Elliptic Differential Operators on Manifolds with Edges

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Abstract

On a manifold with edge we construct a specific class of (edge-degenerate) elliptic differential operators. The ellipticity refers to the principal symbolic structure $\sigma = (\sigma_\psi, \sigma_\wedge)$ of the edge calculus consisting of the interior and edge symbol, denoted by σ_ψ and $\sigma_\wedge$, respectively. For our choice of weights the ellipticity will not require additional edge conditions of trace or potential type, and the operators will induce isomorphisms between the respective edge spaces.

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Contents

Introduction

Ellipticity of a (pseudo-) differential operator A on a manifold M with edge Y (see Section 1.1) refers to a principal symbolic structure $\sigma(A) = (\sigma_\psi(A), \sigma_\wedge(A))$. The first component, the so-called interior symbol, is the usual homogeneous principal symbol as a function on $T^*(M \setminus Y) \setminus 0$, though edge-degenerate in stretched variables close to the edge (see Section 1.2). The second component, the so-called edge symbol, is an operator-valued function on $T^*Y \setminus 0$, pointwise acting in weighted Sobolev spaces on the infinite model cone transversal to the edge. The situation is similar to operators on a C^∞ manifold M with boundary Y; in this case the boundary plays the role of the edge, and $\overline{\mathbb{R}}_+$, the inner normal (with respect to some Riemannian metric), of the model cone. From the calculus of boundary value problems it is well known that the ellipticity is determined by a pair of principal symbols, consisting of the interior and boundary symbol $\sigma_\psi(\cdot)$ and $\sigma_\partial(\cdot)$, respectively. For the case of operators A with the transmission property at the boundary (see, for instance, Boutet de Monvel [1], or Rempel and Schulze [4]) the boundary symbol is a family of continuous operators

$$\sigma_\partial(A)(y,\eta) : H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+) \tag{0.1}$$

(here $H^s(\mathbb{R}_+) := H^s(\mathbb{R})|_{\mathbb{R}_+}$ with $H^s(\mathbb{R})$ being the standard Sobolev space of smoothness s on $\mathbb{R}$), $(y,\eta) \in T^*Y \setminus 0$. The ellipticity of A with respect to $\sigma_\psi(A)$ (i.e., $\sigma_\psi(A) \neq 0$ on $T^*M \setminus 0$) entails the Fredholm property of (0.1) for all $(y,\eta) \in T^*Y \setminus 0$ (and all sufficiently large s). This explains the role of elliptic boundary conditions (in general of trace and potential type), namely, on the level of boundary symbols, to fill up the operators (0.1) to a 2×2 block matrix family $\sigma_\partial(\mathcal{A})(y,\eta)$ of isomorphisms by corresponding entries of finite rank, with $\sigma_\partial(A)(y,\eta)$ in the upper left corner. The invertibility of $\sigma_\partial(\mathcal{A})(y,\eta)$ is just a second ellipticity condition, namely, of the extra boundary data with respect to the given elliptic operator A, also called the Shapiro-Lopatinskij condition. Recall that when A is a differential operator, (0.1) is surjective, and then it suffices to pose trace (i.e., boundary) conditions, while for pseudo-differential operators we need, in general, both trace and potential conditions (since both $\ker \sigma_\partial(A)$ and $\operatorname{coker} \sigma_\partial(A)$ may be non-trivial).

In the case of a manifold with edge there is an analogue of the boundary symbol, namely, the edge symbol

$$\sigma_\wedge(A)(y,\eta) : \mathcal{K}^{s,\gamma}(X^\Delta) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\Delta), \tag{0.2}$$

$(y,\eta) \in T^*Y \setminus 0$, operating between weighted spaces on an infinite (so-called model) cone $X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ with base X which is (in our case here) a closed compact C^∞ manifold. The spaces $\mathcal{K}^{s,\gamma}(X^\Delta)$ of smoothness $s \in \mathbb{R}$ and weight $\gamma \in \mathbb{R}$ will be defined below. The model cone X^Δ appears in the description of the given manifold M with edge Y locally near Y as a wedge $X^\Delta \times \Omega$ for some open $\Omega \subseteq \mathbb{R}^q$, $q = \dim Y$. Instead of $X^\Delta \times \Omega$ we often refer to the open stretched wedge $X^\wedge \times \Omega$, $X^\wedge := \mathbb{R}_+ \times X$, with the variables (r, x, y), and we then write $\mathcal{K}^{s,\gamma}(X^\wedge)$ rather than $\mathcal{K}^{s,\gamma}(X^\Delta)$.

Also in the edge case the 'interior' ellipticity of an edge-degenerate operator A (i.e., ellipticity with respect to $\sigma_\psi(A)$) entails the Fredholm property of (0.2), however, with the exception of a discrete set of weights γ. Moreover, the ellipticity 'up to the edge' requires additional edge data, encoded by a 2×2 block matrix family of isomorphisms $\sigma_\wedge(\mathcal{A})(y,\eta)$ with $\sigma_\wedge(A)(y,\eta)$ in the upper left corner. Although in 'abstract terms' it is easy (under fairly general assumptions on A) to construct extra conditions of that kind, (see, for instance, [6]), the characterisation of the admissible weight γ as well as the computation of the number of those edge conditions can be extremely difficult. It is therefore an interesting task of the edge calculus to explicitly construct elliptic edge operators in which those data are known. This is just the program of the present paper. We construct a variety of differential operators of even order in the edge calculus which are elliptic with respect to σ_ψ and $\sigma_\wedge$ and where no additional edge conditions are necessary for the $\sigma_\wedge$-ellipticity. Our operators will induce isomorphisms between the chosen weighted edge spaces and as such represent reductions of orders.

1 Edge-degenerate differential operators

1.1 Manifolds with edges

A manifold M with (smooth) edge can be defined as the quotient space $M := \mathbb{M}/\sim$, where $\mathbb{M}$ is a manifold with C^∞ boundary $\partial\mathbb{M}$ which has the structure of an X-bundle over a C^∞ manifold Y, the edge, where the fibre X is assumed to be a closed compact C^∞ manifold. Denoting the bundle projection by $\pi : \partial\mathbb{M} \to Y$ the equivalence $m \sim m'$ of points $m, m' \in \mathbb{M}$ means that $m = m'$ when $m, m' \in \operatorname{int}\mathbb{M}$ and $\pi m = \pi m'$ when $m, m' \in \partial\mathbb{M}$. We call $\mathbb{M}$ the stretched manifold associated with M. A simple example is $M := X^\Delta \times \Omega$ for an open set $\Omega \subseteq \mathbb{R}^q$, where $X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ is the infinite cone with base X. In that case we have $Y = \Omega$, and $\mathbb{M} = \overline{\mathbb{R}}_+ \times X \times \Omega$. In general, if M is a manifold with smooth edge Y, then $M \setminus Y$ is C^∞, and every $y \in Y$ has a neighbourhood V such that there is a homeomorphism

$$V \to X^\Delta \times \Omega \tag{1.1}$$

which restricts to a diffeomorphism $V \setminus Y \to X^\wedge \times \Omega$ for $X^\wedge := \mathbb{R}_+ \times X$ and $Y \cap Y \to \Omega$. Maps of the kind (1.1) will be called singular charts. Analogously, denoting by $\mathbb{V}$ the preimage of V with respect to the above mentioned quotient map $\mathbb{M} \to M$, we have a diffeomorphism

$$\chi : \mathbb{V} \to \overline{\mathbb{R}}_+ \times X \times \Omega \tag{1.2}$$

in the sense of C^∞ manifolds with boundary, where (1.2) is induced by (1.1) in an obvious manner.

By $\mathrm{Diff}^\mu(\cdot)$ we denote the space of all differential operators of order μ with smooth coefficients on the smooth manifold in parenthesis. (For simplicity all manifolds in consideration here are assumed to be countable unions of compact sets; then $\mathrm{Diff}^\mu(\cdot)$ is a Fréchet space in a natural way.)

1.2 Edge-degenerate operators

An operator $A \in \mathrm{Diff}^\mu(M \setminus Y)$ will be called edge-degenerate, written $A \in \mathrm{Diff}^\mu_{\mathrm{deg}}(\mathbb{M})$, if the push forwards of $A|_{V \setminus Y}$ with respect to $V \setminus Y \to X^\wedge \times \Omega$ have the form

$$r^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j\alpha}(ry) \left(-r \frac{\partial}{\partial r} \right)^j (rD_y)^\alpha \tag{1.3}$$

with coefficients $a_{j\alpha}(r,y) \in C^\infty\big(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X)\big)$. Variables on $\mathbb{R}_+ \times X \times \Omega$ and covariables are denoted by (r,x,y) and (ρ,ξ,η), respectively. Let $\sigma_\psi(A)$ denote the homogeneous principal symbol of A of order μ, also called the interior symbol (which is a smooth function on $T^*(\mathrm{int}\,\mathbb{M}) \setminus 0$). Locally near Y in the splitting of variables (r,x,y) we also write $\sigma_\psi(A)(r,x,y,\rho,\xi,\eta)$. Then, by virtue of the special form of (1.3) the function

$$\widetilde{\sigma}_\psi(A)(r,x,y,\rho,\xi,\eta) := r^\mu \sigma_\psi(A)(r,x,y,r^{-1}\rho,\xi,r^{-1}\eta) \tag{1.4}$$

is C^∞ up to $r=0$. Let us call (1.4) the reduced interior symbol.

Moreover, with an operator $A \in \mathrm{Diff}^\mu_{\mathrm{deg}}(\mathbb{M})$ we associate the so-called homogeneous principal edge symbol which is defined in the local splittings of variables close to the edge by

$$\sigma_\wedge(A)(y,\eta) := r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(0,y)(-r\partial_r)^j (r\eta)^\alpha, \tag{1.5}$$

$(y,\eta) \in T^*Y \setminus 0$. As we see it takes values in $\mathrm{Diff}^\mu(X^\wedge)$. Thus it acts on functions $u(r,x)$ on $X^\wedge$. The homogeneity refers to the one-parameter group of transformations

$$(\kappa_\lambda u)(r,x) = \lambda^{\frac{n+1}{2}+g} u(\lambda r, x), \quad \lambda \in \mathbb{R}_+, \tag{1.6}$$

$g \in \mathbb{R}$; the choice of g will be fixed below. Then we have

$$\sigma_\wedge(A)(y, \lambda\eta) = \lambda^\mu \kappa_\lambda \sigma_\wedge(A)(y, \eta)\kappa_\lambda^{-1} \tag{1.7}$$

for all $\lambda \in \mathbb{R}_+$.

1.3 Weighted spaces on a manifold with edges

We now consider weighted spaces on cones and on manifolds with edges. The definition will rely on certain weighted spaces on the infinite stretched cone $X^\wedge$. By $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s \in \mathbb{N}$, $\gamma \in \mathbb{R}$, we denote the set of all $u(r,x) \in r^{\gamma-\frac{n}{2}}L^2(\mathbb{R}_+ \times X)$ (for $n = \dim X$ and $L^2(\mathbb{R}_+ \times X)$ based on $drdx$ with a measure dx associated with a fixed Riemannian metric on X) such that $(r\partial_r)^j D^\alpha u(r,x) \in r^{\gamma-\frac{n}{2}}L^2(\mathbb{R}_+ \times X)$ for arbitrary $D^\alpha \in \mathrm{Diff}^\alpha(X)$, $j \in \mathbb{N}$, $j + |\alpha| \leq s$. Observe that then

$$\mathcal{H}^{s,\gamma}(X^\wedge) = r^\gamma \mathcal{H}^{s,0}(X^\wedge) \tag{1.8}$$

for all $\gamma \in \mathbb{R}$. Defining $\mathcal{H}^{s,0}(X^\wedge)$ for $s \in \mathbb{Z}$ by duality (with respect to the scalar product of $\mathcal{H}^{0,0}(X^\wedge)$) and then for $s \in \mathbb{R}$ by complex interpolation, we obtain the spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ by (1.8) for all $s, \gamma \in \mathbb{R}$.

By a cut-off function on the half-axis we understand any real-valued $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\omega \equiv 1$ in a neighbourhood of 0. In order to define suitable spaces on $X^\wedge$ for the action of the edge symbol (1.5) we denote by $H^s_{\mathrm{cone}}(X^\wedge)$ the subspace of all $u \in H^s_{\mathrm{loc}}(\mathbb{R} \times X)|_{\mathbb{R}_+ \times X}$ such that for every coordinate neighbourhood U of X, every chart $\sigma_1 : U \to U_1$ to an open set $U_1 \subset S^n$ and for any $\varphi \in C_0^\infty(U)$ and a cut-off function ω the push forward of $(1-\omega)\varphi u$ with respect to $1 \times \sigma_1 : \mathbb{R}_+ \times U \to \{\tilde{x} \in \mathbb{R}^{1+n} : \tilde{x}/|\tilde{x}| \in U_1\}$, $(1 \times \sigma_1)(r,x) := r\sigma_1(x)$, belongs to $H^s(\mathbb{R}^{1+n})$. Observe, in particular, that when $X = S^n$ and $\mathbb{R}_+ \times X$ is identified with $\mathbb{R}^{1+n} \setminus \{0\}$ via polar coordinates, we have $(1-\omega)H^s_{\mathrm{cone}}(X^\wedge) = (1-\omega)H^s(\mathbb{R}^{1+n})$. We now define the weighted cone spaces

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{\omega u + (1-\omega)v : u \in \mathcal{H}^{s,\gamma}(X^\wedge), v \in H^s_{\mathrm{cone}}(X^\wedge)\} \tag{1.9}$$

where ω is a cut-off function (the choice of ω does not affect (1.9)). In (1.9) we can fix a scalar product such that $\mathcal{K}^{s,\gamma}(X^\wedge)$ is a Hilbert space. For $s = \gamma = 0$ we choose it in such a way that $\mathcal{K}^{0,0}(X^\wedge) = r^{-\frac{n}{2}}L^2(\mathbb{R}_+ \times X)$ (which is also equal to $\mathcal{H}^{0,0}(X^\wedge)$). We also define

$$\mathcal{K}^{s,\gamma;g}(X^\wedge) := \langle r \rangle^{-g} \mathcal{K}^{s,\gamma}(X^\wedge) \tag{1.10}$$

for $g \in \mathbb{R}$, $\langle r \rangle := (1+r^2)^{1/2}$, which is a variant of (1.9) with an extra weight g at $r = \infty$. We endow (1.10) with the action of the one-parameter group of isomorphisms (1.6). We then have an example of a Hilbert space E with group action in the following sense.

We say that a Hilbert space E is endowed with a group action $\kappa = \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, if $\kappa_\lambda : E \to E$, $\lambda \in \mathbb{R}_+$, is a family of isomorphisms, $\kappa_\lambda\kappa_\nu = \kappa_{\lambda\nu}$ for all $\lambda, \nu \in \mathbb{R}_+$, and $\lambda \to \kappa_\lambda e$ defines an element of $C(\mathbb{R}_+, E)$ for every $e \in E$ (i.e., κ is strongly continuous).

If E is a Hilbert space with group action $\kappa = \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, then the 'abstract' edge space $\mathcal{W}^s(\mathbb{R}^q, E)$ of smoothness $s \in \mathbb{R}$ is defined to be the completion of $\mathcal{S}(\mathbb{R}^q, E)$ (the Schwartz space on $\mathbb{R}^q$ of functions with values in E) with respect to the norm $\{\int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle}\widehat{u}(\eta)\|^2_E d\eta\}^{1/2}$, $\widehat{u}(\eta) = \int_{\mathbb{R}^q} e^{-iy\eta}u(y)dy$, $\langle\eta\rangle := (1+|\eta|^2)^{1/2}$. If necessary we also write $\mathcal{W}^s(\mathbb{R}^q, E)_\kappa$ when we want to indicate the dependence of the space on the group action κ. Spaces of that kind are well known for the case $\kappa_\lambda = \mathrm{id}_E$, $\lambda \in \mathbb{R}_+$, under the notation $H^s(\mathbb{R}^q, E)$. The spaces for general κ were introduced in [5]. In connection with operators on manifolds with edge we can take $E := \mathcal{K}^{s,\gamma;g}(X^\wedge)$ and $\kappa = \{\kappa^g_\lambda\}_{\lambda\in\mathbb{R}_+}$, cf. the formula (1.6). We then have the property

$$H^s_{\mathrm{comp}}(X^\wedge \times \mathbb{R}^q) \subset \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma;g}(X^\wedge)) \subset H^s_{\mathrm{loc}}(X^\wedge \times \mathbb{R}^q)$$

for all $s, \gamma, g \in \mathbb{R}$, cf. [2, Section 7.1.2]. A particularly natural choice is obtained for the case $g = s - \gamma$, cf. [8] or [7]. We then write

$$K^{s,\gamma}(X^\wedge) := \mathcal{K}^{s,\gamma;s-\gamma}(X^\wedge)$$

and

$$W^{s,\gamma}(X^\wedge \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, K^{s,\gamma}(X^\wedge))_{\kappa^{s-\gamma}}.$$

Given a compact manifold M with edge Y with its associated stretched manifold $\mathbb{M}$ we define $W^{s,\gamma}(\mathbb{M})$ to be subspace of all $u \in H^s_{\mathrm{loc}}(\mathrm{int}\,\mathbb{M})$ such that for every $\mathbb{V}$ from the above mentioned covering of $\mathbb{W}$, and $\chi : \mathbb{V} \to \overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q$ in the sense of (1.2) we have

$$\chi_*(\psi u) \in W^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$$

for every $\psi \in C^\infty_0(\mathbb{V})$.

Observe that every $A \in \mathrm{Diff}^\mu_{\mathrm{deg}}(\mathbb{M})$ induces continuous operators

$$A : W^{s,\gamma}(\mathbb{M}) \to W^{s-\mu,\gamma-\mu}(\mathbb{M})$$

for all $s, \gamma \in \mathbb{R}$. Moreover, the homogeneous principal edge symbol represents a family of continuous operators

$$\sigma_\wedge(A)(y,\eta) : K^{s,\gamma}(X^\wedge) \to K^{s-\mu,\gamma-\mu}(X^\wedge),$$

$s, \gamma \in \mathbb{R}$, $(y, \eta) \in T^*Y \setminus 0$.

2 Elliptic operators and isomorphisms in weighted edge spaces

2.1 Edge-degenerate operators of special form

Let $\mathbb{M}$ be the stretched manifold associated with a compact manifold M with edge Y. From the definition we have a finite system of singular charts close to $\partial\mathbb{M}$

$$\chi : \mathbb{V} \to \overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q, \tag{2.1}$$

cf. the formula (1.2), where $\mathbb{V}$ denotes a neighbourhood in $\mathbb{M}$ with $\mathbb{V} \cap \partial\mathbb{M} \neq \emptyset$. For simplicity we assume here that the transition maps to different maps of the kind (2.1) are independent of r for small r. Furthermore, we choose coordinate neighbourhoods U on $\mathbb{M}_{\text{reg}} = \mathbb{M} \setminus \partial\mathbb{M}$ together with charts

$$\kappa : U \to \mathbb{R}^{1+n+q}, \tag{2.2}$$

$n = \dim X$, $q = \dim Y$, such that the sets $\mathbb{V}$ and U form a finite covering of $\mathbb{M}$. On every $\mathbb{V}$ we have an $\omega_{\mathbb{V}} \in C_0^\infty(\mathbb{V})$ and on U a $\varphi_U \in C_0^\infty(U)$ such that the system of functions $\omega_{\mathbb{V}}$ and φ_U form a partition of unity subordinate to the covering of $\mathbb{M}$ by the sets $\mathbb{V}, U$. Let

$$\omega := \chi_* \omega_{\mathbb{V}}, \quad \varphi := \kappa_* \varphi_U$$

for any fixed χ and κ as in (1.2) and (2.2), respectively. We will choose the functions ω in such a way that they only depend on $(r, y) \in \overline{\mathbb{R}}_+ \times \mathbb{R}^q$. Let us fix operators $D_j \in \mathrm{Diff}^1(X)$, $j = 1, \ldots, N$ (for a suitable N), expressed by vector fields on X which span the tangent space of X at every $x \in X$. Moreover, let $H_j \in \mathrm{Diff}^1(\mathbb{M}_{\text{reg}})$, $j = 1, \ldots, L$ (for a suitable L), be operators expressed by vector fields on $\mathbb{M}_{\text{reg}}$ which span the tangent space of $\mathbb{M}_{\text{reg}}$ at every point of $\mathbb{M}_{\text{reg}}$.

Let us fix an $s \in \mathbb{N}$ and form the differential operators depending on a parameter $\lambda \in \mathbb{R}^l$,

$$(\chi^{-1})_* \omega(r, y) r^{|\beta|} (-r\partial_r)^j D^\alpha D_y^\beta r^{-s} \lambda^\iota \tag{2.3}$$

for every $\mathbb{V}$ with the associated χ and ω, for arbitrary $j \in \mathbb{N}$, $\alpha \in \mathbb{N}^N$, $\beta \in \mathbb{N}^q$, $\iota \in \mathbb{N}^l$, such that $j + |\alpha| + |\beta| + |\iota| \leq s$, $D^\alpha = D_1^{\alpha_1} \cdot \ldots \cdot D_N^{\alpha_N}$, and

$$(\kappa^{-1})_* \varphi \kappa_* H^\delta \lambda^\iota \tag{2.4}$$

for every U with the associated κ and φ, for arbitrary $\delta \in \mathbb{N}^L$, $\iota \in \mathbb{N}^l$, such that $|\delta| + |\iota| \leq s$, $H^\delta = H_1^{\delta_1} \cdot \ldots \cdot H_L^{\delta_L}$. This gives us a column vector of parameter-dependent differential operators

$$^{\mathrm{t}}(B_k(\lambda))_{k=0,\ldots,K} \tag{2.5}$$

for some K, determined by all possible combinations of the involved multi-indices and the number of the sets $\mathbb{V}, U$ in the open covering of $\mathbb{M}$.

Remark 2.1. *In* (2.3) *we understand the r-variables as operators of multiplication from the left, with the exception of the factor r^{-s}. More precisely,* (*apart from* $(\chi^{-1})_*$) *the operator has the form*

$$u(r,\cdot) \to \omega(r,y) r^{|\beta|}(-r\partial_r)^j D^\alpha D_y^\beta \lambda^\iota (r^{-s} u(r,\cdot)). \tag{2.6}$$

In formal adjoints of those operators the factor r^{-s} becomes an operator from the left, while the other r-variables are acting from the right.

The operators (2.5) induce continuous operators

$$B(\lambda) : W^{t,\frac{n}{2}+\gamma}(\mathbb{M}) \to \bigoplus_{k=0}^{K} W^{t-s,\frac{n}{2}+(\gamma-s)}(\mathbb{M})$$

for every $t, \gamma \in \mathbb{R}$, $\lambda \in \mathbb{R}^l$. The formal adjoints with respect to the scalar product of $L^2(\mathbb{M}) = \mathcal{W}^{0,\frac{n}{2}}(\mathbb{M})$ induce continuous operators

$$B^*(\lambda) : \bigoplus_{k=0}^{K} W^{t,\frac{n}{2}+\delta}(\mathbb{M}) \to W^{t-s,\frac{n}{2}+(\delta-s)}(\mathbb{M})$$

for every $t, \delta \in \mathbb{R}$, $\lambda \in \mathbb{R}^l$. Let us now interpret

$$B^*(\lambda)B(\lambda) = \sum_{k=0}^{K} B_k^*(\lambda)B_k(\lambda) \tag{2.7}$$

as a family of continuous operators

$$B^*(\lambda)B(\lambda) : W^{2t,\frac{n}{2}+s}(\mathbb{M}) \to W^{2(t-s),\frac{n}{2}-s}(\mathbb{M}). \tag{2.8}$$

It is clear that we have $B^*(\lambda)B(\lambda) \in \mathrm{Diff}^{2s}_{\mathrm{deg}}(\mathbb{M})$ for every $\lambda \in \mathbb{R}^l$. We are interested in the question of ellipticity of (2.8) with respect to the principal symbolic structure $\sigma = (\sigma_\psi, \sigma_\wedge)$ of the edge calculus, especially for single operators, say, when $\lambda = 0$. Here we will employ the fact that

$$B^*B := B^*(0)B^*(0)$$

is embedded in a parameter-dependent family of operators; this will provide additional useful information.

2.2 Ellipticity with respect to the principal edge symbols

Let us first observe that the operators $B^*(\lambda)B(\lambda)$ are parameter-dependent elliptic of order $2s$ on $\mathbb{M}_{\text{reg}}$, i.e., the parameter-dependent principal symbol $\sigma_\psi(B^*B)(\cdot,\lambda)$ with $\lambda \in \mathbb{R}^l$ as an additional covariable does not vanish on $T^*\mathbb{M}_{\text{reg}} \times \mathbb{R}^l \setminus 0$. Moreover, in the local splitting of variables (r,x,y) close to $\partial\mathbb{M}$ we can write

$$\sigma_\psi(B^*B)(\cdot,\lambda) = r^{-2s}\widetilde{\sigma}_\psi(B^*B)(r,x,y,r\rho,\xi,r\eta,\lambda)$$

for a function $\widetilde{\sigma}_\psi(B^*B)(r,x,y,\widetilde{\rho},\xi,\widetilde{\eta},\lambda)$ which is homogeneous of order $2s$ in $(\widetilde{\rho},\xi,\widetilde{\eta},\lambda) \neq 0$ and non-vanishing up to $r=0$. In particular, the operators $B^*(\lambda)B(\lambda)$ are σ_ψ-elliptic in the sense of the edge calculus for every fixed $\lambda \in \mathbb{R}^l$.

Let us now turn to the principal edge symbol $\sigma_\wedge(B^*B)$ as a family of continuous operators

$$\sigma_\wedge(B^*B)(y,\eta,\lambda) : K^{t,\frac{n}{2}+s}(X^\wedge) \to K^{t-2s,\frac{n}{2}-s}(X^\wedge), \tag{2.9}$$

$t \in \mathbb{R}$. It can be written as the composition of the operator functions

$$\sigma_\wedge(B)(y,\eta,\lambda) : K^{t,\frac{n}{2}+s}(X^\wedge) \to \bigoplus_{j+|\alpha|+|\beta|+|\iota|\leq s,\omega} K^{t-s,\frac{n}{2}}(X^\wedge),$$

and

$$\sigma_\wedge(B^*)(y,\eta,\lambda) : \bigoplus_{j+|\alpha|+|\beta|+|\iota|\leq s,\omega} K^{t-s,\frac{n}{2}}(X^\wedge) \to K^{t-2s,\frac{n}{2}-s}(X^\wedge),$$

where $\sigma_\wedge(B)(y,\eta,\lambda)$ is a vector of operator functions of the form

$$\omega(0,y)r^{|\beta|}(-r\partial_r)^j D^\alpha \eta^\beta \lambda^\iota r^{-s},$$

$j+|\alpha|+|\beta|+|\iota| \leq s$, cf. the formula (2.6). (The notation with subscript 'ω' in the latter expressions means that ω indicates the set $\mathbb{V} \cap Y$ where $\mathbb{V}$ runs over the above mentioned system of neighbourhoods on $\mathbb{M}$.) For the subordinate conormal symbols we have

$$\sigma_{\mathrm{c}}\sigma_\wedge(B)(y,w,\lambda) = {}^{\mathrm{t}}\big(\omega(0,y)(w+s)^j D^\alpha \lambda^\iota\big)_{j+|\alpha|+|\iota|\leq s,\omega},$$

$$\sigma_{\mathrm{c}}\sigma_\wedge(B^*)(y,w,\lambda) = \big(\omega(0,y)(1-(w+s))^j (D^\alpha)^* \lambda^\iota\big)_{j+|\alpha|+|\iota|\leq s,\omega}.$$

These are families of continuous maps

$$\sigma_c\sigma_\wedge(B)(y,w,\lambda) : H^t(X) \to \bigoplus_{j+|\alpha|+|\iota|\le s,\omega} H^{t-s}(X),$$

$$\sigma_c\sigma_\wedge(B^*)(y,w,\lambda) : \bigoplus_{j+|\alpha|+|\iota|\le s,\omega} H^{t-s}(X) \to H^{t-2s}(X),$$

$t \in \mathbb{R}$. The operators

$$\sigma_c\sigma_\wedge(B^*B)(y,w,\lambda) : H^t(X) \to H^{t-2s}(X) \tag{2.10}$$

are parameter-dependent elliptic on X for every fixed $y \in Y$, with the parameter $(\operatorname{Im} w, \lambda) \in \mathbb{R}^{1+l}$, $w \in \Gamma_\beta$ for every real β (here $\Gamma_\beta := \{w \in \mathbb{C} : \operatorname{Re} w = \beta\}$). In particular, we consider (2.10) on the weight line $\Gamma_{\frac{1}{2}-s}$, where

$$\begin{aligned}\sigma_c\sigma_\wedge(B^*B)&(y, \frac{1}{2} - s + i\rho, \lambda)\\ &= \sum_{\omega} \sum_{j+|\alpha|+|\iota|\le s} \omega^2(0,y)\big(\frac{1}{2} - i\rho\big)^j (D^\alpha)^*\lambda^\iota \big(\frac{1}{2} + i\rho\big)^j D^\alpha \lambda^\iota \end{aligned} \tag{2.11}$$

(similarly as before the first sum means summation over the contributions from $\mathbb{V} \cap Y$ for $\mathbb{V}$ varying over our system of neighbourhoods on $\mathbb{M}$).

Proposition 2.2. *The operators* (2.10) *form a family of isomorphisms for all $y \in Y$, $w \in \Gamma_{\frac{1}{2}-s}$, $\lambda \in \mathbb{R}^l$, and all $t \in \mathbb{R}$.*

Proof. Since $B^*(\lambda)B(\lambda)$ is σ_ψ-elliptic in the sense of the edge calculus, the conormal symbol $\sigma_c\sigma_\wedge(B^*B)(y,w,\lambda)$ is a family of parameter-dependent elliptic operators on X with the parameters $(w,\lambda) \in \Gamma_\beta \times \mathbb{R}^l$ for every $\beta \in \mathbb{R}$ (and every fixed $y \in Y$). Therefore, the operators (2.10) are Fredholm for all those (w,λ), and they are isomorphisms for all $|w,\lambda|$ sufficiently large. Therefore, the index of (2.10) is zero for all $(w,\lambda) \in \Gamma_\beta \times \mathbb{R}^l$. Thus, for the bijectivity of (2.10) for $w \in \Gamma_{\frac{1}{2}-s}$, $\lambda \in \mathbb{R}^l$, it is enough to show the injectivity. Now $u \in \ker \sigma_c\sigma_\wedge(B^*B)(y,w,\lambda)$ show that

$$\begin{aligned} 0 &= (\sigma_c\sigma_\wedge(B^*B)(y,w,\lambda)u, u)_{L^2(X)} \\ &= (\sigma_c\sigma_\wedge(B)(y,w,\lambda)u, \sigma_c\sigma_\wedge(B)(y,w,\lambda)u)_{\oplus L^2(X)} \end{aligned}$$

where the direct sum in the subscript means the number of components enumerated by ω and the triples (j,α,ι) of length $\le s$. Since all summands are non-negative, the summands vanish separately, especially those with $j + |\alpha| + |\iota| = 0$. This entails $u = 0$. □

Theorem 2.3. *The operators* (2.9) *from a family of isomorphisms for all* $(y, \eta) \in T^*Y \setminus 0$*,* $\lambda \in \mathbb{R}^l$*, and for all* $t \in \mathbb{R}$*.*

Proof. First we show that (2.9) is a family of Fredholm operators. To this end we have to verify that these operators are elliptic in the cone calculus on the infinite (stretched) cone $X^\wedge$, where $r \to \infty$ is treated as a conical exit to infinity. There are three symbolic contributions, namely, the conormal symbol which is elliptic on the weight line $\Gamma_{\frac{1}{2}-s}$ by Proposition 2.2, moreover, the interior symbol $\sigma_\psi \sigma_\wedge(B^*B)(r, x, \rho, \xi)$ which is non-vanishing for $(\rho, \xi) \neq 0$ and its reduced version $r^{2s}\sigma_\psi\sigma_\wedge(B^*B)(r, x, r^{-1}\rho, \xi)$ which is non-vanishing for $(\rho, \xi) \neq 0$, up to $r = 0$, and the exit symbol $\sigma_E\sigma_\wedge(B^*B)$, responsible for $r \to \infty$. The exit symbol refers to conical sets in $\mathbb{R}^{1+n} \ni \widetilde{x}$ with the covariable $\widetilde{\xi} \in \mathbb{R}^{1+n}$ and contains two conditions, namely non-vanishing of the homogeneous component in $\widetilde{x}$ of order 0 for $|\widetilde{x}| \to \infty$ and all $\widetilde{\xi} \in \mathbb{R}^{1+n}$ and non-vanishing of its principal homogeneous part in $\widetilde{\xi}$ of order $2s$ for $\widetilde{\xi} \neq 0$. This property is satisfied for $\eta \neq 0$, cf. [3, Section 3.3.8], or [2, Section 2.4.5]. Thus we obtain the Fredholm property of (2.9). The parameter $\lambda \in \mathbb{R}^l$ behaves like an extra covariable in the exit calculus, and we have parameter-dependent ellipticity. In particular, the above mentioned homogeneous component in $\widetilde{x}$ of order 0 for $|\widetilde{x}| \to \infty$ is non-vanishing for all $(\widetilde{\xi}, \lambda) \in \mathbb{R}^{1+n+l}$ (including $(\widetilde{\xi}, \lambda) = 0$) and its homogeneous principal part in $(\widetilde{\xi}, \lambda)$ of order $2s$ does not vanish for $(\widetilde{\xi}, \lambda) \neq 0$. Of course, λ is also an extra covariable in the pseudo-differential calculus on the open manifold $X^\wedge$, and we have parameter-dependent ellipticity. Finally, the conormal symbol is a bijective family in the sense of parameter-dependent elliptic operators on the base X. This has the consequence that we can construct a parametrix of (2.9) in the cone algebra on $X^\wedge$ with exit property at ∞, where the left-over terms are Schwartz functions in $\lambda \in \mathbb{R}^l$ with values in Green operators on the infinite cone. In other words we conclude that the operators (2.9) are isomorphisms for all $|\lambda| \geq C$ for some $C > 0$. Thus, since they are Fredholm for all $\lambda \in \mathbb{R}^l$, we obtain

$$\operatorname{ind} \sigma_\wedge(B^*B)(y, \eta, \lambda) = 0 \tag{2.12}$$

for all $\lambda \in \mathbb{R}^l$, $(y, \eta) \in T^*Y \setminus 0$. To complete the proof of Theorem 2.3 we show that (2.9) has a trivial kernel for all $\lambda \in \mathbb{R}^l$, $(y, \eta) \in T^*Y \setminus 0$. Assuming $u \in \ker \sigma_\wedge(B^*B)(y, \eta, \lambda)$, we know $u \in \mathcal{K}^{\infty, \frac{n}{2}+s}(X^\wedge)$ which is contained in $L^2(X^\wedge)$. Thus we can form

$$\begin{aligned} 0 &= (\sigma_\wedge(B^*B)(y, \eta, \lambda)u, u)_{L^2(X^\wedge)} \\ &= (\sigma_\wedge(B)(y, \eta, \lambda)u, \sigma_\wedge(B)(y, \eta, \lambda)u)_{\oplus_{L^2(X^\wedge)}} \end{aligned}$$

where the direct sum in the subscript concerns different copies of $L^2(X^\wedge)$ enumerated by ω and triples $(j, \alpha, \beta, \iota) \in \mathbb{N} \times \mathbb{N}^N \times \mathbb{N}^q \times \mathbb{N}^l$ of length $\leq s$. In other words, all terms

$$\big(\omega(0,y)r^{|\beta|}(-r\partial_r)^j D^\alpha \eta^\beta \lambda^\iota r^{-s}u, \omega(0,y)r^{|\beta|}(-r\partial_r)^j D^\alpha \eta^\beta \lambda^\iota r^{-s}u\big)_{L^2(X^\wedge)}$$

vanish. This is the case, in particular, for $j+|\alpha|+|\beta|+|\iota| = 0$, and it follows that $u \equiv 0$. □

Corollary 2.4. *The operators $B^*(\lambda)B(\lambda)$ are elliptic of order $2s$ in the edge calculus on for every $\lambda \in \mathbb{R}^l$. As such they are Fredholm as operators* (2.8) *for all $t \in \mathbb{R}$, $\lambda \in \mathbb{R}^l$.*

2.3 Order reducing families

In this section we modify the construction of operators $B^*(\lambda)B(\lambda)$ in order to obtain isomorphisms (2.8) in the edge algebra, for all $t \in \mathbb{R}$, $\lambda \in \mathbb{R}^l$. To this end we start from families of operators

$$(\chi^{-1})_* \omega(r,y) r^{|\beta|}(-r\partial_r)^j D^\alpha (D_y, \vartheta)^\beta r^{-s} \lambda^\iota \tag{2.13}$$

for every $\mathbb{V}$ with the associated χ and ω, where $\vartheta = (\vartheta_1, \ldots, \vartheta_p)$, $\beta \in \mathbb{N}^{q+p}$, $(D_y, \vartheta)^\beta := D_{y_1}^{\beta_1} \ldots D_{y_q}^{\beta_q} \vartheta_1^{\beta_{q+1}} \ldots \vartheta_p^{\beta_{q+p}}$, and $j \in \mathbb{N}$, $\alpha \in \mathbb{N}^N$, $\iota \in \mathbb{N}^l$, such that $j + |\alpha| + |\beta| + |\iota| \leq s$ and

$$(\kappa^{-1})_* \varphi \kappa_* H^\delta (\lambda, \vartheta)^\iota \tag{2.14}$$

for every U with the associated κ and φ, for arbitrary $\delta \in \mathbb{N}^M$, $\iota \in \mathbb{N}^{l+p}$, such that $|\delta| + |\iota| \leq s$, $(\lambda, \vartheta)^\iota := \lambda_1^\iota \ldots \lambda_l^{\iota_l} \vartheta_1^{\iota_{l+1}} \ldots \vartheta_p^{\iota_{l+p}}$. The considerations of Section 2.1 and 2.2 can be carried out in analogous form, now with the extra parameter ϑ. Instead of (2.8) we therefore obtain operators

$$B^*(\lambda, \vartheta)B(\lambda, \vartheta) : W^{2t, \frac{n}{2}+s}(\mathbb{M}) \to W^{2(t-s), \frac{n}{2}-s}(\mathbb{M}) \tag{2.15}$$

which agree with (2.8) for $\vartheta = 0$.

Remark 2.5. *The operators $B^*(\lambda, \vartheta)B(\lambda, \vartheta)$ are elliptic of order $2s$ in the edge calculus on $\mathbb{M}$. As such they induce Fredholm operators* (2.15) *for all $t \in \mathbb{R}$ and $(\lambda, \vartheta) \in \mathbb{R}^{l+p}$.*

In fact, the arguments for Corollary 2.4 can easily be modified for the case of (λ, ϑ)-depending families.

Theorem 2.6. *The operators* (2.15) *are isomorphisms for all $t \in \mathbb{R}$, $(\lambda, \vartheta) \in \mathbb{R}^{l+p}$.*

Proof. The operators (2.15) form a parameter-dependent elliptic family of the edge calculus. In particular, the analogue of (2.8)

$$\sigma_\wedge(B^*B)(y,\eta,\lambda,\vartheta) : K^{t,\frac{n}{2}+s}(X^\wedge) \to K^{t-2s,\frac{n}{2}-s}(X^\wedge) \tag{2.16}$$

is a family of isomorphisms for all $(y,\eta,\vartheta) \in T^*Y \times \mathbb{R}^p \setminus 0$ (where 0 means $(\eta,\vartheta) = 0$) and all $\lambda \in \mathbb{R}^l$. This allows us the construction of a parameter-dependent parametrix of (2.15) with parameter $\vartheta \in \mathbb{R}^p$, for every fixed $\lambda \in \mathbb{R}^l$, where the left-over terms are Schwartz functions in $\vartheta \in \mathbb{R}^p$ with values in the space of smoothing operators of the edge calculus on $\mathbb{M}$. This shows that the operators (2.15) become invertible when $|\vartheta|$ is sufficiently large. Thus the Fredholm operators (2.15) are of index 0 for all $\vartheta \in \mathbb{R}^p$ and $\lambda \in \mathbb{R}^l$. To show that the operators are isomorphisms it is enough to check that the kernel is trivial for all λ, ϑ. This can be done in a similar manner as in the last part of the proof of Theorem 2.3. In fact, $u \in \ker(B^*(\lambda,\vartheta)B(\lambda,\vartheta))$ shows that $u \in \mathcal{W}^{\infty,\frac{n}{2}+s}(\mathbb{M})$; this is contained in $\mathcal{W}^{0,\frac{n}{2}}(\mathbb{M})$. We have

$$0 = (B^*(\lambda,\vartheta)B(\lambda,\vartheta)u,u)_{L^2(\mathbb{M})} = \sum_{k=0}^{K}(B_k(\lambda,\vartheta)u, B_k(\lambda,\vartheta)u)_{L^2(\mathbb{M})}$$

which gives us $(B_k(\lambda,\vartheta)u, B_k(\lambda,\vartheta)u)_{L^2(\mathbb{M})} = 0$ for every k. Going back to the original manning of the operators $B_k(\lambda,\vartheta)$, namely (2.13) or (2.14), we see that, when we insert, for instance, $\lambda = 0, \vartheta = 0$, $j + |\alpha| + |\beta| + |\iota| = 0$ or $|\delta| + |\iota| = 0$, the functions $\omega_{\mathbb{V}} r^{-s} u$ and $\varphi_U u$ have a vanishing $L^2(\mathbb{M})$-norm for all $\mathbb{V}$ and U, and this gives us $u \equiv 0$. □

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