The Index Problem on Manifolds with Singularities[1]

Vladimir Nazaikinskii

Institute for Problems in Mechanics, Russian Academy of Sciences
and Institute for Mathematics, Potsdam University
e-mail: nazaik@math.uni-potsdam.de

Anton Savin

Independent University of Moscow
and Institute for Mathematics, Potsdam University
e-mail: antonsavin@mail.ru

Bert-Wolfgang Schulze

Institute for Mathematics, Potsdam University
e-mail: schulze@math.uni-potsdam.de

Boris Sternin

Independent University of Moscow
and Institute for Mathematics, Potsdam University
e-mail: sternin@math.uni-potsdam.de

[1]Supported by the DFG via a project with the Arbeitsgruppe “Partielle Differentialgleichungen und Komplexe Analysis,” Institut für Mathematik, Universität Potsdam and by RFBR grants Nos. 02-01-00118, 02-01-00928 and 03-02-16336.

Differential Operators on Manifolds with Singularities

Analysis and Topology

This is a draft version of Chapter VII of the book “*Differential Operators on Manifolds with Singularities. Analysis and Topology*” to be published by *Francis and Taylor.*

Contents

Chapter 7

The Index Problem on Manifolds with Singularities

Preface

In Part II of the book, we have studied operators on manifolds with singularities. In particular, we have found the ellipticity conditions under which the operators in question are Fredholm. A manifold with singularities, which we assume to be a stratified manifold, consists of the smooth part and the set of singularities. Accordingly, the principal symbol of an operator consists of several components corresponding to the strata. The ellipticity condition is stated in terms of these symbols, and it turns out that the index of an elliptic operator on a manifold with singularities is also determined by these symbols.

> *The index problem is the problem of writing out an index formula in the form of a functional on the set of symbols corresponding to the stratification.*

It follows that a natural index formula is an index formula in which the contributions of symbols corresponding to different strata are represented independently for each stratum. It is also natural to require that these contributions be homotopy invariants.

Such an index formula is especially important in the theory of manifolds with singularities, since, in contrast to the smooth case, here the index contributions of the smooth and singular parts of the manifold differ in their very nature: the contribution of the smooth part is determined by the finite-dimensional principal symbol, and the singular part is usually represented in the index formula by the contribution of an infinite-dimensional operator-valued symbol.

Unfortunately, the index of an arbitrary elliptic operator on a manifold with singularities in general *does not admit* a homotopy invariant splitting into contributions of the symbols corresponding to the strata. In fact, there is a *topological obstruction* to such a splitting, and the main theorem of this chapter provides a criterion for the existence of such a splitting in topological terms. Specific index formulas are discussed in the next chapter.

In conclusion of the introduction, note that splittings of invariants into contributions of parts of the manifold are well known in topology. For example, the Euler characteristic of an even-dimesional manifold $\mathcal{M}$ with a subset Σ of singularities has the form

$$\chi(\mathcal{M}) = \chi(\mathcal{M} \setminus \Sigma) + \chi(\Sigma).$$

Hence index splittings can be viewed as a far-reaching generalization of this relation in elliptic theory.

Let us briefly outline the contents of the chapter. In the first section, we establish some simplest index formulas. We also present so-called *relative index formulas*, which express the variation of the index as only one of the symbol components varies. A criterion for the existence of index splittings into a sum of symbol contributions in the case of manifolds with isolated singularities is given in the second section. Section 3 deals with a similar criterion for operators on manifolds with edges.

7.1. The Simplest Index Formulas

In this section, we obtain index formulas on manifolds with singularities in some simplest situations. We deal with operators on manifolds with isolated singular points except for the last two subsections, where edge problems are treated.

We shall assume that all operators in question are differential. Although all results remain valid for pseudodifferential operators, the proofs in some cases require more complicated techniques (e.g., in Proposition 7.3). For brevity, we consider only the conical case. Using the cylindrical representation introduced in Chapter 5, the reader can readily extend all the results to other types of isolated singularities, say, cusps.

7.1.1. General properties of the index

We recall some definitions and theorems in Chapter 5.

Representations of a manifold with singularities. Consider a manifold $\mathcal{M}$ with an isolated conical point. The blow-up of $\mathcal{M}$ is denoted by M. Recall that M is a manifold with boundary and a homeomorphism $\mathcal{M} \simeq M/\partial M$ is defined. The base of the cone will be denoted by Ω.

Operators. On $\mathcal{M}$ we consider a cone-degenerate operator

$$\mathbf{D} = r^{-m} D\left(\omega, r, -i\frac{\partial}{\partial\omega}, ir\frac{\partial}{\partial r}\right) : H^{s,\gamma}(\mathcal{M}, E) \longrightarrow H^{s-m,\gamma-m}(\mathcal{M}, F) \tag{7.1}$$

of order m.

Symbols. The operator $\mathbf{D}$ is determined modulo compact operators by two symbols:

1. The *principal symbol* $\sigma(\mathbf{D})$ defined on the compressed cotangent bundle $T_0^*\mathcal{M}$ without the zero section;

2. The *conormal symbol* $\sigma_c(\mathbf{D})$ defined on the weight line $L_\gamma = \{p \in \mathbb{C} \mid \operatorname{Im} p = \gamma\}$ in the complex plane with coordinate p.

We especially point out that the principal symbol is defined on a manifold with boundary, i.e., is continuous *up to the boundary* $\partial T^*\mathcal{M} \simeq T^*\partial M \times \mathbb{R}$. The restriction of the principal symbol to the boundary of the compressed cotangent bundle denoted by $\sigma_\partial(\mathbf{D})$ is called the *boundary symbol* of $\mathbf{D}$.

Ellipticity. The operator (7.1) is Fredholm in the Sobolev spaces $H^{s,\gamma}$ if and only if it is *elliptic*, i.e., both symbols (the principal symbol and the conormal symbol) are invertible. (See the finiteness theorem in Chapter 5).

In what follows, we consider only elliptic operators.

The index is independent of the smoothness exponent s of the Sobolev space, just as in the smooth case. However, the choice of the weight γ affects the index. The index of an operator $\mathbf{D}$ for the weight γ will be denoted by $\operatorname{ind}_\gamma \mathbf{D}$. Consider the simplest example.

EXAMPLE 7.1. The *Cauchy–Riemann operator*

$$\frac{\partial}{\partial t} + i\frac{\partial}{\partial \varphi}$$

on the infinite cylinder $\mathbb{R} \times \mathbb{S}^1$ can be viewed as a cone-degenerate operator. To see this, it suffices to make logarithmic changes of variables as $t \to \pm\infty$. (For example, for $t \to +\infty$ one sets $t = -\log r$.)

EXERCISE 7.2. Show that the operator

$$\frac{\partial}{\partial t} + i\frac{\partial}{\partial \varphi} : H^{s,\gamma_1,-\gamma_2}(\mathbb{R} \times \mathbb{S}^1) \longrightarrow H^{s-m,\gamma_1,-\gamma_2}(\mathbb{R} \times \mathbb{S}^1)$$

(where $\gamma_{1,2}$ are the weights at $\mp\infty$) is Fredholm provided that $\gamma_1, \gamma_2 \notin \mathbb{Z}$ and its index is given by the expression

$$\operatorname{ind}_{\gamma_1,\gamma_2}\left(\frac{\partial}{\partial t} + i\frac{\partial}{\partial \varphi}\right) = \operatorname{sgn}(\gamma_2 - \gamma_1)\,\{\text{the number of integer points between } \gamma_1 \text{ and } \gamma_2\}\,.$$

In particular, by varying the weight at infinity, one obtains an arbitrary prescribed value of the index.

◀ *Hint.* Use separation of variables to give a closed-form description of the kernel and cokernel of the operator in the weighted spaces: for $\gamma_1 < \gamma_2$, the cokernel is trivial and the kernel is spanned by the exponentials $e^{n(t+i\varphi)}$. In the computations, it is convenient to use the fact that the norm in $H^{0,\gamma_1,-\gamma_2}$ is equivalent to the norm given by the square root of the integral

$$\int_{\mathbb{R}} \|u(t)\|^2_{L^2(\Omega)}\varphi(t)dt, \text{ where } \varphi(t) = \begin{cases} e^{-2\gamma_1 t} & \text{at } -\infty, \\ e^{-2\gamma_2 t} & \text{at } +\infty. \end{cases}$$

▶

Let us discuss some other properties of the index which are due to the singular points and have no analogs on smooth manifolds. Recall that the index on smooth closed manifolds is invariant under perturbations (deformations) of the operator. More precisely, the index remains unchanged under the following transformations:

- an arbitrary variation in lower-order terms of an elliptic operator;
- a continuous deformation[1] of the coefficients of an elliptic operator.

These assertions follow from standard theorems of functional analysis stating the invariance of the index under compact perturbations and continuous deformations of a Fredholm operator.

Needless to say, these two properties remain valid on manifolds with singularities (by the same theorems of functional analysis).[2]

[1] Preserving the ellipticity.

[2] We note that *lower-order terms* in the singular case can be treated as the coefficients of the operator which occur in neither the principal symbol nor the conormal symbol.

However, if we weaken the condition and require the ellipticity of the principal symbol alone, then both assertions about the invariance of the index (as well as the assertion that the operator is Fredholm) fail. Formulas describing the variation of the index under the above-mentioned transformations are called *relative index formulas*.

To obtain relative index formulas, we need to compute the index in a model situation. This will be done in the next subsection.

7.1.2. The index of invariant operators on the cylinder

Consider the infinite cylinder $\mathbb{R}\times\Omega$ with smooth closed base Ω. We represent the cylinder as the smooth part of a manifold with two conical singular points corresponding to two infinite parts of the cylinder. The corresponding compact manifold with singularities is known in topology as the *suspension* over Ω

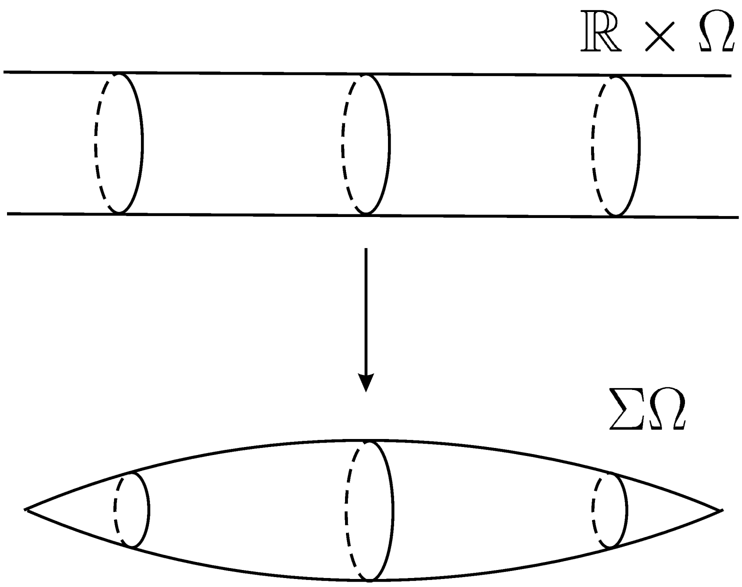

Figure 7.1. Infinite cylinder and the suspension.

and is denoted by $\Sigma\Omega$ (see Fig. 7.1). The suspension can also be represented as the space obtained by gluing two identical cones over Ω along their bases.

On the cylinder, we consider the operator

$$D\left(-i\frac{\partial}{\partial t}\right), \tag{7.2}$$

obtained from a family $D(p)$ of elliptic operators with parameter on the base of the cylinder by the substitution $p\mapsto -i\partial/\partial t$. This operator is obviously invariant under the translations $t\mapsto t+a$, and the index on the closed smooth manifold $\mathbb{S}^1\times\Omega$ would be necessarily zero. However, in the nonsmooth case in question the index may be nonzero, and we give a closed-form expression for it in the following assertion. Suppose that the operator (7.2) is elliptic with weight γ_1 at $-\infty$ and weight $-\gamma_2$ at $+\infty$ (cf. Exercise 7.2).

PROPOSITION 7.3. *For $\gamma_1 < \gamma_2$, the index is equal to the sum of multiplicities* (*see Chapter* 2) *of singular points of the family $D(p)$ in the strip between the weight lines*:

$$\operatorname{ind}_{\gamma_1,\gamma_2} D\left(-i\frac{\partial}{\partial t}\right) = \sum_{\gamma_1<\operatorname{Im} p_j<\gamma_2} m_D(p_j). \tag{7.3}$$

If $\gamma_2 < \gamma_1$, then the sign on the sum should be opposite (*and the singular points are taken in the strip* $\gamma_2 < \operatorname{Im} p_j < \gamma_1$).

Proof. For simplicity, we compute the index for the case in which all zeros of the family $D(p)$ are simple (i.e., have the unit multiplicity). By applying the Euler method to the homogeneous equation

$$D\left(-i\frac{\partial}{\partial t}\right) u = 0,$$

we find that its solutions are represented as linear combinations of functions of the form

$$e^{it\lambda} u_\lambda, \tag{7.4}$$

where $\lambda \in \mathbb{C}$ is a zero of the family $D(p)$ and u_λ is a nontrivial solution of the corresponding equation $D(\lambda)u_\lambda = 0$. Furthermore, the condition that the solution (7.4) must belong to the weighted space H^{s,γ_1,γ_2} results in the additional requirement $\gamma_1 < \operatorname{Im}\lambda < \gamma_2$. Thus if $\gamma_1 < \gamma_2$, then the dimension of the kernel of the operator is equal to the number of zeros of $D(p)$ in the strip. If $\gamma_1 > \gamma_2$, we conclude that the kernel is trivial.

Considering the adjoint operator, we see that the situation with the cokernel is opposite: the cokernel is trivial if the kernel is nontrivial, and vice versa. Thus we have proved the formula (7.3). $\square$

Remark 7.4. The proof of this assertion for general differential operators is given in (Sternin 1972).

An expression like the sum of multiplicities of the zeroes of a family occurring in the index formula (7.3) has already occurred in Chapter 2 in the definition of the spectral flow for families of parameter-dependent elliptic operators. It turns out that the spectral flow also occurs in index formulas if arbitrary operators on the cylinder are considered. The index formula in this case will be obtained as a corollary to the relative index formulas in the next subsection.

7.1.3. Relative index formulas

Consider the following operations over operators on manifolds with conical singularities:

1) change of the weight exponent γ of the Sobolev space $H^{s,\gamma}$;

2) change of the conormal symbol;

3) homotopy of the operator.

These operations can violate the ellipticity. Hence the index is no longer preserved. The aim of this subsection is to obtain formulas for the variation of the index of elliptic operators under these operations.

THEOREM 7.5 (relative index formulas). 1 (change of the weight exponent). *For an operator* $\mathbf{D}$ *elliptic with respect to two weights* $\gamma_1 < \gamma_2$, *the index variation under the passage from one weight to the other is equal to the sum of multiplicities of singular points of the conormal symbol* $\sigma_c(\mathbf{D})$ *in the strip between the weight lines*:

$$\operatorname{ind}_{\gamma_1}\mathbf{D} - \operatorname{ind}_{\gamma_2}\mathbf{D} = \sum_{\gamma_1 < \operatorname{Im} p_j < \gamma_2} m_{\sigma_c(\mathbf{D})}(p_j). \tag{7.5}$$

2 (change of the conormal symbol). *Let* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *be elliptic operators with the same interior symbols. Then the difference of their indices is equal to the spectral flow* (*see Chapter* 2) *across the weight line* $\operatorname{Im} p = \gamma$ *of the linear homotopy* $t\sigma_c(\mathbf{D}_1) + (1-t)\sigma_c(\mathbf{D}_2)$ *with parameter* t *joining the conormal symbols*:

$$\operatorname{ind}_\gamma \mathbf{D}_2 - \operatorname{ind}_\gamma \mathbf{D}_1 = \operatorname{sf}_\gamma\{t\sigma_c(\mathbf{D}_1) + (1-t)\sigma_c(\mathbf{D}_2)\}_{t\in[0,1]}. \tag{7.6}$$

3 (homotopy of operators). *Let* $\{\mathbf{D}_a\}_{a\in[0,1]}$ *be a continuous homotopy of cone-degenerate operators with elliptic interior symbols. If the operators at the beginning and the end of the homotopy are elliptic with respect to some weight* γ, *then the index variation is equal to the spectral flow across the weight line* $\operatorname{Im} p = \gamma$ *of the family of conormal symbols*

$$\operatorname{ind}_\gamma \mathbf{D}_0 - \operatorname{ind}_\gamma \mathbf{D}_1 = \operatorname{sf}_\gamma\{\sigma_c(\mathbf{D}_a)\}_{a\in[0,1]}. \tag{7.7}$$

Proof. Although the homotopy formula (7.7) implies all preceding formulas as simple corollaries, it will be convenient to us to prove first item 1, then item 3, and finally the formula in item 2.

1. Let us prove (7.5). Consider an operator $\mathbf{D}$ elliptic with respect to two weights γ_1 and γ_2. By the index locality property (see Chapter 1), the variation in the index of the operator under the change of weight is independent of the principal symbol of the operator outside the singular point. In other words, the difference of the indices is determined by the conormal symbol. To compute the difference, we use the surgery technique described in Chapter 1.

We deal with a neighborhood of the conical singularity in the cylindrical representation. Consider the surgery diagram shown in Fig. 7.2. Geometrically, the "vertical" surgeries correspond to a change of the weight at the conical point (from γ_2 to γ_1) and the horizontal surgeries correspond to the replacement of the smooth part of the manifold by a conical point with weight $-\gamma_1$. Then on the cylinder we obtain the translation-invariant operator $\sigma_c(\mathbf{D})\,(-i\partial/\partial t)$. An application of the relative index theorem proved in Chapter 1 to this commutative diagram of surgeries gives an expression for the difference of indices of the operator $\mathbf{D}$ with respect to different weights:

$$\operatorname{ind}_{\gamma_2}\mathbf{D} - \operatorname{ind}_{\gamma_1}\mathbf{D} = \operatorname{ind}_{-\gamma_1,\gamma_2}\sigma_c(\mathbf{D})\left(-i\frac{\partial}{\partial t}\right). \tag{7.8}$$

The index of the resulting invariant operator on the cylinder is just minus the sum of multiplicities of the poles in (7.5) (see Proposition 7.3).

2. Now let us obtain a formula for the change in the index under homotopies. To this end, we show that the index and the spectral flow vary by the same rule.

Consider a homotopy $\mathbf{D}_a$ of operators. For some parameter values, these operators fail to be Fredholm. (The conormal symbol $\sigma_c(\mathbf{D}_a)$ may be noninvertible on the weight line $\operatorname{Im} p = \gamma$.) To obtain a

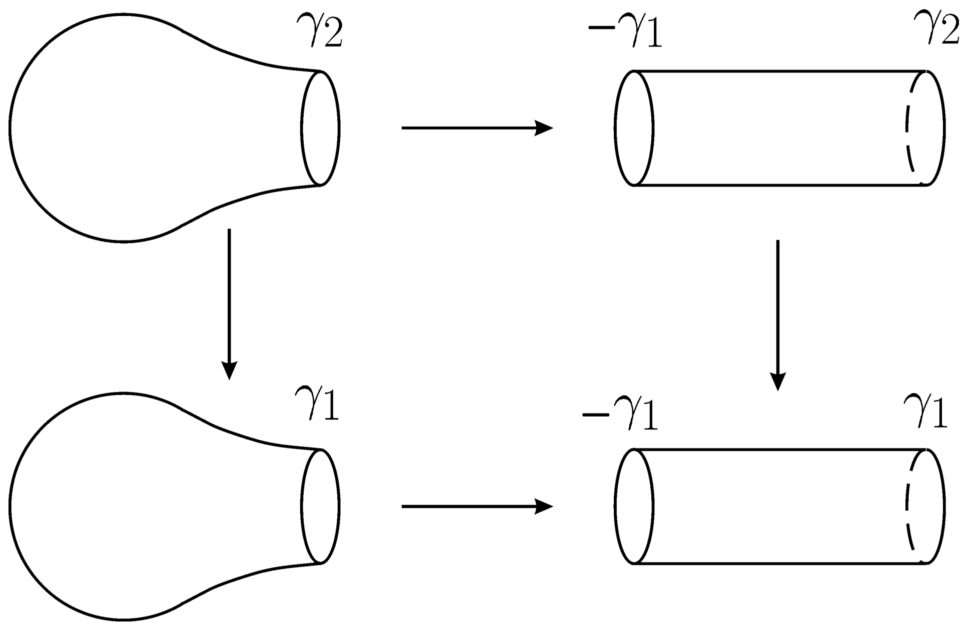

Figure 7.2. Surgery corresponding to a change in the weight.

family of *Fredholm* operators, just as in the definition of the spectral flow in Chapter 2, for the family of conormal symbols $\sigma_c(\mathbf{D}_a)$ we take some *admissible broken line* with vertices (a_i, λ_i) such that

$$a_0 = 0, \lambda_0 = \gamma, \qquad a_N = 1, \lambda_N = \gamma.$$

(Recall that a broken line is said to be admissible for a family $\sigma_c(\mathbf{D}_a)$ if for all parameter values on the interval $a_i \le a \le a_{i+1}$ there are no singular points of the family on the weight line $\operatorname{Im} p = \lambda_i$.)

Using an admissible broken line, we can readily construct a family of Fredholm operators: we choose the weight λ_i for the operator $\mathbf{D}_a$ on the half-open interval $a_i \le a < a_{i+1}$.

We know that only the vertices of the broken line contribute to the spectral flow. The contribution at $a = a_i$ is the sum of multiplicities of singular points of the family in the strip between the weight lines corresponding to λ_i and λ_{i+1}.

Thus to prove (7.7) it suffices to verify that

- the index $\operatorname{ind}_{\lambda_i} \mathbf{D}_a$ is constant on each interval $a_i \le a \le a_{i+1}$;
- the variation of the index as the parameter a passes through some vertex a_i is equal to the sum of multiplicities of zeros of the family in the strip between the weight lines (this corresponds to a change of weight at the singular point).

In the first case, the index is preserved by virtue of the continuity and ellipticity of the family on the intervals, and the index variation in the second case is described by the formula (7.5) proved earlier.

Thus we arrive at (7.7).

3. Finally, the formula (7.6) is obtained by the substitution of the linear homotopy $(1-t)\mathbf{D}_0 + t\mathbf{D}_1$ of operators on the manifold $\mathcal{M}$ into (7.7). $\square$

Relative index formulas can be used to compute the index in some special classes of operators. Examples of such applications will be considered in the next two subsections

Remark 7.6. For operators on manifolds with conical points, one also has an analog of the Agranovich formula (see Chapter 1): if two elliptic operators $\mathbf{D}_1$ and $\mathbf{D}_2$ coincide in a neighborhood of the singularity, then the difference of their indices is determined by the principal symbols outside the neighborhood and

$$\operatorname{ind}_\gamma \mathbf{D}_1 - \operatorname{ind}_\gamma \mathbf{D}_2 = \operatorname{ind} \mathbf{D}_1 \mathbf{D}_2^{-1}, \tag{7.9}$$

where $\mathbf{D}_2^{-1}$ is an almost inverse operator. The product $\mathbf{D}_1\mathbf{D}_2^{-1}$ on the right-hand side of the formula acts as the identity operator in a neighborhood of the singular point. Hence its index can be computed by the ordinary Atiyah–Singer formula.[3] The formula (7.9) follows from the logarithmic property of the index.

Note that this construction remains valid if we require only that the conormal symbols of the two operators coincide.

7.1.4. The index of general operators on the cylinder

As a first application of relative index formulas, we give a formula for the index of an arbitrary elliptic operator $\mathbf{D}$ stabilizing at infinity on the infinite cylinder $\mathbb{R} \times \Omega$. In particular, the stabilization condition holds for the cone-degenerate operators corresponding to the suspension $\Sigma\Omega$. In contrast to the case considered in Subsection 7.1.2, here we do not assume that the operator is translation-invariant with respect to the variable t. Without loss of generality, we can assume that the principal symbol of the operator is independent of t for large t.

EXERCISE 7.7. Show that the last condition can be ensured by a small deformation and addition of compact operators. Moreover, the deformation can also be chosen in such a way that the conormal symbol remains constant.

◀ *Hint.* Use that fact that the principal symbol of a stabilizing operator on the infinite cylinder can be viewed as the ordinary symbol over the compact manifold $\Omega \times [0, 1]$ with boundary. The latter symbol can readily be deformed to a symbol constant with respect to the normal variable in a neighborhood of the boundary. ▶

It turns out that for the case of the infinite cylinder the interior principal symbol and the conormal symbols at $\pm\infty$ can be considered together. To this end, we treat the operator $\mathbf{D}$ as an operator on the line $\mathbb{R}$ with an operator-valued symbol (in the sense of Chapter 4) acting in function spaces on Ω. The operator-valued symbol, which we denote by $D_t(p)$, is a homotopy $D_{t'}(p)$ of parameter-dependent elliptic operators. Here the parameter of the family is $p \in L_\gamma$, and the homotopy parameter is $t' \in (-\infty, +\infty)$. Moreover, the values of the homotopy at infinity are determined by the conormal symbols:

$$D_{t'} = \begin{cases} \sigma_c(\mathbf{D})|_{t\to-\infty} & \text{for } t' \le -N, \\ \sigma_c(\mathbf{D})|_{t\to+\infty} & \text{for } t' \ge N. \end{cases}$$

[3]The Atiyah–Singer theorem remains valid for manifolds with boundary if in a neighborhood of the boundary the operator is induced by a bundle isomorphism, i.e., "does not contain differentiations."

EXERCISE 7.8. Show that, conversely, to the family $D_t(p), t \in [-N, N]$ one can assign an operator on the infinite cylinder. Moreover, the correspondence

$$\text{(operator on the cylinder)} \;\leftrightarrow\; \text{(homotopy of families with parameter)}$$

is one-to-one if operators on the cylinder are considered modulo compact operators and homotopies modulo compact-valued homotopies vanishing for $|t| \geq N$.

◀ *Hint.* It suffices to note that a family of parameter-dependent pseudodifferential operators is uniquely determined by its principal symbol modulo families ranging in the set of compact operators. ▶

PROPOSITION 7.9. *The index of an operator on the infinite cylinder is equal to the spectral flow of the corresponding family of parameter-dependent elliptic operators*:

$$\operatorname{ind}_{\gamma,\gamma} \mathbf{D} = -\operatorname{sf}\{D_t\}_{t\in[-N,N]},$$

where sf *stands for the spectral flow across the weight line* $\operatorname{Im} p = \gamma$.

Proof. The idea of the proof is to use the fact that the cylinder $\Omega \times [-N, N]$ can be retracted to its base and thus deform $\mathbf{D}$ to an operator with coefficients independent of t.

Specifically, we equip the family D_t with an additional parameter a by the formula $D_{-N+a(t+N)}$. This deformation of families continuously joins the homotopy D_t corresponding to the original operator $\mathbf{D}$ (for $a = 1$) with the constant family D_0 ($a = 0$), which corresponds to an operator with coefficients independent of t; the index of the latter operator is zero (see Subsection 7.1.2). Hence the index of $\mathbf{D}$ is equal to the index variation under the homotopy and coincides with the spectral flow of the corresponding family of conormal symbols at $+\infty$, where the conormal symbol varies from D_{-N} to D_N. (The conormal symbol at $-\infty$ does not vary.) The proof is complete. □

7.1.5. The index of operators of the form $1 + \mathbf{G}$ with a Green operator $\mathbf{G}$

By a *Green operator* we mean a cone-degenerate pseudodifferential operator whose principal symbol is zero. Our aim is to compute the index of elliptic operators of the form $1 + \mathbf{G}$, where $\mathbf{G}$ is a Green operator. Such operators arise, say, as quotients of elliptic differential operators with coinciding principal symbols. The conormal symbol of the Green operator will be denoted by $g = \sigma_c(\mathbf{G})$. The ellipticity condition means in this case that the function $1 + g$ is invertible on the corresponding weight line.

The elliptic symbol $1 + g$ defines a topological invariant. Indeed, the conormal symbol g of the Green operator is compact-valued and tends to zero at infinity. Thus the symbol $1 + g$ can be treated as a continuous function on the circle $\mathbb{S}^1 = L_\gamma \cup \{\infty\}$ obtained as the one-point compactification of the weight line. By

$$w_\gamma(1 + g) \in \mathbb{Z}$$

we denote the *winding number* of this invertible operator function.

Recall that the space of invertible operator functions of the form $1 + K$, where K is compact-valued, has a fundamental group isomorphic to $\mathbb{Z}$, the isomorphism just being given by the winding number.

If the symbol of the Green operator is finite-dimensional, then the winding number can be computed by the well-known formula

$$w_\gamma(1 + g) = \frac{1}{2\pi i} \int_{\operatorname{Im} p=\gamma} \operatorname{Tr}((1 + g)^{-1} dg).$$

PROPOSITION 7.10. *The index of the operator* $1 + \mathbf{G}$ *is equal to the winding number of its conormal symbol* $1 + g$:

$$\operatorname{ind}_\gamma(1 + \mathbf{G}) = w_\gamma(1 + g).$$

Proof. The operator $1 + \mathbf{G}$ is linearly homotopic to the identity operator. By the relative index formula (7.6), the index of $1 + \mathbf{G}$ is equal to the spectral flow of the homotopy $1 + tg$ as t varies from zero to unity. The relation

$$\operatorname{sf}_\gamma\{1 + tg\}_{t\in[0,1]} = w_\gamma(1 + g) \tag{7.10}$$

can be verified by standard methods. □

EXERCISE 7.11. Prove (7.10).

◀ *Hint.* 1. First, using the homotopy invariance of both sides, replace g by a finite-dimensional family.

2. For finite-dimensional families, verify the relation for a family of diagonal matrices with the only nonunit diagonal entry equal to $(p + i)/(p - i)$.

3. The preceding computation suffices for establishing the formula in the general case, since finite-dimensional invertible families modulo homotopy form the fundamental group $\pi_1(\mathbf{GL}(N)) \simeq \mathbb{Z}$ of the group of invertible matrices, the generator being the family indicated in item 2. ▶

7.1.6. The index of operators of the form $1 + \mathbf{G}$ on manifolds with edges

In this and the next subsections, we consider some simplest index formulas on manifolds with edges.

Let $\mathcal{M}$ be a manifold with edge $X \subset \mathcal{M}$. Its blow-up will be denoted by M. Recall that the edge structure of $\mathcal{M}$ means that the boundary of the blow-up is the total space of a smooth locally trivial bundle $\pi : \partial M \to X$ with smooth base. The underlying topological space of $\mathcal{M}$ can be obtained from M if we identify points in the fibers of π on ∂M. If the fiber of π is denoted by Ω, then the space $\mathcal{M}$ in a neighborhood of a point of X looks like a neighborhood of an (arbitrary) singular point in the product $\mathbb{R}^n \times K_\Omega$. (Here n is the dimension of the edge and K_Ω is a cone over Ω.)

A *Green operator* is a zero-order edge-degenerate operator whose principal symbol is zero. It follows that the edge symbol of a Green operator treated as an operator function on the cotangent bundle T^*X of the edge has a compact fiber variation.

In this subsection, we give an index formula for operators of the form $1 + \mathbf{G}$, where $\mathbf{G}$ is a Green operator. The edge symbol of $\mathbf{G}$ will be denoted by g. The ellipticity condition in this case means that the operator function $1 + g(x, \xi)$ is invertible for $(x, \xi) \in T_0^*X$.

Recall (see Chapter 4) that an elliptic symbol on T_0^*X with compact fiber variation determines an elliptic Fredholm operator on X acting in the space of sections of an infinite-dimensional bundle. The fiber of this bundle is the space where the edge symbol acts. (In our case, the fiber is a function space on the infinite cone.) On the other hand, such a symbol determines a difference element in the K-group with compact supports of the cotangent bundle. This element is denoted by

$$\operatorname{ind}(1 + g) \in K_c(T^*X),$$

and the Luke formula

$$\operatorname{ind}\left(1 + g\left(x, -i\frac{\partial}{\partial x}\right)\right) = p_!(\operatorname{ind}(1 + g)) \tag{7.11}$$

is valid, where $p_! : K_c(T^*X) \to K_c(pt) = \mathbb{Z}$ is the direct image map induced by the projection into a point.

PROPOSITION 7.12. *The index of an elliptic edge-degenerate operator* $1+\mathbf{G}$ *with a Green operator* $\mathbf{G}$ *is equal to the direct image of the difference element determined by the edge symbol of the operator:*

$$\operatorname{ind}(1+\mathbf{G}) = p_!(\operatorname{ind}(1+g)).$$

Proof. The action of $\mathbf{G}$ far from the edge can be assumed to be zero. By the index locality principle, the index of $1+\mathbf{G}$ is equal to the index of an operator with the same principal and edge symbols on the bundle with fiber the infinite cone K_Ω. The latter operator on the bundle of infinite cones can be treated as a pseudodifferential operator on X with operator-valued symbol $1+g(x,\xi)$ (see Chapter 6). Hence our index formula is a special case of the Luke formula (7.11). □

Remark 7.13. If we additionally assume that not only the principal symbol of the Green operator is zero but also the conormal symbol $\sigma_c(g)$ of the edge symbol is zero, then the edge symbol will be compact-valued, and hence g can be approximated by a finite-dimensional symbol in such a way that the sum $1+g$ remains elliptic. Up to this approximation, the computation of the index in this case is reduced to the ordinary Atiyah–Singer formula.

7.1.7. The index on bundles with smooth base and fiber having conical points

In the preceding subsection, we have given an index formula for elliptic edge problems with unit principal symbol. Geometrically, the key point in the computation is the fact that it was carried out on a special manifold with edges, namely, on an infinite wedge, which is globally a bundle over the edge X, and the Green operator itself is represented as a pseudodifferential operator acting in sections of infinite-dimensional vector bundles on the edge.

In this subsection, we apply this technique in the more general case in which the manifold $\mathcal{M}$ with edge is globally represented as a bundle

$$\pi : \mathcal{M} \longrightarrow X$$

whose fiber is a closed manifold $\mathcal{N}$ with isolated conical singularities. Consider the simplest example.

EXAMPLE 7.14. Let $\pi : E \to X$ be a smooth bundle with closed base and fiber Ω. Then there is an associated *suspension bundle*

$$\widetilde{\pi} : \Sigma E \longrightarrow X$$

whose fiber is the suspension over Ω,

$$\Sigma\Omega = \{\{\Omega \times [0,2]\}/\{\Omega \times \{0\}\}\}/\{\Omega \times \{2\}\},$$

obtained from the finite cylinder $\Omega \times [0,2]$ with base Ω by shrinking both bases into points (which are conical singularities of the resulting manifold). The total space ΣE is a manifold with edge that is the disjoint union of two copies of the manifold X.

Let $\mathbf{D}$ be an elliptic edge problem on $\mathcal{M}$. We assume that its principal symbol $\sigma(\mathbf{D})$ is independent of the radial variable r in the neighborhood $\{r < \varepsilon\}$ of the edge.[4] For brevity, we restrict ourselves to the case of a zero-order scalar operator. The reader can readily generalize this to operators of arbitrary order acting in sections of vector bundles.

[4] The following proposition remains valid without this assumption, but the proof becomes technically more complicated.

PROPOSITION 7.15. *An elliptic zero-order edge-degenerate operator* $\mathbf{D}$ *can be represented modulo compact operators as an operator*

$$\mathbf{D} = D\left(x, -i\frac{\partial}{\partial x}\right) \tag{7.12}$$

on the edge X *with elliptic*[5] *operator-valued symbol*

$$D(x,\xi) \in S^0_{CV}(T^*_0 X)$$

with compact fiber variation ranging in the space of cone-degenerate pseudodifferential operator on the fiber $\mathcal{N}$.

In particular, the index is given by the Luke formula

$$\operatorname{ind}\mathbf{D} = p_!(\operatorname{ind} D). \tag{7.13}$$

Proof. We need only to prove the existence of the representation (7.12) and construct the symbol $D(x,\xi)$. To avoid clumsy notation, we assume that the manifold with edges is the Cartesian product $X \times \mathcal{N}$.

Under our conditions on the symbol, the operator $\mathbf{D}$ in a neighborhood of the edge is determined by the operator-valued symbol (see Chapter 6)

$$\big(\chi_1(r)\sigma_\wedge(\mathbf{D})\chi_1(r)\big)(x,\xi), \quad (x,\xi) \in T^*_0 X, \tag{7.14}$$

where $\sigma_\Lambda(\mathbf{D})$ is the edge symbol of our operator and the function χ_1 is identically equal to unity for small r and zero for large r.

Far from the edge, the operator $\mathbf{D}$ is a usual pseudodifferential operator on the smooth bundle $X \times N \to X$ and, as is well-known, is a pseudodifferential operator on X with operator-valued symbol

$$\chi_2(r)\sigma(\mathbf{D})\left(x, n, \xi, -i\frac{\partial}{\partial n}\right)\chi_2(r),$$

where n stands for the coordinates on the blow-up N and the smooth function $\chi_2(r)$ is equal to zero for small r and unity for large r.

Taking a partition of unity

$$1 = \chi_1^2(r) + \chi_2^2(r), \quad \operatorname{supp}\chi_1 \in [0,\varepsilon], \quad \operatorname{supp}\chi_2 \in [\varepsilon/2, \infty),$$

we arrive at the desired representation (7.12) with symbol $D(x,\xi)$ equal to

$$D(x,\xi) = \big(\chi_1\sigma_\wedge(\mathbf{D})\chi_1\big)(x,\xi) + \left(\chi_2\sigma(\mathbf{D})\left(x, n, \xi, -i\frac{\partial}{\partial n}\right)\chi_2\right). \tag{7.15}$$

Both terms in the last expression are zero-order symbols with compact fiber variation with respect to ξ. Indeed, for the first symbol this condition is ensured by the multiplication of the edge symbol by the cut-off function. The second term has a compact fiber variation as a parameter-dependent pseudodifferential operator. Thus the sum $D(x,\xi)$ also has a compact fiber variation:

$$D \in S^0_{CV}(T^*X).$$

[5] Recall that an operator-valued symbol defined on $T^*_0 X$ is said to be elliptic if it is Fredholm everywhere and uniformly invertible outside a compact set.

It remains to show that the symbol $D(x,\xi)$ is elliptic, i.e., is invertible for large $|\xi|$. Indeed, we can construct an almost inverse $C(x,\xi)$ of $D(x,\xi)$ by substituting the inverse symbols into (7.15):

$$C(x,\xi) = \left(\chi_1\sigma_\wedge(\mathbf{D})^{-1}\chi_1\right)(x,\xi) + \left(\chi_2\sigma(\mathbf{D})^{-1}\left(x,n,\xi,-i\frac{\partial}{\partial n}\right)\chi_2\right).$$

One can show that the products $C(x,\xi)D(x,\xi)$ and $D(x,\xi)C(x,\xi)$ have the form $1+K_j(x,\xi)$, $j=1,2$, where the norm of the remainder $K_j(x,\xi)$ tends to zero as $|\xi|\to\infty$. Hence $D(x,\xi)$ is invertible for large $|\xi|$. □

EXERCISE 7.16. Check the last part of the proof. To this end, use the twisted homogeneity of the edge symbol and the fact that the second terms in the symbols $C(x,\xi)$ and $D(x,\xi)$ are pseudodifferential operators with parameter ξ.

7.2. The Index Problem for Manifolds with Isolated Singularities

In this section, we consider operators in weighted spaces only for $\gamma = 0$.

We have seen that the index of an elliptic operator on a manifold with conical singularities essentially depends not only on the principal symbol but also on the conormal symbol, which represents the "infinite-dimensional" part of the problem data. On the other hand, it is clear that homotopy invariants of a finite-dimensional object are much easier to compute than those of an infinite-dimensional object. For this reason, we are interested only in index formulas in which these substantially different index contributions of the symbol components are separated:

$$\operatorname{ind}\mathbf{D} = f_1(\sigma(\mathbf{D})) + f_2(\sigma_c(\mathbf{D})). \tag{7.16}$$

Moreover, we assume that f_1 and f_2 are homotopy invariants of their respective arguments.

We have already seen an index formula similar to (7.16) in the theory of index defects for spectral problems in Chapter 3. Indeed, if (7.16) holds, then, by transposing the functional f_2 to the left-hand side, we find that the difference $\operatorname{ind}\mathbf{D} - f_2(\sigma_c(\mathbf{D}))$ is determined by the principal symbol of $\mathbf{D}$. In other words, f_2 is an index defect.

This analogy with spectral problems is very deep, and all facts of the theory of index defects developed earlier in this book can be transferred to the case of manifolds with singularities. For example, there is an obstruction to the index splitting (7.16) in the general case, and index formulas of this kind exist only for certain classes of operators rather than for all elliptic operators simultaneously. In other words, one can speak of index splittings only if one does not consider the (huge) space of elliptic operators on a manifold with singularities but restricts oneself to subspaces specified by certain conditions. We consider only conditions imposed on the principal symbol in a neighborhood of the singular point, since all main effects of the theory occur near the singularity. Let us give the simplest examples of conditions on the principal symbol.

EXAMPLE 7.17. 1. (the trivial example: all symbols) If there are no restrictions, then we consider the space of all symbols and accordingly all elliptic operators on $\mathcal{M}$.

2. (symbols constant with respect to the covariables) One can consider symbols independent of the covariables in a neighborhood of the singularity. This is a very severe restriction. The reader can verify

that the operators corresponding to such symbols are essentially[6] compositions of operators induced in a neighborhood of the singular point by bundle isomorphisms and operators $1+\mathbf{G}$, where $\mathbf{G}$ is a Green operator.

3. (symbols with symmetry) Consider the set of symbols whose restriction to the boundary of the compressed cotangent bundle satisfies some relation, say, the symmetry condition

$$\sigma(\xi,p)=\sigma(\xi,-p).$$

(Here (ξ,p) are coordinates in the fibers of the restriction $T^*\mathcal{M}|_{\partial M}$ of the compressed cotangent bundle to the boundary.)

Let us proceed to the precise statement of the index problem.

7.2.1. Statement of the index splitting problem

By Ell_Σ we denote the set of all elliptic operators on a manifold $\mathcal{M}$ with conical points whose boundary symbols lie in some given subset Σ.

We say that the class Ell_Σ admits an *index splitting* if there exist two homotopy invariant functionals

$$f_1,f_2:\mathrm{Ell}_\Sigma\longrightarrow\mathbb{R}$$

such that the first functional is determined by the principal symbol, the second functional is determined by the conormal symbol, and relation (7.16) holds for each operator $\mathbf{D}\in\mathrm{Ell}_\Sigma$.

We assume that Σ contains only symbols corresponding to elliptic theory on $\mathcal{M}$: each element of Σ can be realized as the boundary symbol of some elliptic operator $\mathbf{D}$.

7.2.2. The obstruction to the index splitting

The theorem stated below gives a necessary and sufficient condition for the existence of an index splitting in the class Ell_Σ in terms of the vanishing of some obstruction.

THEOREM 7.18 (on the index splitting). *The class* Ell_Σ *admits an index splitting* (7.16) *into homotopy invariant contributions of the symbols if and only if the spectral flow of any periodic family* $\{D_\tau\}_{\tau\in\mathbb{S}^1}$ *of elliptic operators with parameter whose symbols belong to* Σ *is zero*:

$$\operatorname{sf}\{D_\tau\}_{\tau\in\mathbb{S}^1}=0. \tag{7.17}$$

Remark 7.19. Informally speaking, one can single out a homotopy invariant contribution of the smooth part of the manifold to the index if and only if a nonzero periodic spectral flow cannot leak through the singularity.

Proof. 1) *Necessity.* We proceed by contradiction. Suppose that there exists an index splitting but the spectral flow of some periodic homotopy $D_t, t\in\mathbb{S}^1$, of parameter-dependent elliptic operators with principal symbols in Σ is nonzero.

The homotopy D_t can be lifted to a homotopy (not necessarily periodic) of operators $\mathbf{D}_t$ elliptic in the interior of the manifold and compatible with D_t: the conormal symbol of $\mathbf{D}_t$ is equal to D_t.

[6]Modulo small homotopies of operators.

EXERCISE 7.20. Construct such a homotopy $\{\mathbf{D}_t\}$ on $\mathcal{M}$.

◀ *Hint.* 1. It follows from the condition imposed on Σ that there exists an elliptic symbol σ_0 defined on $T^*\mathcal{M}$ and compatible with D_0 at the boundary: $\sigma_0|_{\partial T^*\mathcal{M}} = \sigma(D_0)$.

2. Starting from σ_0, we can define a homotopy σ_t of elliptic symbols on $T^*\mathcal{M}$ consistent with D_t for all t.

3. The desired homotopy $\mathbf{D}_t$ of operators is obtained by the quantization of the homotopy of the compatible pairs (σ_t, D_t) of principal and conormal symbols. ▶

Then the difference of the indices at the beginning and the end of the homotopy[7] is equal to the spectral flow and is nonzero by assumption. On the other hand, the indices of the operators at the beginning and the end of the homotopy necessarily coincide, since both components of the index splitting (7.16) for these operators are the same: the interior symbols are homotopic, and the conormal symbols merely coincide. This contradiction shows that the condition imposed on the spectral flow is necessary.

2) *Sufficiency.* Under condition (7.17), we construct some index splitting. To this end, for each arcwise connected component $\Sigma_\alpha \subset \Sigma$ we choose a point $\sigma_\alpha \in \Sigma_\alpha$ and an invertible family D_α with elliptic symbol σ_α. The families D_α play the role of origins to which we assign the zero value of the defect (the second component in the index splitting).

Now the value of f_2 on an elliptic family D can be defined as the spectral flow

$$f_2\,(D) \stackrel{\text{def}}{=} \operatorname{sf}\{D_t\}_{t\in[0,1]},$$

where by $\{D_t\}_{t\in[0,1]}$ we denote a homotopy joining D with the family D_α whose principal symbol lies in the connected component of the boundary symbol of D. This is well defined (independent of the choice of a homotopy) by condition (7.17) in the theorem. Indeed, the difference of two spectral flows with the same beginnings and ends is equal to the spectral flow of a periodic family.

Finally, the reader can verify (using the formula (7.7) for the index variation under changes in the conormal symbol) that if we take the first functional in the form of the difference

$$f_1(\sigma(\mathbf{D})) \stackrel{\text{def}}{=} \operatorname{ind}\mathbf{D} - f_2(\sigma_c(\mathbf{D})), \tag{7.18}$$

then it is indeed determined by the principal symbol alone. We have obtained an index splitting and proved the sufficiency. □

Remark 7.21. For the functional of the principal symbol, one can give an expression that does not use the functional of the conormal symbol. Namely,

$$f_1(\sigma(\mathbf{D})) = \operatorname{ind}\widetilde{\mathbf{D}}, \tag{7.19}$$

where $\widetilde{\mathbf{D}}$ is the elliptic operator on $\mathcal{M}$ obtained according to Fig. 7.3. More precisely, on the infinite cylindrical part we replace the operator by the expression $D_\alpha(-i\partial/\partial t)$, and in the finite part of the cylinder the symbol is determined by a homotopy joining the boundary symbol $\sigma(\mathbf{D})|_{\partial T^*\mathcal{M}}$ with the symbol $\sigma(D_\alpha)$. The fact that the expressions (7.18) and (7.19) are equal can readily be verified with the use of the formula expressing the index variation under homotopies.

[7] For these operators to be Fredholm, the family $D_0(p)$ must be invertible on the weight line; this can be achieved by a small deformation.

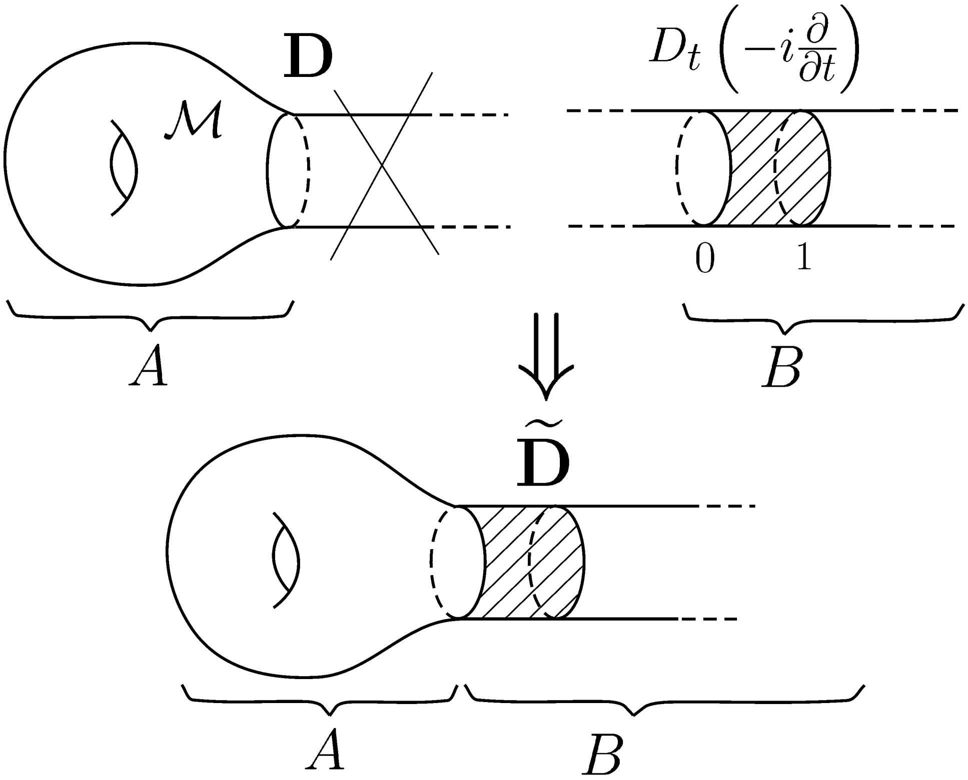

Figure 7.3. The construction of the operator $\widetilde{\mathbf{D}}$.

EXERCISE 7.22. One can also consider the (apparently more general) problem on the representation of the index in the form

$$\operatorname{ind}\mathbf{D} = f\Big([\sigma(\mathbf{D})], [\sigma_c(\mathbf{D})]\Big),$$

in terms of some functional f defined on pairs consisting of homotopy classes of elliptic symbols (the principal symbol and the conormal symbol taken separately). Show that the existence of an index splitting (7.16) is equivalent to the existence of the latter representation.

◀ *Hint.* The equivalence readily follows from the proof of the theorem: to prove that the spectral flow is zero, we have used only the equality of homotopy classes of the symbols. ▶

The simplest examples. Let us return to the operator classes introduced in Example 7.17.

1. The class of all elliptic symbols (operators) contains families with nonzero periodic spectral flow (e.g., see Chapter 3), and so *the index splitting for all operators simultaneously is impossible.*

2. For the class Σ of symbols independent of the covariables in a neighborhood of the singular point, the periodic spectral flow is zero and the index splitting has the form

$$\operatorname{ind}\mathbf{D} = \operatorname{ind}\widetilde{\mathbf{D}} + \operatorname{ind}\mathbf{D}\widetilde{\mathbf{D}}^{-1},$$

where $\widetilde{\mathbf{D}}$ is the elliptic operator whose principal symbol coincides with that of the original operator $\mathbf{D}$ and whose conormal symbol is induced by the *bundle isomorphism* $\sigma(\mathbf{D})|_{\partial M}$.

The indices of the operators in this decomposition can be computed. The first term contains an operator whose index can be computed by the Atiyah–Singer formula, and the second term involves an operator with unit principal symbol. The index of such an operator is equal to the winding number of its conormal symbol (see Subsection 7.1.5).

3. Finally, consider the class of symbols invariant with respect to the change $p \mapsto -p$ of the sign of the conormal variable p. In this case, the obstruction to the index splitting also vanishes. Indeed, consider a periodic homotopy $D_t(p)$ of families with parameter. Then we can assume that not only the principal symbol but also the family itself is symmetric:[8]

$$D_t(p) = D_t(-p).$$

A distinguishing feature of the symmetric family is that its singular points in the complex plane are symmetric with respect to the origin. Hence the spectral flow across the weight line $\operatorname{Im} p = 0$ is zero: as some zero of the family crosses the line downwards, there is a zero of the same multiplicity crossing the line upwards. The index formula in this case will be obtained in the next chapter with the use of the surgery technique.

7.2.3. Computation of the obstruction in topological terms

To verify that the obstruction vanishes in more complicated examples, a formula expressing the obstruction in topological terms would be useful. One can obtain such a formula by expressing the spectral flow via the index on a closed manifold. The latter can be expressed in topological terms by the Atiyah–Singer formula.

PROPOSITION 7.23. *The spectral flow of a periodic family $\{D_\varphi\}_{\varphi\in\mathbb{S}^1}$ of elliptic operators with parameter on a smooth compact closed manifold Ω is given by the formula*

$$\operatorname{sf}\{D_\varphi\}_{\varphi\in\mathbb{S}^1} = -\operatorname{ind} D_\varphi\left(-i\frac{\partial}{\partial\varphi}\right), \tag{7.20}$$

where the elliptic operator on the right-hand side is considered on the torus $\mathbb{S}^1 \times \Omega$.

Proof. We represent the spectral flow of the family $D_\varphi, 0 \le \varphi \le 2\pi$ as minus the index of an elliptic operator on the infinite cylinder (see Proposition 7.9).

Now we use the periodicity of D_φ. Consider the surgery diagram shown in Fig. 7.4. The upper left corner shows the suspension with the operator $\mathbf{D}$. The horizontal arrows stand for the surgery in which neighborhoods of the singularities are cut away and the corresponding bases of the cylinder are glued together to form a torus. The vertical arrows stand for the surgery in which the operator is replaced on the dashed cylindrical part by an operator with coefficients independent of t. The indices of the operators in the lower row are zero (the operator is translation invariant, and the weight is zero), and so the relative index theorem in Chapter 1 provides the desired equality (7.20) of indices for the operators in the upper row. □

[8]One can achieve this by replacing the family by the half-sum $(D_t(p) + D_t(-p))/2$.

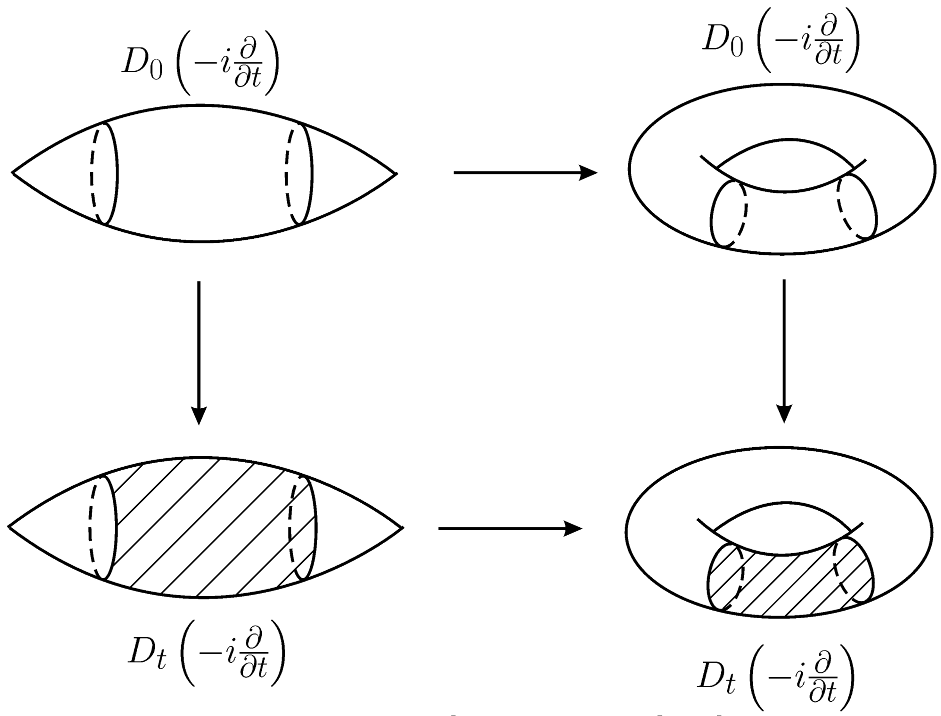

Figure 7.4. Surgery on the suspension and on the torus

7.2.4. Examples. Operators with symmetries

Theorem 7.18 and Proposition 7.23 taken together express the obstruction in terms of the index on a closed manifold. Let us try to find a class of operators for which the index in (7.20) vanishes. This index can be computed by the Atiyah–Singer formula. This formula expresses the index as the integral over the cotangent bundle of some expression depending on the principal symbol of the operator. The index will be zero if we consider symbols such that the contribution to the integral of a point of the cotangent bundle is cancelled by the contribution of some other point. By way of example, we give a specific implementation of this idea.

$\mathbb{Z}_2$-equivariant symbols. Suppose that the restriction $\partial T^*\mathcal{M}$ of the compressed cotangent bundle to the boundary is equipped with a smooth fiberwise involution

$$\alpha : \partial T^*\mathcal{M} \longrightarrow \partial T^*\mathcal{M}, \quad \alpha^2 = 1.$$

Here standard examples are

$$(p,\xi) \longmapsto (-p,\xi); \quad (p,\xi) \longmapsto (p,-\xi); \quad (p,\xi) \longmapsto (-p,-\xi),$$

where (p,ξ) are coordinates in the fibers of the bundle $\partial T^*\mathcal{M} \simeq \mathbb{R} \times T^*\Omega$.

By Σ_α we denote the class of $\mathbb{Z}_2$-*equivariant* boundary symbols on $\partial T^*\mathcal{M}$ with respect to the involution α.

This means that the group $\mathbb{Z}_2$ acts on $\partial T^*\mathcal{M}$ via the involution α, and the $\mathbb{Z}_2$-equivariance condition means that the vector bundles E and F are equipped with some involutions e and f and the symbol σ

satisfies the condition

$$f(\alpha^*\sigma) = \sigma e \tag{7.21}$$

everywhere on $\partial T^*\mathcal{M}$.

PROPOSITION 7.24. *The class* $\mathrm{Ell}_{\Sigma_\alpha}$ *admits an index splitting provided that the involution* α *reverses the orientation in the fibers of the bundle* $\partial T^*\mathcal{M}$.

Here a remark is in order. The notion of an equivariant symbol involves the choice of involutions in the vector bundles. Hence a *continuous homotopy of equivariant symbols* should be understood as a homotopy such that the involutions also vary continuously. Strictly speaking, we have not stated the index splitting theorem in this case of symbols with an additional structure. The reader can verify that Theorem 7.18 (together with the proof) can be transferred to this case without any modifications.

Proof of Proposition 7.24. It suffices to verify that the obstruction vanishes by using the formula in Theorem 7.23. This theorem involves an operator on $\mathbb{S}^1 \times \Omega$ corresponding to some homotopy of $\mathbb{Z}_2$-equivariant boundary symbols. This operator will be denoted by D. Its principal symbol is also $\mathbb{Z}_2$-equivariant. Hence for the difference element

$$[\sigma(D)] \in K_c(T^*(\mathbb{S}^1 \times \Omega))$$

of the elliptic symbol $\sigma(D)$ we obtain the following relation in K-theory:

$$\alpha^*[\sigma(D)] = [\sigma(D)]. \tag{7.22}$$

On the other hand, the involution α reverses the orientation of the bundle $T^*(\mathbb{S}^1 \times \Omega)$ by assumption. Hence from the Atiyah–Singer formula for the topological index we readily obtain the relation[9]

$$\mathrm{ind}_t[\alpha^*\sigma(D)] = -\,\mathrm{ind}_t[\sigma(D)].$$

This, together with (7.22), implies that $\mathrm{ind}\, D = \mathrm{ind}_t[\sigma(D)] = 0$. ☐

In the next chapter, we obtain index formulas for operators with $\mathbb{Z}_2$-equivariant symbols.

In closing of the subsection, we note that we have described only $\mathbb{Z}_2$-equivariant symbols, which can be viewed as a generalization, to this class of operators, of Gilkey's parity conditions in the theory of spectral boundary value problems (see Chapter 3). By way of exercise, the reader might wish to generalize another examples given in Chapter 3 (in particular, examples related to flat bundles).

[9]It suffices to substitute the identity $-[T^*N] = \alpha^*[T^*N]$ (orientation reversal) into the expression for the topological index (for brevity, we set $N = \mathbb{S}^1 \times \Omega$)

$$\mathrm{ind}_t[\alpha^*\sigma(D)] = \langle \mathrm{ch}\,[\alpha^*\sigma(D)]\,\mathrm{Td}\,(T^*N \otimes \mathbb{C})\,, [T^*N]\rangle$$

and cancel α^*.

7.3. The Index Problem for Manifolds with Edges

This section deals with the index problem for elliptic problems on manifolds with edges. The exposition will be rather brief. The reader can naturally ask why, for manifolds with edges in particular include manifolds with conical points, and so it would be logical to expect that the theory of the former is more informative.

This is of course true as long as one speaks of specific index formulas. However, the index splitting problem can be transferred from the case of conical points to the case of edges without serious modifications. This simplicity is due to the fact that the index problem is the problem of singling out the contribution of the principal symbol, which, just as in the conical case, is essentially defined over a manifold with boundary.

We shall first describe the so-called index excision property, which will then be used in the computation of the obstruction to the existence of index splittings.

7.3.1. The index excision property

Recall that the index splitting theorem for manifolds with isolated singularities, considered in the preceding section, is based on the notion of spectral flow and a formula relating the spectral flow to the index variation under deformations of operators. We have not introduced the notion of spectral flow (of the corresponding edge symbols) for the case of manifolds with edges. Hence we need some version of the index splitting theorem in the case of edges. Instead of the formula describing the index variation under deformations of the operator, we us the *index excision property*, which is described in what follows.

Let Σ be a two-sided hypersurface in a smooth closed compact manifold M. We take a tubular neighborhood $\Sigma \times [-1, 1] \subset M$. Then we can define two smooth compact manifolds (see Fig. 7.5):

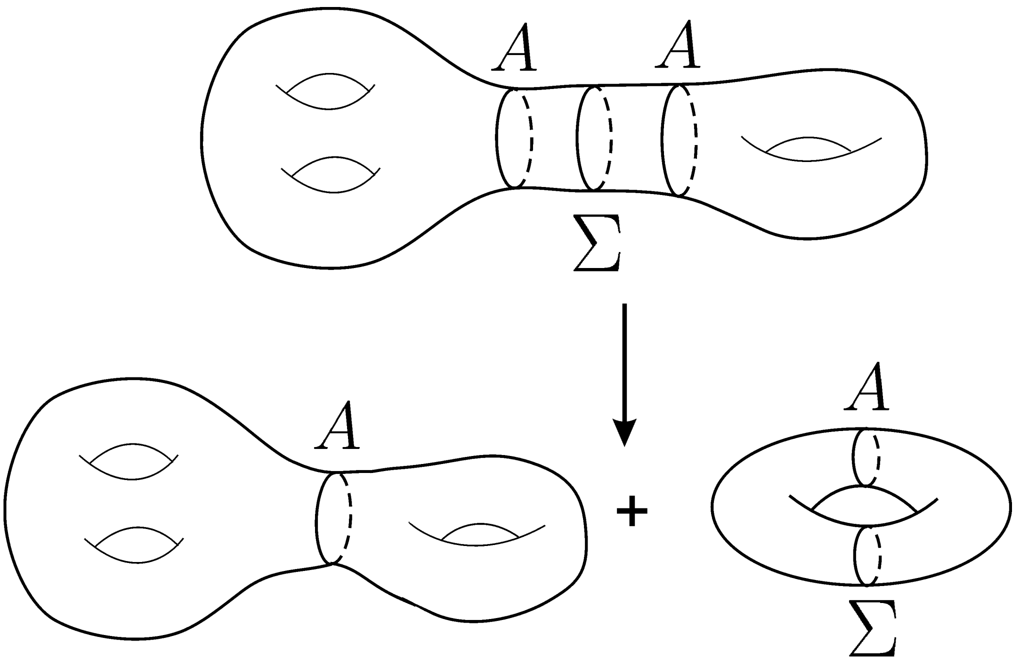

Figure 7.5. Cutting away a neighborhood of a hypersurface.

- the first manifold $M \setminus \Sigma \times (-1,1)$ is obtained by cutting $\Sigma \times (-1,1)$ away from M and then by gluing the components $\Sigma \times \{\pm 1\}$ of the boundary of the resulting manifold together;[10]
- the second manifold is the torus $\Sigma \times \mathbb{S}^1$ obtained by gluing together the boundary components of the deleted part $\Sigma \times [-1,1]$.

Let us carry out a similar surgery in elliptic theory.

Let D_M be an elliptic operator on M such that its principal symbol is the same on both components of the boundary of the tubular neighborhood:

$$\sigma(D_M)|_{\Sigma\times\{-1\}} = \sigma(D_M)|_{\Sigma\times\{1\}}$$

Under this condition, the symbol $\sigma(D_M)$ naturally determines symbols on the above-mentioned manifolds $M\setminus\Sigma\times(-1,1)$ and $\Sigma\times\mathbb{S}^1$. We denote the operators corresponding to the symbols by $D_{M\setminus\Sigma\times(-1,1)}$ and $D_{\Sigma\times\mathbb{S}^1}$.

INDEX EXCISION PROPERTY.

$$\operatorname{ind} D_M = \operatorname{ind} D_{M\setminus\Sigma\times(-1,1)} + \operatorname{ind} D_{\Sigma\times\mathbb{S}^1}. \tag{7.23}$$

EXERCISE 7.25. Prove the excision property.

◀ *Hint.* See the surgery diagram on Fig. 7.6. ▶

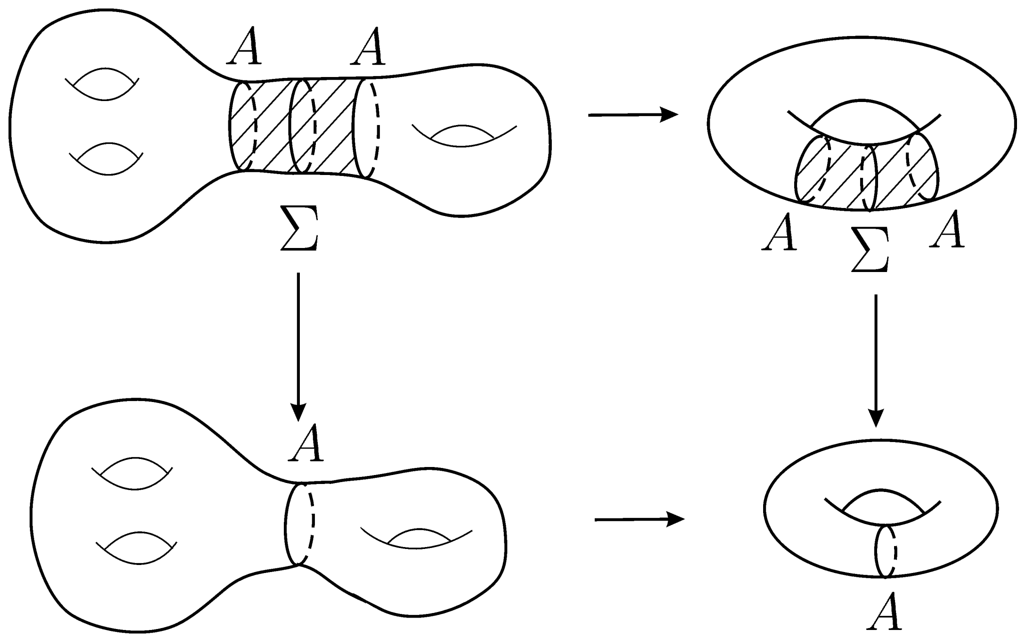

Figure 7.6. The index excision property.

Note that the index excision property remains valid (together with the proof) even if M has singularities. We should only require that the hypersurface Σ together with its tubular neighborhood lie in the smooth part of the manifold. A typical example of such a hypersurface is the submanifold $r = \varepsilon$ of a manifold with edge (a *section parallel to the edge*).

[10]Thus we obtain a manifold diffeomorphic to M.

7.3.2. The obstruction to the index splitting

The main result of this section is a *necessary and sufficient condition for the existence of index splitting* for elliptic problems on manifolds with edges (an analog of Theorem 7.18 for isolated singularities).

Just as before, we take a class Ell_Σ of edge-degenerate elliptic operators whose boundary symbols lie in some given subset Σ of the space of symbols on $\partial T^*\mathcal{M}$. Here, as before, the *boundary symbol* is defined as the restriction of the principal symbol to the boundary $\partial T^*\mathcal{M}$ of the cotangent bundle.

THEOREM 7.26 ((on the index splittings)). *The class* Ell_Σ *admits an index splitting*

$$\operatorname{ind}\mathbf{D} = f_1(\sigma(\mathbf{D})) + f_2(\sigma_\Lambda(\mathbf{D})), \tag{7.24}$$

where f_1 is a homotopy invariant of an invertible principal symbol and f_2 is a homotopy invariant of an invertible edge symbol if and only if for each periodic family of parameter-dependent elliptic operators $\{D_\varphi(p)\}_{\varphi\in\mathbb{S}^1}$ with symbols in Σ the index of the corresponding operator on the torus $\partial M\times\mathbb{S}^1$ is zero:

$$\operatorname{ind} D_\varphi\left(-i\frac{\partial}{\partial\varphi}\right) = 0. \tag{7.25}$$

Proof. The proof reproduces that of 7.18; the main difference is related to the fact that instead of the formula expressing the index variation via the spectral flow of a homotopy we use the index excision property.

1) *Necessity of condition* (7.25) *for the existence of an index splitting.*

The periodic homotopy of boundary symbols mentioned in the theorem can also be viewed as an elliptic symbol σ on the torus $\partial M\times\mathbb{S}^1$. We take an elliptic operator $\mathbf{D}$ on $\mathcal{M}$ with boundary symbol equal to $\sigma|_{\varphi=0}$. We cut $\mathcal{M}$ along the submanifold $\{r=\varepsilon\}$ and glue in the cylinder $\partial M\times[-1,1]$ with the symbol σ defined on it between the coasts of the cut. The resulting operator will be denoted by $\mathbf{D}'$. Then by the index excision property we have

$$\operatorname{ind}\mathbf{D}' = \operatorname{ind}\mathbf{D} + \operatorname{ind}\widehat{\sigma}, \tag{7.26}$$

where $\widehat{\sigma}$ is the elliptic operator with symbol σ on the torus $\partial M\times\mathbb{S}^1$.

If there exists an index splitting, then the indices of $\mathbf{D}$ and $\mathbf{D}'$ are the same, since they have homotopic principal symbols and coinciding edge symbols, which, together with (7.26) gives the desired relation $\operatorname{ind}\widehat{\sigma}=0$.

2) *Sufficiency.* Just as before, for each connected component $\Sigma_\alpha\subset\Sigma$ we take an elliptic edge symbol D_α with principal symbol in Σ_α.

Now for an arbitrary elliptic operator $\mathbf{D}$ on the manifold $\mathcal{M}$ the first term in the index splitting can be defined as the index

$$f_1(\sigma(\mathbf{D})) = \operatorname{ind}\widetilde{\mathbf{D}}$$

of an elliptic operator $\widetilde{\mathbf{D}}$ on $\mathcal{M}$ whose edge symbol is equal to D_β (we assume that the boundary symbol $\sigma_\partial(\mathbf{D})$ lies in the component Σ_β) and whose principal symbol is obtained, by analogy with the preceding part of the proof, by attaching a homotopy $\sigma_\partial(\mathbf{D})\sim\sigma(D_\beta)$ joining the boundary symbol and the principal symbol of the edge symbol D_β to the principal symbol $\sigma(\mathbf{D})$ in a neighborhood of the edge.

The second term is given by the formula

$$f_2\left(\sigma_\Lambda(\mathbf{D})\right) \stackrel{\text{def}}{=} \operatorname{ind}\mathbf{D} - \operatorname{ind}\widetilde{\mathbf{D}}. \tag{7.27}$$

Now the relation $f_1+f_2=\operatorname{ind}$ holds automatically. It remains to show that the functionals f_1 and f_2 are well defined by the corresponding symbols.

The first functional f_1 is *a priori* determined by the principal symbol $\sigma(\mathbf{D})$ and the homotopy. It is actually independent of the homotopy, which can readily be derived from the excision property and the vanishing of the index of the operator on the torus.

The independence of the functional f_2 defined in (7.27) of the form of the operator in the interior part of the manifold is actually a restatement of the index locality property under two surgeries, one changing the operator far from the edge and the other gluing in a homotopy in a neighborhood of the edge (the passage from $\mathbf{D}$ to $\widetilde{\mathbf{D}}$). □

We see that the obstruction to the representation of the index in the form of a sum of invariants of the symbol components in the edge theory coincides with that in the conical case. Hence the examples considered in Section 7.2.4 can be transferred to manifolds with edges word for word.

Hence there are no new effects in edge theory as long as the existence of index splittings is concerned. The differences arise once we deal with the derivation of specific index formulas. This is the subject of the next chapter.

7.4. Bibliographical Remarks

Index formulas in the form of separate contributions of the principal symbol and the conormal symbol were considered for the first time in (Schulze, Sternin and Shatalov 1998). A criterion for the existence of an index splitting was obtained in the framework of spectral problems in (Savin, Schulze and Sternin 1999). In the conical case, the criterion is given in (Nazaikinskii, Schulze and Sternin Oktober 1999). For manifolds with edges, the corresponding results were obtained in (Nazaikinskii, Savin, Schulze and Sternin 2003).

Bibliography

Nazaikinskii, V., Savin, A., Schulze, B.-W. and Sternin, B. (2003), *Elliptic Theory on Manifolds with Nonisolated Singularities. V. Index Formulas for Elliptic Problems on Manifolds with Edges*, Univ. Potsdam, Institut für Mathematik, Potsdam. Preprint N 2003/02.

Nazaikinskii, V., Schulze, B.-W. and Sternin, B. (Oktober 1999), *On the Homotopy Classification of Elliptic Operators on Manifolds with Singularities*, Univ. Potsdam, Institut für Mathematik, Potsdam. Preprint N 99/21.

Savin, A., Schulze, B.-W. and Sternin, B. (1999), 'On invariant index formulas for spectral boundary value problems', *Differentsial'nye uravnenija* **35**(5), 709–718.

Schulze, B.-W., Sternin, B. and Shatalov, V. (1998), 'On the index of differential operators on manifolds with conical singularities', *Annals of Global Analysis and Geometry* **16**(2), 141–172.

Sternin, B. Yu. (1972), *Quasielliptic Operators on an Infinite Cylinder*, MIEM, Moscow [in Russian].

Potsdam 2004