ON CHEMOTAXIS SYSTEMS WITH SATURATION GROWTH*

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Abstract: In this paper, we discuss the global existence of solutions for Chemotaxis models with saturation growth. If the coefficients of the equations are all positive smooth T-periodic functions, then the problem has a positive T-periodic solution, and meanwhile we discuss here the stability problems for the T-periodic solutions.

Key words: Saturation model, Chemotaxis, global solution, asymptotic stable.

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1 Introduction

Chemotaxis is a widespread phenomenon in biological systems (cf. [1-9]). Cells or organisms respond to chemical substance by motion and rearrangement (cf. [1]). They may move toward the higher concentration of the chemical substance or away from it, to search for food, to endure starvation conditions, to explore new regions, to reproduce or to give each other shelter. For example, the fruiting body cycle begins with the development of spores which germinate and develop in vegetative growth until starved of nutrients. In this latter case the vegetative growth aggregates to form a new fruiting body and to start the cycle once more. The myxobacteria are ubiquitous soil bacteria, which glide on suitable surfaces or at air-water interfaces. During gliding the myxobacteria produce so-call slime trails on which they prefer to glide. When a myxobacterium glides on bare substrate and encounters another slime trail at a relatively shallow angle, it will typically glide onto it. Under starvation conditions they tend to glide close to one another. During gliding they form different patterns and finally they aggregate to build so-called fruiting bodies (cf. [1, 6, 12]). Inside these fruiting bodies they survive as dormant myxospores. The mechanisms by which myxobacteria glide on the substrate and aggregate are still not understood and thus theoretical analysis of different

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mechanism is useful.

A chemotaxis process occurs also in the growth of a tumor. The tumor secretes chemical species that attract the nearby endothelial cells, which form the surface of capillary blood vessels. In this way new blood vessels sprout towards the tumor and begin to provide it with additional nourishment. The phenomenon of sprouting of new blood vessels is called angiogenesis.

Recently, Othmer and Stevens have developed some models [10] to describe these complicated processes which are far from completely understood. These kinds of models in generally consist of two main parts: One is called " master equation " as defined by

$$\frac{\partial p}{\partial t} = D\nabla \cdot (p\nabla(\ln \frac{p}{w})),$$

and the other is called the local dynamics for the control species:

$$\frac{\partial w}{\partial t} = F(x, t, p, w),$$

where $p(x,t)$ is the particle density of a particular species and $w(x,t)$ is the concentration of the " active agent " on some domain $\Omega \times (0,T)$. Recently many authors have discussed these models ([11-16]), and they found that from these models, many processes of aggregation, blow-up, as well as collapse can be described. Even though, we have just obtained very little information about the solutions for these kinds of models. In these papers, authors have studied the models for which the control species w, from the second equation, is linear or exponential growth, namely

(1) $\frac{\partial w}{\partial t} = \beta p - \mu w$, or

(2) $\frac{\partial w}{\partial t} = (\beta p - \mu)w$.

In fact, saturation with respect to p in the production of the control species is certainly more realistic in the biological context. It is also important to investigate the following model with saturation growth, which appeared in Othmer-Steven [10]:

$$\begin{cases} \frac{\partial p}{\partial t} = D \bigtriangledown \cdot (p \bigtriangledown (\ln(\frac{p}{w}))) & \\ & \text{for } x \in \Omega, t > 0, \\ \frac{\partial w}{\partial t} = \frac{\alpha_1(t)p}{\alpha_2(t) + \alpha_3(t)p} - \mu(t)w & \\ \bigtriangledown (\ln(\frac{p}{w})) \cdot \mathbf{n} = 0, & \text{for } x \in \partial\Omega, t > 0, \\ p(x,0) = p_0(x) > 0, & \\ & \text{for } x \in \bar{\Omega}, \\ w(x,0) = w_0(x) > 0, & \end{cases}$$

where $\alpha_i(t)$ $(i = 1, 2, 3)$, and $\mu(t)$ are smooth bounded positive functions, n is the outer normal of $\partial\Omega$. In this paper, we shall prove the existence of global solutions in section 2, and the problems for periodic solutions and stability of the solutions will be discussed in section 3.

2 The existence of global solutions

In this section we consider the following problem:

$$\begin{cases} \dfrac{\partial p}{\partial t} = D \bigtriangledown \cdot (p \bigtriangledown (\ln(\dfrac{p}{w}))) & \text{for } x \in \Omega, t > 0, \\ \dfrac{\partial w}{\partial t} = \dfrac{\alpha_1(t)p}{\alpha_2(t) + \alpha_3(t)p} - \mu(t)w & \\ \bigtriangledown (\ln(\dfrac{p}{w})) \cdot \mathbf{n} = 0, & \text{for } x \in \partial\Omega, t > 0, \\ p(x,0) = p_0(x) > 0, & \text{for } x \in \bar{\Omega}, \\ w(x,0) = w_0(x) > 0, & \end{cases} \tag{2.1}$$

where $\alpha_i(t)$ $(i = 1, 2, 3)$, and $\mu(t)$ are all smooth bounded positive functions. Observe that $p(x,t)$ and $w(x,t)$ are positive solutions in their definition domains. This is because that the initial data $(p_0(x), w_0(x))$ is positive, thus there exists $T_1 > 0$ such that $p(x,t) > 0$ for $0 < t < T_1$. Because of $\dfrac{\partial w}{\partial t} = \dfrac{\alpha_1(t)p}{\alpha_2(t) + \alpha_3(t)p} - \mu(t)w$, we have $(e^{\int_0^t \mu(s)ds} w)_t = \dfrac{\alpha_1(t)pe^{\int_0^t \mu(s)ds}}{\alpha_2(t) + \alpha_3(t)p}$, which implies that $(e^{\int_0^t \mu(s)ds} w)_t > 0$ for $0 < t < T_1$. If there exist $t_0 > 0$, $x_0 \in \bar{\Omega}$ such that $p(x,t) > 0$ for $(x,t) \in Q_{t_0}$ and $p(x_0,t_0) = 0$, then $w(x,t) > e^{-\int_0^t \mu(s)ds} w_0(x) > 0$, for $(x,t) \in \bar{Q}_{t_0}$. So we have that $\dfrac{p(x,t)}{w(x,t)} > 0$ for $(x,t) \in Q_{t_0}$ and $\dfrac{p(x_0,t_0)}{w(x_0,t_0)} = 0$. Thus we can introduce that $u(x,t) = \dfrac{p(x,t)}{w(x,t)}$, and $u_t = \dfrac{p_t}{w} - \dfrac{u}{w} w_t$, then from

$$\begin{aligned} p_t &= D \bigtriangledown \cdot (p \bigtriangledown \ln(\frac{p}{w})) = D \bigtriangledown \cdot (p \bigtriangledown \ln u) \\ &= D \bigtriangledown \cdot (p \frac{\bigtriangledown u}{u}) = D \bigtriangledown \cdot (w \bigtriangledown u) = Dw\Delta u + D(\bigtriangledown w) \cdot (\bigtriangledown u), \\ w_t &= \frac{\alpha_1(t)p}{\alpha_2(t) + \alpha_3(t)p} - \mu(t)w = \frac{\alpha_1(t)uw}{\alpha_2(t) + \alpha_3(t)uw} - \mu(t)w, \end{aligned}$$

we have

$$\begin{aligned} u_t &= D\Delta u + D\frac{1}{w}(\bigtriangledown w) \cdot (\bigtriangledown u) - \frac{u}{w}(\frac{\alpha_1(t)uw}{\alpha_2(t) + \alpha_3(t)uw} - \mu(t)w) \\ &= D\Delta u + D\frac{1}{w}(\bigtriangledown w) \cdot (\bigtriangledown u) + \mu(t)u - \frac{\alpha_1(t)u^2}{\alpha_2(t) + \alpha_3(t)uw}. \end{aligned}$$

Thus we know that $(p(x,t),w(x,t))$ is a solution of the system (2.1), if and only if $(u(x,t),w(x,t))$ is the solution of following system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\Delta u - D\dfrac{1}{w}(\bigtriangledown w)\cdot(\bigtriangledown u) - \mu(t)u + \dfrac{\alpha_1(t)u^2}{\alpha_2(t)+\alpha_3(t)uw} = 0 & \\ \dfrac{\partial w}{\partial t} = \dfrac{\alpha_1(t)uw}{\alpha_2(t)+\alpha_3(t)uw} - \mu(t)w, & \text{for } x\in\Omega, t>0, \\ \dfrac{\partial u}{\partial n} = 0 & \text{for } x\in\partial\Omega, t>0, \\ u(x,0) = u_0(x) = \dfrac{p_0(x)}{w_0(x)} & x\in\bar{\Omega}, t>0. \\ w(x,0) = w_0(x) & \end{cases} \tag{2.2}$$

We have

Theorem 2.1. *There exists a unique global solution for the problem (2.1).*

Proof. From the result of [15], the problem (2.1) has a unique local solution $(p(x,t),w(x,t))$ for $(x,t)\in Q_T=\{(x,t)\mid x\in\Omega, t\in(0,T)\}$, where $T>0$ is maximum existence time. We know

$$\frac{\partial w}{\partial t} = \frac{\alpha_1(t)p}{\alpha_2(t)+\alpha_3(t)p} - \mu(t)w,$$

thus

$$(e^{\int_0^t \mu(s)ds}w)_t = \frac{\alpha_1(t)pe^{\int_0^t \mu(s)ds}}{\alpha_2(t)+\alpha_3(t)p} \le \frac{\bar{\alpha}_1}{\underline{\alpha}_3}e^{\int_0^t \mu(s)ds},$$

where $\bar{\alpha}_i = \sup_{t\ge 0}\alpha_i(t) < +\infty$, $\underline{\alpha}_i = \inf_{t\ge 0}\alpha_i(t) > 0, (i=1,2,3)$. Thus

$$e^{\int_0^t \mu(s)ds}w(x,t) - w_0(x) \le \frac{\bar{\alpha}_1}{\underline{\alpha}_3}\int_0^t e^{\int_0^s \mu(s_1)ds_1}ds.$$

So we have, for $t>0$,

$$\begin{aligned} w(x,t) &\le e^{-\int_0^t \mu(s)ds}(w_0(x) + \frac{\bar{\alpha}_1}{\underline{\alpha}_3}\int_0^t e^{\int_0^s \mu(s_1)ds_1}ds) \\ &\le w_0(x) + \frac{\bar{\alpha}_1}{\underline{\alpha}_3\underline{\mu}}e^{-\int_0^t \mu(s)ds}\int_0^t \mu(s)e^{\int_0^s \mu(s_1)ds_1}ds \\ &\le w_0(x) + \frac{\bar{\alpha}_1}{\underline{\alpha}_3\underline{\mu}}(1-e^{-\int_0^t \mu(s)ds}) \\ &\le w_0(x) + \frac{\bar{\alpha}_1}{\underline{\alpha}_3\underline{\mu}} < +\infty \end{aligned}$$

where $\underline{\mu} = \inf_{t\ge 0}\mu(t) > 0$, that implies that $w(x,t)$ is finite in any $t>0$.

Secondly

$$\begin{aligned} u_t &= D\Delta u + D\frac{1}{w}(\bigtriangledown w)\cdot(\bigtriangledown u) + \mu(t)u - \frac{\alpha_1(t)u^2}{\alpha_2(t)+\alpha_3(t)uw} \\ &\geq D\Delta u + D\frac{1}{w}(\bigtriangledown w)\cdot(\bigtriangledown u) + \underline{\mu}u - \frac{\bar{\alpha}_1}{\underline{\alpha}_2}u^2. \end{aligned}$$

Obviously $u(x,t)$ is a super-solution of the following dynamics:

$$\begin{cases} \frac{\partial v}{\partial t} - D\Delta v - D\frac{1}{w}(\bigtriangledown w)\cdot(\bigtriangledown v) - \underline{\mu}v + \frac{\bar{\alpha}_1}{\underline{\alpha}_2}v^2 = 0 & \text{for } x\in\Omega, t>0, \\ \frac{\partial v}{\partial n} = 0, & \text{for } x\in\partial\Omega, t>0, \\ v(x,0) = \min_{x\in\bar{\Omega}} u_0(x) & \text{for } x\in\bar{\Omega}. \end{cases}$$

Similar to the argument in [16, section 2], we can prove that the problem above has a positive global solution $v(x,t)$, and $\lim_{t\to+\infty} v(x,t) = \frac{\underline{\mu}\underline{\alpha}_2}{\bar{\alpha}_1}$. Thus, by comparison principle, we can find a positive constant $\delta > 0$, such that $u(x,t) \geq v(x,t) \geq \delta > 0$, which implies that $u(x,t)$ is a strictly positive function. That ensures both functions $p(x,t)$ and $w(x,t)$ are positive in their definition domains.

Finally we prove that the problem (2.1) has a global solution. Since

$$\begin{aligned} u_t &= D\Delta u + D\frac{1}{w}(\bigtriangledown w)\cdot(\bigtriangledown u) + \mu(t)u - \frac{\alpha_1(t)u^2}{\alpha_2(t)+\alpha_3(t)uw} \\ &< D\Delta u + D\frac{1}{w}(\bigtriangledown w)\cdot(\bigtriangledown u) + \mu(t)u, \end{aligned}$$

we know that $u(x,t)$ is a sub-solution of the following problem:

$$\begin{cases} \frac{\partial v}{\partial t} - D\Delta v - D\frac{1}{w}(\bigtriangledown w)\cdot(\bigtriangledown v) - \mu(t)v = 0 & \text{for } x\in\Omega, t>0, \\ \frac{\partial v}{\partial n} = 0 & \text{for } x\in\partial\Omega, t>0, \\ v(x,0) = \overline{u_0} = \max_{x\in\bar{\Omega}} u_0(x) & x\in\bar{\Omega}, t>0. \end{cases}$$

It is easy to check that the function

$$v(x,t) = v(t) = \bar{u}_0 e^{\int_0^t \mu(s)ds} < +\infty$$

is the unique solution for the problem above. From the comparison principle, we have

$$u(x,t) < v(t) = \bar{u}_0 e^{\int_0^t \mu(s)ds},$$

so $u(x,t)$ does not blow up in finite time. Since $p(x,t) = u(x,t)w(x,t)$ and $w(x,t)$ is bounded above, which implies that the solution of (2.1) is the global solution.

For the problem (2.2), we can prove

Theorem 2.2 . *For any pair of positive initial data* $(u(x,0), w(x,0)) = (u_0(x), w_0(x))$, *there exist positive constants* m *and* M, $0 < m < M < +\infty$, *such that,* $m \le u(x,t) \le M$ *and* $m \le w(x,t) \le M$.

Proof For any fixed positive initial data $(u(x,0), w(x,0)) = (u_0(x), w_0(x))$, there exists a unique solution $(u(x,t), w(x,t))$ of the initial boundary problem which is positive and global. Let $\bar{u}_0 = \max_{x\in\bar{\Omega}} u_0(x)$, and $\bar{w}_0 = \max_{x\in\bar{\Omega}} w_0(x)$, for w fixed, we denote $(f(x,t), g(x,t))$ as solution of following initial boundary problem:

$$\begin{cases} \dfrac{\partial f}{\partial t} = D\Delta f - D\dfrac{1}{w}(\bigtriangledown w)\cdot(\bigtriangledown f) + \mu f - \dfrac{\alpha_1}{\alpha_2+\alpha_3 fg} f^2, & \\ \dfrac{\partial g}{\partial t} = \dfrac{\alpha_1 fg}{\alpha_2+\alpha_3 fg} - \mu g, & \text{for } x\in\Omega, t>0, \\ \dfrac{\partial f}{\partial n} = 0, & \text{for } x\in\partial\Omega, t>0, \\ f(x,0) = \bar{u}_0, & \\ g(x,0) = \bar{w}_0 & \text{for } x\in\bar{\Omega}. \end{cases} \tag{2.3}$$

It is obvious that $(f(x,t), g(x,t))$ is a super-solution for the problem (2.2), hence $0 < u(x,t) \le f(x,t)$ and $0 < w(x,t) \le g(x,t)$.

Next the problem (2.3) has constant initial data, $(f(x,t), g(x,t))$ is also a solution of the initial problem of the ordinary differential system

$$\begin{cases} \dfrac{\partial f}{\partial t} - \mu f + \dfrac{\alpha_1}{\alpha_2+\alpha_3 fg} f^2 = 0, & \\ \dfrac{\partial g}{\partial t} = \dfrac{\alpha_1 fg}{\alpha_2+\alpha_3 fg} - \mu g, & \text{for } t>0, \\ f(x,0) = \bar{u}_0, & \\ g(x,0) = \bar{w}_0. & \end{cases} \tag{2.4}$$

From (2.4), we can deduce $(fg)_t = 0$, which implies $fg = f(0)g(0) = \bar{u}_0\bar{w}_0$. So $(f(x,t), g(x,t))$ is also a solution of the initial problem of the ordinary differential system

$$\begin{cases} \dfrac{\partial f}{\partial t} - \mu f + \dfrac{\alpha_1}{\alpha_2+\alpha_3 k} f^2 = 0, & \\ \dfrac{\partial g}{\partial t} = \dfrac{\alpha_1 k}{\alpha_2+\alpha_3 k} - \mu g, & \text{for } t>0, \\ f(x,0) = \bar{u}_0, & \\ g(x,0) = \bar{w}_0, & \end{cases} \tag{2.5}$$

where $k = \bar{u}_0\bar{w}_0$. It is well known that

$$\lim_{t\to+\infty} f(t) \le \frac{\bar{\mu}(\bar{\alpha}_2+\bar{\alpha}_3 k)}{\underline{\alpha}_1}, \quad \lim_{t\to+\infty} g(t) \le g(0) + \frac{\bar{\alpha}_1 k}{\underline{\mu}(\underline{\alpha}_2+\underline{\alpha}_3 k)},$$

which implies that there exists a positive constant $M > 0$, such that $f(x,t) \le M, g(x,t) \le M$. So the solution of the problem (2.2) has a upper bound.

Let $(\hat{f}(t), \hat{g}(t))$ be the solution of the following initial problem:

$$\begin{cases} \dfrac{\partial f}{\partial t} - \mu + \dfrac{\alpha_1}{\alpha_2 + \alpha_3 fg} f^2 = 0, & \\ \dfrac{\partial g}{\partial t} = \dfrac{\alpha_1 fg}{\alpha_2 + \alpha_3 fg} - \mu g, & \text{for } t > 0, \\ f(x,0) = \underline{u}_0, & \\ g(x,0) = \underline{w}_0 & \end{cases} \tag{2.6}$$

By the similar argument above, we can get a positive constant m such that $0 < m < \hat{f}(t) < u(x,t)$, and $0 < m < \hat{g}(t) < w(x,t)$. The proof is completed.

Since $u(x,t) = \frac{p(x,t)}{w(x,t)}$, for the solutions of (2.1) we also have

Corollary 2.1 . *For any pair of positive initial data* $(p(x,0), w(x,0)) = (p_0(x), w_0(x))$, *there exist positive constants* $0 < m_1 < M_1 < +\infty$, *such that* $m_1 \le p(x,t) \le M_1$ *and* $m_1 \le w(x,t) \le M_1$.

3 The systems with positive T-periodic coefficients

We have already proved that the problem (2.1) has a unique global solution. In order to understand the asymptotic behavior of the solution, we first consider the following initial-boundary problem:

$$\begin{cases} \dfrac{\partial p}{\partial t} = D\nabla \cdot \Big(p\nabla(\ln(\dfrac{p}{w}))\Big) & \text{for } x \in \Omega, t > 0, \\ \dfrac{\partial w}{\partial t} = \dfrac{\alpha_1(t)p}{\alpha_2(t) + \alpha_3(t)p} - \mu(t)w, & \\ (p\nabla(\ln(\dfrac{p}{w}))) \cdot n = 0, & \text{for } x \in \partial\Omega,\ t > 0, \\ p(x,0) = p_0(x) > 0, & \text{for } x \in \bar{\Omega}, \\ w(x,0) = w_0(x) > 0, & \end{cases} \tag{3.1}$$

where $\alpha_i(t)$ $(i = 1, 2, 3)$ and $\mu(t)$ are all positive smooth T-periodic functions.

Let $u(x,t) = \dfrac{p}{w}$, then $(p(x,t), w(x,t))$ is a solution for (3.1) if and only if $(u(x,t), w(x,t))$

is a solution for the following initial-boundary problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\triangle u - D\dfrac{1}{w}(\nabla w)\cdot(\nabla u) = \mu(t)u - \dfrac{\alpha_1(t)u^2}{\alpha_2(t)+\alpha_3(t)uw} & \\ \dfrac{\partial w}{\partial t} = \dfrac{\alpha_1(t)uw}{\alpha_2(t)+\alpha_3(t)uw} - \mu(t)w, & \text{for } x\in\Omega, t>0, \\ \dfrac{\partial u}{\partial n} = 0, & \text{for } x\in\partial\Omega, t>0, \\ u(x,0) = u_0(x) = \dfrac{p_0(x)}{w_0(x)} > 0, & \\ w(x,0) = w_0(x) > 0, & \text{for } x\in\bar{\Omega}, \end{cases} \tag{3.2}$$

In order to solve the system (3.2) clearly, we consider following ODE:

$$u_t = \mu(t)u(t) - \beta(t)u^2(t), \tag{3.3}$$

where $\mu(t)$ and $\beta(t)$ are positive smooth T-periodic functions.

Lemma 3.1. *(1) There exists a unique positive T-periodic solution $u^*(t)$ for the ordinary differential equation (3.3);*
(2) The positive T-periodic solution $u^*(t)$ is monotone decreasing in $\beta(t)$.

Proof First, we prove that there exists a positive T-periodic solution for the equation (3.3). Let $M \geq \dfrac{\bar{\mu}}{\underline{\beta}}, 0 < m \leq \dfrac{\underline{\mu}}{\overline{\beta}}$, where

$$\overline{\mu} = \sup_{t\geq 0}\mu(t), \underline{\mu} = \inf_{t\geq 0}\mu(t), \overline{\beta} = \sup_{t\geq 0}\beta(t), \text{ and } \underline{\beta} = \inf_{t\geq 0}\beta(t).$$

Denote $\hat{u}_0 = m$, and $\tilde{u}_0 = M$, then $\hat{u}_0$ is a sub-solution of

$$\begin{cases} u_t = \mu(t)u - \beta(t)u^2, & \text{for } t>0, \\ u(0) = m. \end{cases} \tag{3.4$_1$}$$

and $\tilde{u}_0$ is a super-solution of

$$\begin{cases} u_t = \mu(t)u - \beta(t)u^2, & \text{for } t>0, \\ u(0) = M. \end{cases} \tag{3.4^1}$$

Denote the solutions of the problems (3.4$_1$) and (3.4^1) by $\hat{u}_1$, and $\tilde{u}_1$ respectively, we have $0 < m = \hat{u}_0(t) \leq \hat{u}_1(t) \leq \tilde{u}_1(t) \leq \tilde{u}_0(t) = M$ for $t \geq 0$. Especially we have $m = \hat{u}_0(0) = \hat{u}_0(T) = \hat{u}_1(0) \leq \hat{u}_1(T) \leq \tilde{u}_1(T) \leq \tilde{u}_1(0) = \tilde{u}_0(T) = \tilde{u}_0(0) = M$. Denote the solutions of the following problems by $\hat{u}_2$, and $\tilde{u}_2$:

$$\begin{cases} u_t = \mu(t)u - \beta(t)u^2, & \text{for } t>0, \\ u(0) = \hat{u}_1(T). \end{cases} \tag{3.4$_2$}$$

and

$$\begin{cases} u_t = \mu(t)u - \beta(t)u^2, & \text{for } t > 0, \\ u(0) = \tilde{u}_1(T). \end{cases} \tag{3.4^2}$$

It is obvious that $\hat{u}_1(t) \leq \hat{u}_2(t) \leq \tilde{u}_2(t) \leq \tilde{u}_1(t)$. Especially we obtain that $m \leq \hat{u}_1(T) \leq \hat{u}_2(T) \leq \tilde{u}_2(T) \leq \tilde{u}_1(T) \leq M$. Denote the solutions of the following problems by $\hat{u}_3$, and $\tilde{u}_3$, respectively:

$$\begin{cases} u_t = \mu(t)u - \beta(t)u^2, & \text{for } t > 0, \\ u(0) = \hat{u}_2(T). \end{cases} \tag{3.4$_3$}$$

and

$$\begin{cases} u_t = \mu(t)u - \beta(t)u^2, & \text{for } t > 0, \\ u(0) = \tilde{u}_2(T). \end{cases} \tag{3.4^3}$$

Also we have $\hat{u}_1(t) \leq \hat{u}_2(t) \leq \hat{u}_3(t) \leq \tilde{u}_3(t) \leq \tilde{u}_2(t) \leq \tilde{u}_1(t)$, and $m \leq \hat{u}_1(T) \leq \hat{u}_2(T) \leq \hat{u}_3(T) \leq \tilde{u}_3(T) \leq \tilde{u}_2(T) \leq \tilde{u}_1(T) \leq M$.

By using the same technique, we can get a series of solutions: $\{\hat{u}_i(t)\}$, $\{\tilde{u}_i(t)\}$, such that

(a) $\hat{u}_i(t)$, $\tilde{u}_i$ are solutions for the following problems:

$$\begin{cases} u_t = \mu(t)u - \beta(t)u^2, & \text{for } t > 0, \\ u(0) = \hat{u}_{i-1}(T). \end{cases} \tag{3.4$_i$}$$

and

$$\begin{cases} u_t = \mu(t)u - \beta(t)u^2, & \text{for } t > 0, \\ u(0) = \tilde{u}_{i-1}(T). \end{cases} ; \tag{3.4^i}$$

(b) For any $i > 0$, we have $0 < m \leq \hat{u}_{i-1}(t) \leq \hat{u}_i(t) \leq \cdots \leq \tilde{u}_i(t) \leq \tilde{u}_{i-1}(t) \leq M$ and $\hat{u}_{i-1}(T) \leq \hat{u}_i(T) = \hat{u}_{i+1}(0) \leq \hat{u}_{i+2}(0) = \hat{u}_{i+1}(T) \leq \tilde{u}_{i+1}(T) \leq \tilde{u}_i(T) = \tilde{u}_{i+1}(0) \leq \tilde{u}_i(0) = \tilde{u}_{i-1}(T) = \tilde{u}_i(0) \leq \tilde{u}_{i-1}(0)$.

Since $\{\hat{u}_i(t)\}$ (resp. $\{\tilde{u}_i(t)\}$) are monotonic increasing (resp. decreasing), bounded, and smooth, there exists a unique limit function, say u_* (resp. u^*), which are bounded and smooth. We can easily check that both $u_*(t)$ and $u^*(t)$ satisfy the equation

$$u_t = \mu(t)u(t) - \beta(t)u^2(t). \tag{3.3}$$

Since the coefficients of the equation (3.3) are T-periodic, we have that for any $i > 1$ and $t > 0$,

$$\hat{u}_{i-1}(t+T) = \hat{u}_i(t), \qquad \tilde{u}_{i-1}(t+T) = \tilde{u}_i(t).$$

So we get

$$\begin{aligned} u_*(t+T) &= \lim_{i\to+\infty} \hat{u}_i(t+T) = \lim_{i\to+\infty} \hat{u}_{i+1}(t) = u_*(t), \\ u^*(t+T) &= \lim_{i\to+\infty} \tilde{u}_i(t+T) = \lim_{i\to+\infty} \tilde{u}_{i+1}(t) = u^*(t), \qquad \text{for } t > 0, \end{aligned}$$

that means $u_*(t)$ and $u^*(t)$ are T-periodic functions.

Next, we can prove that the T-periodic solution of (3.3) is unique. In fact if there exist two positive T-periodic solutions $u_1(t)$ and $u_2(t)$. Choose $0 < m \leq \min_{0\leq t\leq T}\{u_1(t), u_2(t)\}$ $\leq \max_{0\leq t\leq T}\{u_1(t), u_2(t)\} \leq M < +\infty$. By the same method as we used before, we can obtain two positive T-periodic solutions, $u_*(t)$ and $u^*(t)$ for the equation (3.3) and

$$0 < m \leq u_*(t) \leq u_1(t), u_2(t) \leq u^*(t) \leq M < +\infty.$$

Let $\lambda = \min_{t\in[0,T]}\{\frac{u^*(t)}{u_*(t)}\} \geq 1$, and define the function $h(t) = u^*(t) - \lambda u_*(t)$, then the function $h(t) \geq 0$ is a T-periodic function, and $h(t_0) = 0$ for some $t_0 \in (0, 2T)$. Since $h(t)$ gets its minimum at the point $t = t_0 \in (0, 2T)$, we have $h_t(t_0) = h(t_0) = 0$. So at the point $t = t_0$, we get

$$0 = h_t - \mu h = \lambda\beta u_*^2 - \beta u^{*2} = \lambda\beta(1-\lambda)u_*^2.$$

Because of $u_*(t) \neq 0, \beta(t_0) \neq 0$, and $\lambda \geq 1$, the above equality implies that $\lambda = 1$. So we obtain that $h(t) = u^*(t) - u_*(t) \geq 0$, and $h(t_0) = u^*(t_0) - u_*(t_0) = 0$. Notice that for any $t > t_0$, we have

$$\begin{aligned} 0 &= h_t(t) - \mu(t)h(t) + \beta(t)u^{*2}(t) - \lambda\beta(t){u_*}^2(t) \\ &= h_t(t) - \mu(t)h(t) + \beta(t)u^{*2}(t) - \beta(t){u_*}^2(t) \\ &= h_t(t) - \mu(t)h(t) + \beta(t)\{u^*(t) + u_*(t)\}h(t), \end{aligned}$$

and $h(t_0) = h(t_0 + T) = 0$, which leads to $h(t) = 0$ for any $t \geq t_0$. Keeping in mind that $h(t)$ is T-periodic function, we know that $h(t) = 0$ for all $t \geq 0$, which means that $u_*(t) = u^*(t)$ for all $t \geq 0$. Combining with the condition:

$$0 < m \leq u_*(t) \leq u_1(t), u_2(t) \leq u^*(t) \leq M < +\infty,$$

we have

$$0 < m \leq u_*(t) = u_1(t) = u_2(t) = u^*(t) \leq M < +\infty.$$

Thus the T-periodic solution is unique.

(2) In order to prove that the positive T-periodic solution of (3.3) is monotonic decreasing in $\beta(t)$, we assume that $\beta_2(t) \geq \beta_1(t) > 0$ for all $t \geq 0$, and $u_i(t)$ ($i = 1$, 2) are corresponding periodic solutions. Since $u_i(t) > 0$, the functions $w_i = \frac{1}{u_i(t)}$ ($i = 1, 2$) are well defined for all $t \geq 0$. It is easy to know that $w_i(t)$ is the positive T-periodic solution of following equation:

$$(w_i)_t(t) + \mu(t)w_i(t) = \beta_i(t), \qquad \text{for } t > 0. \tag{3.4}$$

Let $w(t) = w_2(t) - w_1(t)$, then $w(t)$ is a T-periodic function, and satisfies the following equation:

$$w_t(t) + \mu(t)w(t) = \beta_2(t) - \beta_1(t) \geq 0, \qquad \text{for } t > 0.$$

Suppose that the T-periodic function $w(t)$ get its minimum at a point $t = t_0 > 0$, thus $w_t(t_0) = 0$. Since

$$\mu(t_0)w(t_0) = \beta_2(t_0) - \beta_1(t_0) \geq 0,$$

and $\mu(t_0) > 0$, we have $w(t_0) \geq 0$, which implies $w(t) \geq 0$, for all $t \geq 0$. Hence $w_2(t) \geq w_1(t)$, and $u_2(t) \leq u_1(t)$ for all $t \geq 0$. The proof is completed.

Lemma 3.2. *For any positive constant $c > 0$ the solution $u(t)$ of the initial problem:*

$$\begin{cases} u_t = \mu(t)u - \beta(t)u^2, \text{ for } t > 0, \\ u(0) = c > 0, \end{cases} \tag{3.5}$$

has the asymptotic behavior $\lim\limits_{t\to\infty}[u(t) - u^*(t)] = 0$, where $u^*(t)$ is the positive T-periodic solution for the related ordinary differential equation (3.3).

Proof For $t > 0$, we choose positive constants m and M satisfying $m = \min\{c, \frac{\underline{\underline{\mu}}}{\beta}\}$ and $M = \max\{c, \frac{\overline{\mu}}{\underline{\beta}}\}$. Denote $\hat{u}_0$ by m, and $\tilde{u}_0$ by M. Then $\hat{u}_0$ and $\tilde{u}_0$ satisfy the following problems:

$$\begin{cases} \hat{u}_{0_t} \leq \mu(t)\hat{u}_0 - \beta(t)\hat{u}_0^2, & \text{for } t > 0, \\ \hat{u}_0(0) = m. \end{cases} \tag{3.5$_1$}$$

and

$$\begin{cases} \tilde{u}_{0_t} \geq \mu(t)\tilde{u}_0 - \beta(t)\tilde{u}_0^2, & \text{for } t > 0, \\ \hat{u}_0(0) = M. \end{cases} \tag{3.5^1}$$

Thus $\hat{u}_0$ and $\tilde{u}_0$ are sub- and super-solutions for the systems (3.4_1) and (3.4^1) respectively. By using the same method, we can get series of solutions: $\{\hat{u}_i(t)\}$, $\{\tilde{u}_i(t)\}$, and there exist a unique positive T-periodic solution , say $u^*(t)$ as before, such that $\hat{u}_i(t)$ converges increasingly to $u^*(t)$ as $i \to +\infty$, and $\tilde{u}_i(t)$ converges decreasingly to $u^*(t)$ as $i \to +\infty$. For the solution of (3.5), we also know for $i \geq 1$, $\hat{u}_i(t) \leq u(t) \leq \tilde{u}_i(t)$, thus

$$\begin{array}{rcl} \hat{u}_i(t) - u^*(t) & = & \hat{u}_i(t+iT) - u^*(t+iT) \\ & \leq & u(t+iT) - u^*(t+iT) \\ & \leq & \tilde{u}_i(t+iT) - u^*(t+iT) \\ & = & \tilde{u}_i(t) - u^*(t). \end{array} \quad \text{for all } t \geq 0, i \geq 1$$

Let $i \to +\infty$, we have

$$\lim_{t\to+\infty}(u(t) - u^*(t)) = 0,$$

which completes the proof.

Consider another ordinary differential equation:

$$w_t = \beta(t) - \mu(t)w, \qquad \text{for } t > 0. \tag{3.6}$$

By using the same technique, we can also get the following result:

Lemma 3.3. *For the equation (3.6), there exists a unique positive T-periodic solution $w^*(t)$, which is increasing in $\beta(t)$. Furthermore, for every positive initial datum the corresponding solution $w(t)$ satisfies $\lim_{t\to+\infty} |w(t) - w^*(t)| = 0$, which implies $w^*(t)$ is a global attractor.*

In the following, we want to study the positive T-periodic solution of the ordinary differential system:

$$\begin{cases} u_t = \mu(t)u - \dfrac{\alpha_1(t)u^2}{\alpha_2(t) + \alpha_3(t)uw}, \\ w_t = \dfrac{\alpha_1(t)uw}{\alpha_2(t) + \alpha_3(t)uw} - \mu(t)w, \end{cases} \quad \text{for } t > 0, \tag{3.7}$$

and we investigate the asymptotic behavior of the solution for the following initial problem:

$$\begin{cases} u_t = \mu(t)u - \dfrac{\alpha_1(t)u^2}{\alpha_2(t) + \alpha_3(t)uw}, & \text{for } t > 0, \\ w_t = \dfrac{\alpha_1(t)uw}{\alpha_2(t) + \alpha_3(t)uw} - \mu(t)w, \\ u(0) = k_1 > 0, \\ w(0) = k_2 > 0. \end{cases} \tag{3.8}$$

Let $k = k_1 k_2$, then it is similar to (2.4), we know that if $(u(t), w(t))$ is the solution of the initial problem (3.8), then $(u(t), w(t))$ is the solution of the following initial problem:

$$\begin{cases} u_t = \mu(t)u - \dfrac{\alpha_1(t)u^2}{\alpha_2(t) + \alpha_3(t)k}, & \text{for } t > 0, \\ w_t = \dfrac{\alpha_1(t)k}{\alpha_2(t) + \alpha_3(t)k} - \mu(t)w, \\ u(0) = k_1 > 0, \\ w(0) = k_2 > 0. \end{cases} \tag{3.9}$$

It is obvious that we have

Theorem 3.1. *(1) For any positive constant $k > 0$, there exists a unique positive T-periodic solution $(u^*(t), w^*(t))$ for the system (3.7), which satisfies $u^*(t)w^*(t) = k$ for all $t \geq 0$; and*

$$\begin{cases} u_t = \mu(t)u - \dfrac{\alpha_1(t)u^2}{\alpha_2(t) + \alpha_3(t)k}, \\ w_t = \dfrac{\alpha_1(t)k}{\alpha_2(t) + \alpha_3(t)k} - \mu(t)w. \end{cases}$$

(2) For any initial data $(u(0), w(0)) = (k_1, k_2)$, there exists a unique solution $(u(t), w(t))$ of the system (3.8). Furthermore, the solution $(u(t), w(t))$ has the following asymptotic behavior:

$$\begin{cases} \lim_{t\to+\infty} (u(t) - u^*(t)) = 0, \\ \lim_{t\to+\infty} (w(t) - w^*(t)) = 0. \end{cases} \tag{3.10}$$

For $k > 0$, we know that the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\triangle u - D\dfrac{1}{w}(\nabla w)\cdot(\nabla u) = \mu(t)u - \dfrac{\alpha_1(t)u^2}{\alpha_2(t)+\alpha_3(t)uw} & \text{for } x \in \Omega, t > 0, \\ \dfrac{\partial w}{\partial t} = \dfrac{\alpha_1(t)uw}{\alpha_2(t)+\alpha_3(t)uw} - \mu(t)w, & \\ \dfrac{\partial u}{\partial n} = 0, & \text{for } x \in \partial\Omega. \end{cases} \tag{3.11}$$

has a positive T-periodic solution $(u^*(t), w^*(t))$ which is spatial independent and satisfies $u^*(t)w^*(t) = u^*(0)w^*(0) = k$. How about the positive T-periodic solutions of (3.11) which depends on the spatial variable? The following result will give us a complete answer.

Theorem 3.2. *There is no spatial dependent positive T-periodic solution for the problem (3.11).*

Proof Assume that the system (3.11) has a positive T-periodic solution $(u(x,t), w(x,t))$ which depends on spatial variable. Then there exist some points $x_i \in \bar{\Omega}$, $(i = 1, 2)$, such that $u(x_1, 0) = u_0(x_1) = \max_{t\in\bar{\Omega}} u_0(x) = M_1$, $w(x_2, 0) = w_0(x_2) = \max_{t\in\bar{\Omega}} w_0(x) = M_2$. Suppose that $(\tilde{u}(t), \tilde{w}(t))$ is the pair of solution for the system (3.11) with the initial data $(u(x,0), w(x,0)) = (M_1, M_2)$. Then by the maximum principle for parabolic equations, we obtain that $u(x,t) \le \tilde{u}(t), w(x,t) \le \tilde{w}(t)$. Also we know that there exists a unique positive T-periodic solution of the system (3.11), say $(u^*(t), w^*(t))$, such that

$$\begin{cases} {u^*}_t &= \mu(t)u^* - \dfrac{\alpha_1(t){u^*}^2}{\alpha_2 + \alpha_3 u^* w^*} \\ &= \mu(t)u^* - \dfrac{\alpha_1(t){u^*}^2}{\alpha_2 + \alpha_3 M_1 M_2}, \\ {w^*}_t &= \dfrac{\alpha_1(t)u^* w^*}{\alpha_2 + \alpha_3 u^* w^*} - \mu(t)w^* \\ &= \dfrac{\alpha_1(t)M_1 M_2}{\alpha_2 + \alpha_3 M_1 M_2} - \mu w^* \end{cases} \quad \text{for } t > 0 \tag{3.12}$$

and

$$\lim_{t\to+\infty}(\tilde{u}(t) - u^*(t)) = \lim_{t\to+\infty}(\tilde{w}(t) - w^*(t)) = 0.$$

Now we can prove that $u^*(0) \ge M_1$, $w^*(0) \ge M_2$. If not, we can assume $u^*(0) < M_1$, then there exists a positive constant, say $\delta > 0$, such that $u^*(0) < M_1 - 2\delta$. Because of $\lim_{t\to+\infty}(\tilde{u}(t) - u^*(t)) = 0$, there exists a positive integer N, such that $u^*(t) - \delta < \tilde{u}(t) < u^*(t) + \delta$, for $t \ge NT$. Since $u^*(t)$ is a T-periodic function, we have $\tilde{u}(NT) < u^*(NT) + \delta = u^*(0) + \delta$. Also for $x \in \bar{\Omega}$, $u(x,0) = u(x, NT) \le \tilde{u}(NT) < u^*(NT) + \delta = u^*(0) + \delta$. Furthermore we have

$$M_1 = u(x_1, 0) = u(x_1, NT) < u^*(NT) + \delta = u^*(0) + \delta < M_1 - 2\delta + \delta = M_1 - \delta,$$

which implies $u^*(0) \geq M_1$. It is similar that we have $w^*(0) \geq M_2$.

Since $u^*(0)w^*(0) = M_1M_2, u^*(0) \geq M_1$, and $w^*(0) \geq M_2$, we have $u^*(0) = M_1, w^*(0) = M_2$. From the uniqueness of the solution, we can deduce that $\tilde{u}(t) = u^*(t), \tilde{w}(t) = w^*(t)$.

In fact, both solutions $(u(x,t), w(x,t))$ and $(u^*(t), w^*(t))$ are positive T-periodic functions and $u(x_1, NT) = u^*(NT)$, $w(x_2, NT) = w^*(NT)$, for all integers $N \geq 0$. Let $h(x,t) = u^*(t) - u(x,t)$, then $h(x,t)$ is spatial dependent, and $h(x,t) \geq 0$, $h(x_1, NT) = 0$. Suppose that there exists at least one point $x_3 \in \bar{\Omega}$, such that $h(x_3, NT) > 0$, thus from $u(x,t)w(x,t) \leq u^*(t)w^*(t)$, we have

$$\begin{aligned}
& \frac{\partial h}{\partial t} - D\triangle h - D\frac{1}{w}(\nabla w)\cdot(\nabla h) \\
= \ & \mu(t)h - \frac{\alpha_1(t)u^{*2}}{\alpha_2(t) + \alpha_3(t)u^*w^*} + \frac{\alpha_1(t)u^2}{\alpha_2(t) + \alpha_3(t)uw} \\
\geq \ & \mu(t)h - \frac{\alpha_1(t)u^{*2}}{\alpha_2(t) + \alpha_3(t)u^*w^*} + \frac{\alpha_1(t)u^2}{\alpha_2(t) + \alpha_3(t)u^*w^*} \\
= \ & \mu(t)h + \frac{\alpha_1(t)(u^* + u)}{\alpha_2(t) + \alpha_3(t)u^*w^*}(-h).
\end{aligned}$$

Then $h(x,t)$ satisfies

$$\begin{cases}
\dfrac{\partial h}{\partial t} - D\triangle h - D\dfrac{1}{w}(\nabla w)\cdot(\nabla h) + \{\dfrac{\alpha_1(t)(u^* + u)}{\alpha_2(t) + \alpha_3(t)u^*w^*} - \mu(t)\}h \geq 0, \\
\qquad\qquad \text{for } x \in \bar{\Omega}, t > 0, \\
\dfrac{\partial h}{\partial n} = 0, \qquad \text{for } x \in \partial\Omega, t > 0, \\
h(x,0) = u^*(0) - u_0(x), \qquad \text{for } x \in \bar{\Omega}.
\end{cases}$$

Notice that $h(x_1, NT) = 0$ for $N = 1, 2, 3, \cdots$, we must have $h(x,t) = 0$ for all $x \in \bar{\Omega}, t > 0$, by the strong maximum argument, which contradicts to $h(x_3, NT) > 0$. Hence, there is no positive T-periodic solution for the system (3.11), which is spatial dependent.

According to the results above, we know that, for the problem (3.1), even for constant initial data, the asymptotic behavior of the solution would have a lot of changes. We can not expect stable steady-state solutions for this problem in the general meaning of small perturbation for initial data. In despite of that, we still want to pay more attention to the problem on the special feature for large t to the positive solutions of the problem (3.11) which depend on space variable.

Let $\phi_1(t) = \min_{x\in\bar{\Omega}} u(x,t)$, $\phi_2(t) = \min_{x\in\bar{\Omega}} w(x,t)$, $\psi_1(t) = \max_{x\in\bar{\Omega}} u(x,t)$, and $\psi_2(t) = \max_{x\in\bar{\Omega}} w(x,t)$. Denote by A the value of $\underline{\lim}_{t\to+\infty} \phi_1(t)\phi_2(t)$, and by B the value of $\overline{\lim}_{t\to+\infty} \psi_1(t)\psi_2(t)$, thus $0 < A \leq B < +\infty$.

Theorem 3.3. *For the problem (3.1), if at least one of two functions $u_0(x)$ and $w_0(x)$ of initial data is not constant, then $\phi_1(t)\phi_2(t)$ converges to A increasingly, as $t \to +\infty$, and $\psi_1(t)\psi_2(t)$ converges to B decreasingly, as $t \to +\infty$.*

Proof We divide our proof into several steps. First, we can prove that $\psi_1(0)\psi_2(0) \geq B$. This is because if $\psi_1(0)\psi_2(0) < B$, then we can find two positive constants $c_i > 0$, $(i = 1, 2)$ such that $\psi_1(0) < c_1$, $\psi_2(0) < c_2$ and $\psi_1(0)\psi_2(0) < c_1c_2 < B$. Consider following problem:

$$\begin{cases} \dfrac{\partial f}{\partial t} - D\triangle f - D\dfrac{1}{w}(\nabla w)\cdot(\nabla f) = \mu(t)f - \dfrac{\alpha_1(t)f^2}{\alpha_2(t)+\alpha_3(t)fg} & \text{for } x \in \Omega, t > 0, \\ \dfrac{\partial g}{\partial t} = \dfrac{\alpha_1(t)fg}{\alpha_2(t)+\alpha_3(t)fg} - \mu(t)g, & \\ \dfrac{\partial f}{\partial n} = 0, & \text{for } x \in \partial\Omega. \\ f(x,0) = c_1, & \text{for } x \in \bar{\Omega}. \\ g(x,0) = c_2. & \end{cases} \tag{3.13}$$

We know that if (3.13) has a solution $(f(t), g(t))$, then $f(t)g(t) = c_1c_2$, and $u(x,t) \leq f(t)$, $w(x,t) \leq g(t)$ for $x \in \bar{\Omega}$, $t > 0$ thus we have $\psi_1(t) \leq f(t)$, and $\psi_2(t) \leq g(t)$. That means

$$\overline{\lim_{t\to+\infty}}\, \psi_1(t)\psi_2(t) \leq \overline{\lim_{t\to+\infty}}\, f(t)g(t) \equiv c_1c_2 < B,$$

which contradicts to the definition of B, so we have $\psi_1(0)\psi_2(0) \geq B$.

Secondly, we can prove that $\psi_1(t)\psi_2(t) \geq B$ for all $t > 0$. In fact, for any fixed $t_0 > 0$, we define functions $\breve{u}(x,t) = u(x, t+t_0)$, $\breve{w}(x,t) = w(x,t+t_0)$, $\breve{\mu}(t) = \mu(t+t_0)$, and $\breve{\alpha}_i(t) = \alpha_i(t+t_0)$, then $\breve{\mu}(t)$, and $\breve{\alpha}_i(t)$ $(i = 1, 2, 3)$ are all positive T-periodic functions, and $(\breve{u}(x,t), \breve{w}(x,t))$ is the solution of the following problem:

$$\begin{cases} \dfrac{\partial \breve{u}}{\partial t} - D\triangle \breve{u} - D\dfrac{1}{\breve{w}}(\nabla \breve{w})\cdot(\nabla \breve{u}) = \breve{\mu}(t)\breve{u} - \dfrac{\breve{\alpha}_1(t)\breve{u}^2}{\breve{\alpha}_2(t)+\breve{\alpha}_3(t)\breve{u}\breve{w}} & \text{for } x \in \Omega, t > 0, \\ \dfrac{\partial \breve{w}}{\partial t} = \dfrac{\breve{\alpha}_1(t)\breve{u}\breve{w}}{\breve{\alpha}_2(t)+\breve{\alpha}_3(t)\breve{u}\breve{w}} - \breve{\mu}(t)\breve{w}, & \\ \dfrac{\partial \breve{u}}{\partial n} = 0, & \text{for } x \in \partial\Omega. \\ \breve{u}(x,0) = u(x,t_0), & \text{for } x \in \bar{\Omega}. \\ \breve{w}(x,0) = w(x,t_0). & \end{cases}$$

By the argument above, we can deduce that

$$\begin{aligned}
\psi_1(t_0)\psi_2(t_0) &= \max_{x\in\bar\Omega} u(x,t_0)\max_{x\in\bar\Omega} w(x,t_0)\\
&= \max_{x\in\bar\Omega} \check u(x,0)\max_{x\in\bar\Omega} \check w(x,0)\\
&\geq \overline{\lim_{t\to+\infty}}\{\max_{x\in\bar\Omega} \check u(x,t)\max_{x\in\bar\Omega} \check w(x,t)\}\\
&= \overline{\lim_{t\to+\infty}}\{\max_{x\in\bar\Omega} u(x,t)\max_{x\in\bar\Omega} w(x,t)\}\\
&= B.
\end{aligned}$$

Next we can prove that the function $\psi_1(t)\psi_2(t)$ is monotonic decreasing in t. Actually, from the processes above, we only need to prove that $\psi_1(0)\psi_2(0) \geq \psi_1(t)\psi_2(t)$ for all $t > 0$. In fact, let $(f(t), g(t))$ be the solution of following problem:

$$\begin{cases}
\dfrac{\partial f}{\partial t} - D\triangle f - D\dfrac{1}{w}(\nabla w)\cdot(\nabla f) = \mu(t)f - \dfrac{\alpha_1(t)f^2}{\alpha_2(t)+\alpha_3(t)fg} & \\
\dfrac{\partial g}{\partial t} = \dfrac{\alpha_1(t)fg}{\alpha_2(t)+\alpha_3(t)fg} - \mu(t)g, & \text{for } x\in\Omega, t>0,\\
\dfrac{\partial f}{\partial n} = 0, & \text{for } x\in\partial\Omega.\\
f(x,0) = \psi_1(0), & \\
g(x,0) = \psi_2(0). & \text{for } x\in\bar\Omega.
\end{cases}$$

Then $f(t)g(t) \equiv \psi_1(0)\psi_2(0) \geq B$, and $f(t) \geq u(x,t)$, $g(t) \geq w(x,t)$. Thus we get $\psi_1(0)\psi_2(0) = f(0)g(0) = f(t)g(t)) \geq \psi_1(t)\psi_2(t)$, for $t > 0$. therefore the function $\psi_1(t)\psi_2(t)$ is convergent to B as $t \to +\infty$.

It is similar to deduce that $\phi_1(t)\phi_2(t)$ converges to A increasing as $t \to +\infty$.

If all coefficients of the system (3.1) are positive constants, then (3.1) has a trivial constant solution $(p_0, \frac{\alpha_1 p_0}{(\alpha_2+\alpha_3 p_0)\mu})$, in this case we have

Theorem 3.4. *Assume that all the coefficients of the system (3.1) are positive constants, then for any positive constant $c_1 > 0$, the positive constant solution $(c_1, c_2) = (c_1, \dfrac{\alpha_1 c_1}{\mu(\alpha_2+\alpha_3 c_1)})$ is asymptotic stable in the space $\Lambda = \{(p,w) : \int_\Omega p(x,0)dx = c_1,\ \int_\Omega w(x,0)dx = \frac{\alpha_1 c_1}{\mu(\alpha_2+\alpha_3 c_1)}\}$.*

Proof For any positive constants c_1, $c_2 = \dfrac{\alpha_1 c_1}{\mu(\alpha_2+\alpha_3 c_1)}$, then near the constant solution $(p, w) = (c_1, c_2)$, the linearized equation with small perturbations is

$$\begin{cases} \dfrac{\partial \xi}{\partial t} = D\triangle\xi - D\dfrac{c_1}{c_2}\triangle\eta & \\ \dfrac{\partial \eta}{\partial t} = \dfrac{\alpha_1\alpha_2}{(\alpha_2+\alpha_3 c_1)^2}\xi - \mu\eta & \text{for } x \in \Omega, t > 0 \\ \dfrac{\partial \xi}{\partial n} = 0, & \\ \dfrac{\partial \eta}{\partial n} = 0 & \text{for } x \in \partial\Omega, t > 0 \\ \xi(x,0) = \sum_{n=0}^{+\infty} a_n X_n(x) & \\ \eta(x,0) = \sum_{n=0}^{+\infty} b_n X_n(x) & \text{for } x \in \Omega, \end{cases} \tag{3.14}$$

where the functions X_n $(n \geq 0)$ are eigenfunctions of the following eigenvalue problem possessing the n-th positive eigenvalue $\lambda = \lambda_n$:

$$\begin{cases} -D\triangle u = \lambda_n u, & \text{for } x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & \text{for } x \in \partial\Omega. \end{cases}$$

We can denote ξ, η as follows:

$$\xi(x,t) = \sum_{i=0}^{+\infty} X_i(x)T_i(t), \qquad \eta(x,t) = \sum_{i=0}^{+\infty} X_i(x)H_i(t), \text{ for } t > 0.$$

In light of the conservation condition, we have $a_0 = b_0 = 0$, and for any $i \geq 1$,

$$\begin{cases} T_i'(t)X_i(x) = T_i[D\triangle X_i(x)] - H_i[D\frac{c_1}{c_2}\triangle X_i(x)] = -\lambda_i\{T_i - \frac{c_1}{c_2}H_i\}X_i(x), \\ H_i'(t)X_i(x) = \dfrac{\alpha_1\alpha_2}{(\alpha_2+\alpha_3 c_1)^2}T_i X_i(x) - \mu H_i X_i(x). \end{cases}$$

From the first equation, we obtain

$$\{T_i(t)exp\{\lambda_i t\}\}_t = \lambda_i \frac{c_1}{c_2} H_i e^{\lambda_i t}$$

Putting the form of the function $T_i(t)$ into the second equation, we can find

$$\{H_i e^{\lambda_i t}\}_t = \frac{\alpha_1\alpha_2}{(\alpha_2+\alpha_3 c_1)^2} T_i e^{\lambda_i t} + [\lambda_i - \mu] H_i e^{\lambda_i t}$$

$$\{H_i e^{\lambda_i t}\}_{tt} + [\mu - \lambda_i]\{H_i e^{\lambda_i t}\}_t - \frac{\alpha_1\alpha_2}{(\alpha_2+\alpha_3 c_1)^2}\lambda_i \frac{c_1}{c_2} H_i e^{\lambda_i t} = 0$$

We may assume the solution of the form

$$H_i(t) = A_i e^{(k_i^+ - \lambda_i)t} + B_i e^{(k_i^- - \lambda_i)t},$$

where A_i, B_i are constants and

$$k_i^{\pm} = \frac{1}{2}\{\lambda_i - \mu \pm \sqrt{(\mu - \lambda_i)^2 + \frac{4\lambda_i c_1 \alpha_1 \alpha_2}{c_2(\alpha_2 + \alpha_3 c_1)^2}}\}.$$

When $t \to +\infty$, then $H_i(t)$ tends to 0 if and only if $k_i^+ - \lambda_i < 0$, which is equivalent to

$$\lambda_i + \mu > \sqrt{(\mu - \lambda_i)^2 + \frac{4\lambda_i c_1 \alpha_1 \alpha_2}{c_2(\alpha_2 + \alpha_3 c_1)^2}},$$

i.e.

$$\mu > \frac{c_1 \alpha_1 \alpha_2}{c_2(\alpha_2 + \alpha_3 c_1)^2}.$$

Since $c_2 = \dfrac{\alpha_1 c_1}{\mu(\alpha_2 + \alpha_3 c_1)}$, and $\alpha_i > 0, c_1 > 0$, we have

$$\mu > \frac{\alpha_2}{\alpha_2 + \alpha_3 c_1}\mu,$$

which is always valid for any $\mu > 0$, that means $k_i^+ - \lambda_i < 0$ is true.

Theorem 3.4 is proved.

Next, we have

Theorem 3.5. *If at least one of the coefficients for the system (3.1) is non-constant positive T-periodic function, then the positive T-periodic solution of (3.1) is stable, but non-asymptotic stable.*

Proof By Theorem 3.2, the positive T-periodic solution $(p^*, w^*(t))$ of (3.1) is spatial independent. In order to investigate the asymptotic behavior of the solution, we consider the following linearized equation at the point $(p(x,t), w(x,t)) = (p^*, w^*(t))$

$$\begin{cases} \xi_t - D\Delta\xi = -D\dfrac{p^*}{w^*(t)}\Delta\eta & \\ \eta_t = \dfrac{\alpha_1(t)\alpha_2(t)}{[\alpha_2(t) + \alpha_3(t)p^*]^2}\xi - \mu(t)\eta & \text{for } x \in \Omega, t > 0 \\ \dfrac{\partial\xi}{\partial n} = \dfrac{\partial\eta}{\partial n} = 0 & \text{for } x \in \partial\Omega, t > 0 \\ \xi(x,0) = a(x) & \\ \eta(x,0) = b(x) & \text{for } x \in \Omega \end{cases}$$

We can write

$$\xi(x,t) = \sum_{n=0}^{+\infty} f_n(t)X_n(x), \qquad \eta(x,t) = \sum_{n=0}^{+\infty} g_n(t)X_n(x),$$

where $X_n(x)$ is eigenfunctions for the following eigenvalue problem possessing the n-th positive eigenvalue λ_n:

$$\begin{cases} -D\Delta X(x) = \lambda X(x) & \text{for } x \in \Omega \\ \dfrac{\partial X}{\partial n} = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

In light of the conservation condition, we get $f_0(t) = 0$, and

$$\begin{cases} g_0'(t) + \mu(t) g_0(t) = 0, & \text{for } t > 0, \\ g_0(0) = b_0, \end{cases}$$

and

$$\begin{cases} f_n'(t) + \lambda_n f_n(t) = \lambda_n \dfrac{p^*}{w^*(t)} g_n(t) & \\ g_n'(t) = \dfrac{\alpha_1(t)\alpha_2(t)}{[\alpha_2(t) + \alpha_3(t)p^*]^2} f_n(t) - \mu(t) g_n(t) & \text{for } t > 0 \\ f_n(0) = a_n & \\ g_n(0) = b_n & \end{cases} \tag{3.15}$$

for $n = 1, 2, 3, \cdots$, where

$$a(x) = \sum_{n=0}^{+\infty} a_n X_n(x), \qquad b(x) = \sum_{n=0}^{+\infty} b_n X_n(x).$$

We know that the solution $(p^*, w^*(t))$ is asymptotic stable if and only if for any initial data $(a(x), b(x))$, we have the limits

$$\lim_{t\to+\infty} \xi(x,t) = 0, \qquad \lim_{t\to+\infty} \eta(x,t) = 0, \text{ for } x \in \bar{\Omega}.$$

So if the solution $(p^*, w^*(t))$ is asymptotic stable, then for any $n = 1, 2, \cdots$, we have

$$\lim_{t\to+\infty} f_n(t) = 0, \qquad \lim_{t\to+\infty} g_n(t) = 0.$$

In order to prove our statement on the non-asymptotic stable of the solution $(p^*, w^*(t))$, we need only to show that for some integer n_0 , we can choose functions $a(x) = \sum_{n=1}^{+\infty} f_n(0) X_n(x)$ and $b(x) = \sum_{n=1}^{+\infty} g_n(0) X_n(x)$ such that the solution of the ordinary differential equations (3.15) $(f_n(t), g_n(t))$ does not tend to 0 as $t \to +\infty$.

It is obvious that $\lim_{t\to+\infty} g_0(t) = 0$ because $\mu(t)$ is a positive, T-periodic smooth function. We need only to consider the case $n > 0$. According to the theory on the structure of the solutions for the linear ordinary differential equations with T-periodic smooth coefficients, we know that for each integer n fixed, the solution $(f_n(t), g_n(t))$ of (3.15) would be as follows:

$$f_n(t) = aF_1(t)e^{\beta_1 t} + bF_2(t)e^{\beta_2 t}, \qquad g_n(t) = cG_1(t)e^{\beta_1 t} + dG_2(t)e^{\beta_2 t},$$

where a, b, c, and d are any constants, $F_i(t)$, $G_i(t)$ $(i = 1, 2)$ are all T-periodic smooth functions, and β_i $(i = 1, 2)$ are eigenvalues for the matrix $K = (h_1(t), h_2(t))$ where $h_i(t)$ is the solution of the following systems:

$$\begin{cases} f_n'(t) + \lambda_n f_n(t) = \lambda_n \dfrac{p^*}{w^*(t)} g_n(t) \\ g_n'(t) = \dfrac{\alpha_1(t)\alpha_2(t)}{[\alpha_2(t) + \alpha_3(t)p^*]^2} f_n(t) - \mu(t) g_n(t), \end{cases} \quad \text{for } t > 0$$

with the initial data $f_n(0) = 1$, $g_n(0) = 0$ and $f_n(0) = 0$, $g_n(0) = 1$ respectively. Now we can point out that at least one of the eigenvalues of the matrix K has non-negative real part. That implies that the trivial solution $(f(t), g(t)) = (0, 0)$ of the systems (3.15) is not asymptotic stable, which implies that the solution $(p^*, w^*(t))$ for the systems (3.1) is not asymptotic stable, which is what we want to obtain.

Assume that it is false, then the real parts of both eigenvalues are negative. We can suppose that the real part of β_1 is negative and is not smaller than the real part of another eigenvalue β_2. Thus, for every pair of initial function $(a(x), b(x))$ there exists some positive constant $M > 0$ such that

$$\|f_n(t)\| \leq Me^{\mathrm{Re}\beta_1 t}, \quad \|g_n(t)\| \leq Me^{\mathrm{Re}\beta_1 t} \quad \text{for } t > 0. \tag{3.16}$$

Also, from the first equation of the system (3.15), we have

$$(f_n(t)e^{\lambda_n t})' = \lambda_n \frac{p^*}{w^*(t)} g_n e^{\lambda_n t},$$

that means

$$f_n(t)e^{\lambda_n t} = f_n(0) + \lambda_n p^* \tilde{H}_n(t), \tag{3.17}$$

where $\tilde{H}_n(t) = \displaystyle\int_0^t \frac{1}{w^*(s)} g_n(s) e^{\lambda_n s} ds$. Putting that equation into the second equation of the system (3.15), we obtain that

$$(g_n e^{\lambda_n t})' = \frac{\alpha_1(t)\alpha_2(t)}{(\alpha_2(t) + \alpha_3(t)p^*)^2} f_n(t)e^{\lambda_n t} + (\lambda_n - \mu(t)) g_n(t) e^{\lambda_n t}$$

Notice that $(w^*)' = \dfrac{\alpha_1(t)p^*}{\alpha_2(t) + \alpha_3(t)p^*} - \mu(t)w^*$, we can deduce that

$$\begin{aligned} &(\frac{g_n(t)e^{\lambda_n t}}{w^*(t)})' + [\frac{\alpha_1(t)p^*}{(\alpha_2(t) + \alpha_3(t)p^*)w^*(t)} - \lambda_n] \frac{g_n(t)e^{\lambda_n t}}{w^*(t)} \\ = &\frac{\alpha_1(t)\alpha_2(t)}{(\alpha_2(t) + \alpha_3(t)p^*)^2 w^*(t)} [f_n(0) + \lambda_n p^* \tilde{H}_n(t)]. \end{aligned}$$

It means that $\tilde{H}_n(t)$ must be the solution for the following system:

$$\begin{cases} -\tilde{H}_n(t)'' + (\lambda_n - \dfrac{\alpha_1(t)p^*}{(\alpha_2(t)+\alpha_3(t)p^*)w^*})\tilde{H}_n(t)' + \dfrac{\lambda_n\alpha_1(t)\alpha_2(t)p^*}{[\alpha_2(t)+\alpha_3(t)p^*]^2 w^*(t)}\tilde{H}_n(t) \\ = -\dfrac{\alpha_1(t)\alpha_2(t)}{(\alpha_2(t)+\alpha_3(t)p^*)^2 w^*(t)} f_n(0) \qquad \text{for } t > 0 \\ \tilde{H}_n(0) = 0 \\ \tilde{H}_n'(0) = \dfrac{g_n(0)}{w^*(0)}. \end{cases} \tag{3.18}$$

Choose $0 < \dfrac{p^*}{w^*(0)} g_n(0) < \dfrac{\alpha_2(t)}{\alpha_2(t)+\alpha_3(t)p^*} f_n(0)$, then there exists constant $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, we have

$$\varepsilon \frac{g_n(0)}{w^*(0)} + \frac{\alpha_1(t)}{(\alpha_2(t)+\alpha_3(t)p^*)w^*(t)}[\frac{p^*}{w^*(0)} g_n(0) - \frac{\alpha_2(t)}{\alpha_2(t)+\alpha_3(t)p^*} f_n(0)] \leq 0,$$

and

$$\varepsilon \frac{g_n(0)}{w^*(0)} + \frac{\alpha_1(t) g_n(0)}{(\alpha_2(t)+\alpha_3(t)p^*)w^*(t)} \frac{p^*}{w^*(0)} (\frac{\lambda_n}{\lambda_n - \varepsilon} \frac{\alpha_2(t)}{\alpha_2(t)+\alpha_3(t)p^*} - 1) \leq 0.$$

Let $H_n(t) = \dfrac{g_n(0)}{(\lambda_n - \varepsilon)w^*(0)}[e^{(\lambda_n - \varepsilon)t} - 1]$, then we have

$$\begin{aligned} & -H_n''(t) + (\lambda_n - \frac{\alpha_1(t)p^*}{(\alpha_2(t)+\alpha_3(t)p^*)w^*(t)})H_n'(t) + \frac{\alpha_1(t)\alpha_2(t)\lambda_n p^*}{(\alpha_2(t)+\alpha_3(t)p^*)^2 w^*(t)} H_n(t) \\ & + \frac{\alpha_1(t)\alpha_2(t)}{(\alpha_2(t)+\alpha_3(t)p^*)^2 w^*(t)} f_n(0) \\ = \; & [-(\lambda_n - \varepsilon)\frac{g_n(0)}{w^*(0)} + (\lambda_n - \frac{\alpha_1(t)p^*}{(\alpha_2(t)+\alpha_3(t)p^*)w^*(t)})\frac{g_n(0)}{w^*(0)} \\ & + \frac{\alpha_1(t)\alpha_2(t)\lambda_n p^*}{(\alpha_2(t)+\alpha_3(t)p^*)^2 w^*(t)} \frac{g_n(0)}{(\lambda_n - \varepsilon)w^*(0)}] e^{(\lambda_n - \varepsilon)t} \\ & - \frac{\alpha_1(t)\alpha_2(t)\lambda_n p^*}{(\alpha_2(t)+\alpha_3(t)p^*)^2 w^*(t)} \frac{g_n(0)}{(\lambda_n - \varepsilon)w^*(0)} \\ & + \frac{\alpha_1(t)\alpha_2(t) f_n(0)}{(\alpha_2(t)+\alpha_3(t)p^*)^2 w^*(t)} \\ = \; & [\varepsilon \frac{g_n(0)}{w^*(0)} - \frac{\alpha_1(t)p^*}{(\alpha_2(t)+\alpha_3(t)p^*)w^*(t)} \frac{g_n(0)}{w^*(0)} \\ & + \frac{\alpha_1(t)\alpha_2(t)}{(\alpha_2(t)+\alpha_3(t)p^*)^2 w^*(t)} \frac{\lambda_n}{(\lambda_n - \varepsilon)} \frac{p^*}{w^*(0)} g_n(0)] e^{(\lambda_n - \varepsilon)t} \\ & + \frac{\alpha_1(t)\alpha_2(t)}{(\alpha_2(t)+\alpha_3(t)p^*)^2 w^*(t)} [f_n(0) - \frac{\lambda_n}{\lambda_n - \varepsilon} \frac{p^*}{w^*(0)} g_n(0)] \end{aligned}$$

$$
\begin{aligned}
&= [\varepsilon\frac{g_n(0)}{w^*(0)} + \frac{\alpha_1(t)g_n(0)}{(\alpha_2(t)+\alpha_3(t)p^*)w^*(t)}\frac{p^*}{w^*(0)}(\frac{\lambda_n}{\lambda_n-\varepsilon}\frac{\alpha_2(t)}{\alpha_2(t)+\alpha_3(t)p^*}-1)]e^{(\lambda_n-\varepsilon)t} \\
&\quad + \frac{\alpha_1(t)\alpha_2(t)}{(\alpha_2(t)+\alpha_3(t)p^*)^2w^*(t)}[f_n(0)-\frac{\lambda_n}{\lambda_n-\varepsilon}\frac{p^*}{w^*(0)}g_n(0)] \\
&\leq [\varepsilon\frac{g_n(0)}{w^*(0)} + \frac{\alpha_1(0)g_n(0)}{(\alpha_2(0)+\alpha_3(0)p^*)w^*(0)}\frac{p^*}{w^*(0)}(\frac{\lambda_n}{\lambda_n-\varepsilon}\frac{\alpha_2(0)}{\alpha_2(0)+\alpha_3(0)p^*}-1)] \\
&\quad + \frac{\alpha_1(0)\alpha_2(0)}{(\alpha_2(0)+\alpha_3(0)p^*)^2w^*(0)}[f_n(0)-\frac{\lambda_n}{\lambda_n-\varepsilon}\frac{p^*}{w^*(0)}g_n(0)] \\
&= \varepsilon\frac{g_n(0)}{w^*(0)} + \frac{\alpha_1(0)}{(\alpha_2(0)+\alpha_3(0)p^*)w^*(0)}[\frac{p^*}{w^*(0)}g_n(0)-\frac{\alpha_2(0)}{\alpha_2(0)+\alpha_3(0)p^*}f_n(0)] \\
&\leq 0,
\end{aligned}
$$

since the function defined by

$$
[\varepsilon\frac{g_n(0)}{w^*(0)} + \frac{\alpha_1(t)g_n(0)}{(\alpha_2(t)+\alpha_3(t)p^*)w^*(t)}\frac{p^*}{w^*(0)}(\frac{\lambda_n}{\lambda_n-\varepsilon}\frac{\alpha_2(t)}{\alpha_2(t)+\alpha_3(t)p^*}-1)]e^{(\lambda_n-\varepsilon)t}+
\frac{\alpha_1(t)\alpha_2(t)}{(\alpha_2(t)+\alpha_3(t)p^*)^2w^*(t)}[f_n(0)-\frac{\lambda_n}{\lambda_n-\varepsilon}\frac{p^*}{w^*(0)}g_n(0)],
$$

is monotonic decreasing in t, notice that $H_n(0)-\tilde{H}_n(0)=0$, $H_n'(0)-\tilde{H}_n'(0)=0$, and

$$
\begin{aligned}
&H_n''(0)-\tilde{H}_n''(0) \\
&= (\lambda_n-\varepsilon)\frac{g_n(0)}{w^*(0)} + [\frac{\alpha_1(0)p^*}{(\alpha_2(0)+\alpha_3(0)p^*)w^*(0)}-\lambda_n]\frac{g_n(0)}{w^*(0)} - \\
&\quad \frac{\alpha_1(0)\alpha_2(0)}{(\alpha_2(0)+\alpha_3(0)p^*)^2w^*(0)}f_n(0) \\
&= -\varepsilon\frac{g_n(0)}{w^*(0)} + \frac{\alpha_1(0)}{(\alpha_2(0)+\alpha_3(0)p^*)w^*(0)}[\frac{p^*}{w^*(0)}g_n(0)-\frac{\alpha_2(0)}{\alpha_2(0)+\alpha_3(0)p^*}f_n(0)] \\
&< 0.
\end{aligned}
$$

We can get the result: $H_n(t)\leq\tilde{H}_n(t)$. Therefore we have

$$
\begin{aligned}
&\frac{g_n(0)}{(\lambda_n-\varepsilon)w^*(0)}[e^{(\lambda_n-\varepsilon)t}-1] \\
&\leq \int_0^t \frac{g_n(s)}{w^*(s)}e^{\lambda_n s}ds \\
&\leq \frac{M}{\min\limits_{0\leq t\leq T} w^*(t)}\int_0^t e^{(\lambda_n+\mathrm{Re}\beta_1)s}ds \\
&= \frac{M}{\min\limits_{0\leq t\leq T} w^*(t)(\lambda_n+\mathrm{Re}\beta_1)}[e^{(\lambda_n+\mathrm{Re}\beta_1)t}-1].
\end{aligned}
$$

That implies that

$$
\begin{aligned}
0 &< \frac{g_n(0)}{(\lambda_n-\varepsilon)w^*(0)}[1-e^{-(\lambda_n-\varepsilon)t}] \\
&\leq \frac{M}{\min\limits_{0\leq t\leq T} w^*(t)(\lambda_n+\mathrm{Re}\beta_1)}[e^{(\varepsilon+\mathrm{Re}\beta_1)t}-e^{-(\lambda_n-\varepsilon)}].
\end{aligned}
$$

Let $t \to +\infty$, since $\mathrm{Re}\beta_1 < 0$, it must be

$$\frac{g_n(0)}{(\lambda_n - \varepsilon)w^*(0)} \leq 0.$$

However this is impossible. The result leads to that at least one of the real parts for the eigenvalues of the matrix K is non-negative, which implies that the solution $(p^*, w^*(t))$ of the system (3.1) is not asymptotic stable. The result is proved.

Comparing with Theorem 3.4 and Theorem 3.5, we know that the different choice for the coefficients will change the asymptotic behavior of the solution, i.e. the solution will tend to the constant solution $(c_1, \dfrac{\alpha_1 c_1}{\alpha_2 + \alpha_3 c_1})$ or T-periodic solution $(p^*, w^*(t))$. It means that these kinds of mathematical models are very unstable. So it is not so strange that the biological systems have different behavior even they have many same biological action.

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References

[1] W. Alt, *Biased random walk models for chemotaxis and related diffusion approximations*, J. Math. Biol., 9(1980),147-173.

[2] S. Childress, *Chemotactic collapse in two dimensions*, Lecture Motes in Biomathematics, Springer-Verlag, 55(1984), 61-66.

[3] S. Childress and J. K. Percus, *Nonlinear aspects of chemotaxis*, Math. Bioisc., 56(1981), 217-237.

[4] H. Gajewski and K. Zacharias, *Global behavior of a reaction-diffusion system modeling chemotaxis*, Math. Nachr. 195(1998), 77-114.

[5] M. A. Herrero and J. J. L. Velazquez, *Chemotactic collapse for the Leller-Segel model*, J. Math. Biol. 35(1996), 177-194.

[6] M. A. Herrero and J. J. L. Velazquez, *Singularity patterns in a chemotaxis model*, Math. Ann. 306(1996), 583-623.

[7] D. Horstmann and G. Wang, *Blowup in a chemotaxis model without symmetry assumptions*, preprint

[8] W. Jäger and S. Luckhaus, *On explosions of solutions to a system of partial differential equations modeling chemotaxis*, Trans. Amer. Math. Soc. 329(1992), 819-824.

[9] M. Rascle and C. Ziti, *Finite time blow-up in some models of chemotaxis*, J. Math. Biol. 33(1995), 388-414.

[10] H.G. Othmer and A. Stevens, *Aggregation, blow-up and collapse: the ABC's of taxis and reinforced random walks*, SIAM J. Appl. Math. 57(3)(1997), 1044-1081.

[11] H.A. Levine and B.D. Sleeman, *A system of reaction diffusion equations arising in the theory of reinforced random walks*, SIAM J.Appl. Math. 57(3)(1997), 683-730.

[12] H.A. Levine and B.D. Sleeman, *Mathematical Modeling of the onset of capillary formation initiating angiogenesis*, preprint.

[13] B.D.Sleeman and H.A.Levine, *Partial differential equations of chemotaxis and angiogenesis*, Math. Meth. Appl. Sci. 24(2001), 405-426.

[14] B.D. Sleeman and E.Tuma, *Comparison principles for strongly coupled reaction-diffusion equation*, Proceedings of the Royal Society of Edinburgh, 106A(1987), 209-219.

[15] Yang Yin, Chen Hua, Liu Weian and B.D.Sleeman, *The solvability of some chemotaxis systems*, J. Differential Equations, 212(2005), 432-451.

[16] Yang Yin, Chen Hua and Liu Weian, *On existence of global solutions and blow-up to a system of reacton-diffusion equations modelling chemotaxis*, SIAM. J. Math. Anal. 33(4) (2001), 736-785.