. People with a higher level of education save more and those saving the highest amounts have either a degree from a technical college[7], a degree from a university of applied science[8] or a university degree. For each group with a higher qualification the hump moves a bit more to the right (see Figure 3.7).

The EVS figures present a very similar picture, with a higher level of education leading to higher saving amounts. The steps between the saving amounts of the different groups amount to about 1200 Euros each, except for the step between the university-educated and those holding a degree from a university of applied sciences, where the difference amounts to about 2040 Euros. The saving rates also increase with the level of education, but not as much. While the saving amount of a university degree-holding household is five times higher than the saving amount of a household with no educational qualification, the saving rate is only about twice as high (see Table 3.1).

[7]In German: Fachschule.
[8]In German: Fachhochschule.

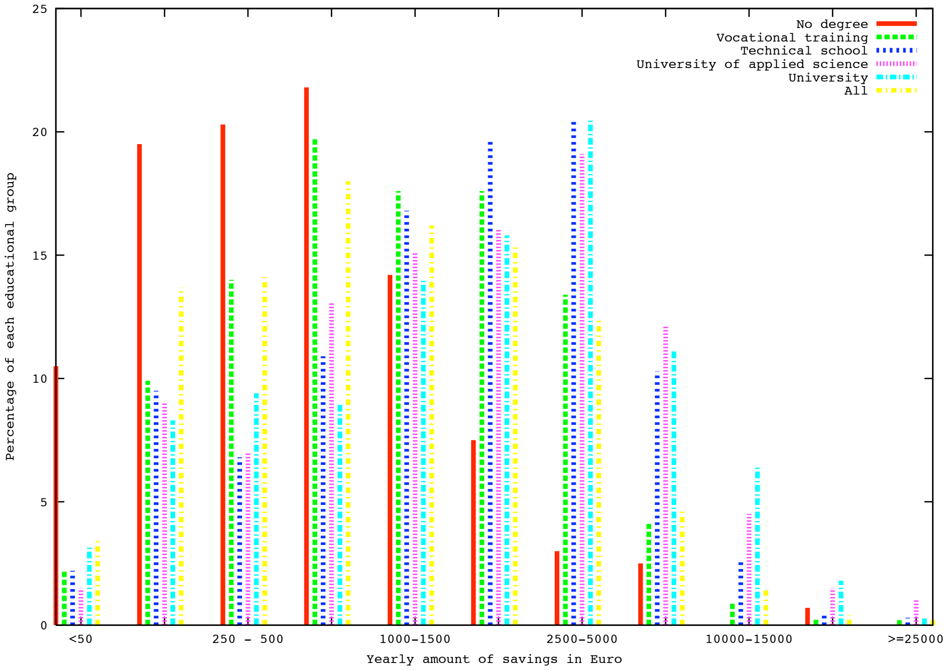

Figure 3.7: Yearly saving amounts according to educational degree in percent of each group (2003)

Source: Spiegel (2004)

It is a stylized fact that people with a higher level of education and a higher life time income not only save more than people with a lower level of education but also save a larger fraction of their income. This is not only true of Germany but has been discovered in other countries as well. Dynan et al. (2004), p.1 [59] have analyzed this question using several data sets from the US and found:

> *"...a strong positive relationship between personal saving rates and lifetime income".*

Poterba (1994), p.9 [143] also concluded in his book on household saving in different countries that:

> *“”. . . various country studies yield consistent evidence that saving rates rise sharply with income”.*

	Saving amounts (in Euros)	**Saving rates**
All households	321	11.1%
University	594	15.0%
University of applied science	421	11.7%
Technical school	333	11.2%
Vocational training	247	9.5%
No qualifications	137	8.3%

Table 3.1: Monthly saving rates and saving amounts according to level of education (2003)

Source: EVS 2003

3.3.3 The employment status-saving profile

A further interesting fact concerns the saving behavior of households with different employment status (data showing the relationship of saving behavior and employment status is only available from the EVS). Unsurprisingly, the unemployed in every educational attainment group save less than the employed. The amounts employed and unemployed people save, however, is different for each educational group. While unemployed persons with a degree from an university of applied science have a yearly saving rate of two thirds of what they saved when being employed, unemployed persons with an university degree have a saving rate which is five times as high as it was when they were employed (see Figures 3.8 and 3.9).

In general, however, people in Germany still can and do save while they are unemployed. An average employed person in Germany saves 6816 Euros yearly and 14.5 percent of its income, while the average unemployed person saves 1068 Euros and 5.5 percent of its income.

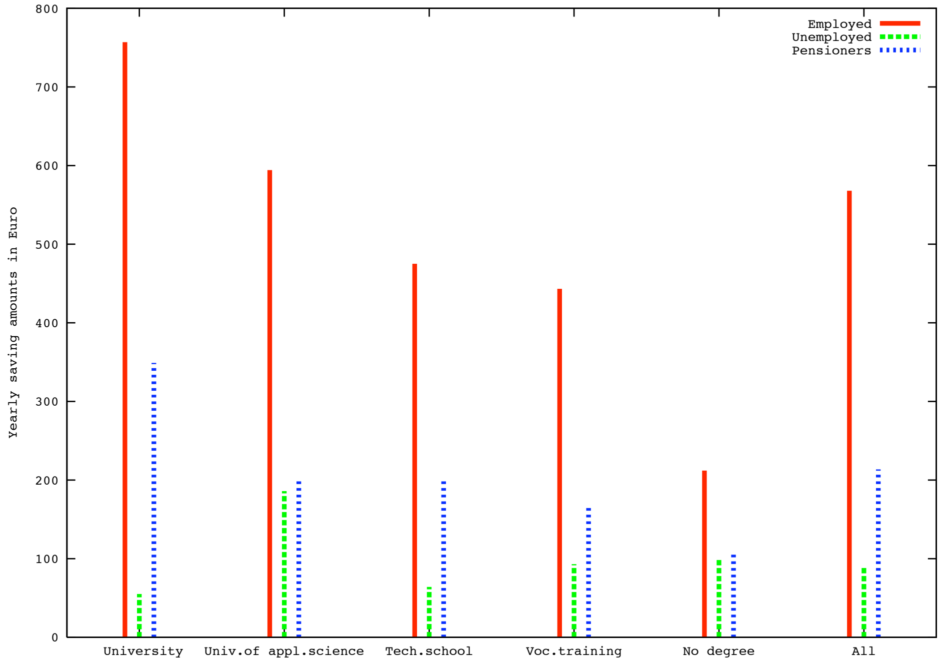

Figure 3.8: Monthly saving amounts according to employment status and educational degree (2003)

Source: EVS 2003

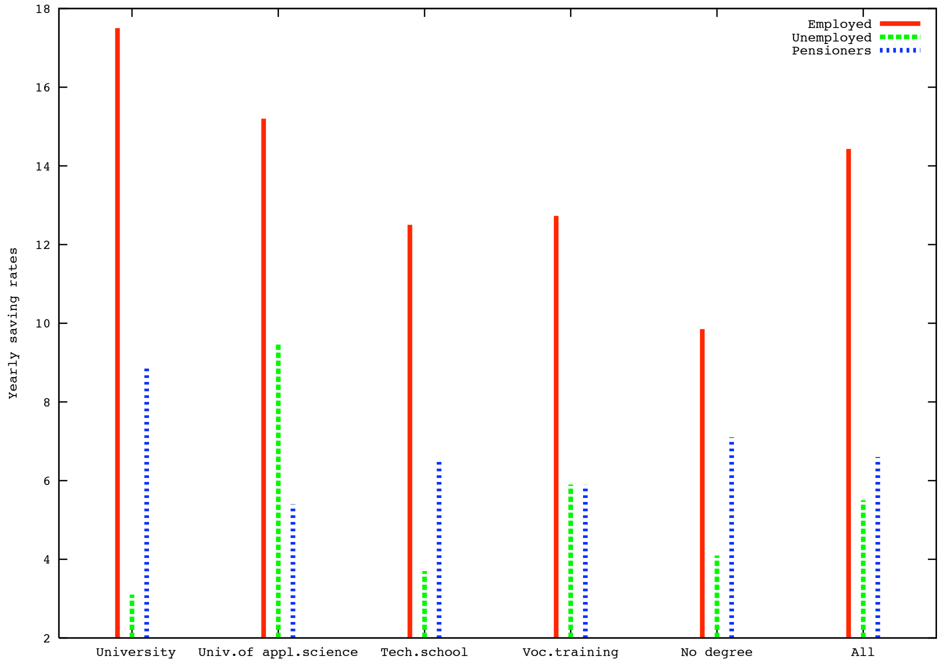

Figure 3.9: Monthly saving rates according to employment status and educational degree (2003)

Source: EVS 2003

Chapter 4

The CRRA-CES-Stone-Geary utility function

The utility function[1] is at the heart of a consumer model. It is the deciding factor for the realism of the consumption and saving behavior of the agents. This is the reason why the utility function of the German life cycle model is build so carefully and has many features which diverge from the standard life cycle models.

To create a utility function which is able to display many features of consumption and saving behavior, several simple utility functions have been combined. In the following sections the elementary utility functions of the composite utility function will be listed, explained and then combined. The chapter concludes with a paragraph which puts the composite utility function used in this model into the context of the current literature.

Utility functions can represent the behavior of an individual consumer as well as that of a representative consumer. But a utility function which can be used for both types of agents must fulfill the condition that the redistribution of income from one agent to the next does does not change the aggregate demand. This requirement is fulfilled for the composite utility function in the German life cycle model; a complete proof is shown in Appendix D.

4.1 Building the composite function

The goal of this section is to describe the utility function of the German life cycle model. In order to give a plausible and approximately realistic picture

[1]For an introduction to utility theory see Carlo C. Jaeger et al. (2001), p.45 [94].

of the behavior of an individual, this complex utility function combines several utility functions. It consists of the following: the intertemporal utility U, the CRRA function u, and the S-branch utility $\bar{x}$ which is a combination of a CES and a Stone-Geary function.

The composite utility function has the following features: the agent optimizes over his entire life, the function ensures that the agent prefers balanced consumption over time; it displays the risk aversion of the agents when faced with uncertainty; it includes different goods; it displays preferences for a variety of goods; and it reproduces the tendency that, as the income changes, the agent will demand goods in different proportions.

4.1.1 The intertemporal utility

The intertemporal utility sums the utility of every period of time over the agent's finite lifetime. The utility of future periods is discounted, conforming to the notion that people prefer present consumption to future consumption. The intertemporal utility is given by:

$U(u_t, u_{t+1}, \ldots, u_T, b_{T-t+1}) = \sum_{s=t}^{T} \beta^{s-t} u_s + \beta^{T-t+1} b_{T-t+1}$

$U : \mathbb{R}_+^{T-t+1} \to \mathbb{R}$, $0 < \beta < 1$

where β is the discount factor, u_s the utility at time s, t the first and T the last period in time, and b_{T+1} the bequest (or scrap) term.

4.1.2 The CRRA function

The CRRA (constant relative risk aversion) function relates the utility of a composite good to itself in different periods with each other and models two different preferences of the household, its preference for a smooth consumption path and the risk aversion of an agent.

This is the CRRA utility function for a specific point in time for deterministic cases:

$u(\bar{x}_s) = \frac{\bar{x}_s{}^{1-\sigma}}{1-\sigma}$

$u : \mathbb{R}_+ \longrightarrow \mathbb{R}$, for $\sigma \neq 1$, $\sigma > 0$

where σ is the the coefficient of relative risk aversion in case of uncertainty, $1/\sigma$ is the elasticity of substitution between consumption at any two points in time (or intertemporal substitution) and $\bar{x}$ is the bundle of goods The whole term represents the per period utility u from the intertemporal utility.

The two preferences mentioned above are specified through the constant σ. The shape of the indifference curves shows the relationship between consumption at different points of time and express the willingness of the consumer to shift the demand between any two periods. For $\sigma > 0$, the CRRA utility function is concave (with convex indifference curves) (see Figure 4.1), which means that the agent prefers a smooth path of consumption over time to an erratic one. The smaller σ is, the less concave is the utility function and the slower the marginal utility falls as consumption rises. For $\sigma = 0$, the utility function is linear in $\bar{x}_s$ and the indifference curves are straight lines; in this case the consumer is indifferent between a unit of a good tomorrow and a unit of a good today (or any two other points in time) (see Figure 4.2). For $\sigma < 0$, the indifference curves are concave and the agent would be willing to accept large swings in consumption from one period to the next[2]. The term $\bar{x}_s^{1-\sigma}$ is divided by $1 - \sigma$ to ensure that the marginal utility of consumption is always positive. In the special case of $\sigma \rightarrow 1$ the utility function simplifies to $\ln \bar{x}_s$.[3]

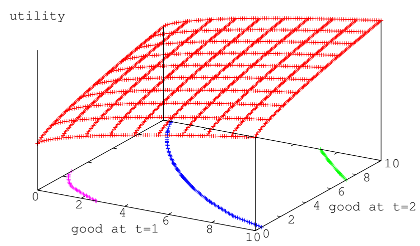

Figure 4.1: The indifference curves and the curve of the utility of the CRRA for $\sigma = 0.3$

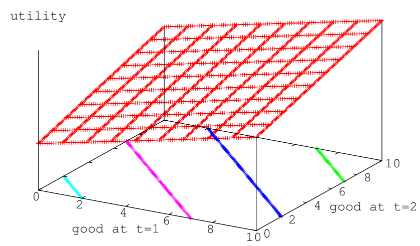

Figure 4.2: The indifference curves and the curve of the utility of the CRRA for $\sigma \rightarrow 0$

[2]This case is excluded in the function above because it is implausible.
[3]This result can be shown by using the rule of l'Hôpital.

Since the CRRA function is sometimes used in the non-stochastic case, some authors also call it the constant elasticity of substitution (CES) (Blanchard, Fischer (2000), p.43 [19])[4] or the constant intertemporal elasticity of substitution (CIES) (de la Croix, Michel (2002), p.5 [51]).

The name CRRA is derived from the fact that the agent, equipped with this function, demonstrates constant relative risk aversion when faced with uncertainty. In the German life cycle model, the agent is faced with an uncertain income (that is to say with different employment states) and, therefore, with uncertainty concerning his future consumption.

The function[5] which relates the goods at different points of time in case of uncertainty has the following form:

$$f(\bar{x}_{t,1}, \bar{x}_{t,2}, \ldots, \bar{x}_{t,n}, \bar{x}_{t+1,1}, \bar{x}_{t+1,2}, \ldots, \bar{x}_{t+1,n}, \bar{x}_{T,1}, \bar{x}_{T,2}, \ldots, \bar{x}_{T,n}) =$$

$$\sum_{z=1}^{n} \sum_{s=t}^{T} p(\bar{x}_{s,z}) \beta^{s-t} \frac{\bar{x}_{s,z}{}^{1-\sigma}}{1-\sigma}$$

$$f : \mathbb{R}_{+}^{(T-t+1)\times(n)} \to \mathbb{R},\ 0 < \beta < 1,\ \sigma \neq 1,\ \sigma > 0$$

where z is the index for the state of the world, n is the number of states and $p(\bar{x}_{s,z})$ is the probability of an income at time s in state z, which is another way of saying that this is the probability of buying a certain quantity of the bundle of goods x in state z at time s.[6] This means that the variables in the CRRA function are bundles of goods which are characterized by a time period and a state of the world.

The agent optimizes his expected utility over the possible incomes in the different states. Using different possible incomes in different states to depict uncertainty is deduced from the state preference approach. This approach to uncertainty was introduced by Kenneth J. Arrow (1953) [5] and Gerard Debreu (1959) [55]). In the textbook case of this approach: *"the same physical goods available at different states of the world are treated as distinct commodities"* (see Silberberg, Suen (2001), p.399 [158]) (these goods are also called state contingent commodities). The utility here depends on the income

[4]This designation should not be confused with the intratemporal elasticity of substitution.

[5]This function is also called the Von Neumann-Morgenstern utility function after John von Neumann and Oskar Morgenstern, who developed the expected utility theory (von Neumann, Morgenstern (1957) [173]).

[6]Provided, of course, that the prices of that bundle of goods stay the same.

through the possible consumption; a high income guarantees high consumption, whereas with a lower income only little consumption is possible.

If the indifference curves between two different states of the world are straight lines, the agent is indifferent to any two combinations of possible incomes as long as the expected income stays the same, meaning that a risky future with a high income in one state and a low income in the other state is the same to the agent as a future where the income is equally distributed between all the states. If the indifference curves are convex, however, (and the utility curve of the expected utility is concave) the agent prefers a certain income in both possible states to a risky high income in only one of the possible states. This preference of the agent is called risk aversion (see Figure 4.3).

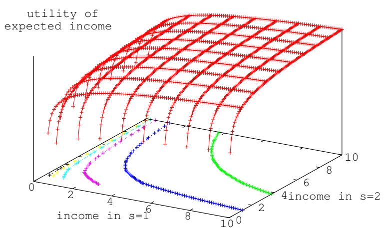

Figure 4.3: Indifference curves for two different states with $\sigma = 1.2$

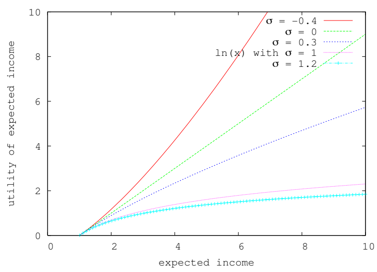

Figure 4.4: Different curves of the expected income for different σs

A measure of the degree of risk aversion is the degree of convexity of the indifference curves: the more convex the indifference curves, the more risk averse is the agent. The coefficient of relative risk aversion[7], here σ, determines the curvature of the indifference curves in the same way as explained above, meaning that the greater σ becomes, the more risk averse is the agent; for σ=0 the agent is risk neutral; and for $\sigma < 0$ he is risk loving (see Figure 4.4).

[7]This coefficient is also called the Arrow-Pratt coefficient of relative risk aversion after Kenneth Arrow and John W. Pratt (Arrow (1965) and Pratt (1964) [6], [145]) and is $-\frac{u''(x)x}{u'(x)}$. When the Von Neumann-Morgenstern utility function is a logarithm ($u = ln(x)$), the Arrow-Pratt-coefficient is the constant 1, because $u'(x) = \frac{1}{x}$ and $u''(x) = -\frac{1}{x^2}$ and $-\frac{u''(x)x}{u'(x)} = \frac{x^2}{x^2} = 1$.

Relative risk aversion expresses the risk aversion of an agent proportional to his already accumulated wealth. In the special case of the CRRA function, the relative risk aversion is a constant which is independent of the already accumulated wealth. The coefficient of relative risk aversion can also be chosen in such a way as to make the relative risk aversion increase or decrease with the income (it is a widespread assumption that the relative risk aversion of people decreases as income increases).

Another way of modeling risk aversion would be to use the coefficient of absolute risk aversion[8]. A constant coefficient of absolute risk aversion would mean that the aversion of an agent to having an extra risky income of 10 Euros does not change, no matter how high his income is (or, to put it differently, the risk premium the agent is willing to pay to avoid an uncertain income is the same for every original level of wealth). The equivalent of the CRRA function with a coefficient of risk aversion that is absolute and not relative is the CARA (constant absolute risk aversion) function[9].

The CRRA also possesses another important feature: it has a positive coefficient of relative prudence[10], which leads to a precautionary motive for saving, one of the most important features of life cycle models.

4.1.3 The CES function

The CES (constant elasticity of substitution) function relates different goods at one point in time with each other and determines the bundle of goods that maximize utility (which is then used as an argument in the CRRA).

$$\bar{x}(x_1, \ldots, x_n) = \left(\sum_{j=1}^{n} x_j^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}$$

$$\bar{x} : \mathbb{R}_+^n \to \mathbb{R},\ \theta > 1$$

where x_j is the amount of good j, n the number of consumption goods and θ the parameter which regulates how willing the agent is to substitute one good for another. In the standard reference of utility and production functions by Jae Wan Chung, θ is called the Allen-Uzawa crosspartial elasticity

[8] The coefficient of absolute risk aversion is: $-\frac{u''(x)}{u'(x)}$.

[9] $u(x) = -(\frac{1}{\sigma})e^{-\sigma x}$

[10] The coefficient of relative prudence is the third derivative of the utility function: $-\frac{u'''(x)}{u''(x)}$ and has been defined by Kimball (1990) [98].

of substitution (Chung (1994) [45]).

The CES function is based on the concept of the elasticity of substitution. This concept was originally introduced by Hicks (1932) [88] and Robinson (1933) [147] independently, both working in the area of production theory. It served the purpose of measuring the percentage change in factor proportions due to a change in the marginal rate of technical substitution, which can also be interpreted as a *"measure of ease with which the varying factors can be substituted for each other"* Hicks (1932) [87]. The CES production function, based on that concept, was then introduced almost 30 years later by Arrow, Chenery, Minhas and Solow (1961) [7] and generalized to the n-factor case by Uzawa (1962) [171] and Mcfadden (1963) [122].

The CES utility function is a direct analogue to the CES production function, with goods instead of factors. The elasticity of substitution measures here, at a constant utility, the ease with which two goods can be substituted for each other, either being substitutes or complements depending on the parameter θ. With this feature of the CES function, it is possible to change the "taste of variety" of the consumer in accordance with empirical findings. The elasticity of substitution is effectively a measure of the curvature of the indifference curve[11].

In the literature, the exponent of the CES function is often written in another, equivalent form, using for θ:

$\theta = \frac{1}{1+\tau}, \tau \neq 0, \tau > -1$

Solving for τ:

$\tau = (1-\theta)/\theta$

the CES utility function then has the following form:

$(\sum_{j=1}^{n} x_j^{-\tau})^{-1/\tau}$

The parameter τ (or θ) influences the indifference curves (which describe the relation of the goods) and the utility curve. If $\tau \rightarrow -1$ (and $\theta \rightarrow \infty$) the indifference curve is linear and the two goods are perfect substitutes (see Figure 4.5). If $\tau \rightarrow 0$ (and $\theta \rightarrow 1$), the CES utility function is a Cobb-Douglas

[11]For a more detailed analysis of the CES utility function, see Chung (1994), pp.57 [45].

function[12] (see Figure 4.6). If $\tau \rightarrow +\infty$ (and $\theta \rightarrow 0$) the two goods are perfect complements and the CES function converges to a Leontief function (see Figure 4.8) (at the limit, the indifference curve is given by $\min[q_1, q_2] =$ const.).

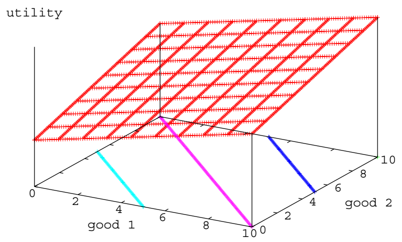

Figure 4.5: The indifference curves and the utility for τ = -1

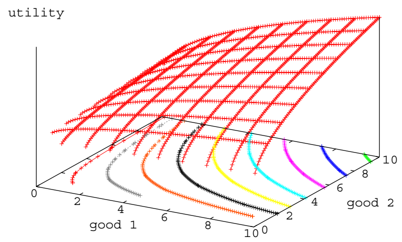

Figure 4.6: The indifference curves and the utility for $\tau \rightarrow 0$

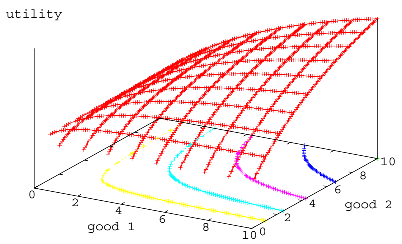

Figure 4.7: The indifference curves and the utility for $\tau = 1$

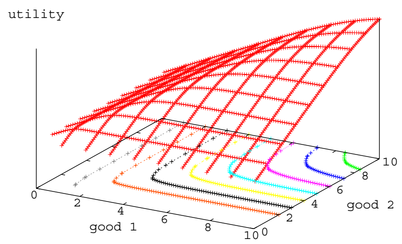

Figure 4.8: The indifference curves and the utility for $\tau = 5.9$

The relation of the intra- and intertemporal elasticity of substitution determines the behavior of the agent. If θ is larger than σ, the agent would rather substitute between goods in one period than between consumption bundles in different periods. If, on the other hand, $\sigma > \theta$, the agent will try to have a smooth consumption path and the diversity of goods in one period is not as important to him.

[12] For a proof of this see, for example, David de la Croix and Philippe Michel (2002), p.310 [51].

4.1.4 The S-branch function (Stone-Geary)

The S-branch utility is the CES function combined with the Stone-Geary function. It relates goods at a point of time (CES) and gives every good a utility factor and a subsistence level (Stone-Geary).

The S-Branch utility function is defined by:

$$\bar{x}(x_1, \ldots, x_n) = \left(\sum_{j=1}^{n} \alpha_j (x_j - \gamma_j)^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}$$

$$\bar{x} : \mathbb{R}_+^n \rightarrow \mathbb{R},\ 0 > \alpha_j > 1,\ \gamma_j > 0,\ x_j - \gamma_j > 0,\ \theta > 1$$

where γ_j is the subsistence level of good j as in the Stone-Geary function, θ is the elasticity of substitution between goods in one period, α_j is the utility factor of good j and n is the amount of goods. The whole expression is the bundle of goods then used in the CRRA.

Since the features of the CES utility function have been explained above, only the Stone-Geary function will be characterized below.

The Stone-Geary function is a further development of the Cobb-Douglas function. The latter has been widely used in production and consumption theory because of its minimal requirements of parameters and its rendition of real world facts (Cobb, Douglas (1928) [47]). With the Cobb-Douglas utility function, income elasticities always have the constant one; this means that when income increases, the demand for every good increases proportionally, not displaying the fact that consumers might, for example, buy more luxury goods and spend relatively less money on food when their income increases. This behavior of the Cobb-Douglas function has also been characterized as the *"absence of money illusion"* (see, for example, Chung (1994), p.25 [45]).

On the basis of a paper introducing "A Linear Demand System" (Klein and Rubin (1947-1948) [102]), Samuelson (1947-1948) [151] and two years later Geary (1949-1950) [76] developed a utility function which was later on empirically investigated by Stone and is now most commonly referred to as the Stone-Geary utility function (Neary (1947) [130]). Only then *"minimal changes were made needed to break this parameter rigidity by displacing the Cobb-Douglas indifference map away from the origin"* (Powell et al. (2002) [144]). This has been done by introducing a subsistence level for every good of the utility function. The Cobb-Douglas function is therefore a Stone-Geary function with subsistence levels of zero. The Stone-Geary utility function can now be interpreted as a function in which the arguments are not the actual quantities consumed but the extent to which these quantities exceed their respective subsistence levels.

In the following, the Stone-Geary function will be derived from the Cobb-Douglas function. The Cobb-Douglas utility function is:

$\prod_j (x_j)^{\alpha_j}, \sum_j \alpha_j = 1$

Subtracting the subsistence level γ_j from good j, the function becomes a Stone-Geary function:

$\prod_j (x_j - \gamma_j)^{\alpha_j}$

The Stone-Geary function in its commonly used form is obtained by taking the logarithm:

$\sum_{j=1}^{n} \alpha_j \log(x_j - \gamma_j), \sum_j \alpha_j = 1$

This function then combines with the CES function to form the S-Branch function.

The most important feature of the Stone-Geary function is that it is not necessarily homothetic[13], since this allows that changes in income not only change the amount of goods demanded (as in the Cobb-Douglas function), but also the type of good demanded. The Cobb-Douglas utility function is homothetic and in this form with $\sum_j \alpha_j = 1$ even homogenous of degree one:

$\Theta \prod_j q_j = \prod_j (\Theta q_j)^{\alpha_j}$

Subtracting subsistence levels from the goods forms the resulting Stone-Geary function to a non-homothetic function ($\sum_j \alpha_j = 1$ for that case too):

$\Theta \sum_j \log(q_j - \gamma_j) \neq \sum_j \alpha_j \log(\Theta q_j - \gamma_j)$

For the Cobb-Douglas function, the indifference curves in the commodity plane are magnified or reduced versions of each other (see Figure 4.9); with

[13]Homotheticity is a more general concept of homogeneity. A function is homothetic if it is a monotonic transformation of a homogenous function. A function is homogenous of degree k if scalar multiplication of each argument in the function changes the value of the function exactly by that scalar to the power of k: $u(\Theta q_1, \Theta q_2, \ldots . q_n) = \Theta^k u(q_1, q_2, \ldots, q_n)$.

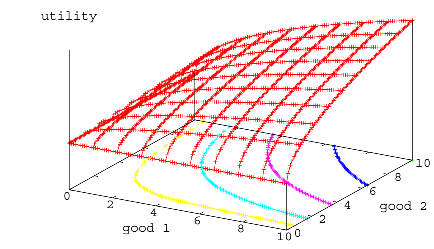

Figure 4.9: The Cobb-Douglas function (with no subsistence levels)

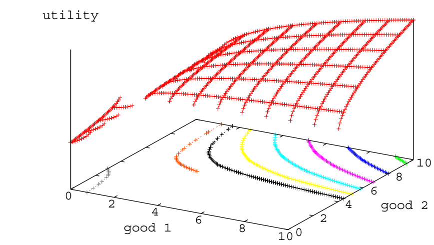

Figure 4.10: The Stone-Geary function with subsistence levels 1 and 9

a subsistence level subtracted from one or several of the goods, the Cobb-Douglas function becomes a Stone-Geary function, the preferences are no longer homothetic and the indifference curves are displaced from the origin (see Figure 4.10)[14].

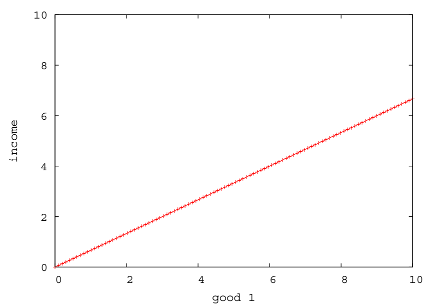

Figure 4.11: Engel curve without subsistence level

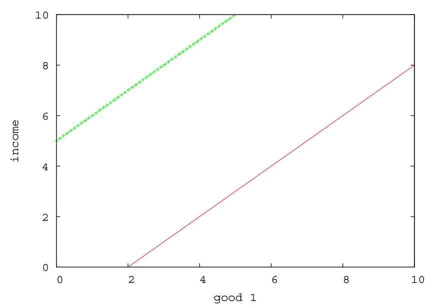

Figure 4.12: Engel curves with subsistence levels -5 and 2

The existence of a subsistence level not only influences the indifference curves, but also the Engel curves. An Engel curve is the demand for one of the goods as a function of income with all prices held constant[15]. For a homothetic Cobb-Douglas function the Engel curve is a straight line through the origin[16]

[14]The area below the subsistence levels in 4.10 is not defined but displayed in the graphic for a better understanding of the issue.

[15]The Engel curve was named after the German statistician Ernst Engel, who, studying working-class households, found that food expenditure increases with income but that the relative part of the income spent on food decreases (Engel (1857) [61], (1895) [62]).

[16]The demand function for a good q_1 derived from the Cobb-Douglas function is: $q_1 =$

(see Figure 4.11). For the Stone-Geary function the Engel curve is a straight line as well, but need not go through the origin[17]. The Engel curve for the Stone-Geary function can have a positive or a negative intercept, positive for normal goods (the household must buy a certain amount of that good before it can distribute its income according to its preferences) and negative for luxury goods (the household will not buy any of those good unless its income exceeds a certain threshold) (see Figure 4.12).

When using the Stone-Geary function instead of the Cobb-Douglas function, household expenditure shares become income dependent, but expenditure patterns still do not vary with income growth. The Stone-Geary function still has its shortcomings, the most important being the fact that marginal budget shares are constant. There have been many developments of demand systems which take this problem into consideration (see Deaton and Muellbauer (1980) [54], Boer and Jensen (2005) [50], Powell et al. (2002) [144]). For the purposes of the German life cycle model, however, the Stone-Geary function is more suitable and sufficient.

4.1.5 The composite function with bequest

The one detail lacking to complete the utility function for the German life cycle model is the bequest function. The agent in this model gains utility from leaving something for the next generation. He will not bequest all the goods x_j from his utility function, but only one. There will be an explicit description of all the arguments in the utility function in Section 6.1, but for now we will introduce only one of them, namely the one used in the bequest function: money. The bequest function is a CRRA function for one period[18]:

$b(M_{T+1}) = \frac{(\psi M_{T+1})^{1-\sigma}}{1-\sigma}$

$b : \mathbb{R}_+ \rightarrow \mathbb{R}$, $\sigma \neq 1$, $\sigma > 0$

where ψ is the utility factor for money and M is money.

All the functions explained above result in the following composite utility function when combined:

---

m/p_1, with m = income and p_1 = price of q_1.

[17]The demand function for the Stone-Geary is $q_1 = (m/p_1) - \gamma_1$.

[18]Similar bequest functions have been used by Christopher Carroll (2000) [35] and Marco Cagetti (2003) [30].

$U(x_{t,1}, x_{t,2}, \ldots, x_{t,n}, \ldots x_{t+1,1}, x_{t+1,2}, \ldots, x_{t+1,n}, \ldots, x_{T,1}, x_{T,2}, \ldots, x_{T,n};$
$M_t, M_{t+1}, \ldots, M_T, M_{T+1}) =$

$$\sum_{s=t}^{T} \beta^{s-t} \frac{\left(\left(\sum_{j=1}^{n} \alpha_j (x_{j,s}-\gamma_j)^{\theta-1/\theta} + \psi M_s^{\theta-1/\theta}\right)^{\theta/\theta-1}\right)^{1-\sigma}}{1-\sigma} + \frac{(\psi M_{T+1})^{1-\sigma}}{1-\sigma}$$

$$U : \mathbb{R}_+^{(T-t+1)\times n} \times \mathbb{R}_+ \rightarrow \mathbb{R}$$

$$\theta > 1, 0 < \beta < 1, \sigma \neq 1, \sigma > 0, 0 > \alpha_j > 1,\ \gamma_j > 0,\ x_{j,s} - \gamma_j > 0$$

where β is the discount factor, T the last time period, $x_{j,s}$ the good j at time s, α_j the utility factor of each good, γ_j the subsistence level of the single goods, θ the parameter of constant elasticity of substitution and σ the parameter of constant relative risk aversion (and the inverse of the parameter of constant intertemporal elasticity of substitution).

The composite utility function displays the same characteristics as the individual utility functions. The graphs below show the composite utility function for two goods at one point in time (see Figure 4.13) and for the case of one single good and two periods with a discount factor (see Figure 4.14).

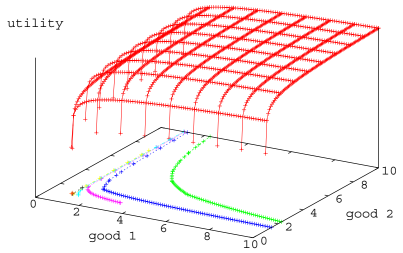

Figure 4.13: The composite utility function with two goods at one point in time

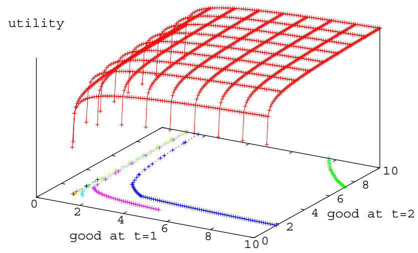

Figure 4.14: The composite utility function with a single good at two points in time

4.2 Relation to literature

Combining utility functions to achieve more realistic simulations is not exclusive to the German life cycle model. Other authors have combined the elements of different functions and implemented subsistence levels. In the following section, first, some models will be introduced which implemented several goods and subsistence levels and combine the CRRA function with the CES function and, second, the existing literature on life cycle models is investigated to find models which also use subsistence levels.

4.2.1 Other combined utility functions

The idea of combining the CRRA function with a utility function with subsistence levels was introduced by Kongsamut et al. (2001), p.15 [104] in their paper "Beyond Balanced Growth". Their utility function combines goods with subsistence levels through multiplication and uses the CRRA function with a discount factor for intertemporal utility. They use three different subsistence levels for three groups of goods; agricultural goods, manufacturing goods and services. For each of these groups, he uses a different subsistence level: a negative level for agricultural goods, which implies an income elasticity of less than one; a level of zero for the manufacturing goods, which implies an income elasticity of one; and a positive subsistence level for the services, which implies an income elasticity greater than one. If the income elasticity is smaller than zero (as in the case of the agricultural goods) the relative demand for this good decreases as the income increases; if the income elasticity is one, the relative demand stays the same; if the elasticity is greater than one, the relative demand decreases.

The utility function used by Kongsamut et al. has the following form:

$$\int_0^\infty e^{-\rho t} \frac{[(A_t-\bar{A})^\beta M^\gamma (S_t+\bar{S})^\theta]^{1-\sigma}-1}{1-\sigma} dt$$

$$\beta+\gamma+\theta=1,\ \sigma,\beta,\gamma,\rho,\bar{A},\bar{S}>0$$

where $\bar{A}$ is the level of subsistence consumption, $\bar{S}$ are domestic production services, β, γ and θ are the utility exponents and σ is the inverse elasticity of intertemporal substitution.

Following Kongsamut et al. Meckl has also used the Stone-Geary function to combine the goods in his model (Meckl (1999), p.5 [124]):

$$\sum_{j=1}^n \beta_j \ln(x_j-\gamma_j), \sum_{j=1}^n = 1,$$

where x_j is the good j, γ_j the subsistence level of good j, n the number of goods and β_j the utility factor of good j. Meckl has then implemented this expression in a CRRA function.

Combining a CRRA and a CES function in order to link different goods at one point of time as well as different periods with each other is something other economists have done as well. Ngai and Pissarides, two authors in the field of multi-sectoral modelling, have combined a CRRA and a CES function as described above (Ngai, A. Pissarides (2004), p.5 [131]). Ogaki and Reinhart (1998) [134] use a CRRA-CES function with two goods, durables and nondurables, in order to estimate the intertemporal elasticity of substitution. Smets and Wouters (2002) [161] also use a CRRA-CES function in their "open-economy model with sticky domestic and imported goods prices" and they also introduce uncertainty. However, they implement uncertainty about the survival of households instead of the income uncertainty used in the German life cycle model. None of the models above combine the CRRA-CES function with subsistence levels.

Introducing different goods into an intertemporal utility function can also be done in different ways. A way was tried early on is nesting a CES utility function for several goods inside an intertemporal CES function. This was done by Keller (1977) [96], for example, who implements three goods (necessities, durables and services) in his saving and consumption model. Another possibility was explored by Farr and Luengo-Prado (2001) [66], who use an individual CRRA function for each good they are modeling, in their case: durables and nondurables, and combine these CRRA functions in each period through addition.

4.2.2 Nonhomothetic utility functions in life cycle models

Nonhomothetic utility functions such as the one used in the German life cycle model are uncommon in life cycle models (and in consumption models in general), but some authors have used such utility functions to explain certain phenomena.

Ait-Sahalia et al. (2004) [1] try to find an explanation for the equity premium puzzle by using a luxury and a basic good in their utility function combined with a subsistence level. They use two CRRA functions, each for one good, and define utility as the sum of both:

The utility function the authors use in their model is the following:

$$u(C, L) = \frac{(C - a)^{1-\phi}}{1 - \phi} + \frac{(L + b)^{1-\psi}}{1 - \psi} \tag{4.1}$$

where C is the basic consumption good, a its subsistence level and L is the luxury good with the negative subsistence level b. ϕ and ψ are the parameters of the CRRA utility function. The additive concatenation of the two goods means that C and L are perfect substitutes at every point in time. This means that poor consumers will first satisfy their basic needs before they start to consume luxuries and that very rich consumers only need to spend a basic amount on necessities and then spend the rest on luxury goods. Since the authors are only interested in the behavior of the very rich, the existence of the subsistence level can be neglected and does not play any further role in the model. The authors find that risk aversion is much lower for luxury goods than for basic consumption and thus can solve the equity premium puzzle for the very rich.

A later paper by one of the authors of the article uses a nonhomothetic utility function with subsistence level to solve the contradiction between empirical facts and the results of the standard lifecycle model with regard to consumption growth (Yogo (2006) [183]). For US households the consumption level is positively related to the variance of consumption growth. The standard life cycle model with a CRRA utility function, as used by Carroll and many others, predicts that the consumption level is negatively related to the variance of subsequent consumption growth, meaning that people with low wealth are very responsive to income shocks.

To solve this puzzle, the author introduces a subsistence level and, with it a decreasing relative risk aversion function. With this feature, the households in his model are more risk averse when they have a small income (and will therefore save relatively more) and less risk averse when they have a high income, so that the consumption will vary more with the increasing amount of money available. The author uses a simple CRRA function with one good and a subsistence level:

$$u(C) = \frac{(C - X)^{1-\gamma}}{1 - \gamma} \tag{4.2}$$

where C is the consumption good, X the subsistence level and γ the CRRA parameter. The subsistence level is defined as 40% of the income for all income groups, which is, according to the author, the part of the income which is spent by households on subsistence goods.

An article by Wachter and Yogo (2007) [174] tries to solve the antagonism between empirical facts and the results of the standard life cycle models. In life cycle models, portfolio shares decline as the wealth of households rises, whereas according to empirical facts, households with higher wealth have more risky assets. To solve this antagonism, Wachter and Yogo introduce a nonhomothetic utility function, where an increase in total expenditure leads to a relative decrease in basic consumption and to a relative increase in the consumption of luxuries:

$$u(B,L) = \left(B^{1-\lambda} + \frac{\alpha(1-\lambda)}{1-\phi}L^{1-\phi}\right) \tag{4.3}$$

L and B are again the luxury and the basic consumption goods, α is the utility factor and λ and ϕ the curvature parameters. When $\lambda = \phi$ the utility function is a CES function:

$$u(B,L) = (B^{1-\lambda} + \alpha L^{1-\lambda})^{1/1-\lambda} \tag{4.4}$$

Because of the specific characteristics of the function, no subsistence level is needed.

Christopher Carroll, who authored one of the three standard SDP life cycle models described in Chapter 2, has also built an extended Life Cycle Model for the very rich, using a Stone-Geary function (Carroll (2000) [35]). The idea behind his model of "Capitalist spirit" is that the very rich obtain utility not only from consumption but also from owning wealth; therefore he has introduced wealth into his utility function:

$$u(C,W) = \frac{(C)^{1-\alpha}}{1-\alpha} + \frac{(W+\gamma)^{1-\beta}}{1-\beta} \tag{4.5}$$

where C is the consumption good, W is wealth, γ is the negative subsistence level (which means that wealth has a utility of its own only after a certain level of consumption is already achieved) and α and β are the parameters of the CRRA functions.

Although utility functions have been deployed to allow for several goods and subsistence levels, and others combine CRRA and CES functions, a combination of all three functions, the CRRA, the CES and the Stone-Geary, has not been conducted in any of the papers above (and to the knowledge of the author nowhere else). In the life cycle literature, the subsistence level has mainly been introduced to explain the behavior of the rich and the very rich. Only Yogo (2006) [183] uses the Stone-Geary function to solve a puzzle concerning the whole population (as is done in the German life cycle model), but he uses a relative subsistence level, which seems implausible, considering that rich people will spend proportionally much less on basic goods than poor people.

Chapter 5

The model and its implementation

At present, optimization problems posed by models similar to the German life cycle model cannot be solved analytically (see Carroll (1997) [34], Hubbard et al. (1994)[91]). For this reason, this model, like all other models in the life cycle model literature, are solved numerically. The optima found through a numerical method are not necessarily also the global optima. A sufficient condition to ensure that the optimum found is indeed global is the singularity of the optimum. For the German life cycle model this has been shown by proving that the goal function is concave, and since a critical point on a concave function is always a global maximum, the optimum found must be the global maximum (the complete proof can be found in Appendix C).

5.1 Description of the model

The following section describes the mathematical model. First the optimization problem under constraints is considered, then uncertainty is introduced and finally the optimization problem is transformed in preparation for the computational implementation.

5.1.1 The optimization problem under constraints

The agent considered here maximizes expected, discounted utility over his finite life time. He is faced with uncertainty concerning his future employment and the amount of money he can spend in the succeeding periods. The agent spends money in a way that guarantees that there is money left after his death; this behaviour is ensured by the bequest term.

The optimization problem for the agent is:

$$\max_{\{\bar{x}_s\}_t^T} E_t \left(\sum_{s=t}^{T} \beta^{s-t} u(\bar{x}_s, M_s) + \beta^{T-t+1} b(M_{T+1}) \right) \tag{5.1}$$

$0 < \beta < 1, s = t, \ldots, T$

where $\bar{x}_s$ is the vector of different types of goods at time s, M_s is the money at time s, β is the discount factor, t the starting time period and T the last period. The per period utility function u and the bequest function b have been described in Chapter 4.

The first transition constraint describes the savings (the amount of money left after consumption) for every period. M_t, the money in the starting period, is given exogenously.

$$A_s = M_s - \sum_{j=1}^{n} x_{j,s} \Pi_j \tag{5.2}$$

$\Pi_j > 0, j = 1, \ldots, n$

where Π_j is the price for good j.

The second transition constraint defines the money the agent can spend in the next period.

$$M_{s+1} = RA_s + Y_{s+1} \tag{5.3}$$

where R is the interest rate, A_s are the savings in period s and Y_s is the income in period s.

The control variable in the model is:

$x_{j,s}$ - purchased quantity of good j at time s

The state variables in the deterministic case are:

M_s - money in period s for consumption or saving

s - time period

The initial condition in the deterministic case is:

M_t - the money of the agent in the starting period t (given exogenously)

5.1.2 Introducing uncertainty

In this model, uncertainty is introduced with regard to income. The agent does not know if he is going to be employed or unemployed in the succeeding periods. This means that it is uncertain whether he will earn an income or receive the smaller government subsidy.

This uncertainty is treated as a Markov chain[1,2]. The Markov chain which describes the uncertainty in the German life cycle model consists of three objects: the transition matrix P, which contains the conditional probabilities of becoming employed or unemployed, the vector $\bar{p}_t$ with the probabilities for being unemployed and employed in the starting period t, and the income vector $\bar{Y}_{s+1}$ with different incomes for the two different states.

The conditional probabilities for being employed (e) and earning the employed income, and being unemployed (u) and earning the subsidy income

[1] Markov chains have the property that probabilities of future events only depend on the current state. For more explicit explanations of Markov chains, see Hillier and Liebermann (2005), pp.732 [89], Ljungqvuist and Sargent (2004), pp.29 [113], Simon and Blume (1994), pp.615 [159].

[2] The Russian mathematician Andrei Markov developed the fundamentals of the Markov chain theory at the beginning of last century, Markov(1906) [118].

are the following:

$$Prob\,\{e, s+1 \mid e, s\} = P_{ee}$$
$$Prob\,\{u, s+1 \mid e, s\} = P_{eu}$$
$$Prob\,\{e, s+1 \mid u, s\} = P_{ue}$$
$$Prob\,\{u, s+1 \mid u, s\} = P_{uu}$$

The stochastic matrix of the transition probabilities is:

$$\mathrm{P} = \begin{bmatrix} P_{ee} & P_{ue} \\ P_{eu} & P_{uu} \end{bmatrix}$$

The Matrix P has to satisfy the conditions:

$$P_{ee} + P_{ue} = 1$$
$$P_{eu} + P_{uu} = 1$$

The vector of the probabilities for being employed or unemployed in period s is:

$$\bar{p}_s = [p_{e,s}\; p_{u,s}]$$

These probabilities also satisfy:

$$p_{e,s} + p_{u,s} = 1$$

The transition from the probability vector of period s to period $s+1$ is:

$$\bar{p}_{s+1} = \bar{p}_s P$$

The income vector for $s+1$ with the incomes for being employed and unemployed is:

$$\bar{Y}_{s+1} = [Y_{e,s+1}\; Y_{u,s+1}]$$

with:

$Y_{u,s} = I$ if the agent is unemployed

$Y_{e,s} = B(\mu + 1)^s$ if the agent is employed

where I is the government subsidy which the agent receives during his working life when being unemployed, while the employed agent receives the basic income B whose increase with age is ensured through the growth factor μ.

The expected income of period $s + 1$ is then:

$E[Y_{s+1}] = \bar{p}_{s+1}\bar{Y}'_{s+1}$

where $\bar{Y}'_{s+1}$ is the transpose of the vector $_{s+1}$.

The stochastic state variable is:

$\bar{p}_s$ - vector with the probabilities of being employed and unemployed in period s

The exogenously given stochastic variables are:

$\bar{Y}_s$ - vector of possible incomes in period s
P - transition matrix

The stochastic initial condition is the status of being employed or unemployed in period t.

The life of the agent is divided into working and retirement years. For the retirement period, the agent has no income uncertainty, but instead receives a secure pension. The optimization problem for the time of retirement is therefore equivalent to the deterministic case described in the previous section.

5.1.3 Transforming the problem into a dynamic programming problem

The structure of the dynamic stochastic optimization problem can now be defined in terms of the objective function, the time horizon, the control and the state variables, and the transition functions. The objective function is the expected and discounted sum of the per period utilities. Time is discrete and $T > t$. The goods in the vector $\bar{x}_s$ are the control variables; the state variables are M_s, $\bar{p}_s$ and s, the amount of money the agent can spend in every period, the possible employment states which determine the income and the time period. The transition functions are $A_s = M_s - \sum_{j=1}^{n} x_{j,s}\Pi_j$, the amount of money the agent saves in every period, and $M_{s+1} = RA_s + \bar{p}_{s+1}\bar{Y}'_{s+1}$, the amount of money the agent has in period $s+1$.

The optimization problem is now rewritten in recursive form in order to apply dynamic programming[3]. Dynamic programming simplifies the optimization procedure by transforming a complex optimization problem into many (easier to solve) interrelated small ones.

To make things simpler, a new function $U(\bar{x}_s, M_s, s)$ is defined which splits the optimization problem into the per period utilities for $s < T$ and the per period utility for T with the bequest term.

$$U(\bar{x}_s, M_s, s) \begin{cases} u(\bar{x}_s, M_s) & \text{if } s < T \\ u(\bar{x}_T, M_T) + \beta b(M_{T+1}) & \text{otherwise} \end{cases}$$

The optimal value of the optimization problem is the maximum utility over T periods, which depends on the starting period, the starting probabilities for being employed and unemployed and the amount of money the agent has in the first period:

$$V_t(M_t, Y_t) = \max_{\{\bar{x}_t^T\}} E_t \left(\sum_{s=t}^{T} \beta^{s-t} U(\bar{x}_s, M_s, s) \right) \tag{5.4}$$

which, if $t < T$, can be transformed into:

$$V_t(M_t, Y_t) = \max_{\{\bar{x}_t^T\}} E_t \left(u(\bar{x}_t, M_t) + \beta \sum_{s=t+1}^{T} \beta^{s-(t+1)} U(\bar{x}_s, M_s, s) \right) \tag{5.5}$$

[3]Richard Bellman discovered the Bellmann equation, on which dynamic programming is based, in 1952 and also invented the term and the field of "dynamic programming" (Bellmann (1952) [17]). For a more thorough explanation of dynamic programming see: Hillier and Liebermann (2005), pp.440 [89], Bertsekas (1995) [18], Sargent (1987) [93], Ross (1983) [149], Stokey and Lucas (1989) [168].

$$V_t(M_t, Y_t) = \max_{\{\bar{x}_t\}} \left(u(\bar{x}_t, M_t) + \beta \max_{\{\bar{x}_{t+1}^T\}} E_{t+1} \sum_{s=t+1}^{T} \beta^{s-(t+1)} U(\bar{x}_s, M_s, s) \right) \quad (5.6)$$

The second term on the right hand side is now equivalent to the value function of period $t+1$, so the value function for t can now be written in recursive form:

$$V_t(M_t, Y_t) = \max_{\{\bar{x}_t\}} \left(U(\bar{x}_t, M_t, t) + \beta E_t V_{t+1}(M_{t+1}, Y_{t+1}) \right) \quad (5.7)$$

This equation is also called the Bellman equation. It formalizes the principle of optimality, which says that the optimal behavior of the agent is independent of decisions made in the previous periods.

The constraint to this problem is as shown in Section 5.1.1:

$M_{s+1} = R(M_s - \sum_{j=1}^{n} x_{j,s}\Pi_j) + Y_{s+1}$

5.2 Description of the program

With the introduction of dynamic programming and the availability of more computational power, it has become possible to solve complex dynamic stochastic optimization problems[4].

First the value function for the last period, T, must be specified. For the German life cycle model this is the utility function with the bequest term. Once the optimization for the final period is solved, the other periods can be solved recursively by repeatedly applying the Bellman equation.

The problem is solved in each period for a vector of possible money values[5]

[4]The model is written in Matlab, to see the entire code see Appendix B.

[5]These discrete elements are evenly spaced between the lowest possible income for each agent and a large amount of 300 000 Euros. This vector is divided between a part with a fine grid which goes up to 50 000, where the results are of economic interest, and a coarse grid up to the highest element, which is added to improve the extrapolation.

and for both states of employment. After execution the program returns two matrices, one containing the optimal utility values, the other the optimal consumption for each good. Both matrices have the dimensions of number of periods of time × number of states of employment × number of elements in the money vector.

The finite life of the agents is divided into working periods and retirement. The retired agents receive a fixed pension, the working agents are faced with income uncertainty. The probability of being employed is not stationary for every period, but depends on the current state of employment.

In the first step, the program initiates the two matrices with zeros. A Matlab optimization routine[6] then solves the utility function for the last period:

$$\max_{\bar{x}_T} E_T \left(\beta^T u(\bar{x}_T, M_s) + \beta^{T+1} b(M_{T+1})\right)$$

The function is optimized for each element of the money vector, but only for one state of employment since in the last period, as in all retirement periods, the income is fixed. The results of the optimization are then entered into the two matrices.

In the next step, the program maximizes the Bellmann equation starting with the next to last period:

$$\max_{\{\bar{x}_s\}} \left(U(\bar{x}_s, M_s, s) + \beta E_s V_{s+1}(M_{s+1}, Y_{s+1})\right)$$

To be able to solve this equation for period s, the program needs the utility for each available money value in period $s+1$. The available money consists of the income in period $s+1$ (which is either a labor income, an unemployment benefit or a pension) and the money left from period s. With the available money for $s+1$, the program can look up the utility values in the matrix. Since the elements in the money vector are discrete and the available money is not necessarily identical with the values in the money vector, a Matlab interpolation routine[7] is called to find intermediate values. For the agents with income uncertainty, who have two possible incomes in period $s+1$, the utility of period s is the expected utility based on these two incomes.

[6] The optimization routine is called "fmincon" and is able to optimize over different variables with constraints.

[7] After several interpolation routines were tested, cubicspline has been found to work best and is therefore used here.

The program now optimizes the Bellman equation and the results are entered element by element into the matrices. This procedure is done for every period until the first period of the agents working lives.

Equipped with the two matrices, it is now possible to simulate lifetime trajectories for the average agent of each educational qualification. The employment status in the first period is chosen randomly based on the unconditional probabilities of becoming unemployed, while in the other working periods the employment status is chosen based on the conditional probabilities. As starting money values, the agents receive the starting income according to their educational qualifications. In order to obtain representative statistics for the agent's optimal behavior, the same initial values are used to run a large number of simulations (50 000 is the number of runs carried out for the simulations shown later on), which are then averaged over.

Chapter 6

Adapting the model to Germany

In the following chapter, the arguments and parameters of the utility function are specified and the heterogeneous agents defined.

6.1 The arguments in the utility function

Many economic models, complete macroeconomic models as well as life cycle models, use only one generic good in their utility function. In the German life cycle model, several groups of goods are introduced. Two facts make this economic assumption plausible: first, some goods, such as food, clothes and housing, are essential, and money has to be spent on them before it can be spent on any other goods; second, people like diversity, a combination of goods being almost always preferred to only one kind of good. Additionally, the implemented subsistence levels make sense only if the agents have the choice between different goods.

The three groups of goods used in the German life cycle model are: necessities, which include food, drinks, tobacco, clothes and shoes; durables, including rent, water, electricity and heating, as well as furniture and devices for the household; and nondurables, including transportation, communication (such as telephones), leisure time, entertainment, culture, vacations, and restaurant visits.[1]

[1]The division of goods into these three groups follows the Federal Statistical Office of Germany's method of data collection.

6.1.1 Money in the utility function

In addition to the three groups of consumption goods introduced above, money will be implemented in the utility function. The role of money in utility functions and in economic models in general has been discussed controversially. The following section gives an overview of the most important contributions and developments in the literature and explains the decision to include money in the German life cycle model.

In the original general equilibrium model,[2] labor is traded directly against goods, and goods against goods. Even if money were introduced, the model is built in such a way that money would not enhance utility for the agents, which is unrealistic since people enjoy owning money. Walras had seen this problem and developed a money-demand theory as part of his theory of equilibrium (Walras (1874) [178]). Walras introduced money by making a distinction between the stock of money, which does not have a utility of its own, and the "services of availability" of money, which enter the utility function of the agent. However, money in this sense is not an inherent part of the model; neither does its implementation change the outcome of the model:

> *"Walras clearly made monetary phenomenon an optional add-on rather than an integral component of the mechanism of exchange."*
> (Ostroy (1987), p.516 [137])

But Walras was not the only one interested in implementing money in an economic model. In 1936, Keynes published "The General Theory of Employment, Interest and Money" [97], in which he developed his own theory about money: the liquidity preference theory. This theory is based on Keynes's belief that the future is uncertain and that agents have expectations. If an agent expects the interest rates of bonds to rise and the prices to fall, he will prefer abstaining from spending his money and wait until the interest rates have risen before buying bonds. But if the agent expects the interest rates to fall and the prices of bonds to rise, he will buy bonds now, hoping to sell his bonds profitably later. Keynes's idea has inspired many economists, but he himself has never incorporated liquidity preference into a model.

[2] Which was first developed by Walras (1874) [178] to model prices for the whole economy and then further developed by different economists to become the modern general equilibrium model of today.

After Walras, many economists tried to implement money in the general equilibrium model. Three approaches are used in modern economic theory today: the cash-in-advance model, the money-in-the-utility-function model, and the overlapping-generations model. The way money is implemented in these models is strongly dependent on how their authors value the three functions of money: medium of exchange, unit of account and storage of value.[3]

The cash-in-advance model, developed by Clower (1967) [46], emphasizes the importance of the role of money as a medium of exchange. The model introduces transaction costs by using the cash-in-advance constraint. This constraint requires every agent to have some money before he starts exchanging goods since the exchange is costly (hence cash-in-advance). However, the model does not display all the functions of money; the storage function of money is completely neglected, and the rate of interest does not influence the outcome of the model (Engels (2004), pp.34-35 [63], Söderlin (2003), p.15 [162]). A further problem is that the money the agents hold at the beginning of the period limits the amount of goods that can be consumed in that period:

> *" . . . , it* (the cash-in-advance model) *failed to provide a convincing explanation why people use money or what objects circulate as money; in short, it could not provide the microfoundations for money which it intended to do."* (Sriram (1999), pp.11 [166]).

The cash-in-advance model by Clower was a further development of the early models of money demand by Baumol (1952) [15] and Tobin (1956) [170] in which the transaction costs which give rise to the demand for money are introduced by making asset exchanges costly. In the Baumol/Tobin model there are two stores of value: money and an interest-paying alternative asset. The household has to choose between money, which yields no interest but is the only means of paying transaction costs, and assets which yield interest but cannot be used for payments.

Another model which emphasizes the importance of money as a medium of exchange is the model by Kyotaki and Wright (1989) [101], in which money is indirectly incorporated by making the direct barter of commodities costly.

[3] These are the definitions used in standard economic textbooks (e.g. Walsh (1998) [179], Heijdra and van der Ploeg (2002) [86], and Mankiw (2007) [116]).

After Walras' initial attempt at introducing money, important contributions to the money-in-utility-function models have been made by Patinkin (1965) [140] and Sidrauski (1967) [157]. This approach assumes that money yields utility because of its function as a medium of exchange. This model has been criticized because an agent who exchanges money against an equivalent amount of goods will still have the same utility, so money yields utility, even when it is not used to purchase goods (Walsh (1998), p.46 [179]). McCallum (1984) [121] defends the money-in-utility-function approach:

> *"It is my impression that most economists would not consider such a specification to be inconsistent with intrinsic uselessnes; even the store of value role* (and not only the medium of exchange role) *can, after all, be utility enhancing."* (McCallum (1984), p.151 [121])

A class of models called overlapping-generations (OLG) models emphasizes the store-of-value function of money. The first OLG model which considered money was created by Samuelson (1958) [152] and then further developed by Wallace (1980) [176], Wallace (1981) [177], Sargent and Wallace (1982) [154] and Kareken and Wallace (1981) [95], to name just a few authors.

A simple OLG has two generations living at the same time, with each agent living for two periods. The agents receive a certain endowment of consumption goods at birth, which are nondurable and cannot be stored for consumption in the next period. But the goods can be exchanged for money, which then can be kept until the consecutive period. In each period, the young exchange some of their goods for money with the older generation and then change the money back to goods when they are old themselves. This intergenerational trade benefits all agents concerned. The OLG only works with the assumption of an infinite time horizon. If the world would end one day and the agents would know about it, young agents would not trade their goods for money because they would be afraid that in the next period there would be no new young agents to give goods to in exchange for their money. But the OLG models have also been criticized:

> *" . . . they fail to explain the observed tendency for agents to hold money when other assets exist which are devoid of nominal risks but pay positive interest rates."* (McCallum (1984) [121])

One of the three approaches presented above has been implemented in the German life cycle model: money is part of the utility function since, for the consuming and saving agent in the German life cycle model, the most important aspect of the money theories presented above is that owning money already yields utility.

6.2 Structuring the agents

The goal of this section is to give distinct and realistic features to the heterogeneous agents representing the German population[4]. The data used originate almost entirely from the Federal Statistical Office of Germany: all figures used are from the year 2000 and after. Unfortunately, it is not possible to use data from a single year only, since some specialized data are collected only every few years. This is also the reason why no time series are employed; for such specialized purposes, for example finding conditional probabilities of becoming unemployed for different eductional groups, no data over several years are available.

Being faced with the risk of being or staying unemployed is at the core of the German life cycle model, and unemployment is therefore an important feature to be considered in the data. A striking fact about German unemployment is the uneven distribution among people with different education levels. To reproduce this fact, agents in this model will be distinguished according to their educational background. The qualities differing among agents in this model are their probabilities of becoming unemployed, the durations of their working lives, their wages, unemployment benefits, and pensions.

Germany has a population of 82.5 million, with 37.9 million of the inhabitants being employable and 44.6 million not (Statistisches Bundesamt (2005) [26]). The part of the population which is here called unemployable consists of children, students, housewives and pensioners. The division of the population into employable and unemployable follows the data of the Federal Statistical Office of Germany. This division indicates as employable those that work or are registered as looking for a job. People who would work if they found a job have not gone to the trouble of registering are counted as unemployable.

[4]The German life cycle model is not validated with time series, but only with benchmark data, which are given as starting points to the agents.

6.2.1 Dividing the agents into groups

The group of interest for the German life cycle model is the group of employable Germans. For the reasons discussed above, we will distinguish three subgroups. The employable without educational training (neither vocational training nor a university or comparable degree) form the low educational group. Employable persons with vocational training are called the intermediate educational group. Finally, persons with either a university degree or a degree from a university of applied science constitute the high educational group. The low educational group comprises 6.5 million people, the intermediate educational group 25.5 million people and the high eductional group 5.9 million people (Mikrozensus (2005) [26]).

The first distinction between the three subgroups discussed here is the length of their working lives. The time a person enters his working life depends on different things: the age at which the person finishes his educational training or respectively, stops going to school, how long his transition into working life takes, and if the person does military or community service. In the German life cycle model, the beginning of the working life will be defined as the moment the agent starts looking for work.

Since Germany is a federal state with different federal laws governing education, compulsory education ends at different ages in different states. In some states the students are allowed to leave school at the age of 15 while in others they have to attend some kind of school until they are 18; this includes schools providing vocatonal training. The members of the low educational group consist of those who have dropped out of school as early as possible, as well as others who went to school until they got a school-leaving certificate, which took some of them until the age of 20. The average entry age used in this model for the low educational group will therefore be 18. Determining the age of the intermediate educational group is straightforward. Most of the people in this group obtain a school leaving certificate in secondary school (Hauptschule/ Realschule) at an age of about 16, and some of them finish high school with a university-entrance diploma (Abitur) at about 19 (depending on the federal state). Vocational training in Germany takes about three years; the estimated average age for starting working life is therefore about 20. An average member from the high educational group will enter his working life at about 30 years of age. This late entrance into working life is due to the fact that studying in Germany takes relatively long, longer than in most other European countries.

While the entry age into working life is quite different among these groups, the retirement age is not. The average German goes into retirement at the age of 60 (Steinmann (2005) [167]). The statutory retirement age is 65, but since there has been high unemployment in the past decades, many people were encouraged to retire early.

The agents in this model have finite lives, which are divided into a working and a retirement period. The retirement period is the same for each agent: it starts with the average retirement age of 60 and ends at the average life expectancy of the German population of about 80 years.[5] The working period will be of different length for each group because of their different working entry ages. The three groups of agents are divided by their age in working and retired as shown in Table 6.1.

	low educational	**intermediate educational**	**high educational**
working	18-59	20-59	30-59
retired	60-79	60-79	60-79

Table 6.1: Age distribution for each group

Source: Own calculations

To find out how many members each of these age groups has, it is important to consider the age structure of the German population. The Federal Statistical Office of Germany has published data for the year 2005 (see Table 6.2).

There is no data available for the age structure in the different educational groups therefore the age structure data for the whole German population will be used to estimate the number of people in each age group and in each educational group. Within each educational group people of different ages are equally distributed (e.g., there are as many one-year-olds as there are 15-year-olds.). With this assumption, it is possible to derive the age structure in the employable population for each educational group. In order to find the number of old people in each educational group, we assume an evenly

[5]The statistical value of life expectancy today is 78.75 (Statistisches Bundesamt (2006) [29]), but for reasons of convenience, and because it is to be assumed that life expectancy will increase in the future, a value of 80 will be used in the model.

below 20	20-40	40-60	60-80	over 80
20 %	26 %	29.1 %	20.5 %	4.5 %

Table 6.2: Age distribution of the German population (%) (2005)

Source: Statistisches Bundesamt 2007 [27]

distributed age structure in the whole population (i.e., the number of old people relates to the total number of people in the educational group they come from as the number of old people relates to the total number of people in the rest of the population). Now it is possible to derive the number of retired people within each group. The age structure of the educational groups is shown in Table 6.3.

	low educational	intermediate educational	high educational
young	3.04	12.0	2.6
middle aged	3.46	13.5	3.3
old	2.85	11.31	3.78

Table 6.3: Number of people belonging to each group in millions (2005)

Source: Statistisches Bundesamt (2004) [25], (2005) [26] and own calculations

6.3 The parameters of the utility function

Estimations of elasticities and other important empirical parameters are a difficult and complex matter, which has triggered a lot of controversial discussion in the literature. These estimations are not a subject of this thesis, but they will be used as a tool to model German households as realistically as possible. As Mansur and Whalley (1984), p.119 [117] put it:

> *"...it does not seem reasonable to suggest modelers suspend their work in order to devote themselves to prior estimation of elasticities. The accommodation might be to clearly display the absence of estimates where this occurs and to limit modeling efforts where elasticities are the bottleneck."*

With this in mind, in the following section some of the relevant papers are reviewed and values for the German life cycle model determined.

6.3.1 The discount factor

There are several reasons for using a discount factor in an economic model. First, people are myopic, i.e., they value future utility less than present utility. Second, the future of the agents is uncertain: they do not know if they will still be alive in future periods to use their money (there is no stochastic lifetime in the model considered here, but this uncertainty is already integrated in the empirically estimated discount factor). And, third, inflation is another incentive for agents to devalue their future money (inflation is not a part of this model; we only mention it here for completeness). The discount factor is closely connected to the propensity to save: If people have a low discount factor, their savings will be low; with a high discount factor they will value the future highly and tend to save more.

Instead of a discount factor, some papers use the time preference rate, which is a simple transformation of the discount factor.

$$\text{discount factor} = 1/(1 + \text{time preference rate}).$$

The discount factor used in models with individual agents is not as disputed as the discount factor used to model entire societies (where a small difference determines the fate of entire future generations). In the literature, a discount factor for individual agents of 0.96 is standard, which implies a time preference rate of about 4 percent (Constantinides et al. (2002) [48], Carroll (2001) [37], Carroll (1997) [41]) (this is mostly done under the assumption of a 4 percent interest rate). Very similar to the standard value of 0.96 are Deaton (1991) [53] with a discount factor of 0.95 and Hubbard et al. (1994) [91] with a factor of 0.97. Smets and Wouters (2002) [161] and Kremer et al. (2003) [106], on the other hand, use estimates of 0.99 in their sophisticated macroeconomic models. Barsky et al. (1997) [13], who have actually estimated values of discount factors by interviewing people about their time preference, also found a discount factor of 0.99.

There is no consensus in the literature concerning the question of different discount factors in different population groups. Some authors find it consistent with empirical results that the discount factor is constant across rich and poor housholds (Ogaki and Atkeson (1997) [133]), whereas others, e.g. Cagetti (2003) [30], come to the conclusion that the discount factor increases with education and is low for lower educational groups. Cagetti estimates a value of around 0.98 for the educated and a value of 0.95 to 0.93 for the less educated.

For the purpose of this thesis the standard discount factor of 0.96 for all population groups seems to be a reasonable choice (which gives a time preference rate of 0.042).

6.3.2 The rate of risk aversion and the intertemporal elasticity of substitution

The rate of risk aversion σ used in the CRRA utility function[6] is the inverse of the intertemporal elasticity of substitution. This connection is convenient since it requires only one empirical estimation for the determination of two parameters, but it is also problematic since the behavior toward risk and the degree of substitution between consumption at different times are not primarily connected, and there has been no empirical evidence so far that they are (see Selden (1987) [156], Epstein and Zin (1991) [64], Barsky et al. (1997) [13], Blanchard, Fischer (2000), p.82 [19], Giuliano and Turnovsky (2003) [77] and Garcia et al. (2006) [75]). Epstein and Zin (1991) [64] think that the bad empirical performance of many models might be caused by the widespread use of the CRRA and the linkage of the rate of risk aversion with the intertemporal elasticity of substitution.

Problems occur when the estimated intertemporal elasticity of substitution is close to zero (which does not contradict any widely held beliefs about consumer behavior), since this would mean that the rate of risk aversion would go towards infinity (Hall (1988) [85]). The connection between risk aversion and intertemporal elasticity of substitution has been a much debated point in the literature, with as yet no final conclusion.

[6]For an explanation of the different utility functions see Chapter 4.

There have been approaches to disentangle risk aversion from the intertemporal elasticity of substitution (e.g., Hall (1985) [83], Zin (1987) [185], Attanasio and Weber (1989) [10], Weil (1990) [180] or Garcia et al. (2006) [75]), but none of them have been satisfactory enough to spread in the literature. Hall (1987) [84] observes:

> "*There does not seem to be a convenient class of utility functions in which the two parameters are clearly separated.*".

Due to the lack of a better alternative and following the literature, the CRRA function will be used in this thesis, and with it, the combination of the rate of relative risk aversion and intertemporal rate of substitution.

A widely used method of estimating the rate of risk aversion has been the linear approximation to dynamic Euler equations. Results from these estimations have been between zero and one. This has been criticized in many papers as being too low to be consistent with the widely accepted beliefs about risk aversion (Campell and Mankiw (1990) [31], Dynan (1993) [57], Kuehlwein (1991) [107], Attanasio and Weber (1995) [11]). Ludvigson and Paxson (2001) [114] and Carroll (2001) [36] have shown why these estimations are poor and often yield too low a value. Other authors have tried to use different estimation methods; Cagetti (2003) [30], for example, has matched the simulated median wealth profile with observed data. Other estimation methods have been used by Gourinchas and Parker (2002) [81] and Samwick (1998) [153]. These estimations of the rate of risk aversion have all been higher than the estimations carried out with the approximation of the Euler equations. In the case of Cagetti, estimates of the coefficient of risk aversion had a range of between three and four.

Another way of finding the parameter values necessary for the utility function is to interview people about their risk aversion. Applying this method, Barsky et al. (1997) [13] found rates of risk aversion ranging from 4.2 to 12.1. The phenomenon of exceptionally high risk aversion also arises when looking at the low demand for risky but profitable stocks relative to government bonds. This is known as the "equity premium puzzle".[7]

[7]See Mehra and Prescott (1985) [125] for the original article on this paradox, and Kocherlakota (1996) [103] for a good summary of the problem.

Finding a good estimation of the rate of risk aversion is even more difficult when one considers that there could be different rates for rich and poor households. If, for example, a poor household already has a consumption which is very close to the subsistence level, this household may not be willing to bear any risk and therefore has a very high rate of risk aversion (Ogaki and Zhang (2001) [136]), whereas the risk aversion for the rich household would be low.

Estimates of the intertemporal elasticity of substitution also have a high degree of variation. Robert E. Hall (1988) [85] estimates that the elasticity of substitution equals one over the rate of risk aversion for the US, using data from 1945 to 1959, and obtains values ranging from -0.03 to 0.98. The author comes to the conclusion that:

> "*A detailed study of data for the twentieth-century United States shows no strong evidence that the elasticity of intertemporal substitution is positive.*"

Campbell and Mankiw (1990) [31] conclude that the intertemporal elasticity is close to zero, Paul Beaudry and Eric van Wincoop (1996) [16] find an intertemporal elasticity for the US which is "*significantly different from 0 with point estimates close to 1*" and Epstein and Zin (1991) [64] find estimates from 0.05 to 1. In widely used DSGE (Dynamic Stochastic General Equilibrium) models, the estimated intertemporal elasticity of substitution is close to one. Smets and Wouters (2002) [160] use an estimated elasticity of substitution slightly below one, and Casares (2001) [44] estimates a value of 0.8, both for the Euro zone.

The studies mentioned above have been done with one-good models. This has led to criticism from some economists who argue that:

> "*. . . the magnitude of the intratemporal substitution directly affects the measure of the intertemporal substitution*" (Pakson (2003) [139]).

Having only one good in the model neglects the important fact that households which are faced with a shock, such as an increase in the interest rate, would substitute from durable to nondurable goods. Since the estimation of the elasticity is often based on the growth rate of nondurables, the intertemporal elasticity of substitution will be biased down. Such studies have been carried out by Ogaki and Reinhart (1998) [134], [135], who suggest values which cluster around 0.4; by Fauvel and Samson (1991) [67], who have found values between 1.5 and 2.3; and by the above-mentioned Pakson, who estimates an intertemporal rate of substitution of 0.381.

The already mentioned study carried out by Barsky et al. (1997) [13] yields an average elasticity of 0.18 and concludes that:

> *"Virtually no respondents have intertemporal substitution as elastic as that implied by log utility."*

(which means a rate of risk aversion and an intertemporal elasticity of one and is, for reasons of convenience, used in many models in the literature).

Another empirical finding is that the intertemporal elasticity of substitution is larger for rich households than for poor, an assumption which is made by many economists[8] and has been empirically shown by Ogaki and Atkeson (1997) [133] using household level data from India and the US.

After looking at these results, it seems plausible to use a rather high rate of risk aversion, since people have shown to be very risk averse, implying a low intertemporal elasticity of substitution, which is also well in the range of empirical findings. In conclusion, we will follow Carroll (2001) [37] and use a rate of risk aversion of 2, which gives an intertemporal elasticity of substitution of 0.5. Due to the lack of good data and following conventions, all households groups will be given the same two parameters in the default case.

6.3.3 The intratemporal elasticity of substitution

Since most of the economic models have only one composite good, the research effort put into the estimation of intratemporal rates of substitution

[8]See, for example, King and Rebelo (1995) [99], Easterly (1991) [60] or Ostry et al. (1995) [138], who use different elasticities for different groups of households

between different goods has been relatively small. However, the above-mentioned studies on intertemporal elasticity of substitution, which have found that introducing a durable good and a nondurable good changes the elasticity of substitution over time, have also analyzed the intratemporal elasticity of substitution. Fauvel and Samson (1991) [67], as well as Mankiw (1985) [115], have estimated a value of one for the intratemporal rate of substitution between durables and nondurables, Ogaki and Reinhart (1998) [134] have found a slightly lower value of 0.97, and Pakos (2003) [139] has found a value of 0.25.

The intratemporal elasticity of substitution has been estimated for other goods as well. Amano (1999) [3] has estimated the elasticity for domestic and imported goods and found a value of 1.09. There have also been estimations for smaller groups of goods, such as food, alcohol, clothes, etc. The most comprehensive study of such elasticities has been conducted by Houthakker and Taylor (1970) [90], who have, however, not considered the elasticity between these goods but only their own-price elasticity. A later study by Deaton (1981) [52] estimated the elasticities of eleven goods including houses, food, tobacco, clothes, fuel, and others and found that the only cross-elasticities which are significant are the ones between housing (which is very similar to durables) and food, and between housing and transport/-communication.

Three different kinds of goods will be used in the utility function of the German life cycle model: durables, nondurables, necessities, and money. The elasticity of substitution chosen will be the same between each of these three goods and between money. There is no empirical estimation which addresses this problem exactly. Therefore, an intratemporal elasticity of 1.1 is chosen, which is slightly higher than the value chosen by some studies dealing with durables.

This choice also seems valid when taking into consideration that it is plausible to have a rate of intratemporal substitution which is smaller than the rate of intertemporal substitution because this means that agents, faced with higher prices of durables, would rather substitute across time and buy the durables later than substitute durables for necessities in that period.

6.3.4 The utility factors

Utility factors[9] give an indication of how much relative weight an agent assigns to each group of goods. Therefore the utility factors can also be seen as the proportion of income an agent spends on each of these groups of goods. The Federal Statistical Office of Germany has collected data on how much all private households in Germany annually spend on necessities, durables and nondurables annually. In 2006, total consumption expenditures were 1 282.09 billion Euros; of these 257.08 billions were spent on necessities, 405.12 billion on durables and 619.89 billion on nondurables (Statistisches Bundesamt (2007) [28]). For the purposes of the German life cycle model it will be assumed that the way all households distribute their income over these goods is roughly the same as that of an average person living in Germany. Subsequently, the utility factors of the three goods are: 0.2 for necessities, 0.32 for durables and 0.48 for nondurables.

The agents in the model also need a utility factor for money. How much an average person in Germany values having liquid money (placed in a saving account from which it can be withdrawn at any time) compared to other, more risky investment strategies can be derived by considering the monetary aggregate compared to all capital assets possessed in Germany. The German Central Bank distinguishes between three definitions of monetary aggregates. The first is M1, which consists of liquid assets and demand deposits. The second is M2, which is composed of M1, time deposits with a duration of up to 2 years, and savings deposits with a period of notice of up to three months. Finally, there is M3, which comprises everything which can be defined as money and includes, in addition to M2, money market fonds and securities, obligations with a duration of up to 2 years, and stocks. In the year 2002, the monthly monetary aggregate of M1 in Germany added up to 0.5415 billion Euros, the monthly average of M2 amounted to 1 291.06 billion Euros, and the monthly average of M3 came to 1 394.46 billion Euros (Deutsche Bundesbank (2003) [56]).

The utility factor of money specifies the value a German agent designates to the amount of money he has in his bank account compared to all other investments. To put it differently, this is the relation of the monetary aggregate M1 (money in bank accounts) compared to all capital assets in Germany. Capital assets consist of estates, durable goods, insurance and everything considered to belong to M3. In Germany in 2002, the capital assets added up to 8 400 billion Euros (McKinsey (2003) [123]).

[9] They are also called share parameters.

Comparing the capital assets M3 to the monetary aggregate M1 of 6.498 billion Euros, the utility factor of money is 0.038. If the basis for the utility factor were M2, the factor would increase to 0.1536.

6.3.5 The subsistence levels

The subsistence levels in the utility function ensure that the basic needs of an agent are satisfied before he starts to distribute his income according to his preferences. The German Federal Ministry of Labour and Social Affairs has published a table defining the basic needs of a person living in Germany (Bundesministerium für Arbeit und Soziales (2005) [73]). These figures are calculated by taking a sample survey of income and expenditure of the lowest 10 percent of all income classes.[10] From 2006 on, the basic needs are estimated as shown in Table 6.4.

	necessities	**durables**	**nondurables**
in Euros	1 987.2	4 442.4	1 490.4
in %	25.1	56.1	18.8

Table 6.4: Basic needs of an average German per year, according to "Book Two of the Social Code", in Euros

Source: Bundesministerium für Arbeit und Soziales (2005) [73]

The rent is included in the durables and depends on the region the person lives in and on the number of members the family consists of. The German Federal Ministry of Transport, Building and Urban Affairs has published principles for accommodation allowances. These are used as a guideline for the allocation of money for rent under unemployment benefit II by most municipalities (Bundesministerium für Verkehr, Innovation und Technologie (2006) [74]). The average amount for rent and heating over all German regions for a single person adds up to 315 Euros monthly according to the guidelines. We will assume here, as well as in all following cases, a single agent

[10]These data are the basis for computing the unemployment benefit II (also called Hartz IV), which is the social welfare people receive when being unemployed for longer than one year.

with no children. The basic needs in Euros, as calculated by the German Federal Ministry of Labour and Social Affairs, will be used as subsistence levels in the German life cycle model.

6.4 The parameters of the dynamic constraint

6.4.1 The interest rate

The interest rate, in combination with the time preference rate, determines the saving rates of the agents. If the time preference rate equals the interest rate, people will distribute their wealth equally over their entire lives. If the time preference rate is higher than the interest rate, people will still save some money for later, but keep a relatively larger share for today and the immediately following time periods. A time preference rate lower than the interest rate means that the agents want to save a larger share of their wealth for the future than for today. Since the market interest rates influence people's saving behavior, they are used by many authors to determine the time preference rate in their model.[11]

Since the relation of the time preference rate and the interest rate influences the saving behavior of the agents, the interest rate has to be set accordingly.[12] The time preference rate in the German life cycle model is 0.042. As has been discussed in Chapter 2, an interest rate smaller than the time preference rate is the most plausible choice. Therefore, this model follows Hubbard et al. (1995) [92], who chose an interest rate of 3 percent. Similar models have chosen a interest rate of four percent (Deaton (1991) [53] or even zero percent (Carroll (1992) [32], (1997) [41], (1997a) [34]).

6.4.2 The probabilities of becoming unemployed

In 2004, an average of 4.1 million people in Germany were unemployed and 33.8 million were employed. The distribution over the three educational groups can be seen in Table 6.5.

From these data and from the figures of employable persons in each educational group it is possible to deduce the annual probabilities of becoming

[11]For a thorough discussion of the relation of the time preference rate and the interest rate see Chapter 2.

[12]The interest rate in the German life cycle model is nominal, and no inflation is implemented.

	low educational	**intermediate educational**	**high educational**
in millions	1.1	2.8	0.2
in %	27	68	5

Table 6.5: Unemployed in Germany distributed over the three groups (2004)

Own calculations, based on Mikrozensus (2004)

unemployed for the agents in each group. There is a probability of 16.9 percent of becoming unemployed for the members of the low educational group, for an agent from the intermediate educational group it is 11 percent and for the high educational group it is only 3.4 percent.

The time spent in unemployment is unevenly distributed. Some people find a new job in less than a year, the so called short-term unemployed, while others stay unemployed for far longer (the long-term unemployed). Of unemployed people in Germany in 2004, 37.5 percent stayed in this state for one year or longer, and 62.5 percent found a new job within one year (Sozialpolitik-Aktuell (2005) [163]). The relation between long-term unemployed and short-term unemployed is assumed to be the same in each of the educational groups. The distribution of employment and the two kinds of unemployment in the different groups are shown in Table 6.6.

One important piece of information about the employment situation in Germany is the flow from the unemployed to the employed and vice versa. In the year 2004, 8.03 million people left the group of unemployed, but only 3.106 million of these became employed. The others went into retirement, gave up looking for a job, etc. In the same period, 8.179 million became unemployed, but only 4.219 of these had been unemployed before (Bundesagentur für Arbeit (2005) [72]). For the purpose of finding the conditional probabilities of becoming unemployed when currently employed, and of becoming employed when currently unemployed, the direct transitions from being employed to unemployed and vice versa as well as the number of unemployed and employed are needed. Since 33.8 million people in Germany were employed in 2004 and 4.2 million left this group to become unemployed, the conditional probability of becoming unemployed when employed is 12.42. The same

	low educational	intermediate educational	high educational
unemployed			
one year and less	0.69	1.75	0.125
over one year	0.41	1.05	0.075
employed	5.4	22.7	5.7

Table 6.6: Unemployed and employed in each group in millions of people (2004)

Own calculations based on Sozialpolitik-Aktuell (2005) [163]

calculation also works for the group of the unemployed, which consisted of 4.1 million people in 2004, of which 3.1 million people found a job. The conditional probability of becoming employed when unemployed is therefore 75.8 percent. Unfortunately, transition data from the unemployed to the employed and vice versa is not available for the different educational groups, so a different way of calculating their conditional probabilities is used and explained below.

Finding the conditional probabilities for becoming unemployed has also been a subject in the literature. Bachmann (2005) [12] calculates the conditional probabilities by using the monthly flows between the states of employment and unemployment in Western Germany from 1980 to 2000. He finds that becoming unemployed when employed has a probability of 0.63 percent and becoming employed when unemployed has a probability of 7.1 percent. For a time period from 1983 to 1994 for Western Germany, Schmidt (1999) [155], also using monthly transition data, calculates a probability of becoming unemployed in the next month when having been employed of 0.39%, and of staying unemployed of 9.36%. The differences from the probabilities found above can be understood when looking at the time periods. The probabilities of finding or losing a job are much higher in the course of one year than in the course of one month. Schmidt (1999) [155] also looks at a more detailed data set and finds numbers for people with a low, intermediate and high level of education (see Table 6.7).

A similar analysis for the more dynamic employment market in the US gives a conditional monthly probability of 1.3 percent of becoming unemployed when

	low educational	intermediate educational	high educational
Prob (e|u)	6.616	8.66	11.37
Prob (u|e)	0.525	0.395	0.265

Table 6.7: Conditional monthly probabilities of becoming unemployed, 1983-1994, West Germany (%)

u: unemployed, e: employed
Source: Schmidt (1999)[155]

employed, and of 28.3 percent of becoming employed when unemployed (Fallick and Fleischmann (2004) [65]).

Unfortunately, there is no detailed data set for the three educational groups as presented by Schmidt (1999) [155] for yearly transitions in Germany. However, for the German life cycle model it will be assumed that the way the annual conditional transition probabilities of the three groups relate to each other is the same as in the paper published by Schmidt. To find the conditional probabilities of finding or losing a job in one year for each group, the conditional probabilities for all employed and unemployed persons found by Schmidt are taken as a point of reference and the ratios between the probabilities for each group and the reference probabilities are calculated. These ratios can now be multiplied by the annual conditional probabilities found from the in- and outflows in the year 2004, which gives us the conditional probabilities for each group, as shown in Table 6.8. These are the probabilities which will be used in the German life cycle model.

6.4.3 Income growth

Wages grow with age and as a consequence of promotion, but at a certain point in life income stagnates. For different educational groups, income grows at different rates. In most life cycle models, the assumed growth of income for all agents lies between 2 and 3 percent, which is consistent with the growth of the GDP of most industrialized countries. Carroll (1997) [34] differentiates the income growth in his model according to his three income groups, and calibrates these growth rates using US income data. Income growth in the

	low educational	intermediate educational	high educational
Prob (u|u)	50.03	30.1	06.04
Prob (e|u)	49.97	69.9	93.96
Prob (u|e)	16.8	12.64	8.48
Prob (e|e)	83.2	87.36	91.52

Table 6.8: Conditional yearly probabilities of becoming unemployed and employed (%)

u: unemployed, e: employed
Source: Own calculations, based on Schmidt (1999)[155], and Bundesagentur für Arbeit (2005) [72]

German life cycle model follows these parameters. A member of the high educational group will have an increase in wages through age and promotion of 3 percent per year, a member of the intermediate educational group will have an increase of 2.5 percent per year, and for the low educational group the wage grows at the rate of 2 percent per year. Income will stop growing ten years before retirement, when the agents are fifty.

These growth rates are higher than in Hubbard et al. (1995) [92], for example. The reason for this is that they also incorporate the generally rising level of productivity in the economy.

6.5 The initial conditions

6.5.1 Wages

The statistical wage data are given as gross, but we are interested in the wage the agent can actually use to consume and to save. For this we need the wage after deductions (which include the payments for social security, health insurance and the government-based pension scheme). How large the deductions are depends on different parameters: marital status, number of children, membership in the Christian church,[13] etc. We will assume here a very simple agent who is unmarried and has no children, but pays church taxes. Since all agents have these features in common, this specification does not influence the saving behavior among the different agents. Table 6.9 displays the annual wages before and after the deductions.

	low educational	**intermediate educational**	**high educational**
gross income	25 200	31 320	52 356
net income	15 921	18 721	27 824

Table 6.9: Average yearly wage in Euros (2002)

Source: Weißhuhn and Rövekamp (2004) [181]
with data from the German Socio-Economic Panel

6.5.2 Starting wages

The German life cycle model requires the wage with which the agent starts his working life. A simple way of providing this is to take the average wage for each agent, assume that he will earn this wage in the middle of his life, and then decrease this salary by the wage growth factor for the number of years he has worked until then. This gives us starting salaries as shown in Table 6.10.

6.5.3 Unemployment benefit

The agents optimize not only over the time they are working but also over their times of unemployment. In Germany, a two-level social security net exists. In the first year of involuntary unemployment, an unemployment compensation of 60 percent of the former salary is paid (for a single person without children): this is the unemployment benefit I. After that one year, everybody receives the same amount of unemployment compensation, the so-called unemployment benefit II (also called Hartz IV), which is independent

[13]Since the German government collects taxes only for the Christian churches, this is the only relevant religion for computing wages.

	low educational	intermediate educational	high educational
gross income	16 626	19 114	33 605
net income	10 504	11 425	17 859

Table 6.10: Average yearly starting wage in Euros (2002)

Source: Own calculations based on table 6.9

of any former wages. This second form of compensation, however, is only paid when all savings are used up. The amount for a single German without children, which is supposed to enable him to pay all living expenses except rent and heating, amounts to 345 Euros per month. For rent and heating, the German state pays a rate depending on the region. As explained in Section 6.3.5, the average amount of rent and heating for all German regions adds up to 315 Euros for a single person. The first period of unemployment benefit is neglected in the German life cycle model, meaning that the agent optimizes knowing that in case of unemployment, he will receive unemployment benefit II of 660 Euros a month, which amounts to 7 920 Euros a year, for all the following years.

6.5.4 Pensions

We will assume that all agents in our scheme will receive a statutory pension. The statutory pension is calculated by multiplying pension points obtained during the working life of an agent by an entrance factor, a pension factor and a pension value.

The pension points are calculated by dividing the yearly gross wage by an average wage for all pension insurance payers. The predicted yearly average wage for 2007 is 29 488 Euros. For the years a person has obtained unemployment benefit I, he receives pension points amounting to 80 percent of the points he would receive if employed. For each year of unemployment benefit II, he is credited 0.25 points, i. e. 25 percent of the points an average insurance payer receives per year.

The entrance factor is determined by the age at which the insured person starts to receive his pension. A person who starts his pension at the pre-

	low educational	intermediate educational	high educational
working	0.85	1.06	1.77
unemployment benefit I	0.68	0.85	1.42
unemployment benefit II	0.25	0.25	0.25

Table 6.11: Average pension points per year (2007)

Source: Own calculations

scribed age of 65 is assigned a factor of one. Persons who retire later obtain a higher factor, while retiring earlier leads to a factor lower than one. For each month an insured person starts in his pension life earlier, the factor is decreased by 0.3 percent. For retiring at 60, like the agents in the German life cycle model, the entrance factor is 0.82.

The pension factor is determined by the family status of the insured. For persons who are not married and have no children, like the agents in the German life cycle model, the factor is one. The pension value is determined by the German pensions regulatory authority, and amounts to 24.55 Euros for the year 2007. The average pension points for working people and the agents receiving unemployment benefit I and II are displayed in Table 6.11.

To find the average pension for each agent, it is necessary to know the average time each agent is employed or unemployed. Using the probabilities of becoming unemployed from section 6.4.2 and the number of years each agent works, we find that on average an agent of the low educational group is unemployed for 7.1 years, an agent of the intermediate educational group for 4.4 years, and an agent of the high educational group is without work for 1.02 years.

To derive the average time each agent received unemployment benefit I or unemployment benefit II, it is necessary to know how many persons are long-term and how many are short-term unemployed. This information can also be obtained from the data presented in Section 6.4.2. In 2005, 37.5 percent of the unemployed in Germany were categorized as long-term unemployed

	low educational	intermediate educational	high educational
unemployment benefit I	4.46	2.75	0.64
unemployment benefit II	2.68	1.65	0.38

Table 6.12: Average duration of receiving unemployment benefit I and II in years (2005)

Source: Sozialpolitik-Aktuell (2005) [163] and own calculations

(longer than a year without interruption), while the remaining 62.5 percent were classified as short-term unemployed (unemployed for less than a year) (Sozialpolitik-Aktuell (2005) [163]). Assuming that the short-term and long-term unemployed are equally distributed over all educational groups, it is possible to calculate how long a member of each group receives the different unemployment benefits (see Table 6.12).

The average pension for the agents of each group can be calculated by multiplying the pension points for each state of unemployment and employment by the average time the agents spend in these states and then multiplying the resulting sum by the entrance factor (0.82), the pension factor (1) and the pension value (24.55 Euros). A member of the low educational group will thus receive a yearly pension of 8 053 Euros, a member of the intermediate educational group a pension of 9 780 Euros and a member of the high educational group 12 634 Euros. All of the agents in the German life cycle model will also have the money they have saved during their working life available for their retirement.

6.6 Summary of all parameters

In the following, the parameters and figures which have been introduced in the text are summarized in two tables. First, all the parameters needed for the utility function are listed (see Table 6.13), and then the figures which specify the agents in the German life cycle model (Table 6.14).

Parameters	Default
Discount factor (β)	0.96
Rate of risk aversion (σ)	2
Intratemporal rate of substitution (θ)	1.1
Interest rate (R)	0.04
Utility factor necessities (α_1)	0.2
Utility factor nondurables (α_2)	0.48
Utility factor durables (α_3)	0.32
Utility factor money (ψ)	0.038
Subsistence level necessities (γ_1)	1987.2
Subsistence level nondurables (γ_2)	1490.4
Subsistence level durables (γ_3)	4442.4

Table 6.13: Parameters used in the utility function and in the dynamic constraints

Parameters	**low educational**	**intermediate educational**	**high educational**
Starting Income (Euros)	10 504	11 425	17 859
Unemployment benefit (Euros)	7 920	7 920	7 920
Pension (Euros)	8 053	9 780	12 634
Growth rate of income	0.02	0.025	0.03
Working years	42	40	30
Years in pension	20	20	20
Prob(u|u)	0.5003	0.301	0.0604
Prob(e|u)	0.4997	0.699	0.9396
Prob(u|e)	0.168	0.1264	0.0848
Prob(e|e)	0.832	0.8736	0.9152

Table 6.14: Parameters and initial conditions which specify the agents in the model

The probabilities, the growth rate and the amount of money the agents receive are per year, u: unemployed, e: employed

Chapter 7

Simulation results and interpretation

In this chapter, simulations of the economic behavior of the single agents of the German life cycle model are presented.[1] In the first section, simulations for different versions of the model are shown in order to explore how features such as uncertainty, bequest, several goods and subsistence levels influence the saving and consumption behavior of the agents. In the second section, the model output of parameter variations is displayed. These figures help to understand how robust the behavior of a single agent towards changes in parameter variations is and how variations in certain parameters change the model output. In the third section, simulations for the three different agents are illustrated; these results depict the differences in the saving and consumption behavior of the agents and their preferences for different kinds of goods. In the last section, the results are compared to stylized facts and empirical German saving data in order to see how well the model has performed in explaining these facts and reproducing data. And, finally, the model results of the German life cycle model are compared to the current life cycle model literature.

The simulated agents displayed on the following pages represent microeconomic units.[2] In the following two sections, all parameters are set to default values. The default parameters are: the rate of intratemporal substitution

[1]For the direct results of the optimization i. e. the policy function, see Appendix A, and for a description of how a single agent is generated see Chapter 5.2 and the commented code in Appendix B.

[2]The German life cycle model can also be used to simulate aggregate agents. See Appendix D for the complete proof that the requirements necessary for aggregation are fulfilled.

$\theta = 1.1$, the parameter of risk aversion $\sigma = 2$, the discount factor $\beta = 0.69$ and the interest rate $R = 0.03$.[3] The agent displayed in the next two sections is the one with the intermediate education (university degree of applied science), which ensures that his income growth is $\mu = 0.025$, and that the agent works for forty years and spends twenty years in retirement.

7.1 Presentation of the results

7.1.1 Implementing different features of the model

In this section, a very simplified model without uncertainty and with only a simple utility function is extended by uncertainty, bequest, several goods and subsistence levels in order to demonstrate how these features, taken individually, influence the behavior of an agent.

The first two figures, Figures 7.1 and 7.2, show wealth accumulation and consumption behavior of an agent using the basic utility function in a deterministic setting without uncertainty.

The behavior of the agent is similar to that of an agent in the classical life cycle model. The agent's consumption is approximately balanced over his life; he saves money during working years and spends it during retirement (Figure 7.1). Savings and dissavings are almost symmetrically centered around the start of retirement at age sixty. The agent starts to save for retirement at an age of forty-two years and exhausts all his savings by the time he dies at age 80. The agent's consumption closely follows his income until he starts to save for retirement (Figure 7.2). It departs from the income at forty and then slowly decreases until the end of his life. The reason the agent in the deterministic case can afford to start saving relatively late lies not only in the absence of uncertainty but is also based on the fact that the agent has little income at the beginning but knows that his income is going to grow until he is fifty.

In Figures 7.3 and 7.4, the agent is faced with the risk of unemployment. The agent reacts to this uncertainty by starting to save at the beginning of his working life. This kind of saving is called buffer-stock saving and is an inherent characteristic of stochastic, dynamic life cycle models. Just as in

[3] For a description of all parameters and an explanation of why they are chosen in such a way see Chapter 6.

the graphs with no uncertainty, the agent also starts to save for retirement around the age of 42. That is the reason why the consumption diverges from the income. The reason the agents in the deterministic case nevertheless have a higher saving peak than the agents that are faced with uncertainty lies in the way income is modeled. While the agent facing uncertainty also has to accept periods of unemployment, and therefore less money, the agent who is not faced with uncertainty always receives the full income.

In Figure 7.5, the consumption and the accumulated wealth for the utility function with bequest in the deterministic case are shown. The agent reacts to the implementation of the bequest function by not exhausting all his wealth by the end of his life but by leaving some money for his heirs instead.

In Figure 7.6, an agent's consumption of all three goods is shown, still in the deterministic case without bequest. The agent distributes his money among the three different types of goods according to the utility factors.[4] For nondurables, the share parameter is largest and for necessities smallest. No subsistence level is implemented as yet.

To demonstrate the impact of the subsistence levels, a simulation is shown in Figure 7.7, in which the agent receives only the minimum unemployment benefit over his entire life. In this case, the agent will always spend his money exactly according to the subsistence levels. As soon as his income increases, he will change his consumption pattern towards the share parameters.

When subsistence levels are implemented (see Figure 7.8), the distribution of money over the different goods consumed changes. Most money is now spent on the good with the highest subsistence level (durables) instead of nondurables, but the low spending on necessities remains. How much the subsistence levels influence the demand for the single types of goods depends on the accumulated wealth and the income of the agent. When the income is low (as it is at the beginning of the agent's working life), the good with the highest subsistence level (durables) is consumed almost four times as much as necessities or nondurables. As the income of the agent increases, he spends an increasing amount of money on nondurables until he reaches a point at which the consumption of nondurables lies barely below the consumption of durables. This is a very plausible reaction. When income is low, basic needs like housing and heating (durables) have to be satisfied first before the agent can spend his money on vacations, restaurant visits and the

[4]For an explanation see Chapter 4 and Section 6.3.4.

like (nondurables); the more money an agent has, the more he can spend proportionally on "luxuries". The reason that in our particular case the consumption of necessities hardly changes when introducing the subsistence level is that the agent with the intermediate level of education has enough money to satisfy his need for necessities. If the agent were very rich, the introduction of subsistence levels would not change his consumption behavior at all.

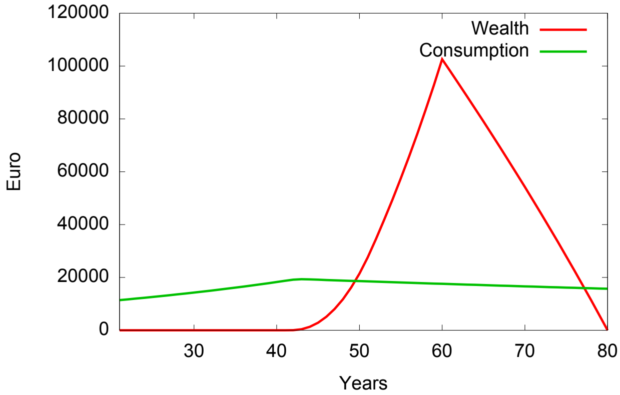

Figure 7.1: The basic model (deterministic case): Wealth and consumption

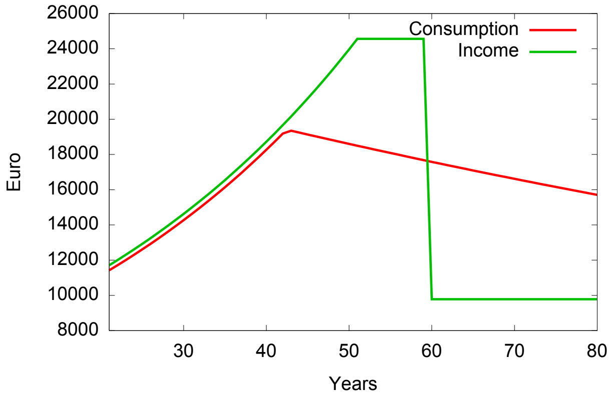

Figure 7.2: The basic model (deterministic case): Consumption and income

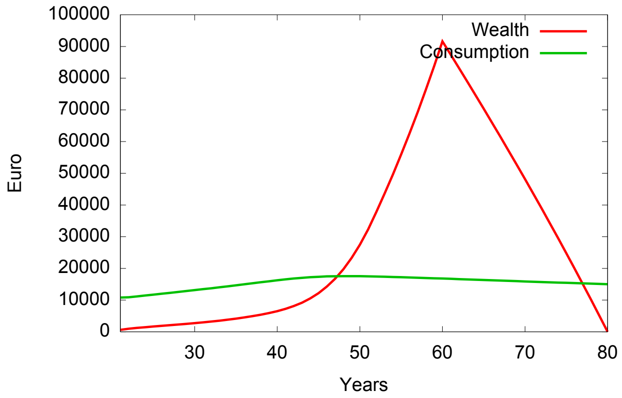

Figure 7.3: The basic model and uncertainty: Wealth and consumption

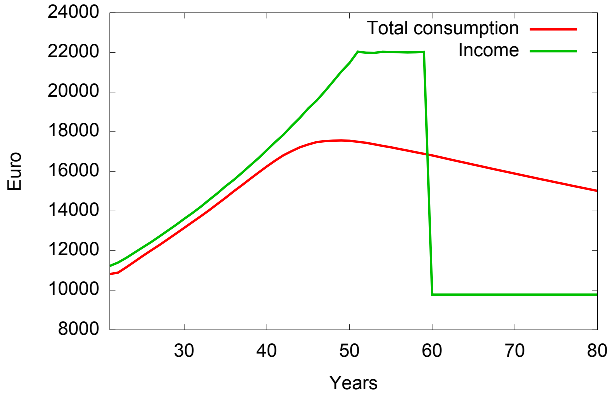

Figure 7.4: The basic model and uncertainty: Consumption and income

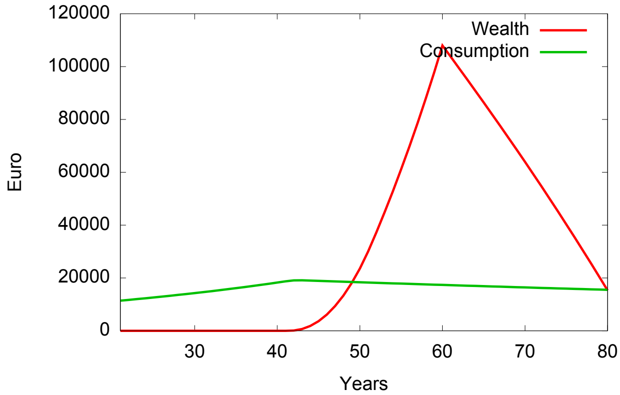

Figure 7.5: The basic model with bequest: Wealth and consumption

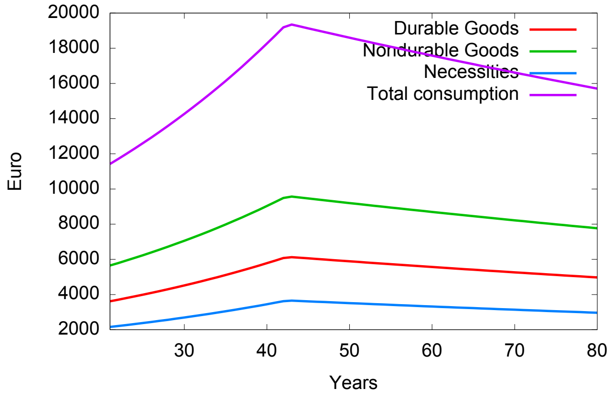

Figure 7.6: The basic model with three goods

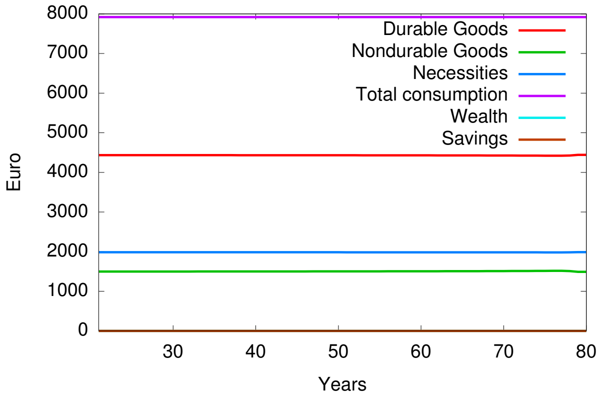

Figure 7.7: The basic model with three goods, subsistence levels and unemployment benefit

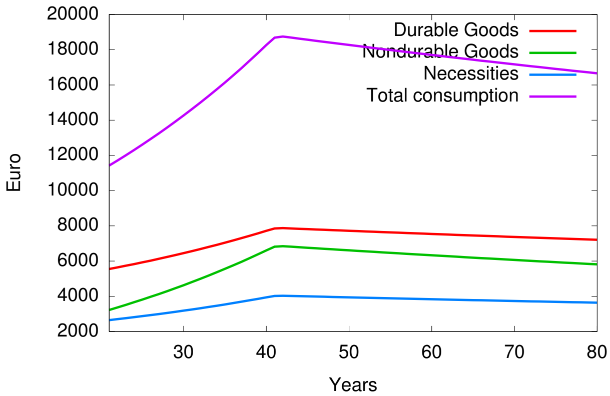

Figure 7.8: The basic model with three goods and subsistence levels

7.1.2 Variations in the input

In this section, one parameter is varied at a time, while the others are kept constant. These simulations are of interest since they show the robustness of the model outputs and how and to what extent the parameters influence the behavior of the average agent.

In Figures 7.9 and 7.10, the consumption and the accumulation of wealth are presented for different discount factors. A discount factor of one (equivalent to a time preference rate of zero) means that the agent values the future as much as the present. When agents are patient (i.e. their discount factor is close to 1), they accumulate more money over their life than their low discount factor counterparts (Figure 7.10). Impatient agents tend to consume more early in their lives, particularly between ages 40 and 60, and must consequently consume less towards the end of their lives (Figure 7.9).

The lower the discount factor, the higher is the agent's consumption at the height of his income and the less wealth is available to him at the end of his life. The greedy behavior in the middle of an agent's life influences retirement saving. An agent with a low discount factor will start accumulating wealth for retirement later in his life and will therefore have to get along with less money. The bequest behavior, however, is not influenced by the discount factor. All agents leave the same amount of money behind. Although hardly visible, the buffer-stock saving in the first twenty years of an agent's life is also influenced by the discount factor. An agent with a high discount factor will save higher amounts when faced with the risk of unemployment than a similar agent with a lower discount factor, but only when retirement saving starts (and with his higher income) will the patient agent start to save money massively.

Variations in income growth (see Figures 7.11 and 7.12) also influence the simulation. Strong income growth leads to a larger accumulation of wealth (Figure 7.14), to more consumption over life, and also to a steeper increase in consumption during the working years (Figure 7.11). With an income growth of zero, the agent will have an almost constant consumption over life (since the retirement benefit and the average income are about the same) and accumulate almost no money. Income growth does not influence the bequest behavior of the agents.

The variation of the parameter of risk aversion influences consumption and wealth accumulation only on a small scale (see Figures 7.13 and 7.14). The higher the parameter of risk aversion, the more wealth the agent will accumulate in his working years since, being risk averse, he prefers a larger buffer of money to a larger consumption (Figure 7.14). Consumption develops accordingly; the agent, being very risk averse, will always consume a little less during his working years and increase consumption during retirement, when he is no longer confronted with uncertainty (Figure 7.13). At the end of his life, an agent with a high risk aversion has still accumulated more wealth than an agent with low risk aversion, meaning that the bequest behavior is not influenced by the parameter for risk aversion.

It comes as no surprise that a high utility factor for money influences the agent in such a way as to accumulate more wealth over his lifetime; therefore it will also influence the agent's consumption behavior (Figure 7.16). The agent with the high utility factor consumes less during working years than the agents with the lower factors (he uses this money for wealth accumulation), but during retirement he spends a part of the additional money accumulated and consumes more than his counterparts with lower utility factors (Figure 7.15). But having a high utility factor for money, means having a high preference for owning money and therefore the agent will keep a larger accumulated wealth until the end of his life and leave more money to his heirs than the agents with the lower utility factor (see Figures 7.15 and 7.16).

A high interest rate means accumulated wealth increases fast, but the larger accumulated wealth and the prospect of more consumption in the future seems to be no incentive for the agent to consume less in the beginning. During the first 20 to 25 years of his working life, when the agent saves only to buffer himself against unemployment, he consumes almost the same at every interest rate. Only when the buffer-stock saving switches to retirement saving does he use the extra income from the higher interest rate to consume more (see Figure 7.17). The agent then consumes more until the last period, in which he leaves almost the same bequest as the agents with the lower interest rates (see Figure 7.18).

A change in the intratemporal rate of substitution θ does not influence wealth accumulation or overall consumption, but rather the way the money is distributed between the different goods. In Figures 7.19 and 7.20, consumption for two different θs is shown. The rate of intratemporal substitution determines how important it is for the agents to have a smooth consumption path for each good and how much the agent values having a diversity of several goods versus a large quantity of one good. With a small θ, the agent prefers diversity and a smooth consumption path, while for a large θ his preferences are reversed.

The agents in this model distribute their money according to the share parameters (or utility factors) and the subsistence levels of the goods. The largest share parameter is allocated to the nondurables, while the share allocated to necessities is the smallest. The subsistence level on the other hand is largest for the durables and smallest for the nondurables. At the beginning of the agent's life, when no wealth has been accumulated, the subsistence levels affect consumption most; this is the reason so few nondurables are consumed at this point. The more wealth the agent accumulates, the more nondurables he will consume, until he reaches the point at which he switches from precautionary saving to retirement saving. From then on, any extra money is put into wealth accumulation, and the proportions of the consumption of the different goods stay almost the same.

The reason the consumption of the three goods does not exactly reflect the proportion of the share parameters lies in the existence of the subsistence levels, which assure that agents with a moderate income spend their money in such a way as to make the good with the highest subsistence level (the durable good) the one most money is spent on. The good on which the agent spends the next highest amount is the one with the highest utility factor, the nondurable good. If θ is small, the agent is very interested in smoothing the consumption path over time; this is why for $\theta = 0.6$ the proportion of the three different goods does not change much over time. For a larger θ, the smoothing of consumption is not as important to the agent as approximating the share parameters; therefore, in the case of $\theta = 1.6$, the consumption of nondurables increases over time.

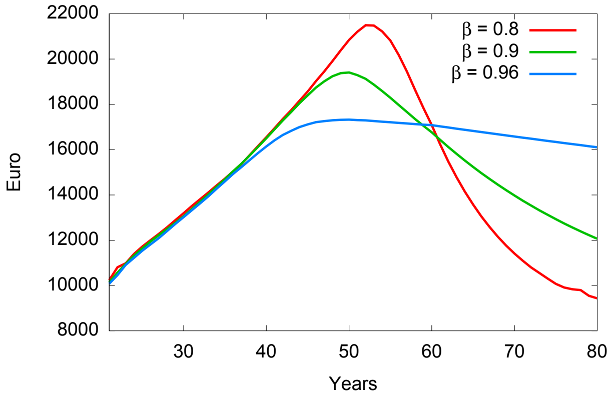

Figure 7.9: Variations in the discount factor: Consumption

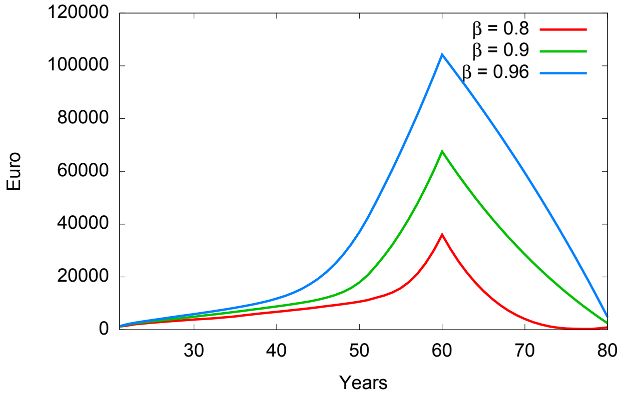

Figure 7.10: Variations in the discount factor: Wealth

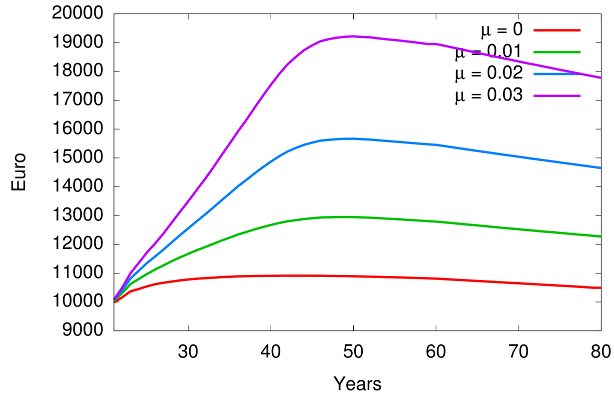

Figure 7.11: Variations in the income growth: Consumption

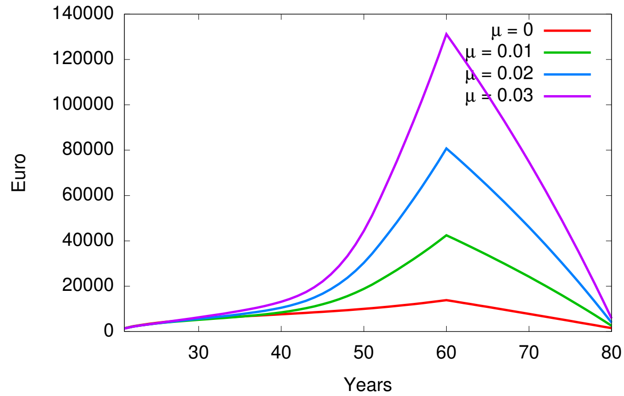

Figure 7.12: Variations in the income growth: Wealth

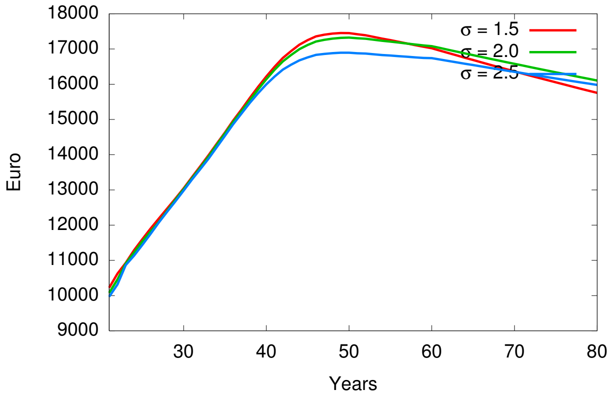

Figure 7.13: Variations in the risk aversion: Consumption

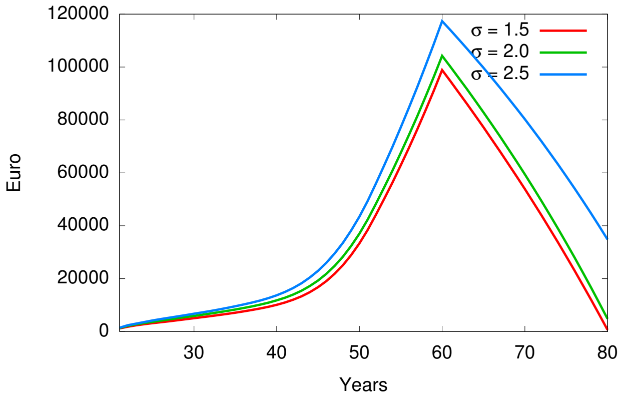

Figure 7.14: Variations in the risk aversion: Wealth

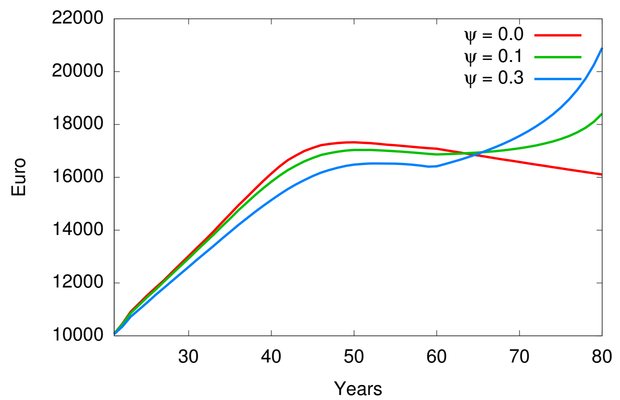

Figure 7.15: Variations in the utility factor for money: Consumption

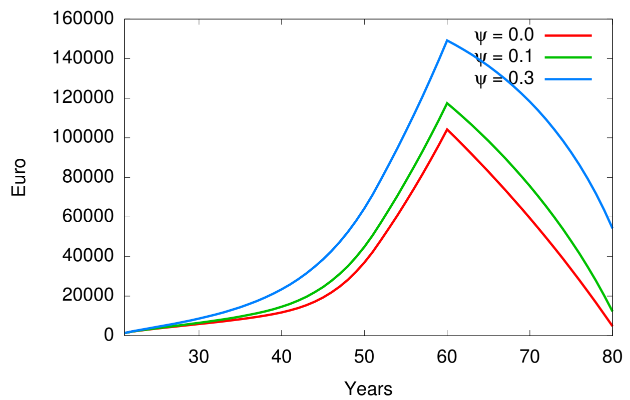

Figure 7.16: Variations in the utility factor for money: Wealth

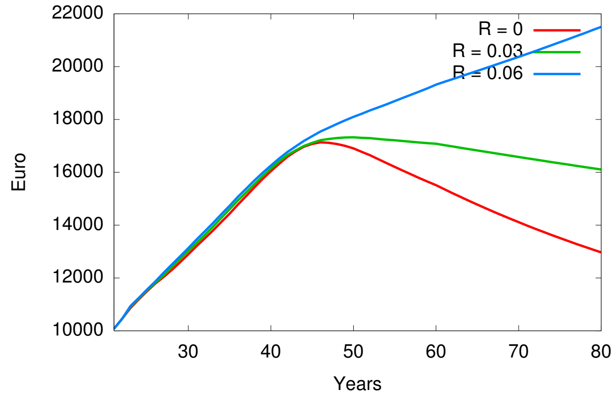

Figure 7.17: Variations in the interest rate: Consumption

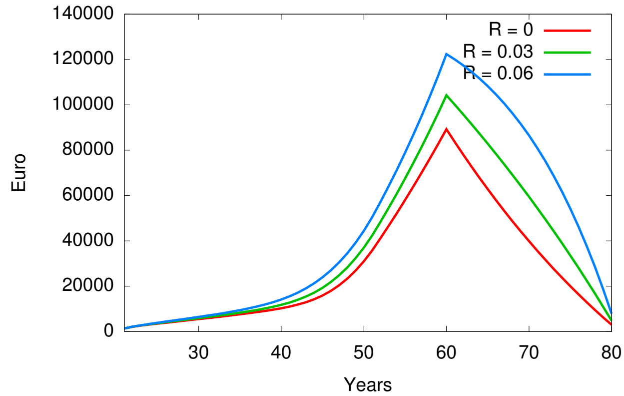

Figure 7.18: Variations in the interest rate: Wealth

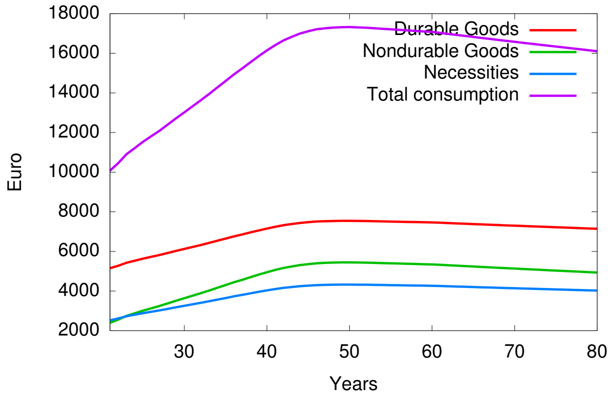

Figure 7.19: Intratemporal rate of substitution $\theta = 0.6$: Consumption

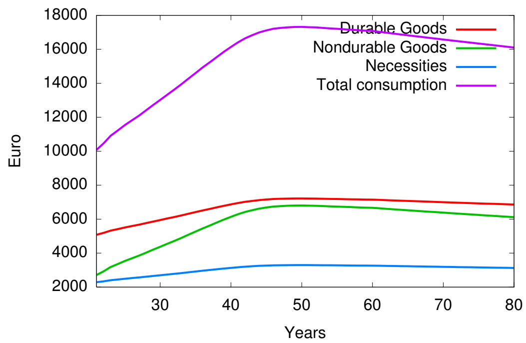

Figure 7.20: Intratemporal rate of substitution $\theta = 1.6$: Consumption

7.1.3 The three different agents

In this section, simulations of the three types of agents with different educational backgrounds are shown. The agents are distinguished by income, pension, length of working lives, probabilities of becoming unemployed and the growth rate of their income.[5] At the age of sixty, all agents start their 20 years of retirement. The income of all agents grows with a constant rate until they are fifty, stagnates after this point, and falls to a constant income in retirement.

The simulations show that all agents attempt to smooth their consumption over their lifetime by saving during their working years and dissaving during retirement (see Figures 7.21 – 7.26). All three types of agents show a peak in accumulated wealth in the early periods of retirement. Consumption grows with income, reaches its highest level when income is highest, and then decreases slowly towards the end of their lives. There is no sudden increase in consumption in the last periods since the agents in the simulations know when they will die and plan to bequeath their money. All three agents save from the beginning of their working lives in order to create a buffer-stock against unemployment, and at age sixty start saving for retirement.

There are two distinct differences between the three agents. First, the agents with more money save and consume more in total. The agents with the high level of education[6] save and consume twice as much as the agents with the low level of education[7] (Figures 7.25 and 7.26).

The second distinction lies in the way the agents consume: the agent with the low level of education and the lowest income distributes his small income guided mostly by the subsistence levels.[8] Over his entire lifetime he spends most of his money on durables (housing, heating etc.), which is the good with the highest subsistence level. In the middle of his life, when his income is highest, he increases the consumption of nondurables, which is the good with the highest share parameter and the lowest subsistence level. The agent with the highest income and a high level of education spends more on each of the three goods, but in particular spends more on the nondurable good

[5] For a summary of all parameters used to the configure the three agents see Table 6.6.

[6] University degree or a degree from a university of applied sciences.

[7] No vocational training and no school leaving certificate.

[8] Since the agents with the low level of education earn more than only the sum of the subsistence levels, they will also spend their money in such a way that the distribution across the three goods only approximates the subsistence levels.

(which is the good with the highest share parameter).

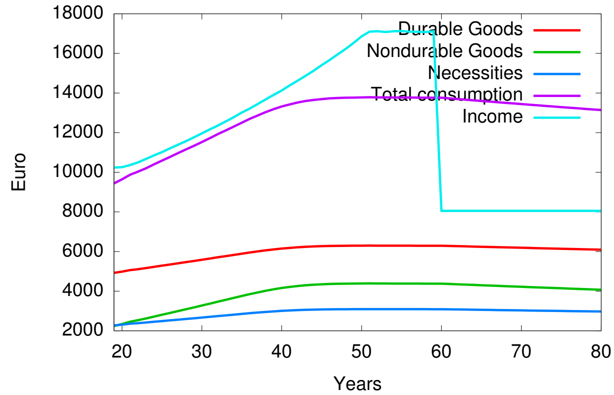

Figure 7.21: Agent with low education level: Consumption

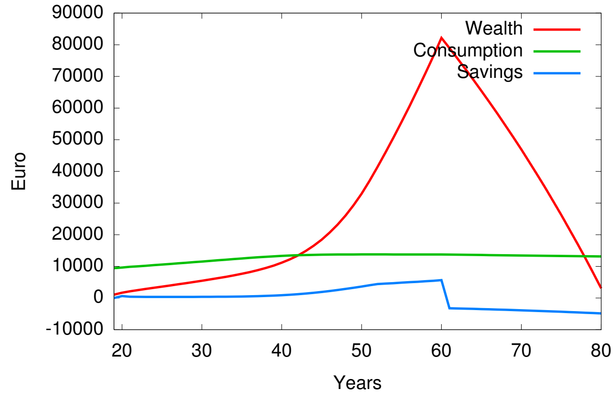

Figure 7.22: Agent with low education level: Wealth and consumption

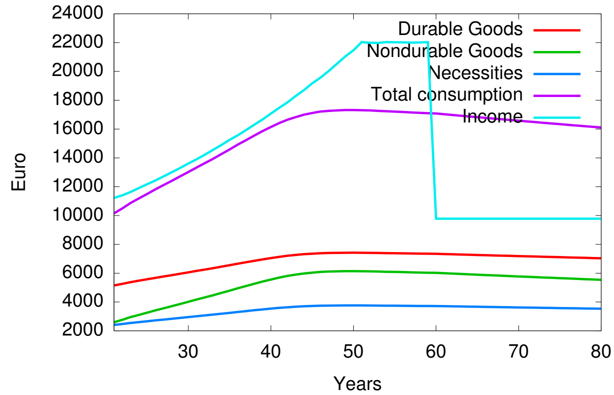

Figure 7.23: Agent with intermediate education level: Consumption

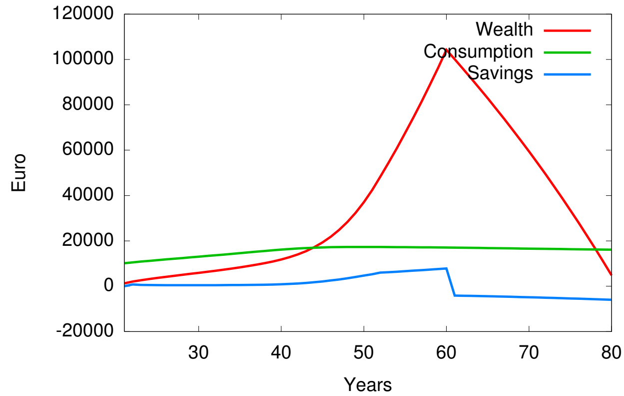

Figure 7.24: Agent with intermediate education level: Wealth and consumption

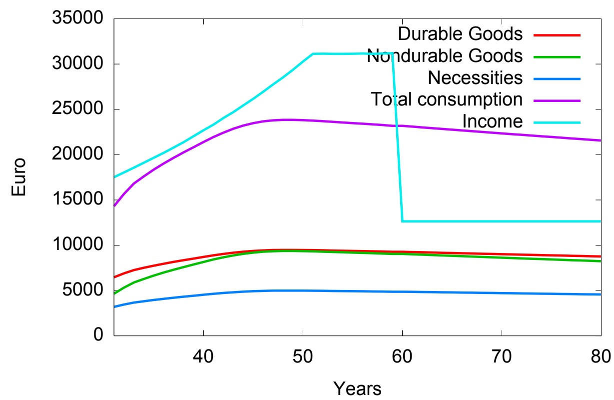

Figure 7.25: Agent with high education level: Consumption

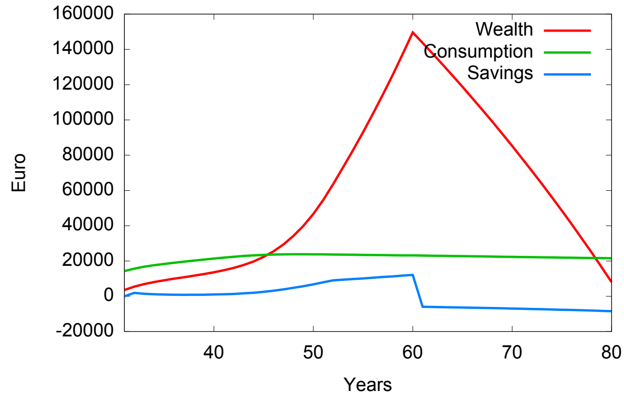

Figure 7.26: Agent with high education level: Wealth and consumption

7.2 The simulation results in a wider context

The stylized facts and the saving patterns introduced in Chapter 3 will now be compared to the simulation results, and the most important outcomes of the model will be discussed in the context of the life cycle model literature.

7.2.1 The model and the stylized facts

The estimation of empirical saving behavior shows that consumption responds to uncertainty in future income. This can be seen clearly in the simulations. When no uncertainty is implemented, the agents start to save approximately 20 years before their retirement (see Figure 7.1), but if the agents are faced with the possibility of unemployment and the consequential income shock, they will start saving at the beginning of their working lives (see Figure 7.3). This stylized fact can be confirmed for each of the three different groups of agents (see Section 7.1.3).

The second stylized fact states that households with high income growth have higher saving rates than households which have the same income but lower income growth. In Figure 7.27 the savings rate of an agent with an income without growth is compared to the savings rate of an agent with normal income growth (both having an intermediate education level and the same average income); and in Figure 7.28, the saving amounts of the two incomes are displayed against each other. The agent with no income growth has a relatively constant saving rate and saving amounts over his working life, while the agent with the growing income starts with lower savings rates and amounts, but overtakes the savings of the agent with no income growth as his income grows. Over his lifetime, the agent with no income growth accumulates more wealth first but is then overtaken by the agent with income growth (see Figure 7.29). That means the second stylized fact can be explained with the German life cycle model only to a limited extent. The agent with the low income at the beginning of his working life will save little at the start and then increase his savings rate with growing income while the agent with the constant income will keep the savings rate fairly equal over his lifetime. But one fact can be reproduced: the agent with the higher income growth will provide better for retirement, his accumulated wealth being higher shortly before retirement; therefore he is able to spend more money during his retirement (see Figure 7.29).

The third stylized fact concerns the relative and the absolute income. The consumption of most people does not follow the income closely but diverges from the income when income is prone to shocks. This can be seen when looking at a single random run of an agent. In Figure 7.30, one agent with an intermediate education level has gone through several stages of unemployment and during that time received only the unemployment benefit. The consumption of the agent goes down in the periods following unemployment, but not as much as the income; the agent smoothes his consumption with his accumulated wealth.

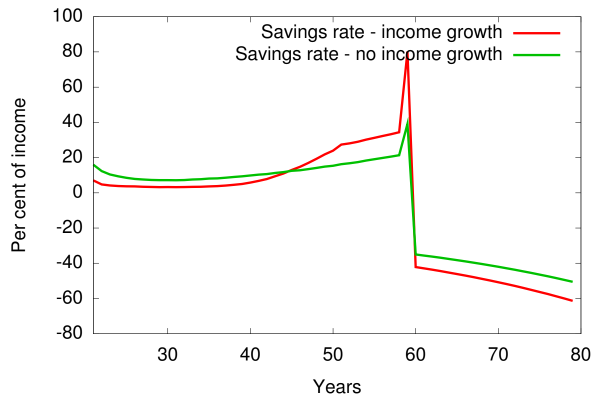

Figure 7.27: Comparison of two agents, one with a growing and one with a constant income: Savings rate

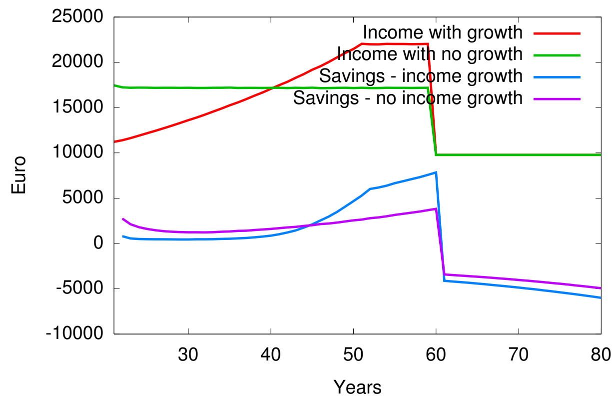

Figure 7.28: Comparison of two agents, one with a growing and one with a constant income: Savings amount and income

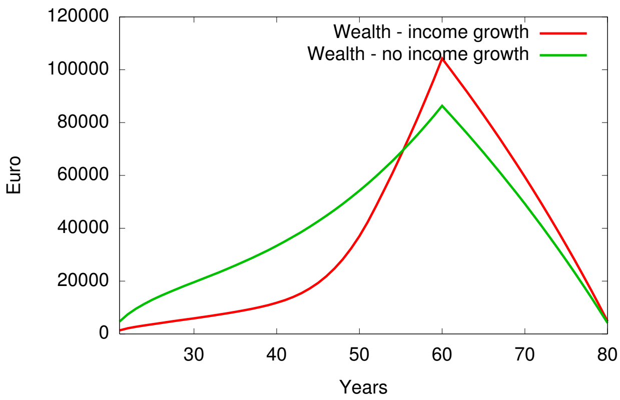

Figure 7.29: Comparison of two agents, one with a growing and one with a constant income: Wealth

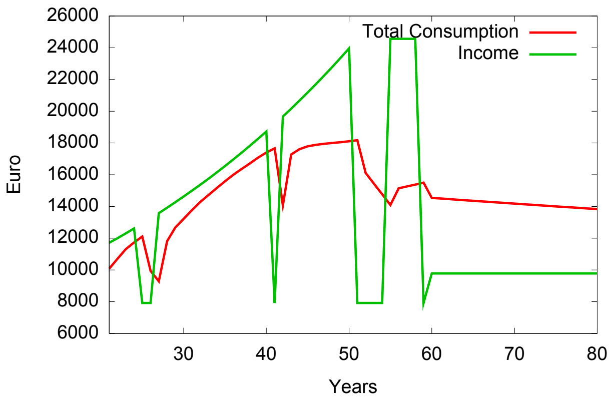

Figure 7.30: Random agent: Consumption and income

7.2.2 The model and the empirical data

Trying to reproduce the behavior of people[9] with a computer model is difficult for at least two reasons: first, people are very different from each other and it is not possible to model each single individual; one can only model certain individuals or an aggregation of all of them.[10] Second, persons have many preferences, live in different environments and are influenced by many factors, and models can reproduce this complexity only to a very limited extent. Although the German life cycle model already tries to implement many important features of German individuals, it cannot capture them all.

The average German household increases its yearly saving amounts during its working years until the last decade before retirement (45–55 years),[11] the lowest saving amounts are accumulated in the decade when retirement starts. The average household increases its savings again after retirement has begun (see Chapter 3, Figure 3.4).

The simulated agents[12] save little (compared to periods shortly before retirement) until they are about forty, and start then to accumulate wealth. The agents' savings drop suddenly on retirement, and from this point on they spend their accumulated wealth[13] (see Figure 7.31).

The highest savings are made in the time before retirement[14] while young people, at the beginning of their working life, save the least: these two facts are consistent with the empirical data and the simulation results. One difference between the empirical data and the model results is the point at which the agents and households start to increase their savings. While the empirical data suggest that people already save four times as much in the second decade (25 – 35 years) as in the beginning and in the third decade (35 – 45 years) almost as much as in the decade before retirement, the simulated

[9]In the following I abstract from the educational backgrounds since no data has been collected to show saving behavior according to education levels.

[10]For a discussion of the aggregation problem see Appendix D.

[11]The average yearly saving amounts are only given for ten year periods in the data collected by the EVS.

[12]Keep in mind that the agents simulated here are individuals and not households. The comparison between households and individuals is necessary due to a lack of data on single persons.

[13]The savings per period in Figure 7.31 are the differences between the wealth from the current period and the wealth from the previous period.

[14]This is an empirical observation which has been made in many countries; see Chapter 3.

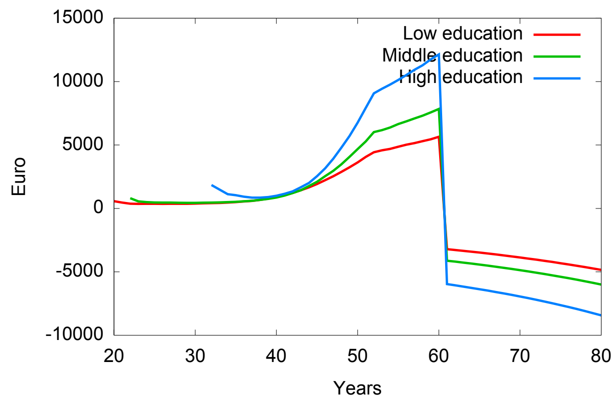

Figure 7.31: Yearly savings of the three agents

agents start to increase savings (depending on their educational status) only about twenty years before retirement.

The German life cycle model basically reproduces agents who save in the first twenty years of their working lives in order to have a buffer-stock for unforeseen events and then, after they turn forty, exhibit retirement saving. The saving behavior of actual people cannot be put this easily into a scheme. The motives for their saving behavior are more complex and more difficult to define.[15] Actual individuals are also not as rational as the agents in the simulations, and therefore do not make a clear distinction between times when they save for retirement and times when they save to create a buffer-stock, or for other reasons.

Another crucial difference between the empirical data and the simulation results is the fact that average households in Germany hardly spend their accumulated wealth during retirement while the agents in the simulations dissave almost their entire wealth. The reason for this large disagreement between data and simulations is the way the German data are collected by the Federal Statistical Office of Germany. In the empirical data, the accumulated wealth cannot be spent by the households since the wealth is turned into a stream of income once retirement starts. The pension plus the income from the accumulated wealth is the income from which the households save. In the simulations, the wealth is not turned into income but is consumed by the agents directly. If the accumulated wealth was turned into a

[15]See the discussion about saving motives in Chapter 3.

positive stream of income in the simulations as well, the profile would be less different from the empirical data, but the phenomenon of the average household in Germany, which saves out of its retirement income instead of spending it, persists. This is a puzzle which has also been stated and left unsolved by Rodepeter and Winter (1998) [148], who analyzed a life cycle model for Germany with an uncertain point of death and income uncertainty.

The most important difference between the empirical data and the simulations lies not in the form of the saving profile over the life cycle, but in the amounts of savings. The average savings in the empirical data for the fourth decade (45 – 55 years) amount to almost 6000 Euros and the average savings of young people (under 25 years) are 840 Euros while the average yearly savings in the simulations for an intermediate educated agent amount to 3 699 Euros per year in the decade from 45 to 55 years and the average yearly savings in the decade from 20 to 30 years to 444 Euros.

The reason for this discrepancy between data and simulations could be the high discount rate used in the model. With a low discount factor of 0.752[16] and a corresponding time preference rate of thirty percent, the saving amounts of the agents are in a more realistic range. The intermediate educated agent in the first decade (20 – 30 years) then has an average yearly saving amount of 232 Euros and in the age decade of 45 – 55 years, the agent saves 391 Euros a year on average[17] (see Figures 7.32 and 7.33).

The time preference rate of thirty percent is not chosen randomly, but was suggested in 1957 by Friedman in his "Theory of the Consumption Function" [70]. Friedman's idea was that people are in general very myopic and not rational enough to already plan in their youth for retirement. Although his high time preference rate is rejected by many economists today, Carroll reinvigorates Friedman's suggestion for a high time preference rate and his model, saying that they can explain many consumption and saving phenomena of individuals well:

> *"Today with the benefit of a further round of mathematical (and computational) advances, Friedman's original analysis looks more prescient than primitive. It turns out that when there is mean-*

[16]The default discount rate used in the German life cycle model is 0.96.

[17]The discount factor is, in fact, chosen so low that the agents will save even less for retirement than in real life. However, the example is supposed to illustrate a point, not to match the data exactly.

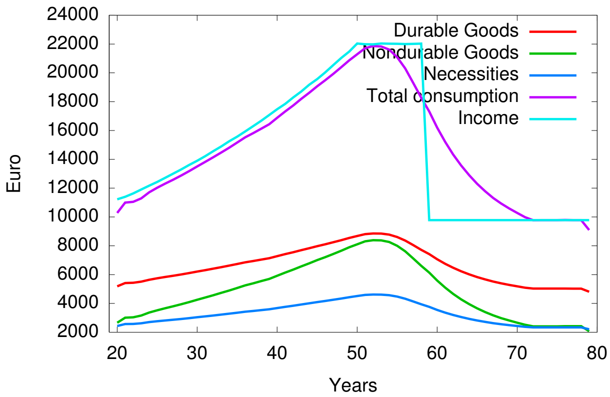

Figure 7.32: The agent with the intermediate education level and a low discount factor: Consumption

> *ingful uncertainty in future labor income, the optimal behavior of moderately impatient consumers is much better described by Friedman's original statement of the permanent income hypothesis than by the later explicit maximizing versions."* (Carroll (2001), p.23 [37])

Carroll himself uses a time preference rate in some of his models which is higher than the standard 4 percent, namely around 10 percent[18] (Carroll (1992) [32], (1997) [41]).[19]

A second reason for the differences between the saving amounts in the data and the simulations might lie in the fact that the empirical data display a cohort of people of all ages in the year 2003 (the young people saving and the older people receiving a pension are taken from the same pool) and the German life cycle model simulates one consistent individual over time (who works today and will be in retirement in the future).

This disagreement in time is decisive because of the difference in the German pension today and in the future. The state-based pension insurance scheme is based on the transfer from employees today (who pay the insurance) to employees from yesterday (who receive the pension), and was invented in a

[18]Which represents a discount factor of 0.9.

[19]Carroll also argues that the low and widely used time preference rates not only make no sense but that the estimation methods used to find them are flawed (Carroll (1997), p.1, [34]).

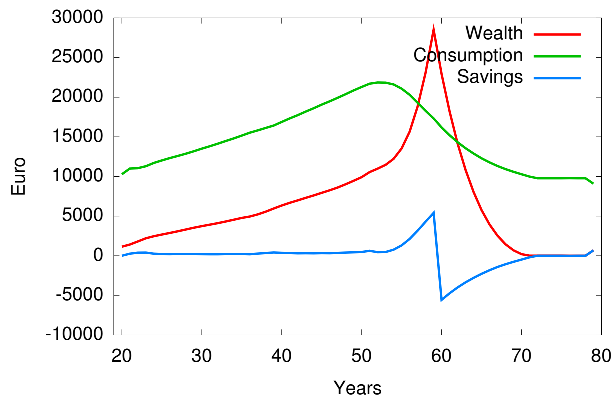

Figure 7.33: Agent with intermediate education level and low discount factor: Wealth and savings

time when many young people worked and few people received a pension. German's demography has been changing for some time now, with the number of old people increasing and the number of young people decreasing.[20] Concerned about this changing demography, the government has started to reform the pension system: while pensioners today still receive relatively generous pensions, young people and pensioners of the future already know that they will have to accept cutbacks.

Therefore, the pensions in the EVS[21] data from 2003 are larger than in the simulations. For the agents in the simulations the expected low future pensions imply that they have to save more during their working years to assure their standard of living during retirement.

A third reason for the large saving rates during working years in the simulations might be that the goal of the agents is to smooth consumption over their lifetime. They attempt to spend the same amount of money on all three goods throughout their entire lifetime. In real life, people might spend more money during their working years on one kind of good, for example in order to buy a house, and are then able during their retirement to save on rent (and accumulate money instead).

[20]For a detailed explanation of the German "Umlagesystem" and a discussion of the question of whether taking care of children should increase the parents' pension see Spill and Fuhrmann (2000) [165].

[21]Sample survey of income and expenditure carried out by the Federal Statistical of Office Germany.

7.2.3 The model and the life cycle literature

One of the most important papers in the life cycle model literature is Carroll's "Buffer-stock Saving and the Life Cycle/Permanent Income Hypothesis"[22] (Carroll (1997) [34]). In this paper, Carroll argues the case for the life cycle model with impatient and prudent agents against the then widely used saving and consumption models based on Modigliani:

> *"This paper argues that a version of the LC/PIH* (life cycle/permanent income hypothesis) *model in which buffer-stock saving emerges is closer both to the behavior of the typical household and to Friedman's original conception of the Permanent Income Hypothesis model."* (Carroll (1997), p.1 [34])

The most important outcomes of Carroll's model can also be demonstrated with the German life cycle model. Carroll's model explains the consumption/income parallel for aggregated consumers as well as the fact that consumption often diverges from income at the individual level. This convergence or divergence between consumption and income can be reproduced using the German life cycle model. For an individual who is faced with the risk of unemployment, income is erratic while consumption develops relatively smoothly (see Figure 7.30). On the other hand, when the consumption and income of a large number of such individuals is averaged out, the consumption follows the income smoothly (see figures in Section 7.1.3).

Another result of Carroll's work, and also of the other life cycle modelers, is the switch from precautionary saving to retirement saving. In Carroll's model this switch takes place at an age between 45 and 50, and five years earlier in the paper by Gourinchas and Parker (2002) [81]. The model of Hubbard et al. (1995) [92] simulates the switch from precautionary to retirement saving shortly after the middle of the working life; their agents start to save at around the age of 32 to 38 (depending on the income). Carroll (1997) [41] attributes the difference between his results and those of Hubbard et al. to the high time preference rate (and therefore the patient agents) in Hubbard's model, which in his opinion motivates the agents to save for retirement at

[22] A description of the model in this paper can be found at the beginning of this thesis in Chapter 2.

an early age.

In the German life cycle model, the switch from precautionary to retirement saving is simulated at the agent's age of 40. The reason for the early start of saving compared to Carroll's model can be partly attributed to the early start of retirement.[23] Carroll's (North American) agents start their retirement at 65, while the agents in the German life cycle model enter retirement at age 60; in both cases, the period of retirement adds up to twenty years.

The early start of retirement saving in the German life cycle model is also caused by the low pensions the agents receive compared to Carroll's agents. In Carroll's model the pensions are 70 percent of the income of the last working period while in the German life cycle model the pensions lie below the starting wage and are (for the intermediate educated agent) only about 36 percent of the income in the last working period. This means that the incentive to start saving early is higher for the agents in the German life cycle model than for Carroll's agents.

In the finite version of Carroll's life cycle model, the agents increase their consumption in the last periods of their lives. This behavior does not correspond to the empirical data and is avoided in the German life cycle model by implementing bequest. Another way of avoiding the upward bend in consumption in the last periods is to implement uncertainty about the date of death, as implemented in Hubbard et al. (1995) [92].

The fact that people save less in real life than in the simulations[24] cannot only be observed in the German life cycle model; it is a well-known problem in the community. Carroll (1992) [32], (1997) [41], (1997a) [34] has identified the problem of too much wealth accumulation and has tried to resolve it by giving the agents an interest rate of zero and a high time preference rate of 4 percent. Deaton (1991) [53] solved the problem in a similar way by choosing an interest rate of two percent and a time preference rate of 5 percent.[25] Taking these parameter values, the authors were able to decrease the wealth accumulation, but unable to match the empirical data.

[23]The time preference rate is not responsible since in the German life cycle model, as well as in Carroll's model, it is 0.046.

[24]This problem is discussed in Section 7.2.2.

[25]The German life cycle model uses a time preference rate of 0.046 and an interest rate of 4 percent.

Hubbard et al. (1995) [92] take another approach to solving the problem of too much wealth accumulation in the simulations: they implement not only a security net but also a structure whereby owning accumulated wealth leads to a 100 percent tax on unemployment benefit. This means that agents who earn little enough to justify getting the benefits but have accumulated some wealth are not entitled to these benefits. With the implementation of not only a social net but also these restrictions, it was possible for Hubbard et al. to simulate empirical data (for the US), where lower income groups have an extremely low accumulation rate.

Chapter 8

Conclusions

8.1 Summary of the findings

The purpose of the present work has been to build a life cycle model with two innovative features: first, income uncertainty that depends on current employment status and educational background; and, second, a utility function that implements three different goods, subsistence levels, utility of money, and the ability of agents to bequeath their wealth. Both approaches, the way uncertainty is implemented, as well as the application of the CRRA-CES-Stone-Geary utility function, are novelties in the life cycle model literature.

The German life cycle model has been calibrated to German benchmark data. The unemployed agents receive "Hartz IV" while the employed agents receive an income according to educational qualification and age. The subsistence levels comply with the figures which the German government considers to be basic needs, and the agents calculate their expected pensions according to the current state of the German pension system. The conditional probabilities of becoming unemployed are computed according to German employment data for the different educational groups.

Simulations of the model showed that the educational backgrounds of agents affect their behavior in two ways. First, agents differ in their saving behavior. Similar to findings in other life cycle models, the agents with the highest level of education save and consume twice as much as the agents with the lowest education. Second, agents distribute their money among the three different goods differently; while the agents with the lower income give essentials like housing and food priority, the agents with the high income spend proportionally more on luxury goods like restaurant visits and vacations. This is an outcome which is due to the combination of different educational groups, subsistence levels, and the security net.

The German life cycle model was able to explain the three most important stylized facts about consumption and saving: consumption responds to uncertainty in future income; individuals with high income growth have higher saving rates than individuals with the same average income but lower income growth; and consumption diverges from income which is exposed to shocks.

Although the model was not able to match the empirical data exactly (none of the life cycle models described in Section 2 were able to do so), it explains several characteristics of German saving behavior: young people save little and increase their savings over their working lives until the beginning of retirement; shortly before retirement the saving rates are highest and they drop after the start of retirement.

None of the agents in the model exhaust their accumulated wealth at the end of their lives, which is a fact that also conforms to the German data and is achieved by implementing the bequest function. Another outcome which reflects the data, especially German data, is the fact that agents are never faced with total destitution. In the worst case, they receive unemployment benefit, which allows them to satisfy all basic needs.

The amounts saved by the simulated agents are considerably larger than the amounts saved in the data (this, too, is a problem which could be observed in the other life cycle models). By creating more impatient agents and giving them a low discount factor (as proposed by Milton Friedman (1957) [70] for example), the saving amounts in the simulations approximate the empirical data.

8.2 The prospects of life cycle models

The use of life cycle models pursues three different objectives. The first is to reproduce and then predict empirical consumption and saving behavior in order to understand the entire economy, which is the foundation for giving good policy advice.[1] The second objective seeks to find the best saving and

[1] To be able to use the German life cycle model as a tool for policy analysis, two things have to be done: the model output has to be aggregated over German individuals who belong to each educational group (that the aggregation of the German life cycle model is

consumption strategy for individuals in order to give people advice on how to best prepare for unforeseen events and old age. Finally, the third objective of life cycle models is to explain the empirical data and to learn how people reach their saving and consumption decisions.

Stochastic dynamic programming models are complicated, and the average person making saving and consumption decisions does not use a complicated computer program for this end. But this does not necessarily mean that normal individuals are not able to come up with optimal solutions. Friedman (1953) [69] defends this idea and alludes to the famous example of the billiard player: although the billiard player would not recognize the mathematical formulas describing the spin, the speed and the direction of the ball which are needed to describe his actions; he can nevertheless make the right move and position the ball in a fraction of a second.

The analogy of the billiard player dos not compare well to life cycle saving, however. While the billiard player has some special talent, many hours of training and a coach to give him advice, the life cycle saver is a normal person, not necessarily talented, who has no possibility of training for the optimal outcome since he lives his life only once.

Nevertheless, inspired by Friedman's idea, Allen and Carroll (2001) [2] tried to reproduce the optimal behavior of agents not by using stochastic dynamic programming but by using a simple rule of thumb: save some money when times are good and spend it when times are bad. The agents undergo a special learning process; they select initial values, observe the results, and update their behavior. The outcome was a little disappointing, however; the authors found that, using this algorithm, an agent would need one million years to find the ideal solution.

According to the authors, rules of thumb could be rescued by social learning.[2] But implementing social learning is a difficult undertaking since agents are

possible is also a result of this thesis), and the model has to be coupled to other players in the German economy such as firms, policy, banking, and other countries. Figures such as wages and unemployment rates (and therefore unemployment probabilities) would not be given to the model exogenously but could be derived within it. The different agents of the general model can then interact with each other and thereby simulate a realistic economy.

[2] Although not explicitly mentioned by Allen and Carroll, we assume that social learning is meant as the counterpart to learning by individual trial and error. This means that learning in the sense of studying books, researching the internet for information, etc., is also understood as social learning, even though some people might not consider these activities as social.

faced with very singular risks and can therefore not easily learn from others who are faced with other kinds of uncertainties. When considering learning from previous generations, another problem appears: the other generations were faced with different economic and social circumstances. While 100 years ago hardly any pension system and security net provided people with a basic income, 30 years ago people in Germany were almost completely financially secured. Today, on the other hand, the government benefits are diminishing again and people in Germany are facing government-based financial incentives to provide for their retirement (so-called "Riester" pensions and similar concepts). Finally, learning is expensive; it costs time, energy and money to acquire the necessary information, and there is always the trade-off between the optimal solution reached by having learned all the information at great expense and a close-to-optimal solution reached with fewer learning costs.

Uhlig and Lettau (1999) [111] also tried to solve a life cycle problem by using rules of thumb. They gave the agents two rules between which they could choose from and found a problem which they termed "good state bias". If things go smoothly in one period and the agent chooses the bad rule in this period, he will still evaluate this rule positively since he cannot distinguish between good luck and smart behavior.

However, the issue is more complicated than just trying to reproduce saving and consumption behavior with rules of thumb. Even the models implementing rules of thumb, although trying to imitate people's behavior, are still based on a rational consumer trying to approximate the stochastic dynamic programming solution, and it is not at all clear whether people are able to find the utility maximizing saving and consumption strategy (Friedman's optimism notwithstanding). Otherwise, how can it be explained that there are impoverished elderly people and people who have run into deep debts as a consequence of unforeseen events?[3] The irrationality of people is something that cannot be calculated with stochastic dynamic programming models (even if it were possible to make agents more complex, extend the utility function and find perfectly fitting parameters) or with models implementing rules of thumb. This problem comes down to the well-known criticism of the homo economicus: people who behave irrationally cannot be modeled perfectly by assuming rational agents.

[3]This is not an entirely accurate example: it is perfectly possible to reproduce such "unreasonable" behavior with stochastic dynamic life cycle models by making the agents very risk-loving and impatient. The rational actors in the life cycle model, however, would have no regrets about the bad outcome of their decisions since it was, after all, optimal while people in real life might judge the situation differently.

A further problem is the fact that life cycle models and models implementing rules of thumb neglect the cost of social learning (learning by trial and error is not possible with optimal life cycle models). People might be able to find close-to-optimal saving paths, but they might not want to spend the necessary time studying investment possibilities and calculating their personal financial risk. This problem could be solved by extending the utility function of stochastic dynamic programming models to include the information necessary to reach good saving and consumption decisions. The degree of information acquired by the agents would then determine the optimality of the consumption and saving strategy. Only with completely (and very expensively) acquired information would the agent be able to find the perfect consumption strategy.

Looking again at the three purposes for life cycle models, there is no plausible reason why the way the data is reproduced and the optimal saving strategy is calculated has to mirror the way people reach their saving decision (assuming, of course, that the results are otherwise satisfying). It is very well imaginable that a computer program can reproduce data perfectly by calculating the optimal saving path, although normal people would not understand the code. For reproducing data, simulations with life cycle models are a good approximation: one possibility of improving the results might be to implement learning costs in the stochastic dynamic programming models or to build a rules-of-thumb model with reasonable social learning features. If one wants to find out how people reach their decisions, stochastic dynamic programing models are not sufficient, but rules of thumb might be an interesting method of approximating the optimization procedures people use to reach their daily saving and consumption decisions. Financial experts, on the other hand, whose task it is to advise people on how to use their money, are well advised to use dynamic programming models because these models give optimal saving paths.

The rational actor who plays the main role in life cycle models, as in all economic models, is only an approximation to reality. Accepting this fact, one can see what progress has been made over the last decades in modeling people's the consumption and saving behavior. Although the results of life cycle models are far from perfect, they are the best we have at present. Rather than dismissing the stochastic dynamic life cycle models, we should try to improve them and see how much closer we can come to approximating reality.

Appendix A

Analysis of the policy function

As explained in Chapter 5.2, the model is divided into two parts. In the first part, the optimization problem is solved using dynamic programming, and a policy function is produced. In the second part, the results of the policy function are processed and the behavior of an average agent is simulated. So far, simulation results have only been shown for the average agent, but the underlying policy function is in some sense the real result of the optimization. To see that the individual agent behaves in a proper way, one first has to verify the policy function.

This chapter describes the results of the policy function,[1] followed by a sensitivity analysis of the policy function. As a representative case, the agent with the intermediate education level is chosen.

A.1 The default case

The way the utility, as well as the consumption and saving behavior of an agent, develops when the given "cash on hand" increases is an important test of the model. In Figures A.1 - A.6, utility, consumption and wealth are shown for the first and the thirtieth period in an agent's life (both for the state of employment and the state of unemployment), and for the last period, when the agent is already in retirement (and therefore neither employed nor unemployed).

[1]Remember that the policy function is a matrix which contains the optimal consumption for each period, each employment state and each element in the money vector (see Chapter 5.2).

The utility of an agent increases with the available money but at a diminishing rate for all periods and for both states of employment. The increase in utility up to an amount of "cash on hand" of approximately 1 000 Euros (which is the range of money close to the sum of all subsistence levels) is very steep; after this point the increase in utility slows down (see Figure A.1). The utility is always higher when the agent is employed.

The consumption of the three goods also increases with the given money (Figures A.2 - A.5). The consumption curves for all periods except for the last one are concave, which, since the agent is risk averse and faced with uncertainty, means that he will put more money aside for every additional Euro of income. In the final period, the money/consumption curve is a straight line. This is due to the fact that the agent knows with certainty that there is no period after the last for which he has to save money. The money/consumption curve in the last period does not follow a 45° angle (as it would be if the agent exhausted his wealth) since the agent keeps a part of his money in order to bequeath it.

The money/consumption curves are influenced by the share parameters and the subsistence levels. If the agent only has the lowest amount of "cash on hand" displayed here available (which is close to the sum of the subsistence levels), he will spend about 4 500 Euros on the good with the highest subsistence level (durables) and about 1 500 Euros on the one with the lowest subsistence level and the highest share parameter (nondurables). When the agent is provided with 30 000 Euros, he will spend (in period 30 and being employed) around 7 000 Euros on durables and around 5 500 Euros on nondurables. This means that the difference between the money spent on the two types of goods narrows down as available money increases, and the demand for nondurables will eventually overtake the demand for durables.

The wealth curves in Figure A.6 are complements to the consumption curves in the sense that an increase in given money also increases wealth and that the wealth curves are convex for the same reasons that the consumption curves are concave.

In Figures A.7 - A.12, utility, consumption and wealth are plotted against time, each for three different amounts of "cash on hand".

The form of the utility function plotted against time (Figure A.7) is influenced by two things:[2] First, because of the way the optimization works, each period includes the utility from the first period up to and including the current period. Accordingly, the last period includes the utility of the entire lifetime and therefore has the highest utility value, with utility constantly decreasing towards the first period. Second, the utility is plotted here for a constant amount of money given each year, but the money available to the agent over his lifetime, including accumulated wealth and income, increases until retirement starts, and then decreases until the end of the agent's life. Since one goal of the optimization is to smooth utility over life, the agent will save money when he has the resources available, that is in the middle of his life when his income his highest, and consumes the savings later, when the income is smaller. This is the reason that his utility resulting from a given constant amount of money in each year is smaller for periods in which he has a lot of money (because he will save a large part of that money) and larger for those in which he has only little accumulated wealth. This means that the utility of a Euro is lowest when the wealth of an agent is highest.

When looking at the utility from the last period of an agent's life, the utility decreases steeply towards the beginning of retirement because both of the causes which decrease utility of the utility decreasing causes, the optimization as well as the wealth increase, work in the same direction. After retirement has begun, utility increases again (as it does towards the beginning of the working years) since now the amount of wealth for each given amount of money decreases. Going back from the age of forty to the beginning of the working life, the decrease in wealth becomes flatter since the retirement saving has not yet started. Utility decreases only as a consequence of the way the optimization works.

At any given period, the utility is higher if the agent is employed (since he is confronted with less uncertainty) and when he has more available money.

The consumption over time (see Figures A.8 - A.11) is also influenced by the fact that the wealth accumulation of a single agent peaks shortly before retirement. Out of a given amount of money, an agent spends least in the periods when his accumulated wealth peaks; with decreasing wealth towards the first period, the consumption out of the given amount increases. In the last period of the agent's life the agent will spend most, but not all, of his wealth since there is no period ahead of him; but he still wants to bequeath

[2]Note that the graphs do not show utility or consumption over one agent's life but display results from the policy function.

money.

For all periods, consumption is higher when the agent is employed and is provided with higher amounts of money. If the agent's available money matches the unemployment benefits, the agent will only consume the subsistence levels.

The wealth over time is the complement to the total consumption; in the last period the agent accumulates the least, and shortly before retirement and in the first period the agent saves the most (see Figure A.12).

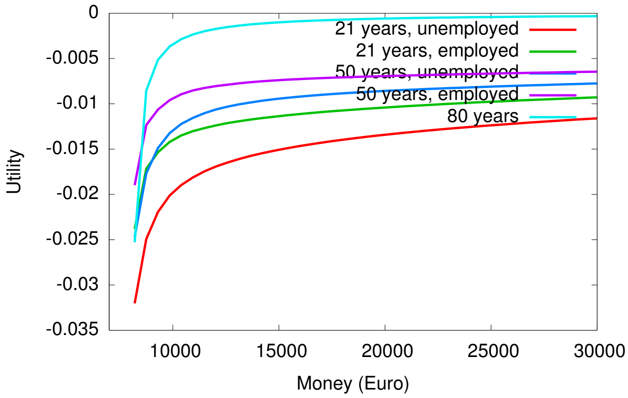

Figure A.1: Utility (versus money)

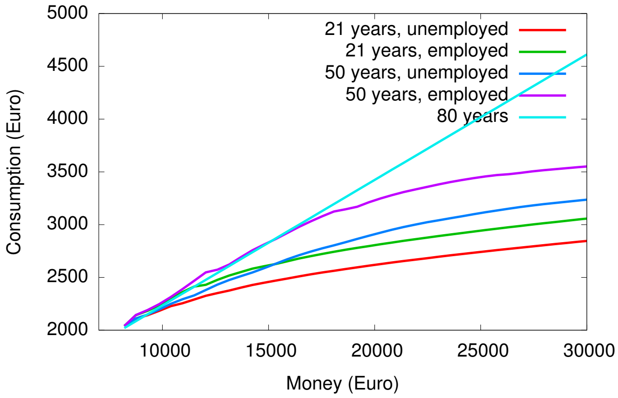

Figure A.2: Consumption of necessities (versus money)

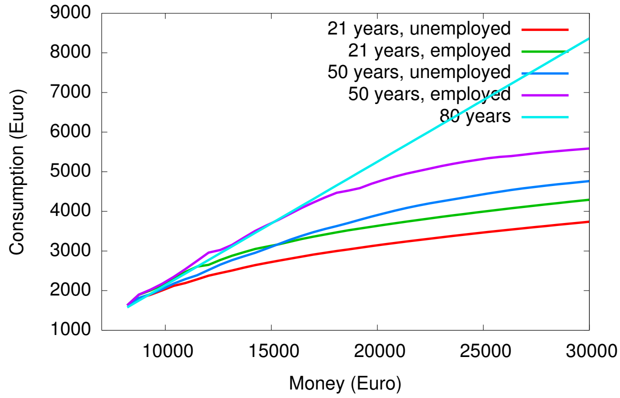

Figure A.3: Consumption of nondurables (versus money)

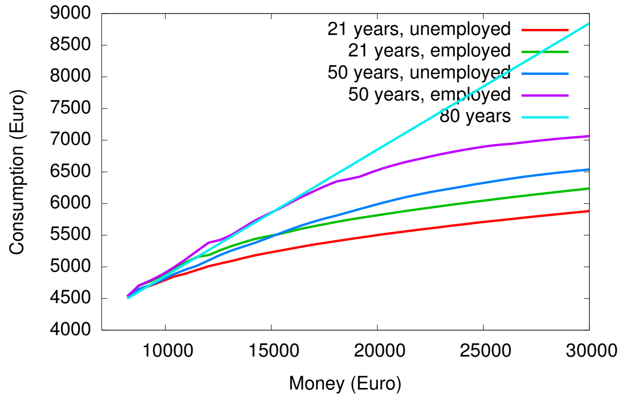

Figure A.4: Consumption of durables (versus money)

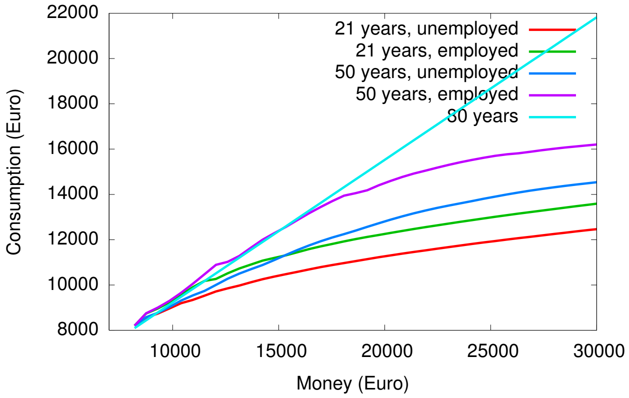

Figure A.5: Total consumption (versus money)

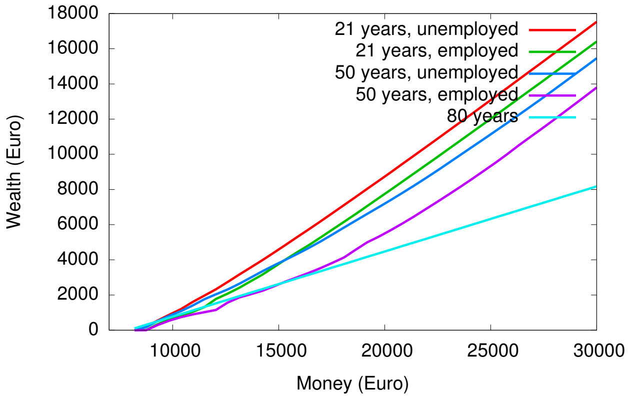

Figure A.6: Wealth (versus money)

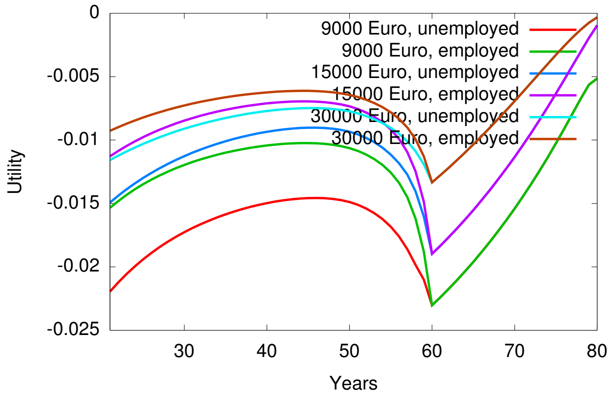

Figure A.7: Utility (over time)

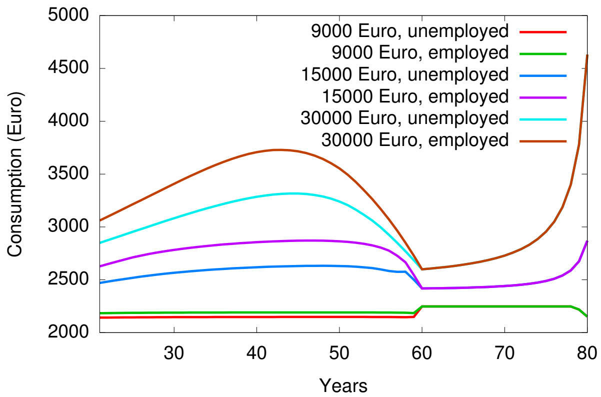

Figure A.8: Consumption of necessities (over time)

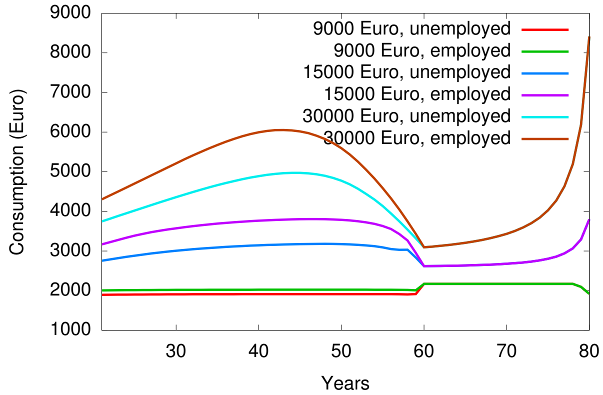

Figure A.9: Consumption of nondurables (over time)

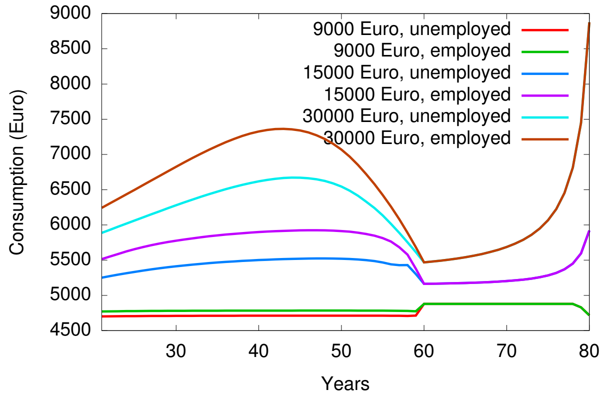

Figure A.10: Consumption durables (over time)

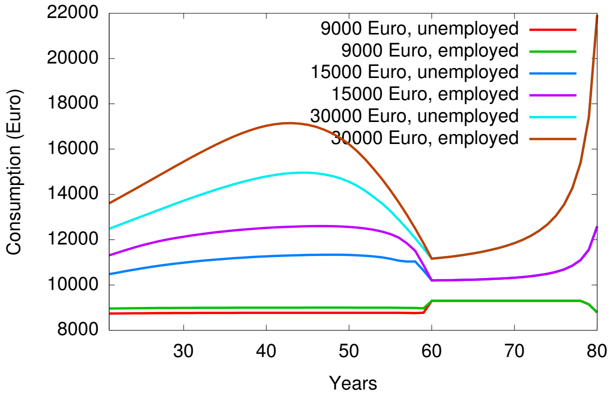

Figure A.11: Total consumption (over time)

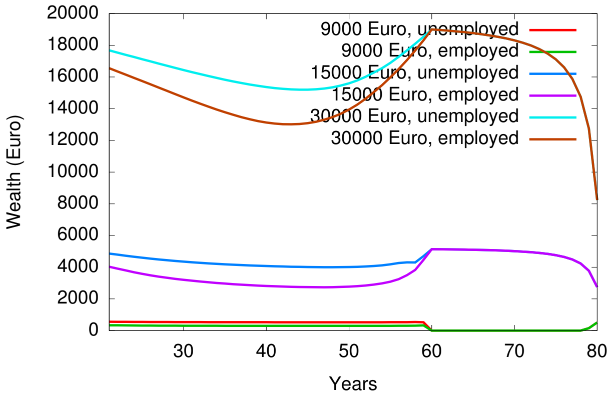

Figure A.12: Wealth (over time)

A.2 Sensitivity analysis

To ensure that the model output is reliable, it is important to know how the model behaves in response to changes in the parameters. One method of examining the relationship between parameters and results is the sensitivity analysis:

> *"Sensitivity analysis is the study of how the variation (uncertainty) in the output of a mathematical model can be apportioned, qualitatively or quantitatively, to different sources of variation in the input of a model."* (Saltelli et al. 2004 [150])

The Data and Computation department of the Potsdam Institute for Climate Impact Research has developed "A Multi-Run Simulation Environment for Quality Assurance and Scenario Analyses" called SimEnv[3]. With the help of this tool it is possible to carry out a sensitivity analysis for the German life cycle model. SimEnv runs the model for every value of a certain parameter in a discrete set. As a result, it is possible to see if the model outputs are robust to changes in the input. Table A.1 shows the default values with which the model is run, and the range in which the parameters are varied.

Parameters	Default	Range
Discount factor (β)	0.96	0.7 - 1
Rate of risk aversion (σ)	2	1.1 - 2.5
Intratemporal rate of substitution (θ)	1.1	0.5 - 1.7
Interest rate (R)	0.03	0.0 - 0.08
Utility factor money (ψ)	0.00006437	0.0 - 0.4
Income growth (μ)[4]	0.02 - 0.03	0.00 - 0.06

Table A.1: Default values and the variation range of the parameters tested in the sensitivity analysis

[3] For a description of the functionality and capabilities of SimEnv, see Flechsig et al. (2008) [68].

The sensitivity analysis has been performed for the results of the policy function.[5] In the following figures the consumption for an employed, intermediate-level educated agent who is provided with "cash on hand" of 15 000 Euros per period is displayed against the permitted ranges of different parameters.[6] To ensure a good intuitive understanding of each analysis, two graphs are shown for each case. The first graph shows the parameter variations in a 2D picture from above, the second figure shows a lateral view.

Figures A.13 and A.14 display the sensitivity analysis for the discount factor. An agent spends less of his 15 000 Euros when a high discount factor is used in the model since a high discount factor and a small degree of myopia constitute a higher incentive to save money for future periods. The difference between high and low discount factors is especially apparent shortly after retirement starts since at this point an agent equipped with a high discount factor expects a high amount of accumulated wealth as preparation for retirement. Since hardly any wealth is accumulated, the agent reacts by consuming even less and saving even more.

The sensitivity analysis for the risk aversion σ is shown in Figures A.15 and A.16. For greater values of σ the risk aversion is high, and the agent will spend relatively little of his 15 000 Euros per period. The difference between high and low parameters of risk aversion is evident shortly before the end of the agent's life. While a high σ leads the agent to consume only at the subsistence level, an agent with risk aversion will spend almost everything.

To see how robust the model is towards changes in the intratemporal rate of substitution θ (see Figures A.17 - A.22), a sensitivity analysis for each kind of good has been performed. The size of θ determines the importance the agents attach to a balanced consumption of all three goods. They are, however, driven away from a balanced consumption by the share parameters, which assign every good a certain weight. When θ is small, the equal distribution of the three goods is very important to the agent. For this reason, the demand for the goods with the low share parameter i. e. the necessities, is accordingly high. When θ increases, the consumption of the necessities

[5] Testing a range of different parameters for the average agent has already been conducted in Section 7.1.2.

[6] Since showing all possible tests would go beyond the scope of this thesis, this sample has been chosen. From a robust consumption, a robust utility function and robust saving behavior can be deduced, and the employed, intermediate-level educated agent can be considered representative for all agents who are based on the same model structure but are parameterized differently.

decreases (see Figures A.17 and A.18). For the nondurable good, the case is reversed, since it is the good with the highest share parameter: for a large θ, the consumption is relatively high and decreases with θ (see Figures A.22 and A.22). For the durables, the consumption hardly changes with the variation of θ. This is due to the fact that the share parameter of the durables lies in between the share parameters of the two other goods; the opposition between the intratemporal rate of substitution and the utility factors therefore has little impact.

As discussed in Section 7.1.2, the interest rate does not influence the consumption of an agent in an isolated period. Only when more money is accumulated as a result of a higher interest rate will the agent consume more. Since the present case only concerns a constant amount of "cash on hand" of 15 000 Euros per period, however, this effect cannot be seen here (see Figures A.23 and A.24).

With a high utility factor for money, an agent will consume less since a high utility factor for money motivates him to save more (see Figures A.25 and A.26). This effect is hardly noticeable during the working years, but in the retirement years, during which there is no uncertainty to provide an additional incentive to accumulate a buffer stock of money, the agent with the high utility factor clearly consumes less and saves his money towards the end of his life to be able to bequath it.

In Figures A.27 and A.28, the income growth has been varied. Income growth influences only the consumption of the working years since only during this time can an agent expect a higher income in the next period. During the working years, high income growth leads to an increase in consumption (since the agent expects a high amount of money in the next period and therefore has no incentive to save) while with an income growth of zero consumption stays approximately constant.

In this section, all parameters have been varied in their permitted ranges. The sensitivity analysis has shown that the model is robust to changes in all these parameters. The graphs presented demonstrate the absence of discontinuities or sudden changes in the model output.

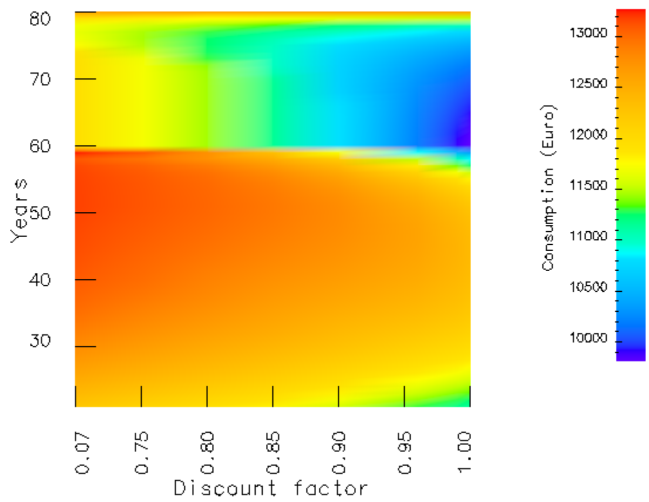

Figure A.13: Sensitivity analysis for the discount factor: Top view

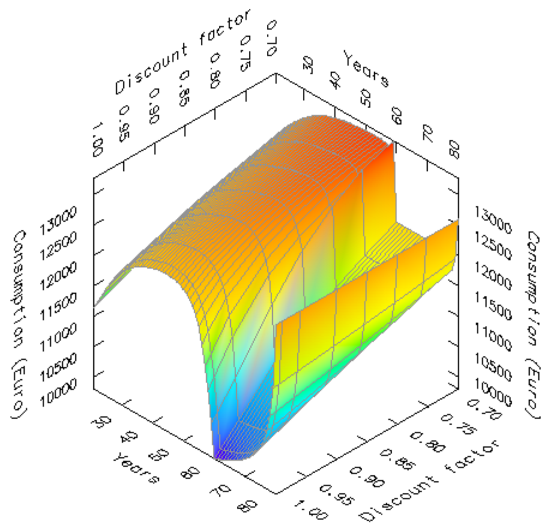

Figure A.14: Sensitivity analysis for the discount factor: Lateral view

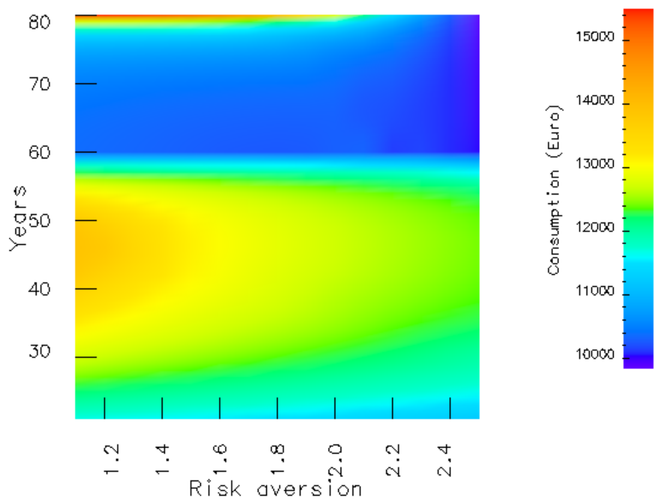

Figure A.15: Sensitivity analysis for the parameter of risk aversion: Top view

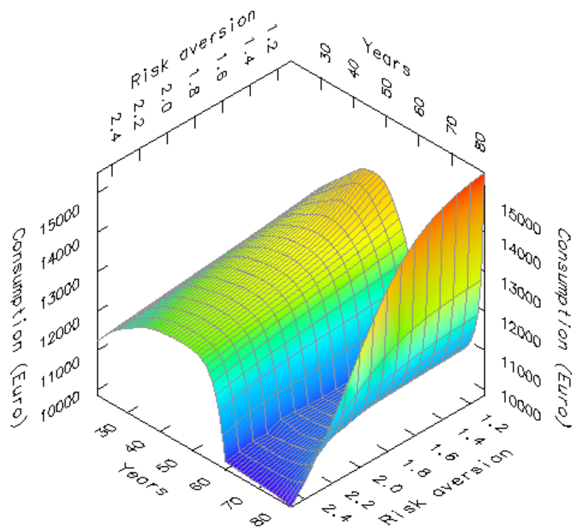

Figure A.16: Sensitivity analysis for the parameter of risk aversion: Lateral view

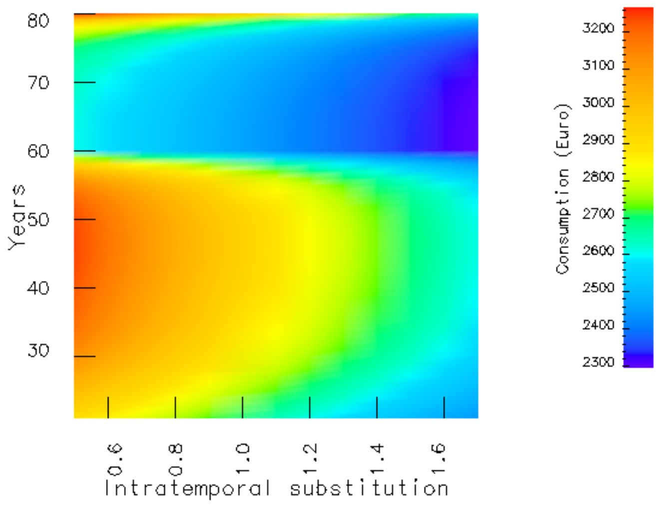

Figure A.17: Sensitivity analysis for the intratemporal substitution (necessities): Top view

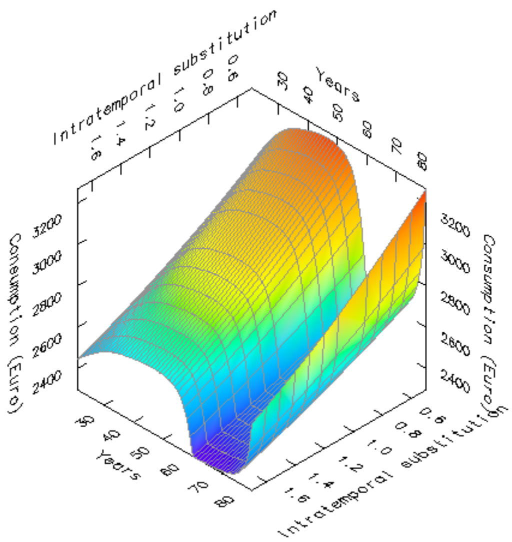

Figure A.18: Sensitivity analysis for the intratemporal substitution (necessities): Lateral view

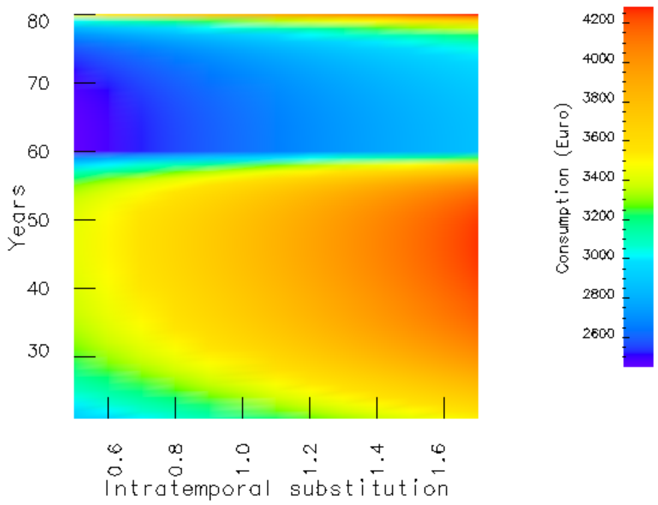

Figure A.19: Sensitivity analysis for the intratemporal substitution (nondurable goods): Top view

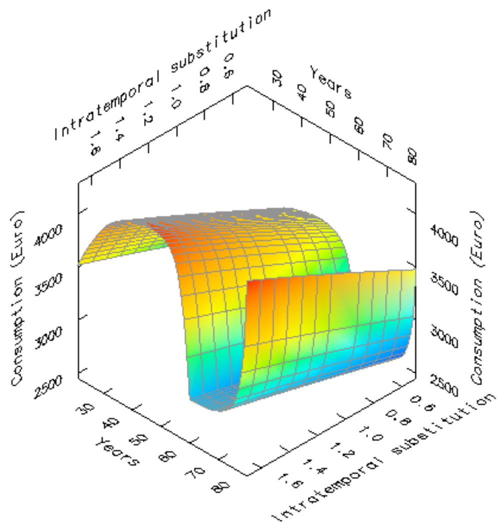

Figure A.20: Sensitivity analysis for the intratemporal substitution (nondurable goods): Lateral view

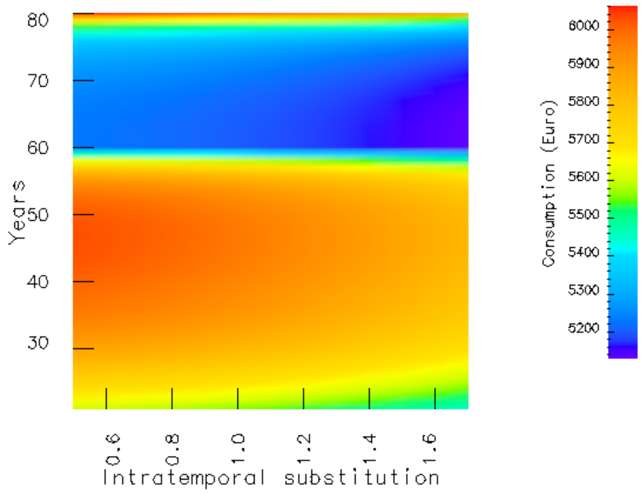

Figure A.21: Sensitivity analysis for the intratemporal substitution (durable goods): Top view

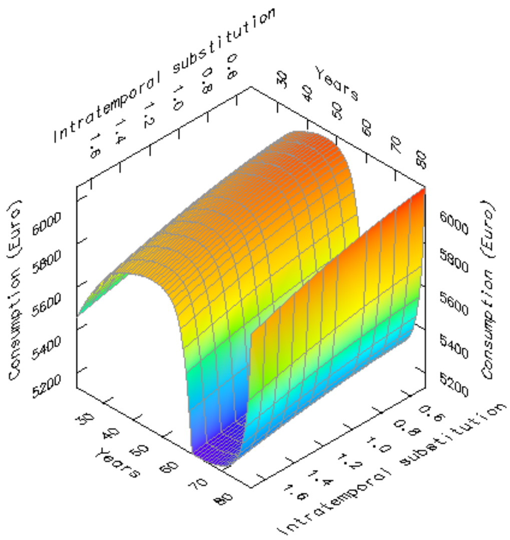

Figure A.22: Sensitivity analysis for the intratemporal substitution (durable goods): Lateral view

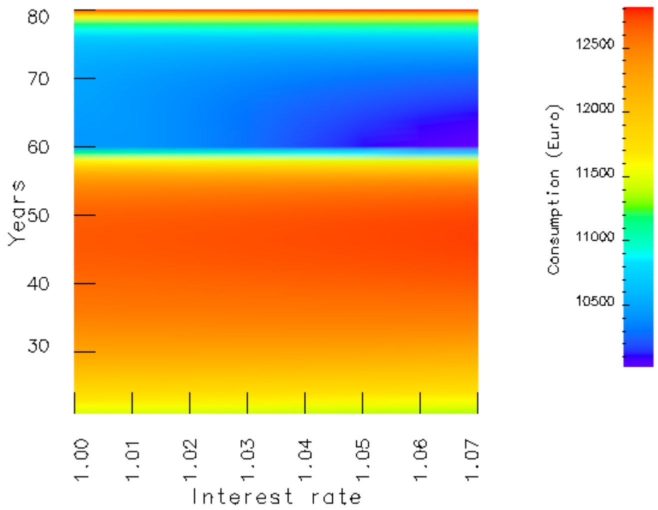

Figure A.23: Sensitivity analysis for the interest rate: Top view

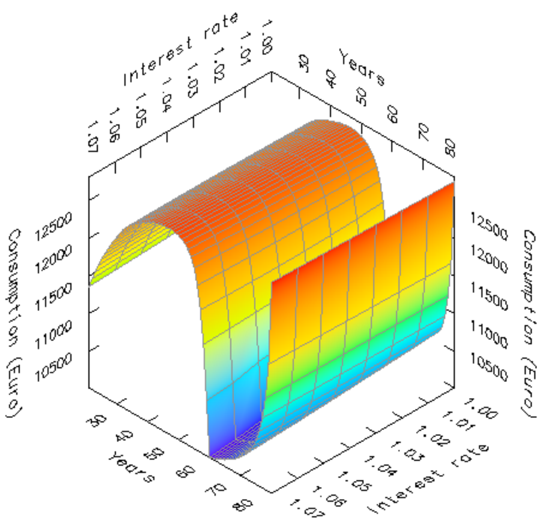

Figure A.24: Sensitivity analysis for the interest rate: Lateral view

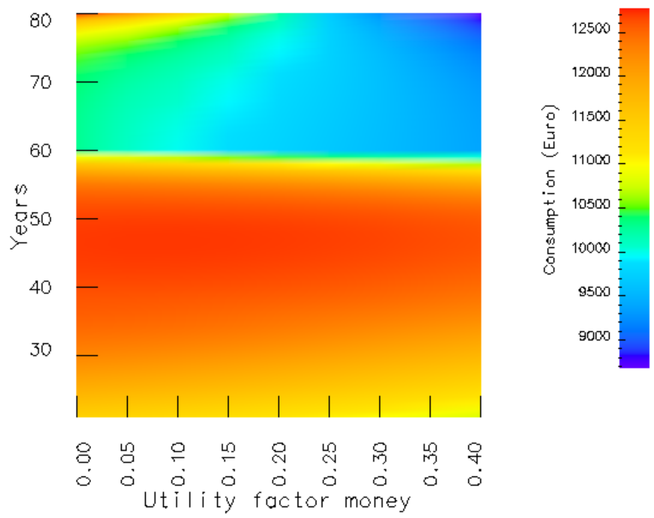

Figure A.25: Sensitivity analysis for the utility factor for money: Top view

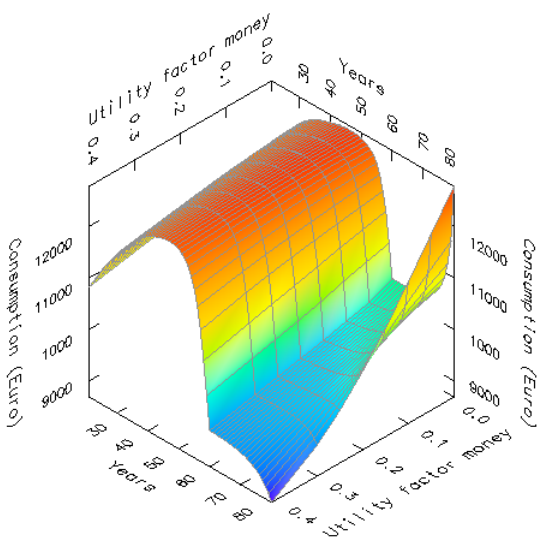

Figure A.26: Sensitivity analysis for the utility factor for money: Lateral view

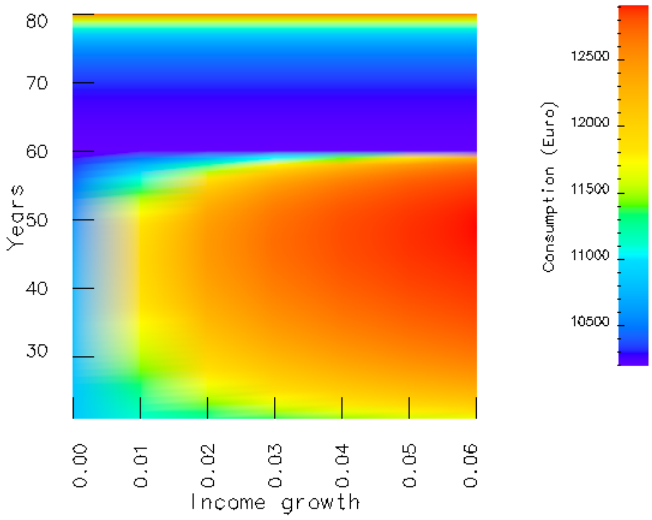

Figure A.27: Sensitivity analysis for the income growth: Top view

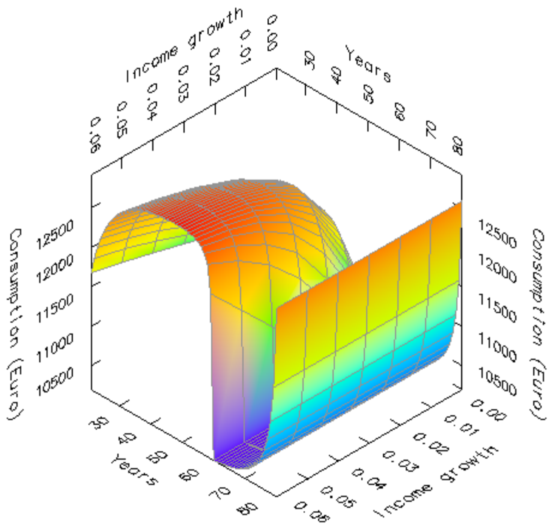

Figure A.28: Sensitivity analysis for the income growth: Lateral view

Appendix B

The Commented Code

In this chapter, the code for all programs which are part of the German life cycle model are displayed. The comments provide detailed explanations for each part of the program.

B.1 The core program

```
% DPTHREEGOODS returns two matrixes, one filled with the
% optimal utility and one filled with the optimal consumption

% Imports parameters from an extra file

parameter;

% Defines all the global variables

global alphavec gammavec t psi R sigma moneyvec mu Ie Iue...
VT beta nrgoods theta P pentime pen scalefactor

% Returns the time, used for measuring the computing time

timestart = clock;

% Creates a three dimensional matrix with zeros, which will
% later be filled with the utility values
% The dimensions are: periods, moneygrid, state of employment

VT = zeros(T-t+1, mgrid, 2);

% Creates a four dimensional matrix with zeroes, which will
% later  be filled with the optimal consumption for the three
```

```
% goods:  necessities, durables, nondurables
% The dimensions are: nrgoods, periods, moneygrid,
% state of employment

pol = zeros(nrgoods, T-t+1, mgrid, 2);

% Sets the options for the optimization
% (default values in brackets)
% MaxIter: maximum number of iterations allowed (400)
% MaxFunEvals: maximum number of function evaluations allowed
% (100*Number of variables)
% TolFun: termination tolerance on the function value (1e-6)
% TolCon: Termination tolerance on the constraint
%  violation (1e-6)
% TolX: Termination tolerance on x (1e-6)
% Diagnostics: Display diagnostic information about the
% function to be minimized or solved (off)
% LargeScale: Use large-scale algorithm if possible (on)
% DiffMaxChange: Maximum change in variables for
% finite differencing  (0.1000)
% DiffMinChange: Minimum change in variables for
% finite differencing (1.0000e-08)

options = optimset('MaxIter', 10000, 'MaxFunEvals',...
120000, 'TolX', 1e-11, 'TolFun', 1e-11, 'TolCon', 1e-11,...
'Diagnostics', 'off', 'LargeScale', 'off',...
'DiffMaxChange',1e-7, 'DiffMinChange', 1e-10);

% x = fmincon(fun,x0,A,b,Aeq,beq,lb,ub): finds the optimal
% utility and consumption
% fun: goal function (utility function for the last period)
% xo: starting point (subsistence levels for each good)
% A*x ≤ b: inequality constraint (income constraint)
% Aeq, beq: equality constraints (do not exist)
% lb, ub: lower bound, upper bound (subsistence levels,
% available money)

% Optimization for the last period

        [pol(:, T-t+1, 1, 1), VT(T-t+1, 1, 1)] = ...
        fmincon(@(c) -(lutility(c, moneyvec(1))),...
        start(1:nrgoods)+ones(1,nrgoods)*0.001,...
        pricevec(1:nrgoods), ...
        [moneyvec(1)-0.001],[], [],...
        gammavec(1:nrgoods)+ones(1,nrgoods)*0.001,...
        ones(1,nrgoods)*moneyvec(1),...
        [], options);

for i = 2:length(moneyvec)
```

```
        [pol(:, T-t+1, i, 1), VT(T-t+1, i, 1)] = ...
        fmincon(@(c) -(lutility(c, moneyvec(i))),...
        pol(:,T-t+1,i-1,1)',...
        pricevec(1:nrgoods), ...
        [moneyvec(i)-0.001],[], [],...
        gammavec(1:nrgoods)+ones(1,nrgoods)*0.001,...
        ones(1,nrgoods)*moneyvec(i),...
        [], options);

% The matrix with the optimal utility values is multiplied by
% minus one; this is necessary since the minus in front of
% the maximization problem changes the sign of the original
% results

    VT(T-t+1, i, 1) = VT(T-t+1, i, 1)*(-1);

% For the last period (the agent is in retirement!) state
% of employment  and state of unemployment are the same

    VT(T-t+1, i, 2) = VT(T-t+1, i, 1);
    pol(:, T-t+1, i, 2) = pol(:, T-t+1, i, 1);

% The optimal saving values are the differences between the
% given  money values and the sum of the optimal consumption
% for all  three goods

    sav(T-t+1, i, 1) = ...
    moneyvec(i)-sum(squeeze(pol(:,T-t+1, i, 1)));
    sav(T-t+1, i, 2) = ...
    moneyvec(i)-sum(squeeze(pol(:,T-t+1, i, 1)));

end

% The other periods

% The three loops are for every period, every money value in
% the grid and for the two states unemployed and employed
% The fitting function is produced and passed on for every
% call

options = optimset('MaxIter', 3000, 'MaxFunEvals', ...
30000, 'TolX', 1e-10, 'TolFun', 1e-10, 'TolCon', 1e-10,...
'Diagnostics', 'off', 'LargeScale', 'off', ...
'DiffMaxChange', 1e-7, 'DiffMinChange', 1e-10);

wg = 1.0;

if T-t>0
pervec = [linspace(T-t, 1, T-t)];
```

```
for s = pervec
    for e = 1:2
    fitfunction = fit([moneyvec, 30000000]',
    [VT(s+1,:, e),0]','cubicspline');
        for m = 1:length(moneyvec)
            [pol(:,s, m, e), VT(s, m, e)] = ...
            fmincon(@(c)...
             -(bellutility(c, moneyvec(m),...
              e, s ,fitfunction)), ...
            (wg*pol(:,s+1,m,e)'+
            (1-wg)*[gammavec(1:nrgoods)]), ...
            [pricevec(1:nrgoods)], ...
            [moneyvec(m)-0.001],[], [],...
            [gammavec(1:nrgoods)]+
            ones(1,nrgoods)*0.001,...
            ones(1,nrgoods)*moneyvec(m),...
            [], options);

            VT(s, m, e) = VT(s, m, e)*(-1);
            sav(s, m, e) = moneyvec(m)-
            sum(squeeze(pol(:,s, m, e)));
        end
   end
end
end


% Dividing VT by the scalefactor is done to undo the
% changes made by the scaling in the utility function

VT = VT/scalefactor;

% Creates matrices for each  good
% (necessities, durables, nondurables)

for i = 1:nrgoods
    if i == 1
        polnec = squeeze(pol(1,:,:,:));
    elseif i == 2
        poldur = squeeze(pol(2,:,:,:));
    else
        polnondur = squeeze(pol(3,:,:,:));
    end
end

% Creates a matrix for total consumption

polall = polnec+poldur+polnondur;

```

```
% Returns the time the program needs to produce results

timeend = clock - timestart;

if timeend(4) < 0
    h = timeend(4) + 24;
else
    h = timeend(4);
end

if timeend(5) < 0
    h = h - 1;
    m = timeend(5) + 60;
else
    m = timeend(5);
end

'hours needed'
h
'minutes needed'
m

% Defines the first, a middle and the last period
% these periods are later used in the graphs

per1 = t;
per2 = (t - 1)+round(T/2);
per3 = T - t + 1;

% Picks three money values to show things on a graph with
% money dimension.

mo = moneyvec;
mo2 = round(mgrid/2);
mo3 = mgrid;
timevec = linspace(t, T, T-t+1);

% Saves the results to files which are later  used
% to plot the graphs in gnuplot

if T > 2
    savdata(timevec, polnec,poldur, polnondur, sav, ...
    polall, mo2, mo3,moneyvec, VT,pol,T, per1, per2, per3);
else
    return
end

% Calls the function evols which creates the random agents
% evols(policy function, vector of money values, starting
```

```
% amount of money, educational status, evolutions,
% version number, income growth)

[gs, savs,ms,emps] =
evols(pol, moneyvec, 50, 2, 100, vers, mu);
```

B.2 The last period

```
% LUTILITY returns the utility with bequest
% for the last period

function lu = lutility(c, m)

global R  sigmabeq scalefactor sigma vers


% The utility is scaled up in order  to enhance the
% performance of the optimization

% Setting vers == 3, means that the bequest term is
% used

el = find(vers == 3);

if  el > 0

lu = utility(c, m) + ...
scalefactor*((R*(m-sum(c)))^(1-sigmabeq))/(1-sigmabeq);

else

lu = utility(c, m);

end
```

B.3 The value function

```
% BELLUTILITY returns the value of the Bellmann utility
% depending on  consumption, money value, employment status
% and time

```

```
function bu  = bellutility(c, m, e, s, fitfunction);

global alphavec gammavec VT R Iue Ie moneyvec mu beta P
pentime pen

% Savings from period s multiplied by the interest rate

capital = R * (m - sum (c));

% The periods where the agent is a pensioner

if s+1 ≥ pentime

% Amount of money the agent has when he is retired in s+1

    mepen = capital + pen;

% Utility value of the agent when he is retired in s+1
% picked from the utility matrix

    valpen = min(fitfunction(mepen),0);

% Bellmann utility for the retired agent

    bu = utility(c, m) + beta*(valpen);

% The periods where the agent is employable

else

% Amount of money the agent receives in period s+1
% when he is unemployed

    mue = capital + Iue;

% Amount of money the agent receives in period s+1
% when he is employed

    me = capital + Ie * (1 + mu) ^ s;

% Utility values for period s+1 taken from the utility matrix
% for the unemployed and the employed agent

    valunemp  = min(fitfunction(mue),0);
    valemp   = min(fitfunction(me),0);

   bu = utility(c,m) + beta*(valunemp * P(e, 1) +
   valemp * P(e,2));

```

```
end
```

B.4 The utility function

```
% UTILITY returns the utility depending on the money and
% consumption value

function util = utility(c, m)

global alphavec gammavec psi sigma nrgoods theta
scalefactor vers

% Multiplying by the scalefactor (10000) is necessary because
% the matlab optimization has problems with the original
% function since the values are so small, but the scaling does
% not change the optimal consumption and is reversed
% in DPthreegoods.

% The function for theta = 1 and money as additional good
% where vers == 4 means money is included in the
% utility function

el = find(vers == 4);

if theta == 1

    if  el > 0
        util = ...
        scalefactor*((((prod((c-gammavec(1:nrgoods)).^...
        (alphavec(1:nrgoods)))*psi*m))^(1-sigma))/(1-sigma));

    else

        util = ...
        scalefactor*((((prod((c-gammavec(1:nrgoods)).^...
        (alphavec(1:nrgoods)))))^(1-sigma))/(1-sigma));
    end

% The function for the variations of theta not equal to one

elseif theta ≠ 1

    if el> 0

        util = ...
```

```
        scalefactor*((((sum(alphavec(1:nrgoods).*...
        (c-gammavec(1:nrgoods)).^((theta-1)/theta))+...
        psi*m^((theta-1)/theta))^(theta/(theta-1)))^...
        (1-sigma))/(1-sigma));

    else

        util = ...
        scalefactor*((((sum(alphavec(1:nrgoods).*...
        (c-gammavec(1:nrgoods)).^((theta-1)/theta)))^...
        (theta/(theta-1)))^(1-sigma))/(1-sigma));

    end
end
```

B.5 Defining the parameters

```

% PARAMETER defines all parameters in the program
% which do not parameterize the agents

% The if clauses around the parameters are necessary
% since parameters can be defined from outside
% (with simenv) for sensitivity analysis

% Parameter of intratemporal elasticity of substitution
% A theta of one means that the CES-function is a
% Cobb-Douglas function

if exist('theta', 'var') == 0
    theta = 1.1;
end

% Coefficient of relative risk aversion

if exist('sigma', 'var') == 0
    sigma    = 2;
end

% Coefficient for the sigma in the bequest function

if exist('sigmabeq', 'var') == 0
sigmabeq = 2;
end

```

```
% Number of goods

nrgoods  = 1;

% Vector of the prices for the constraint:
% [necessities, durables, nondurables]

pricevec = [1,1,1];

% Vector of utility factors for the goods:
% [necessities, durables, nondurables]

if nrgoods == 1
    alphavec = [1];
else
    alphavec = [0.2, 0.32, 0.48];
end

% Utility factor for money

if exist('psi', 'var') == 0
    psi = 0.00006437;
end

% Vector of the starting values used in the optimization
% for the three types of goods

start = [1987.2,4442.4,1490.4];

% Interest rate

if exist('R', 'var') == 0
    R = 1.04;
end

% Amount of money values in the grid

mgrid=50;

% Grid for the money (must have at least two elements)
% The lowest value in the grid is the sum of the
% subsistence levels

moneyvec = [linspace(7920+100, 50000, mgrid),...
60000,70000,80000,90000,100000,200000,300000];

% Used to scale the utility function to help fmincon
% to enhance its performance

```

```
scalefactor = 1000000;

% Discount factor

if exist('beta', 'var') == 0
    beta = 0.96;
end

% Starting period

t = 1;

% Defining the educational status

LOW    = 0;
MIDDLE = 1;
HIGH   = 2;

% Defining different versions of the utility function
% by writing the according numbers into the vector

% 1: subsistence level
% 2: uncertainty
% 3: bequest
% 4: money as good

if exist('pt1', 'var') == 0;
    pt1  = 1;
end

if exist('pt2', 'var') == 0;
    pt2  = 2;
end

if exist('pt3', 'var') == 0;
    pt3  = 3;
end

if exist('pt4', 'var') == 0;
    pt4  = 4;
end

vers = [pt1,pt2,pt3,pt4];

% Vector of the subsistence levels for:
% [necessities, durables, nondurables]
% if the vector consists of zeroes then the utility
% function has no subsistence levels

```

```
el = find(vers == 1);

if   el>0 ;

gammavec = [1987.2,4442.4,1490.4];

else

gammavec = [0,0,0];

end

% The if function takes care that the income growth mu
% will be handed on to get_params, if defined by simenv

if exist('mu', 'var') == 0;
    mu = -42;
end

% Calling get_params  (agent definition)

[P, Iue, Ie, mu, pen, T, pentime,unP] =...
 get_params(MIDDLE, vers, mu);
```

B.6 Specifying the agents

```
% GET_PARAMS returns the parameters defining an agent
% depending on his educational status

function [P, Iue, Ie, mu, pen, T, pentime,unP] = ...
 get_params (edu, vers, mu)

% Markov transition matrix:
% P = [P(u|u), P(e|u); P(u|e), P(e|e)]
% Iue: income when unemployed
% Ie : income when employed
% mu : growth rate of income
% pen: retirement income (pension)
% T: working years plus retirement years
% pentime: first year spent in retirement

el = find(vers == 2);

if edu == 0
```

```
    if el > 0
            P     = [0.5003, 0.4997; 0.168, 0.832];
            unP   = 0.169;
    else
            P = [0,1;0,1];
            unP = 0;
    end

        Iue   = 7920;
        Ie    = 15472;

    if mu == -42
            mu = 0.03;
    end

    pen   = 8053;
    T     = 62;
    pentime = T-19;

elseif edu == 1

    if el > 0
        P     = [0.301, 0.699; 0.1264, 0.8736];
            unP = 0.11;
    else
            P = [0,1;0,1];
            unP = 0;
    end

        unP   = 0.11;
        Iue   = 7920;
        Ie    = 17436;

    if mu == -42
            mu = 0.03;
    end

        pen   = 9780;
        T     =   60;
        pentime = T-19;

elseif edu == 2;

    if el > 0;
            P     = [0.0604, 0.9396; 0.0848, 0.9152];
            unP   = 0.034;
    else
            P = [0,1;0,1];
```

```
            unP = 0;
    end

        Iue  = 7920;
        Ie   = 24684;

    if mu == -42
            mu = 0.03;
    end

        pen  = 12634;
        T    = 50;
        pentime = T-19;

end
```

B.7 Evolution of an individual agent

```

% EVOLS takes the matrices filled with the optimal
% consumption and creates random agents for each
% educational group.

% Input

% pol:  the matrix filled with the optimal consumption
% for each good (policy function)
% moneyvec:  the values of money for which  pol is known
% m:  initial amount of money
% edu:  educational status
% nevs:  number of evolutions to compute

% Output

% gs:  nevs array x periods x  nrgoods: consumption
% savs:  1 x periods x nevs array: savings
% ms:  actual money trajectory
% emps: actual employment trajectory

function [gs, savs,ms,emps] = ...
evols(pol, moneyvec, m, edu, nevs,vers);

[P, Iue, Ie, mu, pen, T, pentime,unP] = ...
get_params (edu,vers, mu);

```

```
% Initialize the random number generator

rand('state');

% Determines the initial employment status using the
% unconditional probability of becoming unemployed
% for each educational group.

r = rand(1);
if r ≤ unP;
    e = 1;
else
    e = 2;
end

% If  pol is known, we T and nrgoods are known
% pol has dimensions nrgoods x T x mgrid x emp

d       = size(pol);
nrgoods = d(1);
T       = d(2);


% These loops build three matrices: ms, emps and savs with
% the dimensions time and nevs, and a matrice for the
% consumption  policies with the additional dimension,
% type of good

for i=1:nevs
    currentM = m;
    currentE = e;
    for t=1:T
        ms(i,t)   = currentM;
        emps(i,t)= currentE;
        s = 0;

% The fitfunction fitg  interpolates the money vector and the
% optimal consumption.

        for g=1:nrgoods
            fitg=fit(moneyvec', ...
             squeeze(pol(g,t,:,currentE)),'cubicspline');
            gs(i,t,g)=max(fitg(currentM),0);
        if gs(i, t, g) < 0
               'gs(', i, t, g, ')=', gs(i, t, g)
                pause
            end
            s = s + gs(i,t,g);
```

```
        end

% If the money s spent on one good is bigger than the
% current money the amount of goods bought is decreased

        if s > currentM
            adj = currentM / s;
            gs(i, t, :) = gs(i, t, :)*adj;
            s = currentM;
        end
        savs(i,t)=currentM - s;
        if t > pentime
            currentE = 1;
            inc      = pen;
        else
            r = rand(1);

% If the time period lies in the pension time the agent
% receives a pension. Otherwise the  employment status
% of the agent is chosen randomly based on the conditional
% probability of becoming  unemployed. The agent then receives
%  either  unemployment benefit  or income. This money
% is the new currentM and an element in the  savs matrix.

            if r <= P(currentE,1)
                currentE = 1;
                inc      = Iue;
            else
                currentE = 2;
                inc      = Ie * (1 + mu * t);
            end
        end
        currentM = savs(i,t)+inc;
     end
end
sumgs = sum(gs);
timev = linspace(1,T,T);

% Saving data for gnuplot

randsavdata(timev, savs, gs, ms, emps, moneyvec, sumgs,nevs);
```

Appendix C

Proof for a global maximum

In this chapter, it will be shown that the composed per period utility function u used in the German life cycle model has only one optimum. Such a proof can be performed by showing that u is concave, since a critical point on a concave function is always the global maximum[1].

Any optimization problem for which a value function can be derived and which can be solved by dynamic programming is subject to the principle of optimality, and can have only one optimum, as long as the per period functions have only one optimum. Since the German life cycle is based on dynamic programming[2], it is sufficient to show that any located optimum for a per period utility is a global optimum if one wants to show that the entire intertemporal optimization problem has only one optimal path.

This proof is Mathematica-based: the derivatives and the determinants have been calculated using Mathematica. All the formulas printed in bold are taken from the Mathematica code.

The utility function for four goods is[3,4]:

$\boldsymbol{u = \frac{1}{1-\delta}}$

[1]For an explanation of ideas underlying the following proof see, for example, "Mathematics for Economists" by Carl P. Simon and Lawrence Blume (1994), pp.513 [159].

[2]For the derivation of the value function, see Chapter 5.

[3]The utility function of the German life cycle model uses three goods and money. For an easier understanding, we subsume all four variables of the utility function under the term "goods".

[4]The utility function must of course be constrained with $x, y, z, w \leq M$, where M is the given amount of money. For a description of the complete model with all constraints, see Chapter 5.

$$(((\alpha\mathrm{x} * (x-\gamma\mathrm{x})^{\frac{\theta-1}{\theta}} + \alpha\mathrm{y} * (y-\gamma\mathrm{y})^{\frac{\theta-1}{\theta}} + \alpha\mathrm{z} * (z-\gamma\mathrm{z})^{\frac{\theta-1}{\theta}} + \alpha\mathrm{w} * (w-\gamma\mathrm{w})^{\frac{\theta-1}{\theta}})^{\frac{\theta}{\theta-1}})^{1-\delta} - 1)$$

$$0 < \alpha\mathrm{x}, \alpha\mathrm{y}, \alpha\mathrm{z}, \alpha\mathrm{w} < 1; \theta > 1; \delta < 1;\ 0 < \gamma\mathrm{x}, \gamma\mathrm{y}, \gamma\mathrm{z}, \gamma\mathrm{a} < 1; x-\gamma\mathrm{x}, y-\gamma\mathrm{y}, z-\gamma\mathrm{z}, w-\gamma\mathrm{w} > 0$$

where:
x, y, z, w are the goods over which the function is optimized
αx, αy, αz, αw are the utility factors of x, y, z, w
θ is the parameter of constant elasticity of substitution
σ is the parameter of constant relative risk aversion

Let H be the Hessian matrix associated with a twice continously differentiable function $u(x, y, z, w), x, y, z, w \,\epsilon\, \mathbb{R}$. Then u is concave when the Hessian matrix is negative semidefinite on $\mathbb{R}^4$. Since each negative definite function is also negative semidefinite, it is sufficient to show that H is negative definite. A Hessian matrix is negative definite, if and only if, its n leading minors alternate in signs, beginning with a negative value for the first order principal minor:

$|H_1| < 0, |H_2| > 0, \ldots, |H_n| > 0$ if n is even and $|H_n| < 0$ if n is odd

To create the Hessian matrix of a function, one needs the second order derivatives of that function. Mathematica calculates the second order derivatives of u as follows.

$$\frac{\partial^2 u}{\partial x \partial y} = -\alpha\mathrm{x}\alpha\mathrm{y}(x-\gamma\mathrm{x})^{-1+\frac{-1+\theta}{\theta}}(y-\gamma\mathrm{y})^{-1+\frac{-1+\theta}{\theta}} (\alpha\mathrm{w}(w-\gamma\mathrm{w})^{\frac{-1+\theta}{\theta}} + \alpha\mathrm{x}(x-\gamma\mathrm{x})^{\frac{-1+\theta}{\theta}} + \alpha\mathrm{y}(y-\gamma\mathrm{y})^{\frac{-1+\theta}{\theta}} + \alpha\mathrm{z}(z-\gamma\mathrm{z})^{\frac{-1+\theta}{\theta}})^{-2+\frac{2\theta}{-1+\theta}} ((\alpha\mathrm{w}(w-\gamma\mathrm{w})^{\frac{-1+\theta}{\theta}} + \alpha\mathrm{x}(x-\gamma\mathrm{x})^{\frac{-1+\theta}{\theta}} + \alpha\mathrm{y}(y-\gamma\mathrm{y})^{\frac{-1+\theta}{\theta}} + \alpha\mathrm{z}(z-\gamma\mathrm{z})^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-1-\delta}\delta + \frac{1}{\theta}(\alpha\mathrm{x}\alpha\mathrm{y}(x-\gamma\mathrm{x})^{-1+\frac{-1+\theta}{\theta}}(y-\gamma\mathrm{y})^{-1+\frac{-1+\theta}{\theta}} (\alpha\mathrm{w}(w-\gamma\mathrm{w})^{\frac{-1+\theta}{\theta}} + \alpha\mathrm{x}(x-\gamma\mathrm{x})^{\frac{-1+\theta}{\theta}} + \alpha\mathrm{y}(y-\gamma\mathrm{y})^{\frac{-1+\theta}{\theta}} + \alpha\mathrm{z}(z-\gamma\mathrm{z})^{\frac{-1+\theta}{\theta}})^{-2+\frac{\theta}{-1+\theta}} ((\alpha\mathrm{w}(w-\gamma\mathrm{w})^{\frac{-1+\theta}{\theta}} + \alpha\mathrm{x}(x-\gamma\mathrm{x})^{\frac{-1+\theta}{\theta}} + \alpha\mathrm{y}(y-\gamma\mathrm{y})^{\frac{-1+\theta}{\theta}} + \alpha\mathrm{z}(z-\gamma\mathrm{z})^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-\delta}(-1+\theta)(-1+\frac{\theta}{-1+\theta}))$$

$\frac{\partial^2 u}{\partial x \partial z} =$
$-\alpha x \alpha z(x-\gamma x)^{-1+\frac{-1+\theta}{\theta}}$
$(\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{-2+\frac{2\theta}{-1+\theta}}$
$((\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-1-\delta}(z-\gamma z)^{-1+\frac{-1+\theta}{\theta}}\delta+\frac{1}{\theta}$
$(\alpha x \alpha z(x-\gamma x)^{-1+\frac{-1+\theta}{\theta}}$
$(\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{-2+\frac{\theta}{-1+\theta}}$
$((\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-\delta}(z-\gamma z)^{-1+\frac{-1+\theta}{\theta}}$
$(-1+\theta)(-1+\frac{\theta}{-1+\theta}))$

$\frac{\partial^2 u}{\partial y \partial z} =$
$-\alpha y \alpha z(y-\gamma y)^{-1+\frac{-1+\theta}{\theta}}$
$(\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{-2+\frac{2\theta}{-1+\theta}}$
$((\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-1-\delta}(z-\gamma z)^{-1+\frac{-1+\theta}{\theta}}\delta+\frac{1}{\theta}$
$(\alpha y \alpha z(y-\gamma y)^{-1+\frac{-1+\theta}{\theta}}$
$(\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{-2+\frac{\theta}{-1+\theta}}$
$((\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-\delta}(z-\gamma z)^{-1+\frac{-1+\theta}{\theta}}$
$(-1+\theta)(-1+\frac{\theta}{-1+\theta}))$

$\frac{\partial^2 u}{\partial w \partial x} =$
$-\alpha w \alpha x(w-\gamma w)^{-1+\frac{-1+\theta}{\theta}}(x-\gamma x)^{-1+\frac{-1+\theta}{\theta}}$
$(\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{-2+\frac{2\theta}{-1+\theta}}$
$((\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-1-\delta}\delta+$
$\frac{1}{\theta}(\alpha w \alpha x(w-\gamma w)^{-1+\frac{-1+\theta}{\theta}}(x-\gamma x)^{-1+\frac{-1+\theta}{\theta}}$
$(\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$

$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{-2+\frac{\theta}{-1+\theta}}$
$((\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-\delta}(-1+\theta)(-1+\frac{\theta}{-1+\theta}))$

$\frac{\partial^2 u}{\partial w \partial y}=$
$-\alpha w \alpha z(w-\gamma w)^{-1+\frac{-1+\theta}{\theta}}$
$(\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{-2+\frac{2\theta}{-1+\theta}}$
$((\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-1-\delta}(z-\gamma z)^{-1+\frac{-1+\theta}{\theta}}\delta+\frac{1}{\theta}$
$(\alpha w \alpha z(w-\gamma w)^{-1+\frac{-1+\theta}{\theta}}$
$(\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{-2+\frac{\theta}{-1+\theta}}$
$((\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-\delta}(z-\gamma z)^{-1+\frac{-1+\theta}{\theta}}$
$(-1+\theta)(-1+\frac{\theta}{-1+\theta}))$

$\frac{\partial^2 u}{\partial w \partial z}=$
$-\alpha w \alpha y(w-\gamma w)^{-1+\frac{-1+\theta}{\theta}}(y-\gamma y)^{-1+\frac{-1+\theta}{\theta}}$
$(\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{-2+\frac{2\theta}{-1+\theta}}$
$((\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-1-\delta}\delta+$
$\frac{1}{\theta}(\alpha w \alpha y(w-\gamma w)^{-1+\frac{-1+\theta}{\theta}}(y-\gamma y)^{-1+\frac{-1+\theta}{\theta}}$
$(\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{-2+\frac{\theta}{-1+\theta}}$
$((\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-\delta}(-1+\theta)(-1+\frac{\theta}{-1+\theta}))$

$\frac{\partial^2 u}{\partial x^2}=$
$\frac{1}{1-\delta}(-\alpha x^2(x-\gamma x)^{-2+\frac{2(-1+\theta)}{\theta}}$
$(\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$
$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{-2+\frac{2\theta}{-1+\theta}}$

$((\alpha\text{w}(w-\gamma\text{w})^{\frac{-1+\theta}{\theta}}+\alpha\text{x}(x-\gamma\text{x})^{\frac{-1+\theta}{\theta}}+\alpha\text{y}(y-\gamma\text{y})^{\frac{-1+\theta}{\theta}}+$
$\alpha\text{z}(z-\gamma\text{z})^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-1-\delta}(1-\delta)\delta+$
$\alpha\text{x}(x-\gamma\text{x})^{-2+\frac{-1+\theta}{\theta}}$
$(\alpha\text{w}(w-\gamma\text{w})^{\frac{-1+\theta}{\theta}}+\alpha\text{x}(x-\gamma\text{x})^{\frac{-1+\theta}{\theta}}+\alpha\text{y}(y-\gamma\text{y})^{\frac{-1+\theta}{\theta}}+$
$\alpha\text{z}(z-\gamma\text{z})^{\frac{-1+\theta}{\theta}})^{-1+\frac{\theta}{-1+\theta}}$
$((\alpha\text{w}(w-\gamma\text{w})^{\frac{-1+\theta}{\theta}}+\alpha\text{x}(x-\gamma\text{x})^{\frac{-1+\theta}{\theta}}+\alpha\text{y}(y-\gamma\text{y})^{\frac{-1+\theta}{\theta}}+$
$\alpha\text{z}(z-\gamma\text{z})^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-\delta}(1-\delta)(-1+\frac{-1+\theta}{\theta})+$
$\frac{1}{\theta}(\alpha\text{x}^2(x-\gamma\text{x})^{-2+\frac{2(-1+\theta)}{\theta}}$
$(\alpha\text{w}(w-\gamma\text{w})^{\frac{-1+\theta}{\theta}}+\alpha\text{x}(x-\gamma\text{x})^{\frac{-1+\theta}{\theta}}+\alpha\text{y}(y-\gamma\text{y})^{\frac{-1+\theta}{\theta}}+$
$\alpha\text{z}(z-\gamma\text{z})^{\frac{-1+\theta}{\theta}})^{-2+\frac{\theta}{-1+\theta}}$
$((\alpha\text{w}(w-\gamma\text{w})^{\frac{-1+\theta}{\theta}}+\alpha\text{x}(x-\gamma\text{x})^{\frac{-1+\theta}{\theta}}+\alpha\text{y}(y-\gamma\text{y})^{\frac{-1+\theta}{\theta}}+$
$\alpha\text{z}(z-\gamma\text{z})^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-\delta}(1-\delta)(-1+\theta)$
$(-1+\frac{\theta}{-1+\theta})))$

$\frac{\partial^2 u}{\partial y^2}=$
$\frac{1}{1-\delta}(-\alpha\text{y}^2(y-\gamma\text{y})^{-2+\frac{2(-1+\theta)}{\theta}}$
$(\alpha\text{w}(w-\gamma\text{w})^{\frac{-1+\theta}{\theta}}+\alpha\text{x}(x-\gamma\text{x})^{\frac{-1+\theta}{\theta}}+\alpha\text{y}(y-\gamma\text{y})^{\frac{-1+\theta}{\theta}}+$
$\alpha\text{z}(z-\gamma\text{z})^{\frac{-1+\theta}{\theta}})^{-2+\frac{2\theta}{-1+\theta}}$
$((\alpha\text{w}(w-\gamma\text{w})^{\frac{-1+\theta}{\theta}}+\alpha\text{x}(x-\gamma\text{x})^{\frac{-1+\theta}{\theta}}+\alpha\text{y}(y-\gamma\text{y})^{\frac{-1+\theta}{\theta}}+$
$\alpha\text{z}(z-\gamma\text{z})^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-1-\delta}(1-\delta)\delta+$
$\alpha\text{y}(y-\gamma\text{y})^{-2+\frac{-1+\theta}{\theta}}$
$(\alpha\text{w}(w-\gamma\text{w})^{\frac{-1+\theta}{\theta}}+\alpha\text{x}(x-\gamma\text{x})^{\frac{-1+\theta}{\theta}}+\alpha\text{y}(y-\gamma\text{y})^{\frac{-1+\theta}{\theta}}+$
$\alpha\text{z}(z-\gamma\text{z})^{\frac{-1+\theta}{\theta}})^{-1+\frac{\theta}{-1+\theta}}$
$((\alpha\text{w}(w-\gamma\text{w})^{\frac{-1+\theta}{\theta}}+\alpha\text{x}(x-\gamma\text{x})^{\frac{-1+\theta}{\theta}}+\alpha\text{y}(y-\gamma\text{y})^{\frac{-1+\theta}{\theta}}+$
$\alpha\text{z}(z-\gamma\text{z})^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-\delta}(1-\delta)(-1+\frac{-1+\theta}{\theta})+$
$\frac{1}{\theta}(\alpha\text{y}^2(y-\gamma\text{y})^{-2+\frac{2(-1+\theta)}{\theta}}$
$(\alpha\text{w}(w-\gamma\text{w})^{\frac{-1+\theta}{\theta}}+\alpha\text{x}(x-\gamma\text{x})^{\frac{-1+\theta}{\theta}}+\alpha\text{y}(y-\gamma\text{y})^{\frac{-1+\theta}{\theta}}+$
$\alpha\text{z}(z-\gamma\text{z})^{\frac{-1+\theta}{\theta}})^{-2+\frac{\theta}{-1+\theta}}$
$((\alpha\text{w}(w-\gamma\text{w})^{\frac{-1+\theta}{\theta}}+\alpha\text{x}(x-\gamma\text{x})^{\frac{-1+\theta}{\theta}}+\alpha\text{y}(y-\gamma\text{y})^{\frac{-1+\theta}{\theta}}+$
$\alpha\text{z}(z-\gamma\text{z})^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-\delta}(1-\delta)(-1+\theta)$
$(-1+\frac{\theta}{-1+\theta})))$

$\frac{\partial^2 u}{\partial z^2}=$
$\frac{1}{1-\delta}(-\alpha\text{z}^2$

$$(\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$$
$$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{-2+\frac{2\theta}{-1+\theta}}$$
$$((\alpha w(w-\gamma w)^{\frac{-1+\theta}{\theta}}+\alpha x(x-\gamma x)^{\frac{-1+\theta}{\theta}}+\alpha y(y-\gamma y)^{\frac{-1+\theta}{\theta}}+$$
$$\alpha z(z-\gamma z)^{\frac{-1+\theta}{\theta}})^{\frac{\theta}{-1+\theta}})^{-1-\delta}(z-\gamma z)^{-2+\frac{2(-1+\theta)}{\theta}}$$

With the second order derivatives derived above, the following Hessian matrix can be constructed:

$$\begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial x \partial z} & \frac{\partial^2 u}{\partial w \partial x} \\ \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial y \partial z} & \frac{\partial^2 u}{\partial w \partial y} \\ \frac{\partial^2 u}{\partial x \partial z} & \frac{\partial^2 u}{\partial y \partial z} & \frac{\partial^2 u}{\partial z^2} & \frac{\partial^2 u}{\partial w \partial z} \\ \frac{\partial^2 u}{\partial w \partial x} & \frac{\partial^2 u}{\partial w \partial y} & \frac{\partial^2 u}{\partial w \partial z} & \frac{\partial^2 u}{\partial w^2} \end{bmatrix}$$

For the function u to be strictly concave the leading minors must have alternating signs, with the first leading principal minor being negative. The leading minors are the determinants of the symmetric submatrices of the Hessian matrix formed by the first n rows and columns.

The first order principal minor is just the determinant of $\frac{\partial^2 u}{\partial x^2}$. Mathematica gives the following simplified expression:

$$\frac{\partial^2 u}{\partial x^2} =$$
$$(\alpha x(x-\gamma x)^{-2/\theta}$$
$$((\alpha w(w-\gamma w)^{1-\frac{1}{\theta}}+\alpha x(x-\gamma x)^{1-\frac{1}{\theta}}+\alpha y(y-\gamma y)^{1-\frac{1}{\theta}}+$$
$$\alpha z(z-\gamma z)^{1-\frac{1}{\theta}})^{\frac{\theta}{-1+\theta}})^{1-\delta}$$
$$(-\alpha x\delta+\frac{\alpha x}{\theta}-\frac{1}{\theta}((x-\gamma x)^{-1+\frac{1}{\theta}}$$
$$(\alpha w(w-\gamma w)^{1-\frac{1}{\theta}}+\alpha x(x-\gamma x)^{1-\frac{1}{\theta}}+\alpha y(y-\gamma y)^{1-\frac{1}{\theta}}+$$
$$\alpha z(z-\gamma z)^{1-\frac{1}{\theta}}))))/$$
$$(\alpha w(w-\gamma w)^{1-\frac{1}{\theta}}+\alpha x(x-\gamma x)^{1-\frac{1}{\theta}}+\alpha y(y-\gamma y)^{1-\frac{1}{\theta}}+$$
$$\alpha z(z-\gamma z)^{1-\frac{1}{\theta}})^2$$

In the domain of definition of u, the terms $x-\gamma x$, $y-\gamma y$, $z-\gamma z$ and $w-\gamma$ as well as $\alpha x, \alpha y, \alpha z$ and αw are positive; therefore all terms of the numerator, up to and including $\alpha z(z-\gamma z)^{1-\frac{1}{\theta}}$, are positive.

In the third line of the numerator we have one negative and one positive additive term, whose sum is $-\alpha x(\delta-\frac{1}{\theta})$, which, since $\delta > 1$ and $\theta > 1$, is negative. Added to the negative bracket in the last part of the numerator and multiplied by the positive first part, the result is a negative expression. Since the denominator is squared and therefore always positive, the first order leading principal minor is negative.

The second order principal minor is the determinant of the 2x2 submatrix:

$$\begin{vmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} \end{vmatrix}$$

Mathematica simplifies the determinant to:

$$\begin{aligned}
&(\alpha \mathrm{x} \alpha \mathrm{y}(w-\gamma \mathrm{w})^{2/\theta}(x-\gamma \mathrm{x})^{-1+\frac{1}{\theta}}(y-\gamma \mathrm{y})^{-1+\frac{1}{\theta}} \\
&(\alpha \mathrm{w}(w-\gamma \mathrm{w})^{1-\frac{1}{\theta}}+\alpha \mathrm{x}(x-\gamma \mathrm{x})^{1-\frac{1}{\theta}}+\alpha \mathrm{y}(y-\gamma \mathrm{y})^{1-\frac{1}{\theta}}+ \\
&\alpha \mathrm{z}(z-\gamma \mathrm{z})^{1-\frac{1}{\theta}})^{\frac{2\theta}{-1+\theta}} \\
&((\alpha \mathrm{w}(w-\gamma \mathrm{w})^{1-\frac{1}{\theta}}+\alpha \mathrm{x}(x-\gamma \mathrm{x})^{1-\frac{1}{\theta}}+\alpha \mathrm{y} \\
&(y-\gamma \mathrm{y})^{1-\frac{1}{\theta}}+\alpha \mathrm{z}(z-\gamma \mathrm{z})^{1-\frac{1}{\theta}})^{\frac{\theta}{-1+\theta}})^{-2\delta} \\
&(z-\gamma \mathrm{z})^{2/\theta} \\
&(\alpha \mathrm{z}(w-\gamma \mathrm{w})^{\frac{1}{\theta}}(x-\gamma \mathrm{x})^{\frac{1}{\theta}}(y-\gamma \mathrm{y})^{\frac{1}{\theta}}(z-\gamma \mathrm{z})+ \\
&(z-\gamma \mathrm{z})^{\frac{1}{\theta}}(\alpha \mathrm{w}(w-\gamma \mathrm{w})(x-\gamma \mathrm{x})^{\frac{1}{\theta}}(y-\gamma \mathrm{y})^{\frac{1}{\theta}}+ \\
&(w-\gamma \mathrm{w})^{\frac{1}{\theta}}(\alpha \mathrm{y}(x-\gamma \mathrm{x})^{\frac{1}{\theta}}(y-\gamma \mathrm{y})+ \\
&\alpha \mathrm{x}(x-\gamma \mathrm{x})(y-\gamma \mathrm{y})^{\frac{1}{\theta}})\delta\theta)))/ \\
&((\alpha \mathrm{z}(w-\gamma \mathrm{w})^{\frac{1}{\theta}}(x-\gamma \mathrm{x})^{\frac{1}{\theta}}(y-\gamma \mathrm{y})^{\frac{1}{\theta}}(z-\gamma \mathrm{z})+ \\
&(\alpha \mathrm{y}(w-\gamma \mathrm{w})^{\frac{1}{\theta}}(x-\gamma \mathrm{x})^{\frac{1}{\theta}}(y-\gamma \mathrm{y})+ \\
&(\alpha \mathrm{x}(w-\gamma \mathrm{w})^{\frac{1}{\theta}}(x-\gamma \mathrm{x})+ \\
&\alpha \mathrm{w}(w-\gamma \mathrm{w})(x-\gamma \mathrm{x})^{\frac{1}{\theta}})(y-\gamma \mathrm{y})^{\frac{1}{\theta}}) \\
&(z-\gamma \mathrm{z})^{\frac{1}{\theta}})^{3}\theta^{2})
\end{aligned}$$

All terms in the numerator are of the form $a(b-c)$, where $(b-c)$ is one of the following: $x-\gamma x, y-\gamma y, z-\gamma z, w-\gamma w$, all of which are positive in the domain of u, and a is $\gamma x, \gamma y, \gamma z$ or γw, which is also defined as always positive. Since the terms are combined by either addition or multiplication the whole numerator is positive. The denominator is composed as above, therefore it is also positive, hence the second order leading principal minor is positive.

The third order principal minor is the determinant of the 3x3 submatrix.

$$\begin{vmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial x \partial z} \\ \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial y \partial z} \\ \frac{\partial^2 u}{\partial x \partial z} & \frac{\partial^2 u}{\partial y \partial z} & \frac{\partial^2 u}{\partial z^2} \end{vmatrix}$$

Mathematica calculates the determinant of this matrix and simplifies it to:

$$\begin{aligned}
&-(\alpha\mathrm{x}\alpha\mathrm{y}\alpha\mathrm{z}(w-\gamma\mathrm{w})^{3/\theta}(x-\gamma\mathrm{x})^{-1+\frac{2}{\theta}}(y-\gamma\mathrm{y})^{-1+\frac{2}{\theta}} \\
&(\alpha\mathrm{w}(w-\gamma\mathrm{w})^{1-\frac{1}{\theta}}+\alpha\mathrm{x}(x-\gamma\mathrm{x})^{1-\frac{1}{\theta}}+\alpha\mathrm{y}(y-\gamma\mathrm{y})^{1-\frac{1}{\theta}}+ \\
&\alpha\mathrm{z}(z-\gamma\mathrm{z})^{1-\frac{1}{\theta}})^{\frac{3\theta}{-1+\theta}} \\
&((\alpha\mathrm{w}(w-\gamma\mathrm{w})^{1-\frac{1}{\theta}}+\alpha\mathrm{x}(x-\gamma\mathrm{x})^{1-\frac{1}{\theta}}+\alpha\mathrm{y}(y-\gamma\mathrm{y})^{1-\frac{1}{\theta}}+ \\
&\alpha\mathrm{z}(z-\gamma\mathrm{z})^{1-\frac{1}{\theta}})^{\frac{\theta}{-1+\theta}})^{-3\delta}(z-\gamma\mathrm{z})^{-1+\frac{2}{\theta}} \\
&(\alpha\mathrm{w}(w-\gamma\mathrm{w})(x-\gamma\mathrm{x})^{\frac{1}{\theta}}(y-\gamma\mathrm{y})^{\frac{1}{\theta}}(z-\gamma\mathrm{z})^{\frac{1}{\theta}}+ \\
&(w-\gamma\mathrm{w})^{\frac{1}{\theta}}(\alpha\mathrm{z}(x-\gamma\mathrm{x})^{\frac{1}{\theta}}(y-\gamma\mathrm{y})^{\frac{1}{\theta}}(z-\gamma\mathrm{z})+ \\
&(\alpha\mathrm{y}(x-\gamma\mathrm{x})^{\frac{1}{\theta}}(y-\gamma\mathrm{y})+\alpha\mathrm{x}(x-\gamma\mathrm{x})(y-\gamma\mathrm{y})^{\frac{1}{\theta}}) \\
&(z-\gamma\mathrm{z})^{\frac{1}{\theta}})\delta\theta))/ \\
&((\alpha\mathrm{z}(w-\gamma\mathrm{w})^{\frac{1}{\theta}}(x-\gamma\mathrm{x})^{\frac{1}{\theta}}(y-\gamma\mathrm{y})^{\frac{1}{\theta}}(z-\gamma\mathrm{z})+ \\
&(\alpha\mathrm{y}(w-\gamma\mathrm{w})^{\frac{1}{\theta}}(x-\gamma\mathrm{x})^{\frac{1}{\theta}}(y-\gamma\mathrm{y})+ \\
&(\alpha\mathrm{x}(w-\gamma\mathrm{w})^{\frac{1}{\theta}}(x-\gamma\mathrm{x})+\alpha\mathrm{w}(w-\gamma\mathrm{w})(x-\gamma\mathrm{x})^{\frac{1}{\theta}}) \\
&(y-\gamma\mathrm{y})^{\frac{1}{\theta}})(z-\gamma\mathrm{z})^{\frac{1}{\theta}})^4\theta^3)
\end{aligned}$$

All additive and multiplicative terms in the denominator are positive; hence the whole denominator is positive. All terms in the bracket within the numerator are positive; a negative sign in front of the bracket turns the fraction negative.

The fourth order principal leading minor is the determinant of the whole matrix:

$$\begin{vmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial x \partial z} & \frac{\partial^2 u}{\partial w \partial x} \\ \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial y \partial z} & \frac{\partial^2 u}{\partial w \partial y} \\ \frac{\partial^2 u}{\partial x \partial z} & \frac{\partial^2 u}{\partial y \partial z} & \frac{\partial^2 u}{\partial z^2} & \frac{\partial^2 u}{\partial w \partial z} \\ \frac{\partial^2 u}{\partial w \partial x} & \frac{\partial^2 u}{\partial w \partial y} & \frac{\partial^2 u}{\partial w \partial z} & \frac{\partial^2 u}{\partial w^2} \end{vmatrix}$$

Mathematica calculates the determinant and simplifies it to:

$$(\alpha w\alpha x\alpha y\alpha z(w-\gamma w)^{-1+\frac{3}{\theta}}(x-\gamma x)^{-1+\frac{3}{\theta}}(y-\gamma y)^{-1+\frac{3}{\theta}}$$
$$(\alpha w(w-\gamma w)^{1-\frac{1}{\theta}}+\alpha x(x-\gamma x)^{1-\frac{1}{\theta}}+\alpha y(y-\gamma y)^{1-\frac{1}{\theta}}+$$
$$\alpha z(z-\gamma z)^{1-\frac{1}{\theta}})^{\frac{4\theta}{-1+\theta}}$$
$$((\alpha w(w-\gamma w)^{1-\frac{1}{\theta}}+\alpha x(x-\gamma x)^{1-\frac{1}{\theta}}+\alpha y(y-\gamma y)^{1-\frac{1}{\theta}}+$$
$$\alpha z(z-\gamma z)^{1-\frac{1}{\theta}})^{\frac{\theta}{-1+\theta}})^{-4\delta}(z-\gamma z)^{-1+\frac{3}{\theta}}\delta)/$$
$$((\alpha z(w-\gamma w)^{\frac{1}{\theta}}(x-\gamma x)^{\frac{1}{\theta}}(y-\gamma y)^{\frac{1}{\theta}}(z-\gamma z)+$$
$$(\alpha y(w-\gamma w)^{\frac{1}{\theta}}(x-\gamma x)^{\frac{1}{\theta}}(y-\gamma y)+$$
$$(\alpha x(w-\gamma w)^{\frac{1}{\theta}}(x-\gamma x)+\alpha w(w-\gamma w)(x-\gamma x)^{\frac{1}{\theta}})$$
$$(y-\gamma y)^{\frac{1}{\theta}})(z-\gamma z)^{\frac{1}{\theta}})^{4}\theta^{3})$$

All additive and multiplicative terms in the numerator and in the denominator are positive; therefore, the fourth order principal minor is positive.

The leading minors associated with the function u alternate in signs starting with a negative first order principal leading minor. Therefore, the function u is concave, and a critical point of u is the global maximizer of u on $\mathbb{R}^4$.

Appendix D

Aggregation of the composite utility function

This chapter tackles the question of whether it is possible to assume an optimizing representative agent for the composite utility function developed in this thesis.

Microeconomic theory analyzes the behavior of individuals. But an interesting question which economic models are supposed to answer is not only how one single consumer behaves but also how a multitude of consumers influence the economy. Therefore, it is important to find a way of applying microeconomic theory to the aggregate data of individuals.

Transferring the behavior of many individuals to a single virtual aggregate individual is called the "aggregation problem" (Deaton and Muellbauer (1980), p.148 [54], Kirman (1992), p.120 [100]). The question underlying the representative agent is: When does a society of utility maximizing individuals behave like a single individual? Or to put it differently, under which conditions do market demand functions exist which share the properties of individual demand functions?

The first economist who analyzed the necessary conditions for aggregation was Gorman (1953) [78], (1961) [79][1]. He showed that:

> *"If and only if all consumers have preferences that admit indirect utility functions of the Gorman form[2] with equal wealth coeffi-*

[1] Gorman's results were independently obtained by Antonelli (1886) [4] and Nataf (1951) [129], but Gorman was the first to state the general conditions under which aggregation is possible.

[2] A more detailed explanation of this function comes later in this text.

cients, aggregate demand can be written as a function of aggregate wealth" (Mas-Colell (1995), p.108 [119])

D.1 The general case

Using Gorman's results, it is possible to prove that demand functions can be aggregated. Let $q_{i,h}$ be the demand for good i of household h, y_h the income of household h and p a vector of prices. The demand function $g_{i,h}$ is:

$q_{i,h} = g_{i,h}(y_h, p)$

Let H be the number of all households and $\bar{q}_i$ the average demand:

$\bar{q}_i = \frac{1}{H}\sum_h g_{i,h}(y_h, p)$

and let $\bar{y}$ be the average income:

$\bar{y} = \frac{1}{H}\sum_h y_h$

Then there exists an aggregated demand function if $\bar{q}_i$ can be written as follows:

$\bar{q}_i = g_i(\bar{y}, p)$

Any redistribution of income between two households which does not change the aggregated income does not change the average income either. When taking income from household 1 and giving it to household 2, the fall in $g_{i,1}(p, y_1)$ must equal the rise in $g_{i,2}(p, y_2)$ for all p, y_1, y_2. It follows therefore:

$\frac{\partial g_{i,1}(p,y_1)}{\partial y_1} = \frac{\partial g_{i,2}(p,y_2)}{\partial y_2}$

The latter expression shows that the two demand functions of income[3] have the same slope. Since this is true for all pairs of income, the slope of these functions is independent of the income. For all households h there is now a function which depends only on p (this function is also an expression for the marginal propensity to consume, which has to be identical for all consumers):

[3]These are the equivalents of the Engel curves, which are explained in Chapter 4.

$\frac{\partial g_{i,h}(p,y_h)}{\partial y^h} = b_i(p)$

Integrating over y_h gives:

$g_{i,h}(p, y_h) = a_{i,h}(p) + b_i(p)y_h$

with $a_{i,h}(p)$ being the constant of integration. Any function $g_{i,h}$ must be linear (or homogeneous of degree one) in income and have the same slope with respect to y_h. That in turn means that every demand function which yields Engel curves that are straight lines and parallel to Engel curves of the other households can be aggregated (this outcome will be used later).

Applying Roy's identity[4] to $g_{i,h}$ yields the polar Gorman form[5] of indirect utility:

$v_{i,h}(p, y_h) = \alpha_{i,h}(p) + \beta_i(p)y_h$

> *"It can be shown that the Gorman form is the most general of the indirect utility functions that will allow for aggregation in the sense of the representative consumer model. Hence, the Gorman form is not only sufficient for the representative consumer model to hold, but it is also necessary."* (Varian (1992), p.154 [172])[6]

D.2 The composite utility function

This section will show that the demand function of the CRRA-CES-Stone-Geary utility function can be aggregated. The utility function in the German life cycle model includes subsistence levels and is therefore a special case, but Gorman's theory can be extended to demand systems general enough to include this case. To show the possibility of using a representative agent in the German life cycle model, first the demand function will be derived using the

[4]This identity relates an ordinary demand function to the derivatives of the indirect utility function, see Takayma (1985), p.139 [169] for a more detailed explanation.

[5]Gorman himself called this function *"the polar form of the underlying utility function"* Gorman (1961), p.54 [79].

[6]For a closer analysis of the proof for necessary conditions for aggregation and the application of Roy's identity see Deaton and Muellbauer (1980), pp.148 [54], Lewis (2003) [112], Gorman (1953) [78], Gorman (1961) [79], Mas-Colell (1995), p.109 [119] and Varian, p.152 [172].

Lagrange optimization method, followed by the proof that the CRRA-CES-Stone-Geary utility function is homogeneous of degree one and has identical slopes for different average incomes.

These are the utility function and the constraints for n goods.

$$u(q_j) = \frac{\left(\left(\sum_{j=1}^{n} \beta_j (q_j - \gamma_j)^{\theta-1/\theta}\right)^{\theta/\theta-1}\right)^{1-\sigma}}{1-\sigma}, \theta > 1, \gamma_j > 0, q_j - \gamma_j > 0, \sigma \neq 1$$

s.t.: $y = \sum_{j=1}^{n} p_j q_j$

For the utility function with two goods the demand function will be derived; the CRRA extension can be ignored since it is a constant which neither influences the result of the optimization nor the demand function.

For two goods q_1 and q_2 the utility function and the constraint are:

$$u(q_1, q_2) = (\beta_1(q_1 - \gamma_1)^{\theta-1/\theta}) + \beta_2(q_2 - \gamma_2)^{\theta-1/\theta}))^{\theta/\theta-1},$$
$$\theta > 1, \gamma_j > 0, q_j - \gamma_j > 0$$

s.t.: $y = p_1 q_1 + p_2 q_2$

The Lagrange-function is:

$$L = (\beta_1(q_1 - \gamma_1)^{\theta-1/\theta}) + \beta_2(q_2 - \gamma_2)^{\theta-1/\theta}))^{\theta/\theta-1} + \lambda(p_1 q_1 + p_2 q_2 - y)$$

Differentiating with respect to q_1, q_2, λ and setting the equations to zero:

$$\frac{\delta L}{\delta q_1} = \beta_1(q_1 - \gamma_1)^{\frac{\theta-1}{\theta}-1}(\beta_1(q_1 - \gamma_1)^{\frac{\theta-1}{\theta}} + \beta_2(q_2 - \gamma_2)^{\frac{\theta-1}{\theta}})^{\frac{\theta}{\theta-1}-1} + p_1\lambda = 0$$

$$\frac{\delta L}{\delta q_2} = \beta_2(q_2 - \gamma_2)^{\frac{\theta-1}{\theta}-1}(\beta_1(q_1 - \gamma_1)^{\frac{\theta-1}{\theta}} + \beta_2(q_2 - \gamma_2)^{\frac{\theta-1}{\theta}})^{\frac{\theta}{\theta-1}-1} + p_2\lambda = 0$$

$$\frac{\delta L}{\lambda} = p_1 q_1 + p_2 q_2 - y = 0$$

Solving the first two equations for λ:

$$\lambda = \frac{\beta_1(q_1-\gamma_1)^{\frac{\theta-1}{\theta}-1}(\beta_1(q_1-\gamma_1)^{\frac{\theta-1}{\theta}} + \beta_2(q_2-\gamma_2)^{\frac{\theta-1}{\theta}})^{\frac{\theta}{\theta-1}-1}}{p_1}$$

$$\lambda = \frac{\beta_2(q_2-\gamma_2)^{\frac{\theta-1}{\theta}-1}(\beta_1(q_1-\gamma_1)^{\frac{\theta-1}{\theta}} + \beta_2(q_2-\gamma_2)^{\frac{\theta-1}{\theta}})^{\frac{\theta}{\theta-1}-1}}{p_2}$$

Setting them equal and solving for q_1:

$q_1 = p_1^{-\theta}\beta_2^{-\theta}(q_2 p_2^\theta \beta_1^\theta + \gamma_1 p_1^\theta \beta_2^\theta - \gamma_2 p_2^\theta \beta_1^\theta)$

Solving $\frac{\delta L}{\lambda}$ for q_1:

$q_1 = \frac{y - p_2 q_2}{p_1}$

Setting the two equations above equal and solving for q_2:

$q_2 = \frac{y p_1^\theta \beta_1^\theta - p_1^{\theta+1}\gamma_1 \beta_2^\theta + \gamma_2 p_1 p_2^\theta \beta_1^\theta}{p_2 p_1^\theta \beta_1^\theta + p_1 p_2^\theta \beta_2^\theta}$

Since the function is symmetric for the two goods, the demand function for q_1 is:

$q_1 = \frac{y p_2^\theta \beta_1^\theta - p_2^{\theta+1}\gamma_1 \beta_1^\theta + \gamma_1 p_2 p_1^\theta \beta_2^\theta}{p_1 p_2^\theta \beta_1^\theta + p_2 p_1^\theta \beta_2^\theta}$

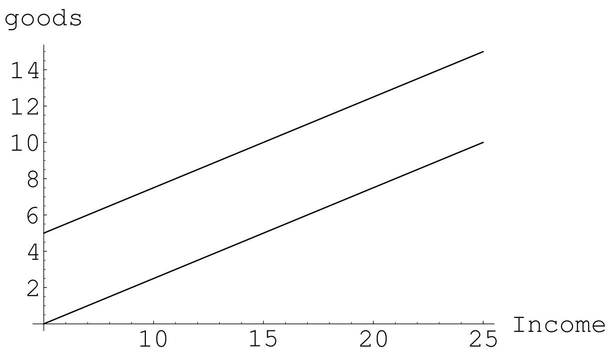

Figure D.1: The Engel curve for a good with and without subsistence level

Figure D.1 shows the Engel curves for one good with a subsistence level of 5 and for one good without a subsistence level. Since the demand function for the good with subsistence level is only defined for $y > \gamma_1 + \gamma_2$, the basic income has to be five and the Engel curve intersects the origin at an income of five. The Engel curve for the demand curve without subsistence levels has already intersected the origin at an income of zero, shown in Figure D.1. An income of five corresponds to a demand of five units of the good.

A function f is homogeneous of degree k if

$f(sx_1, sx_2, \ldots, sx_n) = s^k\, f(x_1, x_2, \ldots, x_n)$

For the case considered here:

$g_2(p_1, p_2, sy) = \frac{syp_1^\theta \beta_1^\theta - p_1^{\theta+1}\gamma_1\beta_2^\theta + \gamma_2 p_1 p_2^\theta \beta_1^\theta}{p_2 p_1^\theta \beta_1^\theta + p_1 p_2^\theta \beta_2^\theta} = s\, g_2(p_1, p_2, y),$

therefore the demand function is homogeneous of degree one and the Engel curve linear (equivalent for $g_1(p_1, p_2, y)$).

Differentiating $g_2(p_1, p_2, y)$ with respect to y:

$\frac{\partial g_2(p_1,p_2,y)}{\partial y} = \frac{p_1}{p_1^\theta p_2 + p_1 p_2^\theta}$

This derivation can be generalized for every good. The second derivative of $g_2(p_1, p_2, y)$ with respect to y is zero. The slope of the demand functions does, therefore, not depend on the income. This means all Engel curves are parallel lines.

D.3 Objections and solutions

The use of a representative agent is object to criticisms and discussions.

> *"The necessary and sufficient condition quoted above is intuitively reasonable. It says, in effect, that an extra unit of purchasing power should be spent in the same way no matter to whom it is given."* Gorman (1953), p.64 [78]

What Gorman found "intuitively reasonable" is today widely debated. If an extra unit of income were taken from a rich and given to a poor consumer, the poor consumer would probably spend it in a way different from the rich consumer. This means that the distribution of income does play a role for the goods demanded, a fact that is not taken into account in the representative consumer model.[7] Kirman (1992) critizes:

[7] Many authors have considered this problem; See for example: Atkinson (1970) [8], Kirman [100], Cowell (1998) [49] or Wong(2002) [182].

> *"The reduction of the behavior of a group of heterogenous agents even if they are all themselves utility maximizers, is not simply an analytical convenience as often explained, but is both unjustified and leads to conclusions which are usually misleading and often wrong. ..., there is no plausible formal justification for the assumption that the aggregate of individuals, even maximizers, acts itself like an individual maximizer."* Kirman(1992), p.107 [100]

Muellbauer (1975) [128] and Muellbauer and Deaton (1980), pp.154 [54] developed a system which avoids the implausible restrictions on income effects. They introduced aggregation with nonlinear Engel curves by aggregating over expenditure patterns of different consumers, rather than over different consumers.

Another problematic issue arises when considering the CRRA-CES-Stone-Geary utility function used in the German life cycle model: no consumer is allowed to have an income smaller than the sum of his subsistence levels. This means that the representative agent has subsistence levels which are so low that rich consumers aggregated in this function wouldn't even take any notice of them, and very poor consumers could not be included in the aggregation.

In order to lessen the consequences of this problem in the German life cycle model, different groups of consumers with different incomes have been defined. When these groups are used as representative agents, the income distribution influences the outcome of the model, and poor agents who receive incomes above the subsistence level are able to spend their money in ways different from rich agents.

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