The Zaremba Problem in Edge Sobolev Spaces

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Abstract

Mixed elliptic boundary value problems are characterised by conditions which have a jump along an interface of codimension 1 on the boundary. We study such problems in weighted edge Sobolev spaces and show the Fredholm property and the existence of parametrices under additional conditions of trace and potential type on the interface. Our methods from the calculus of boundary value problems on a manifold with edges will be illustrated by the Zaremba problem and other mixed problems for the Laplace operator.

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Introduction

By a mixed elliptic problem for an elliptic differential operator A we understand a boundary value problem on a C^∞ manifold X with boundary Y of the form

$$Au = f \quad \text{in} \quad \operatorname{int} X, \quad T_\pm u = g_\pm \quad \text{on} \quad Y_\pm. \tag{1}$$

Here Y is subdivided into C^∞ submanifolds $Y_\pm$ with common boundary Z such that $Y_+ \cup Y_- = Y$ and $Y_+ \cap Y_- = Z$. The operators $T_\pm$ are boundary conditions of the form $\mathrm{r}^\pm B_\pm$, where $B_\pm$ are differential operators in an open neighbourhood of $Y_\pm$ with smooth coefficients, $\mathrm{r}^\pm$ are the operators of restriction to $\operatorname{int} Y_\pm$, and the boundary conditions on the plus and minus side are assumed to satisfy the Shapiro-Lopatinskij condition (up to Z from the respective sides). The general task is to characterise the solvability of such problems in suitable distribution spaces, to construct parametrices and to establish asymptotics of solutions near the interface Z.

This paper is aimed at studying the Zaremba problem for the Laplace operator, i.e., T_- is the Dirichlet and T_+ the Neumann condition. Our approach is completely general and will show that other mixed elliptic problems can be analysed in a similar way.

We assume that X is compact. Then, if $H^s(\operatorname{int} X)$ and $H^s(\operatorname{int} Y_\pm)$ denote the standard Sobolev spaces of smoothness s on $\operatorname{int} X$ and $\operatorname{int} Y_\pm$, respectively, the Zaremba problem represents a continuous operator

$$\mathcal{A} = \begin{pmatrix} \Delta \\ T_- \\ T_+ \end{pmatrix} : H^s(\operatorname{int} X) \to \begin{matrix} H^{s-2}(\operatorname{int} X) \\ \oplus \\ H^{s-1/2}(\operatorname{int} Y_-) \\ \oplus \\ H^{s-3/2}(\operatorname{int} Y_+) \end{matrix} \tag{2}$$

for every real $s > 3/2$. It is clear that when we ask solutions of the problem (1) with arbitrary boundary data $g_\pm$ we cannot expect the existence in $H^s(\operatorname{int} X)$. Therefore, we need another category of spaces which describe the solvability in a more adequate way. These are in the present paper the so called weighted edge Sobolev spaces. Nevertheless, as we shall see, also the behaviour of mixed problems in standard Sobolev spaces is a useful information, cf. Harutjunjan and Schulze [10].

Mixed boundary value problems have a long history, cf. [31]. Methods from singular integral operators and pseudo-differential analysis have been applied in many specific situations, see the work of Gohberg and Krupnik [6], Vishik and Eskin [30], [4] and of many other authors. Mixed elliptic problems with additional interface conditions have been studied in Rempel and Schulze [18], [19], based on the calculus of [17]. However, in contrast to pseudo-differential scenarios for boundary value problems with the transmission property, cf. Boutet de Monvel [2] or [16], the structure of parametrices within a suitable operator algebra with smooth and complete symbolic structures remained to a large extent obscure. The present paper gives a transparent description of parametrices by interpreting mixed problems as operators on a suitable manifold with edges and boundary. What we obtain is that mixed problems belong to a corresponding edge algebra with a hierarchy

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}), \sigma_\wedge(\mathcal{A})) \tag{3}$$

of principal symbolic components (and also with complete amplitude functions); here σ_ψ indicates the interior, σ_∂ the boundary, and $\sigma_\wedge$ the edge symbol (σ_∂ consists of two components $\sigma_{\partial,\pm}$ belonging to $Y_\pm$). We show in which way (and how many) elliptic extra conditions along the interface Z have to be posed and how the operators (2) are linked to corresponding operators in weighted edge Sobolev spaces. In contrast to [10] we avoid here reductions to the boundary but formulate everything directly in the edge calculus.

Moreover, we characterise those weights such that our mixed problems become Fredholm in weighted Sobolev spaces. As a corollary of the approach we obtain parametrices within the edge algebra belonging to $\sigma^{-1}(\mathcal{A})$.

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1 Mixed problems in edge Sobolev spaces

1.1 Edge Sobolev spaces

In this paper we assume X to be compact; the submanifold $Z \subset Y = \partial X$ will be treated as an edge. Throughout this paper we assume $\dim Z > 0$. By a manifold W with edge (first without boundary) we understand a quotient space $W = \mathbb{W}/\sim$, where $\mathbb{W}$ is a (in our case compact) C^∞ manifold with boundary $\partial\mathbb{W}$, and $\partial\mathbb{W}$ is a fibre bundle over another smooth manifold Z with a closed C^∞ manifold N as fibre. The equivalence relation $w_1 \sim w_2$ for the quotient space means $w_1 = w_2$ for $w_1, w_2 \in \mathbb{W} \setminus \partial\mathbb{W}$ and $\pi w_1 = \pi w_2$ for $w_1, w_2 \in \partial\mathbb{W}$, where $\pi : \partial\mathbb{W} \to Z$ is the canonical projection.

Let us set $N^\Delta := (\overline{\mathbb{R}}_+ \times N)/(\{0\} \times N)$ which is a cone with base N. Then an equivalent definition of a manifold with edge starts from a topological space W with a subspace $Z \subset W$ which is the edge such that $W \setminus Z$ and Z are smooth manifolds, and every $z \in Z$ has a neighbourhood modelled on a wedge $N^\Delta \times \Omega$ for some open set $\Omega \subseteq \mathbb{R}^q$ which corresponds to a chart on Z, $q = \dim Z$. The associated stretched manifold $\mathbb{W}$ is then as before, i.e., locally near $\partial\mathbb{W}$ modelled on $\overline{\mathbb{R}}_+ \times N \times \Omega$, and $N \times \Omega$ is a trivialisation of the abovementioned N-bundle over Z. Note that spaces of the kind W are not necessarily manifolds in the standard sense; nevertheless, for convenience we use this terminology.

For our constructions we need the variant that the base N of the model cone is not closed but a compact smooth manifold with boundary. We then talk about a manifold with edge and boundary.

If X is a smooth compact manifold with boundary we first consider the double $2X$ (obtained by gluing together two copies X_+ and X_- of $X =: X_+$ along the common boundary Y). Then $W := 2X$ can be interpreted as a manifold with edge Z, and there is the corresponding stretched manifold $\mathbb{W}$ such that $W = \mathbb{W}/\sim$.

The base N of the model cone for W near Z in this case is the unit circle S^1 in the fibre $\cong \mathbb{R}^2$ of the (trivial) normal bundle of Z in $2X$. For the associated stretched manifold $\mathbb{W}$ we have $\partial\mathbb{W} = Z \times S^1$. Moreover, $\mathbb{W}$ is the union of two subspaces $\mathbb{W}_\pm$ which are locally near $\mathbb{W}_\pm \cap \partial\mathbb{W}$ modelled on $\overline{\mathbb{R}}_+ \times I_\pm$ for the half-circles $I_+ = \{0 \le \phi \le \pi\}$, $\quad I_- := \{\pi \le \phi \le 2\pi\}$.

Summing up, X can be interpreted as a manifold with edge Z and boundary, and the associated stretched manifold $\mathbb{X}$ is equal to $\mathbb{W}_+$. Let us set

$$\mathbb{X}_{\text{sing}} := \mathbb{X} \cap \partial\mathbb{W}, \quad \mathbb{X}_{\text{reg}} := \mathbb{X} \setminus \mathbb{X}_{\text{sing}},$$

where $\mathbb{X}_{\text{sing}} = Z \times I$ for $I := I_+$ and $\mathbb{X}_{\text{reg}} \cong X \setminus Z$ (diffeomorphic in the sense of C^∞ manifolds with boundary).

In order to define weighted cone and edge Sobolev spaces we return for a moment to the general case. Let N be closed and compact, and let $N^\wedge := \mathbb{R}_+ \times N$ be the (open stretched) cone with base N. Let $L^\mu_{\text{cl}}(N; \mathbb{R}^l)$ denote the space of all classical parameter-dependent pseudo-differential operators $A(\lambda)$ of order $\mu \in \mathbb{R}$ on N, with parameter $\lambda \in \mathbb{R}^l$, that is, the local amplitude functions $a(x, \xi, \lambda)$ formally contain (ξ, λ) as an $(m+l)$-dimensional covariable, $m := \dim N$, and $L^{-\infty}(N; \mathbb{R}^l) = \mathcal{S}(\mathbb{R}^l, L^{-\infty}(N))$, where $L^{-\infty}(N)$ is the space of all smoothing operators on N (all manifolds in question are assumed to be equipped with a Riemannian metric; then we have $L^{-\infty}(N) \cong C^\infty(N \times N)$ with its natural Fréchet topology). We employ the well known fact that $L^\mu_{\mathrm{cl}}(N; \mathbb{R}^l)$ for every $\mu \in \mathbb{R}$ contains a parameter-dependent elliptic family $R^\mu(\lambda)$ which induces isomorphisms $H^s(N) \to H^{s-\mu}(N)$ for all $s \in \mathbb{R}$ and $\lambda \in \mathbb{R}^l$.

Let M be the Mellin transform on $\mathbb{R}_+$, i.e., $Mu(w) = \int_0^\infty r^{w-1} u(r) dr$, first for $u \in C_0^\infty(\mathbb{R}_+)$ (which yields holomorphic functions in $\mathbb{C}$) and then extended to more general function and distribution spaces (also vector-valued ones). Then the space $\mathcal{H}^{s,\gamma}(N^\wedge)$ for $s, \gamma \in \mathbb{R}, N^\wedge := \mathbb{R}_+ \times N$, is defined as the completion of $C_0^\infty(\mathbb{R}_+, C^\infty(N))$ with respect to the norm

$$\left\{ (2\pi i)^{-1} \int_{\Gamma_{\frac{m+1}{2}-\gamma}} \| R^s(\mathrm{Im}\, w)(Mu)(w) \|^2_{L^2(N)} dw \right\}^{1/2},$$

with the space $L^2(N) \cong H^0(N)$ in its standard norm and

$$\Gamma_\beta := \{ w \in \mathbb{C} : \mathrm{Re}\, w = \beta \}.$$

In this paper a cut-off function $\omega(r)$ on the half-axis means any real valued $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ which is equal to 1 near $r = 0$. Let us set

$$\mathcal{K}^{s,\gamma}(N^\wedge) := \{ \omega u + (1-\omega) v : u \in \mathcal{H}^{s,\gamma}(N^\wedge), \quad v \in H^s_{\mathrm{cone}}(N^\wedge) \}.$$

Here $H^s_{\mathrm{cone}}(N^\wedge)$ denotes the subspace of all $v = \tilde{v}|_{N^\wedge}$, $\tilde{v} \in H^s_{\mathrm{loc}}(\mathbb{R} \times N)$, such that for every coordinate neighbourhood U on N, every diffeomorphism $\chi : U \to \tilde{U}$ to an open set $\tilde{U} \subset S^m$, $\chi(x) = \tilde{x}$, and every $\varphi \in C_0^\infty(U)$ the function $\varphi(\chi^{-1}(\tilde{x}))(1-\omega(r))v(r, \chi^{-1}(\tilde{x}))$ belongs to the space $H^s(\mathbb{R}^{m+1})$ (where $(r, \tilde{x})$ has the meaning of polar coordinates in $\mathbb{R}^{m+1} \setminus \{0\}$).

The spaces $\mathcal{K}^{s,\gamma}(N^\wedge)$ are independent of the specific choice of ω. They are Hilbert spaces with natural scalar products which we choose for $s = \gamma = 0$ in such a way that

$$\mathcal{K}^{0,0}(N^\wedge) = \mathcal{H}^{0,0}(N^\wedge) = r^{-\frac{m}{2}} L^2(\mathbb{R}_+ \times N)$$

with $L^2(\mathbb{R}_+ \times N)$ referring to $dr dx$.

For the case $N = 2M$ for a compact C^∞ manifold M with smooth boundary we set

$$\mathcal{K}^{s,\gamma}(M^\wedge) := \{ u|_{(\mathrm{int}\, M)^\wedge} : u \in \mathcal{K}^{s,\gamma}(N^\wedge) \} \tag{4}$$

endowed with the quotient topology corresponding to the identification with $\mathcal{K}^{s,\gamma}((2M)^\wedge)/\mathcal{K}_0^{s,\gamma}(M_-^\wedge)$. The manifold $2M$ is obtained by gluing together two copies $M_\pm$ of M with M_+ being identified with M, and $\mathcal{K}_0^{s,\gamma}(M_-^\wedge)$ denotes the set of all elements of $\mathcal{K}^{s,\gamma}((2M)^\wedge)$ supported in $\mathbb{R}_+ \times M_-$ which is a closed subspace. We endow the space $\mathcal{K}^{s,\gamma}(M^\wedge)$ with a group of isomorphisms

$$\kappa_\lambda : \mathcal{K}^{s,\gamma}(M^\wedge) \to \mathcal{K}^{s,\gamma}(M^\wedge), \tag{5}$$

defined by $(\kappa_\lambda u)(r, x) := \lambda^{\frac{m+1}{2}} u(\lambda r, x)$ for $\lambda \in \mathbb{R}_+$, $m = \dim M$, which is strongly continuous for every $s, \gamma \in \mathbb{R}$.

Let us now introduce the notion of 'abstract' edge Sobolev spaces

$$\mathcal{W}^s(\mathbb{R}^q, E), \quad s \in \mathbb{R}, \tag{6}$$

with a Hilbert space E which is endowed with a strongly continuous group $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ of isomorphisms $\kappa_\lambda : E \to E$, such that $\kappa_\lambda\kappa_{\lambda'} = \kappa_{\lambda\lambda'}$ for all $\lambda, \lambda' \in \mathbb{R}_+$ (in such a case we simply say that the space E is endowed with a group action). The space (6) is defined as the completion of $\mathcal{S}(\mathbb{R}^q, E)$ (the Schwartz space of E-valued functions) with respect to the norm

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,E)} = \Big\{ \int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle}\hat{u}(\eta)\|^2_E d\eta \Big\}^{1/2};$$

here $\hat{u}$ is the Fourier transform of u, and $\langle\eta\rangle = (1+|\eta|^2)^{1/2}$. (Instead of $\langle\eta\rangle$ we can equivalently take any function $\eta \to [\eta]$, smooth and strictly positive, satisfying $[\eta] = |\eta|$ for $|\eta| > c$ for some constant $c > 0$). We will apply this to $E = \mathcal{K}^{s,\gamma}(M^\wedge)$ with the group (5) as well as to some other cases, for instance, $E = H^s(\mathbb{R}^p)$ with $(\kappa_\lambda u)(x) = \lambda^{p/2}u(\lambda x)$, $\lambda \in \mathbb{R}_+$, $u \in H^s(\mathbb{R}^p)$. In the latter case we have the identity

$$\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^p)) = H^s(\mathbb{R}^{q+p})$$

for every $s \in \mathbb{R}$. Similarly, if $E = H^s(\mathbb{R}_+)$ with $(\kappa_\lambda u)(t) = \lambda^{\frac{1}{2}}u(\lambda t)$, $\lambda \in \mathbb{R}_+$, we have

$$H^s(\mathbb{R}^n_+) = \mathcal{W}^s(\mathbb{R}^{n-1}, H^s(\mathbb{R}_+)) \tag{7}$$

for $H^s(\mathbb{R}^n_+) = \big\{u|_{\mathbb{R}^n_+} : u \in H^s(\mathbb{R}^n)\big\}$, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$.

Concerning the space $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(M^\wedge)) =: \mathcal{W}^{s,\gamma}(M^\wedge \times \mathbb{R}^q)$, the property

$$H^s_{\text{comp}}((\text{int}\, M)^\wedge \times \mathbb{R}^q) \subset \mathcal{W}^{s,\gamma}(M^\wedge \times \mathbb{R}^q) \subset H^s_{\text{loc}}((\text{int}\, M)^\wedge \times \mathbb{R}^q)$$

allows us to define corresponding weighted edge Sobolev spaces

$$\mathcal{W}^{s,\gamma}(\mathbb{W}), \quad s, \gamma \in \mathbb{R}, \tag{8}$$

globally on a stretched manifold $\mathbb{W}$ associated with a compact manifold W with edge Z as the subspace of all elements of $H^s_{\text{loc}}(\text{int}\, \mathbb{W})$ which locally near $\partial\mathbb{W}$ belong to $\mathcal{W}^{s,\gamma}(M^\wedge \times \mathbb{R}^q)$, $q = \dim Z$; here $\mathbb{R}^q$ belongs to a chart on Z and $\overline{\mathbb{R}}_+ \times M \times \mathbb{R}^q$ is the local representation of a corresponding subset of a collar neighbourhood of $\partial\mathbb{W}$. (It would be more consistent to write $\mathcal{W}^{s,\gamma}(\text{int}\, \mathbb{W})$ rather than $\mathcal{W}^{s,\gamma}(\mathbb{W})$ but we hope our notation will not cause confusion.) In the global definitions of spaces on $\mathbb{W}$ we refer to local representations of $\mathbb{W}$ by (stretched) wedges $\overline{\mathbb{R}}_+ \times M \times \mathbb{R}^q$ where the transition maps respect a chosen collar neighbourhood of ∂M in the sense that the normal variable remains unchanged and where also the axial variable $r \in \overline{\mathbb{R}}_+$ remains preserved. The spaces (8) are Hilbert spaces with natural scalar products, chosen for $s = \gamma = 0$ in such a way that $\mathcal{W}^{0,0}(\mathbb{W}) = r^{-\frac{m}{2}}L^2(\mathbb{W})$, $m = \dim M$.

Applying this to $W = 2X$, cf. the notation of Section 1.1, we obtain the spaces

$$\mathcal{W}^{s,\gamma}(\mathbb{X}) := \Big\{ u\big|_{\text{int}\, \mathbb{X}_{\text{reg}}} : u \in \mathcal{W}^{s,\gamma}(\mathbb{W}) \Big\} \tag{9}$$

endowed with the quotient topology from the identification with $\mathcal{W}^{s,\gamma}(\mathbb{W})/\mathcal{W}^{s,\gamma}_0(\mathbb{X}_-)$; here $\mathcal{W}^{s,\gamma}_0(\mathbb{X}_-)$ denotes the set of all elements of $\mathcal{W}^{s,\gamma}(\mathbb{W})$ supported in $\mathbb{W} \setminus \text{int}\, \mathbb{X}_{\text{reg}}$.

Recall that X is interpreted as a manifold with edge Z and boundary, and $\mathbb{X}$ as its stretched space. The parts $Y_\pm$ of the boundary of X as compact smooth manifolds with boundary Z are themselves manifolds with edge Z (they are equal to their own stretched spaces), and they are contained as subsets in $\mathbb{X}$. In particular, we then have the spaces $\mathcal{W}^{s,\gamma}(Y_\pm)$, $s, \gamma \in \mathbb{R}$. Let us consider the restriction operators

$$\mathrm{r}^\pm : C^\infty_0(\mathbb{X}_{\text{reg}}) \to C^\infty_0(\text{int}\, Y_\pm). \tag{10}$$

For every $s, \gamma \in \mathbb{R}$ we have $C^\infty_0(\mathbb{X}_{\text{reg}}) \subset \mathcal{W}^{s,\gamma}(\mathbb{X})$ and $C^\infty_0(\text{int}\, Y_\pm) \subset \mathcal{W}^{s,\gamma}(Y_\pm)$.

Remark 1.1 *The operator* (10) *extends to a continuous operator*

$$\mathrm{r}^{\pm} : \mathcal{W}^{s,\gamma}(\mathbb{X}_{\mathrm{reg}}) \to \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(Y_{\pm})$$

for every $s, \gamma \in \mathbb{R},\ s > \frac{1}{2}$.

1.2 Elements of the edge calculus

By edge calculus we understand a specific pseudo-differential algebra on a (stretched) manifold $\mathbb{W}$ with edge, cf. Sections 4.1 and 4.2 below. We recall some basic structures here which we need in our approach to mixed problems.

Parallel to the concept of abstract Sobolev spaces we have operator-valued symbols of the class

$$S^{\mu}(U \times \mathbb{R}^q; E, \widetilde{E}), \quad \mu \in \mathbb{R}, \tag{11}$$

$U \subseteq \mathbb{R}^p$ open, where E and $\widetilde{E}$ are Hilbert spaces endowed with group actions $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively. An $a(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(E, \widetilde{E}))$ is said to belong to (11) if

$$\sup_{y\in K, \eta\in\mathbb{R}^q} \langle\eta\rangle^{-\mu+|\beta|} \left\| \tilde{\kappa}^{-1}_{\langle\eta\rangle} \{D^\alpha_y D^\beta_\eta a(y,\eta)\} \kappa_{\langle\eta\rangle} \right\|_{\mathcal{L}(E,\widetilde{E})} < \infty$$

for every $K \Subset U$, and multi-indices $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$. Denoting by $S^{(\mu)}(U \times (\mathbb{R}^q \setminus \{0\}); E, \widetilde{E})$ the space of all $a_{(\mu)}(y,\eta) \in C^\infty(U \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(E, \widetilde{E}))$ satisfying

$$a_{(\mu)}(y, \lambda\eta) = \lambda^\mu \tilde{\kappa}_\lambda a_{(\mu)}(y,\eta) \kappa^{-1}_\lambda \ \text{ for all } \lambda \in \mathbb{R}_+, (y,\eta) \in U \times (\mathbb{R}^q \setminus \{0\}),$$

we have $\chi(\eta) S^{(\mu)}(U \times (\mathbb{R}^q \setminus \{0\}); E, \widetilde{E}) \subset S^\mu(U \times \mathbb{R}^q; E, \widetilde{E})$ for every excision function χ (i.e., $\chi \in C^\infty(\mathbb{R}^q)$, $\chi(\eta) = 0$ for $|\eta| < c_0$, $\chi(\eta) = 1$ for $|\eta| > c_1$ for certain $0 < c_0 < c_1$). An element $a(y,\eta)$ of (11) is said to be classical, if there are homogeneous components $a_{(\mu-j)}(y,\eta) \in S^{(\mu-j)}(U \times (\mathbb{R}^q \setminus \{0\}); E, \widetilde{E})$, $j \in \mathbb{N}$, such that $a(y,\eta) - \chi(\eta) \sum_{j=0}^N a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(U \times \mathbb{R}^q; E, \widetilde{E})$ for all $N \in \mathbb{N}$.

Let $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \widetilde{E})$ denote the space of all classical symbols; if a consideration is valid both for the classical or non-classical case we write $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \widetilde{E})$. The spaces of symbols with constant coefficients, i.e., $a = a(\eta)$, will be denoted by $S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; E, \widetilde{E})$.

If $\Omega \subseteq \mathbb{R}^q$ is some open set and $U = \Omega \times \Omega$ we also write $(y, y') \in \Omega \times \Omega$ rather than $y \in U$. Analogously to standard Sobolev spaces, for every open $\Omega \subseteq \mathbb{R}^q$ we have the spaces $\mathcal{W}^s_{\mathrm{comp}}(\Omega, E), \mathcal{W}^s_{\mathrm{loc}}(\Omega, E)$, where $\mathcal{W}^s_{\mathrm{comp}}(\Omega, E)$ can be identified with the space of all $u \in \mathcal{W}^s(\mathbb{R}^q, E)$ of compact support in Ω, while $\mathcal{W}^s_{\mathrm{loc}}(\Omega, E)$ is the subspace of all $u \in \mathcal{D}'(\Omega, E)$ $(= \mathcal{L}(C^\infty_0(\Omega), E))$ such that $\varphi u \in \mathcal{W}^s_{\mathrm{comp}}(\Omega, E)$ for every $\varphi \in C^\infty_0(\Omega)$. The space $\mathcal{W}^s_{\mathrm{loc}}(\Omega, E)$ is Fréchet, while $\mathcal{W}^s_{\mathrm{comp}}(\Omega, E)$ is an inductive limit of Fréchet spaces.

Let $\mathrm{Op}_y(a)u(y) = \iint e^{i(y-y')\eta} a(y, y', \eta) u(y') dy' đ\eta$, $đ\eta = (2\pi)^{-q} d\eta$, $a(y, y', \eta) \in S^\mu(\Omega \times \Omega \times \mathbb{R}^q; E, \widetilde{E})$. We will need the fact that $\mathrm{Op}(a) : C^\infty_0(\Omega, E) \to C^\infty(\Omega, \widetilde{E})$ $(\mathrm{Op}(a) := \mathrm{Op}_y(a))$ extends to continuous operators

$$\mathrm{Op}(a) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, E) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \widetilde{E}) \tag{12}$$

for every $s \in \mathbb{R}$. This is an immediate analogue of a corresponding continuity of scalar pseudo-differential operators. If the amplitude function has constant coefficients we have

$$\mathrm{Op}(a) : \mathcal{W}^s(\mathbb{R}^q, E) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{E}) \tag{13}$$

for every $s \in \mathbb{R}$.

If E is a Fréchet space, written as a projective limit of Hilbert spaces E^j, $j \in \mathbb{N}$, with continuous embeddings $E^{j+1} \hookrightarrow E^j$ for all j, we say that E is endowed with a group action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$, if there is given a group action on E^0 which restricts to group actions on E^j for all j.

Remark 1.2 *Spaces of symbols as well as abstract Sobolev spaces can be formulated in an analogous manner for Fréchet spaces E and $\widetilde{E}$ with group action, cf.* [25].

An example is the space $\mathcal{S}^\gamma_\varepsilon(M^\wedge) := \varprojlim_{k \in \mathbb{N}} \langle r \rangle^{-k} \mathcal{K}^{k,\gamma+\varepsilon-(1+k)^{-1}}(M^\wedge)$, $\varepsilon > 0$, for a compact C^∞ manifold M (with or without boundary), where $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ is induced by $(\kappa_\lambda u)(r,x) = \lambda^{\frac{m+1}{2}} u(\lambda r, x)$, $m = \dim M$. In particular, for $m = 0$ we obtain the spaces

$$\mathcal{S}^\gamma_\varepsilon(\mathbb{R}_+), \ \gamma \in \mathbb{R}, \ \varepsilon > 0. \tag{14}$$

As noted before mixed boundary value problems will be interpreted as elements of an edge algebra of boundary value problems on the stretched manifold with edge Z and boundary. The edge algebra in the present case consists of 4×4 block matrix operators which define continuous operators

$$\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,\dots,4} : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{X}) \\ \oplus \\ \mathcal{W}^{s-\frac{1}{2}-\nu_-,\gamma-\frac{1}{2}-\nu_-}(Y_-) \\ \oplus \\ \mathcal{W}^{s-\frac{1}{2}-\nu_+,\gamma-\frac{1}{2}-\nu_+}(Y_+) \\ \oplus \\ H^s(Z,\mathbb{C}^d) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{X}) \\ \oplus \\ \mathcal{W}^{s-\frac{1}{2}-\mu_-,\gamma-\frac{1}{2}-\mu_-}(Y_-) \\ \oplus \\ \mathcal{W}^{s-\frac{1}{2}-\mu_+,\gamma-\frac{1}{2}-\mu_+}(Y_+) \\ \oplus \\ H^{s-\mu}(Z,\mathbb{C}^e) \end{array} \tag{15}$$

for certain $\mu, \nu_\pm, \mu_\pm \in \mathbb{R}$ and dimensions d, e. We assume this form only for simplicity; all spaces may also be considered in the variant of distributional sections in vector bundles on the respective spaces, e.g., trivial bundles of some dimensions, and the operators referring to Z may also have other orders in the sense that instead of the spaces $H^s(Z,\mathbb{C}^d)$ $\quad (H^{s-\mu}(Z,\mathbb{C}^e))$ we have $\bigoplus_{j=1}^J H^{s_j}(Z,\mathbb{C}^{d_j})$ $\quad (\bigoplus_{k=1}^K H^{s_k}(Z,\mathbb{C}^{e_k}))$, or, instead of $\mathcal{W}^{s-\frac{1}{2}-\mu_+,\gamma-\frac{1}{2}-\mu_+}(Y_+)$ the spaces $\bigoplus_{l=1}^L \mathcal{W}^{s-\frac{1}{2}-\mu_{+,l},\gamma-\frac{1}{2}-\mu_{+,l}}(Y_+, \mathbb{C}^{m_{+,l}})$, for certain dimensions $m_{+,l}$, etc. Such generalisations are straightforward (some variants of that kind will occur below).

Example 1.3 *Let $A = \sum_{|\alpha| \le \mu} a_\alpha(x) D^\alpha_x$ be an arbitrary differential operator with coefficients $a_\alpha(x) \in C^\infty(X)$, and let $T_\pm = r^\pm B_\pm$ be vectors of boundary operators where $B_\pm = {}^{\mathrm{t}}(B_{\pm,l})_{l=1,\dots,L}$ are differential operators in an open neighbourhood of $Y_\pm$ of order $\mu_{\pm,l}$. Then the corresponding mixed boundary value problem* (1) *induces continuous operators*

$$\mathcal{A} = \begin{pmatrix} A \\ T_- \\ T_+ \end{pmatrix} : \mathcal{W}^{s,\gamma}(\mathbb{X}) \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{X}) \\ \oplus \\ \bigoplus_{l=1}^L \mathcal{W}^{s-\frac{1}{2}-\mu_{-,l},\gamma-\frac{1}{2}-\mu_{-,l}}(Y_-) \\ \oplus \\ \bigoplus_{l=1}^L \mathcal{W}^{s-\frac{1}{2}-\mu_{+,l},\gamma-\frac{1}{2}-\mu_{+,l}}(Y_+) \end{array}$$

for all $s > \max\{\mu_{\pm,l} + \frac{1}{2}\}$ and all $\gamma \in \mathbb{R}$.

Other examples are various kinds of trace, potential and Green operators.

Let us fix weights $\gamma, \delta \in \mathbb{R}$ and consider an operator function

$$g(z,\zeta) \in \bigcup_{\varepsilon>0} \bigcap_{s>-\frac{1}{2}} C^\infty(U \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(I^\wedge), \mathcal{S}^\delta_\varepsilon(I^\wedge))). \tag{16}$$

In this context we denote variables and covariables by (z,ζ) where z plays the role of local coordinates on Z, and $q = n-2$ $(= \dim Z)$. We call (16) a Green symbol of order $\mu \in \mathbb{R}$ and type 0 if there is an $\varepsilon > 0$ such that $g(z,\zeta)$ induces elements

$$g(z,\zeta) \in S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(I^\wedge), \mathcal{S}^\delta_\varepsilon(I^\wedge)), \;\; g^*(z,\zeta) \in S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; \mathcal{K}^{s,-\delta}(I^\wedge), \mathcal{S}^{-\gamma}_\varepsilon(I^\wedge)) \tag{17}$$

for all $s > -\frac{1}{2}$. Here '$*$' means the (z,ζ)-wise formal adjoint with respect to the $\mathcal{K}^{0,0}(I^\wedge)$-scalar product. The symbol spaces in (17) refer to the abovementioned group actions in the corresponding spaces, cf. also Remark 1.2.

According to the 4×4 block matrix structure of our edge operators we have similar symbols $g_{ij}(z,\zeta)$ for all $i,j = 1,2,3,4$, where orders μ and weights γ, δ will depend on i,j (let us set, for simplicity $d = e = 1$). Set

$$E_1 = \mathcal{K}^{s,\gamma}(I^\wedge), E_2 = \mathcal{K}^{s,\gamma}(\mathbb{R}_+), E_3 = \mathcal{K}^{s,\gamma}(\mathbb{R}_-), E_4 = \mathbb{C},$$
$$F_1 = \mathcal{S}^\delta_\varepsilon(I^\wedge), F_2 = \mathcal{S}^\delta_\varepsilon(\mathbb{R}_+), F_3 = \mathcal{S}^\delta_\varepsilon(\mathbb{R}_-), F_4 = \mathbb{C}.$$

For $j = 1$ the conditions are

$$g_{k1}(z,\zeta) \in S^{\mu_{k1}}_{\mathrm{cl}}(U \times \mathbb{R}^q; E_1, F_k), k = 2,3,4, \tag{18}$$

for certain $\varepsilon > 0$ and all $s > -\frac{1}{2}$, and similar conditions for the formal adjoints (now without any restriction on s). Similarly as (5) the group action on $\mathcal{S}^\delta_\varepsilon(\mathbb{R}_\pm)$ is given by $(\kappa_\lambda u)(x_{n-1}) = \lambda^{\frac{1}{2}} u(\lambda x_{n-1})$ for $\lambda \in \mathbb{R}_+$, while on $\mathbb{C}$ we take the identity map for all $\lambda \in \mathbb{R}_+$. Symbols of the form (18) are also said to be of type 0.

For the entries in general we require $g_{ij}(z,\zeta) \in S^{\mu_{ij}}_{\mathrm{cl}}(U \times \mathbb{R}^q; E_j, F_i)$ for a certain $\varepsilon > 0$ and all s (where $s > -\frac{1}{2}$ for $j = 1$), and analogous conditions for the formal adjoints.

The elements g_{ij} for $i = j = 2,3$ have the interpretation of Green symbols in the calculus of boundary value problems on $Y_\pm$ without the transmission property; the elements g_{ij} for $i > j$ $(i < j)$ have the meaning of various kinds of trace (potential) symbols, while g_{44} is simply a classical scalar pseudo-differential symbol.

Concerning the symbols $g_{k1}(z,\zeta)$ we also consider elements of arbitrary type $\mathrm{d} \in \mathbb{N}$ defined as operator functions of the form $g_{k1}(z,\zeta) = \sum_{l=0}^{\mathrm{d}} g_{k1;l}(z,\zeta)\partial^l_\phi$ for arbitrary trace symbols $g_{k1,l}(z,\zeta)$ of order μ_{k1} and type 0 (∂_ϕ is the differentiation in the angular variable $\phi \in I$). We then obtain classical operator-valued symbols

$$g_{k1}(z,\zeta) \in S^{\mu_{k1}}_{\mathrm{cl}}(U \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(I^\wedge), F_k)$$

for all $s > \mathrm{d} - \frac{1}{2}$, $k = 1, \ldots, 4$.

All these (operator-valued) amplitude functions give rise to associated pseudo-differential operators, first with respect to local coordinates on Z and then globally on $\mathbb{X}$ near $\mathbb{X}_{\mathrm{sing}}$ by a straightforward construction with local expressions and partitions of unity, cf. also Section 4.2 below. Modulo global smoothing operators in the edge algebra of boundary value problems we thus have defined the Green, trace (of type $\mathrm{d} \in \mathbb{N}$) and potential parts of operators (15).

The entries of the 4×4 matrices of Green symbols

$$\mathfrak{g}(z,\zeta) = (g_{ij}(z,\zeta))_{i,j=1,\dots,4} \tag{19}$$

have orders μ_{ij} and refer to weights γ_{ij} and δ_{ij}. From (15) we see that we have to set

$$\begin{aligned}
&\mu_{11} = \mu_{41} = \mu_{14} = \mu_{44} = \mu, \\
&\mu_{12} = \mu_{42} = \mu - \frac{1}{2} - \nu_-, \quad \mu_{13} = \mu_{43} = \mu - \frac{1}{2} - \nu_+, \\
&\mu_{22} = \mu_- - \nu_-, \quad \mu_{32} = \mu_+ - \nu_-, \quad \mu_{23} = \mu_- - \nu_+, \quad \mu_{33} = \mu_+ - \nu_+, \\
&\mu_{21} = \mu_{24} = \frac{1}{2} + \mu_-, \quad \mu_{31} = \mu_{34} = \frac{1}{2} + \mu_+,
\end{aligned} \tag{20}$$

$$\gamma_{i1} = \gamma, \qquad \gamma_{i2} = \gamma - \frac{1}{2} - \nu_-, \quad \gamma_{i3} = \gamma - \frac{1}{2} - \nu_+ \quad \text{for} \quad i = 1,2,3, \tag{21}$$

$$\delta_{1j} = \gamma - \mu, \quad \delta_{2j} = \gamma - \frac{1}{2} - \mu_-, \quad \delta_{3j} = \gamma - \frac{1}{2} - \mu_+ \quad \text{for} \quad j = 1,2,3. \tag{22}$$

Let

$$\mathfrak{g}_\wedge(z,\zeta) := (g_{ij,\wedge}(z,\zeta))_{i,j=1,\dots,4} \tag{23}$$

denote the matrix of homogeneous principal components of $g_{ij} \in S^{\mu_{ij}}_{\mathrm{cl}}(U \times \mathbb{R}^q; E_j, F_i)$.

1.3 The Zaremba problem as an edge problem

Let us first observe that the operator $\mathcal{A}$ which represents the Zaremba problem, extends from C_0^∞-functions on $\mathbb{X}_{\mathrm{reg}}$ to a continuous operator

$$\mathcal{A} : \mathcal{W}^{s,\gamma}(\mathbb{X}) \to \begin{matrix} \mathcal{W}^{s-2,\gamma-2}(\mathbb{X}) \\ \oplus \\ \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\operatorname{int} Y_-) \\ \oplus \\ \mathcal{W}^{s-\frac{3}{2},\gamma-\frac{3}{2}}(\operatorname{int} Y_+) \end{matrix} \tag{24}$$

for all $s, \gamma \in \mathbb{R}$, $s > 3/2$. Instead of $\mathcal{A}$ we will also write $\mathcal{A}(\gamma)$ if we consider different weights γ. We want to express the principal symbolic structure of $\mathcal{A}$ from the edge calculus. The principal symbol consists of a triple (3). Here $\sigma_\psi(\mathcal{A}) = -|\xi|^2$ is the standard homogeneous principal symbol of the Laplace operator, $\sigma_\partial(\mathcal{A}) = (\sigma_{\partial,-}(\mathcal{A}), \sigma_{\partial,+}(\mathcal{A}))$ is the pair of boundary symbols on the $\mp$ sides of Y, and $\sigma_\wedge(\mathcal{A})$ is the edge symbol.

Choose a collar neighbourhood $\cong Y \times [0,1)$ of Y in X with the variables $x = (y, x_n)$ and covariables $\xi = (\eta, \xi_n)$. Moreover, fix a tubular neighbourhood $\cong Z \times (-1,1)$ of Z in Y with the variables $y = (z, x_{n-1})$ and covariables (ζ, ξ_{n-1}). We also use y and z as local coordinates in corresponding open sets $U \subseteq \mathbb{R}^{n-1}_y$ and $\Omega \subseteq \mathbb{R}^{n-2}_z$, respectively.

The boundary symbols of $\binom{\Delta}{T_\mp}$ over $Y_\mp$ have the form

$$\sigma_{\partial,\mp}\begin{pmatrix} \Delta \\ T_\mp \end{pmatrix}(\eta) : H^s(\mathbb{R}_+) \to \begin{matrix} H^{s-2}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{matrix},$$

for $\eta \neq 0$, where $\sigma_\partial(\Delta)(\eta) = -|\eta|^2 + \frac{\partial^2}{\partial x_n^2}$ and $\sigma_{\partial,-}(T_-) = \mathrm{r}'_-$, $\sigma_{\partial,+}(T_+) = \mathrm{r}'_+ \frac{\partial}{\partial x_n}$.

In the following we use the fact that the Laplacian Δ in $\mathbb{R}^n$ can be reformulated (in suitable local coordinates near the boundary Y or the interface Z) in the form

$$\Delta = \frac{\partial^2}{\partial x_n^2} + \Delta_Y + L_1 = \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial x_{n-1}^2} + \Delta_Z + L_2,$$

where Δ_Y and Δ_Z are the Laplacians on Y and Z, respectively, belonging to the metrics induced by the Euclidean metric and certain first order operators L_1 and L_2 in x_n and (x_{n-1}, x_n), respectively, cf. [5]. Here x_n is a normal variable to Y in $\mathbb{R}^n$ and x_{n-1} a normal variable to Z in Y.

To express the principal edge symbol we introduce polar coordinates $(r, \phi) \in \mathbb{R}_+ \times S^1$ in the (x_{n-1}, x_n)-plane normal to Z and write the Laplacian in $\mathbb{R}^{n-2}_z \times \mathbb{R}^2_{x_{n-1}, x_n}$ in the form

$$\Delta = r^{-2}((-r\frac{\partial}{\partial r})^2 + \frac{\partial^2}{\partial \phi^2} + r^2\Delta_Z + L_2).$$

Then

$$\sigma_\wedge(\mathcal{A})(\zeta) = \begin{pmatrix} \sigma_\wedge(\Delta)(\zeta) \\ \sigma_\wedge(T_-) \\ \sigma_\wedge(T_+) \end{pmatrix} : \mathcal{K}^{s,\gamma}(I^\wedge) \to \begin{matrix} \mathcal{K}^{s-2,\gamma-2}(I^\wedge) \\ \oplus \\ \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_-) \\ \oplus \\ \mathcal{K}^{s-\frac{3}{2},\gamma-\frac{3}{2}}(\mathbb{R}_+) \end{matrix} \tag{25}$$

for $\zeta \neq 0$ is given by

$$\sigma_\wedge(\Delta)(\zeta) = r^{-2}((-r\frac{\partial}{\partial r})^2 + \frac{\partial^2}{\partial \phi^2} - r^2|\zeta|^2), \tag{26}$$

$$\sigma_\wedge(T_-)u := u|_{\phi=0}, \quad \sigma_\wedge(T_+)u := r^{-1}\frac{\partial}{\partial \phi}u|_{\phi=\pi}. \tag{27}$$

The notation $\mathbb{R}_\mp$ for the components of $\partial I^\wedge$, $I = [0, 2\pi]$, is motivated by the identification between $I^\wedge = \mathbb{R}_+ \times I$ and $\overline{\mathbb{R}}^2_+ \setminus \{0\}$ for $\overline{\mathbb{R}}^2_+ = \{\tilde{x} = (x_{n-1}, x_n) \in \mathbb{R}^2,\ x_n \geq 0\}$, where $\partial\overline{\mathbb{R}}^2_+ = \mathbb{R}_- \cup \mathbb{R}_+$ for $\mathbb{R}_\mp = \{x_{n-1} \in \mathbb{R}, x_{n-1} \lesseqgtr 0\}$.

$\sigma_\wedge(\mathcal{A})$ is a family of boundary value problems on the infinite stretched cone $I^\wedge$. For every fixed $\zeta \neq 0$ it has a conormal symbol from the cone algebra of boundary value problems (which is independent of ζ), namely,

$$\sigma_M\sigma_\wedge(\mathcal{A})(w) = \begin{pmatrix} \sigma_M\sigma_\wedge(\Delta) \\ \sigma_M\sigma_\wedge(T_-) \\ \sigma_M\sigma_\wedge(T_+) \end{pmatrix}(w) : H^s(\operatorname{int} I) \to \begin{matrix} H^{s-2}(\operatorname{int} I) \\ \oplus \\ \mathbb{C} \oplus \mathbb{C} \end{matrix} \tag{28}$$

where $\sigma_M\sigma_\wedge(\Delta)(w) = w^2 + \frac{\partial^2}{\partial\phi^2}$, $\sigma_M\sigma_\wedge(T_-)u = u|_{\phi=0}$, $\sigma_M\sigma_\wedge(T_+)u = \frac{\partial}{\partial\phi}u|_{\phi=\pi}$.

Let us observe the homogeneities in $\sigma(\mathcal{A})$. For the interior symbol we have $\sigma_\psi(\mathcal{A})(\lambda\xi) = \lambda^2\sigma_\psi(\mathcal{A})(\xi)$, $\xi \neq 0$. Concerning the pair of boundary symbols the homogeneities are

$$\sigma_{\partial,-}(\mathcal{A})(\lambda\eta) = \lambda^2 \operatorname{diag}(\kappa_\lambda, \lambda^{-3/2})\sigma_{\partial,-}(\mathcal{A})(\eta)\kappa_\lambda^{-1},$$
$$\sigma_{\partial,+}(\mathcal{A})(\lambda\eta) = \lambda^2 \operatorname{diag}(\kappa_\lambda, \lambda^{-\frac{1}{2}})\sigma_{\partial,+}(\mathcal{A})(\eta)\kappa_\lambda^{-1}$$

for all $\lambda \in \mathbb{R}_+$, $\eta \neq 0$. For the edge symbol we have

$$\sigma_\wedge(\mathcal{A})(\lambda\zeta) = \lambda^2 \operatorname{diag}(\kappa_\lambda^\wedge, \lambda^{-3/2}\kappa_\lambda, \lambda^{-1/2}\kappa_\lambda)\sigma_\wedge(\mathcal{A})(\zeta)(\kappa_\lambda^\wedge)^{-1} \tag{29}$$

for all $\lambda \in \mathbb{R}$, $\zeta \neq 0$.

Theorem 1.4 *The operators* (25) *for* $\zeta \neq 0$ *form a family of Fredholm operators for all* $s > 3/2$ *and all* $\gamma \notin \mathbb{Z} + \frac{1}{2}$.

The dimensions of kernels and cokernels are independent of s *and* ζ.

Proof. The operators (25) as elements in the cone algebra on $I^\wedge$ have the following principal symbolic components

$$\sigma_\psi \sigma_\wedge(\mathcal{A}), \sigma_{\partial,\mp}\sigma_\wedge(\mathcal{A}), \sigma_M \sigma_\wedge(\mathcal{A}), \sigma_{\text{exit}}\sigma_\wedge(\mathcal{A})(\zeta), \tag{30}$$

and we verify their ellipticities which entail the Fredholm property of (25) for every $\zeta \neq 0$. By virtue of (29) the dimensions of kernel and cokernel are independent of ζ. Moreover, the independence of these dimensions of s is a general property of elliptic operators in the cone algebra. We identify $I^\wedge$ with $\mathbb{R}^2_+ \setminus \{0\} \ni (x_{n-1}, x_n)$ where the Dirichlet condition is given on $\mathbb{R}_- = \{x_{n-1} < 0\}$, the Neumann condition on $\mathbb{R}_+ = \{x_{n-1} > 0\}$. We have $\sigma_\psi\sigma_\wedge(\mathcal{A})(\xi_{n-1}, \xi_n) = -|\xi_{n-1}|^2 - |\xi_n|^2$ which is obviously elliptic. Moreover,

$$\sigma_{\partial,\mp}\sigma_\wedge(\mathcal{A})(x_{n-1}, \xi_{n-1}) = \begin{pmatrix} \sigma_{\partial,\mp}\sigma_\wedge(\Delta) \\ \sigma_{\partial,\mp}\sigma_\wedge(T_\mp) \end{pmatrix} (x_{n-1}, \xi_{n-1}) : H^s(\mathbb{R}_+) \to \begin{matrix} H^{s-2}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{matrix} \tag{31}$$

is given by

$$\begin{aligned} \sigma_{\partial,\mp}\sigma_\wedge(\Delta)(x_{n-1}, \xi_{n-1}) &= -|\xi_{n-1}|^2 + \frac{\partial^2}{\partial x_n^2} \quad \text{on} \quad x_{n-1} \lessgtr 0, \\ \sigma_{\partial,-}\sigma_\wedge(T_-)(x_{n-1}, \xi_{n-1})u(x_n) &= u(0) \quad \text{for} \quad x_{n-1} < 0, \\ \sigma_{\partial,+}\sigma_\wedge(T_+)(x_{n-1}, \xi_{n-1})u(x_n) &= \frac{\partial}{\partial x_n} u(0) \quad \text{for} \quad x_{n-1} > 0. \end{aligned}$$

The operators (31) are isomorphisms for $\xi_{n-1} \neq 0$ on $x_{n-1} < 0$ and $x_{n-1} > 0$, respectively, even up to $x_{n-1} = 0$ from the respective sides. The conormal symbol, given by (28) will be discussed below.

The exit symbolic structure consists of two components, both for the boundary symbols on $x_{n-1} < 0$ and $x_{n-1} > 0$ as well as for the interior symbol. In the interior we have the pair

$$\sigma_{\text{exit}}\sigma_\wedge(\Delta)(\zeta, \xi_{n-1}, \xi_n) = (-|\zeta|^2 - |\xi_{n-1}|^2 - |\xi_n|^2, -|\xi_{n-1}|^2 - |\xi_n|^2),$$

where the first component is the complete symbol, non-vanishing for all $(\xi_{n-1}, \xi_n) \in \mathbb{R}^2$ (including 0) (since $\zeta \neq 0$), and the second component, the homogeneous principal part of the first one, is $\neq 0$ for $(\xi_{n-1}, \xi_n) \in \mathbb{R}^2 \setminus \{0\}$. Moreover, on the minus and plus sides of the boundary we have the pairs

$$\sigma_{\text{exit}}\sigma_\wedge \begin{pmatrix} \Delta \\ T_\mp \end{pmatrix} (\zeta, \xi_{n-1}) = \left(\begin{pmatrix} -|\zeta|^2 - |\xi_{n-1}| + \frac{\partial^2}{\partial x_n^2} \\ \sigma_{\partial,\mp}\sigma_\wedge(T_\mp) \end{pmatrix}, \begin{pmatrix} |\xi_{n-1}| + \frac{\partial^2}{\partial x_n^2} \\ \sigma_{\partial,\mp}\sigma_\wedge(T_\mp) \end{pmatrix} \right),$$

where the first components are bijective as operators $H^s(\mathbb{R}_+) \to \begin{matrix} H^{s-2}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{matrix}$ for all $\xi_{n-1} \in \mathbb{R}$ (because of $\zeta \neq 0$) while the second component as the κ_λ-homogeneous principal part of the first component is bijective for all $\xi_{n-1} \in \mathbb{R} \setminus \{0\}$.

In this way we have verified the exit ellipticity of the ζ-dependent Zaremba problem $\sigma_\wedge(\mathcal{A})(\zeta)$ in the infinite half-plane $\mathbb{R}^2_+ \setminus \{0\}$ (concerning the general theory of boundary value problems on manifolds with conical exits to infinity we also refer to [12, Chapter 3]). For the ellipticity in the cone algebra it remains to check the bijectivity of the operators (28) for $w \in \mathbb{C}$, $\mathrm{Re}\, w \notin \mathbb{Z} + \frac{1}{2}$. What we know from the ellipticity of the original problem, i.e., ellipticity of Δ together with the Shapiro-Lopatinskij ellipticity of $T_{\mp}$ is that the operators (28) form a parameter-dependent elliptic family of boundary value problems on the interval I with the parameter $\mathrm{Im}\, w$. At the same time this family is holomorphic in $w \in \mathbb{C}$, and we know the bijectivity of (28) for large $|\,\mathrm{Im}\, w|$. Generalities on holomorphic Fredholm families tell us that there is a discrete set $D \subset \mathbb{C}$ such that (28) is bijective for all $w \in \mathbb{C} \setminus D$.

In the present case we have an explicit information, namely,

$$D = \{w \in \mathbb{C} : \mathrm{Re}\, w \in \mathbb{Z} + \frac{1}{2},\ \mathrm{Im}\, w = 0\}.$$

In fact, first note that for $w = 0$ the problem $\sigma_M \sigma_\wedge(\mathcal{A})(0)u = 0$ has only the trivial solution $u \equiv 0$. For $w = a + ib \neq 0$ a simple argument gives us

$$\ker(\sigma_M \sigma_\wedge(\Delta))(w) = \{c_1 e^{-b\phi} e^{ia\phi} + c_2 e^{b\phi} e^{-ia\phi} : c_1, c_2 \in \mathbb{C}\}. \tag{32}$$

Then from the boundary conditions $u|_{\phi=0} = 0$ and $\frac{\partial u}{\partial \phi}|_{\phi=\pi} = 0$ we have

$$\begin{aligned} c_1 + c_2 &= 0, \\ c_1 e^{-b\pi}(\cos a\pi + i \sin a\pi) - c_2 e^{b\pi}(\cos a\pi - i \sin a\pi) &= 0. \end{aligned}$$

Assuming $c_1 \neq 0$ (otherwise $u \equiv 0$) we obtain $(e^{-b\pi} + e^{b\pi}) \cos a\pi + i(e^{-b\pi} - e^{b\pi}) \sin a\pi = 0$.

Since $e^{-b\pi} + e^{b\pi} \neq 0$ for all $b \in \mathbb{R}$ it follows that $e^{-b\pi} - e^{b\pi} = 0$ and $\cos a\pi = 0$, i.e., $b = 0$ and $a = k + \frac{1}{2}$, $k \in \mathbb{Z}$. □

Remark 1.5 *The non-bijectivity points of $\sigma_M \sigma_\wedge(\mathcal{A})$ are simple.*

In fact, we have $\ker(\sigma_M \sigma_\wedge(\mathcal{A}))(k + \frac{1}{2}) = \{c \sin(k + \frac{1}{2})\phi : c \in \mathbb{C}\}$, and the corresponding root functions (in the terminology of [9]) at the point $k + \frac{1}{2}, k \in \mathbb{Z}$, are $c \sin(w\phi)$. It is now easy to show that $k + \frac{1}{2}, k \in \mathbb{Z}$, is a simple zero for the holomorphic function $\sigma_M \sigma_\wedge(\mathcal{A})(w) \sin(w\phi) = {}^{\mathrm{t}}(\ 0 \quad 0 \quad w \cos(w\pi)\ )$.

2 A relation to standard Sobolev spaces

2.1 Spaces on the boundary

We now construct a relation between the spaces $H^s(\mathrm{int}\, Y_\pm)$ and the weighted Sobolev spaces $\mathcal{W}^{s,s}(Y_\pm)$, where $Y_\pm$ is regarded as a manifold with edge Z.

Let us first consider the local situation with $\mathbb{R}^{n-2} \times \overline{\mathbb{R}}_+$ in place of $Y_\pm$. In this case we have $H^s(\mathbb{R}^{n-2} \times \mathbb{R}_+) = \mathcal{W}^s(\mathbb{R}^{n-2}, H^s(\mathbb{R}_+))$ for all $s \in \mathbb{R}$, cf. the relation (7).

Let $H^s_0(\overline{\mathbb{R}}_+)$ denote the subspace of all $u \in H^s(\mathbb{R})$ with $\mathrm{supp}\, u \subseteq \overline{\mathbb{R}}_+$. Then for every $s > -\frac{1}{2}$ we have a canonical isomorphism $H^s_0(\overline{\mathbb{R}}_+) = \mathcal{K}^{s,s}(\mathbb{R}_+)$, cf. [24, Theorem 2.1.39] or [12, Section 2.1.2].

For $s \notin \mathbb{N} + \frac{1}{2}$ we have isomorphisms $\begin{matrix} \mathcal{K}^{s,s}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^\sigma \end{matrix} \to H^s(\mathbb{R}_+)$ for $\sigma := [s - \frac{1}{2}] + 1$.

As usual, $[s]$ for some $s \in \mathbb{R}$ denotes the integer part of s, i.e., the maximal integer $\leq s$. Let us also write $s = [s] + \{s\}$ for the corresponding $0 \leq \{s\} < 1$.

For purposes below we choose these isomorphisms parameter-dependent with parameters $(\zeta,\lambda)\in\mathbb{R}^{n-2+l}$. Set

$$g(\zeta,\lambda)c:=[\zeta,\lambda]^{\frac{1}{2}}\sum_{j=0}^{\sigma-1}c_j([\zeta,\lambda]x_{n-1})^j\omega([\zeta,\lambda]x_{n-1}), \tag{33}$$

$c=(c_1,\dots,c_\sigma)$, for a cut-off function $\omega(t)$ on $\overline{\mathbb{R}}_+$. Moreover, let

$$b(\zeta,\lambda)u:=\Big\{[\zeta,\lambda]^{-j-\frac{1}{2}}\frac{1}{j!}\frac{\partial^j u(0)}{\partial x_{n-1}^j}\Big\}_{j=0,\dots,\sigma-1}.$$

Then we have

$$b(\zeta,\lambda)g(\zeta,\lambda)=\mathrm{id}_{\mathbb{C}^\sigma}\,. \tag{34}$$

Observe that the functions $g(\zeta,\lambda):\mathbb{C}^\sigma\to H^s(\mathbb{R}_+)$ for $s\in\mathbb{R}$ and $b(\zeta,\lambda):H^s(\mathbb{R}_+)\to\mathbb{C}^\sigma$ for $s\in\mathbb{R}$, $s>\frac{1}{2}$, represent operator-valued symbols

$$g(\zeta,\lambda)\in S^0_{\mathrm{cl}}(\mathbb{R}^{n-2+l};\mathbb{C}^\sigma,H^s(\mathbb{R}_+))\ \text{ and }\ b(\zeta,\lambda)\in S^0_{\mathrm{cl}}(\mathbb{R}^{n-2+l};H^s(\mathbb{R}_+),\mathbb{C}^\sigma),$$

respectively (the group action in $\mathbb{C}^\sigma$ is trivial, i.e., the identity for all $\lambda\in\mathbb{R}_+$).

The composition $g(\zeta,\lambda)b(\zeta,\lambda):H^s(\mathbb{R}_+)\to H^s(\mathbb{R}_+)$ is then a family of continuous projections to $\operatorname{im}g(\zeta,\lambda)\subset C_0^\infty(\overline{\mathbb{R}}_+)$. If $e:\mathcal{K}^{s,s}(\mathbb{R}_+)\to H^s(\mathbb{R}_+)$ denotes the canonical embedding, the operator $(e\quad g(\zeta,\lambda)):\begin{matrix}\mathcal{K}^{s,s}(\mathbb{R}_+)\\ \oplus\\ \mathbb{C}^\sigma\end{matrix}\to H^s(\mathbb{R}_+)$ is an isomorphism with inverse $\begin{pmatrix}1-(gb)(\zeta,\lambda)\\ b(\zeta,\lambda)\end{pmatrix}$ for every $(\zeta,\lambda)\in\mathbb{R}^{n-2+l}$, $s>\frac{1}{2}, s\notin\frac{1}{2}+\mathbb{N}$.

In the following global constructions for a manifold M with boundary we choose a collar neighbourhood of the boundary with a fixed global normal coordinate $t\in[0,1)$.

Let us set for a moment $M=Y_+$ which is a smooth compact manifold with boundary Z, and fix a system of charts $\chi_j:U_j\to\overline{\mathbb{R}}^{n-1}_+$, $j=1,\dots,L$, $\chi_j:U_j\to\mathbb{R}^{n-1}$, $j=L+1,\dots,N$, with coordinate neighbourhoods U_j on M, where we assume that $U_j\cap Z\neq\varnothing$ for $1\le j\le L$ and $U_j\cap Z=\varnothing$ for $L+1\le j\le N$. The charts χ_j for $1\le j\le L$ can (and will) be chosen in such a way that the restrictions $\chi_j:=\chi_j|_{U_j'}$ to $U_j':=U_j\cap Z$ form an atlas $\chi_j':U_j'\to\mathbb{R}^{n-2}$ on Z. In addition the transition maps $\chi_j'\circ(\chi_k')^{-1}$ will assumed to be independent of t (the normal coordinate to the boundary Z) for $0\le t\le\frac{1}{2}$, $j,k=1,\dots,L$. Choose functions $\varphi_j,\psi_j\in C_0^\infty(U_j)$ for $1\le j\le L$ such that $\sum_{j=1}^L\varphi_j=1$ in a neighbourhood of Z and $\psi_j\equiv1$ on $\operatorname{supp}\varphi_j$. Then the functions $\varphi_j':=\varphi_j|_Z\in C_0^\infty(U_j')$, $1\le j\le L$, form a partition of unity on Z subordinate to the covering $\{U_1',\dots,U_L'\}$ of Z, and $\psi_j':=\psi_j|_Z\in C_0^\infty(U_j')$ are equal to 1 on $\operatorname{supp}\varphi_j'$ for all $1\le j\le L$.

Let us form the parameter-dependent pseudo-differential operators

$$G(\lambda):=\sum_{j=1}^L\varphi_j(\chi_j')_*^{-1}\operatorname{Op}_z(g)(\lambda)\psi_j':H^s(Z,\mathbb{C}^\sigma)\to H^s(\operatorname{int}M),$$

$$B(\lambda):=\sum_{j=1}^L\varphi_j(\chi_j')_*^{-1}\operatorname{Op}_z(b)(\lambda)\psi_j:H^s(\operatorname{int}M)\to H^s(Z,\mathbb{C}^\sigma).$$

$G(\lambda)$ is a family of potential operators, $B(\lambda)$ a family of trace operators (of type σ) in the algebra of boundary value problems on M with the transmission property at the boundary Z, and $G(\lambda)B(\lambda)$ is a Green operator in that algebra (of type σ).

Let $E : \mathcal{W}^{s,s}(M) \to H^s(\operatorname{int} M)$ denote the canonical embedding. We then have the following result:

Theorem 2.1 *There is a constant $C > 0$ such that the operator*

$$\mathcal{A}(\lambda) := (E \quad G(\lambda)) : \begin{array}{c} \mathcal{W}^{s,s}(M) \\ \oplus \\ H^s(Z, \mathbb{C}^\sigma) \end{array} \to H^s(\operatorname{int} M)$$

is an isomorphism for $|\lambda| \geq C$ for every $s > \frac{1}{2}$, $s \notin \mathbb{N} + \frac{1}{2}$.

Proof. Let us form the operators

$$\mathcal{B}(\lambda) := \begin{pmatrix} 1 - (GB)(\lambda) \\ B(\lambda) \end{pmatrix} : H^s(\operatorname{int} M) \to \begin{array}{c} \mathcal{W}^{s,s}(M) \\ \oplus \\ H^s(Z, \mathbb{C}^\sigma) \end{array}. \tag{35}$$

Then, according to the rules on operators with operator-valued symbols the composition $\mathcal{A}(\lambda)\mathcal{B}(\lambda)$ has locally near Z a (parameter-dependent) symbol of the form $1 + c(z, \zeta, \lambda)$ where $c(z, \zeta, \lambda)$ is a Green symbol of order -1. A similar observation is true of the composition $\mathcal{B}(\lambda)\mathcal{A}(\lambda)$ with respect to their operator-valued symbols as (z, ζ, λ)-depending families of maps $\begin{array}{c} \mathcal{K}^{s,s}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^\sigma \end{array} \to \begin{array}{c} \mathcal{K}^{s,s}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^\sigma \end{array}$. By Leibniz inverting $1 + c(z, \zeta, \lambda)$ we obtain symbols $d(z, \zeta, \lambda)$ of order -1 such that $(1 + c(z, \zeta, \lambda)) \# (1 + d(z, \zeta, \lambda)) = 1$ modulo a symbol of order $-\infty$ in (ζ, λ) (here $\#$ denotes the Leibniz multiplication of symbols in (z, ζ)). On the level of operators we find an operator family $\mathcal{P}(\lambda)$ such that $\mathcal{A}(\lambda)\mathcal{P}(\lambda) = 1$ modulo a family of smoothing operators which is a Schwartz function in λ. This yields that $\mathcal{A}(\lambda)$ has a right inverse for large $|\lambda|$. In a similar manner we can proceed for $\mathcal{B}(\lambda)\mathcal{A}(\lambda)$, now in terms of the edge algebra on M (regarded as a manifold with edge Z, cf. [25]). Thus, $\mathcal{A}(\lambda)$ also has a left inverse for large $|\lambda|$ and hence $\mathcal{A}(\lambda)$ is invertible for $|\lambda| \geq C$ for a C sufficiently large. □

2.2 Edge spaces in the stretched domain

In order to reinterpret the operator (2) in edge Sobolev spaces we also have to establish corresponding relations between $H^s(\operatorname{int} X)$ and $\mathcal{W}^{s,s}(\mathbb{X})$. They are analogues to a result of [3] for a closed compact C^∞ manifold M with an embedded submanifold Z of codimension 2, where Z has a trivial normal bundle in M.

On M and Z we fix Riemannian metrics and assume that the metric on Z is induced by the one on M. We interpret M as a manifold W with edge Z and then have the weighted edge Sobolev spaces $\mathcal{W}^{s,\gamma}(\mathbb{W})$ on the corresponding stretched manifold $\mathbb{W}$, cf. the notation in Sections 1.1 and 1.2. The proof of the following result is formally analogous to that of Theorem 2.1:

Theorem 2.2 *For every $s \in \mathbb{R}$, $s > 1$, $s \notin \mathbb{N}$, there exists a family of isomorphisms*

$$(E \quad K(\lambda)) : \begin{array}{c} \mathcal{W}^{s,s}(\mathbb{W}) \\ \oplus \\ H^s(Z, \mathbb{C}^{N(s)}) \end{array} \to H^s(M) \tag{36}$$

for $N(s) = \sum_{\substack{\alpha \in \mathbb{N}^2 \\ |\alpha| < s-1}} 1$ for all $\lambda \in \mathbb{R}^l$.

Let $\begin{pmatrix} 1-K(\lambda)T(\lambda) \\ T(\lambda) \end{pmatrix}$ denote the family of operators inverse to (36) (the meaning of $T(\lambda)$ is analogous to that of $B(\lambda)$ in the formula (35)).

As noted before we need an analogue of such a result for $\mathbb{X}$ in place of $\mathbb{W}$, where the base of the model cone is equal to $I = I_+$ instead S^1. This requires a corresponding modification of arguments of [3] for the case of a manifold with boundary. The interval I corresponds to the angular interval $0 \leq \phi \leq \pi$ in polar coordinates (r, ϕ) in the normal plane $\mathbb{R}^2$ transversal to Z. Let us first locally model the manifold Z by $\mathbb{R}^q$ (for $q = n-2$).

Let $\tilde{x} = (x_{n-1}, x_n)$ be the coordinates in $\mathbb{R}^2$, and consider the family of trace operators

$$t^\alpha(\zeta, \lambda)u := [\zeta, \lambda]^{-1-|\alpha|} D^\alpha_{\tilde{x}} u(0), \;\; t^\alpha(\zeta, \lambda) : \mathcal{S}(\mathbb{R}^2) \to \mathbb{C}, \quad \alpha \in \mathbb{N}^2,$$

depending on the parameters $(\zeta, \lambda) \in \mathbb{R}^{q+l}$. We can apply $t^\alpha(\zeta, \lambda)$ also to $u \in H^s(\mathbb{R}^2_+)$ for $s > 1 + |\alpha|$, $\mathbb{R}^2_+ = \{\tilde{x} \in \mathbb{R}^2 : x_n > 0\}$, and we then obtain a symbol

$$t^\alpha(\zeta, \lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^{q+l}; H^s(\mathbb{R}^2_+), \mathbb{C}).$$

The group action in $H^s(\mathbb{R}^2_+)$ is defined by $(\kappa_\lambda u)(\tilde{x}) = \lambda u(\lambda \tilde{x})$, $\lambda \in \mathbb{R}_+$, and on $\mathbb{C}$ by the identity for all λ (as always when the respective space is of finite dimension).

Moreover, choose a cut-off function $\omega(\tilde{x}) \in C^\infty_0(\overline{\mathbb{R}}^2_+)$ which is equal to 1 in a neighbourhood of $\tilde{x} = 0$. Form the potential operators $k^\alpha(\zeta, \lambda)c := [\zeta, \lambda]\frac{1}{\alpha!}([\zeta, \lambda]\tilde{x})^\alpha \omega([\zeta, \lambda]\tilde{x})c$ for any $\alpha \in \mathbb{N}^2$, acting on $c \in \mathbb{C}$. Then it follows that

$$k^\alpha(\zeta, \lambda) \in S^0_{\mathrm{cl}}(\mathbb{R}^{q+l}; \mathbb{C}, H^s(\mathbb{R}^2_+))$$

for arbitrary $s \in \mathbb{R}$. For every $\alpha \in \mathbb{N}^2$ we then have

$$t^\alpha(\zeta, \lambda) k^\alpha(\zeta, \lambda) = \mathrm{id}_{\mathbb{C}}, \tag{37}$$

$(\zeta, \lambda) \in \mathbb{R}^{q+l}$. Let us set $H^s_0(\mathbb{R}^2_+) := \{u \in H^s(\mathbb{R}^2_+) : D^\alpha_{\tilde{x}} u(0) = 0 \quad \text{for all} \quad |\alpha| < s-1\}$ for $s > 1$. In a similar manner we define the space $H^s_0(\mathbb{R}^2)$.

In Section 1.1 we have defined the weighted spaces $\mathcal{K}^{s,\gamma}(N^\wedge)$ for any compact smooth manifold N with or without boundary. We apply this to $N = S^1$ or $N = I$ (as a subinterval of S^1) and obtain the spaces

$$\mathcal{K}^{s,\gamma}((S^1)^\wedge) \quad \text{and} \quad \mathcal{K}^{s,\gamma}(I^\wedge), \tag{38}$$

respectively. Identifying $(S^1)^\wedge$ with $\mathbb{R}^2 \setminus \{0\}$ and $I^\wedge$ with $\mathbb{R}^2_+ \setminus \{0\}$ via polar coordinates instead of (38) we also write

$$\mathcal{K}^{s,\gamma}(\mathbb{R}^2 \setminus \{0\}) \quad \text{and} \quad \mathcal{K}^{s,\gamma}(\mathbb{R}^2_+ \setminus \{0\}),$$

respectively.

Proposition 2.3 *For every $s \geq 0$, $s \notin \mathbb{N}$ we have canonical isomorphisms*

$$\mathcal{K}^{s,s}(\mathbb{R}^2_+ \setminus \{0\}) = H^s_0(\mathbb{R}^2_+), \quad \mathcal{K}^{s,s}(\mathbb{R}^2 \setminus \{0\}) = H^s_0(\mathbb{R}^2).$$

This has the consequence that $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,s}(\mathbb{R}^2_+ \setminus \{0\})) = \mathcal{W}^s(\mathbb{R}^q, H^s_0(\mathbb{R}^2_+))$ and a similar relation for the spaces for $\mathbb{R}^2$ instead of $\mathbb{R}^2_+$. Note that

$$\mathcal{W}^s(\mathbb{R}^q, H^s_0(\mathbb{R}^2_+)) = \{u(z, \tilde{x}) \in H^s(\mathbb{R}^{q+2}) : D^\alpha_{\tilde{x}} u(z, 0) = 0 \quad \text{for all} \quad |\alpha| < s-1\}. \tag{39}$$

Let us form the vectors of symbols

$$t(\zeta,\lambda) := {}^{\mathrm{t}}(t^{\alpha}(\zeta,\lambda) : |\alpha| < s-1) \in S^0_{\mathrm{cl}}(\mathbb{R}^{q+l}; H^s(\mathbb{R}^2_+), \mathbb{C}^{N(s)}),$$
$$k(\zeta,\lambda) := (k^{\alpha}(\eta,\lambda) : |\alpha| < s-1) \in S^0_{\mathrm{cl}}(\mathbb{R}^{q+l}; \mathbb{C}^{N(s)}, H^s(\mathbb{R}^2_+)),$$

where $N(s)$ is given in Theorem 2.2.

From (37) it follows that $t(\zeta,\lambda)k(\zeta,\lambda) = \mathrm{id}_{\mathbb{C}^{N(s)}}$ for all $(\zeta,\lambda) \in \mathbb{R}^{q+l}$, while $1-k(\zeta,\lambda)t(\zeta,\lambda) : H^s(\mathbb{R}^2_+) \to H^s(\mathbb{R}^2_+)$ is a family of continuous projections to $H^s_0(\mathbb{R}^2_+)$. We now pass to parameter-dependent operators

$$T(\lambda) := \mathrm{Op}_z(t)(\lambda) : \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^2_+)) \to H^s(\mathbb{R}^q, \mathbb{C}^{N(s)}),$$
$$K(\lambda) := \mathrm{Op}_z(k)(\lambda) : H^s(\mathbb{R}^q, \mathbb{C}^{N(s)}) \to \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^2_+)).$$

The continuities are a special case of (13) where we use $\mathcal{W}^s(\mathbb{R}^q, \mathbb{C}^k) = H^s(\mathbb{R}^q, \mathbb{C}^k)$ for any k. Observe that $\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^2_+)) = H^s(\mathbb{R}^n_+)$ for $q = n-2$, $\mathbb{R}^n_+ = \{(z,\tilde{x}) \in \mathbb{R}^n : x_n > 0\}$, cf. similarly, the relations (7).

Proposition 2.4 *Let $E : \mathcal{W}^s(\mathbb{R}^q, H^s_0(\mathbb{R}^2_+)) \to \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^2_+))$ be the canonical embedding. Then*

$$(E \quad K(\lambda)) : \begin{array}{c} \mathcal{W}^s(\mathbb{R}^q, H^s_0(\mathbb{R}^2_+)) \\ \oplus \\ H^s(\mathbb{R}^q, \mathbb{C}^{N(s)}) \end{array} \to \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^2_+))$$

is an isomorphism for every $\lambda \in \mathbb{R}^l$ and has the inverse ${}^{\mathrm{t}}(1-K(\lambda)T(\lambda) \quad T(\lambda))$.

Proof. If $a(\zeta,\lambda) \in S^{\mu}(\mathbb{R}^{q+l}; E, \widetilde{E})$ is a symbol which is invertible for all $(\zeta,\lambda) \in \mathbb{R}^{q+l}$ such that $a^{-1}(\zeta,\lambda) \in S^{-\mu}(\mathbb{R}^{q+l}; \widetilde{E}, E)$, then the family of pseudo-differential operators $\mathrm{Op}(a)(\lambda) : \mathcal{W}^s(\mathbb{R}^q, E) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{E})$ is invertible for all $\lambda \in \mathbb{R}^l$, and we have $\mathrm{Op}(a)^{-1}(\lambda) = \mathrm{Op}(a^{-1})(\lambda)$. This holds for all $s \in \mathbb{R}$.

We apply this to $\mu = 0$ for the case $a(\zeta,\lambda) := (e \quad k(\zeta,\lambda)) : \begin{array}{c} H^s_0(\mathbb{R}^2_+) \\ \oplus \\ \mathbb{C}^{N(s)} \end{array} \to H^s(\mathbb{R}^2_+)$ where $e : H^s_0(\mathbb{R}^2_+) \to H^s(\mathbb{R}^2_+)$ is the canonical embedding, and $a^{-1}(\zeta,\lambda) = {}^{\mathrm{t}}(1-k(\zeta,\lambda)t(\zeta,\lambda) \quad t(\zeta,\lambda))$. □

Theorem 2.5 *For every $s > 1$, $s \notin \mathbb{N}$, there exists a family of isomorphisms*

$$(E \quad K(\lambda)) : \begin{array}{c} \mathcal{W}^{s,s}(\mathbb{X}) \\ \oplus \\ H^s(Z, \mathbb{C}^{N(s)}) \end{array} \to H^s(\mathrm{int}\, X)$$

for all $\lambda \in \mathbb{R}^l$, $N(s) = \sum_{\substack{\alpha \in \mathbb{N}^2 \\ |\alpha| < s-1}} 1$.

Remark 2.6 *By virtue of a global version of (39) on X the space $\mathcal{W}^{s,s}(\mathbb{X})$ for $s \geq 0$, $s-1 \notin \mathbb{N}$ can be identified with the subspace $H^s_0(X)$ of all $u \in H^s(\mathrm{int}\, X)$ such that $Bu|_Z = 0$ for every differential operator B of order $< s-1$.*

Then, if $A : H^s(\mathrm{int}\, X) \to H^{s-m}(\mathrm{int}\, X)$ is a differential operator of order m with smooth coefficients, A also induces (by restriction) a continuous map

$$A_0 : H^s_0(\mathrm{int}\, X) \to H^{s-m}_0(\mathrm{int}\, X)$$

for $s \geq m$. Let $(E_s \quad K_s)$ denote the operator of Theorem 2.5 for any fixed $s \in \mathbb{R}$ and $\lambda \in \mathbb{R}^l$, and denote by ${}^{\mathrm{t}}(P_s \quad T_s)$ its inverse. Then we have the identification $A_0 = P_{s-m} A E_s$, where $E_s : H^s_0(\mathrm{int}\, X) \to H^s(\mathrm{int}\, X)$ is the canonical embedding and $P_{s-m} : H^s(\mathrm{int}\, X) \to H^s_0(\mathrm{int}\, X)$ a projection.

2.3 A reformulation of mixed problems from standard Sobolev spaces

In [10] we completed the operator (2) by additional potential operators $L_\pm$ to a Fredholm operator of index 0

$$\mathcal{Z} := \begin{pmatrix} \Delta & 0 \\ T_- & L_- \\ T_+ & L_+ \end{pmatrix} : \begin{matrix} H^s(\mathrm{int}\,X) \\ \oplus \\ H^s(Z,\mathbb{C}^l) \end{matrix} \to \begin{matrix} H^{s-2}(\mathrm{int}\,X) \\ \oplus \\ H^{s-\frac{1}{2}}(\mathrm{int}\,Y_-), \\ \oplus \\ H^{s-\frac{3}{2}}(\mathrm{int}\,Y_+) \end{matrix} \tag{40}$$

$s > \frac{3}{2}, s \notin \mathbb{N} + \frac{1}{2}$, where $l = [s - \frac{1}{2}]$. From Theorem 2.1 applied to $M = Y_\pm$ we have isomorphisms

$$(E_\pm \quad G_\pm) : \begin{matrix} \mathcal{W}^{s,s}(Y_\pm) \\ \oplus \\ H^s(Z,\mathbb{C}^{\sigma(s)}) \end{matrix} \to H^s(\mathrm{int}\,Y_\pm) \tag{41}$$

for $s > \frac{1}{2}$, $s \notin \mathbb{N} + \frac{1}{2}$, where $G_\pm$ is obtained from $G(\lambda)$ by fixing λ sufficiently large, $\sigma(s) = [s - \frac{1}{2}] + 1$; here $E_\pm : \mathcal{W}^{s,s}(Y_\pm) \to H^s(\mathrm{int}\,Y_\pm)$ are the respective embeddings.

Moreover, Theorem 2.5 gives us isomorphisms

$$(E \quad K) : \begin{matrix} \mathcal{W}^{s,s}(\mathbb{X}) \\ \oplus \\ H^s(Z,\mathbb{C}^{N(s)}) \end{matrix} \to H^s(\mathrm{int}\,X) \tag{42}$$

for every $s > 1$, $s \notin \mathbb{N}$ when we set $K = K(\lambda)$ for any fixed λ.

If we want to observe the smoothness s in the operators (41) and (42) we also write $(E_{\pm,s} \quad G_{\pm,s})$ and $(E_s \quad K_s)$, respectively.

Let us set $\begin{pmatrix} R_{\pm,s} \\ B_{\pm,s} \end{pmatrix} = (E_{\pm,s} \quad G_{\pm,s})^{-1}$ and $\begin{pmatrix} P_s \\ T_s \end{pmatrix} = (E_s \quad K_s)^{-1}$. From (42) we obtain an isomorphism

$$\mathcal{K} := \begin{pmatrix} E_s & K_s & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix} : \begin{matrix} \mathcal{W}^{s,s}(\mathbb{X}) \\ \oplus \\ H^s(Z,\mathbb{C}^{N(s)}) \\ \oplus \\ H^s(Z,\mathbb{C}^{[s-\frac{1}{2}]}) \end{matrix} \to \begin{matrix} H^s(\mathrm{int}\,X) \\ \oplus \\ H^s(Z,\mathbb{C}^{[s-\frac{1}{2}]}) \end{matrix}$$

for any fixed $s > 1$, $s \notin \mathbb{N}$. Moreover, (42) together with (41) gives us an isomorphism

$$\mathcal{L} := \begin{pmatrix} P_{s-2} & 0 & 0 \\ T_{s-2} & 0 & 0 \\ 0 & R_{-,s-\frac{1}{2}} & 0 \\ 0 & B_{-,s-\frac{1}{2}} & 0 \\ 0 & 0 & R_{+,s-\frac{3}{2}} \\ 0 & 0 & B_{+,s-\frac{3}{2}} \end{pmatrix} : \begin{matrix} H^{s-2}(\mathrm{int}\,X) \\ \oplus \\ H^{s-\frac{1}{2}}(\mathrm{int}\,Y_-) \\ \oplus \\ H^{s-\frac{3}{2}}(\mathrm{int}\,Y_+) \end{matrix} \to \begin{matrix} \mathcal{W}^{s-2,s-2}(\mathbb{X}) \\ \oplus \\ H^{s-2}(Z,\mathbb{C}^{N(s-2)}) \\ \oplus \\ \mathcal{W}^{s-\frac{1}{2},s-\frac{1}{2}}(Y_-) \\ \oplus \\ H^{s-\frac{1}{2}}(Z,\mathbb{C}^{\sigma(s-\frac{1}{2})}) \\ \oplus \\ \mathcal{W}^{s-\frac{3}{2},s-\frac{3}{2}}(Y_+) \\ \oplus \\ H^{s-\frac{3}{2}}(Z,\mathbb{C}^{\sigma(s-\frac{3}{2})}) \end{matrix}$$

for all $s > 3$, $s \notin \mathbb{N}$. This allows us to transform (40) to a Fredholm operator $\mathcal{LZK}$ for $s > 3, s \notin \mathbb{N} \cup \{\mathbb{N} + \frac{1}{2}\}$ between corresponding edge Sobolev spaces plus standard Sobolev spaces on the interface Z.

After an appropriate change of rows and columns in the block matrix $\mathcal{LZK}$ we obtain an equivalent operator

$$\widetilde{\mathcal{A}}(s) : \begin{array}{c} \mathcal{W}^{s,s}(\mathbb{X}) \\ \oplus \\ H^s(Z, \mathbb{C}^d) \end{array} \to \begin{array}{c} \mathcal{W}^{s-2,s-2}(\mathbb{X}) \\ \oplus \\ \mathcal{W}^{s-\frac{1}{2},s-\frac{1}{2}}(Y_-) \\ \oplus \\ \mathcal{W}^{s-\frac{3}{2},s-\frac{3}{2}}(Y_+) \\ \oplus \\ \bigoplus_{j=1}^{3} H^{s_j}(Z, \mathbb{C}^{d_j}) \end{array} \tag{43}$$

for $s > 3, s \notin \mathbb{N} \cup \{\mathbb{N} + \frac{1}{2}\}$ and $d = N(s) + [s - \frac{1}{2}]$, $s_1 = s - 2$, $s_2 = s - \frac{1}{2}$, $s_3 = s - \frac{3}{2}$, $d_1 = N(s-2)$, $d_2 = \sigma(s - \frac{1}{2})$, $d_3 = \sigma(s - \frac{3}{2})$. By construction we have

$$\operatorname{ind} \mathcal{Z} = \operatorname{ind} \widetilde{\mathcal{A}}(s) = 0.$$

Remark 2.7 *Similarly as in Remark* 2.6 *the* 3×1 *upper left corner of* $\tilde{\mathcal{A}}(s)$ *is nothing other but the restriction of* (2) *to* $\mathcal{W}^{s,s}(\mathbb{X})$ *regarded as a subspace of* $H^s(\operatorname{int} X)$.

Proposition 2.8 *The operator* (43) *for any fixed* $s \in \mathbb{R}$, $s > 3$, $s \notin \mathbb{N} \cup \{\mathbb{N} + \frac{1}{2}\}$ *is an element of the edge operator algebra on* $\mathbb{X}$ (*with edge* Z) *in the sense of Definition* 4.6 *below, belonging to the weight* $\gamma = s$ *and with the boundary orders* $\mu_- = 0$, $\mu_+ = 1$.

Proof. The 3×1 upper left corner of the operator $\tilde{\mathcal{A}}(s)$ belongs to the edge algebra, cf. Example 1.3 and Remark 2.7. The lower right corner which defines a map $H^s(Z, \mathbb{C}^d) \to \oplus_{j=1}^{3} H^{s_j}(Z, \mathbb{C}^{d_j})$ is obviously a matrix of classical pseudo-differential operators on Z. The remaining entries consist of operators

$$\begin{pmatrix} R_{-,s-\frac{1}{2}} L_- & R_{-,s-\frac{1}{2}} T_- K_s \end{pmatrix} : \begin{array}{c} H^s(Z, \mathbb{C}^{[s-\frac{1}{2}]}) \\ \oplus \\ H^s(Z, \mathbb{C}^{N(s)}) \end{array} \to \mathcal{W}^{s-\frac{1}{2},s-\frac{1}{2}}(Y_-),$$

$$\begin{pmatrix} R_{+,s-\frac{3}{2}} L_+ & R_{+,s-\frac{3}{2}} T_+ K_s \end{pmatrix} : \begin{array}{c} H^s(Z, \mathbb{C}^{[s-\frac{1}{2}]}) \\ \oplus \\ H^s(Z, \mathbb{C}^{N(s)}) \end{array} \to \mathcal{W}^{s-\frac{3}{2},s-\frac{3}{2}}(Y_+).$$

Let us characterise the operators for the plus side (those on the minus side are analogous).

From the constructions of [10] it follows that the operator L_+ has the form

$$L_+ = Q_+ C_+ : L^2(Z) \to H^{s-\frac{3}{2}}(\operatorname{int} Y_+),$$

where $C_+ : L^2(Z) \to L^2(Y_+)$ is a potential operator from the calculus of pseudo-differential boundary value problems on Y_+ without the transmission property at Z, see also [24] or [26], and Q_+ is a reduction of orders in that calculus of order $-s + \frac{3}{2}$, see [11].

As is known from [26] all these operators belong to the edge calculus (here to the substructure of boundary value problems without the transmission property). The Green, potential etc. operators are formulated in the category of $\mathcal{S}_\varepsilon^\gamma(\mathbb{R}_+)$-spaces, cf. the notation in Section 1.2. Since the compositions in question belong again to the edge calculus, we immediately obtain the desired characterisation. The composition $R_{+,s-\frac{3}{2}} L_+$ is a potential operator of order $\frac{3}{2}$, while $B_{+,s-\frac{3}{2}} L_+$ is a $\sigma(s-\frac{3}{2}) \times [s-\frac{1}{2}]$ matrix of classical pseudo-differential operators on Z of order $\frac{3}{2}$. □

Let us now consider the principal edge symbolic structure of the operator $\widetilde{\mathcal{A}}(s)$ with its subordinate conormal symbol. According to the general notation the principal edge symbol consists of an operator family

$$\sigma_\wedge(\widetilde{\mathcal{A}}(s))(z,\zeta):\quad \begin{array}{c}\mathcal{K}^{s,s}(\mathbb{R}^2_+\setminus\{0\})\\ \oplus\\ \mathbb{C}^d\end{array} \quad\to\quad \begin{array}{c}\mathcal{K}^{s-2,s-2}(\mathbb{R}^2_+\setminus\{0\})\\ \oplus\\ \mathcal{K}^{s-\frac{1}{2},s-\frac{1}{2}}(\mathbb{R}_-)\\ \oplus\\ \mathcal{K}^{s-\frac{3}{2},s-\frac{3}{2}}(\mathbb{R}_+)\\ \oplus\\ \mathbb{C}^{d_1+d_2+d_3}\end{array} \tag{44}$$

parametrised by $(z,\zeta)\in T^*Z\setminus 0$ with a corresponding scheme of DN homogeneities. Because of the Fredholm property of (43) the operator function (44) is bijective for every (z,ζ). That means that the upper left 3×1 corner

$$\sigma_\wedge(\mathcal{A}(s))(z,\zeta):\mathcal{K}^{s,s}(\mathbb{R}^2_+\setminus\{0\})\to \begin{array}{c}\mathcal{K}^{s-2,s-2}(\mathbb{R}^2_+\setminus\{0\})\\ \oplus\\ \mathcal{K}^{s-\frac{1}{2},s-\frac{1}{2}}(\mathbb{R}_-)\\ \oplus\\ \mathcal{K}^{s-\frac{3}{2},s-\frac{3}{2}}(\mathbb{R}_+)\end{array} \tag{45}$$

is a family of Fredholm operators for any fixed $s>3$, $s\notin\mathbb{N}\cup\{\mathbb{N}+\frac{1}{2}\}$. Since (44) is bijective we have

$$\operatorname{ind}\sigma_\wedge(\mathcal{A}(s))(z,\zeta)=d_1+d_2+d_3-d=N(s-2)+\sigma(s-\frac{1}{2})+\sigma(s-\frac{3}{2})-N(s)-[s-\frac{1}{2}].$$

Using the relations $N(s)=\frac{1}{2}\{[s]^2+[s]\}$ and $\sigma(s)=[s-\frac{1}{2}]+1$ it follows that

$$\operatorname{ind}\sigma_\wedge(\mathcal{A}(s))(z,\zeta)=-[s-\frac{1}{2}]. \tag{46}$$

3 Elliptic interface conditions

3.1 Mixed problems in spaces of arbitrary weights

As noted in the introduction the discussion of solvability of the mixed problem (2) in standard Sobolev spaces rules out 'most of the interesting' solutions when we prescribe independent boundary data $g_\pm$ on $Y_\pm$. On the other hand, because of the elliptic regularity for boundary value problems (in this case with the transmission property at $\operatorname{int}Y_\pm$) solutions with independently given boundary data in $H^{s-\frac{1}{2}}_{\mathrm{loc}}(\operatorname{int}Y_-)$ and $H^{s-\frac{3}{2}}_{\mathrm{loc}}(\operatorname{int}Y_+)$ belong to $H^s_{\mathrm{loc}}(X\setminus Z)$ regardless of any possible jump of solutions close to Z. The role of weighted edge Sobolev spaces $\mathcal{W}^{s,\gamma}(\mathbb{X})\subset H^s_{\mathrm{loc}}(X\setminus Z)$ is to reflect the standard elliptic regularity outside Z and to admit adequate discontinuities near Z. For large s the regularity in $\mathcal{W}^{s,s}(\mathbb{X})$ is not really different from that in $H^s(\operatorname{int}X)$ as we saw in Section 2.3, but the 'realistic' situation corresponds to weights $\gamma<s$. In other words the main task will be to pass from the $\mathcal{W}^{s,s}(\mathbb{X})$-case to $\mathcal{W}^{s,\gamma}(\mathbb{X})$ for small weights γ. This is just the program of the present section.

From Example 1.3 we know that the operator $\mathcal{A}={}^{\mathrm{t}}(\ \Delta\quad T_-\quad T_+\ )$ induces an operator $\mathcal{A}(\gamma)$ in the edge algebra on $\mathbb{X}$ for an arbitrary weight $\gamma\in\mathbb{R}$. For the ellipticity it is important to identify those γ such that the associated principal edge symbol

$$\sigma_\wedge(\mathcal{A}(\gamma))(z,\zeta) : \mathcal{K}^{s,\gamma}(\mathbb{R}^2_+ \setminus \{0\}) \to \begin{array}{c} \mathcal{K}^{s-2,\gamma-2}(\mathbb{R}^2_+ \setminus \{0\}) \\ \oplus \\ \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_-) \\ \oplus \\ \mathcal{K}^{s-\frac{3}{2},\gamma-\frac{3}{2}}(\mathbb{R}_+) \end{array} \tag{47}$$

represents a family of Fredholm operators (parametrised by $(z,\zeta) \in T^*Z \setminus 0$; in the present case (47) is independent of z).

The operators (47) belong to the cone algebra on the infinite cone $\mathbb{R}^2_+ \setminus \{0\}$ with corresponding mixed Dirichlet and Neumann conditions on $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively. From the cone algebra it is known that the Fredholm property of an operator in $\mathcal{K}^{s,\gamma}$-spaces is equivalent to the bijectivity of all its principal symbols, cf. the formula (30).

The components $\sigma_\psi\sigma_\wedge(\mathcal{A}(\gamma))$, $\sigma_{\partial,\pm}\sigma_\wedge(\mathcal{A}(\gamma))$ and $\sigma_{\text{exit}}\sigma_\wedge(\mathcal{A}(\gamma))(\zeta)$ are independent of the weight γ. For $\gamma = s$ their bijectivities have been checked in the proof of Theorem 1.4. In other words, it remains to recall the properties of the conormal symbol and to identify those $w \in \mathbb{C}$ such that the operators (28) are bijective. This has been answered by Theorem 1.4.

Theorem 3.1 *For every $s > \frac{3}{2}$ and $\gamma \notin \mathbb{Z} + \frac{1}{2}$ there are dimensions $d(\gamma)$, $e(\gamma)$ such that the operator*

$$\mathcal{A}(\gamma) : \mathcal{W}^{s,\gamma}(\mathbb{X}) \to \begin{array}{c} \mathcal{W}^{s-2,\gamma-2}(\mathbb{X}) \\ \oplus \\ \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(Y_-) \\ \oplus \\ \mathcal{W}^{s-\frac{3}{2},\gamma-\frac{3}{2}}(Y_+) \end{array} =: \mathcal{W}^{s-2,\gamma-2}(\mathbb{X}; Y_-, Y_+) \tag{48}$$

can be completed by extra conditions $\mathcal{K}, \mathcal{T}$ and $\mathcal{Q}$ with respect to the interface Z to an elliptic operator in the edge algebra

$$\widetilde{\mathcal{A}}(\gamma) := \begin{pmatrix} \mathcal{A}(\gamma) & \mathcal{K}(\gamma) \\ \mathcal{T}(\gamma) & \mathcal{Q}(\gamma) \end{pmatrix} : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{X}) \\ \oplus \\ H^s(Z, \mathbb{C}^{d(\gamma)}) \end{array} \to \begin{array}{c} \mathcal{W}^{s-2,\gamma-2}(\mathbb{X}; Y_-, Y_+) \\ \oplus \\ H^{s-2}(Z, \mathbb{C}^{e(\gamma)}) \end{array}. \tag{49}$$

Proof. The existence of elliptic interface conditions depends on some topological property of $\sigma_\wedge(\mathcal{A}(\gamma))$. In this proof we interpret $\sigma_\wedge(\mathcal{A}(\gamma))(z,\zeta)$ as a family of Fredholm operators parametrised by the points (z,ζ) of the unit cosphere bundle S^*Z induced by T^*Z. The criterion is an analogue of the Atiyah-Bott condition for the existence of Shapiro-Lopatinskij elliptic boundary value problems, cf. [1]. An analogous condition for the existence of elliptic edge conditions in the edge algebra (for a closed base of the model cone) is obtained in [23], see also the papers [27] or [14]. In the present case the base $I = [0,\pi]$ of the model cone has a boundary, but the situation is very similar.

First recall that a continuous family $F : X \to \mathcal{F}(H, \widetilde{H})$ of Fredholm operators between Hilbert spaces H and $\widetilde{H}$, where X is a compact topological space (for simplicity, arcwise connected), generates an index element $\operatorname{ind}_X F \in K(X)$ in the K-group of X. If $\dim\ker F(x)$ and $\dim\operatorname{coker} F(x)$ are constant the families $\ker_X F$ and $\operatorname{coker}_X F$ of kernels and cokernels, respectively, are (continuous complex) vector bundles on X, and we have $\operatorname{ind}_X F = [\ker_X F] - [\operatorname{coker}_X F]$, where $[\ldots]$ denotes the class in $K(X)$ represented by the bundle in the brackets. If the dimensions are not constant it suffices to pass from F to a surjective operator family

$$(F \quad C) : \begin{matrix} H \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \to \widetilde{H}$$ for a suitable constant map $C : \mathbb{C}^{N_-} \to \widetilde{H}$ for a sufficiently large choice of N_-. Then we can set

$$\operatorname{ind}_X F := [\ker(F \quad C)] - [N_-]$$

where N_- stands for the trivial bundle on X with fibre N_- (it is well known that this construction does not depend on the choice of N_- or C).

In our case we have $X = S^*Z$ and $F = \sigma_\wedge(\mathcal{A}(\gamma))$. The condition for the existence of 'Shapiro-Lopatinskij' elliptic interface conditions is now

$$\operatorname{ind}_{S^*Z} \sigma_\wedge(\mathcal{A}(\gamma)) \in \pi_1^* K(Z), \tag{50}$$

where $\pi_1^* : K(Z) \to K(S^*Z)$ is the pull back of the corresponding K-groups under the canonical projection $\pi_1 : S^*Z \to Z$ (induced by the bundle pull back). More precisely, the relation (50) is necessary and sufficient. For $s = \gamma > \frac{3}{2}$, $s \notin \mathbb{N} \cup \{\mathbb{N} + \frac{1}{2}\}$, the operator (43) as an element of the edge algebra (cf. Proposition 2.8) is Fredholm and hence elliptic (different orders of smoothness in the Sobolev spaces on Z do not affect this; we can always unify the orders by composing our operators by suitable elliptic order reducing pseudo-differential operators on Z). Thus the property (50) holds for $\sigma_\wedge(\mathcal{A}(s))$. To discuss arbitrary weights γ we recall that (47) is a family of elliptic operators in the cone algebra of boundary value problems on the infinite cone $\overline{\mathbb{R}}_+^2 \setminus \{0\}$. In this situation the property (50) is independent of γ as soon as (47) is Fredholm for different γ. The technique to prove this is similar to that in boundary value problems without the transmission property, cf. [24], or [28]. An inspection of the details shows immediately that the ideas also apply in the present situation. In other words, (50) is satisfied for all $\gamma \notin \mathbb{Z} + \frac{1}{2}$. This completes the proof. □

3.2 Construction of elliptic interface conditions

Our next objective is to obtain more information on the numbers $d(\gamma)$ and $e(\gamma)$ of extra interface conditions in the sense of Theorem 3.1.

Theorem 3.2 *Let $s > 3, s \notin \mathbb{N} \cup \{\mathbb{N} + \frac{1}{2}\}, \gamma < s$ and $\gamma \notin \mathbb{Z} + \frac{1}{2}$, and let $n(s,\gamma)$ denote the number of non-bijectivity points of $\sigma_M \sigma_\wedge(\mathcal{A})(w)$ in the strip*

$$\{w \in \mathbb{C} : 1 - s < \operatorname{Re} w < 1 - \gamma\}.$$

Then we have

$$\operatorname{ind} \sigma_\wedge(\mathcal{A}(\gamma)) = \operatorname{ind} \sigma_\wedge(\mathcal{A}(s)) + n(s,\gamma)$$

(*which is independent of s*).

Proof. The assertion of this theorem can be interpreted as a relative index result on boundary value problems in an infinite cone when the weight $\beta = s$ is replaced by γ. The strategy of the proof is based on index formulas of boundary value problems B_β and B_γ on the infinite cone which are of Fuchs type both with respect to $r = 0$ and $r = \infty$, operating in weighted Sobolev spaces with weight at zero and infinity, cf. Gohberg and Sigal [7], and the applications of [29], [9]. However, the original operators (for convenience composed with suitable powers of r) act in other scales of spaces; we denote them by K_β and K_γ, respectively, cf. the formula (51) below. We therefore compare $\operatorname{ind} B_\gamma - \operatorname{ind} B_\beta$ with $\operatorname{ind} K_\gamma - \operatorname{ind} K_\beta$. The details are as follows:

For abbreviation let $\mathcal{X} := H^t(\operatorname{int} I)$, $\widetilde{\mathcal{X}} := \begin{array}{c} H^{t-2}(\operatorname{int} I) \\ \oplus \\ \mathbb{C} \oplus \mathbb{C} \end{array}$ for any fixed $t > \frac{3}{2}$ and $a(w) := \sigma_M \sigma_\wedge(\mathcal{A})(w)$, cf. the formula (28). Then $a(w)$ is a holomorphic family of Fredholm operators $\mathcal{X} \to \widetilde{\mathcal{X}}$ with $\mathbb{Z} + \frac{1}{2}$ as the set of points where $a(w)$ is not bijective. Since the index of a continuous Fredholm function is constant, we have $\operatorname{ind} a(w) = 0$ for all $w \in \mathbb{C}$. From Section 1.3 it follows that $\dim \ker a(w) = \dim \operatorname{coker} a(w) = 1$ for all $w \in \mathbb{Z} + \frac{1}{2}$. These properties are independent of t.

Let us fix $\zeta \neq 0$, $t > 3$, and write

$$K_\gamma := \operatorname{diag}(r^2, r^{\frac{1}{2}}, r^{\frac{3}{2}}) \sigma_\wedge(\mathcal{A}(\gamma))(\zeta). \tag{51}$$

This induces a Fredholm operator

$$K_\gamma : \mathcal{K}^{t,\gamma} \to \tilde{\mathcal{K}}^{t-2,\gamma}$$

for the spaces $\mathcal{K}^{t,\gamma} := \mathcal{K}^{t,\gamma}(\mathbb{R}^2_+ \setminus \{0\})$ and

$$\tilde{\mathcal{K}}^{t-2,\gamma} := r^2 \mathcal{K}^{t-2,\gamma-2}(\mathbb{R}^2_+ \setminus \{0\}) \oplus r^{\frac{1}{2}} \mathcal{K}^{t-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_-) \oplus r^{\frac{3}{2}} \mathcal{K}^{t-\frac{3}{2},\gamma-\frac{3}{2}}(\mathbb{R}_+).$$

We then have

$$\operatorname{ind} \sigma_\wedge(\mathcal{A}(\gamma))(\zeta) = \operatorname{ind} K_\gamma.$$

By definition the operator K_γ has the form $K_\gamma = \operatorname{op}_M^{\gamma-1}(a) + \begin{pmatrix} -r^2|\zeta|^2 \\ 0 \\ 0 \end{pmatrix}$ with the Mellin symbol $a(w) = \begin{pmatrix} w^2 + \frac{\partial^2}{\partial\phi^2} \\ \mathrm{r}_{\{\phi=0\}} \\ \mathrm{r}_{\{\phi=\pi\}} \frac{\partial}{\partial\phi} \end{pmatrix}$, where $\mathrm{r}_{\{\phi=\alpha\}}$ denotes the restriction operator to $\phi = \alpha, \alpha = 0, \pi$.

Let us now set $B_\gamma := \operatorname{op}_M^{\gamma-1}(a) + \omega(r) \begin{pmatrix} -r^2|\zeta|^2 \\ 0 \\ 0 \end{pmatrix}$ for some cut-off function ω. Then

$$B_\gamma : \mathcal{H}^{t,(\gamma,\delta)} \to \tilde{\mathcal{H}}^{t-2,(\gamma,\delta)}$$

is continuous when we set $\mathcal{H}^{t,(\gamma,\delta)} := \omega_1 \mathcal{H}^{t,\gamma}(\mathbb{R}^2_+ \setminus \{0\}) + (1-\omega_1)\mathcal{H}^{t,\delta}(\mathbb{R}^2_+ \setminus \{0\})$ and

$$\tilde{\mathcal{H}}^{t-2,(\gamma,\delta)} := \omega_1 \begin{pmatrix} \mathcal{H}^{t-2,\gamma}(\mathbb{R}^2_+ \setminus \{0\}) \\ \oplus \\ \mathcal{H}^{t-\frac{1}{2},\gamma}(\mathbb{R}_-) \\ \oplus \\ \mathcal{H}^{t-\frac{3}{2},\gamma}(\mathbb{R}_+) \end{pmatrix} + (1-\omega_1) \begin{pmatrix} \mathcal{H}^{t-2,\delta}(\mathbb{R}^2_+ \setminus \{0\}) \\ \oplus \\ \mathcal{H}^{t-\frac{1}{2},\delta}(\mathbb{R}_-) \\ \oplus \\ \mathcal{H}^{t-\frac{3}{2},\delta}(\mathbb{R}_+) \end{pmatrix}$$

for every $\delta \geq \gamma$ and any cut-off function ω_1 (the choice of the cut-off function does not affect the spaces). The second summand in the expression for B_γ is a compact operator in these spaces. Therefore, ellipticity and Fredholm property are determined by $a(w)$ alone. By assumption $a(w)$ has no non-bijectivity points on the weight line $\Gamma_{1-\gamma}$ (in the sense of the weight shift corrections below, cf. the formula (78)). Since the non-bijectivity points form a discrete set, here the real half-integers, we can choose δ in such a way that $a(w)$ is also bijective on $\Gamma_{1-\delta}$.

In a similar manner we now form the operators

$$K_\beta : \mathcal{K}^{t,\beta} \to \tilde{\mathcal{K}}^{t-2,\beta}, \; B_\beta : \mathcal{H}^{t,(\beta,\delta)} \to \tilde{\mathcal{H}}^{t-2,(\beta,\delta)} \tag{52}$$

for another weight $\beta \in \mathbb{R}$ and $\delta \geq \max(\gamma, \beta)$, such that $\Gamma_{1-\beta}$ does not contain non-bijectivity points of $a(w)$. Then the operators (52) are also Fredholm.

We are now in a situation of Fredholm theory on manifolds as is studied by Nazaikinskij and Sternin in [15]. The operators K_γ, B_γ and K_β, B_β satisfy the compatibility conditions $K_\gamma|_{0<r<R} = B_\gamma|_{0<r<R}$, $K_\beta|_{0<r<R} = B_\beta|_{0<r<R}$ and $K_\gamma|_{R<r<\infty} = K_\beta|_{R<r<\infty}$, $B_\gamma|_{R<r<\infty} = B_\beta|_{R<r<\infty}$ for every $R > 0$ such that $\omega \equiv 1$ on $[0, R)$. Here $0 < r < R$ ($R < r < \infty$) indicates those points $(x_{n-1}, x_n) \in \overline{\mathbb{R}}^2_+ \setminus 0$ such that $0 < |x_{n-1}, x_n| < R$ ($R < |x_{n-1}, x_n| < \infty$). The corresponding result from [15] now reads as follows:

$$\operatorname{ind} K_\gamma - \operatorname{ind} K_\beta = \operatorname{ind} B_\gamma - \operatorname{ind} B_\beta. \tag{53}$$

As noted before the operators B_γ and B_β coincide modulo compact operators with $\operatorname{op}_M^{\gamma-1}(a)$ and $\operatorname{op}_M^{\beta-1}(a)$, respectively. This gives us

$$\begin{aligned} \operatorname{ind} B_\gamma - \operatorname{ind} B_\beta &= \operatorname{ind} \operatorname{op}_M^{\gamma-1}(a) - \operatorname{ind} \operatorname{op}_M^{\beta-1}(a) \\ &= n(\delta, \gamma) - n(\delta, \beta) = n(\beta, \gamma). \end{aligned} \tag{54}$$

The second equation is a consequence of the results of [8] which is a version of [29] for boundary value problems, for the technique see also [9], using the fact that the non-bijectivity points of a in the respective weight strip are all simple. To complete the proof it suffices to combine the relations (53) and (54). □

Corollary 3.3 *For every $k \in \mathbb{Z}$ we have*

$$\operatorname{ind} \sigma_\wedge(\mathcal{A}(\gamma)) = k. \tag{55}$$

for all $\gamma \in (\frac{1}{2} - k, \frac{3}{2} - k)$, i.e., $e(\gamma) - d(\gamma) = k$ for the dimensions $e(\gamma), d(\gamma)$ of Theorem 3.1.

In fact, for a given weight $\gamma \notin \mathbb{Z} + \frac{1}{2}$ we can choose any $s > 3$, $s \notin \mathbb{N} \cup \{\mathbb{N} + \frac{1}{2}\}$, and apply Theorem 3.2 combined with the relation (46). This gives us $\operatorname{ind} \sigma_\wedge(\mathcal{A}(\gamma)) = -[s - \frac{1}{2}] + n(s, \gamma)$. Then (55) follows from $n(s, \gamma) = k + [s] - 1$, $-[s - \frac{1}{2}] = -[s] + 1$ for $0 < \{s\} < \frac{1}{2}$ and $n(s, \gamma) = k + [s]$, $-[s - \frac{1}{2}] = -[s]$ for $\frac{1}{2} < \{s\} < 1$.

3.3 Parametrices and regularity of solutions for the Zaremba problem

We now pass to parametrices in the edge calculus and obtain regularity of solutions to our mixed problems.

Theorem 3.4 *For every fixed $\gamma \notin \mathbb{Z} + \frac{1}{2}$ the operator $\tilde{\mathcal{A}} := \tilde{\mathcal{A}}(\gamma)$ of Theorem* 3.1 *has a parametrix $\tilde{\mathcal{P}}$ in the edge calculus, cf. Definition* 4.6 *below, i.e., we have $\tilde{\mathcal{P}}\tilde{\mathcal{A}} = \mathcal{I} - \mathcal{C}$, $\tilde{\mathcal{A}}\tilde{\mathcal{P}} = \mathcal{I} - \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are smoothing operators as in Definition* 4.6 (iv)*, and $\mathcal{I}$ is the identity operator in the corresponding spaces. The operator $\tilde{\mathcal{P}}$ is (ψ, ∂) regular* (*cf. the terminology of Section* 4.2 *below*) *and has the type* 0 (*both on the Dirichlet and the Neumann sides*).

Proof. By Theorem 3.1 the operator $\tilde{\mathcal{A}}$ is elliptic in the calculus of Section 4.2, i.e., all symbolic components are bijective. Thus the existence of a parametrix is a consequence of Theorem 4.11 below. The resulting type follows from the corresponding generalities on boundary value problems, cf. [12, Section 1.2.7]. In fact, the type of the parametrix of an elliptic boundary value problem of order μ and type d is equal to $\max(\mathrm{d} - \mu, 0)$. In the present case we have $\mu = 2$ and $\mathrm{d} = 1$ on the Dirichlet and $d = 2$ on the Neumann side. The (ψ, ∂) regularity of the parametrix follows from the fact that $\tilde{\mathcal{A}}$ itself is (ψ, ∂) regular and that the inversion of symbols from the edge calculus is compatible with the Leibniz inversion of smooth complete symbols, relevant for the (ψ, ∂) regularity. ☐

Corollary 3.5 *The operator* (49) *is Fredholm for every $\gamma \notin \mathbb{Z} + \frac{1}{2}, s > \frac{3}{2}$, and kernel and cokernel are independent of s. Moreover, $\tilde{\mathcal{A}}u \in \mathcal{W}^{s-2,\gamma-2}(\mathbb{X}; Y_-, Y_+) \oplus H^{s-2}(Z, \mathbb{C}^{e(\gamma)})$ and $u \in \mathcal{W}^{-\infty,\gamma}(\mathbb{X}) \oplus H^{-\infty}(Z, \mathbb{C}^{d(\gamma)})$ implies $u \in \mathcal{W}^{s,\gamma}(\mathbb{X}) \oplus H^{s}(Z, \mathbb{C}^{d(\gamma)})$.*

3.4 Jumping oblique derivatives and other examples

Let us now consider other examples for the Laplace operator Δ , namely mixed conditions with jumping oblique derivatives on $Y_\pm$. In this case we have

$$\mathcal{A} = \begin{pmatrix} \Delta \\ T_- \\ T_+ \end{pmatrix} : H^s(\operatorname{int} X) \to \begin{array}{c} H^{s-2}(\operatorname{int} X) \\ \oplus \\ H^{s-\frac{3}{2}}(\operatorname{int} Y_-) \\ \oplus \\ H^{s-\frac{3}{2}}(\operatorname{int} Y_+) \end{array} \qquad (56)$$

$s > \frac{3}{2}$, for $T_- := \mathrm{r}^- B_-,\ T_+ := \mathrm{r}^+ B_+$. Here $B_\pm$ are locally of the form

$$B_- = \sum_{i=1}^{n-2} \alpha_i D_{z_i} + \alpha D_{x_{n-1}} + \lambda D_{x_n},\ \ B_+ = \sum_{i=1}^{n-2} \beta_i D_{z_i} + \beta D_{x_{n-1}} + \delta D_{x_n}.$$

with coefficients α_i, β_i smoothing depending on $z = (z_1, \ldots, z_{n-2})$, and constants $\alpha, \beta, \lambda, \delta$ such that $\lambda \neq 0, \delta \neq 0$. We always assume $n \geq 3$; the operators $T_\pm$ then satisfy the Shapiro-Lopatinskij condition. In [10] for the case $\sum_{i=1}^{n-2} \alpha_i(z) D_{z_i}|_Z = \sum_{i=1}^{n-2} \beta_i(z) D_{z_i}|_Z = 0$ and $\alpha = \beta = \lambda = \delta = 1$ we completed the operator (56) by additional potential operators $L_\mp$ to a Fredholm operator of index zero

$$\mathcal{Z} := \begin{pmatrix} \Delta & 0 \\ T_- & L_- \\ T_+ & L_+ \end{pmatrix} : \begin{array}{c} H^s(\operatorname{int} X) \\ \oplus \\ H^s(Z, \mathbb{C}^l) \end{array} \to \begin{array}{c} H^{s-2}(\operatorname{int} X) \\ \oplus \\ H^{s-\frac{3}{2}}(\operatorname{int} Y_-), \\ \oplus \\ H^{s-\frac{3}{2}}(\operatorname{int} Y_+) \end{array} \qquad (57)$$

for $s > \frac{3}{2}, s \notin \mathbb{N}$, where $l = 0$ for $\frac{3}{2} < s < 2$ and $l = [s-1]$ for $s > 2$. The operator (57) can equivalently be reformulated as an operator $\tilde{\mathcal{A}}(s)$ of index zero in the edge algebra, by applying the same technique as before for (43). In the present case this holds for $s > 3, s \notin \mathbb{N}$, and we have $d = N(s) + [s-1], s_2 = s - \frac{3}{2}, d_2 = \sigma(s - \frac{3}{2})$, and $d_i, s_i, i = 1, 3$, are as in (43). Similarly as in Section 2.3 we can express the principal edge symbol of the upper left 3×1

corner $\mathcal{A}(s)$ of $\tilde{\mathcal{A}}(s)$

$$\sigma_\wedge(\mathcal{A}(s))(z,\zeta) = \begin{pmatrix} \sigma_\wedge(\Delta)(\zeta) \\ \sigma_\wedge(T_-)(z,\zeta) \\ \sigma_\wedge(T_+)(z,\zeta) \end{pmatrix} : \mathcal{K}^{s,s}(\mathbb{R}^2_+ \setminus \{0\}) \to \begin{matrix} \mathcal{K}^{s-2,s-2}(\mathbb{R}^2_+ \setminus \{0\}) \\ \oplus \\ \mathcal{K}^{s-\frac{3}{2},s-\frac{3}{2}}(\mathbb{R}_-) \\ \oplus \\ \mathcal{K}^{s-\frac{3}{2},s-\frac{3}{2}}(\mathbb{R}_+) \end{matrix}$$

for $\sigma_\wedge(\Delta)(\zeta)$ as in (26) and

$$\sigma_\wedge(T_-(z,\zeta) = r^{-1}\{-\frac{\alpha}{i}(-r\frac{\partial}{\partial r}) + \frac{\lambda}{i}\frac{\partial}{\partial\phi}\}|_{\phi=0},$$
$$\sigma_\wedge(T_+(z,\zeta) = r^{-1}\{\frac{\beta}{i}(-r\frac{\partial}{\partial r}) - \frac{\delta}{i}\frac{\partial}{\partial\phi}\}|_{\phi=\pi},$$

is a family of Fredholm operators for any fixed $s > 3, s \notin \mathbb{N} \cup \{\mathbb{N} + \frac{1}{2}\}$ with index

$$\operatorname{ind} \sigma_\wedge(\mathcal{A}(s))(z,\zeta) = -[s].$$

The conormal symbol

$$\sigma_M\sigma_\wedge(\mathcal{A})(w) = \begin{pmatrix} \frac{\partial^2}{\partial\phi^2} + w^2 \\ \frac{1}{i}(\frac{\partial}{\partial\phi} - w)|_{\phi=0} \\ \frac{1}{i}(-\frac{\partial}{\partial\phi} + w)|_{\phi=\pi} \end{pmatrix} : H^s(I) \to \begin{matrix} H^{s-2}(I) \\ \oplus \\ \mathbb{C} \oplus \mathbb{C} \end{matrix} \tag{58}$$

defines a family of bijective operators for all $w \notin \mathbb{Z}$. In fact, first observe that $w = 0$ is a simple non-bijectivity point of $\sigma_M\sigma_\wedge(\mathcal{A})(w)$. Now let $w = a + ib \neq 0$. Then the boundary conditions give us

$$c_1(i-1) = c_2(i+1),$$
$$c_1(1-i)e^{-b\pi}(\cos a\pi + i \sin a\pi) + c_2(1+i)e^{b\pi}(\cos a\pi - i \sin a\pi) = 0.$$

Assume that $c_1 = 0$. Then we obtain $(e^{b\pi} - e^{-b\pi}) \cos a\pi - i(e^{b\pi} + e^{-b\pi}) \sin a\pi = 0$, and hence $b = 0, a = k, k \in \mathbb{Z}$. We have $\ker \sigma_M\sigma_\wedge(\mathcal{A})(k) = \{c(\cos k\phi + \sin k\phi) : c \in \mathbb{C}\}$.

Using $\sigma_M\sigma_\wedge(\mathcal{A})(w)(\cos w\phi + \sin w\phi) = {}^t(\ 0 \quad 0 \quad \frac{2}{i} w \sin w\pi\ )$ we see that the non-bijectivity points are simple. Similarly as for the Zaremba problem we thus obtain for the edge symbol

$$\operatorname{ind} \sigma_\wedge(\mathcal{A}(\gamma)) = \operatorname{ind} \sigma_\wedge(\mathcal{A}(s)) + n(s,\gamma)$$

for all $\gamma < s, \gamma \notin \mathbb{Z}$. More precisely, we have

$$\operatorname{ind} \sigma_\wedge(\mathcal{A}(\gamma)) = k$$

for all $\gamma \in (-k, -k+1)$. This follows from $n(s,\gamma) = k + [s]$.

As another example we take the Laplace operator with the Dirichlet condition T_- on Y_- and the condition T_+ on Y_+ as above in this section. Applying a result of [10] for the case $\sum_{i=1}^{n-2} \beta_i(z) D_{z_i}|_Z = 0$ and $\beta = \delta = 1$ the corresponding operator $\mathcal{A}$ can be completed to a Fredholm operator (40) for all $s > \frac{3}{2}, s \notin \mathbb{N} + \frac{3}{4}$, where in this case $l = [s - \frac{3}{4}]$. Introducing again $\mathcal{A}(s)$ as a realisation of $\mathcal{A}$ in weighted Sobolev spaces for $s > 3, s \notin \mathbb{N} \cup \{\mathbb{N} + \frac{3}{4}\}$ we obtain in this case $\sigma_\wedge(\mathcal{A}(s))(\zeta) = -[s - \frac{3}{4}]$. The set of non-bijectivity points of $\sigma_M\sigma_\wedge(\mathcal{A})$ coincides with $\{w \in \mathbb{C} : w \in \mathbb{Z} + \frac{1}{4}\}$, and the points are simple. In fact, for $w = 0$ the operator

$\sigma_M\sigma_\wedge(\mathcal{A})(0)u = 0$ has only trivial solution. If $w = a + ib \neq 0$ the boundary conditions give us

$$\begin{cases} c_1 + c_2 = 0 \\ c_1(1-i)e^{-b\pi}e^{ia\pi} + c_2(1+i)e^{b\pi}e^{-ia\pi} = 0 \end{cases},$$

cf. (32). Let us assume that $c_1 \neq 0$ (otherwise $u = 0$). Then we obtain

$$(\cos a\pi + \sin a\pi)(1 - e^{2b\pi}) + i(\sin a\pi - \cos a\pi)(1 + e^{2b\pi}) = 0$$

which implies that $b = 0$ and $a = k + \frac{1}{4}, k \in \mathbb{Z}$. We have $\ker \sigma_M\sigma_\wedge(\mathcal{A})(k+\frac{1}{4}) = \{c \sin(k+\frac{1}{4})\phi : c \in \mathbb{C}\}, k \in \mathbb{Z}$, and using

$$\sigma_M\sigma_\wedge(\mathcal{A})(w)\sin(w\phi) = {}^{\mathrm{t}}\!\left(\, 0 \quad 0 \quad \frac{1}{i}(-w\cos(w\pi) + w\sin(w\pi)) \,\right)$$

we see that the non-bijectivity points are simple. Similarly as before it follows that

$$\operatorname{ind}\sigma_\wedge(\mathcal{A}(\gamma))(\zeta) = \operatorname{ind}\sigma_\wedge(\mathcal{A}(s))(\zeta) + n(s,\gamma)$$

for all $\gamma < s, 1 - \gamma \notin \mathbb{Z} + \frac{1}{4}$. More precisely,

$$\operatorname{ind}\sigma_\wedge(\mathcal{A}(\gamma))(\zeta) = k$$

for $\gamma \in (\frac{3}{4} - k, \frac{7}{4} - k)$. This follows from $n(s,\gamma) = k + [s] - 1,\ [s - \frac{3}{4}] = [s] - 1$ for $\{s\} < \frac{3}{4}$ and $n(s,\gamma) = k + [s],\ [s - \frac{3}{4}] = [s]$ for $\{s\} > \frac{3}{4}$.

4 Boundary value problems on manifolds with edges

4.1 Edge amplitude functions

The specific nature of parametrices of mixed elliptic problems is determined (modulo suitable smoothing operators) by a category of amplitude functions of the classes $S^{\mu_{ij}}_{\mathrm{cl}}(U \times \mathbb{R}^q; E_j, F_i)$ in the variables and covariables (z,ζ), $U \subseteq \mathbb{R}^q$ open, $q = \dim Z$ (cf. Section 1.2). The orders μ_{ij} are defined by the spaces

$$E_1 = \mathcal{K}^{s,\gamma}(\mathbb{R}^2_+ \setminus \{0\}),\ E_2 = \mathcal{K}^{s-\frac{1}{2}-\nu_-,\gamma-\frac{1}{2}-\nu_-}(\mathbb{R}_-),\ E_3 = \mathcal{K}^{s-\frac{1}{2}-\nu_+,\gamma-\frac{1}{2}-\nu_+}(\mathbb{R}_+),$$
$$F_1 = \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}^2_+ \setminus \{0\}),\ F_2 = \mathcal{K}^{s-\frac{1}{2}-\mu_-,\gamma-\frac{1}{2}-\mu_-}(\mathbb{R}_-),\ F_3 = \mathcal{K}^{s-\frac{1}{2}-\mu_+,\gamma-\frac{1}{2}-\mu_+}(\mathbb{R}_+)$$

for certain $\mu, \nu_\pm, \mu_\pm \in \mathbb{R}$ and $E_4 = \mathbb{C}^d, F_4 = \mathbb{C}^e$ for some dimensions d, e (recall that the latter spaces are endowed with the trivial group actions). As already noted in connection with the operators (15) we can also start from vector-valued functions with different orders of the components; for simplicity we content ourselves with the scalar case. In Section 1.2 we already introduced Green, trace and potential amplitude functions. The remaining ingredients of the complete edge symbolic calculus which we discuss in the present section only refer to the spaces E_j, F_i for $i, j = 1, 2, 3$; so we may set $d = e = 0$.

In order to have a convenient approach we first ignore the orders μ_{ij} but formulate symbols in terms of parameter-dependent families

$$\tilde{p}(r, z, \tilde{\varrho}, \tilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times U, B^{\mu,\mathrm{d}}(I; \mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\zeta}}))$$

for $\mu \in \mathbb{Z}$, $\mathrm{d} \in \mathbb{N}$, or

$$\tilde{g}(r, z, \tilde{\varrho}, \tilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times U, \mathcal{B}^{\mu,\mathrm{d}}_G(I; \mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\zeta}}))$$

for $\mu \in \mathbb{R}$, $\mathrm{d} \in \mathbb{N}$, where we then insert $\tilde{\varrho} = r\varrho$, $\tilde{\zeta} = r\zeta$. For the definition of $B^{\mu,\mathrm{d}}(I;\mathbb{R}^{1+q})$ and $\mathcal{B}_G^{\mu,\mathrm{d}}(I;\mathbb{R}^{1+q})$ the specific meaning of $(\tilde{\varrho},\tilde{\zeta}) \in \mathbb{R}^{1+q}$ is unimportant; so we first denote these parameters for a while by (ϱ,ζ).

Let $\mathcal{B}_G^{-\infty,0}(I)$ defined to be the space of all 3×3 block matrix operator functions

$$g = (g_{ij})_{i,j=1,2,3} : \begin{array}{c} H^s(\mathrm{int}\, I) \\ \oplus \\ \mathbb{C}^2 \end{array} \rightarrow \begin{array}{c} C^\infty(I) \\ \oplus \\ \mathbb{C}^2 \end{array}$$

$s > -\frac{1}{2}$, where g_{11} is an integral operator with kernel in $C^\infty(I \times I)$, $g_{1j}c := f_{1j}(\phi)c$ for $j = 2,3$, $c \in \mathbb{C}$, $g_{i1}u := \int_0^\pi f_{i1}(\phi)u(\phi)d\phi$ for $i = 2,3$, with arbitrary functions $f_{1j}, f_{i1} \in C^\infty(I)$ for $j = 2,3$ and $i = 2,3$, and $(g_{ij})_{i,j=2,3}$ is an arbitrary 2×2 matrix with entries in $\mathbb{C}$. To avoid confusion let us note that the components of $(c_0, c_\pi) \in \mathbb{C}^2$ are related to the end points $\{0\}$ and $\{\pi\}$ of the interval I.

The space $\mathcal{B}_G^{-\infty,0}(I)$ is Fréchet in a natural way (as a direct sum of its 9 components), and we set $\mathcal{B}_G^{-\infty,0}(I;\mathbb{R}^{1+q}) := \mathcal{S}(\mathbb{R}^{1+q},\ \mathcal{B}_G^{-\infty,0}(I))$. Moreover, let $\mathcal{B}_G^{-\infty,\mathrm{d}}(I;\mathbb{R}^{1+q})$ for any $\mathrm{d} \in \mathbb{N}$ be the space of all operator families $g(\varrho,\zeta) := g_0(\varrho,\zeta) + \sum_{j=1}^{\mathrm{d}} g_j(\varrho,\zeta)\,\mathrm{diag}(\partial_\phi^j, 0, 0)$ for arbitrary $g_j \in \mathcal{B}_G^{-\infty,0}(I;\mathbb{R}^{1+q})$.

Let us now consider 2×2 block matrix symbols $g(\varrho,\zeta)$ of the class

$$S_{\mathrm{cl}}^\mu(\mathbb{R}^{1+q}; L^2(\mathbb{R}_+) \oplus \mathbb{C},\ \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}) \tag{59}$$

(with group actions $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ being defined by $\kappa_\lambda(u(\phi) \oplus c) := \lambda^{\frac{1}{2}} u(\lambda\phi) \oplus c$ for $u \oplus c$ in $L^2(\mathbb{R}_+) \oplus \mathbb{C}$ or $\mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}$) such that also $g^*(\varrho,\zeta)$ (the pointwise adjoint with respect to the $L^2(\mathbb{R}_+) \oplus \mathbb{C}$ scalar product) belong to the space (59).

With every such $g_{(0)}(\varrho,\zeta)$ we can associate an operator family

$$a(\varrho,\zeta) := \omega g_{(0)}(\varrho,\zeta)\tilde{\omega} : \begin{array}{c} H^s(\mathrm{int}\, I) \\ \oplus \\ \mathbb{C} \end{array} \rightarrow \begin{array}{c} C^\infty(\mathrm{int}\, I) \\ \oplus \\ \mathbb{C} \end{array}, \tag{60}$$

$s > -\frac{1}{2}$, for any fixed choice of cut-off functions $\omega, \tilde{\omega}$ on $\overline{\mathbb{R}}_+$ supported by $[0,\varepsilon)$ for some $0 < \varepsilon < \pi$. Here I is assumed to be embedded in $\overline{\mathbb{R}}_+$ with $\{0\} \in I$ corresponding to the origin in $\overline{\mathbb{R}}_+$.

In a similar manner we can form operators $\omega g_{(\pi)}(\varrho,\zeta)\tilde{\omega}$ for another symbol $g_{(\pi)}(\varrho,\zeta)$ of the abovementioned kind and then obtain operators

$$b(\varrho,\zeta) := \chi_*(\omega g_{(\pi)}(\varrho,\zeta)\tilde{\omega}) : \begin{array}{c} H^s(\mathrm{int}\, I) \\ \oplus \\ \mathbb{C} \end{array} \rightarrow \begin{array}{c} C^\infty(I) \\ \oplus \\ \mathbb{C} \end{array}, \tag{61}$$

where χ_* is the push forward under the map $\chi : I \rightarrow I$ for $\chi(\phi) := -\phi + \pi$. Observe that the direct summands $\mathbb{C}$ in the spaces of (60) belong to $\{0\} \in I$, those of (61) to $\{\pi\} \in I$. Writing (60) and (61) as 2×2 block matrices with entries a_{ij} and b_{ij}, respectively, we now form

$$g(\varrho,\zeta) := \begin{pmatrix} a_{11} + b_{11} & a_{12} & b_{12} \\ a_{21} & a_{22} & 0 \\ b_{21} & 0 & b_{22} \end{pmatrix} (\varrho,\zeta) : \begin{array}{c} H^s(\mathrm{int}\, I) \\ \oplus \\ \mathbb{C} \\ \oplus \\ \mathbb{C} \end{array} \rightarrow \begin{array}{c} C^\infty(I) \\ \oplus \\ \mathbb{C} \\ \oplus \\ \mathbb{C} \end{array}. \tag{62}$$

More generally, we consider operator families

$$g(\varrho,\zeta)=g_0(\varrho,\zeta)+\sum_{j=1}^{\mathrm{d}} g_j(\varrho,\zeta)\,\mathrm{diag}(\partial_\phi^j,0,0) \tag{63}$$

for any $\mathrm{d}\in\mathbb{N}$, where $g_j(\varrho,\zeta)$ are of the kind (62), of order $\mu-j$ (with μ from (59)).

Definition 4.1 *The space $\mathcal{B}_G^{\mu,\mathrm{d}}(I;\mathbb{R}^{1+q}_{\varrho,\zeta})$ for $\mu\in\mathbb{R}$, $d\in\mathbb{N}$ is defined as the space of all operator functions $g(\varrho,\zeta)+c(\varrho,\zeta)$ for arbitrary families of the form* (63) *and $c(\varrho,\zeta)\in\mathcal{B}_G^{-\infty,\mathrm{d}}(I;\mathbb{R}^{1+q}_{\varrho,\zeta})$.*

Let $B_G^{\mu,\mathrm{d}}(I;\mathbb{R}^{1+q})$ denote the space of upper left corners of elements of $\mathcal{B}_G^{\mu,\mathrm{d}}(I;\mathbb{R}^{1+q})$.

Remark 4.2 *The space $\mathcal{B}_G^{\mu,\mathrm{d}}(I;\mathbb{R}^{1+q})$ has a natural Fréchet topology. So we can form spaces of the kind $C^\infty(\overline{\mathbb{R}}_+\times U,\mathcal{B}_G^{\mu,\mathrm{d}}(I;\mathbb{R}^{1+q}))$ or $\mathcal{A}(D,\mathcal{B}_G^{\mu,\mathrm{d}}(I;\mathbb{R}^{1+q}))$; here $\mathcal{A}(D,E)$ for an open set $D\subseteq\mathbb{C}$ and a Fréchet space E denotes the space of all holomorphic functions in D with values in E.*

Let $S_{\mathrm{cl}}^{\mu}(I\times\mathbb{R}_\vartheta\times\mathbb{R}^{1+q}_{\varrho,\zeta})_{\mathrm{tr}}$ denote the space of all classical symbols of order $\mu\in\mathbb{Z}$ in the variable $\phi\in I$ and covariables $(\vartheta,\varrho,\zeta)$ (with ϑ being the dual variable to ϕ) and constant coefficients with respect to the variables (r,z) which have the transmission property at the end points $\{0\}$ and $\{\pi\}$ of the interval I. Recall that the transmission property (for instance, at $\phi=0$) of a symbol $a(\phi,\vartheta,\varrho,\zeta)$ requires from the homogeneous components $a_{(\mu-j)}(\phi,\vartheta,\varrho,\zeta)$ that

$$D_\phi^k D_{\varrho,\zeta}^\alpha\{a_{(\mu-j)}(\phi,\vartheta,\varrho,\zeta)-(-1)^{\mu-j}a_{(\mu-j)}(\phi,-\vartheta,-\varrho,-\zeta)\}=0$$

on the set $\{(\phi,\vartheta,\varrho,\zeta):\phi=0,\,\vartheta\in\mathbb{R}\setminus\{0\},\,(\varrho,\zeta)=0\}$ for all $k\in\mathbb{N}$, $\alpha\in\mathbb{N}^{1+q}$ and all $j\in\mathbb{N}$.

Given a symbol $a\in S_{\mathrm{cl}}^{\mu}(I\times\mathbb{R}_\vartheta\times\mathbb{R}^{1+q}_{\varrho,\zeta})_{\mathrm{tr}}$ we set

$$\mathrm{op}^I(a)(\varrho,\zeta)u(\phi):=\mathrm{r}\,\mathrm{op}(\tilde a)(\varrho,\zeta)\mathrm{e}u(\phi), \tag{64}$$

where $\tilde a(\phi,\vartheta,\varrho,\zeta)$ is any element of $S_{\mathrm{cl}}^{\mu}(\mathbb{R}_\phi\times\mathbb{R}^{2+q}_{\vartheta,\varrho,\zeta})$ such that $a=\tilde a|_{I\times\mathbb{R}^{2+q}}$ and e the operator of extension by zero to $\mathbb{R}\setminus(\mathrm{int}\,I)$, r the operator of restriction to $\mathrm{int}\,I$, $\mathrm{op}(\tilde a)(\varrho,\zeta)u(\phi)=\iint e^{i(\phi-\phi')\vartheta}a(\phi,\vartheta,\varrho,\zeta)u(\phi')d\phi'đ\vartheta$. As is known, (64) represents a (ϱ,ζ)-dependent family of continuous operators $\mathrm{op}^I(a)(\varrho,\zeta):H^s(\mathrm{int}\,I)\to H^{s-\mu}(\mathrm{int}\,I)$ for every real $s>-\frac{1}{2}$ (which is independent of the extension $\tilde a$ of a).

Definition 4.3 *The space $B^{\mu,\mathrm{d}}(I;\mathbb{R}^{1+q}_{\varrho,\zeta})$ for $\mu\in\mathbb{Z}$, $d\in\mathbb{N}$ is defined to be the set of all operator families of the form*

$$\mathrm{op}^I(a)(\varrho,\zeta)+g(\varrho,\zeta)$$

for arbitrary $a\in S_{\mathrm{cl}}^{\mu}(I\times\mathbb{R}^{2+q})_{\mathrm{tr}}$ and $g\in B_G^{\mu,\mathrm{d}}(I;\mathbb{R}^{1+q})$.

Let us set $\mathcal{B}^{\mu,\mathrm{d}}(I;\mathbb{R}^{1+q}):=\{\mathrm{diag}(p,0,0)+\mathfrak{g}:p\in B^{\mu,\mathrm{d}}(I;\mathbb{R}^{1+q}),\mathfrak{g}\in\mathcal{B}_G^{\mu,\mathrm{d}}(I;\mathbb{R}^{1+q})\}$. In the case $p\neq 0$ we assume $\mu\in\mathbb{Z}$, otherwise $\mu\in\mathbb{R}$.

Also the space $\mathcal{B}^{\mu,\mathrm{d}}(I;\mathbb{R}^{1+q})$ is Fréchet in a natural way.

Remark 4.4 *The space $\mathcal{B}^{\mu,\mathrm{d}}(I;\mathbb{R}^{1+q})$ is a particular case of the parameter-dependent algebra $\mathcal{B}^{\mu,d}(M;\mathbb{R}^l)$ of pseudo-differential boundary value problems on a smooth manifold M with boundary ∂M, here for $I=M,\partial M=\{0\}\cup\{\pi\}$, and $\mathbb{R}^{1+q}\ni(\varrho,\zeta)$ as the parameter space. For $l=0$ this corresponds to Boutet de Monvel's algebra* [2]*; material for arbitrary l may be found in* [16]*, see also* [12]*. We will use here elements of this calculus in a variety of other special cases, in particular, for $M=I^\wedge$ (or $M=\overline{\mathbb{R}}^2_+\setminus\{0\}$) and for $M=X\setminus Z$. In both cases M is not compact, and the boundary has two components, namely $\mathbb{R}_\pm$ and $\mathrm{int}\,Y_\pm$, respectively, with different boundary conditions.*

Let $\mathcal{M}_{\mathcal{O}}^{\mu,\mathrm{d}}(I;\mathbb{R}^q_\zeta)$ denote the space of all $f(w,\zeta)\in\mathcal{A}(\mathbb{C},\mathcal{B}^{\mu,\mathrm{d}}(I;\mathbb{R}^q_\zeta))$ such that

$$f(\beta+i\varrho,\zeta)\in\mathcal{B}^{\mu,\mathrm{d}}(I,\mathbb{R}^{1+q}_{\varrho,\zeta})$$

for every $\beta\in\mathbb{R}$, uniformly in $c\leq\beta\leq c'$ for every $c\leq c'$.

In order to express edge amplitude functions we employ the Mellin transform, cf. the notation of Section 1.1, and associated (weighted) Mellin pseudo-differential operators

$$\mathrm{op}_M^\gamma(f)u(r)=\int_{\mathbb{R}}\int_0^\infty\left(\frac{r}{r'}\right)^{-(\frac{1}{2}-\gamma+i\varrho)}f(r,r',\frac{1}{2}-\gamma+i\varrho)u(r')\frac{dr'}{r'}đ\varrho$$

for a weight $\gamma\in\mathbb{R}$, with scalar or operator-valued amplitude functions $f(r,r',w)$. Their precise nature will be explained below; we also will have several parameter-dependent variants, e.g., $f(r,r',z,w,\zeta)$; in that case we write $\mathrm{op}_M^\gamma(f)(z,\zeta)$.

For pseudo-differential actions on $\mathbb{R}_+$ based on the Fourier transform we write $\mathrm{op}(p)(z,\zeta)$ (or also $\mathrm{op}_r(p)(z,\zeta)$). Concerning the amplitude functions we take them in the form

$$p(r,z,\varrho,\zeta)=\tilde{p}(r,z,\tilde{\varrho},\tilde{\zeta})|_{\tilde{\varrho}=r\varrho,\tilde{\zeta}=r\zeta}\tag{65}$$

for $\tilde{p}\in C^\infty(\overline{\mathbb{R}}_+\times U,\mathcal{B}^{\mu,\mathrm{d}}(I;\mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\zeta}})),U\subseteq\mathbb{R}^q$ open.

Theorem 4.5 *For every $p(r,z,\varrho,\zeta)$ of the form (65) there exists a Mellin symbol*

$$h(r,z,w,\zeta)=\tilde{h}(r,z,w,\tilde{\zeta})|_{\tilde{\zeta}=r\zeta}\tag{66}$$

associated with an $\tilde{h}\in C^\infty(\overline{\mathbb{R}}_+\times U,\mathcal{M}_{\mathcal{O}}^{\mu,\mathrm{d}}(I;\mathbb{R}^q_{\tilde{\zeta}}))$ such that

$$\mathrm{op}_r(p)(z,\zeta)=\mathrm{op}_M^\gamma(h)(z,\zeta)\mod C^\infty(U,\mathcal{B}^{-\infty,d}(I^\wedge;\mathbb{R}^q_\zeta))\tag{67}$$

for every $\gamma\in\mathbb{R}$. Similarly, for $p_0(r,z,\varrho,\zeta):=\tilde{p}(0,z,r\varrho,r\zeta)$, $h_0(r,z,w,\zeta):=\tilde{h}(0,z,w,r\zeta)$ we have $\mathrm{op}_r(p_0)(z,\zeta)=\mathrm{op}_M^\gamma(h_0)(z,\zeta)\mod C^\infty(U,\mathcal{B}^{-\infty,\mathrm{d}}(I^\wedge;\mathbb{R}^q_\zeta))$.

If (66) is related to (65) as in the previous theorem we will say that h is a (holomorphic) Mellin quantisation of p.

Let us now choose arbitrary cut-off functions $\sigma(r),\tilde{\sigma}(r)$ and $\omega_0(r),\omega_1(r),\omega_2(r)$ such that $\omega_1\equiv1$ on $\mathrm{supp}\,\omega_0$ and $\omega_0\equiv1$ on $\mathrm{supp}\,\omega_2$. Moreover, let $\zeta\to[\zeta]$ denote any strictly positive function in $C^\infty(\mathbb{R}^q)$ such that $[\zeta]=|\zeta|$ for all $|\zeta|>c$ for some $c>0$. We now form operator functions of the kind

$$a(z,\zeta):=\sigma(r)\{a_M(z,\zeta)+a_\psi(z,\zeta)\}\tilde{\sigma}(r)\tag{68}$$

for

$$a_M(z,\zeta):=r^{-\mu}\omega_0(r[\zeta])\mathrm{op}_M^{\gamma-1}(h)(z,\zeta)\omega_1(r[\zeta]),$$
$$a_\psi(z,\zeta):=r^{-\mu}(1-\omega_0(r[\zeta]))\mathrm{op}_r(p)(z,\zeta)(1-\omega_2(r[\zeta])),$$

where h is a Mellin quantisation of p.

Let us set

$$\sigma_\wedge(a)(z,\zeta):=r^{-\mu}\{\omega_0(r|\zeta|)\mathrm{op}_M^{\gamma-1}(h_0)(z,\zeta)\omega_1(r|\zeta|)+(1-\omega_0(r|\zeta|))\mathrm{op}_r(p_0)(z,\zeta)(1-\omega_2(r|\zeta|))\}.\tag{69}$$

Observe that

$$\sigma_\wedge(a)(z,\lambda\zeta)=\lambda^\mu\boldsymbol{\kappa}_\lambda\sigma_\wedge(z,\zeta)\boldsymbol{\kappa}_\lambda^{-1}\tag{70}$$

for $\lambda \in \mathbb{R}_+, (z,\zeta) \in U \times (\mathbb{R}^q \setminus \{0\})$, where $\boldsymbol{\kappa}_\lambda := \operatorname{diag}(\kappa_\lambda^\wedge, \kappa_\lambda, \kappa_\lambda)$, cf. the notation in the formula (29). Comparing the relations (29) and (70) we see a difference in homogeneities which comes from the assumed unified orders in the construction of (68). Moreover, set

$$\sigma_M \sigma_\wedge(a)(z,w) = h_0(0,z,w,0), \;\; w \in \Gamma_{1-\gamma}. \tag{71}$$

By definition (68) is a 3×3 matrix of operator functions $a(z,\zeta) = (a_{ij}(z,\zeta))_{i,j=1,2,3}$ associated with an order $\mu \in \mathbb{R}$ and a weight $\gamma \in \mathbb{R}$.

For our calculus we choose orders and weights for the entries a_{ij} individually, namely, as in Section 1.2 by (20) and (21), respectively, for $i,j = 1,2,3$. We then obtain

$$a_{ij}(z,\zeta) \in S^{\mu_{ij}}(U \times \mathbb{R}^q; E_j, F_i) \tag{72}$$

for the abovementioned spaces E_j, F_i, for $s > d - \frac{1}{2}$ (cf. the general definition (11). The technique for proving relations of the kind (72) may be found in [22].

In order to express the complete edge amplitude functions we need a further category of operator-valued symbols, the so-called smoothing Mellin symbols. They also only occur for the indices $i,j = 1,2,3$. Similarly as before they refer to orders μ_{ij} and weights γ_{ij}, but again for convenience we first consider ij-independent orders and weights.

Let $\mathcal{M}_\beta^{-\infty,\mathrm{d}}(I)$ for $\beta \in \mathbb{R}, \mathrm{d} \in \mathbb{N}$, denote the space of all operator functions

$$f(w) \in \mathcal{A}(\{\beta - \varepsilon < \operatorname{Re} w < \beta + \varepsilon\}, \mathcal{B}^{-\infty,\mathrm{d}}(I))$$

for some $\varepsilon > 0$ which may depend on f, such that $f(\eta + i\varrho) \in \mathcal{B}^{-\infty,\mathrm{d}}(I;\mathbb{R}_\varrho)$ for every $\eta \in (\beta - \varepsilon, \beta + \varepsilon)$, uniformly in compact subintervals. Denoting for a moment by $\mathcal{M}_\beta^{-\infty,\mathrm{d}}(I)_\varepsilon$ the subspace of all $f \in \mathcal{M}_\beta^{-\infty,\mathrm{d}}(I)$ belonging to a fixed $\varepsilon > 0$, we obtain a Fréchet space. This gives us $C^\infty(U, \mathcal{M}_\beta^{-\infty,\mathrm{d}}(I)) = \bigcup_{\varepsilon > 0} C^\infty(U, \mathcal{M}_\beta^{-\infty,\mathrm{d}}(I)_\varepsilon)$.

Setting

$$m(z,\zeta) := r^{-\mu}\omega(r[\zeta])\operatorname{op}_M^{\gamma-1}(f)(z)\tilde{\omega}(r[\zeta])$$

for an $f \in C^\infty(U, \mathcal{M}_{1-\gamma}^{-\infty,\mathrm{d}}(I))$ and any choice of cut-off functions we obtain a 3×3 block matrix of classical operator-valued symbols.

The homogeneous principal component of $m(z,\zeta)$ of order μ has the form

$$\sigma_\wedge(m)(z,\zeta) = r^{-\mu}\omega(r|\zeta|)\operatorname{op}_M^{\gamma-1}(f)(z)\tilde{\omega}(r|\zeta|). \tag{73}$$

Let us set

$$\sigma_M \sigma_\wedge(m)(z,w) := f(z,w), \; w \in \Gamma_{1-\gamma}. \tag{74}$$

We now assume μ as well as the weight γ to depend on i,j as in (20) and (21) which yields a matrix $m(z,\zeta) = (m_{ij}(z,\zeta))_{i,j=1,2,3}$ of elements

$$m_{ij}(z,\zeta) \in S_{\mathrm{cl}}^{\mu_{ij}}(U \times \mathbb{R}^q; E_j, F_i) \tag{75}$$

for arbitrary $s > d - \frac{1}{2}$ (clearly in the target spaces F_i the finite smoothness may be replaced by ∞ in this case).

An edge amplitude function is defined as a 4×4 block matrix operator function of the form

$$\mathfrak{a}(z,\zeta) := \mathfrak{p}(z,\zeta) + \mathfrak{m}(z,\zeta) + \mathfrak{g}(z,\zeta) \tag{76}$$

for $\mathfrak{p}(z,\zeta) = \begin{pmatrix} a(z,\zeta) & 0 \\ 0 & 0 \end{pmatrix}$, $\mathfrak{m}(z,\zeta) = \begin{pmatrix} m(z,\zeta) & 0 \\ 0 & 0 \end{pmatrix}$ with $a(z,\zeta) = (a_{ij}(z,\zeta))_{i,j=1,2,3}$ being given by (72), $m(z,\zeta) = (m_{ij}(z,\zeta))_{i,j=1,2,3}$ by (75), and $\mathfrak{g}(z,\zeta) = (g_{ij})_{i,j=1,\dots,4}$ as in (19).

From (23), (69) and (73) we obtain a 4×4 matrix of homogeneous principal symbols $\sigma_\wedge(\mathfrak{a})(z,\zeta), (z,\zeta) \in U \times (\mathbb{R}^q \setminus \{0\})$,

$$\sigma_\wedge(\mathfrak{a})(z,\zeta) : \oplus_{j=1}^4 E_j \to \oplus_{i=1}^4 F_i \tag{77}$$

with the abovementioned spaces $E_j, F_i, i,j = 1,\dots,4$, for $s > d - \frac{1}{2}$, with entries of order μ_{ij} as in (72). The 3×3 upper left corners of the operators (77) form a family of boundary value problems on the infinite cone $\mathbb{R}^2_+ \setminus \{0\}$ with the boundary components $\mathbb{R}_\pm$. As such they have a 3×3 matrix of subordinate conormal symbols

$$\sigma_M\sigma_\wedge(\mathfrak{a})(z,w) : \begin{array}{c} H^s(\operatorname{int} I) \\ \oplus \\ \mathbb{C} \oplus \mathbb{C} \end{array} \to \begin{array}{c} H^{s-\mu}(\operatorname{int} I) \\ \oplus \\ \mathbb{C} \oplus \mathbb{C} \end{array}$$

depending on $z \in Z$ and the complex variable w. According to (71) and (74) the entries $\sigma_M\sigma_\wedge(\mathfrak{a}_{ij})$ of $\sigma_M\sigma_\wedge(\mathfrak{a})$ are given for w on the weight lines $\Gamma_{1-\gamma_{ij}}$ for $i,j = 1,2,3$. It is convenient to normalise the representation by setting

$$\sigma_M^\gamma\sigma_\wedge(a_{ij})(z,w) := \sigma_M\sigma_\wedge(a_{ij})(z, w + \gamma_{ij} - \gamma) \tag{78}$$

such that all entries are defined on $\Gamma_{1-\gamma}$.

4.2 Operators on manifolds with edges

The following definition and the subsequent remarks introduce a new variant of edge boundary value problems. The role of these operators is to express parametrices of mixed elliptic problems. Compared with the calculus of [12] the main difference is that we employ here a more general class of Green and smoothing Mellin amplitude functions, cf. Section 4.1, encoded by 'small weight improvements' $\varepsilon > 0$ (depending on the operator) rather than asymptotics. From this point of view the operators of [12] with (discrete or continuous) asymptotics in the Green and smoothing Mellin terms form a subcalculus, cf. also the remarks in Section 4.3 below. It is not our intention to present a full edge calculus here because the formal part is close to that of [12], see also [25] for the case without boundary.

Definition 4.6 *The edge algebra of boundary value problems on $\mathbb{X}$ is defined as the space of all 4×4 block matrix operators* (15), *i.e., $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,\dots,4}$ with orders $(\mu_{ij})_{i,j=1,\dots,4}$, cf.* (20), *which have* (*modulo smoothing operators to be described below under* (iv)) *the following properties:*

(i) *$\mathcal{A}$ is locally in coordinates $(z, x_{n-1}, x_n) \in \mathbb{R}^{n-2} \times \mathbb{R}^2$ of the form $\mathrm{Op}_z(\mathfrak{a})$ for an edge amplitude function as in* (76),

(ii) *The 2×2 submatrices $(\mathcal{A}_{ij})_{i,j=1,2}$ $((\mathcal{A}_{ij})_{i,j=1,3})$ locally near $\operatorname{int} Y_-$ $(\operatorname{int} Y_+)$ belong to Boutet de Monvel's calculus of boundary value problems with the transmission property at $\operatorname{int} Y_-$ $(\operatorname{int} Y_+)$ and with the respective DN order conventions for the trace and potential parts.*

(iii) *$\mathcal{A}_{11}|_{\operatorname{int} X}$ belongs to $L^\mu_{\mathrm{cl}}(\operatorname{int} X)$.*

(iv) *In the sequel, for convenience, in contrast to* (15) *we formulate the operators for spaces over Z for the dimension 1; the general case is completely analogous.*

An operator $\mathcal{C}$ is called regularising Green operator in the edge algebra of boundary value problems and of type 0 *if $\mathcal{C}$ induces continuous operators*

$$\mathcal{C}: \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{X}) \\ \oplus \\ \mathcal{W}^{s',\gamma-\frac{1}{2}-\nu_-}(Y_-) \\ \oplus \\ \mathcal{W}^{s'',\gamma-\frac{1}{2}-\nu_+}(Y_+) \\ \oplus \\ H^{s'''}(Z) \end{array} \to \begin{array}{c} \mathcal{W}^{\infty,\gamma-\mu+\varepsilon}(\mathbb{X}) \\ \oplus \\ \mathcal{W}^{\infty,\gamma-\frac{1}{2}-\mu_-+\varepsilon}(Y_-) \\ \oplus \\ \mathcal{W}^{\infty,\gamma-\frac{1}{2}-\mu_++\varepsilon}(Y_+) \\ \oplus \\ H^{\infty}(Z) \end{array}$$

for some $\varepsilon = \varepsilon(\mathcal{C}) > 0$, for all $s, s', s'', s''' \in \mathbb{R}, s > -\frac{1}{2}$, and if also the formal adjoint $\mathcal{C}^*$ has analogous continuity properties, now with respect to the modified weights $-\gamma + \mu, -\gamma - \frac{1}{2} + \mu_-, -\gamma - \frac{1}{2} + \mu_+$ in the preimage and $-\gamma, -\gamma - \frac{1}{2} + \nu_-, -\gamma - \frac{1}{2} + \nu_+$ in the target spaces. The formal adjoint is defined via

$$(\mathcal{C}u, v) = (u, \mathcal{C}^* v)$$

for all $u, v \in \mathcal{W}^{\infty,\infty}(\mathbb{X}) \oplus \mathcal{W}^{\infty,\infty}(Y_-) \oplus \mathcal{W}^{\infty,\infty}(Y_+) \oplus H^{\infty}(Z)$ with the scalar products of $\mathcal{W}^{0,0}(\mathbb{X}) \oplus \mathcal{W}^{0,-\frac{1}{2}}(Y_-) \oplus \mathcal{W}^{0,-\frac{1}{2}}(Y_+) \oplus H^0(Z)$.
An operator $\mathcal{C}$ is called smoothing and of type $\mathrm{d} \in \mathbb{N}$ *if $\mathcal{C}$ has the form*

$$\mathcal{C} = \mathcal{C}_0 + \sum_{j=1}^{\mathrm{d}} \mathcal{C}_j \mathrm{diag}\,(T^j, 0, 0, 0)$$

for arbitrary smoothing operators $\mathcal{C}_j$ of type 0 *as before and a differential operator T on X of first order (with smooth coefficients up to the boundary) which is equal to $\frac{\partial}{\partial x_n}$ in a collar neighbourhood of $Y = \partial X$ (where x_n is a global coordinate in normal direction).*

Remark 4.7 *Definition* 4.6 *is adapted to the case of mixed problems for second order elliptic operators A with conditions $T_\pm$ of arbitrary order; this is just the structure of the Zaremba problem for the Laplacian (where some of the entries are simply zero). There is also a more general version with block matrices $\mathcal{A}_{ij}$ rather than 'scalar' entries and vectors of orders $\nu_\pm$ and $\mu_\pm$, respectively, and we can also have schemes of DN orders for the operators on Z as in* (43)*. As before some components may be zero, so there are also row and column matrix versions, cf. Example* 1.3*. Another variant concerns operators between distributional sections of vector bundles on the various components $\mathbb{X}, Y_\pm$ and Z of the configuration. Such generalisations may be subsumed under the notation 'edge algebra'.*

The upper left corners $\mathcal{A}_{11}$ of our edge algebra have a rich structure. By Definition 4.6 (iii) they induce elements of $L^{\mu}_{\mathrm{cl}}(\mathrm{int}\, X)$, and by (ii) those operators have the transmission property at $\mathrm{int}\, Y_\pm$. They also contain Green terms near $\mathrm{int}\, Y_\pm$ with the same behaviour as Green operators of some type d in Boutet de Monvel's calculus. Near the interface Z the operators $\mathcal{A}_{11}$ have an edge-degenerate non-smoothing part, cf. the requirements (65) for the local symbols, and they contain also the other 'smoothing' ingredients such as Green and Mellin operators in $X \setminus Z$ from the edge algebra, cf. (76). In our application the original operator (the Laplace operator) has a smooth symbol across the interface Z; the symbol becomes edge-degenerate of the form (65) when we transform it to polar coordinates in the (x_{n-1}, x_n) plane normal to Z. In other words, there is a 'smooth' symbol behind the edge-degenerate structure with the transmission property everywhere at $Y = \partial X$. Let us call an element $\mathcal{A}$ of the edge algebra ψ regular at Y if it has such a classical interior symbol which is smooth across Z. The requirement of regularity singles out a particularly convenient substructure. Those operators $\mathcal{A}$ have a standard homogeneous principal symbol of order μ

$$\sigma_\psi(\mathcal{A})(x,\xi) := \sigma_\psi(\mathcal{A}_{11})(x,\xi), \tag{79}$$

$(x,\xi) \in T^*X \setminus 0$.

In general, near Z the operators $\mathcal{A}$ of Definition 4.6 have edge-degenerate homogeneous principal interior symbols, coming from the representation of local symbols in the upper left corners which have the form

$$r^{-\mu} a(r,\phi,z,\varrho,\vartheta,\zeta)$$

where

$$a(r,\phi,z,\varrho,\vartheta,\zeta) = \tilde{a}(r,\phi,z,r\varrho,\vartheta,r\zeta)$$

for an $\tilde{a}(r,\phi,z,\tilde{\rho},\vartheta,\tilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times U, S^\mu_{\mathrm{cl}}(I \times \mathbb{R}_\vartheta \times \mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\zeta}})_{\mathrm{tr}})$, cf. Section 4.1. In this case we set

$$\sigma_\psi(\mathcal{A})(r,\phi,z,\varrho,\vartheta,\zeta) = r^{-\mu}\tilde{\sigma}_\psi(\mathcal{A})(r,\phi,z,r\varrho,\vartheta,r\zeta)$$

for

$$\tilde{\sigma}_\psi(\mathcal{A})(r,\phi,z,\tilde{\varrho},\vartheta,\tilde{\zeta}) := \tilde{a}_{(\mu)}(r,\phi,z,\tilde{\varrho},\vartheta,\tilde{\zeta})$$

($\tilde{a}_{(\mu)}$ denotes the homogeneous principal component of $\tilde{a}$ of order μ in $(\tilde{\varrho},\vartheta,\tilde{\zeta}) \neq 0$).

Let us also consider the operators $\mathcal{B}_- := (\mathcal{A}_{ij})_{i,j=1,2}$ and $\mathcal{B}_+ := (\mathcal{A}_{ij})_{i,j=1,3}$ of Definition 4.6 (ii). In differential mixed problems such as the Zaremba problem they can be regarded as restrictions to $\operatorname{int} Y_\mp \times [0,1)$ of corresponding operators $\tilde{\mathcal{B}}_\mp$ in Boutet de Monvel's calculus in a collar neighbourhood $\cong Y \times [0,1)$ of the boundary Y. In the pseudo-differential case we can ask a similar property, modulo the contributions of the smoothing Mellin and Green operators near Z and the smoothing operators from Definition 4.6 (iii). Let us call an element $\mathcal{A}$ of the edge algebra (ψ,∂) regular if it has this property (clearly any such operator is necessarily ψ regular). In that case we have pairs of boundary symbols

$$\sigma_{\partial,\mp}(\mathcal{A})(y,\eta) := \sigma_\partial(\tilde{\mathcal{B}}_\mp)(y,\eta) : \begin{array}{c} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{n_\mp} \end{array} \to \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{m_\mp} \end{array} \tag{80}$$

for $(y,\eta) \in T^*Y_\mp \setminus 0$, smooth up to Z. In the latter relations we assume on the operator $\mathcal{A}$ that the boundary problems $\mathcal{B}_\mp$ contain $n_\mp$ potential and $m_\mp$ trace conditions with respect to $\operatorname{int} Y_\mp$.

In general, near Z the operators $\mathcal{A}$ of Definition 4.6 have edge-degenerate homogeneous principal boundary symbols, coming from (65) together with the corresponding weight factors. For instance, for the upper left corner $\mathcal{A}_{11}$ we have

$$\sigma_{\partial,+}(\mathcal{A}_{11})(r,z,\varrho,\zeta) = r^{-\mu}\tilde{\sigma}_{\partial,+}(\mathcal{A}_{11})(r,z,r\varrho,r\zeta)$$

for

$$\tilde{\sigma}_{\partial,+}(\mathcal{A}_{11})(r,z,\tilde{\varrho},\tilde{\zeta}) := \mathrm{r}^+\tilde{a}_{(\mu)}(0,z,r\varrho,D_\phi,r\zeta)\mathrm{e}^+ : H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+) \tag{81}$$

and similarly for the $-$ sign, where $\tilde{a}_{(\mu)}$ is to be frozen at $\phi = \pi$ and D_ϕ replaced by $-D_\phi$ (because of the opposite orientation of the ϕ half-axis in this case).

For the other components of the boundary symbols near Z (i.e, trace and potential parts as well as standard pseudo-differential symbols on the boundary) we have similar expressions (with exponents in the weight factors being linked to the orders of the boundary operators).

Operators in the edge algebra of type d define continuous operators

$$\mathcal{A} : \oplus_{j=1}^{4}\mathcal{E}_j \to \oplus_{i=1}^{4}\mathcal{F}_i \tag{82}$$

for $\mathcal{E}_1 := \mathcal{W}^{s,\gamma}(\mathbb{X}),\ \ \mathcal{F}_1 := \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{X}),$

$$\mathcal{E}_2 := \mathcal{W}^{s-\frac{1}{2}-\nu_-,\gamma-\frac{1}{2}-\nu_-}(Y_-,\mathbb{C}^{n_-}),\ \ \mathcal{F}_2 := \mathcal{W}^{s-\frac{1}{2}-\mu_-,\gamma-\frac{1}{2}-\mu_-}(Y_-,\mathbb{C}^{m_-}),$$

$$\mathcal{E}_3 := \mathcal{W}^{s-\frac{1}{2}-\nu_+,\gamma-\frac{1}{2}-\nu_+}(Y_+,\mathbb{C}^{n_+}),\ \ \mathcal{F}_3 := \mathcal{W}^{s-\frac{1}{2}-\mu_+,\gamma-\frac{1}{2}-\mu_+}(Y_+,\mathbb{C}^{m_+}),$$

$\mathcal{E}_4 := H^s(Z,\mathbb{C}^d)$ $\mathcal{F}_4 := H^{s-\mu}(Z,\mathbb{C}^e), s > \mathrm{d} - \frac{1}{2}$. Here, for simplicity, the upper left corners are assumed to be scalar, while the other entries $\mathcal{A}_{ij}$ of $\mathcal{A}$ are now block matrices, e.g., $\mathcal{A}_{22}$ an $m_- \times n_-$ matrix, etc. Clearly, as in Example 1.3, the orders $\nu_\mp$ and $\mu_\mp$ may also assumed to be vectors, according to the components of $\mathbb{C}^{n_\mp}$ and $\mathbb{C}^{m_\mp}$, respectively.

$\mathcal{A}$ is called elliptic, if the components of

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_{\partial,\pm}(\mathcal{A}), \sigma_\wedge(\mathcal{A}))$$

are bijective in the following sense:

(i) $\sigma_\psi(\mathcal{A}) := \sigma_\psi(\mathcal{A}_{11})$ does not vanish on $T^*\mathbb{X}_{\mathrm{reg}} \setminus 0$, and $\tilde{\sigma}_\psi(\mathcal{A})$ does not vanish for $(\tilde{\varrho}, \vartheta, \tilde{\zeta}) \neq 0$ up to $r = 0$.

(ii) The boundary symbols (80) are isomorphisms for all $(y, \eta) \in T^*(\mathrm{int}\, Y_\mp) \setminus 0$, and $\tilde{\sigma}_{\partial,\mp}(\mathcal{A})(r, z, \tilde{\varrho}, \tilde{\zeta}) : \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{n_\mp} \end{matrix} \to \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{m_\mp} \end{matrix}$ are isomorphisms for $(\tilde{\varrho}, \tilde{\zeta}) \neq 0$ up to $r = 0$; these conditions are required for any $s > \mathrm{d} - \frac{1}{2}$.

(iii) The edge symbol (77) is an isomorphism for every $(z, \zeta) \in T^*Z \setminus 0$ and $s > \mathrm{d} - \frac{1}{2}$; in the present notation the spaces E_j, F_i are vector-valued, and of different smoothness and weight, i.e., $E_1 = \mathcal{K}^{s,\gamma}(\mathbb{R}^2_+ \setminus \{0\}),\ \ F_1 = \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}^2_+ \setminus \{0\}),$

$$E_2 = \mathcal{K}^{s-\frac{1}{2}-\nu_-,\gamma-\frac{1}{2}-\nu_-}(\mathbb{R}_-,\mathbb{C}^{n_-}),\ \ F_2 = \mathcal{K}^{s-\frac{1}{2}-\mu_-,\gamma-\frac{1}{2}-\mu_-}(\mathbb{R}_-,\mathbb{C}^{m_-}),$$

$$E_3 = \mathcal{K}^{s-\frac{1}{2}-\nu_+,\gamma-\frac{1}{2}-\nu_+}(\mathbb{R}_-,\mathbb{C}^{n_+}),\ \ F_3 = \mathcal{K}^{s-\frac{1}{2}-\mu_+,\gamma-\frac{1}{2}-\mu_+}(\mathbb{R}_-,\mathbb{C}^{m_+}),$$

$E_4 = \mathbb{C}^d,\ \ F_4 = \mathbb{C}^e$, and the homogeneities in ζ correspond to the scheme of *DN* ordes.

Remark 4.8 *Every operator $\mathcal{A}$ in the edge algebra is determined by $\sigma(\mathcal{A})$ modulo a compact operator $\oplus_{j=1}^{4}\mathcal{E}_j \to \oplus_{i=1}^{4}\mathcal{F}_i, s > \mathrm{d} - \frac{1}{2}$. This is an analogue of a corresponding property in edge algebras in the boundaryless case, cf.* [25].

Remark 4.9 *The principal symbols $\sigma(\mathcal{A})$ of operators $\mathcal{A}$ in the edge algebra form an algebra under componentwise composition. For the components coming from boundary problems with the transmission property this is known from the calculus of* [2]. *The edge symbols are families of boundary value problems on the infinite cone $\mathbb{R}^2_+ \setminus \{0\}$ with the boundary components $\mathbb{R}_\mp$. Their composition behaviour is known from the corresponding calculus, cf.* [21].

Remark 4.10 *The notation 'algebra' in Definition* 4.6 *is justified by the fact that the composition of two elements $\mathcal{A}$ and $\mathcal{B}$ belongs to the calculus again (provided that the spaces in the image of $\mathcal{A}$ fit to those of the domain of $\mathcal{B}$), and we then have*

$$\sigma(\mathcal{A}\mathcal{B}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$$

with componentwise composition. This result is analogous to a corresponding composition property for the subclass with asymptotics as is analysed in [12, Chapter 4].

If $\mathcal{A}$ and $\mathcal{P}$ are operators in the edge algebra we call $\mathcal{P}$ a parametrix of $\mathcal{A}$ if $\mathcal{I}-\mathcal{P}\mathcal{A}$ and $\mathcal{I}-\mathcal{A}\mathcal{P}$ are regularising Green operators in the sense of Definition 4.6 (iv).

Theorem 4.11 *Every elliptic operator $\mathcal{A}$ has a parametrix $\mathcal{P}$ in the edge algebra where $\sigma(\mathcal{P}) = \sigma^{-1}(\mathcal{A})$ with componentwise inversion.*

Proof. If $\mathcal{A}$ is elliptic we can first pass to $\sigma^{-1}(\mathcal{A})$ and associate it with an element $\mathcal{P}_0$ in the edge calculus, i.e., $\sigma(\mathcal{P}_0) = \sigma^{-1}(\mathcal{A})$. We then obtain $\mathcal{P}_0\mathcal{A} = \mathcal{I} - \mathcal{C}_0$ for an operator $\mathcal{C}_0$ in the edge calculus such that $\sigma(\mathcal{C}_0) = 0$. Let us consider the formal Neumann series $(\mathcal{I}-\mathcal{C}_0)^{-1} \sim \mathcal{I}-\mathcal{C}_1$ for $\mathcal{C}_1 \sim -\sum_{j=0}^{\infty} \mathcal{C}_0^j$. Similarly as in [12] the asymptotic summation for $\mathcal{C}_1$ can be carried out within the calculus; so we may set $\mathcal{P} := (\mathcal{I}-\mathcal{C}_1)\mathcal{P}_0$ which has the property that $\mathcal{P}\mathcal{A} - \mathcal{I}$ is smoothing. The same can be done from the right; thus $\mathcal{P}$ is a parametrix as desired. □

Corollary 4.12 *The ellipticity of $\mathcal{A}$ entails the Fredholm property of* (82) *for every $s > \mathrm{d} - \frac{1}{2}$ (where d is the type of $\mathcal{A}$), and kernel and cokernel are independent of s. Furthermore, we have elliptic regularity in our weighted Sobolev spaces.*

4.3 Asymptotics of solutions

From elliptic boundary value problems in domains with conical singularities it is known (cf. Kondratyev [13]) that solutions have asymptotics of the form $\sum_j \sum_{k=0}^{m_j} c_{jk}(x) r^{-p_j} \log^k r$ for $r \to 0$ modulo flat remainders. Here $r \in \mathbb{R}_+$ is the distance variable to the singularity. The coefficients c_{jk} are smooth on the base of the local cone. The exponents $-p_j \in \mathbb{C}$ and the number of logarithmic terms are determined by the non-bijectivity points of the conormal symbol of the given elliptic operator. For the Zaremba problem these points are calculated in Section 1.3 and for other mixed problems in Section 3.4. In general we have to expect a dependence on the interface variable z (also the multiplicities may change under varying z). Such phenomena can be described in terms of continuous asymptotics (cf. [25] and the references there). Here we content ourselves with the case of constant discrete asymptotics.

The structure is as follows. Let M be a compact C^∞ manifold with boundary. Denote by $P = \{(p_j, m_j)\}_{j\in\mathbb{N}}$ the sequence of data which characterise asymptotics for $r \to 0$, with $p_j \in \mathbb{C}$, $\mathrm{Re}\, p_j \to -\infty$ as $j \to \infty, m_j \in \mathbb{N}$. Then the space $\mathcal{K}_P^{s,\gamma}(M^\wedge)$ is defined to be the set of all $u(r,x) \in \mathcal{K}^{s,\gamma}(M^\wedge)$ having such asymptotics with coefficients $c_{jk} \in C^\infty(M)$ for all $0 \le k \le m_j, j \in \mathbb{N}$; (for the given weight γ this implies $\mathrm{Re}\, p_j < \frac{1}{2}(1+m) - \gamma, m = \dim M$, for all j). For every $\nu \ge 0$ there is then an $N = N(\nu)$ such that

$$\mathcal{K}_P^{s,\gamma}(M^\wedge) = \Big\{ \sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk}(x) r^{-p_j} \log^k r\, \omega(r) : c_{jk} \in C^\infty(M), 0 \le k \le m_j, j \in \mathbb{N} \Big\} + \mathcal{K}^{s,\gamma+\nu}(M^\wedge)$$

for a cut-off function ω. Applying the construction (6) to $E = \mathcal{K}_P^{s,\gamma}(M^\wedge)$, cf. Remark 1.2, we obtain weighted spaces

$$\mathcal{W}_P^{s,\gamma}(M^\wedge \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}_P^{s,\gamma}(M^\wedge)),$$

or spaces $\mathcal{W}_P^{s,\gamma}(\mathbb{W})$ globally on a (stretched, say compact) manifold $\mathbb{W}$ with edge. In this construction we assume that the transition diffeomorphisms between local wedges are independent of the axial variable r for $0 \le r \le \varepsilon$ for some $\varepsilon > 0$. The dimension of M may be zero; so the same can be done for the half-space $\mathbb{R}_+ \times \mathbb{R}^q$, or, globally, on a manifold with boundary, such as the manifolds $Y_\pm$.

Note that the singular functions of edge asymptotics in the space $\mathcal{W}^{s,\gamma}_P(\mathbb{R}_+ \times \mathbb{R}^q)$ are a generalisation of the 'Taylor' edge asymptotic terms of standard Sobolev distributions $H^s(\mathbb{R}_+ \times \mathbb{R}^q)$, cf. the formula (33). In the present case from the definition of (6) we see that

$$\begin{aligned}\mathcal{W}^{s,\gamma}_P(\mathbb{R}_+ \times \mathbb{R}^q) &= \{F^{-1}_{\zeta\to z}[\zeta]^{\frac{1}{2}} \sum_{j=0}^{N}\sum_{k=0}^{m_j} c_{jk}(\zeta)(r[\zeta])^{-p_j}\log^k(r[\zeta])\omega(r[\zeta]) : c_{jk}(\zeta) \in \hat{H}^s(\mathbb{R}^q)\} \\ &+ \mathcal{W}^{s,\gamma+\nu}(\mathbb{R}_+ \times \mathbb{R}^q), \qquad (83)\end{aligned}$$

where $\hat{H}^s(\mathbb{R}^q) := \{\hat{v}(\zeta) : v(z) \in H^s(\mathbb{R}^q)\}$. In a similar manner we can express the asymptotic terms of the edge asymptotics for a non-trivial (stretched) wedge $M^\wedge \times \mathbb{R}^q$; it suffices in (83) to formally replace $\mathbb{R}_+$ by $M^\wedge$ and $[\zeta]^{\frac{1}{2}}$ by $[\zeta]^{\frac{m+1}{2}}$; the coefficients c_{jk} now belong to $C^\infty(M, \hat{H}^s(\mathbb{R}^q))$.

In the case of our mixed problems we have $M = I = [0,\pi]$. The result on elliptic regularity from Corollary 3.5 then specifies to spaces with asymptotics as follows:

Theorem 4.13 *Set*

$$\mathcal{W}^{s-2,\gamma-2}_P(\mathbb{X}; Y_-, Y_+) := \mathcal{W}^{s-2,\gamma-2}_{P_0}(\mathbb{X}) \oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{P_-}(Y_-) \oplus \mathcal{W}^{s-\frac{3}{2},\gamma-\frac{3}{2}}_{P_+}(Y_+) \oplus H^{s-2}(Z, \mathbb{C}^{e(\gamma)}) \qquad (84)$$

for a triple $P = (P_0, P_-, P_+)$ of discrete asymptotic types (constant in $z \in Z$). Then $\tilde{\mathcal{A}}u \in$(84) for $\gamma \notin \mathbb{Z} + \frac{1}{2}$ and $s > \frac{3}{2}$ and $u \in \mathcal{W}^{-\infty,\gamma}(\mathbb{X}) \oplus H^{-\infty}(Z, \mathbb{C}^{d(\gamma)})$ entails $u \in \mathcal{W}^{s,\gamma}_{Q_0}(\mathbb{X}) \oplus H^s(Z, \mathbb{C}^{d(\gamma)})$ for some resulting (constant in $z \in Z$) asymptotic type Q_0.

Proof. First we apply a parametrix $\tilde{\mathcal{P}}$ on both sides of $\tilde{\mathcal{A}}u = f$. This yields $\tilde{\mathcal{P}}\tilde{\mathcal{A}} = \tilde{\mathcal{P}}f$ where $\mathcal{C} = \mathcal{I} - \tilde{\mathcal{P}}\tilde{\mathcal{A}}$ is a smoothing operator. We then obtain asymptotics of solutions if $\tilde{\mathcal{P}}$ can be chosen in such a way that $\tilde{\mathcal{P}}f \in \mathcal{W}^{s,\gamma}_{R_0}(\mathbb{X})$ and $\mathcal{C}(\mathcal{W}^{-\infty,\gamma}(\mathbb{X})) \subset \mathcal{W}^{\infty,\gamma}_{S_0}(\mathbb{X})$ for certain asymptotic types R_0 and S_0, respectively. Parametrices of that kind can be constructed in a refined version of the edge algebra, namely with Mellin and Green symbols with asymptotics. Such a calculus in the framework of the so called continuous asymptotics is developed in [12, Chapter 4]. This concept contains asymptotic types of the present constant discrete type as a special case. If we know that the non-bijectivity points of the conormal symbol $\sigma_M\sigma_\wedge(\tilde{\mathcal{A}})$ of the given operator $\tilde{\mathcal{A}}$ are independent of $z \in \mathbb{C}$, the poles of the conormal symbol $\sigma_M\sigma_\wedge(\tilde{\mathcal{P}})$ of the parametrix $\tilde{\mathcal{P}}$ are also independent of $z \in Z$ because $\sigma_M\sigma_\wedge(\tilde{\mathcal{P}})$ is simply the inverse of $\sigma_M\sigma_\wedge(\tilde{\mathcal{A}})$ (up to a translation in the complex w-plane). An inspection of the arguments of [12] shows that then the operators $\tilde{\mathcal{P}}$ and $\mathcal{C}$ have the desired mapping properties. In the case of the Zaremba problem we have proved in Section 1.3 that the non-bijectivity points of the conormal symbol are just as we want. □

Remark 4.14 *The mechanism to compute Q_0 in terms of the given (P_0, P_-, P_+) and of the poles of $\sigma_M\sigma_\wedge(\tilde{\mathcal{P}})$ is the same as in the theory of boundary value problems on manifolds with conical singularities, cf.* [20, Theorem 3.3.12]. *The result is that the solutions have the asymptotic types from the data (together with possible translations to the left in the complex plane by integers) and in addition the poles plus multiplicities from $\sigma_M\sigma_\wedge(\tilde{\mathcal{P}})$, in this case $j + \frac{1}{2}, j \in \mathbb{Z}$. Thus the specific extra singular functions for the Zaremba problem in the space $\mathcal{W}^{s,\gamma}$ are locally near Z of the form*

$$F^{-1}_{\zeta\to z}\sum_{j=0}^{N}\{[\zeta]c_{jk}(\phi,\zeta)(r[\zeta])^{j+\frac{1}{2}}\omega(r[\zeta])\},\ -j - \frac{1}{2} < 1 - \gamma,$$

$c_{jk}(\phi,\zeta) \in C^\infty(I, \hat{H}^s(\mathbb{R}^q))$, modulo remainders of flatness $\gamma + \nu, N = N(\nu)$ (cf. the formula (83)). Analogous relations hold for the mixed problems of Section 3.4.

References

[1] M.F. Atiyah and R. Bott, *The index problem for manifolds with boundary*, Coll. Differential Analysis, Tata Institute Bombay, Oxford University Press, Oxford, 1964, pp. 175–186.

[2] L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math. **126** (1971), 11–51.

[3] N. Dines and B.-W. Schulze, *Mellin-edge-representations of elliptic operators*, Preprint 2003/18, Institut für Mathematik, Potsdam.

[4] G.I. Eskin, *Boundary value problems for elliptic pseudodifferential equations*, Math. Monographs, vol. 52, Amer. Math. Soc., Providence, Rhode Island, 1980, Transl. of Nauka, Moskva, 1973.

[5] L. Fontana, S.G. Krantz, and M.M. Peloso, *Hodge theory in the Sobolev topology for the de Rham complex*, Mem. Am. Math. Soc. **131**, 622 (1998).

[6] I.C. Gohberg and N. Krupnik, *Singular integral operators with piecewise continuous coefficients and their symbols*, Izv. Akad. Nauk SSR **35**, 4 (1971), 940–964.

[7] I.C. Gohberg and E.I. Sigal, *An operator generalization of the logarithmic residue theorem and the theorem of Rouché*, Math. USSR Sbornik **13**, 4 (1971), 603–625.

[8] G. Harutjunjan and B.-W. Schulze, *Boundary problems with meromorphic symbols in cylindrical domains*, Izvestiya NAN Armenii. Mathematika, (to appear).

[9] G. Harutjunjan and B.-W. Schulze, *Asymptotics and relative index on a cylinder with conical cross section*, Preprint 2002/27, Institut für Mathematik, Potsdam, 2002.

[10] G. Harutjunjan and B.-W. Schulze, *Parametrices of mixed elliptic problems*, Preprint 2002/08, Institut für Mathematik, FU, Berlin, 2002, Math. Nachr. (to appear).

[11] G. Harutjunjan and B.-W. Schulze, *Reduction of orders in boundary value problems without the transmission property*, Preprint 2002/03, Institut für Mathematik, Potsdam, 2002, Journal of the Math. Soc. of Japan (to appear).

[12] D. Kapanadze and B.-W. Schulze, *Crack theory and edge singularities*, Kluwer Academic Publ., Dordrecht, 2003.

[13] V.A. Kondratyev, *Boundary value problems for elliptic equations in domains with conical points*, Trudy Mosk. Mat. Obshch. **16** (1967), 209–292.

[14] V. Nazaikinskij, A. Savin, B.-W. Schulze, and B. Ju. Sternin, *Elliptic theory on manifolds with nonisolated singularities: II. Products in elliptic theory on manifolds with edges*, Preprint 2002/15, Institut für Mathematik, Potsdam, 2002.

[15] V. Nazaikinskij and B. Ju. Sternin, *The index locality principle in elliptic theory*, Funct. Anal. and its Appl. **35** (2001), 37–52.

[16] S. Rempel and B.-W. Schulze, *Index theory of elliptic boundary problems*, Akademie-Verlag, Berlin, 1982.

[17] S. Rempel and B.-W. Schulze, *Parametrices and boundary symbolic calculus for elliptic boundary problems without transmission property*, Math. Nachr. **105** (1982), 45–149.

[18] S. Rempel and B.-W. Schulze, *A theory of pseudo-differential boundary value problems with discontinuous conditions I, II*, Ann. Global Anal. Geom. **2**, 3 (1984), 163–251, 289–384.

[19] S. Rempel and B.-W. Schulze, *Asymptotics for elliptic mixed boundary problems (pseudo-differential and Mellin operators in spaces with conical singularity)*, Math. Res., vol. 50, Akademie-Verlag, Berlin, 1989.

[20] E. Schrohe and B.-W. Schulze, *Boundary value problems in Boutet de Monvel's calculus for manifolds with conical singularities II*, Advances in Partial Differential Equations (Boundary Value Problems, Schrödinger Operators, Deformation Quantization), Akademie Verlag, Berlin, 1995, pp. 70–205.

[21] E. Schrohe and B.-W. Schulze, *Mellin operators in a pseudodifferential calculus for boundary value problems on manifolds with edges*, IWOTA 95 Proceedings, "Differential and Integral Operators" (C. Mennicken and C. Tretter, eds.), Operator Theory: Advances and Applications, vol. 102, Birkhäuser Verlag, Basel, 1997, pp. 257–285.

[22] E. Schrohe and B.-W. Schulze, *A symbol algebra for pseudodifferential boundary value problems on manifolds with edges*, Differential Equations, Asymptotic Analysis, and Mathematical Physics, Math. Research, vol. 100, Akademie Verlag, Berlin, 1997, pp. 292–324.

[23] B.-W. Schulze, *Pseudo-differential operators on manifolds with singularities*, North-Holland, Amsterdam, 1991.

[24] B.-W. Schulze, *Pseudo-differential boundary value problems, conical singularities, and asymptotics*, Akademie Verlag, Berlin, 1994.

[25] B.-W. Schulze, *Boundary value problems and singular pseudo-differential operators*, J. Wiley, Chichester, 1998.

[26] B.-W. Schulze and J. Seiler, *The edge algebra structure of boundary value problems*, Annals of Global Analysis and Geometry **22** (2002), 197–265.

[27] B.-W. Schulze and J. Seiler, *Edge operators with conditions of Toeplitz type*, Preprint 2002/13, Institut für Mathematik, Potsdam, 2002.

[28] B.-W. Schulze and J. Seiler, *Pseudodifferential boundary value problems with global projection conditions*, J. Funct. Anal. **206**, 2 (2004), 449–498.

[29] B.-W. Schulze and N.N. Tarkhanov, *Euler solutions of pseudodifferential equations*, Integral Equations Operator Theory **33** (1999), 98–123.

[30] M.I. Vishik and G.I. Eskin, *Convolution equations in a bounded region*, Uspekhi Mat. Nauk **20**, 3 (1965), 89–152.

[31] S. Zaremba, *Sur un problème mixte relatif à l'equation de Laplace*, Bull. de l'Academie des Sciences de Cracovie, Class. math. et nat., Sér. A (1910), 313–344.