# Manifolds with Isolated Singularities[1]

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# Differential Operators on Manifolds with Singularities

## Analysis and Topology

This is a draft version of Chapter V of the book "*Differential Operators on Manifolds with Singularities. Analysis and Topology*" to be published by *Francis and Taylor.*

# Contents

# Chapter 5

# Manifolds with Isolated Singularities

## 5.1. Differential Operators and the Geometry of Singularities

### 5.1.1. How do isolated singularities arise? Examples

First of all, let us give some simple examples in which manifolds with isolated singularities and the corresponding degenerate or “singular” differential operators arise.

***The transition to polar coordinates.*** Possibly the simplest way to obtain a degenerate operator and an isolated singularity is to take a usual differential operator with smooth coefficients in Cartesian coordinates and proceed to polar coordinates. For obvious reasons, the singularity thus obtained should be called *fictitious*. However, it becomes quite real if we slightly perturb the coefficients of the equation in the new coordinates. Indeed, consider, say, the Laplace operator

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

on the plane and proceed to the polar coordinates by the formulas

$$x = r\cos\varphi, \quad y = r\sin\varphi.$$

In these coordinates, the $xy$-plane is represented by the cone

$$K = \left(\overline{\mathbb{R}}_+ \times S^1\right) \Big/ \left(\{0\} \times S^1\right),$$

whose base is the circle $S^1 \ni \varphi$ and whose generator is the half-line $\overline{\mathbb{R}}_+ = [0, +\infty) \ni r$.

◀ The passage to the quotient in the above formula means that all points of the form $(0, \varphi)$, $\varphi \in S^1$, are identified with one another. The resulting common point of all generators is the cone vertex (corresponding to the origin in the variables $(x, y)$). ▶

The Laplacian is given in the polar coordinates by the well-known expression

$$\triangle = \frac{1}{r^2}\left[\left(r\frac{\partial}{\partial r}\right)^2 + \frac{\partial^2}{\partial\varphi^2}\right]$$

and (after the omission of the common factor $r^{-2}$) can be viewed as an operator with smooth coefficients on the half-cylinder

$$K^\wedge = \overline{\mathbb{R}}_+ \times S^1$$

obtained from $K$ by blowing up the vertex to a circle. However, the new operator

$$\left(r\frac{\partial}{\partial r}\right)^2 + \frac{\partial^2}{\partial\varphi^2},$$

in contrast to the original operator, is not elliptic but *degenerate* elliptic. More precisely, it has a Fuchsian degeneration at $r = 0$ with respect to the variable $r$ (i.e., can be represented as a polynomial in $r\partial/\partial r$). Of course, this is only an "apparent" degeneration in that the operator has no singularity or degeneracy in the original Cartesian coordinates. However, if we slightly perturb the coefficients in the class of smooth functions on the half-cylinder and consider, say, the operator

$$\triangle_\varepsilon = \frac{1}{r^2}\left[\left(r\frac{\partial}{\partial r}\right)^2 + (1+\varepsilon)\frac{\partial^2}{\partial\varphi^2}\right],$$

then we see that the perturbed operator has singularities in the coefficients even in the original coordinates. (Note that one can hardly tell the qualitative difference between $\triangle = \triangle_0$ and $\triangle_\varepsilon$ with $\varepsilon \neq 0$ by examining the expressions for these operators in polar coordinates.)

EXERCISE 5.1. Write out the expression of $\triangle_\varepsilon$ in the Cartesian coordinates.

Consider the equation

$$\triangle_\varepsilon u = 0.$$

We require that this equation hold only outside the singularity, in other words, for $r > 0$ in the polar coordinates. Moreover, we assume that the function $u$ is also defined in general only for $r > 0$. Then for all $\varepsilon$ the equation in question has not only the trivial solution $u = \mathrm{const}$ but also the solution $u = C\ln r$, where $C$ is an arbitrary constant.

EXERCISE 5.2. Do there exist other solutions? What solutions would we obtain if we started from the three-dimensional Laplacian and passed to spherical coordinates?

***The Laplacian on the cone.*** Let us now give the simplest example in which the singularity is "genuine," i.e., geometric. In space $\mathbb{R}^3$ with coordinates $(x, y, z)$, consider the right circular cone $K_\Theta$ given by the relations

$$x^2 + y^2 = z^2\sin^2\Theta, \quad z \geq 0 \tag{5.1}$$

(see Fig. 5.1). Here $\Theta$ is the angle between the axis and the generator of the cone. As coordinates on $K_\Theta$, we use the functions $(r, \varphi)$, where $r$ is the distance from the vertex along the generator and $\varphi$ is the polar angle corresponding to the projection of a point of the cone on the $xy$-plane. Thus

$$r^2 = x^2 + y^2 + z^2 = (x^2+y^2)(1+\sin^2\Theta), \quad \sin\varphi = \frac{y}{\sqrt{(x^2+y^2)}}.$$

The cone $K_\Theta$ is equipped with the natural metric given by the restriction of the Euclidean metric $dx^2 + dy^2 + dz^2$ of the ambient space $\mathbb{R}^3$. It has the form

$$ds^2 = dr^2 + r^2\sin^2\Theta\, d\varphi^2. \tag{5.2}$$

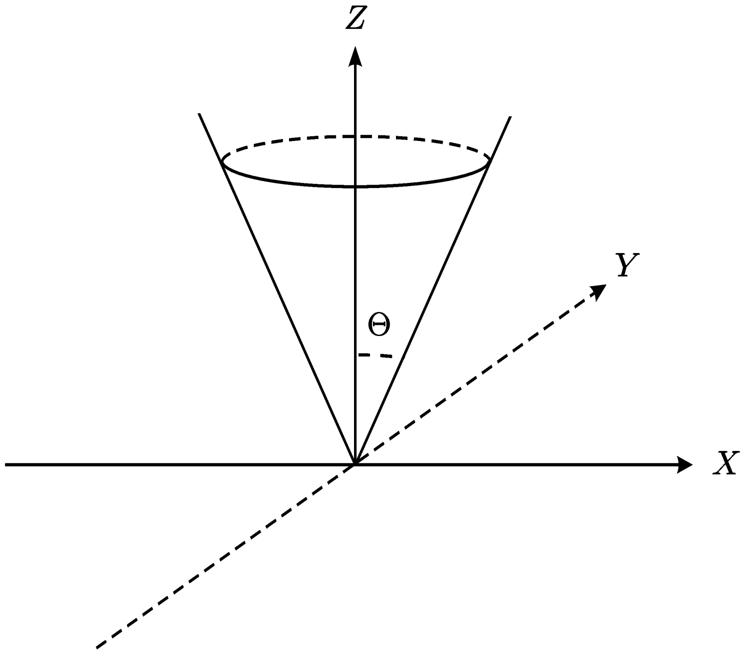


Figure 5.1. A circular cone

Consider the corresponding Beltrami–Laplace operator

$$\triangle = \frac{1}{r^2}\left[\left(r\frac{\partial}{\partial r}\right)^2 + c^2\frac{\partial^2}{\partial \varphi^2}\right], \tag{5.3}$$

where the positive parameter $c^2 = \sin^{-2}\Theta$ is determined by the angle $\Theta$ at the cone vertex. (By the way, note that this operator coincides with $\triangle_\varepsilon$, where $\varepsilon = c^2 - 1$, and is reduced to the usual Laplace operator for $\Theta = \pi/2$, i.e., for the case in which the cone is just the $xy$-plane.)

EXERCISE 5.3. Prove the formula (5.3).

◀ Use the general formula

$$\triangle_g = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial \alpha^i}\left(g^{ij}\sqrt{\det g}\frac{\partial}{\partial \alpha^j}\right),$$

which expresses the Beltrami–Laplace operator $\triangle_g$ corresponding to the metric $g$ in local coordinates $\{\alpha_j\}$ via the components and the determinant of the metric tensor $g = (g_{ij})$. ▶

We see that a natural geometric operator on the cone, the Beltrami–Laplace operator, is a differential operator with Fuchsian degeneration. The same is true for other geometric operators.

EXERCISE 5.4. Show that the exterior differentiation operator and the Euler operator on the cone are also operators with Fuchsian degeneration.

***The Laplacian on the cuspidal cone.*** Now consider the cuspidal cone $\Upsilon_k \subset \mathbb{R}^3$ given by the relations

$$x^2 + y^2 = z^{2(k+1)}, \quad z \geq 0 \tag{5.4}$$

(see Fig. 5.2), where $k$ is a positive integer[1] known as the *order* of the cusp.

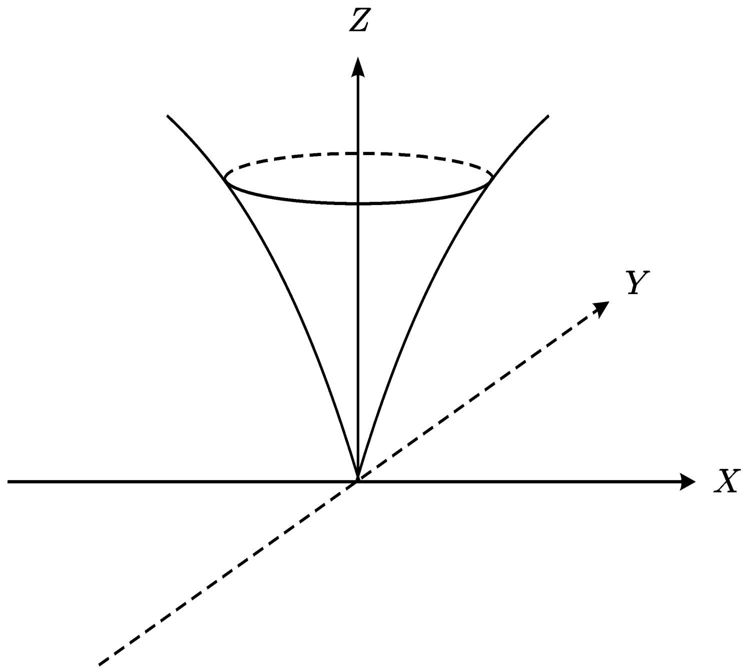


Figure 5.2. A circular cusp of order $k$

It is convenient to take coordinates on $\Upsilon_k$ in the form $(z, \varphi)$, where $\varphi$ is the polar angle of the projection of a point of the cusp in the $xy$-plane. The restriction of the Euclidean metric of $\mathbb{R}^3$ to $\Upsilon_k$ has the form

$$ds^2 = \big(1 + (k+1)^2 z^{2k}\big) dz^2 + z^{2k+2} d\varphi^2, \tag{5.5}$$

and the corresponding Beltrami–Laplace operator can be represented as

$$\triangle = z^{-2(k+1)} \left[ a(z)^{-3/2} \left( z^{k+1} \frac{\partial}{\partial z} \right) a(z)^{1/2} \left( z^{k+1} \frac{\partial}{\partial z} \right) + \frac{\partial^2}{\partial \varphi^2} \right], \tag{5.6}$$

where $a(z) = \big(1 + (k+1)^2 z^{2k}\big)$ is a smooth nonvanishing function, and (up to the unessential factor $z^{-2(k+1)}$) is an operator with degeneration of order $k+1$: the operator $\partial/\partial z$ occurs only in the combination $z^{k+1}\partial/\partial z$.

EXERCISE 5.5. Derive the formula (5.6).

One can verify that the other geometric operators (like the exterior differentiation operator of the Euler operator) are also operators with degeneration of order $k+1$.

[1] For $k = 0$, we return to the conical case. One can also consider fractional-order cusps (e.g., see (Schulze, Sternin and Shatalov 1998)), but we do not touch this matter in the present book.

### 5.1.2. Definition and methods for the description of manifolds with isolated singularities

Generalizing the preceding examples, we shall consider manifolds with isolated singularities such that each singular point has a neighborhood homeomorphic to a cone (whose base is not necessarily a circle or a sphere but can be an arbitrary compact closed manifold of class $C^\infty$). On these manifolds, we deal with differential operators with degeneration of order $k+1$, where $k \in \{0\} \cup \mathbb{N}$ is a given number. In this subsection, we formalize the relevant notions.

***A geometric description.*** Let us pass to a definition of a manifold with isolated singularities. We are interested only in the case of finitely many singular points. In what follows, we assume that there is only one singular point so as to avoid additional subscripts. First, let us consider the manifolds themselves without any relationship with differential operators.

The following definition describes a local model of a manifold in a neighborhood of the singular point.

DEFINITION 5.6. Let $\Omega$ be a smooth compact manifold without boundary. The *cone with base* $\Omega$ is the topological space[2]

$$K \equiv K_\Omega = \big([0,1) \times \Omega\big) \;/\; \big(\{0\} \times \Omega\big).$$

(We identify all points of the form $(0,\omega)$, $\omega \in \Omega$.) The point $\beta = [\Omega \times \{0\}] \in K$ is called the cone *vertex*.

The set $\overset{\circ}{K} = K \backslash \{\beta\}$ possesses the natural differentiable structure inherited from $\Omega \times (0,1)$. The variable $r \in [0,1)$ specifies a well-defined function on $K$. This function will be referred to as the *radial variable*.

The *blow-up* of the cone $K_\Omega$ is defined as the cylinder

$$K_\Omega^\wedge = \Omega \times [0,1).$$

Now we can give a formal definition of a manifold with isolated singularities.

DEFINITION 5.7. Let $M$ be a separable topological space, and let $\alpha \in M$ be a given point. One says that $M$ is a *manifold with isolated singularities* if the set $M \setminus \{\alpha\}$ is equipped with the structure of an open manifold of class $C^\infty$ and if for some neighborhood $U$ of the point $\alpha$ there is a given homeomorphism

$$\psi : U \longrightarrow K_\Omega, \tag{5.7}$$

where $\Omega$ is a smooth compact closed manifold, such that

(a) $\psi(\alpha) = \beta$, i.e., the mapping (5.7) takes the singular point to the cone vertex;

(b) the mapping $\psi$ and the inverse mapping are smooth outside $\alpha$ (respectively, outside the cone vertex).

The point $\alpha$ will be called the *singular point* of $M$, and the neighborhood $U$ together with the mapping (5.7) will be called the *chart* or *coordinate neighborhood* of the singular point. The manifold $\Omega$ is called the *base of the cone* at the singular point.

[2] Sometimes we shall use the infinite cone $\big([0,\infty) \times \Omega\big) \;/\; \big(\{0\} \times \Omega\big)$, which will be denoted by $K_\Omega$. This will not lead to a misunderstanding, since the meaning will be always clear from the context.

EXERCISE 5.8. Let $M$ be a manifold with an isolated singular point $\alpha$. Then there is a well-defined manifold $M^\wedge$ with boundary $\partial M^\wedge = \Omega$ called the *blow-up* of $M$ at the singular point. The interior of $M^\wedge$ can be identified with $M \setminus \alpha$, and one can reconstruct $M$ itself from $M^\wedge$ by identifying all points of the boundary.

◀ *Hint*. To construct the desired blow-up, use the coordinate mapping (5.7) and blow up the cone $K_\Omega$. ▶

***Differential operators and the type of the singular point.*** Definition 5.7 is essentially coarse. For example, it does not distinguish between the cone (5.1) and the cusp (5.4), which prove to be isomorphic manifolds with isolated singularities under this definition. To specify the *type* of the singular point (cone, cusp of order $k$, ...), one should introduce some *additional structure* on $M$. Following the examples considered above, one could try to specify the type of the singular point by taking a smooth Riemannian metric on $M \setminus \{\alpha\}$ with a given type of singularity (or degeneration) at $\alpha$. This approach would suffice if we were interested only in geometric operators, which are determined by the metric. However, we wish to study differential operators of sufficiently general form. Hence the most direct way of defining the type of singularity is to explicitly describe the supply of differential operators to be considered without resorting to a Riemannian metric or specific embeddings of the manifold in question in Euclidean space. This method was suggested in (Schulze, Sternin and Shatalov 1998). We point out that when studying operators on manifolds with isolated singularities, we omit the common factor $r^l$ occurring in operators that naturally arise in geometric situations (e.g., see (5.6)).

***The algebra of cone-degenerate operators.*** Let $M$ be a manifold with singular point $\alpha$; the base of the corresponding cone will be denoted by $\Omega$. Consider the set $\mathcal{D}_0(M)$ of differential operators that can be arbitrary operators with smooth coefficients away from the singular point and are representable in the coordinate neighborhood $U$ by finite sums[3]

$$D = \sum a_{j\sigma}(r,\omega) \left( ir\frac{\partial}{\partial r} \right)^j \left( -i\frac{\partial}{\partial \omega} \right)^\sigma , \tag{5.8}$$

where $r$ and $\omega$ are coordinates on $[0,1)$ and $\Omega$, respectively, and the coefficients $a_{j\sigma}(r,\omega)$ are smooth functions (up to $r=0$).

EXERCISE 5.9. Prove that $\mathcal{D}_0(M)$ is an algebra (with respect to standard operators of addition and multiplication of operators).

The algebra $\mathcal{D}_0$ consists of operators with Fuchsian degeneration with respect to the variable $r$ as $r \to 0$. It will be called the algebra of *cone-degenerate* operators.

DEFINITION 5.10. A *manifold with conical singularities* is a manifold $M$ with isolated singularities equipped with the algebra $\mathcal{D}_0(M)$. The algebra $\mathcal{D}_0(M)$ will be referred to as the *structure ring* of $M$.

***The algebra of cusp-degenerate operators of order $k$.*** Let $M$ again be a manifold with singular point $\alpha$ and base $\Omega$ of the cone. By analogy with the conical case, we define the algebra of cusp-degenerate operators as the set of differential operators that have the form (cf. (5.6))

$$D = \sum a_{j\sigma}(r,\omega) \left( ir^{k+1}\frac{\partial}{\partial r} \right)^j \left( -i\frac{\partial}{\partial \omega} \right)^\sigma \tag{5.9}$$

[3]The factors $\pm i$ multiplying the partial derivatives have been introduced, as usual, for convenience in the definition of the symbol in what follows.

in the coordinate neighborhood $U$ of the singular point and can be arbitrary differential operators with smooth coefficients away from the singular point.

DEFINITION 5.11. A *manifold with cusp singularities of order* $k$ is a manifold $M$ with isolated singularities equipped with the algebra $\mathcal{D}_k(M)$ of differential operators that have the form (5.9) in $U$ and are arbitrary differential operators with smooth coefficients outside the singular point. The algebra $\mathcal{D}_k(M)$ will be called the *structure ring* of $M$.

*Remark* 5.12. Our definitions use a single coordinate neighborhood $U$ of the singular point and a given mapping

$$\psi : U \longrightarrow K_\Omega$$

onto the model cone $K_\Omega$. One could admit a multitude of singular charts

$$\psi_j : U_j \longrightarrow K_\Omega;$$

then the transition mappings

$$\psi_{jl} \equiv \psi_j \psi_l^{-1} : K_\Omega \longrightarrow K_\Omega, \tag{5.10}$$

defined on $\overset{\circ}{K}_\Omega$ in a neighborhood of the vertex, should admit extensions to smooth mappings of the blow-up $K_\Omega^\wedge$ of $K_\Omega$ into itself and, for the case of a cusp of order $k > 0$, satisfy some additional conditions providing that the form (5.9) of cusp-degenerate differential operators is preserved under such mappings.

EXERCISE 5.13. Show that it suffices to require that the transition mappings preserve the appropriate form of vector fields:

$$\psi_{jl*}\left(a\frac{\partial}{\partial\varphi} + br^{k+1}\frac{\partial}{\partial r}\right) = \widetilde{a}\frac{\partial}{\partial\varphi} + \widetilde{b}r^{k+1}\frac{\partial}{\partial r}, \tag{5.11}$$

where $a$ and $b$ are arbitrary smooth functions on $K_\Omega^\wedge$ and $\widetilde{a}$ and $\widetilde{b}$ are some new smooth functions (depending on $a$ and $b$).

EXERCISE 5.14. Explicitly write out conditions that should be satisfied by the transition mappings $\psi_{jl}$ for the validity of (5.11).

***Melrose's description of singularities.*** Melrose (e.g., see (Melrose 1981)) suggested an alternative description of singularities. Let $M^\wedge$ be a smooth manifold with boundary. By $\mathrm{Vect}(M^\wedge)$ we denote the Lie algebra of smooth (up to the boundary) vector fields on $M^\wedge$, and by $\mathrm{Vect}_0(M^\wedge)$ we denote the Lie subalgebra of fields whose support does not contain boundary points. The idea is to describe the singularities by specifying a Lie algebra $\mathcal{V}$ of vector fields such that

$$\mathrm{Vect}_0(M^\wedge) \subset \mathcal{V} \subset \mathrm{Vect}(M^\wedge).$$

EXAMPLE 5.15. Let $\mathcal{V} = \mathcal{V}_0$ be the Lie algebra of vector fields tangent to the boundary $\partial M^\wedge$ everywhere on $\partial M^\wedge$. This algebra describes a manifold $M$ with conical singularities obtained from $M^\wedge$ by shrinking the boundary $\partial M^\wedge$ into a point (the "$b$-differential calculus").

EXAMPLE 5.16. Let $\mathcal{V} = \mathcal{V}_k$ be the Lie algebra of vector fields $V$ on $M^\wedge$ such that

$$Vr = r^{k+1} f(x),$$

where $f(x)$ is a smooth function on $M$ (up to the boundary). It is assumed that a direct product structure $U^\wedge \simeq \Omega\times[0,1)$ is chosen in a collar neighborhood $U^\wedge$ of the boundary of $M^\wedge$, and $r$ is the coordinate on $[0,1)$ smoothly extended into the entire $M^\wedge$ (and nonvanishing away from the boundary). This algebra describes a manifold $M$ with cusp singularities of order $k$ obtained from $M^\wedge$ by shrinking the boundary $\partial M^\wedge$ into a point.

*Remark* 5.17. Needless to say, the algebra $\mathcal{V}_k$ depends for $k>0$ on the choice of the direct product structure in $U^\wedge$.

EXERCISE 5.18. Show that the universal enveloping algebra of the Lie algebra $\mathcal{V}\oplus C^\infty(M)$ in these two examples coincides with the corresponding structure rings of differential operators introduced above, and conversely, the algebras $\mathcal{V}$ can be reconstructed from the structure rings of differential operators as the sets of differential operators of order 1 in these rings.

Thus in our examples Melrose's description is equivalent to ours. In the next chapter, we shall see that nonisolated singularities can also be described in the framework of both approaches.

### 5.1.3. Bundles. The cotangent bundle

***Bundles over manifolds with singularities.*** Bundles are an extremely important notion that is hard to avoid in the theory of elliptic differential equations on manifolds with singularities. First, many natural differential operators (e.g., geometric operators) act in sections of bundles. Second, the principal symbol of an operator is itself a function[4] on the total space the cotangent bundle of the manifold (just as in the case of smooth manifolds).

We shall consider only finite-dimensional vector bundles.

DEFINITION 5.19. A *vector bundle* $E$ over a manifold $M$ with singularities is always understood as a vector bundle $E$ (denoted by the same letter) over the blow-up $M^\wedge$ of the manifold $M$.

Let $M$ be a manifold with isolated singular point $\alpha\in M$ of order $k$ (i.e., a conical singular point for $k=0$ or a cusp point for $k>0$) with base $\Omega$. A collar neighborhood $U$ of the boundary $\partial M^\wedge$ has a given trivialization $U\simeq\Omega\times[0,1)$. Hence there is a well-defined projection

$$\pi : U \cong \Omega\times[0,1) \xrightarrow{\pi} \Omega$$

on the first factor. Since the interval $[0,1)$ is contractible, it follows that there exists an isomorphism

$$E|_U \cong \pi^*(E|_\Omega), \tag{5.12}$$

which we shall use in the description of sections of $E$ in a neighborhood of $\alpha$.

Now let $E$ and $F$ be two vector bundles over $M$. Consider the space $\mathcal{D}_k(M;E,F)\equiv\mathcal{D}_k(E,F)$ of differential operators taking sections of $E$ to sections of $F$ and having the form (5.8) (for $k=0$) or (5.9) (for $k>0$) near the singular point. Such operators can be rewritten in the form

$$\widehat{D} = \sum_j \widehat{a}_j(r)\left(ir^{k+1}\frac{\partial}{\partial r}\right)^j, \tag{5.13}$$

[4] More precisely, for operators action in sections of vector bundles, a section of a bundle.

where the $\widehat{a}_j(r)$ are arbitrary differential operators with smooth coefficients on the base of the cone (and smooth in $r$ up to 0).

In what follows in this chapter, we deal only with scalar operators to simplify the notation. All our assertions can be automatically transferred, *mutatis mutandis*, to the case of operators acting in sections of vector bundles.

***The cotangent bundle. Motivation.*** The cotangent bundle is important in the theory of differential and pseudodifferential operators. Let $M$ be a compact manifold with isolated singularities of order $k$. The cotangent bundle $T^*M$, as well as any other bundle over $M$, is a bundle over the blow-up $M^\wedge$ of $M$. However, it is not just equal to $T^*M^\wedge$. The definition of the cotangent bundle given below have been constructed in such a way that an operator of the form

$$\widehat{D} = \sum_{j+|\sigma|\le m} a_j(r,\omega)\left(ir^{k+1}\frac{\partial}{\partial r}\right)^j\left(-i\frac{\partial}{\partial\omega}\right)^\sigma \tag{5.14}$$

has the principal symbol $\sigma(\widehat{D})$ on $T^*M$ of the following form in canonical local coordinates:

$$\sigma(\widehat{D}) = \sum_{j+|\sigma|=m} a_j(r,\omega)p^j q^\sigma. \tag{5.15}$$

(Should we set $T^*M = T^*M^\wedge$, the symbol in canonical local coordinates would have the form

$$\sigma(\widehat{D}) = \sum_{j+|\sigma|=m} a_j(r,\omega)(r^{k+1}p)^j q^\sigma$$

and be necessarily degenerate on the boundary $\partial T^*M = \{r=0\}$.)

Thus the forthcoming definition is adapted to the study of degenerate operators of the form (5.14). If the principal symbol (5.15) is nonzero for $|p|+|q|\neq 0$, then the operator (5.14) is elliptic (or, more precisely, formally elliptic, see below) in our theory.

How can one ensure that the principal symbol of the operator (5.14) has the form (5.15)? Here the following considerations can be helpful.

In the usual (smooth) case differential operators are arbitrary polynomials in vector fields on the manifold with smooth coefficients. Vector fields are sections of the tangent bundle, and accordingly, the principal symbols of differential operators are functions on the total space of the dual (cotangent) bundle.

Let us transfer this construction to manifolds with singularities.

***The cotangent bundle of a manifold with conical singularities.*** First, consider the case $k=0$ (conical singularities). On the manifold $M^\wedge$ with boundary, differential operators belonging to $\mathcal{D}_0(M)$ are polynomials of vector fields tangent to $\partial M^\wedge$. These fields form a $C^\infty(M^\wedge)$-module denoted by $\mathcal{V}_0$. (In other words, one can add vector fields tangent to the boundary and multiply them by smooth functions, the tangency being preserved.) Consider the dual module $\operatorname{Hom}_{C^\infty(M^\wedge)}(\mathcal{V}_0, C^\infty(M^\wedge))$. (Recall that elements of this module are $C^\infty(M^\wedge$-linear mappings of $\mathcal{V}_0$ into $C^\infty(M^\wedge))$; they can be added and multiplied by functions in a natural way.) This is a locally free $C^\infty(M^\wedge)$-module and hence by the Serre–Swan theorem (Atiyah 1989) the module of sections of some vector bundle over $M^\wedge$. This bundle is denoted by $\widetilde{T}^*M^\wedge$ and called the *compressed cotangent bundle* of $M^\wedge$. This construction is due to Melrose; it is described in detail in (Melrose 1981). We define the *cotangent bundle* of $M$ by setting

$$T^*M \stackrel{\text{def}}{=} \widetilde{T}^*M^\wedge.$$

The space $T^*M$ is a manifold with boundary and a vector bundle over $M^\wedge$. Since in the interior of the manifold the set $\mathcal{V}_0$ coincides with the set of all vector fields, it follows that there is a canonical isomorphism

$$\mu : T^*M\backslash\partial T^*M \to T^*\overset{\circ}{M}.$$

Let us describe the structure of $T^*M$ (and the isomorphism $\mu$) near the boundary $\partial T^*M$. Let $\xi \in \partial T^*M$. The projection $\pi(\xi)$ of $\xi$ on $M^\wedge$ lies in $\Omega$. Under the trivialization $U^\wedge \cong \Omega \times [0,1)$ of a collar neighborhood of $\Omega$ in $M^\wedge$, each vector field tangent to $\Omega$ has the form

$$X = Y(r) - a(\omega, r) r \frac{\partial}{\partial r},$$

where $Y(r)$ is a vector field on $\Omega$ depending on the parameter $r$. Thus $X$ is a section of $T\Omega \times \mathbb{R}$ over $U$ (the second component is given by the coefficient $a(\omega, r)$) and hence $T^*M|_{U^\wedge}$ also has the product decomposition

$$T^*M|_{U^\wedge} \cong T^*\Omega \times ([0,1) \times \mathbb{R}). \tag{5.16}$$

The canonical coordinates on $T^*M$ corresponding to (5.16) will be denoted by $(r, \omega, p, q)$, $r \in [0,1)$, $p \in \mathbb{R}$, $(\omega, q) \in T^*\Omega$.

Let us describe the mapping $\mu$ in terms of the natural embedding $T^*\overset{\circ}{M} \subset T^*M^\wedge$. We denote canonical coordinates on $T^*M^\wedge$ near the boundary by $(r, \omega, \widetilde{p}, q)$, where $(\omega, q) \in T^*\Omega$ and $(r, \widetilde{p}) \in T^*[0,1)$. Then $\mu$ has the form

$$(r, \omega, p, q) \mapsto \left(r, \omega, -\frac{p}{r}, q\right),$$

i.e., is given by the formula $\widetilde{p} = -p/r$. The symplectic form on $T^*M$ induced by the action of $\mu$ on the symplectic form on $T^*\overset{\circ}{M}$ has the form

$$\omega^2 = -\frac{dp \wedge dr}{r} + dq \wedge d\omega. \tag{5.17}$$

Thus $T^*M$ is a smooth manifold equipped with a symplectic form $\omega^2$ that has a singularity of the form (5.17) on $\partial T^*M$.

The decomposition (5.16) in particular implies the representation

$$\partial T^*M \cong T^*\Omega \times \mathbb{R}. \tag{5.18}$$

In other words, $\partial T^*M$ is a bundle over $\Omega$ whose fiber over a point $\omega \in \Omega$ is $T^*_\omega\Omega \times \mathbb{R}$.

EXERCISE 5.20. Prove that the coordinate variable $p$ given by the second component of the decomposition (5.18) is independent of the choice of a direct product structure on $U$.

This variable will be called the *conormal variable* on $\partial T^*M$.

***The cotangent bundle of a manifold with cusps.*** Now consider the case of cusp singularities of order $k > 0$. In this case, the differential operators $\widehat{D} \in \mathcal{D}_k(M)$ are polynomials of vector fields on $M^\wedge$ whose normal component (with respect to a given trivialization $U \simeq \Omega \times [0,1)$) vanishes to the order $1 + k$ at the boundary. Such fields form an $C^\infty(M^\wedge)$-module $\mathcal{V}_k$. Arguing just as in the case of a conical singularity, we define the cotangent bundle $T^*M$ in this case.

This is again a manifold with boundary equipped with a symplectic form that has a singularity at the boundary. In the standard coordinates $(r, \omega, p, q)$, the symplectic form is given by the expression

$$\omega^2 = -k\frac{dp \wedge dr}{r^{k+1}} + dq \wedge d\omega. \tag{5.19}$$

The canonical isomorphism

$$\mu : T^*M \backslash \partial T^*M \to T^*\overset{\circ}{M} \subset T^*M^\wedge$$

is described near the boundary by the formula

$$(r, \omega, p, q) \mapsto \left(r, \omega, -kp/r^{k+1}, q\right).$$

## 5.2. Asymptotics of Solutions, Function Spaces, Conormal Symbols

The main goal of this section is to introduce function spaces on manifolds with isolated singularities, where elliptic theory will then be developed. Our elliptic operators degenerate at singular points (i.e., at $r = 0$ in terms of local coordinates). Hence the solutions of the corresponding homogeneous equations may have singularities at $r = 0$, and one should choose function spaces including such solutions. Moreover, when constructing the asymptotics of solutions as $r \to 0$ one naturally singles out the "leading" part of the operator, which is called (up to the Mellin or Borel–Mellin transform) the *conormal symbol* and (as well as the leading asymptotic term) is invariant with respect to some natural one-parameter group of scaling transformations with respect to the variable $r$. Hence it is natural to require that the spaces introduced here be invariant with respect to this group.

### 5.2.1. Conical singularities

***Asymptotics of solutions of the homogeneous equation.*** Let $\widehat{D} \in \mathcal{D}_0(M)$ be a degenerate differential operator having the form

$$\widehat{D} = \sum_{k+|\sigma|\le m} a_{k\sigma}(r, \omega) \left(ir\frac{\partial}{\partial r}\right)^k \left(-i\frac{\partial}{\partial \omega}\right)^\sigma \tag{5.20}$$

near the conical point, and moreover, let

$$\sigma(\widehat{D}) = \sum_{k+|\sigma|=m} a_{k\sigma}(r, \omega) p^k q^\sigma \neq 0 \tag{5.21}$$

for $|p| + |q| \neq 0$. (This condition is known as the *formal ellipticity* condition for the principal symbol.) Consider the homogeneous equation

$$\widehat{D}u = 0. \tag{5.22}$$

For $r > 0$, this equation is elliptic in the usual sense, and hence its solutions are infinitely smooth. Let us study the asymptotics of solutions as $r \to 0$, as the equation degenerates. We first consider the simplest case in which the coefficients $a_{k\sigma}(r, \omega)$ are independent of $r$,

$$a_{k\sigma}(r, \omega) \equiv a_{k\sigma}(\omega),$$

so the operator has the form

$$\widehat{D} = D\left(ir\frac{\partial}{\partial r}\right), \quad D(p) = \sum_{k+|\sigma|\le m} a_{k\sigma}(\omega)p^k \left(-i\frac{\partial}{\partial \omega}\right)^{\sigma}. \tag{5.23}$$

We seek a solution of Eq. (5.22) by separation of variables, i.e., in the form of a sum of products:

$$u(r,\omega) = \sum_k R_k(r)\Phi_k(\omega). \tag{5.24}$$

◀ One can readily justify this approach by passing from Eq. (5.22) to a $2m \times 2m$ system of first order in $r\partial/\partial r$. The operator of the system has the form of the sum $A + ir\partial/\partial r$, where the operator $A$ contains neither the variable $r$ nor the differentiation with respect to $r$ and hence admits separation of variables in a standard manner. ▶

In the most trivial case, the sum (5.24) contains only one term (we omit the index $k$ in this case), and the functions $R(r)$ and $\Phi(\omega)$ can be found from the system

$$ir\frac{\partial}{\partial r}R = pR, \quad D(p)\Phi = 0, \tag{5.25}$$

where $p$ is a number that should be found simultaneously with $R$ and $\Phi$. Here $D(p)$ is a polynomial pencil (see (5.23)) of elliptic operators on the compact manifold $\Omega$ (the base of the cone at the singular point), and the second equation in the system means precisely that $p$ is an *eigenvalue* of this operator pencil (i.e., a parameter value for which the elliptic operator $D(p)$ has a nontrivial kernel). By virtue of the ellipticity, the kernel consists of smooth functions and is finite-dimensional. Moreover, it follows from the formal ellipticity condition (5.21) that the family $D(p)$ is an operator elliptic with parameter $p$ in the sense of Agranovich–Vishik (Agranovich and Vishik 1964) and hence the eigenvalues of the pencil form a discrete set. The solution of the first equation in system (5.25) has the form

$$R(r) = r^{-ip},$$

and we obtain a particular solution of the homogeneous equation (5.22) in the form

$$u(r,\omega) = r^{-ip}\Phi(\omega),$$

where $\Phi(\omega)$ is an eigenfunction of the operator pencil corresponding to the *eigenvalue* $p$. In the general case, one has to take account of the fact that the operator pencil in question may have not only eigenfunctions but also associated functions. Accordingly, our expansions will involve not only eigenfunctions but also associated functions of $ir\partial/\partial r$, which have the form $(\ln r)^j r^{-ip}$, $j = 0, 1, 2, \ldots$, and the solution of the homogeneous equation becomes

$$u(r,\omega) = \sum_k r^{-ip_k} \sum_{j=0}^{m_k} \Phi_{jk}(\omega)(\ln\, r)^j, \tag{5.26}$$

where the $p_k$ are the *characteristic numbers* (the eigenvalues of the operator pencil), and the $m_j$ are their *algebraic multiplicities* (necessarily finite). Note that here we have found *exact* solutions (which is not surprising, since the coefficients $a_{k\sigma}(r,\omega)$ are independent of $r$). In the general case of $r$-dependent coefficients, we represent the operator $\widehat{D}$ as

$$\widehat{D} = \widehat{D}_0 + r\widehat{D}_1 + r^2\widehat{D}_2 + \dots \tag{5.27}$$

by expanding the coefficients in Maclaurin series with respect to $r$. In (5.27), the operator $\widehat{D}_0$ has the form

$$\widehat{D}_0 = D_0\left(ir\frac{\partial}{\partial r}\right), \quad D_0(p) = \sum_{k+|\sigma|\le m} a_{k\sigma}(0,\omega)p^k\left(-i\frac{\partial}{\partial\omega}\right)^\sigma, \tag{5.28}$$

and the family $D_0(p)$ is again elliptic with parameter $p$ in the sense of Agranovich–Vishik. To obtain an asymptotic expansion of the solution of Eq. (5.22), which now acquires the form

$$\widehat{D}_0 u = -r\widehat{D}_1 u - r^2\widehat{D}_2 u - \dots, \tag{5.29}$$

it is natural to use perturbation theory, taking the solution of the homogeneous equation with the operator $\widehat{D}_0$ as the zero approximation. We omit details (which can be found, say, in (Schulze, Sternin and Shatalov 1998) and the literature cited therein) and give only the definitive result. Asymptotically, the solution can be written in the form

$$u(r,\omega) \simeq \sum_{l=0}^{\infty} r^{-i\mu_l}\sum_{j=0}^{m_l}\Phi_{jl}(\omega)(\ln r)^j, \tag{5.30}$$

where the sequence $\{\mu_l\}$, ordered in descending order of imaginary parts, is a finite union of sequences of the form

$$p_j, p_j - i, p_j - 2i, p_j - 3i, \dots, \tag{5.31}$$

starting from the characteristic numbers $p_j$ of the pencil $D_0(p)$. In the absence of "resonances" (cases in which some characteristic number is contained in a sequence starting from another characteristic number), the multiplicities $m_l$ are the same for all elements of each sequence (5.31) and coincide with the multiplicity of the corresponding characteristic number $p_j$.

***The groups $\varkappa_\lambda$ and the conormal symbol.*** The following two observations are crucial for the subsequent exposition.

1) The standard operator $\widehat{D}_0$, which plays the role of the leading term in the construction of asymptotic solutions of the homogeneous equation $\widehat{D}u = 0$, can be obtained from $\widehat{D}$ by scaling followed by a passage to the limit. Indeed, the expression $ir\partial/\partial r$ is invariant with respect to the change of variables $r \mapsto \lambda r$, and hence this change of variables applied to $\widehat{D}$ gives the operator

$$\widehat{D}_\lambda = \sum_{k+|\sigma|\le m} a_{k\sigma}(\lambda^{-1}r,\omega)\left(ir\frac{\partial}{\partial r}\right)^k\left(-i\frac{\partial}{\partial\omega}\right)^\sigma \tag{5.32}$$

and, in the limit as $\lambda\to\infty$, the operator $\widehat{D}_0$. This limit procedure can also be written as

$$\widehat{D}_0 = \lim_{\lambda\to\infty}\varkappa_\lambda^{-1}\widehat{D}\varkappa_\lambda, \tag{5.33}$$

where

$$\varkappa_\lambda u(r,\omega) = u(\lambda r,\omega), \quad \lambda\in\mathbb{R}_+ \tag{5.34}$$

is the multiplicative group of dilations with respect to the variable $r$ and the limit is understood, say, in the strong sense (see 5.4.2 for details). It is obvious from (5.33) (and by straightforward verification) that *the standard operator $\widehat{D}_0$ commutes with $\varkappa_\lambda$*. Formula (5.33) is similar to Hörmander's formula for the principal symbol of a (pseudo)differential operator (see Chap. 4), and the operator $\widehat{D}_0$ can naturally be called a "secondary" symbol of $\widehat{D}$ corresponding to the singularity at $r = 0$. For historical reasons, the secondary symbol is defined in a slightly different way: one takes the operator family $D_0(p)$ obtained from $\widehat{D}_0$ by the substitution $ir\partial/\partial r \mapsto p$. Traditionally, this family is called the *conormal symbol* of $\widehat{D}$ and is denoted by $\sigma_c(\widehat{D})$:

$$\sigma_c(\widehat{D})(p) = \sum_{k+|\sigma|\le m} a_{k\sigma}(0,\omega)p^k \left(-i\frac{\partial}{\partial\omega}\right)^\sigma . \tag{5.35}$$

2) Consider the leading term

$$u_0(r,\omega) = r^{-\mu}(\ln r)^m \Phi(\omega) \tag{5.36}$$

of the asymptotic expansion (5.30) of the solution of the homogeneous equation. (We have omitted the subscripts for brevity: $\mu = \mu_0$, $m = m_0$, and $\Phi = \Phi_{0m_0}$.) Up to lower-order terms, the leading term is invariant under the representation

$$\lambda \longmapsto \lambda^{i\mu}\varkappa_\lambda \tag{5.37}$$

of the group $\mathbb{R}_+$:

$$\lambda^{i\mu}\varkappa_\lambda u_0 = u_0\left(1 + O\left(\frac{1}{\ln r}\right)\right). \tag{5.38}$$

***Weighted Sobolev spaces.*** So, in what spaces is it natural to study elliptic differential operators of Fuchsian type? The following considerations can be used to define such spaces.

1) Away from the degeneration point $r = 0$, such operators are elliptic in the usual sense, and hence the newly introduced spaces should coincide there with the ordinary Sobolev spaces.

2) In view of the above-mentioned invariance of the standard operator $\widehat{D}_0$ and asymptotic solutions, it is natural to consider spaces in which the norm possesses a similar invariance property:

$$\left\|\lambda^{-\gamma}\varkappa_\lambda u\right\| = \|u\|, \tag{5.39}$$

where $\gamma \in \mathbb{R}$ is a given number. One can readily see that condition (5.39), in conjunction with the requirement that on functions supported in the set $r \ge C > 0$ the norm should be equivalent to the ordinary Sobolev norm of given order $s$, *uniquely determines the new norm modulo equivalence* at least if we require (which we do indeed) that compactly supported functions on $\overset{\circ}{M}$ be dense in our space. Indeed, by applying the norm-preserving operator $\lambda^{-\gamma}\varkappa_\lambda$ with an appropriately chosen $\lambda$ to an arbitrary compactly supported function, we can move the support into the domain $r \ge 1$. In the following, we describe spaces with such norms, which are called weighted Sobolev spaces, in more explicit terms.

First, we introduce weighted Sobolev spaces on model cones. Let $K$ be a model cone with base $\Omega$. It is convenient to assume that $K$ is an *infinite* cone, i.e.,

$$K = \{\Omega \times [0,+\infty)\}/\{\Omega \times \{0\}\}.$$

Let $s$ and $\gamma$ be real numbers.

DEFINITION 5.21. The *weighted Sobolev space* $H^{s,\gamma}(K)$ is the completion of the space $C_0^\infty(K)$ of functions $u(r,\omega)$ with compact support $\operatorname{supp} u \subset \Omega \times (0,+\infty)$ with respect to the norm

$$||u||_{H^{s,\gamma}(K)} \equiv ||u||_{s,\gamma} = \left\{ \int_0^\infty \int_\Omega \left| \left(1 + \left(ir\frac{\partial}{\partial r}\right)^2 - \Delta_\Omega\right)^{s/2} (r^{-\gamma} u(r,\omega)) \right|^2 \frac{dr}{r} d\omega \right\}^{1/2}. \tag{5.40}$$

Here $d\omega$ is a smooth measure on $\Omega$ and $\Delta_\Omega$ is the Beltrami–Laplace operator on $\Omega$ (generated by some Riemannian metric), self-adjoint in $L^2(\Omega, d\omega)$.

Note that the operator

$$1 + \left(ir\frac{\partial}{\partial r}\right)^2 - \Delta_\Omega$$

with domain $C_0^\infty(\mathring{K})$ is strongly positive definite and symmetric in $L^2\left(K, dr\, d\omega/r\right)$. Hence it is essentially self-adjoint, and its closure is a strongly positive definite self-adjoint operator in $L^2\left(K, dr d\omega/r\right)$. Thus the arbitrary real power of this operator occurring in (5.40) is well defined.

EXERCISE 5.22. Check that the norm (5.40) satisfies the invariance condition (5.39).

The number $s$ is called the *order*, and $\gamma$ is called the *weight exponent* of the Sobolev space $H^{s,\gamma}(K)$. Let us give another expression for the norm (5.40). It can be obtained with the help of the *Mellin transform*

$$\widetilde{u}(p) \equiv [\mathcal{M}u](p) = \int_0^\infty r^{ip} u(r)\, \frac{dr}{r}. \tag{5.41}$$

(Formula (5.41) differs from the traditional definition of the Mellin transform in that the factor $i$ is lacking in the exponent.) The inverse transform has the form

$$[\mathcal{M}^{-1}\widetilde{u}](r) = \frac{1}{2\pi} \int_{\mathcal{L}_\gamma} r^{-ip} \widetilde{u}(p)\, dp, \tag{5.42}$$

where $\mathcal{L}_\gamma$ is the weight line

$$\mathcal{L}_\gamma = \{p \in \mathbb{C} \,|\, \operatorname{Im} p = \gamma\}.$$

(The integral (5.42) converges to $u(r)$ for every $\gamma$ provided that $u$ is a compactly supported function whose support does not contain the origin.) In terms of the Mellin transform, the norm (5.40) (up to equivalence) can be represented in the form

$$\|u\|_{s,\gamma} = \frac{1}{\sqrt{2\pi}} \left\{ \int_{\mathcal{L}_\gamma} \left\| (1 + |p|^2 - \Delta_\Omega)^{s/2} \widetilde{u}(p,\omega) \right\|^2_{L^2(\Omega, d\omega)} dp \right\}^{1/2},$$

where $\widetilde{u}(p,\omega)$ is the Mellin transform of the function $u(r,\omega)$ with respect to the variable $r$.

*Remark* 5.23. (1) Although the definition of weighted Sobolev spaces on the model cone deals with an *infinite cone*, we actually need only a neighborhood of the point $r = 0$ as we apply this construction to the definition of weighted Sobolev spaces on a manifold with conical singularities.

(2) It readily follows from Definition 5.21 that the operator of multiplication by $r^a$, $a \in \mathbb{R}$, is an isomorphism of the Sobolev spaces

$$r^a : H^{s,\gamma}(K) \to H^{s,\gamma+a}(K)$$

for any $s$ and $\gamma$.

Now let us construct weighted Sobolev spaces on manifolds with conical singularities. Let $M$ be a compact manifold with singular point $\alpha$. The base of the corresponding cone will be denoted by $\Omega$, and the cone itself by $K$. Consider an open cover

$$M = U \cup U_0$$

of $M$ such that

(i) the domain $U_0$ does not contain the singular point;

(ii) the domain $U$ is the standard coordinate neighborhood of the singular point.

Let $1 = e + e_0$ be a smooth partition of unity on $\overset{\circ}{M}$ subordinate to this cover. For functions $u \in C_0^\infty(\overset{\circ}{M})$, we introduce the Sobolev norm $\|u\|_s \equiv \|u\|_{H^s(\overset{\circ}{M})}$ in the standard way.

DEFINITION 5.24. (i) We define the norm $\|u\|_{s,\gamma}$ on $C_0^\infty(\overset{\circ}{M})$ by the formula

$$\|u\|_{s,\gamma} = \left\{ \|e_0 u\|^2_{H^s(\overset{\circ}{M})} + \|eu\|^2_{H^{s,\gamma}(K)} \right\}^{1/2}. \tag{5.43}$$

(ii) The *weighted Sobolev space* $H^{s,\gamma}(M)$ is the completion of $C_0^\infty(\overset{\circ}{M})$ with respect to the norm $\|\cdot\|_{s,\gamma}$.

The following properties of weighted spaces are obvious.

PROPOSITION 5.25. (i) *The norm* (5.43) *is independent of the ambiguity in the above-described construction up to an equivalence of norms.*

(ii) *For each given* $\gamma$, *the spaces* $H^{s,\gamma}(M)$, $s \in \mathbb{R}$, *form a scale of Hilbert spaces.*

(iii) *The scales* $\{H^{s,\gamma}(M)\}_{s\in\mathbb{R}}$ *with various* $\gamma$ *are isomorphic. Namely, the diagrams*

$$\begin{array}{ccc} H^{s,\gamma}(M) & \longrightarrow & H^{s',\gamma}(M) \\ \downarrow & & \downarrow \\ H^{s,\widetilde{\gamma}}(M) & \longrightarrow & H^{s',\widetilde{\gamma}}(M) \end{array} \tag{5.44}$$

*commute for any* $s > s'$, *where the horizontal arrows are the natural embeddings and the vertical arrows are the isomorphisms given by the operator of multiplication by a function* $f(x)$, $x \in M$, *with the following properties*:

(a) $f(x)$ *does not vanish in* $\overset{\circ}{M}$;

(b) $f(x) = r^{\widetilde{\gamma}-\gamma}$ *for* $x \in U$.

### 5.2.2. Cuspidal singularities

Similar constructions can be carried out for cuspidal singularities of order $k$. We only outline the scheme and omit the details.

Let $\widehat{D} \in \mathcal{D}_0(M)$ be a degenerate differential operator having the form

$$\widehat{D} = \sum_{k+|\sigma|\leq 2m} a_{k\sigma}(r,\omega)\left(ir^{k+1}\frac{\partial}{\partial r}\right)^k\left(-i\frac{\partial}{\partial\omega}\right)^\sigma \tag{5.45}$$

near the singular point.

***Asymptotics of solutions of the homogeneous equation.*** For an operator of the form (5.45), the terms in the asymptotics of solutions of the homogeneous equation as $r \to 0$ have the form

$$u \sim \exp\left\{\sum_{j=1}^{k}\frac{S_j}{r^j}\right\} r^\gamma \sum_j r^j a_j(\omega) \tag{5.46}$$

(see (Schulze, Sternin and Shatalov 1998)). Here the $S_j$ are some constants.

***The group $\varkappa_\lambda$ and the conormal symbol.*** In the construction of the asymptotics, just as in the conical case, the main role is played by the *standard operator*

$$\widehat{D}_0 = \sum_{k+|\sigma|\leq 2m} a_{k\sigma}(0,\omega)\left(ir^{k+1}\frac{\partial}{\partial r}\right)^k\left(-i\frac{\partial}{\partial\omega}\right)^\sigma \tag{5.47}$$

and the *conormal symbol*

$$\sigma_c(\widehat{D}) = \sum_{k+|\sigma|\leq 2m} a_{k\sigma}(0,\omega)p^k\left(-i\frac{\partial}{\partial\omega}\right)^\sigma. \tag{5.48}$$

The standard operator commutes with the local dilation group $\varkappa_\lambda$ obtained as the solution of the ordinary differential equation

$$\frac{dr}{dt} = r^{k+1}, \quad t = \ln\lambda. \tag{5.49}$$

EXERCISE 5.26. Write out the expression for the group $\varkappa_\lambda$.

EXERCISE 5.27. Show that terms in the asymptotic expansion (5.46) are asymptotically invariant under $\lambda^{i\mu}\varkappa_\lambda$ for appropriate $\mu$.

***Weighted Sobolev spaces.*** Using the same invariance argument as in the conical case, one introduces weighted Sobolev spaces on manifolds with cuspidal singularities.

First, consider the weighted Sobolev spaces on the model cusp. Let again

$$K = (\Omega \times [0,+\infty))/\{\Omega \times \{0\}\}$$

be an infinite cone with base $\Omega$, and let $s$ and $\gamma$ be real numbers.

DEFINITION 5.28. The *weighted Sobolev space* $H_k^{s,\gamma}(K)$ is the completion of $C_0^\infty(\overset{\circ}{K})$ with respect to the norm

$$\|u\|_{H_k^{s,\gamma}(K)} = \frac{1}{\sqrt{2\pi}}\left\{\int\limits_{\mathcal{L}_\gamma} \left\|(1+|p|^2+\Delta_\Omega)^{s/2}\widetilde{u}(p,\omega)\right\|_{L^2(\Omega,d\omega)} dp\right\}^{1/2}, \tag{5.50}$$

where

$$\widetilde{u}(p) \equiv [B_k u](p) = \int\limits_0^\infty e^{-\frac{ip}{kr^k}} u(r)\frac{dr}{r^{k+1}} \tag{5.51}$$

is the *$k$-Borel transform* of $u$ (e.g., see (Sternin and Shatalov 1996)).

Note that the inverse transform is given by the formula

$$[B_k^{-1}\widetilde{u}](r) = \frac{1}{2\pi}\int\limits_{\mathcal{L}_\gamma} e^{\frac{ip}{kr^k}}\widetilde{u}(p)\,dp. \tag{5.52}$$

Now the definition of the Sobolev spaces $H_k^{s,\gamma}(M)$ reproduces the above construction of the spaces $H^{s,\gamma}(M)$ word for word; the only difference is that one everywhere replaces $H^{s,\gamma}(K)$ by $H_k^{s,\gamma}(K)$. Proposition 5.25 remains valid for the space $H_k^{s,\gamma}(K)$ if item (iii)(b) is replaced by

$$f(x) = \exp\left\{\frac{\gamma-\widetilde{\gamma}}{kr^k}\right\} \quad \text{for} \quad x \in U.$$

In the sequel, the conical weighted Sobolev space $H^{s,\gamma}(M)$ will sometimes be denoted by $H_0^{s,\gamma}(M)$. In conclusion, we make the following remark.

*Remark* 5.29. The definition of the weighted Sobolev space $H_k^{s,\gamma}(M,E)$, where $E$ is a vector bundle over $M$, is a standard generalization of the above constructions. We leave details to the reader.

## 5.3. A Universal Representation of Degenerate Operators and the Finiteness Theorem

When describing elements of the rings $\mathcal{D}_k$ in the form (5.8) and (5.9), it may seem at first glance that the theories of elliptic differential equations with such operators have little in common: the operator have different form and even act in different Sobolev scales. However, this impression is erroneous, and there is a “unifying” transformation that reveals the relationship between these algebras. This transformation is described below.

### 5.3.1. The cylindrical representation

***The uniformizing change of variables.*** It was noticed in (Schulze, Sternin and Shatalov 1996) and (Schulze, Sternin and Shatalov 1998) that, by using an appropriate change of variables $r = \varphi(t)$ that takes the point $r = 0$ to infinity $t = \infty$, in the new variables one can describe the type of the singular point by specifying the *stabilization rate* as $t \to \infty$ of the coefficients of differential operators forming the structure ring. In particular, for a singular point of order $k$ the change of variables is given by the formulas[5]

$$\begin{cases} r = e^{-t} & \text{for } k = 0, \\ r = (1+kt)^{-1/k} & \text{for } k > 0 \end{cases} \tag{5.53}$$

and maps a deleted neighborhood of the singular point onto the infinite half-cylinder $\Omega \times (0, \infty)$. In other words, $\overset{\circ}{M}$ is equipped with the structure of a manifold with cylindrical end $\overset{\circ}{U} \simeq \Omega \times (0, \infty)$ (see Fig. 5.3, where a manifold with several singular points is shown).

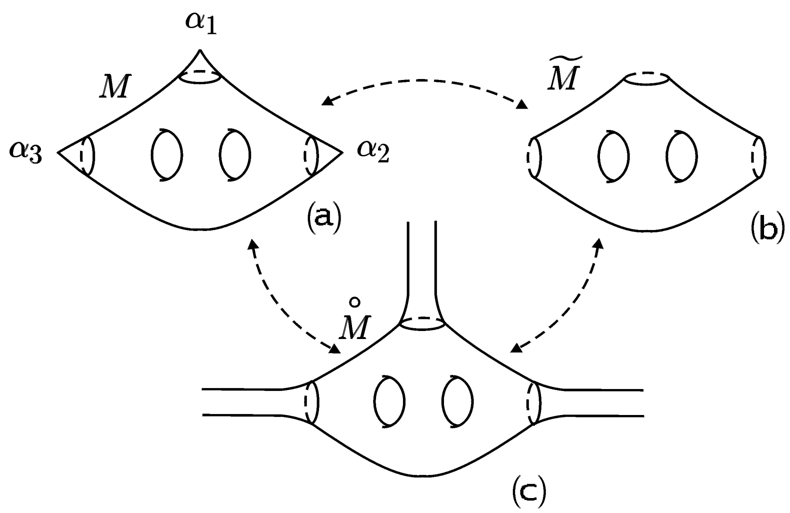


Figure 5.3. (a), a manifold with singularities; (b), its blow-up (a manifold with boundary); (c), a manifold with cylindrical ends

The change of variables (5.53) takes the main operator $ir^{k+1}\partial/\partial r$ to the operator $-i\partial/\partial t$, which is independent of $t$. Accordingly, the elements of the structure ring $\mathcal{D}_k$ on the half-cylinder acquire the form

$$\widehat{D} = D\left(t, -i\frac{\partial}{\partial t}\right),$$

where $D(t,p)$ is a polynomial of $p$ ranging in the ring of differential operators with smooth coefficients on $\Omega$; the dependence of $D(t,p)$ on $t$ as $t \to \infty$ is described by the formula

$$\begin{cases} D(t,p) = F(e^{-t}, p) & \text{for } k = 0, \\ D(t,p) = F(t^{-1/k}, p) & \text{for } k > 0, \end{cases}$$

where $F(r,p)$ smoothly depends on $r$ up to $r = 0$. Thus, the change of variables (5.53) kills the specific dependence of the form of the degenerate differential operator on the degree $k$ of degeneration (the operator argument $ir^{k+1}\partial/\partial r$): in the new variables, all operators in question are polynomials in

[5]The constants used in (5.53) ensure that the power-law change of variables continuously passes into the exponential change of variables as $k \to 0$. This is less important in this book, since we only deal with integer $k$. However, see (Schulze, Sternin and Shatalov 1996) and (Schulze, Sternin and Shatalov 1998).

$-i\partial/\partial t$, and the degeneration itself (vanishing of the coefficients of powers of $-i\partial/\partial t$) is no longer present. From this viewpoint, our change of variables can be treated as uniformization or resolution of singularities. The only thing that reminds of the degeneration is the stabilization of coefficients as $t \to \infty$; a conical point is characterized by an exponential stabilization, and a positive-order cusp, by a power-law stabilization.

In the new variables $(\omega, t)$ (which will be referred to as *cylindrical* variables in what follows), the theory of degenerate differential operators becomes unified. This pertains to the operators themselves as well as to related mathematical notions and object (in particular, to the spaces where the operators act). In the remaining part of this subsection, we clarify this assertion.

***The algebra of stabilizing operators.*** All structure rings $\mathcal{D}_k(M)$[6] are subalgebras of the algebra $\mathcal{D}(M)$ of *stabilizing* differential operators. The elements of this algebra are arbitrary differential operators with smooth coefficients in the interior of $\overset{\circ}{M}$, and in cylindrical coordinates on the cylindrical end $\Omega \times (0, \infty)$ they have the form

$$\widehat{D} = D\left(\omega, t, -i\frac{\partial}{\partial \omega}, -i\frac{\partial}{\partial t}\right), \tag{5.54}$$

where the symbol $D(\omega, t, q, p)$ (polynomial in $(q, p)$) tends as $t \to \infty$ to a limit together with all derivatives uniformly with respect to the parameters $(\omega, q, p)$ on compact sets. (No conditions are imposed on the convergence rate.) In what follows, we prove all assertions of elliptic theory for elements of $\mathcal{D}(M)$. Then they are automatically valid for cone- or cusp-degenerate operators, which belong to specific subalgebras $\mathcal{D}_k(M)$.

The ring $\mathcal{D}(M)$ is the widest ring of differential operators worth consideration (at least in elliptic theory). Hence it is natural to refer to the manifold $M$ equipped with the ring $\mathcal{D}(M)$ as a *general* manifold with isolated singularities. An arbitrary specific type of singularity is given by the choice of some subring of $\mathcal{D}(M)$ containing all differential operators with compactly supported smooth coefficients on $\overset{\circ}{M}$.

***The cotangent bundle and the principal symbol.*** Let $T^*M$ be the compressed cotangent bundle of the manifold $M$ equipped with the structure ring $\mathcal{D}_k(M)$. In cylindrical coordinates, regardless of the value of $k$, it is represented as the union of the ordinary tangent bundle $T^*\overset{\circ}{M}$ of the manifold with cylindrical ends and the "stratum at infinity" $T^*\Omega \times \mathbb{R}$, which represents the boundary $\partial T^*M$. Moreover, the adjacency of $T^*\overset{\circ}{M}$ to $\partial T^*M$ is described as follows: a sequence $(t_l, \omega_l, p_l, q_l) \in T^*\overset{\circ}{M}$ of points lying in the cylindrical end of $T^*\overset{\circ}{M}$ converges to a point $(\omega, q, p) \in T^*\Omega \times \mathbb{R}$ if and only if $t_l \to \infty$, $\omega_l \to \omega$, $p_l \to p$, and $q_l \to q$. Thus the continuous structure on this union is independent of $k$. However, the differentiable structure does depend on $k$ (but is fortunately irrelevant in elliptic theory). Hence we treat the cotangent bundle of $M$ in the cylindrical representation as the union

$$T^*M = \partial T^*M \cup T^*\overset{\circ}{M} \tag{5.55}$$

equipped with the above-mentioned continuous structure as well as the natural differentiable structure on each of the components. This definition does not involve $k$ and hence is suitable for the study of arbitrary stabilizing operators.

[6]As well as the structure rings corresponding to more general type of singularities described in (Schulze, Sternin and Shatalov 1998).

The symplectic form (5.17) (or (5.19)) defined on the interior of the compressed cotangent bundle is represented independently of $k$ by the standard symplectic form on $T^*\overset{\circ}{M}$. Hence the notion of principal symbol adjusted to degenerate operators (see (5.15)) passes in the cylindrical coordinates into the standard notion of principal symbol: if a differential operator $\widehat{D} \in \mathcal{D}(M)$ of order $\leq m$ has the form

$$\widehat{D} = \sum_{|\alpha|+j\leq m} a_{\alpha j}(\omega, t)(-i)^{|\alpha|+j} \frac{\partial^\alpha}{\partial\omega^\alpha}\frac{\partial^j}{\partial t^j}$$

in local cylindrical coordinates, then its principal symbol (of order $m$) is

$$\sigma(\widehat{D}) = \sum_{|\alpha|+j=m} a_{\alpha j}(\omega, t) q^\alpha p^j. \tag{5.56}$$

***The group $\varkappa_\lambda$ and the conormal symbol.*** In cylindrical coordinates, the groups $\varkappa_\lambda$ corresponding to arbitrary (cone or cusp) types of symbol act in the same way; namely, they are represented by shifts in the variable $t$:

$$\varkappa_\lambda u(t, \omega) = u(t + \ln\lambda, \omega).$$

From now on, we shall use the additive notation for the group, taking $\ln\lambda$ as the group parameter (and denoting it again by $\lambda$). Thus now we have $\lambda \in \mathbb{R}$ and the group actions is given by

$$\varkappa_\lambda u(t, \omega) = u(t + \lambda, \omega). \tag{5.57}$$

Let $\widehat{D} \in \mathcal{D}_k(M)$. We can readily compute its conormal symbol (5.35) (for $k = 0$) or (5.48) (for $k > 0$) from the expression (5.54) of $\widehat{D}$ in cylindrical coordinates. Indeed

$$\sigma_c(\widehat{D}) = D\left(\omega, \infty, -i\frac{\partial}{\partial\omega}, p\right), \tag{5.58}$$

where the substitution $t = \infty$ is treated as the limit as $t \to \infty$ of the corresponding expression. The expression (5.58) is independent of the specific value of $k$ and is well defined not only for cone- and cusp-degenerate operators but also for an arbitrary element $\widehat{D} \in \mathcal{D}(M)$ of the algebra of stabilizing operators. In this general case, it will also be called the conormal symbol of the operator $\widehat{D}$. The corresponding standard operator

$$\widehat{D}_0 = D\left(\omega, \infty, -i\frac{\partial}{\partial\omega}, -i\frac{\partial}{\partial t}\right) \tag{5.59}$$

(cf. (5.28) and (5.47)) still can be obtained as the limit

$$\widehat{D}_0 = \lim_{\lambda\to\infty} \varkappa_{-\lambda}\widehat{D}\varkappa_\lambda \tag{5.60}$$

and commutes with $\varkappa_\lambda$:

$$\varkappa_\lambda\widehat{D}_0 = \widehat{D}_0\varkappa_\lambda.$$

Note that the expressions (5.56) and (5.58) imply the following *compatibility condition* for the principal and conormal symbols of a stabilizing operator.

PROPOSITION 5.30. *Let $\widehat{D} \in \mathcal{D}(M)$. Then*

$$\sigma(\sigma_c(\widehat{D})) = \sigma(\widehat{D})\big|_{\partial T^*M},$$

*where the principal symbol on the left-hand side is treated as the principal symbol of an operator with parameter $p$ in the sense of Agranovich–Vishik* (Agranovich and Vishik 1964).

***Weighted Sobolev spaces.*** The change of variables (5.53) takes conical (resp., "cuspidal") weighted Sobolev spaces $H_k^{s,\gamma}(M)$ on a manifold $M$ with conical (respectively, "cuspidal") singularity *to the same* (i.e., independent of $k$) weighted Sobolev spaces $H^{s,\gamma}(\overset{\circ}{M})$ on the manifold with cylindrical ends. In the following, without risk of misunderstanding, we denote these spaces by $H^{s,\gamma}(M)$ (omitting the circle over $M$). The space $H^{s,\gamma}(M)$ is obtained by the standard construction (gluing the norm with the use of a partition of unity) from the standard Sobolev space $H^s_{loc}(\overset{\circ}{M})$ in the interior of $M$ and the space $H^{s,\gamma}(C)$ on the infinite cylinder[7] $C = \Omega \times \mathbb{R}$ with base $\Omega$. The latter space consists of functions $u(\omega, t)$, $\omega \in \Omega$, $t \in \mathbb{R}$, with finite norm

$$\|u\|_{s,\gamma} = \left\{ \int\limits_{-\infty}^{\infty} \int\limits_{\Omega} \left| \left(1 + \Delta_\Omega - \frac{\partial^2}{\partial t^2}\right)^{s/2} (e^{-\gamma t} u(\omega, t)) \right|^2 d\omega\, dt \right\}^{1/2}. \tag{5.61}$$

## 5.3.2. Continuity and compactness

In view of the results of the preceding section, it is convenient to study degenerate elliptic differential operators in the cylindrical representation and consider arbitrary stabilizing operators.

First, let us study the continuity and compactness of such operators in weighted Sobolev spaces. This is a key question in the study of the Fredholm property of elliptic operators.

THEOREM 5.31. *Let $M$ be a manifold with isolated singularities. An arbitrary stabilizing differential operator $\widehat{D} \in \mathcal{D}(M)$ of order $\leq m$ is continuous in the spaces*

$$\widehat{D} : H^{s,\gamma}(M) \to H^{s-m,\gamma}(M)$$

*for any $s, \gamma \in \mathbb{R}$.*

◀ The proof is easy and follows from two facts, which can be verified in a straightforward manner:

- the operator of multiplication by a function bounded together with all derivatives is continuous in the norm (5.61);
- the operators
$$\frac{\partial}{\partial t}, \frac{\partial}{\partial \omega} : H^{s,\gamma}(C) \longrightarrow H^{s-1,\gamma}(C)$$
are continuous.

We leave details to the reader. ▶

Thus $\widehat{D}$ is a continuous operator of order $\leq m$ in the scale $\{H^{s,\gamma}(M)\}_{s\in\mathbb{R}}$ for every $\gamma$. For the ordinary Sobolev spaces $H^s(M)$ on a smooth compact manifold, it is also true that an operator of order $\leq m$ is compact as an operator from $H^s(M)$ into $H^{s-m}(M)$ if its principal symbol (of order $m$) is zero. This is related to the compactness of the embeddings $H^s(M) \subset H^k(M)$ for $s > k$. For weighted Sobolev spaces on manifolds with singularities, the corresponding embeddings are not compact, and nor is the operator

$$\widehat{D} :\ H^{s,\gamma}(M) \to H^{s-m,\gamma}(M)$$

with zero principal symbol compact in the general case. However, there is a necessary and sufficient condition for the compactness of this operator.

[7] Needless to say, the cylinder part with negative $t$ has been added only for convenience, so as not to deal with a boundary at $t = 0$.

THEOREM 5.32. *Let $M$ be a manifold with isolated singularities, and let $\widehat{D} \in \mathcal{D}(M)$ be a differential operator of order $m$ on $M$. The operator $\widehat{D}$*

$$\widehat{D} : H^{s,\gamma}(M) \to H^{s-m,\gamma}(M)$$

*is compact if and only if the following two conditions hold*:

(i) $\sigma(\widehat{D}) = 0$;

(ii) $\sigma_c(\widehat{D}) \equiv 0$.

This theorem is a particular case of a more general theorem stating the same thing for pseudodifferential operators (see further in this chapter).

### 5.3.3. Ellipticity and the finiteness theorem

Here we state one of the main results of this section, namely, the finiteness theorem for elliptic operators on manifolds with isolated singularities. The proof is based on the calculus of pseudodifferential operators on manifolds with isolated singularities, which will be described in the next section.

***Ellipticity.*** Let $\widehat{D} \in \mathcal{D}(M)$ be a stabilizing differential operator of order $m$ on a manifold $M$ with isolated singularities.

DEFINITION 5.33. The operator $\widehat{D}$ is said to be *elliptic with respect to a weight* $\gamma \in \mathbb{R}$ if the following conditions hold:

1) (interior ellipticity): the principal symbol (of order $m$) $\sigma(\widehat{D})$ of the operator $\widehat{D}$ is invertible on $T^*M$ outside the zero section;

2) (conormal ellipticity) the conormal symbol

$$\sigma_c(\widehat{D})(p) : H^s(\Omega) \longrightarrow H^{s-m}(\Omega)$$

is invertible for all $p \in \mathcal{L}_\gamma$, where $\mathcal{L}_\gamma$ is the *weight line*

$$\mathcal{L}_\gamma = \{p \in \mathbb{C} \mid \operatorname{Im} p = \gamma\}.$$

*Remark* 5.34. If the operator $\widehat{D}$ is interior elliptic, then, by the compatibility condition in Proposition 5.30, its conormal symbol $\sigma_c(\widehat{D})$ is a family of differential operators of order $m$ on $\Omega$ elliptic with parameter $p$ in the sense of Agranovich–Vishik in a sector of nonzero angle containing the real axis. By the Gokhberg–Krein theory (e.g., see (Egorov and Schulze 1997)), this family is invertible for all $p$ except for at most countably many points without finite accumulation points, whence it follows that the conormal ellipticity condition holds for all but countably many values of the parameter $\gamma$. Thus under the interior ellipticity condition one can always find a $\gamma$ for which the operator is elliptic.

***The finiteness theorem.*** Now we can state the main assertion.

THEOREM 5.35. *If an operator $\widehat{D} \in \mathcal{D}(M)$ of order $m$ is elliptic with respect to a weight $\gamma \in \mathbb{R}$, then it is Fredholm in the spaces*

$$\widehat{D} : H^{s,\gamma}(M) \longrightarrow H^{s-m,\gamma}(M) \tag{5.62}$$

*for each $s \in \mathbb{R}$. Moreover, the kernel and cokernel of the operator* (5.62) *are independent of $s$.*

The proof involves the construction of a two-sided almost inverse. This almost inverse proves to be a pseudodifferential operator, and hence in the following section we develop the theory of pseudodifferential operators on manifolds with isolated singularities and prove the finiteness theorem directly for elliptic $\Psi$DO, which naturally includes this theorem as a special case.

***Instantaneous stabilization.*** We have described an algebra of differential operators containing all specific algebras corresponding to various types of singularities. An important corollary is the fact that when studying the index problem for operators on manifolds with singularities, one can restrict oneself (from the topological viewpoint) to operators of rather special form. Indeed, if $\widehat{D} \in \mathcal{D}(M)$ is a stabilizing operator, then, using a small deformation that does not affect the conormal symbol, we can always reduce it to an operator whose principal symbol, as well as complete local symbol, is independent of the cylindrical variable $t$ for large $t \gg 0$. The study of such operators is much more convenient than that of cone- or cusp-degenerate operators in their original representation.

## 5.4. Calculus of $\Psi$DO

In this section, we develop the calculus of pseudodifferential operators on manifolds with isolated singularities. Here our interest is twofold. On the one hand, the $\Psi$DO calculus provides one of the possible ways to prove the finiteness theorem stated above. On the other hand, the study of the index problem in the subsequent chapters is based on homotopies: one deforms the symbols to some “simplest” symbols possessing the desired properties. Even if the starting point and the end point of a homotopy are differential operators, the desired homotopy cannot usually be done in the class of symbols of differential operators (just the same situation occurs in the classical index theory on smooth manifolds), and to conclude that the indices of elliptic differential operators at the beginning and the end of the homotopy coincide, one should be able to lift a homotopy of symbols to a homotopy of operators. Here we also see that the study of pseudodifferential operators is necessary. However, we note that if the reader is solely interested in differential operators and/or topological aspects of the index problem, then he or she can skip this section and will still be able to understand the corresponding results completely.

### 5.4.1. General $\Psi$DO

***Notation.*** By $M$ we denote a compact manifold with isolated singular point $\alpha$. The base of the cone is a smooth compact manifold $\Omega$ without boundary. Next, by $U$ we denote a given coordinate neighborhood of the singular point, in which we shall use the cylindrical representation ($U \simeq \mathbb{R}_+ \times \Omega$) and which will be identified with the corresponding subset in the two-way infinite cylinder $C = \mathbb{R} \times \Omega$. Finally, on $M$ we choose and fix a Riemannian metric $d\rho^2$ that is smooth outside the singular point and has the product form

$$d\rho^2 = dt^2 + d\omega^2, \quad \text{where } d\omega^2 \text{ is a smooth Riemannian metric on } \Omega, \tag{5.63}$$

in $U$. The distance between points $x, y \in \overset{\circ}{M}$ in this metric will be denoted by $\rho(x,y)$. The metric $d\rho^2$ generates a Riemannian volume form $d\,\mathrm{vol}$, which can be represented in $U$ in the form

$$d\,\mathrm{vol} = |dt \wedge d\,\mathrm{vol}_\Omega| \tag{5.64}$$

and which will be used to identify distributions with generalized functions. This permits us to avoid the use of half-densities in the description of spaces where $\Psi$DO act and of their integral kernels.

We shall often use cutoff functions, partitions of unity, etc. These functions will be chosen from the class of admissible smooth functions on $M$, described in the following definition.

DEFINITION 5.36. An *admissible smooth function* on $M$ is an infinitely smooth function on $\overset{\circ}{M} = M \setminus \{\alpha\}$ independent of the variable $t$ in a neighborhood of the singular point. An *$R$-function* is an admissible smooth function constant in a neighborhood of the singular point.

Our pseudodifferential operators will act in weighted Sobolev spaces on the manifold $M$ (or on the cylinder $C$). Without loss of generality, we consider only the case of a *zero weight exponent* $\gamma$. Indeed, the case $\gamma \neq 0$ can be reduced to the case $\gamma = 0$ with the use of the isomorphisms

$$e^{-\gamma t} : H^{s,\gamma}(C) \longrightarrow H^{s,0}(C)$$

of Sobolev space on the infinite cylinder and the corresponding isomorphisms of weighted Sobolev spaces on $M$. To simplify the notation, we omit the zero weight in the notation of the spaces and simply write $H^s(C)$ or $H^s(M)$.

***Symbols.*** Let us introduce symbol spaces on $T^*M$. The local model is given by symbol in $\mathbb{R}^{2n}$, and we start from the corresponding definition.

DEFINITION 5.37. By $S^m \equiv S^m(\mathbb{R}^{2n})$ we denote the space of smooth functions $H(x,\xi)$, $x, \xi \in \mathbb{R}^n$, satisfying the estimates

$$\left| \frac{\partial^{\alpha+\beta} H}{\partial x^\alpha \xi^\beta}(x,\xi) \right| \le C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}$$

for all multiindices $\alpha$ and $\beta$. The elements of $S^m(\mathbb{R}^{2n})$ will be called *symbols of order* $m$. Moreover, by

$$S^\infty(\mathbb{R}^{2n}) = \bigcup_m S^m(\mathbb{R}^{2n}), \quad S^{-\infty}(\mathbb{R}^{2n}) = \bigcap_m S^m(\mathbb{R}^{2n})$$

we denote the union and intersection, respectively, of symbol spaces of all possible orders.

Now we are in a position to define symbols on a manifold.

DEFINITION 5.38. By $S^m(T^*\overset{\circ}{M})$ we denote the space of smooth functions on the interior $T^*\overset{\circ}{M} \equiv T^*M \setminus \partial T^*M$ of the (compressed) cotangent bundle $T^*M$ such that in the coordinates on $T^*M$ corresponding to an arbitrary interior chart on $\overset{\circ}{M}$ or to a coordinate neighborhood of the form[8] $W \times (0,\infty) \subset U$, $W \subset \Omega$, they are depicted by elements of $S^m(\mathbb{R}^{2n})$. Next, we set

$$S^\infty(T^*\overset{\circ}{M}) = \bigcup_m S^m(T^*\overset{\circ}{M}), \quad S^{-\infty}(T^*\overset{\circ}{M}) = \bigcap_m S^m(T^*\overset{\circ}{M}). \tag{5.65}$$

[8]Such coordinate systems and coordinate neighborhoods are said to be *admissible*.

By $S^m(T^*M) \subset S^m(T^*\overset{\circ}{M})$ we denote the subspace of symbols $H(x,\xi)$ stabilizing as $t \to \infty$ in each coordinate system of the form $W \times (0,\infty)$, $W \subset \Omega$, in the following sense: the limit relation[9]

$$\lim_{t,\tau\to\infty}(H(t,y,\xi) - H(\tau,y,\xi))(1+|\xi|)^{-m} = 0 \tag{5.66}$$

is valid as $t,\tau \to \infty$ uniformly with respect to $(y,\xi) \in \mathbb{R}^{2n-1}$. (In other words, for each $\varepsilon > 0$ there exists a $T$ such that

$$|H(t,y,\xi) - H(\tau,y,\xi)|(1+|\xi|)^{-m} < \varepsilon \tag{5.67}$$

for $t,\tau > T$.) The spaces $S^\infty(T^*\overset{\circ}{M})$ and $S^{-\infty}(T^*\overset{\circ}{M})$ are defined by analogy with (5.65). We shall use the subscript $cl$ to indicate the corresponding subspaces of classical symbols (i.e., symbols possessing asymptotic expansions in homogeneous function of $\xi$ of decreasing orders as $\xi \to \infty$) in all these spaces.

***Definition of $\Psi$DO.***

DEFINITION 5.39. A continuous linear operator

$$A : H^s(M) \longrightarrow H^{s-m}(M), \quad s \in \mathbb{R} \tag{5.68}$$

of order $\leq m$ on a manifold $M$ with singularities is called a *pseudodifferential operator* (of order $\leq m$) if the following conditions are satisfied:

1) The kernel $K(x,y)$ of $A$ is infinitely smooth outside the diagonal $x = y$ on $\overset{\circ}{M} \times \overset{\circ}{M}$ and satisfies for large $\rho(x,y)$ the estimates

$$|K^{(\alpha\beta)}(x,y)| \leq \operatorname{const}\rho(x,y)^{-l}, \quad \rho(x,y) \to \infty, \tag{5.69}$$

where $\alpha,\beta$, and $l$ are arbitrary and the derivatives with respect to $x$ and $y$ are taken in admissible coordinate systems on $M$.

2) If $V$ is an interior coordinate system on $M$ or a set of the form $W \times (0,\infty) \subset U$, $W \subset \Omega$, on the cylindrical end and $\varphi,\psi$ are smooth functions admissible in the sense of Definition 5.36 and supported in $V$, then the operator $\varphi \circ A \circ \psi$ is a pseudodifferential operator of order $\leq m$ in $\mathbb{R}^n$ whose symbol belongs to $S^m(\mathbb{R}^{2n})$.

The operator $A$ is called a *classical* pseudodifferential operator if all pseudodifferential operators mentioned in item 2) are classical pseudodifferential operators.

THEOREM 5.40. *Pseudodifferential operators of all possible orders on $M$ form an algebra, which will be denoted by $PSD(M)$. A pseudodifferential operator $A$ of order $\leq m$ has a well-defined principal symbol*[10] $\sigma(A) \in S^m(T^*\overset{\circ}{M})/S^{m-1}(T^*\overset{\circ}{M})$. *Further, for each element $\sigma \in S^m(T^*\overset{\circ}{M})/S^{m-1}(T^*\overset{\circ}{M})$ there exists a pseudodifferential operator of order $\leq m$ with principal symbol $\sigma(A) = \sigma$. The product of operators corresponds to the product of their principal symbols. Classical pseudodifferential operators form a subalgebra $PSD_{cl}(M)$.*

[9]Apparently, one should impose similar conditions on the derivatives of $H$. However, they hold automatically by virtue of the following theorem: *If a sequence bounded in $C^\infty$ converges in $C^0$, then it also converges in $C^\infty$.*

[10]More precisely, a principal symbol of order $m$.

◀ Using a partition of unity, one can readily reduce the assertion of the theorem to the case of pseudodifferential operators in coordinate neighborhoods of the form indicated in item 2 of Definition 5.39, and for such pseudodifferential operators the desired assertion follows from the results of Chap. 4 about pseudodifferential operators in $\mathbb{R}^N$. ▶

*Remark* 5.41. Needless to say, one has the relation

$$S^m_{cl}(T^*\overset{\circ}{M})/S^{m-1}_{cl}(T^*\overset{\circ}{M}) \simeq \mathcal{O}^m(T^*\overset{\circ}{M}),$$

whose right-hand side is defined as the space of functions homogeneous of degree $m$ along the fibers and satisfying the estimates occurring in the definition of the space $S^n(T^*\overset{\circ}{M})$ outside a neighborhood of the zero section. Thus the principal symbol of a classical pseudodifferential operator can be treated as a homogeneous function on $T^*\overset{\circ}{M}$.

### 5.4.2. The subalgebra of stabilizing $\Psi$DO

The algebra of pseudodifferential operators on $M$ introduced in the preceding subsection is too wide for elliptic theory. Although the product of operators in this theory corresponds to the product of their principal symbols, an operator with zero principal symbol is not necessarily compact in the corresponding pairs of Sobolev spaces. Hence the calculus of principal symbols alone does not permit one to construct almost inverses and prove the finiteness theorem. To be able to do this effectively, we consider the narrower algebra of stabilizing pseudodifferential operators.

***Stabilizing operators on the infinite cylinder.*** First, we introduce the notion of a stabilizing operator on the cylinder $C$ as $t \to \infty$.

Consider a function $\psi \in C^\infty(\mathbb{R})$ with the following properties: $\psi(t) > 0$ for all $t$, $\psi(t) = 1$ for $t > 1$, and $\psi(t) = -t$ for $t < -1$. Along with the spaces $H^s(C)$, we introduce the auxiliary spaces

$$\widetilde{H}^s(C) = \{u \,|\, \psi(t)u \in H^s(C)\}.$$

In the spaces $H^s(C)$ and $\widetilde{H}^s(C)$, we consider the familiar one-parameter translation group $\varkappa_\lambda$ given by the formula

$$[\varkappa_\lambda u](t) = u(t-\lambda).$$

This group is unitary in any of the spaces $H^s(C)$ (but not in $\widetilde{H}^s(C)$).

DEFINITION 5.42. A continuous operator

$$A:\ H^s(C) \to H^l(C), \quad s, l \in \mathbb{R},$$

is said to be *stabilizing* if the operator family

$$A(\lambda) \equiv \varkappa_{-\lambda} A \varkappa_\lambda :\ \widetilde{H}^s(C) \to H^l(C) \tag{5.70}$$

converges in the uniform operator topology as $\lambda \to \infty$ and the limit operator $A(\infty)$ is continuous in the spaces

$$A(\infty)\colon \widetilde{H}^s(C) \to \widetilde{H}^l(C).$$

Note that the family (5.70) is obtained as the composition of the continuous dense embedding $\widetilde{H}^s(C) \subset H^s(C)$ with the corresponding family (denoted by the same letter)

$$A(\lambda)\colon H^s(C) \to H^l(C), \tag{5.71}$$

which is uniformly bounded as $\lambda \to +\infty$, since the operators $\varkappa_\lambda$ are unitary in the Sobolev spaces on $C$. Since $C_0^\infty(C)$ is dense in both $H^s(C)$ and $\widetilde{H}^s(C)$, we see that the uniformly bounded family (5.71) converges on a dense set, and consequently, the operator $A(\infty)$ can be computed by the uniform boundedness principle as the strong limit

$$A(\infty) = \operatorname*{s-lim}_{\lambda\to\infty} \varkappa_{-\lambda} A \varkappa_\lambda$$

of the family (5.71) and is bounded in the spaces

$$A(\infty)\colon H^s(C) \to H^l(C).$$

DEFINITION 5.43. A continuous operator $A$ of order $\le m$ in the Sobolev scale $\{H^s(C)\}_{s\in\mathbb{R}}$ is said to be *stabilizing* (at $+\infty$) *of stabilization order* $\le m$ if it stabilizes as an operator

$$A:\ H^s(C) \to H^{s-m}(C)$$

for each $s \in \mathbb{R}$.

*Remark* 5.44. Definition 5.43 is consistent: a stabilizing operator of order $\le m$ is also a stabilizing operator of order $\le m'$ for all $m' > m$. This follows from the embeddings $H^s(C) \subset H^{s'}(C)$ and $\widetilde{H}^s(C) \subset \widetilde{H}^{s'}(C)$ valid for $s \ge s'$. Note, however, that the exact order of an operator need not be equal to the exact stabilization order but can be strictly less than the latter.

THEOREM 5.45. *Stabilizing operators in the scale $H^s(C)$ in the sense of Definition* 5.43 *form a filtered algebra.* (*The filtration is determined by the stabilization order.*)

*Proof.* Obviously, the sum of stabilizing operators stabilizes. The same property of the product is not much harder to prove. Let $A$ and $B$ be stabilizing operators. For simplicity, we assume that their stabilization orders are zero. Let us show that the difference

$$A(\lambda)B(\lambda) - A(\infty)B(\infty):\ \widetilde{H}^s(C) \to H^s(C) \tag{5.72}$$

tends to zero. Indeed, we rewrite (5.72) in the form

$$A(\lambda)B(\lambda) - A(\infty)B(\infty) = A(\lambda)(B(\lambda) - B(\infty)) + (A(\lambda) - A(\infty))B(\infty) \tag{5.73}$$

and separately study the two summands on the right-hand side. The first summand can be treated as the product of the operators

$$A(\lambda):\ H^s(C) \to H^s(C) \tag{5.74}$$

and

$$B(\lambda) - B(\infty):\ \widetilde{H}^s(C) \to H^s(C). \tag{5.75}$$

We have already noted that the operator $A(\lambda)$ is bounded uniformly with respect to $\lambda$ and $B(\lambda) - B(\infty)$ tends to zero in the operator norm in the indicated pair of spaces by the definition of a stabilizing operator.

The second term on the right-hand side in (5.73) can be represented as the product of the operators

$$A(\lambda) - A(\infty) : \widetilde{H}^s(C) \to H^s(C) \tag{5.76}$$

and

$$B(\infty) : \widetilde{H}^s(C) \to \widetilde{H}^s(C). \tag{5.77}$$

Indeed, this is the case, since, by the second condition in the definition of a stabilizing operator there is a commutative diagram

$$\begin{array}{ccc} \widetilde{H}^s(C) & \xrightarrow{B(\infty)} & \widetilde{H}^s(C) \\ \downarrow & & \downarrow \\ H^s(C) & \xrightarrow[B(\infty)]{} & H^s(C), \end{array}$$

where the vertical arrows are the natural embeddings. Here the operator (5.76) is bounded, and the operator (5.77) tends to zero. It remains to note that the operator

$$A(\infty)B(\infty) : \widetilde{H}^s(C) \longrightarrow \widetilde{H}^s(C)$$

is continuous as a product of continuous operators. □

The following theorem is the main theorem of the theory of stabilizing operators. It provides a criterion for the compactness of such operators, which will be further specialized to the case of pseudodifferential operators in terms of the principal and conormal symbols.

THEOREM 5.46 ((Nazaikinskii 2000), (Nazaikinskii and Sternin 2001).). (1) *If an operator*

$$A : H^s(C) \to H^l(C)$$

*is compact, then it stabilizes and satisfies the condition* $A(\infty) = 0$.

(2) *Conversely, if* $A$ *stabilizes,* $A(\infty) = 0$*, and the operator* $A \circ \varphi(t)$ *is compact for any function* $\varphi(t)$ *bounded together with all derivatives and vanishing for sufficiently large positive* $t$*, then the operator* $A$ *is compact.*

*Proof.* (1). If $A$ is compact, then it is the uniform limit of a sequence of finite rank operators $A_n$. Hence it suffices to prove (1) for finite rank operators, or, by linearity, for rank 1 operators. Such an operator has the form

$$Au = v(w,u), \quad u \in H^s(C),$$

where $v \in H^l(C)$, $w \in H^{-s}(C)$, and $(\cdot, \cdot)$ is the standard $L^2$-pairing of $H^s(C)$ with $H^{-s}(C)$. The operator of multiplication by $\psi(t)^{-1}$ is an isomorphism of $H^s(C)$ onto $\widetilde{H}^s(C)$, and hence it suffices to show that the operator

$$A(\lambda)\psi(t)^{-1} : H^s(C) \to H^l(C)$$

tends to zero in the operator norm as $\lambda \to +\infty$. Note, however, that

$$A(\lambda)\psi(t)^{-1}u = (\varkappa_{-\lambda}v)(\psi(t-\lambda)^{-1}w, \varkappa_\lambda u),$$

and the desired assertion follows from the fact that $\psi(t-\lambda)^{-1}w$ strongly converges to zero as $\lambda \to +\infty$. Thus we have proved (1).

(2) Let $\varphi(t)$ be an infinitely smooth function vanishing for large positive $t$ and equal to unity for large negative $t$. By assumption, the operator $A \circ \varphi(t-\lambda)$ is compact. Let us show that $A \circ \varphi(t-\lambda) \to A$ as $t \to \infty$ in the operator norm. This readily implies the desired assertion. It suffices to show that $A \circ (\varphi(t-\lambda) - 1) \to 0$. We have

$$A \circ (\varphi(t-\lambda) - 1) = \varkappa_\lambda A(\lambda)(\varphi(t) - 1)\varkappa_{-\lambda},$$

and since the operators $\varkappa_\lambda$ are unitary, it suffices to show that $A(\lambda)(\varphi(t)-1) \to 0$ in the operator norm in Sobolev spaces. Since $\varphi(t) - 1 = 0$ for large negative $t$, it follows that the operator of multiplication by $\varphi$ is continuous from $H^s(C)$ into $\widetilde{H}^s(C)$. Combining this with the definition of a stabilizing operator, we find that $A(\lambda)(\varphi(t)-1) \to A(\infty)(\varphi(t)-1)$ in the operator norm in Sobolev spaces. It remains to recall that $A(\infty) = 0$. □

***The conormal symbol of a stabilizing operator.*** If $A$ is a stabilizing operator of some order $m$ in the Sobolev spaces $H^s(C)$ on the infinite cylinder $C$, then the "limit" operator $A_\infty$ is translation-invariant, i.e., commutes with the translation group:

$$\varkappa_\lambda A_\infty \varkappa_{-\lambda} = A_\infty.$$

It follows that the composition

$$\mathcal{F}_{t\to p} \circ A_\infty \circ \mathcal{F}^{-1}_{p\to t}$$

of this operator with the direct and inverse Fourier transforms with respect to the variable $t$ is an "operator-valued multiplier", i.e., a generalized function of $p$ with values in the space of linear operators in the Sobolev scale on the base $\Omega$ of the cylinder.

DEFINITION 5.47. The generalized operator function

$$\sigma_c(A)(p) = \mathcal{F}_{t\to p} \circ A_\infty \circ \mathcal{F}^{-1}_{p\to t} \tag{5.78}$$

is called the *conormal symbol* of the stabilizing operator $A$.

The study of properties of this function for an arbitrary stabilizing operator $A$ is apparently a difficult task. However, we shall see that the situation is simplified dramatically if $A$ is a stabilizing pseudodifferential operator.

***Stabilizing $\Psi$DO on a manifold.***

DEFINITION 5.48. A pseudodifferential operator $A$ on $M$ is said to be *stabilizing* if so is each operator of the form $\varphi A \psi$ on the infinite cylinder, where $\varphi$ and $\psi$ are $R$-functions on $M$ supported in $U$ and equal to unity near the singular point. The *conormal symbol* of $A$ is defined in this case by the formula

$$\sigma_c(A) \stackrel{\text{def}}{=} \sigma_c(\varphi A \psi). \tag{5.79}$$

LEMMA 5.49. *Definition* 5.48 *is consistent*: *the stabilization condition and definition* (5.79) *of the conormal symbol are independent of the choice of $\varphi$ and $\psi$.*

◀ The validity of the lemma is a consequence of the following simple assertion. If $\chi$ is a smooth compactly supported function on $C$ and $B$ is an arbitrary pseudodifferential operator of order $\leq m$ on $C$, then the products $\chi B$ and $B\chi$ stabilize (with the same order) to zero. Indeed, the difference $\chi B - B\chi$ is compact and hence stabilizes to zero by Theorem 5.46, so that it suffices to prove that $\sigma_c(B\chi) = 0$. In notation (5.70), we have

$$\varkappa_{-\lambda} B\chi\varkappa_\lambda = B(\lambda)\chi(t-\lambda),$$

where the operator can be assumed to act in the spaces

$$\chi(t-\lambda) : \widetilde{H}^s(C) \longrightarrow H^s(C), \quad B(\lambda) : H^s(C) \longrightarrow H^{s-m}(C).$$

The first operator tends to zero in the norm as $\lambda \to \infty$, and the second is uniformly bounded, since the group $\varkappa_\lambda$ is unitary in the Sobolev spaces. Thus we arrive at the desired assertion. ▶

The following theorem summarizes the main properties of stabilizing pseudodifferential operators on manifolds with singularities. To simplify the presentation, from now on we restrict ourselves to the case of classical pseudodifferential operators; the adjective “classical” and the subscript $cl$ will be omitted for brevity. We point out that the assertions given below remain valid for nonclassical symbols, but the statements (especially those concerning the compatibility condition for the principal and conormal symbols) become somewhat more complicated.

THEOREM 5.50. 1) *Let $A$ be a stabilizing pseudodifferential operator on a manifold $M$ with isolated singularities. Then its principal symbol $\sigma = \sigma(A) \in \mathcal{O}^m(T^*\overset{\circ}{M})$ extends to be a continuous function on the space $T^*M$ without the zero section, and the conormal symbol $\sigma_c = \sigma_c(A)(p)$ is a pseudodifferential operator with parameter $p \in \mathbb{R}$ in the sense of Agranovich–Vishik on $\Omega$. The principal and conormal symbols satisfy the compatibility condition*

$$\sigma(\sigma_c) = \sigma_\partial, \tag{5.80}$$

*where $\sigma_\partial$ is the restriction of the principal symbol $\sigma$ to $\partial T^*M$ and $\sigma(\sigma_c)$ is the principal symbol of the conormal symbol viewed as a pseudodifferential operator with parameter. This latter symbol is a homogeneous function on the total space of the bundle $T^*\Omega \times \mathbb{R}$ over $\Omega$ without the zero section.*[11]

2) *Conversely, if a symbol $\sigma \in \mathcal{O}^m(T^*M)$ and a pseudodifferential operator $\sigma_c = \sigma_c(p)$ with parameter $p \in \mathbb{R}$ of order $m$ on $\Omega$ satisfy the compatibility condition* (5.80)*, then there exists a stabilizing pseudodifferential operator $A$ on $M$ such that*

$$\sigma = \sigma(A), \quad \sigma_c = \sigma_c(A). \tag{5.81}$$

3) *Stabilizing pseudodifferential operators on a manifold $M$ with isolated singularities form an algebra. The sum and the product of operator correspond to the sums and the products, respectively, of their symbols and of their conormal symbols.*

4) *A stabilizing pseudodifferential operator $A$ of order $m$ on $M$ is compact in the Sobolev spaces*

$$A : H^s(M) \longrightarrow H^{s-m}(M), \quad s \in \mathbb{R}$$

*if and only if it has zero principal and conormal symbols.*

[11]This total space is just the boundary of $T^*M$, and the compatibility conditions (5.80) live on this space.

*Proof.* 1) Using cutoff functions, we see that it suffices to prove this for stabilizing pseudodifferential operators on the infinite cylinder. Let $A$ be such an operator. Since stabilizing smoothing operators (i.e., operators with smooth kernel) do not contribute to the principal symbol and their conormal symbols are obviously pseudodifferential operators with parameter of order $-\infty$, it suffices to consider a local representation of our operator in some coordinate system $(\omega, t)$ on the cylinder. In the local coordinates, the operator has the form

$$A = H\left(t, \omega, -i\frac{\partial}{\partial t}, -i\frac{\partial}{\partial \omega}\right),$$

where $H(t, \omega, p, q)$ is its complete symbol in these coordinates.

We claim that this symbol stabilizes in the sense of (5.66). Indeed, assume the opposite: the operator $A$ stabilizes, but its symbol does not stabilize. For simplicity, let $m = 0$. Then there exists an $\varepsilon > 0$, a sequence of point $(\omega_k, \xi_k) \in \mathbb{R}^{2n-1}$, and sequences $\lambda_k, \mu_k \to \infty$ such that

$$|H(\lambda_k, \omega_k, \xi_k) - H(\mu_k, \omega_k, \xi_k)| > \varepsilon.$$

We use the following lemma.

LEMMA 5.51. *For any $C > 0$ and $C_1 > 0$, there exists a constant $\varepsilon > 0$ such that if a symbol $H$ of zero order satisfies the estimates*

$$\left|\frac{\partial^\alpha H}{\partial x^\alpha}(x, \xi)\right| \leq C, \quad 1 \leq |\alpha| \leq n + 1$$

*and*

$$|H(x_0, \xi_0)| > C_1$$

*at some point $(x_0, \xi_0) \in \mathbb{R}^{2n}$, then the norm of the pseudodifferential operators $\widehat{H}$ is not less than $\varepsilon$:*

$$\left\| H\left(\overset{2}{x}, -i\overset{1}{\frac{\partial}{\partial x}}\right)\right\| \geq \varepsilon.$$

EXERCISE 5.52. Prove the lemma.

It follows from the lemma that there exists a $\delta > 0$ such that $\|A_{\lambda_k} - A_{\mu_k}\| > \delta$ for all $k$, and so $A$ cannot be a stabilizing operator.

Thus we have proved that the local complete symbol of the operator in question stabilizes. Then, again locally, the conormal symbol has the form

$$\sigma_c(A)(p) = H\left(\infty, \omega, p, -i\frac{\partial}{\partial \omega}\right)$$

and hence is a pseudodifferential operator with parameter in the sense of Agranovich–Vishik. The principal symbol of the conormal symbol is equal to

$$\sigma(\sigma_c(A)) = H_m(\infty, \omega, p, q) = \sigma(A)|_{t=\infty},$$

where $H_m(t, \omega, p, q)$ is the leading homogeneous term of $F$, i.e., the principal symbol of $A$, expressed in the same local coordinates, and coincides with the restriction of the principal symbol to the boundary of the compressed cotangent bundle.

2) Let $\varphi$ be an admissible smooth function on $M$ supported in $U$ and equal to unity in a neighborhood of the singular point. Then the operator

$$B = \varphi\sigma_c\left(-i\frac{\partial}{\partial t}\right)\varphi$$

on the infinite cylinder can also be treated as an operator on $M$ (since the support of its integral kernel is contained $U \times U$). To construct $A$, it now suffices to construct the difference $C = A - B$ from the following data:

$$\sigma_c(C) = 0, \quad \sigma(C) = \widetilde{\sigma} \equiv \sigma - \varphi^2(t)\sigma(\sigma_c);$$

moreover, it is known that $\widetilde{\sigma}_\partial = 0$. This problem is however trivial. Using a partition of unity consisting of smooth admissible functions on $M$, we reduce the problem to the cylinder. Let

$$1 = \sum_j \chi_j^2$$

be a smooth partition of unity on $\Omega$ subordinate to some cover of $\Omega$ by coordinate patches. We set

$$C = \sum_j \chi_j \circ [\widetilde{\sigma}\varphi]\left(t, \omega, -i\frac{\partial}{\partial t}, -i\frac{\partial}{\partial \omega}\right) \circ \chi_j, \tag{5.82}$$

where the middle factor is the pseudodifferential operator in $\mathbb{R}^n$ whose complete symbol $\widetilde{\sigma}\varphi$ is obtained from $\widetilde{\sigma}$ by the multiplication by an excision function $\varphi(p^2 + q^2)$ that vanishes in a neighborhood of zero and is equal to unity in a neighborhood of infinity.

EXERCISE 5.53. Verify that the principal symbol of the operator (5.82) is $\widetilde{\sigma}$ and the conormal symbol is zero.

3) Let us prove the only nontrivial assertion that the product of stabilizing operators stabilizes and the conormal symbol of the product is equal to the product of the conormal symbols of the factors. Let $A$ and $B$ be stabilizing operators. Then the operator $\varphi A\chi$ and $\theta B\psi$ on the infinite cylinder also stabilize, where the extreme factors are admissible smooth functions supported in $U$ and equal to unity in a neighborhood of the singular point. We need to verify that $\varphi AB\psi$ stabilizes. We can write

$$\varphi AB\psi = \varphi A\chi\theta B\psi + \varphi A(1 - \chi\theta)B\psi.$$

The first term on the cylinder stabilizes by Theorem 5.45 to the product of the limit operators of the factors, which gives the product $\sigma_c(A)\sigma_c(B)$ after the Fourier transform. Hence it remains to show that the second term stabilizes to zero. Without loss in generality, we can assume from the very beginning that $\chi\theta = 1$ on $\operatorname{supp}\varphi \cup \operatorname{supp}\psi$. In this case, the second term is an integral operator with smooth kernel of the form

$$K(x, x') = \varphi(t)\psi(t') \int_{\operatorname{supp}(1-\chi\theta)} K_A(x, z)K_B(z, x')\, d\operatorname{vol}(z) \tag{5.83}$$

where $K_A$ and $K_B$ are the integral kernels of $A$ and $B$. The integration in (5.83) is carried out over a compact set, and hence the integral, by virtue of the estimates of kernels of pseudodifferential operators outside the diagonal, is estimated together with derivatives by $\operatorname{const}[(1 + |t|)(1 + |t'|)]^{-N}$ for each $N$, whence we see that it stabilizes to zero.

4) This assertion follows from Theorem 5.46 on the compactness of stabilizing operators on the infinite cylinder.

The proof is complete. ☐

DEFINITION 5.54. The algebra of stabilizing pseudodifferential operators on a manifold $M$ with isolated singularities will be denoted by $\mathcal{PSD}(M) \subset PSD(M)$ and called the *maximal algebra*.

### 5.4.3. Ellipticity and the finiteness theorem

We know (see Chap. 4) that the symbol calculus permits one to prove the finiteness theorem for elliptic operators. These results are very important, and we reproduce the general scheme here in our specific situation. Namely, we shall state and prove a theorem on the Fredholm property of pseudodifferential operators for the algebra $\mathcal{PSD}(M)$.

DEFINITION 5.55. An operator $A \in \mathcal{PSD}(M)$ is said to be *elliptic* if the following conditions hold:

1) (interior ellipticity) the principal symbol $\sigma(A) \in \mathcal{O}^m(T^*M)$ is invertible everywhere outside the zero section;

2) (conormal ellipticity) the conormal symbol $\sigma_c(A)(p)$ is invertible for all $p \in \mathbb{R}$.

THEOREM 5.56 (finiteness). *If $A \in \mathcal{PSD}(M)$ is an elliptic operator, then it is Fredholm.*

*Proof.* The proof is essential trivial in the light of the preceding. Under conditions 1) and 2) one has the embedding $\sigma(A)^{-1} \in \mathcal{O}^{-m}(T^*M)$, the family $\sigma_c^{-1}(A)(p)$ is pseudodifferential with parameter $p \in \mathbb{R}$ in the sense of Agranovich–Vishik of order $-m$, and the compatibility condition

$$\sigma_\partial(A)^{-1} = \sigma(\sigma_c^{-1}(A))$$

holds, since $\sigma(\sigma_c^{-1}) = (\sigma(\sigma_c))^{-1}$.

By Theorem 5.50, 3), it follows that there exists a pseudodifferential operator $B \in \mathcal{PSD}(M)$ of order $-m$ such that

$$\sigma(B) = \sigma^{-1}(A), \quad \sigma_c(B) = \sigma_c^{-1}(A).$$

We see from item 3) of the same theorem that the operators

$$AB - 1, \quad BA - 1$$

have zero principal and conormal symbol and hence are compact by Theorem 5.50, 4). It follows that $A$ is Fredholm. The proof of the theorem is complete. ☐

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