Elliptic Theory on Manifolds with Nonisolated Singularities

V. Index Formulas for Elliptic Problems on Manifolds with Edges

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Abstract

For elliptic problems on manifolds with edges, we construct index formulas in form of a sum of homotopy invariant contributions of the strata (the interior of the manifold and the edge). Both terms are the indices of elliptic operators, one of which acts in spaces of sections of finite-dimensional vector bundles on a compact closed manifold and the other in spaces of sections of infinite-dimensional vector bundles over the edge.

Keywords: manifold with edge, elliptic problem, index formula, symmetry conditions.

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Introduction

The present paper deals with index formulas for elliptic operators on manifolds with edges. It is well known (e.g., see [39]) that the ellipticity of operators on manifolds with edges is determined by two conditions, one of which refers to the interior principal symbol defined on the smooth part of the manifold and the other to the edge symbol, which is a family defined on the cotangent space of the edge. If we treat the manifold with edges as a stratified space, then the ellipticity condition (which ensures the Fredholm property of the corresponding operator) can be stated as a condition on the symbols on the corresponding strata of the manifold. It is therefore natural to expect that an index formula contains two terms, contributed by the respective strata. Moreover, since the ellipticity condition on the interior stratum is determined by the principal symbol (the invertibility on the compressed cotangent space minus the zero section), it is natural to assume that the term corresponding to the interior stratum in the index formula is a homotopy invariant of the principal symbol. As to the second term, corresponding to the edge, it depends on the edge symbol and is its homotopy invariant. In other words, treating the symbol of an elliptic operator with edge degeneracy as a pair of symbols associated to the strata of the manifold, we focus on index formulas comprising two terms, each being a homotopy invariant of the corresponding component of the symbol on the stratum. This will be the idea in our constructions. Needless to say, such a representation of the index is not possible for arbitrary operators, and we start by establishing necessary and sufficient conditions for the existence of an index splitting. Here the situation is similar to that on manifolds with isolated singularities (where the edge is just a point). Recall [42] that in this case the index can be represented as the index of some elliptic operator defined on the double of the manifold plus the index of an elliptic operator on the infinite cylinder. More precisely, this representation holds under some additional symmetry conditions on the principal symbol of the operator.

For the case of an edge of arbitrary dimension, we also impose some conditions like symmetry and obtain index formulas in the desired form. The main problem here is to single out a homotopy invariant of the principal symbol and to present the corresponding operator, which (in contrast to the case of manifolds with isolated singularities) is no longer an operator on the double of the original manifold in general. Moreover, the correction terms in the index formula also are not operators on the infinite cylinder but more complicated objects related to operators acting in sections of infinite-dimensional bundles over the edges. Note that the index of such operators is given by the well-known Luke formula [15] as well as by formulas in [30, 31].

Let us say a few words about papers most related to our investigation. Apart from analytic aspects of the theory, which are well developed by now, there are only a few papers in the literature on index formulas for elliptic operators on manifolds with edges, namely, for model examples (the infinite wedge, see [9, 38, 30, 31]). The first three of these papers pertain to the case in which the edge is an n-dimensional vector space, and the last paper treats the case in which the edge is a smooth closed manifold of arbitrary dimension. Next, in the paper [46], in spirit close to [2], an index formula is obtained for Dirac operators in form of two terms, each of which is however not a homotopy invariant. Finally, in [14] an index formula of Melrose–Nistor type [17] was obtained, which involves not only principal symbols, but also lower-order symbols. Let us point out once again that, in contrast to these formulas, our formula will be a sum of two homotopy invariant terms, one being a homotopy invariant of the principal symbol and the other a homotopy invariant of the edge symbol.

0 Preliminaries

We consider the index problem for elliptic problems on manifolds with edges in the sense of [39, 8]. Analytic aspects of elliptic theory on such manifolds are comprehensively discussed in these books and the literature cited therein. Note, however, that the general theory of elliptic problems on manifolds with edges deals with rather complicated symbolic structures and quantization constructions, because one aims at studying not only principal but also complete symbols of operators and obtaining asymptotic expansions of the latter.[1] In the framework of index theory, there is no need to construct a "high-precision" calculus; it suffices to have composition formulas, the construction of an almost inverse operator, etc. *modulo compact operators*. Moreover, it proves useful to get rid of subtle properties of the original operator algebra, related to the analyticity properties of Mellin symbols in the complex plane, by embedding the original algebra in a wider algebra in which the compactness of remainders in composition formulas is not related to shifts of the weight exponent. The latter algebra is constructed in Appendix 2. The construction is a specialization of a general construction pertaining to the algebra of pseudodifferential operators on a manifold with fibered boundary [25] (see also [34]). Since the order reduction procedure (see § A.6.1) always permits one to reduce an elliptic edge problem for an edge-degenerate operator to a zero-order problem in the main spaces of the corresponding scales, we deal only with zero-order operators in these spaces in the main body of the text. The main distinction of the algebra described in Appendix 2 from those considered in [8] and, say, [9] is the fact that it contains a wider set of Green operators. Namely, Green operators are arbitrary operators with zero principal symbol and compact-valued edge symbol.

The technique of pseudodifferential operators with operator-valued symbols, or, in other words, pseudodifferential operators in spaces of sections of infinite-dimensional vector bundles, plays an important role in the theory of pseudodifferential operators on manifolds with edges. This technique, which was studied in the context of elliptic theory, say, in [15], is recalled in Appendix 1.

[1] Furthermore, in applications it is often important to deal *with spaces with asymptotics* rather than usual Sobolev spaces, which further complicates the symbolic structure. We do not touch these questions in this paper.

Prior to proceeding to the results, we introduce the main notation. More detailed explanations can be found in the appendices.

Manifolds with edges

Let $\mathcal{M}$ be a compact manifold with edge X, which is a smooth compact manifold of dimension n. For simplicity, we assume that the edge is connected. (The passage to the case of several connected components is not difficult.) A neighborhood of an arbitrary point $x \in X$ in $\mathcal{M}$ is homeomorphic to a neighborhood of the point $(0, \beta)$ in the Cartesian product $\mathbb{R}^n \times K_\Omega$ of the n-dimensional space $\mathbb{R}^n$ by the model cone

$$K_\Omega = \{\Omega \times [0, \infty)\}/\{\Omega \times \{0\}\}$$

with base Ω that is a smooth compact manifold of dimension k (thus, the dimension of $\mathcal{M}$ is $n+k+1$) and with vertex β obtained by shrinking $\Omega \times \{0\}$ into a point. The standard coordinates on $\mathcal{M}$ in a neighborhood of a operator on the edge will be denoted by (x, ω, r), where x is a coordinate on the edge, ω is a coordinate on the base of the model cone, and r is the radial coordinate (the distance from the edge). We denote a collar neighborhood of the edge in $\mathcal{M}$ by $U = \{r < 1\}$, and the smooth manifold with boundary obtained by blowing up $\mathcal{M}$ will be denoted by M and called the *stretched manifold*. The boundary $Y = \partial M$ of M is a locally trivial bundle

$$\pi : Y \longrightarrow X$$

with fiber Ω over X. The collar neighborhood of ∂M will also be denoted by U (this will not lead to a confusion); it is isomorphic to the direct product $\partial M \times [0, 1)$. Associated with π is the infinite wedge $W \equiv W_\pi$, whose stretched manifold is $Y \times \mathbb{R}_+$. The wedge W can be thought of as a bundle with fiber K_Ω over X. The collar neighborhood of the edge in W will also be denoted by U and identified with the corresponding neighborhood in $\mathcal{M}$. By

$$\mathring{\mathcal{M}} \equiv \mathring{M} = \mathcal{M} \setminus X = M \setminus Y, \quad \mathring{W} = W \setminus X, \quad \mathring{K} = K \setminus \{\Omega \times \{0\}\},$$

etc. we denote the open manifolds obtained from the corresponding objects without circles by deleting the edge, the boundary, or the conical singular point.

By $T^*\mathcal{M}$ we denote the compressed cotangent bundle of $\mathcal{M}$. We also write

$$T_0^*\mathcal{M} = T^*\mathcal{M} \setminus \{0\} \quad \text{and } T_0^*X = T^*X \setminus \{0\}$$

(the zero section is deleted). The momentum variables dual to the coordinates x, ω, and $t = -\ln r$ are denoted by ξ, q, and $p = -r\zeta$, respectively. (Here ζ is the momentum variable dual to r on T^*M.)

Function spaces

By $\mathcal{K}^{s,\gamma}(K)$ we denote the function space on the model cone described in Definition A.12. Next, $\mathcal{W}^{s,\gamma}(\mathcal{M})$ and $\mathcal{W}^{s,\gamma}(W)$ are the weighted edge Sobolev spaces on the manifold $\mathcal{M}$ and the infinite wedge W, respectively. For $s=\gamma=0$, these spaces will also be denoted by $\mathcal{W}$ and $\mathcal{K}$.

Symbol spaces

By $S^0(T^*\mathcal{M})$ we denote the (Hörmander) space of zero order symbols on the compressed cotangent bundle $T^*\mathcal{M}$, and $S^0_{cl}(T^*\mathcal{M})$ stands for the subspace of classical (polyhomogeneous) symbols. The corresponding space of homogeneous principal symbols will be denoted by $S^{(0)}(T^*\mathcal{M})$. We use similar notation for symbols defined on T^*X. The bundles between which the symbols act are indicated where necessary.

Let $S^0_{CV}(T^*X)$ (respectively, $S^0_{CV}(T^*_0X)$) be the space of zero-order operator-valued symbols with compact fiber variation on T^*X (respectively, T^*_0X) acting in Hilbert bundles over X, and let $S^{(0)}_{CV}(T^*_0X)\subset S^0_{CV}(T^*_0X)$ be the subspace of symbols $f(x,\xi)$ twisted-homogeneous with respect to some strongly continuous group $\varkappa_\lambda$ acting in the fibers of the corresponding Hilbert bundles:

$$f(x,\lambda\xi)=\varkappa_\lambda f(x,\xi)\varkappa_\lambda^{-1},\quad (x,\xi)\in T^*_0X,\quad \lambda\in\mathbb{R}_+.$$

In the Hilbert spaces $\mathcal{K}^{s,\gamma}$, we use the group given by the formula

$$\varkappa_\lambda u(r,\omega)=\lambda^{(k+1)/2}u(\lambda r,\omega).$$

Pseudodifferential morphisms and elliptic edge problems

A pseudodifferential morphism (an edge problem) in edge Sobolev spaces $\mathcal{W}^{s,\gamma}(\mathcal{M})$ on a manifold $\mathcal{M}$ with edge X has the form

$$\mathbf{A}\equiv\begin{pmatrix}A+G & C\\ B & D\end{pmatrix}:\mathcal{W}^{s,\gamma}(\mathcal{M},E_1)\oplus H^s(X,J_1)\longrightarrow\mathcal{W}^{s-m,\gamma-m}(\mathcal{M},E_2)\oplus H^{s-m}(X,J_2),\tag{0.1}$$

where E_1 and E_2 are bundles over $\mathcal{M}$, J_1 and J_2 are bundles over X, and m is the order[2] of the operator $\mathbf{A}$. Here A is a pseudodifferential operator on $\mathcal{M}$, and the matrix

$$\mathbf{G}=\begin{pmatrix}G & C\\ B & D\end{pmatrix}$$

is called a *Green matrix*, or a *matrix Green operator*. (See § A.6 for a more detailed description of its components.) For brevity, we also sometimes write $\mathbf{A}=A+\mathbf{G}$. The

[2]One can also consider the more general case of operators with components of various orders, as in the Douglis–Nirenberg theory.

principal symbol of the problem **A** will be denoted by

$$a = \sigma(\mathbf{A})(y,\zeta) \equiv \sigma(A)(y,\zeta), \quad (y,\zeta) \in T_0^*\mathcal{M}$$

(here $T^*\mathcal{M}$ is the compressed cotangent bundle of $\mathcal{M}$, which is a vector bundle over M, see § A.5), and the *edge symbol* will be denoted by

$$\mathbf{a}_\wedge = \sigma_\wedge(\mathbf{A})(x,\xi) \equiv \begin{pmatrix} \sigma_\wedge(A) + \sigma_\wedge(G) & \sigma_\wedge(C) \\ \sigma_\wedge(B) & \sigma_\wedge(D) \end{pmatrix}(x,\xi) = a_\wedge(x,\xi) + \mathbf{g}_\wedge(x,\xi),$$
$$(x,\xi) \in T_0^*X,$$

where $a_\wedge(x,\xi) = \operatorname{diag}(\sigma_\wedge(A), 0)$. They are related by the *compatibility condition*

$$\sigma_\partial(A) = \sigma(\sigma_\wedge(A)), \tag{0.2}$$

where $\sigma_\partial(A) = \sigma(A)|_{\partial T^*\mathcal{M}}$ is the restriction of the principal symbol to the boundary $\partial T^*\mathcal{M}$, which will be called the *boundary symbol*, and $\sigma(\sigma_\wedge(A))$ is the principal symbol of the edge symbol of the pseudodifferential operator A. A necessary and sufficient condition that an edge problem is Fredholm is that the principal symbol is elliptic (invertible on $T_0^*\mathcal{M}$) and the edge symbol is elliptic (invertible on T_0^*X).

By $\sigma_c(a_\wedge)$ we denote the *conormal symbol* of the edge symbol $a_\wedge$. It satisfies the compatibility condition

$$\sigma(\sigma_c(a_\wedge)) = a_\partial|_{\xi=0}. \tag{0.3}$$

The *order reduction* procedure (see § A.6) permits one to assume when studying index theory issues that the order m of the edge problem is zero and that it acts in spaces with zero smoothness exponent s and weight exponent γ. This is assumed throughout the following, except for examples.

1 Index formulas in the simplest cases

In this section, we present some index formulas for the simplest cases, which will be needed in the proofs of the main results of this paper.

1.1 The index of operators on the infinite wedge

Let $W = W_\pi$ be the infinite wedge associated with the projection $\pi : Y \longrightarrow X$. Consider an edge symbol

$$\mathbf{a}_\wedge(x,\xi) = a_\wedge(x,\xi) + \mathbf{g}_\wedge(x,\xi) : \mathcal{K}(K_x, E_1) \oplus J_1 \longrightarrow \mathcal{K}(K_x, E_2) \oplus J_2, \quad (x,\xi) \in T_0^*X, \tag{1.1}$$

where E_1 and E_2 are bundles over W, J_1 and J_2 are bundles over X, and K_x is the cone over the point $x \in X$ in the infinite wedge W.

Suppose that the principal symbol of the edge symbol is equal to unity:

$$\sigma(\mathbf{a}_\wedge(x,\xi)) \equiv \sigma(a_\wedge(x,\xi)) = 1.$$

By Proposition A.31, the edge symbol $\mathbf{a}_\wedge(x,\xi)$ belongs to the space $S^0_{CV}(T^*_0X)$ of symbols with compact fiber variation on T^*_0X, and consequently, by Definition A.5 there is a well-defined pseudodifferential operator

$$\mathbf{a}_\wedge\left(x, -i\frac{\partial}{\partial x}\right) : \mathcal{W}(W, E_1) \oplus L^2(X, J_1) \longrightarrow \mathcal{W}(W, E_2) \oplus L^2(X, J_2), \tag{1.2}$$

which defines an edge problem on the infinite wedge W. If $\mathbf{a}_\wedge(x,\xi)$ is elliptic (i.e., invertible for all $(x,\xi) \in T^*X \setminus \{0\}$), then the operator (1.2) is Fredholm by Theorem A.9. The following assertion holds.

Proposition 1.1. *Let* $\mathbf{a}_\wedge(x,\xi)$ *be the elliptic edge symbol* (1.1) *with unit principal symbol. Then the index of the Fredholm operator* (1.2) *on the infinite wedge* W *is given by the formula*

$$\operatorname{ind} \mathbf{a}_\wedge\left(x, -i\frac{\partial}{\partial x}\right) = p_!(\operatorname{ind} \mathbf{a}_\wedge), \tag{1.3}$$

where $\operatorname{ind} \mathbf{a}_\wedge \in K(T^*X)$ *is the element of the* K*-group with compact supports of the cotangent bundle* T^*X *and*

$$p_! : K(T^*X) \longrightarrow K(pt)$$

is the direct image mapping corresponding to the projection $p : X \longrightarrow \{pt\}$ *into a point.*

Proof. As explained in § A.4, the index $\operatorname{ind} \mathbf{a}_\wedge \in K(T^*X)$ is well defined as the index of an arbitrary Fredholm family $\widetilde{\mathbf{a}_\wedge}(x,\xi) \in S^0_{CV}(T^*X)$ that coincides with $\mathbf{a}_\wedge(x,\xi)$ for sufficiently large $|\xi|$ (and, in particular, invertible at infinity). Formula (1.3) can be obtained by a straightforward application of Theorem A.10 to the symbol (1.1). □

1.2 The index of operators on the bundle of suspensions

In the preceding subsection, we have expressed the index of an elliptic morphism on a special manifold with edge (the model wedge W) using the fact that this manifold is a bundle over the edge X and representing the morphism in question as a pseudodifferential operator on X acting in sections of infinite-dimensional vector bundles.

There is yet another case in which elliptic morphisms on a manifold with edge can be represented as pseudodifferential operator with operator-valued symbols on the edge and compute the index as the direct image of the corresponding element on the K-theory of the edge. Namely, this is the case in which $\mathcal{M}$ is the bundle over X with fiber the suspension over Ω. Recall that the suspension over Ω, where Ω is a smooth compact manifold, is the manifold with two conical points given by

$$S\Omega = \{\{\Omega \times [0,2]\}/\{\Omega \times \{0\}\}\}/\{\Omega \times \{2\}\};$$

it can be obtained from the finite cylinder $\Omega \times [0,2]$ with base Ω by shrinking both faces into points (conical singularities of the resulting manifold). Let $\pi : Y \longrightarrow X$ be a locally trivial bundle with fiber Ω over a smooth compact manifold X. Naturally associated with this bundle is the bundle

$$\widetilde{\pi} : \mathcal{S} \longrightarrow X, \quad \widetilde{\pi}^{-1}(x) \simeq S\Omega$$

over X with fiber being the suspension over $S\Omega$. The total space $\mathcal{S}$ of this bundle is a manifold with an edge, which is the disjoint union of two copies of X. Let U be the collar neighborhood $\{r < 1\}$ of one of these copies in $\mathcal{S}$ (here r is the coordinate on $[0,2]$); then $\mathcal{S}$ is obtained from two copies of the closure $\bar{U}$ by gluing along the subset $\Omega \times \{r = 1\}$.

Let $\mathbf{A}$ be an elliptic morphism on $\mathcal{S}$. We assume that the principal symbol $\sigma(\mathbf{A})$ is independent of the radial variable r in the neighborhoods $\{r < \varepsilon\}$ and $\{r > 2 - \varepsilon\}$ of respective components of the edge. (The following assertion remains valid without this assumption, but the proof becomes technically more complicated.)

Proposition 1.2. *Under the above-mentioned conditions, the operator* $\mathbf{A}$ *can be represented modulo compact operators as a pseudodifferential operator*

$$\mathbf{A} = F\left(x, -i\frac{\partial}{\partial x}\right)$$

on the manifold X *with elliptic operator-valued symbol*

$$F(x,\xi) \in S^0_{CV}(T^*_0 X)$$

ranging in the space of conically degenerate pseudodifferential operator on the suspension $S\Omega$ *plus the space of matrix Green symbols on each of the two components of the edge.*

The index of this operator is given by the formula

$$\operatorname{ind} \mathbf{A} = p_!(\operatorname{ind} F(x,\xi)), \tag{1.4}$$

where $\operatorname{ind} F(x,\xi) \in K(T^*X)$ *is the element of the* K*-group with compact supports of the cotangent bundle* T^*X*, and*

$$p_! : K(T^*X) \longrightarrow K(pt)$$

is the direct image mapping corresponding to the projection $p : X \longrightarrow \{pt\}$ *into a point.*

Proof. To avoid clumsy notation, we assume that Y is the direct product $X \times \Omega$. The spaces in which the operator $\mathbf{A}$ acts are

$$\mathcal{W}(\mathcal{S}) \oplus L^2(X, J^{(1)}_\pm) \oplus L^2(X, J^{(2)}_\pm) = L^2(X, \mathcal{K}(S\Omega)) \oplus L^2(X, J^{(1)}_\pm) \oplus L^2(X, J^{(2)}_\pm),$$

where $J^{(1,2)}_\pm$ are finite-dimensional bundles over X. Thus, $\mathbf{A}$ can be interpreted as an operator acting on sections of infinite-dimensional bundles over X with fibers being Hilbert spaces of the form

$$\mathcal{K}(S\Omega) \oplus J^{(1)}_\pm \oplus J^{(2)}_\pm = \widetilde{r}^{-n/2} L^2(S\Omega) \oplus J^{(1)}_\pm \oplus J^{(2)}_\pm,$$

where $\widetilde{r}$ is a nonnegative smooth function on the interval $[0,2]$ equal to r near zero, equal to $2-r$ near $r=2$, and nonvanishing at the interior points of the interval. Let us represent it (modulo compact operators) as a pseudodifferential operator on X with operator-valued symbol. Indeed, note that if the principal symbol $\sigma(\mathbf{A})$ is independent of r in an ε-neighborhood of the edges, then $\mathbf{A}$ can be expressed via the principal and edge symbols as follows (cf. Remark A.34). We take a partition of unity

$$1=\sum_{j=1}^{3}\chi_j^2(r),\quad \operatorname{supp}\chi_1\subset[0,\varepsilon],\quad \operatorname{supp}\chi_2\subset[\varepsilon/2,2-\varepsilon/2],\quad \operatorname{supp}\chi_3\subset[2-\varepsilon,2].$$

Then

$$\mathbf{A}=\bigl(\chi_1(r)\sigma_\wedge(\mathbf{A})\chi_1(r)\bigr)\left(x,-i\frac{\partial}{\partial x}\right)+\bigl(\chi_2(r)\sigma(A)\chi_2(r)\bigr)^\wedge \\ +\bigl(\chi_3(r)\widetilde{\sigma_\wedge}(\mathbf{A})\chi_3(r)\bigr)\left(x,-i\frac{\partial}{\partial x}\right), \tag{1.5}$$

where the hat $^\wedge$ in the second term stands for the usual pseudodifferential quantization on the infinite cylinder $X\times\Omega\times\mathbb{R}_r$. Here $\sigma_\wedge(\mathbf{A})$ and $\widetilde{\sigma_\wedge}(\mathbf{A})$ are the edge symbols of $\mathbf{A}$ on the first and second component of the edge, respectively. The second term, which is a pseudodifferential operator on the product $X\times\Omega\times\mathbb{R}_r$, can be rewritten in a standard way as a pseudodifferential operator on X with operator-valued symbol $f(x,\xi)$ that is a pseudodifferential operator with parameter $\xi\in T_x^*X$ (e.g., see [43]), depending on $x\in X$, on the cylinder $\Omega\times\mathbb{R}_r$:

$$\bigl(\chi_2(r)\sigma(A)\chi_2(r)\bigr)^\wedge=f\left(x,-i\frac{\partial}{\partial x}\right).$$

As a result, we obtain

$$\mathbf{A}=\bigl(\chi_1(r)\sigma_\wedge(\mathbf{A})\chi_1(r)\bigr)\left(x,-i\frac{\partial}{\partial x}\right)+f\left(x,-i\frac{\partial}{\partial x}\right) \\ +\bigl(\chi_3(r)\widetilde{\sigma_\wedge}(\mathbf{A})\chi_3(r)\bigr)\left(x,-i\frac{\partial}{\partial x}\right)=F\left(x,-i\frac{\partial}{\partial x}\right),$$

where

$$F(x,\xi)=\bigl(\chi_1(r)\sigma_\wedge(\mathbf{A})\chi_1(r)\bigr)(x,\xi)+f(x,\xi)+\bigl(\chi_3(r)\widetilde{\sigma_\wedge}(\mathbf{A})\chi_3(r)\bigr)(x,\xi).$$

All three terms in the latter expression are zero-order symbols with compact fiber variation, and so the same is true for F: one has $F\in S^0_{CV}(T^*X)$. For large $|\xi|$, the symbol $F(x,\xi)$ is invertible. Indeed, $F(x,\xi)$ has an almost inverse $G(x,\xi)$, which can be constructed in a standard way from local (almost) inverses with the help of a partition of unity:

$$G(x,\xi)=\bigl(\chi_1(r)\sigma_\wedge(\mathbf{A})^{-1}\chi_1(r)\bigr)(x,\xi)+g(x,\xi)+\bigl(\chi_3(r)\widetilde{\sigma_\wedge}(\mathbf{A})^{-1}\chi_3(r)\bigr)(x,\xi),$$

where the operator-valued symbol $g(x,\xi)$ is defined (by analogy with $f(x,\xi)$) from the condition

$$g\left(x,-i\frac{\partial}{\partial x}\right)=\left(\chi_2(r)\sigma(A)^{-1}\chi_2(r)\right)^{\wedge}.$$

Using the twisted homogeneity of the edge symbol and the fact that $f(x,\xi)$ and $g(x,\xi)$ are pseudodifferential operators with parameter ξ, one can readily show that the products $F(x,\xi)G(x,\xi)$ and $G(x,\xi)F(x,\xi)$ have the form $1+K_j(x,\xi)$, $j=1,2$, where the norm of the remainder $K_j(x,\xi)$ tends to zero as $|\xi|\to\infty$. Hence $F(x,\xi)$ is invertible for large $|\xi|$. Now the assertion concerning the index of $\mathbf{A}$ follows from Theorem A.10. □

1.3 The relative index theorem

Let A be a pseudodifferential operator $\mathcal{M}$ with edge. Suppose that the principal symbol of A is invertible on $T_0^*\mathcal{M}$ (that is, A is formally elliptic) and that A admits elliptic edge problems. Next, let $\mathbf{A}$ and $\mathbf{B}$ be two such problems. Then we arrive at the natural question of obtaining an analog of the Agranovich–Dynin theorem [1] in this case, that is, of computing the relative index

$$\operatorname{ind}(\mathbf{A},\mathbf{B})=\operatorname{ind}\mathbf{A}-\operatorname{ind}\mathbf{B}. \tag{1.6}$$

First, suppose that problems $\mathbf{A}$ and $\mathbf{B}$ act in the same spaces. Then the relative index is

$$\operatorname{ind}(\mathbf{A},\mathbf{B})=\operatorname{ind}(\mathbf{A}\mathbf{B}^{-1}),$$

where $\mathbf{B}^{-1}$ is the almost inverse of $\mathbf{B}$. (In particular, the principal and edge symbols of $\mathbf{B}^{-1}$ are the inverses of the corresponding symbols of $\mathbf{B}$.) The morphism $\mathbf{A}\mathbf{B}^{-1}$ has a unit principal symbol; applying the surgery [24], we can assume that it is defined on the infinite wedge W. By applying Proposition 1.1, we arrive at the following assertion.

Theorem 1.3. *The relative index of problems* $\mathbf{A}$ *and* $\mathbf{B}$ *is equal to*

$$\operatorname{ind}(\mathbf{A},\mathbf{B})=p_!\left(\operatorname{ind}(\sigma_\wedge(\mathbf{A})\sigma_\wedge(\mathbf{B})^{-1})\right), \tag{1.7}$$

where $\operatorname{ind}(\sigma_\wedge(\mathbf{A})\sigma_\wedge(\mathbf{B})^{-1})\in K(T^*X)$ *is an element of the* K*-group with compact supports of the cotangent bundle* T^*X *and*

$$p_!:K(T^*X)\longrightarrow K(pt)$$

is the direct image map corresponding to the projection $p:X\longrightarrow\{pt\}$ *into a point.*

Now suppose that problems $\mathbf{A}$ and $\mathbf{B}$ act in distinct spaces, namely,

$$\begin{aligned}\mathbf{A}&:\mathcal{W}(\mathcal{M},E)\oplus L^2(X,J_1)\longrightarrow\mathcal{W}(\mathcal{M},F)\oplus L^2(X,J_2),\\ \mathbf{B}&:\mathcal{W}(\mathcal{M},E)\oplus L^2(X,\widetilde{J}_1)\longrightarrow\mathcal{W}(\mathcal{M},F)\oplus L^2(X,\widetilde{J}_2).\end{aligned}$$

We embed J_1 and J_2 as well as $\widetilde{J}_1$ and $\widetilde{J}_2$ in an infinite-dimensional Hilbert bundle H over X. The orthogonal complements $J_1^\perp$ and $J_2^\perp$ are infinite-dimensional and hence isomorphic. Let

$$\chi : J_1^\perp \longrightarrow J_2^\perp$$

be an isomorphism, and let

$$\widetilde{\chi} : \widetilde{J}_1^\perp \longrightarrow \widetilde{J}_2^\perp$$

be a similar isomorphism for the bundles with tildes. Then problems $\mathbf{A} \oplus \chi$ and $\mathbf{B} \oplus \widetilde{\chi}$ act in the same spaces

$$\mathcal{W}(\mathcal{M}, E) \oplus L^2(X, H) \longrightarrow \mathcal{W}(\mathcal{M}, F) \oplus L^2(X, H),$$

are elliptic, and have the same indices as the original problems. Accordingly, the above argument applies (since the finite dimension of bundles over X was not used in it), which shows that Theorem 1.3 remains valid for this case provided the construction of the operator $\sigma_\wedge(\mathbf{A})\sigma_\wedge(\mathbf{B})^{-1}$ uses the infinite-dimensional bundle H. Moreover, the assertion remains valid even if one is allowed to change the edge symbol of the operator A itself (provided that the edge symbol remains Fredholm) and only the principal symbol of A remains unchanged.

2 Index as the sum of stratum contributions

The index of the elliptic problem (0.1) is a homotopy invariant of the compatible pair $(\sigma(A), \sigma_\wedge(\mathbf{A}))$ of elliptic symbols:

$$\operatorname{ind} \mathbf{A} = F(\sigma(A), \sigma_\wedge(\mathbf{A})). \tag{2.1}$$

The manifold $\mathcal{M}$ with edge X can be viewed as a *stratified space* with two strata, namely, the *interior stratum* $\overset{\circ}{\mathcal{M}} \equiv \overset{\circ}{M} = \mathcal{M} \setminus X$ and the *boundary stratum* X (the edge). Associated with each stratum is a component of the symbol. (The principal symbol is related to the interior, and the edge symbol is related to the edge.) The compatibility condition for the symbol components can be viewed as naturally related to the adjacency of the strata. Since the stratification of the manifold is naturally reflected in the symbolic structure, it is logical to ask if the index of problem (0.1) can be represented as the sum of contributions corresponding to separate strata, or, which is the same, to separate components of the symbol. Hence the following statement of the problem can be considered. Compute the functional on the right-hand side in (2.1) as the sum of two terms, one of which depends only on the principal symbol $\sigma(A)$ and the second only on the edge symbol $\sigma_\wedge(\mathbf{A})$. More precisely, we wish to represent the index in the form

$$\operatorname{ind} \mathbf{A} = f(\sigma(A)) + g(\sigma_\wedge(\mathbf{A})), \tag{2.2}$$

where $f(\sigma(A))$ is a functional depending only on the principal symbol and homotopy invariant on the class of elliptic principal symbols and $g(\sigma_\wedge(\mathbf{A}))$ depends only on the edge symbol and is homotopy invariant on the class of elliptic edge symbols. The problem of representing the index of problem (0.1) in the form (2.2) will be called the *problem on a* (*homotopy invariant*) *splitting of the index* (*into stratum contributions*). This problem is the subject of the present paper.

Remark 2.1. The homotopy invariance requirement in this setting is very natural, since the left-hand side of (2.1) is homotopy invariant. However, this requirement is much stronger, for we assume the homotopy invariance of each term *separately*, so that the sum proves to be homotopy invariant on the set of *all* pairs $(\sigma(A), \sigma_\wedge(\mathbf{A}))$ rather than only pairs satisfying the compatibility condition (0.2). Hence it is not surprising at all that (as we shall see in the next subsection) certain additional conditions are needed for the existence of a homotopy invariant splitting (2.2).

Remark 2.2. The problem stated above can be interpreted as a problem on an *index defect*, that is, the problem of finding a functional that depends only on the edge symbol and restores the homotopy invariance (which anyway holds if the edge is empty) of the index of problem (0.1):

$$\operatorname{ind} \mathbf{A} + \text{ind-def}\, \sigma_\wedge(\mathbf{A}) = f(\sigma(A)). \tag{2.3}$$

In other situations, index defects occurred, say, in [35] and in the survey [33]. Here we do not discuss this interpretation.

2.1 A criterion for the existence of invariant splittings

First, we note that the problem on a homotopy invariant splitting of the index into stratum contributions has no solution in the class of *all* elliptic edge problems in general. A similar situation takes place for spectral boundary value problems [32] or for operators on manifolds with conical singularities (e.g., see [20]). Namely, consider a class Σ of elliptic symbols on $T^*\mathcal{M}$ for which the obstruction (A.44) is zero. By Σ_∂ we denote the set of restrictions of symbols $\sigma \in \Sigma$ to $\partial T^*\mathcal{M}$. We assume that the class Σ is completely determined by these restrictions; in other words, if $\sigma_\partial \in \Sigma_\partial$, then $\sigma \in \Sigma$. Let $\{\sigma_{\partial t}\} \subset \Sigma_\partial$ be a continuous periodic family. It can be interpreted as a symbol on $T^*(\partial M \times \mathbb{S}^1)$ and determines, via the difference construction, an element

$$[\sigma_{\partial t}] \in K(T^*(\partial M \times \mathbb{S}^1)) \tag{2.4}$$

of the K-group with compact supports. The following theorem holds.

Theorem 2.3. *There exists an invariant*[3] *splitting* (2.2) *for the class of Fredholm edge problems with principal symbols in* Σ *if and only if*

$$p_![\sigma_{\partial t}] = 0, \quad p : \partial M \times \mathbb{S}^1 \to \{pt\} \tag{2.5}$$

[3] Here we mean that the second term must be defined and homotopy invariant on the class of elliptic edge symbols with principal symbols in Σ_∂.

for each continuous periodic family $\{\sigma_{\partial t}\}_{t\in\mathbb{S}^1} \subset \Sigma_\partial$.

Proof. Consider a periodic family $\{\sigma_{\partial t}\}_{t\in\mathbb{S}^1} \subset \Sigma_\partial$ and two elliptic principal symbols σ and $\widetilde{\sigma}$ such that $\sigma_\partial = \widetilde{\sigma}_\partial = \sigma_{\partial 0}$, $\sigma = \widetilde{\sigma}$ outside T^*U, σ is independent of the radial variable r in T^*U, and

$$\widetilde{\sigma}(x, \omega, r, \xi, q, p) = \sigma_{\partial, 2\pi r}(x, \omega, \xi, q, p).$$

(In other words, $\widetilde{\sigma}$ can be obtained from σ by attaching the periodic family in the collar neighborhood T^*U.) We can equip σ and $\widetilde{\sigma}$ with the same elliptic edge symbol $\sigma_\wedge$. Then we obtain two Fredholm edge problems, which will be denoted by $\mathbf{A}$ and $\widetilde{\mathbf{A}}$. These problems have principal symbols homotopic in Σ and the same edge symbols. The difference of indices of these problems is given by the formula

$$\operatorname{ind} \widetilde{\mathbf{A}} - \operatorname{ind} \mathbf{A} = \operatorname{ind} \mathbf{B},$$

where $\mathbf{B}$ is the edge problem with principal symbol $\sigma(\mathbf{B}) = \widetilde{\sigma} \cdot \sigma^{-1}$ and with unit edge symbol. Note that the principal symbol is equal to unity on $\partial T^*\mathcal{M}$ as well as outside T^*U. By applying surgery and the relative index theorem in [24], we find that the index of problem $\mathbf{B}$ is equal to the index of an elliptic operator on $\partial M \times \mathbb{S}^1$ whose principal symbol is equal to $\sigma_{\partial t}$ multiplied by the family $\sigma_{\partial 0}^{-1}$, which is independent of t. Applying the Atiyah–Singer theorem, we see that this index is equal to the left-hand side of (2.5). The necessity of condition (2.5) is now clear, and the sufficiency can be proved in the same way as in [32]. $\square$

Remark 2.4. This theorem is essentially the same as the corresponding theorems for spectral boundary value problems and for operators on manifolds with conical singularities [32, 20]. (Indeed, $p_![\sigma_{\partial t}]$ is equal to the spectral flow of the periodic family of operators elliptic with parameter p on ∂M with symbols $\sigma_{\partial t}$. In contrast with the cited papers, we do not use the spectral flow neither in the statement, nor in the proof, nor in applications of this theorem.) We note especially that the edge structure of the manifold (i.e., in terms of M, the projection $\pi : \partial M \longrightarrow X$) plays no role in this theorem at all: *the obstruction to the existence of homotopy invariant splittings of the index is the same for the manifold M with boundary and for the manifolds with conical or edge singularities for which M is the stretched manifold.*

Now if Σ_∂ is the set of restrictions to $\partial T^*\mathcal{M}$ of all elliptic symbols for which the obstruction (A.44) to the existence of Fredholm problems is zero, then the assumption of the theorem is in general not satisfied, and so there are no homotopy invariant splittings. To solve the problem, one has to single out narrower classes of elliptic symbols. In the subsequent sections we explain how this can be done.

2.2 A relationship with the splitting problem on a manifold with conical singularities

Remark 2.4 suggests that the index splitting problem for a manifold with edges and the corresponding index splitting problem for a manifold with conical singularities are closely related. This is indeed the case. In a sense, the former problem can be reduced to the latter. Let us explain this assertion. Along with $\mathcal{M}$, consider the manifold $\mathfrak{M}$ with conical singularities that has the same stretched manifold M as $\mathcal{M}$. The compressed cotangent bundles $T^*\mathcal{M}$ and $T^*\mathfrak{M}$ are (noncanonically) isomorphic to T^*M and hence isomorphic to each other. Let us choose an isomorphism. Then to each class Σ of symbols on $T^*\mathcal{M}$ satisfying the criterion in Theorem 2.3 there corresponds a class of symbols on $T^*\mathfrak{M}$ with the same property. We denote it by the same letter (or, in other words, identify $T^*\mathcal{M}$ with $T^*\mathfrak{M}$ via this isomorphism). Let Σ be such a class, and suppose that the index splitting problem has been solved for $\mathfrak{M}$ in the class of operators with principal symbols from Σ. In other words, we have two explicitly written out homotopy invariant functions $\widetilde{f}$ and $\widetilde{g}$ such that for an elliptic operator

$$A : H^{s,0}(\mathfrak{M}) \longrightarrow H^{s-m,0}(\mathfrak{M}) \tag{2.6}$$

with principal symbol $\sigma(A) \in \Sigma$ and conormal symbol $\sigma_c(A)$ the index formula

$$\operatorname{ind} A = \widetilde{f}(\sigma(A)) + \widetilde{g}(\sigma_c(A)) \tag{2.7}$$

is valid. (The weight exponent in (2.6) is taken to be zero without loss of generality. The passage to any other exponent merely changes the second term on the right-hand side in (2.7) by a summand given in closed form by the relative index theorem for pseudodifferential operators on manifolds with conical singularities; e.g., see [42] and references therein.)

Now let us construct the solution of the problem on a homotopy invariant splitting of the index in the class Σ of principal symbols for a manifold $\mathcal{M}$ with an edge. We take an arbitrary order problem $\mathbf{A}$ with principal symbol $\sigma(A) \in \Sigma$ and edge symbol $\sigma_\wedge(\mathbf{A})$. Using a homotopy if necessary, we can assume that $\sigma(A)$ is independent of the radial variable r over U. We cut U away from $\mathcal{M}$ and shrink the cut hypersurface $\partial M \times \{r = 1\}$ into a conical singular point on both resulting parts. Then one of the parts, $\mathcal{M} \setminus U$, becomes a manifold with conical singularities, which can be identified with $\mathfrak{M}$. The second part, $\overline{U}$, becomes a more complicated manifold with singularities, which will be denoted by $\mathfrak{U}$. The smooth part of $\mathfrak{U}$ is just the direct product $\partial M \times (0, 1)$, but for $r = 0$ is has the same edge X as the original manifold M, and for $r = 1$ it has a conical singular point. Let us take an arbitrary elliptic conormal symbol σ_c compatible with $\sigma(A)$. The pair $(\sigma(A), \sigma_c)$ determines (modulo compact operators) a Fredholm operator $\widetilde{A}$ on $\mathfrak{M}$, and the triple $(\sigma_\partial(A), \sigma_\wedge(\mathbf{A}), \sigma_c)$ determines a Fredholm problem $\widetilde{\mathbf{A}}$ on $\mathfrak{U}$. (Here the boundary symbol $\sigma_\partial(A)$ is extended to the entire $T_0^*\mathfrak{U} = \big((T^*\partial M \times \mathbb{R}) \setminus \{0\}\big) \times [0, 1]$ as a function independent of $r \in [0, 1]$.) We note that, up to the ambiguity in the choice of a compatible conormal symbol σ_c, the problem $\widetilde{\mathbf{A}}$ is determined modulo compact operators by the edge symbol $\sigma_\wedge(\mathbf{A})$, since $\sigma_\partial(A)=\sigma(\sigma_\wedge(A))$ by the compatibility condition.

We apply surgery and the index locality principle in [24], thus obtaining

$$\operatorname{ind}\mathbf{A}=\operatorname{ind}\widetilde{\mathbf{A}}+\operatorname{ind}\widetilde{A}.$$

Next, we combine this with (2.7) and see that

$$\operatorname{ind}\mathbf{A}=\widetilde{f}(\sigma(A))+\bigl[\operatorname{ind}\widetilde{\mathbf{A}}+\widetilde{g}(\sigma_c)\bigr]. \tag{2.8}$$

The first term is a homotopy invariant of the principal symbol, and we set

$$f(\sigma(A))=\widetilde{f}(\sigma(A)).$$

To show that the expression in brackets is an invariant of the edge symbol, we choose an arbitrary quantization that assigns a compatible conormal symbol $\widetilde{\sigma}_c$ to each boundary symbol $\sigma_\partial(A)$ and set

$$\sigma_c(p)=\widetilde{\sigma}_c(p+i\gamma),$$

where $\gamma\in(0,1)$ is so small that $\widetilde{\sigma}_c(p)$ is invertible on the lines $\{\operatorname{Im}p=\delta\}$ for all $\delta\in(0,\gamma]$. By the relative index theorem for pseudodifferential operators on manifolds with conical singularities, neither of the two terms in brackets depends on the specific choice of γ, and we see that each of them, as well as the sum, is completely determined by the edge symbol. Thus, we can set

$$g(\sigma_\wedge(\mathbf{A}))=\operatorname{ind}\widetilde{\mathbf{A}}+\widetilde{g}(\sigma_c). \tag{2.9}$$

The homotopy invariance of this functional follows from the index formula and the homotopy invariance of the first term.

Remark 2.5. Needless to say, our argument in this subsection is somewhat abstract, since the functional (2.9) has a complicated structure depending on an arbitrary (though fixed) quantization

$$\sigma_\partial(A)\mapsto\widetilde{\sigma}_c.$$

Note that the index formulas given further in this preprint are derived from completely different ideas, without using (2.9) even as an intermediate result.

3 The homotopy invariant functional of the principal symbol

In the preceding section, we have shown that under the necessary and sufficient condition given by Theorem 2.3 the index of an elliptic problem on a manifold with edges can be represented as the sum of a homotopy invariant functional of the principal symbol of the problem and a correction term that depends only on the edge symbol. In this section, we compute the main term in the index formula, namely, the above-mentioned homotopy invariant functional. Needless to say, this is hardly a tractable problem for general symbol classes satisfying the criterion given by the theorem. Hence we introduce quite natural *symmetry conditions* guaranteeing that the criterion holds and compute the homot5opy invariant of the principal symbol for symbol classes satisfying these conditions.

3.1 Symmetry conditions for the principal symbol at the edge

The criterion in 2.3 for the existence of a homotopy invariant index splitting is stated in terms of the boundary symbols $\sigma_\partial(A)$. Hence it is natural to single out symbol classes satisfying this criterion with the help of conditions imposed on the boundary symbol. Since the criterion is independent of the structure of the edge, we use the idea in [20] (where the case of manifolds with conical singularities was considered), generalizing the parity conditions in [36, 37], and consider symbols invariant with respect to some involution in the real vector bundle

$$\partial T^*\mathcal{M} \simeq T^*\partial M \times \mathbb{R} \equiv T^*\partial M \oplus \mathbf{1}$$

(here $\mathbf{1}$ is the trivial one-dimensional real bundle) over the base ∂M. Thus, consider an involution

$$\alpha : T^*\partial M \oplus \mathbf{1} \longrightarrow T^*\partial M \oplus \mathbf{1}, \tag{3.1}$$

that is, an automorphism of the vector bundle $T^*\partial M \oplus \mathbf{1}$ over ∂M such that $\alpha^2 = 1$.

We separately consider the following two cases:

- α reverses the orientation ($\det\alpha = -1$);
- α preserves the orientation ($\det\alpha = 1$).

Definition 3.1. Let α reverse the orientation. By Σ_α we denote the set of triples of the form $(a, \varepsilon_E, \varepsilon_F)$, where

$$a : \pi^*_{\mathcal{M}}E \longrightarrow \pi^*_{\mathcal{M}}F \quad \pi_{\mathcal{M}} : T^*_0\mathcal{M} \longrightarrow M$$

is an elliptic symbol acting between the lifts to $T^*\mathcal{M}$ of vector bundles E and F over M and

$$\varepsilon_E : E|_{\partial M} \longrightarrow E|_{\partial M}, \quad \varepsilon_F : F|_{\partial M} \longrightarrow F|_{\partial M}$$

are involutions acting in the restrictions of these bundles to ∂M; moreover, the boundary symbol $a_\partial = a|_{\partial T^*\mathcal{M}}$ satisfies the symmetry condition

$$\varepsilon_F \alpha^*(a_\partial)\varepsilon_E = a_\partial, \tag{3.2}$$

or, in other words, is equivariant with respect to the representations of the group $\mathbb{Z}_2$ given by the involutions α, ε_E, and ε_F.

The set Σ_α will be referred to as the set of *α-symmetric elliptic symbols.* By $\widetilde{\Sigma}_\alpha \subset \Sigma_\alpha$ we denote the subset of symbols for which the obstruction (A.44) is zero.

Remark 3.2. We point out that admissible homotopies in Σ_α are not just homotopies of the symbol a itself but rather homotopies of the entire *triple*; in other words, ε_E and ε_F must also vary continuously, and condition (3.2) must hold for all values of the homotopy parameter. Throughout the following, homotopy invariance is understood as invariance with respect to admissible homotopies. Thus, formally we deal with a situation slightly more general than the one considered in Theorem 2.3. However, the theorem remains valid in this case. We have simplified that statement only on order not to make the exposition awkward.

Definition 3.3. Now let α preserve the orientation. By Σ_α we denote the set of pairs (a, ε), where

$$a : \pi^*_{\mathcal{M}} E \longrightarrow \pi^*_{\mathcal{M}} F \quad \pi_{\mathcal{M}} : T^*_0 \mathcal{M} \longrightarrow M$$

is an elliptic symbol acting between the lifts to $T^*\mathcal{M}$ of vector bundles E and F over M and

$$\varepsilon : E|_{\partial M} \longrightarrow F|_{\partial M}$$

is an isomorphism of the restrictions of these bundles to ∂M; moreover, the boundary symbol $a_\partial = a|_{\partial T^*\mathcal{M}}$ satisfies the antisymmetry condition

$$\alpha^*(a_\partial) = \varepsilon a_\partial^{-1} \varepsilon \tag{3.3}$$

on the unit spheres in $\partial T^*\mathcal{M}$.

The set Σ_α will be called the set of *α-antisymmetric elliptic symbols*. By $\widetilde{\Sigma}_\alpha \subset \Sigma_\alpha$ we denote the subset of symbols for which the obstruction (A.44) is zero.

Remark 3.4. Note that, by passing to the symbol $a\varepsilon^{-1}$, one can assume without loss of generality that ε is the identity isomorphism. (This passage can be made only in a neighborhood of the edge in the general case.)

Remark 3.5. If the bundle E is equipped with an inner product, then one can also consider the class of symbols $a : \pi^*_M E \longrightarrow \pi^*_M E$ satisfying the condition

$$\alpha^* a_\partial = a^*_\partial,$$

where a^*_∂ is the adjoint symbol. This condition obviously coincides with the preceding condition (with identity isomorphism ε) on unitary symbols. In the general case, the relationship between the two conditions is given by the polar decomposition theorem. More precisely, one can verify that the validity of one of these conditions for an elliptic symbol implies the other condition for the isometric part of the symbol.

The following proposition shows that homotopy invariant index splittings for problems with principal symbols in $\widetilde{\Sigma}_\alpha$ exist in the symmetric (α reverses the orientation) as well as the antisymmetric (α is orientation-preserving) case.

Proposition 3.6. *The criterion* (2.5) *in Theorem* 2.3 *holds for an arbitrary admissible periodic homotopy of elliptic symbols in* Σ_α *(and so much the more in* $\widetilde{\Sigma}_\alpha$*).*

The proof reproduces that of Theorem 9 in [20] word for word. □

3.2 The Hirzebruch operator associated with the principal symbol

The desired homotopy invariant of the principal symbol is given by the index of some elliptic operator on a smooth closed manifold, more precisely, the Hirzebruch operator with coefficients in the bundle determined by the principal symbol. In this subsection, we describe this smooth closed manifold and construct the Hirzebruch operator. Note that the construction is defined for arbitrary principal symbols in Σ_α, even though elliptic edge problems exist only for symbols in the subclass $\widetilde{\Sigma}_\alpha$.

The closed manifold. Consider the Atiyah–Bott–Patodi space [5]

$$D^*\mathcal{M} \overset{\text{def}}{\equiv} S\Big(T^*\mathcal{M} \oplus \mathbf{1}\Big)$$

of the compressed cotangent bundle $T^*\mathcal{M}$. The Atiyah–Bott–Patodi space is a bundle over M with fiber $\mathbb{S}^{n+k+1}$ and is a smooth even-dimensional manifold with boundary. (More precisely, the dimension is equal to $2(n+k+1)$). It can also be viewed as the gluing of two unit disk bundles for $T^*\mathcal{M}$ along the unit spheres. The desired closed manifold is the gluing of two copies of the Atiyah–Bott–Patodi space along their common boundary, the gluing being constructed as follows with the use of the involution α. It will be more convenient for us to glue not the Atiyah–Bott–Patodi spaces themselves, but the bundles $T^*\mathcal{M} \oplus \mathbf{1}$, and only then pass to the spheres.

We extend the involution $\alpha : T^*\mathcal{M}|_{\partial M} \to T^*\mathcal{M}|_{\partial M}$ to the orientation-reversing involution

$$\widetilde{\alpha} : T^*\mathcal{M}|_{\partial M} \oplus \mathbf{1} \longrightarrow T^*\mathcal{M}|_{\partial M} \oplus \mathbf{1}, \quad \widetilde{\alpha} = \alpha \oplus \pm 1, \tag{3.4}$$

where the plus sing is taken in the symmetric case (where $\det \alpha = -1$) and the minus sign in the antisymmetric case (where $\det \alpha = 1$). In both cases, we obtain $\det \widetilde{\alpha} = -1$. The involution $\widetilde{\alpha}$ permits one to glue two copies of the vector bundle $T^*\mathcal{M} \oplus \mathbf{1}$ along their common boundary by identifying the points (y, ζ, t) and $(y, \widetilde{\alpha}(\zeta, t))$ in the first and second copies of $T^*\mathcal{M} \oplus \mathbf{1}$, respectively, in the fibers over the same boundary point $y \in \partial M$. As a result, we obtain the vector bundle

$$(T^*\mathcal{M} \oplus \mathbf{1}) \underset{\widetilde{\alpha}}{\cup} (T^*\mathcal{M} \oplus \mathbf{1}) \in \operatorname{Vect}(2M) \tag{3.5}$$

over the double $2M$ of M. Since $\widetilde{\alpha}$ reverses the orientation in the fibers, it follows that the total space of this bundle is oriented by the canonical orientation of the compressed cotangent bundle on each of the two copies of $T^*\mathcal{M}$. (The change in the orientation under α when passing from one copy to the other is compensated for by another change in the orientation caused by the opposite direction of the r-axis.) Finally, consider the bundle

$$D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M} \overset{\text{def}}{\equiv} S\Big((T^*\mathcal{M} \oplus \mathbf{1}) \underset{\widetilde{\alpha}}{\cup} (T^*\mathcal{M} \oplus \mathbf{1})\Big)$$

of unit spheres in the vector bundle $(T^*\mathcal{M} \oplus \mathbf{1}) \underset{\widetilde{\alpha}}{\cup} (T^*\mathcal{M} \oplus \mathbf{1})$. Its total space is also oriented and is an even-dimensional smooth closed manifold.

The vector bundle corresponding to the principal symbol. On $D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M}$, we now construct a vector bundle corresponding to an α-(anti)symmetric principal symbol a. Just as in the standard Atiyah–Bott–Patodi construction [5], the symbol a determines a vector bundle

$$[a] \in \mathrm{Vect}(D^*\mathcal{M})$$

by the gluing of the lifts of E and F to the unit ball bundles in $T^*\mathcal{M}$ via the isomorphism a defined on the unit spheres.

A straightforward computation shows that the restriction of the bundle $[a]$ to the boundary $\partial D^*\mathcal{M}$ is a $\mathbb{Z}_2$-bundle with respect to the natural action of the involution $\widetilde{\alpha}$. Consequently,

$$[a \underset{\widetilde{\alpha}}{\cup} a] \in \mathrm{Vect}(D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M}))$$

is a well-defined vector bundle on $D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M}$.

The Hirzebruch operator. Using the bundle thus constructed, we define an elliptic operator on $D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M}$ corresponding to the elliptic symbol a. By

$$\mathcal{H} = \mathcal{H}_{D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M}}$$

we denote the Hirzebruch (signature) operator (e.g., see [28]) on $D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M}$. We define an operator $\mathcal{H}_a$ by setting

$$\mathcal{H}_a = \mathcal{H} \otimes (\Lambda^*(2M) \otimes \mathbb{C})^{-1} \otimes [a \underset{\widetilde{\alpha}}{\cup} a], \tag{3.6}$$

where the second factor is a virtual bundle (a formal difference of ordinary bundles) determined by the exterior form bundle $\Lambda^*(2M)$ as the right-hand side of the expression

$$\begin{aligned}(\Lambda^*(2M))^{-1} &\equiv \frac{1}{2^{n+k+1} - (2^{n+k+1} - \Lambda^*(2M))} \\ &= \frac{1}{2^{n+k+1}} \left(1 + \sum_{l \geq 1} (1 - 2^{-n-k-1}\Lambda^*(2M))^l \right) \in K(2M) \otimes \mathbb{Z}\left[\frac{1}{2}\right].\end{aligned} \tag{3.7}$$

On a finite-dimensional space, this series has only finitely many nonzero terms in the following sense: for sufficiently large l, the element $(2^{n+k+1} - \Lambda^*(2M))^l$ is zero in the K-group (see [4]). The second factor can also be described in terms of the Grothendieck operations in K-theory:

$$[\Lambda^*(E)]^{-1} = \gamma_{1/2}^{-1}(E), \quad \text{where} \quad \gamma_t(x) = \lambda_{t/(1-t)}(x).$$

Remark 3.7. In fact, our Hirzebruch operator is an element of the group

$$\operatorname{Ell}\big(D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M}\big) \otimes \mathbb{Z}\left[\frac{1}{2}\right],$$

i.e., is represented by a formal difference of two elliptic operators with dyadic coefficients (rationals whose denominators are powers of 2).

3.3 The index of the Hirzebruch operator as a homotopy invariant functional

By construction, the index of $\mathcal{H}_a$ is determined by the symbol a and is a homotopy invariant of a in the class Σ_α. First, let us compute it in cohomological terms.

By$[D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M}] \in H_{2n+2k+2}\big(D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M}\big)$ we denote the fundamental cycle.

Proposition 3.8 (the cohomological formula)**.** *One has*

$$\operatorname{ind}\mathcal{H}_a = \langle ch\big[a \underset{\widetilde{\alpha}}{\cup} a\big]\pi^* Td(T(2M)\otimes\mathbb{C}), [D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M}]\rangle,$$

where $\pi : D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M} \longrightarrow 2M$ *is the natural projection.*

Proof. 1. We us the cohomological expression

$$\langle ch(\Lambda^*(2M)\otimes\mathbb{C})^{-1} ch\big[a \underset{\widetilde{\alpha}}{\cup} a\big] L(T(D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M})), [D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M}]\rangle$$

for the index of the Hirzebruch operator. In view of this formula, it suffices to prove the relation

$$\pi^* Td(T(2M)\otimes\mathbb{C}) = L\big(T(D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M})\big) ch(\Lambda^*(2M)\otimes\mathbb{C})^{-1} \in H^*(D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M})$$

between the Todd class and the L-class of a real vector bundle. (In the Borel–Hirzebruch formalism, the latter class corresponds to the function $x/\tanh(x/2)$; e.g., see [28].) We establish this relation in the subsequent two items.

2. First, we note that the tangent bundle $T(D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M})$ and the bundle $\pi^*(T(2M)\oplus T(2M))$ determine the same element modulo 2-torsion. To prove this, we recall the standard isomorphism

$$TSV \oplus \mathbf{1} \simeq \pi^*(TB \oplus V), \quad \pi : SV \to B,$$

which is valid for an arbitrary vector bundle $V \in \operatorname{Vect}(B)$ over a smooth base B. (To establish the isomorphism, one decomposes tangent vectors into horizontal and vertical components and realizes the one-dimensional trivial bundle as the tangent bundle to the sphere.) Using this result, we obtain the isomorphism

$$T(D^*\mathcal{M} \underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M}) \oplus \mathbf{1} \simeq \pi^*\Big(T(2M) \oplus ((T^*\mathcal{M}\oplus\mathbf{1}) \underset{\widetilde{\alpha}}{\cup} (T^*\mathcal{M}\oplus\mathbf{1}))\Big).$$

But the bundle $(T^*\mathcal{M}\oplus\mathbf{1})\underset{\widetilde{\alpha}}{\cup}(T^*\mathcal{M}\oplus\mathbf{1})$ is obtained from the bundle $T^*(2M)\oplus\mathbf{1}$ by twisting with the use of the involution $\widetilde{\alpha}$. Hence the two bundles coincide in the K-group modulo 2-torsion, and we obtain the desired relation

$$[T(D^*\mathcal{M}\underset{\widetilde{\alpha}}{\cup}D^*\mathcal{M})]=[\pi^*(T(2M)\oplus T(2M))]\in K(D^*\mathcal{M}\underset{\widetilde{\alpha}}{\cup}D^*\mathcal{M})\otimes\mathbb{Z}\left[\frac{1}{2}\right].$$

3. By virtue of item 2, to verify the equality of characteristic classes in item 1, it suffices to prove the following relation on the base $2M$:

$$Td(T(2M)\otimes\mathbb{C})=\big(L(T(2M))\big)^2 ch(\Lambda^*(2M)\otimes\mathbb{C})^{-1}. \tag{3.8}$$

In the Borel–Hirzebruch formalism, the left- and right-hand sides of the formula correspond to the functions

$$\left(\frac{x/2}{\sinh x/2}\right)^2 \quad\text{and}\quad \left(\frac{x}{\tanh x/2}\right)^2\frac{1}{(1+e^x)(1+e^{-x})},$$

which coincide identically. The proof of the proposition is complete. □

Now we shall prove that the homotopy invariant thus constructed (and multiplied by 1/2) is indeed the main term in the index formula for an elliptic edge problem. We need the following lemma.

Lemma 3.9. 1) *Let P be an elliptic pseudodifferential operator on the double $2M$. Then*

$$\operatorname{ind}P=\operatorname{ind}\left(\mathcal{H}_{D^*(2M)}\otimes(\Lambda^*(2M)\otimes\mathbb{C})^{-1}\otimes[\sigma(P)]\right),\quad [\sigma(P)]\in\operatorname{Vect}(D^*(2M)).$$

2) *Let a be an elliptic symbol on $T^*\mathcal{M}$ equivariant with respect to either of the involutions $\alpha_0: p\mapsto -p$ and α. Namely, writing out the equivariance conditions explicitly, $\alpha_0^*a=\varepsilon_{0F}a\varepsilon_{0E}$, and either $\alpha^*a=\varepsilon_F a\varepsilon_E$ (in the symmetric case) or $\alpha^*a=\varepsilon a^{-1}\varepsilon$ (in the antisymmetric case). Furthermore, let one of the following two conditions be satisfied:*

1) *the restriction $a|_{\partial T^*\mathcal{M}}$ is a bundle isomorphism lifted from the base;*

2) *the involutions α_0 and α commute, and the bundle morphisms occurring in the definition of equivariance are compatible in the sense that*

$$\varepsilon_{0E}\varepsilon_E=\varepsilon_E\varepsilon_{0E},\quad \varepsilon_{0F}\varepsilon_F=\varepsilon_F\varepsilon_{0F}$$

in the symmetric case and

$$\varepsilon_{0F}\varepsilon=\varepsilon\varepsilon_{0E}$$

in the antisymmetric case.

Then

$$\operatorname{ind}\big(\mathcal{H}_{D^*(2M)} \otimes (\Lambda^*(2M)\otimes\mathbb{C})^{-1} \otimes [a\cup a]\big)$$
$$= \operatorname{ind}\Big(\mathcal{H}_{D^*\mathcal{M}\underset{\widetilde{\alpha}}{\cup} D^*\mathcal{M}} \otimes (\Lambda^*(2M)\otimes\mathbb{C})^{-1} \otimes \big[a \underset{\widetilde{\alpha}}{\cup} a\big]\Big) \equiv \operatorname{ind}\mathcal{H}_a, \quad (3.9)$$

where $\cup$ stands for the gluing corresponding to the involution $p \mapsto -p$.

Proof. 1) The Atiyah–Singer formula on the smooth closed manifold $2M$ in the (cohomological) Atiyah–Bott–Patodi form reads

$$\operatorname{ind} P = \langle ch[\sigma(P)]\pi^* Td(T(2M)\otimes\mathbb{C}), [D^*(2M)]\rangle .$$

On the other hand, let us apply the cohomological Atiyah–Singer formula to the Hirzebruch operator on the right-hand side of the relation to be proved. The index of the Hirzebruch operator is equal to

$$\Big\langle L\big(TD^*(2M)\big) ch(\Lambda^*(2M)\otimes\mathbb{C})^{-1} ch[\sigma(P)], [D^*(2M)]\Big\rangle.$$

Now the desired relation follows from the equality of characteristic classes in item 1 in the proof of Proposition 3.8 (with the involution $p \mapsto -p$).

2) The manifolds on which the operators on the left- and right-hand sides of the desired relation are defined are obtained from each other by one-codimensional surgery along the submanifold $D^*(2M)|_{\partial M}$. It is well known (e.g., see [7] or [24]) that the difference of indices of operators under such surgery is equal to the index of a certain elliptic operator $T_{\widetilde{\beta}}$ on the mapping torus

$$\widetilde{\beta} : D^*(2M)|_{\partial M} \longrightarrow D^*(2M)|_{\partial M},$$

where $\beta = \alpha_0\alpha$. The principal symbol of $T_{\widetilde{\beta}}$ is constant along the generator of the cylinder and coincides with the restriction of the symbol a of the original operator to the boundary $D^*(2M)|_{\partial M}$. In the first of the two cases indicated in the lemma, this symbol is equal to a bundle isomorphism lifted from the base, and so $\operatorname{ind} T_{\widetilde{\beta}} = 0$. In the second case, we use the relation

$$\operatorname{ind} T_{\widetilde{\beta}^2} = 2 \operatorname{ind} T_{\widetilde{\beta}}.$$

But $\widetilde{\beta}^2 = Id$, since α commutes with α_0. The gluing of the bundles for $T_{\widetilde{\beta}^2}$ is also by the identity morphism by virtue of our conditions. Hence the index of T_β is zero, and the index is not affected by the surgery.

The proof is complete. □

Now we can state and prove the main theorem of this section.

Theorem 3.10. *Let* $\mathbf{A}$ *be an elliptic edge problem of the form* (0.1) *with principal symbol* $\sigma(A) \in \widetilde{\Sigma}_\alpha$. *The difference*

$$\operatorname{ind} \mathbf{A} - \frac{1}{2} \operatorname{ind} \mathcal{H}_{\sigma(A)} \tag{3.10}$$

is independent of $\sigma(A)$ *and is completely determined by the edge symbol* $\sigma_\wedge(\mathbf{A})$. *Thus, half the index of the Hirzebruch operator* $\mathcal{H}_{\sigma(A)}$ (3.6) *is indeed the main term in the index splitting formula* (2.2).

Proof. Consider two edge problems $\mathbf{A}$ and $\widetilde{\mathbf{A}}$ with the same edge symbol

$$\mathbf{a}_\wedge \equiv \sigma_\wedge(\mathbf{A}) = \sigma_\wedge(\widetilde{\mathbf{A}})$$

and with principal symbols $\sigma(A)$ and $\sigma(\widetilde{A}) \in \widetilde{\Sigma}_\alpha$. We must prove that

$$\operatorname{ind} \widetilde{\mathbf{A}} - \operatorname{ind} \mathbf{A} = \frac{1}{2}\{\operatorname{ind} \mathcal{H}_{\sigma(\widetilde{A})} - \operatorname{ind} \mathcal{H}_{\sigma(A)}\}. \tag{3.11}$$

Consider the problem $\mathbf{B} = \widetilde{\mathbf{A}} \oplus \mathbf{A}^{-1}$. (By $\mathbf{A}^{-1}$, as usual, we understand the *almost inverse* problem, i.e., the problem whose principal and edge symbols are the inverses of the corresponding symbols of $\mathbf{A}$.) Its edge symbol is

$$\mathbf{b}_\wedge = \begin{pmatrix} \mathbf{a}_\wedge & 0 \\ 0 & \mathbf{a}_\wedge^{-1} \end{pmatrix}, \tag{3.12}$$

and the boundary symbol is

$$b_\partial = \begin{pmatrix} a_\partial & 0 \\ 0 & a_\partial^{-1} \end{pmatrix}. \tag{3.13}$$

The (anti)symmetry condition for b_∂ can be written in the form

$$\alpha^* b_\partial = \varepsilon_1 b_\partial \varepsilon_2, \quad \text{where } \varepsilon_1 = \begin{pmatrix} \varepsilon_F & 0 \\ 0 & \varepsilon_E \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} \varepsilon_E & 0 \\ 0 & \varepsilon_F \end{pmatrix}, \tag{3.14}$$

in the symmetric case and

$$\alpha^* b_\partial = \varepsilon_0 b_\partial^{-1} \varepsilon_0, \quad \text{where } \varepsilon_0 = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon^{-1} \end{pmatrix}, \tag{3.15}$$

in the antisymmetric case.

Let us perform a homotopy of the problem $\mathbf{B}$ as follows. We consider the compatible homotopies

$$\mathbf{b}_\wedge(t) = \begin{pmatrix} \mathbf{a}_\wedge \cos t & \sin t \\ -\sin t & \mathbf{a}_\wedge^{-1} \cos t \end{pmatrix}, \quad b_\partial(t) = \begin{pmatrix} b_\partial \cos t & \sin t \\ -\sin t & b_\partial^{-1} \cos t \end{pmatrix}, \quad t \in [0, \pi/2], \tag{3.16}$$

of its edge and boundary symbols and take an accompanying homotopy of the principal symbol. (Note that the ellipticity condition is not violated for any t.) The resulting problem will be denoted by $\mathbf{B}(t)$ and the pseudodifferential operator on $\mathcal{M}$ involved in this problem by $B(t)$.

The (anti)symmetry condition (3.14) or (3.15) holds for each t. (This can be shown by straightforward computation.) Thus, we have a have a continuous family of Hirzebruch operators $\mathcal{H}_{\sigma(B(t))}$.

Problem $\mathbf{B}(\pi/2)$ has the property

$$\sigma_\wedge(\mathbf{B}(\pi/2)) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_\partial(B(\pi/2)) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.17}$$

We can assume without loss of generality that $B(\pi/2)$ is a bundle isomorphism in a neighborhood of the edge. By symmetry, we continue $B(\pi/2)$ to the double $2M$, and then

$$\operatorname{ind} \mathbf{B}(\pi/2) = \frac{1}{2} \operatorname{ind} 2B(\pi/2),$$

where $2B(\pi/2)$ is the continuation. Now we apply Lemma 3.9. By item 1 of this lemma, $\operatorname{ind} 2B(\pi/2)$ is equal to the index of a certain Hirzebruch operator on $D^*(2M)$. By item 2, the index of the latter is equal to $\operatorname{ind} \mathcal{H}_{\sigma(B(\pi/2))}$. Thus, we have

$$\operatorname{ind} \mathbf{B}(t) = \operatorname{ind} \mathcal{H}_{\sigma(B(t))} \tag{3.18}$$

for $t = \pi/2$. Now we apply the homotopy invariance of the index and find that (3.18) remains valid for $t = 0$, that is, with regard to the fact that for $t = 0$ our operators are direct sums,

$$\operatorname{ind} \widetilde{\mathbf{A}} - \operatorname{ind} \mathbf{A} = \frac{1}{2} \{ \operatorname{ind} \mathcal{H}_{\sigma(\widetilde{A})} + \operatorname{ind} \mathcal{H}_{\sigma(A)^{-1}} \}. \tag{3.19}$$

Setting $\widetilde{\mathbf{A}} = \mathbf{A}$ in (3.19) shows that

$$\operatorname{ind} \mathcal{H}_{\sigma(A)^{-1}} = -\operatorname{ind} \mathcal{H}_{\sigma(A)},$$

and we substitute this into (3.19) to obtain the desired equation (3.11). The proof is thereby complete. □

4 Classification of boundary symbols

In § 3.1, we introduced symmetry conditions for the restriction of a pseudodifferential operator A on a manifold $\mathcal{M}$ with edge X to the boundary $\partial T^*\mathcal{M} \simeq T^*\partial M \oplus \mathbf{1}$ of the compressed cotangent bundle $T^*\mathcal{M}$. In this section, we study the classification of boundary symbols a_∂ satisfying these condition with respect to stable homotopy equivalence. The results will be needed in the next section in the derivation of index formulas for elliptic operators on manifolds with edges, more precisely, in the proof of Lemma 5.10 and Corollary 5.18. The results of this section will be needed nowhere else, and so the reader interested only in index formulas may well skip this section.

Note that we carry out the homotopy classification of *all* (anti)symmetric boundary symbols a_∂ regardless of whether they can be extended as elliptic symbols to the entire manifold, that is, whether $a_\partial = a|_{\partial T^*\mathcal{M}}$ for some elliptic symbol a on the entire $T_0^*\mathcal{M}$. Thus, thee classification of extendible symbols is a subset (depending on the choice of a manifold M with given boundary ∂M) of our classification. (Indeed, stable homotopy does not violate the extendibility of symbols, since the stabilization is by identity symbols, which are obviously extendible.)

Thus, in this section we consider symbols defined on a vector bundle over an arbitrary compact smooth manifold B. In the above-mentioned lemmas, the results will be used for $B = \partial M$, and the bundle will be $\partial T^*\mathcal{M} \simeq T^*\partial M \oplus \mathbf{1}$.

4.1 Symmetric symbols

4.1.1 $\mathbb{Z}_2$-equivariant symbols

Let $V \in \mathrm{Vect}(B)$ be a real vector bundle with projection π over a compact manifold B. We assume that the bundle is the direct sum

$$V = V_+ \oplus V_-, \quad V_\pm \subset V$$

of two subbundles. On the total space of V, we define the involution

$$\alpha : V \longrightarrow V, \, \alpha|_{V_\pm} = \pm Id.$$

From now on, for a bundle E equipped with an involution by $E_\pm$ we denote the subbundles corresponding to the eigenvalues ± 1 of the involution. If E is not equipped with an involution, then by E_+ (respectively, E_-) we denote the same bundle equipped with the identity (respectively, antipodal) involution.

On the manifold B, we consider $\mathbb{Z}_2$-*equivariant symbol.* These are quintuples

$$(E, F, \varepsilon_E, \varepsilon_F, a),$$

where $E, F \in \mathrm{Vect}(B)$ are vector bundles over B,

$$\varepsilon_E : E \longrightarrow E, \quad \varepsilon_F : F \longrightarrow F$$

are involutions, and

$$a : \pi_0^* E \longrightarrow \pi_0^* F, \quad \pi_0 : SV \longrightarrow B$$

(where SV is the sphere bundle for V with respect to some metric) is a $\mathbb{Z}_2$-equivariant isomorphism:

$$\varepsilon_F \alpha^*(a) \varepsilon_E = a.$$

4.1.2 The difference construction

By $L_{\mathbb{Z}_2}(V)$ we denote the group of stable equivalence classes of such quintuples modulo trivial quintuples. (A quintuple is said to be *trivial* if the isomorphism a is induced by a bundle isomorphism on the base B.)

In a standard way (see [4]), one defines a mapping (the equivariant analog of the difference construction)

$$\chi : L_{\mathbb{Z}_2}(V) \longrightarrow K_{\mathbb{Z}_2}(V),$$

where $K_{\mathbb{Z}_2}(V)$ is the equivariant K-group of the $\mathbb{Z}_2$-space V.

For completeness, we recall the construction. To define the difference mapping, we consider a quintuple of the form[4]

$$(E, \mathbb{C}^{n+m}, \varepsilon_E, 1_n \oplus (-1)_m, a).$$

The clutching of the bundles $\pi^* E$ and $\mathbb{C}^{n+m}$ via a determines a $\mathbb{Z}_2$-bundle on the Thom space V^+ of V. The corresponding element will be denoted by

$$(E, \mathbb{C}^{n+m}, \varepsilon_E, 1_n \oplus (-1)_m, a) \in \mathrm{Vect}_{\mathbb{Z}_2}(V^+).$$

We define a mapping χ by the formula

$$\chi(E, \mathbb{C}^{n+m}, \varepsilon_E, 1_n \oplus (-1)_m, a) = \left[(E, \mathbb{C}^{n+m}, \varepsilon_E, 1_n \oplus (-1)_m, a)\right] - [\mathbb{C}^n_+ \oplus \mathbb{C}^m_-] \in K_{\mathbb{Z}_2}(V^+).$$

This difference is an element of the K-group of V.

Proposition 4.1 ([4]). *The difference construction is an isomorphism*

$$L_{\mathbb{Z}_2}(V) \overset{\chi}{\simeq} K_{\mathbb{Z}_2}(V).$$

4.1.3 Computation of the equivariant K-group

Until the end of this item, we assume that the involution α on the bundle $V = V_+ \oplus V_-$ has a negative determinant. In other words,

$$\dim V_- \equiv 1 \mod 2.$$

This condition is known as the *evenness condition*. We shall show that in this case the equivariant K-group is isomorphic to a usual (nonequivariant) K-group.

[4]An arbitrary quintuple $(E, F, \varepsilon_E, \varepsilon_F, a)$ can be reduced to this form by the addition of a trivial quintuple $(F^\perp, F^\perp, \varepsilon_F^\perp, \varepsilon_F^\perp, Id)$, where $(F^\perp, \varepsilon_F^\perp)$ is the complementary bundle of F ($F \oplus F^\perp \simeq \mathbb{C}^N$) with involution chosen in such a way that the eigenspaces corresponding to the same eigenvalues of the involutions ε_F and $\varepsilon_F^\perp$ are complementary.

The sequence of the pair in K-theory. . The natural embedding $i : V_+ \longrightarrow V$ induces the exact sequence of the pair in equivariant K-theory:

$$\ldots \to K^*_{\mathbb{Z}_2}(V) \xrightarrow{i^*} K^*_{\mathbb{Z}_2}(V_+) \xrightarrow{\partial} K^{*+1}_{\mathbb{Z}_2}(V \setminus V_+) \to K^{*+1}_{\mathbb{Z}_2}(V) \to K^{*+1}_{\mathbb{Z}_2}(V_+) \ldots \tag{4.1}$$

The action on V_+ is trivial, and we obtain the decomposition

$$K^*_{\mathbb{Z}_2}(V_+) = K^*(V_+) \oplus K^*(V_+),$$

corresponding to the trivial and nontrivial irreducible representations. The projections on these direct summands will be denoted by π_+ and π_-. On the other hand, the action on $V \setminus V_+$ is free, and hence the equivariant K-group is equal to the usual K-group of the quotient space:

$$K^*_{\mathbb{Z}_2}(V \setminus V_+) = K^*\big((V \setminus V_+)/\mathbb{Z}_2\big).$$

The quotient space is a bundle over B with fiber over a point $b \in B$ diffeomorphic to the product $V_{+,b} \times (0,\infty) \times P(V_{-,b})$. (Here P stands for the projectivization of a linear space.) Here the diffeomorphism is given by the formula

$$(v_+, v_-) \mapsto \left(v_+, |v_-|, \frac{v_-}{|v_-|}\right).$$

By $p : (V \setminus V_+)/\mathbb{Z}_2 \longrightarrow V_+ \times (0,\infty)$ we denote the projection on the first two components.

Lemma 4.2 ([10, 36])**.** *Under the evenness condition, the projection p induces an isomorphism in K-theory modulo 2-torsion*:

$$K^*(V_+ \times (0,\infty)) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \overset{p^*}{\simeq} K^*((V \setminus V_+)/\mathbb{Z}_2) \otimes \mathbb{Z}\left[\frac{1}{2}\right],$$

where $\mathbb{Z}\left[\frac{1}{2}\right]$ is the ring of dyadic rationals.

Proof. In the special case of zero bundle V_+, the corresponding homomorphism is

$$K^*(B) \to K^*(PV_-).$$

It corresponds to a locally trivial bundle with fiber homeomorphic to an even-dimensional projective space. The isomorphism modulo 2-torsion is well known in this case [10].

The proof in the general case ($V_+ \neq 0$) follows if we note that the fiber of the projection p is still an even-dimensional projective space □

Now let us apply the result to the exact sequence (4.1). We multiply it by $\mathbb{Z}[1/2]$; then one can show that the coboundary mapping ∂ viewed as a homomorphism

$$\partial : K^*(V_+) \oplus K^*(V_+) = K^*_{\mathbb{Z}_2}(V_+) \longrightarrow K^{*+1}_{\mathbb{Z}_2}(V \setminus V_+) \otimes \mathbb{Z}[1/2] \simeq K^*(V_+) \otimes \mathbb{Z}[1/2]$$

is defined by the formula $(\pi_+ x, \pi_- x) \mapsto \pi_+ x + \pi_- x$. This can be shown as follows. By the definition of the coboundary map, one shows that ∂ first takes the element $\pi_\pm x$ to its product by the Bott element $\beta \in K^1(\mathbb{R})$. The product is then lifted to the bundle $V \setminus V_+ \longrightarrow V_+$ with fiber the sphere and is treated on that space as a $\mathbb{Z}_2$-difference element with fiberwise involution ± 1. One can readily see that by applying the isomorphism of Lemma 4.2, we map the element $\pi_\pm x$, $x \in K^*_{\mathbb{Z}_2}$, to the underlying element with the equivariant structure forgotten. It follows that i_* is an isomorphism onto the antidiagonal subgroup. We have thereby proved the following assertion.

Proposition 4.3 (the homotopy classification of symmetric symbols)**.** *If the evenness condition is satisfied, then the mapping*

$$(\pi_+ - \pi_-)i^* : K^*_{\mathbb{Z}_2}(V) \longrightarrow K^*(V_+)$$

is an isomorphism modulo 2-torsion.

Let us construct the inverse mapping modulo 2-torsion. For a sufficiently large N, there exists an element (see [10])

$$[c] \in K_{\mathbb{Z}_2}(V_-),$$

determined by a linear Hermitian symbol

$$c : \pi_1^* \mathbb{C}^{2^N}_+ \longrightarrow \pi_1^* \mathbb{C}^{2^N}_-, \quad \pi_1 : V_- \to B,$$

invertible outside the zero section. (One can define such a symbol by embedding the bundle in a trivial bundle and by considering Clifford multiplication.)

Proposition 4.4. *The mapping*

$$\begin{array}{rcl} K(V_+) & \longrightarrow & K_{\mathbb{Z}_2}(V), \\ \sigma : \pi_2^* E \to \pi_2^* F & \mapsto & [c\#\sigma], \end{array}$$

where $\pi_2 : V_+ \longrightarrow B$ and $\#$ is the external tensor product

$$c\#\sigma = \begin{pmatrix} c \otimes 1 & 1 \otimes \sigma^* \\ 1 \otimes \sigma & -c^* \otimes 1 \end{pmatrix} : \begin{array}{c} \mathbb{C}^{2^N}_+ \otimes E \\ \oplus \\ \mathbb{C}^{2^N}_- \otimes F \end{array} \longrightarrow \begin{array}{c} \mathbb{C}^{2^N}_- \otimes E \\ \oplus \\ \mathbb{C}^{2^N}_+ \otimes F, \end{array}$$

is an isomorphism modulo 2-torsion.

Proof. We denote the mapping by τ. The desired assertion follows from the easy-to-verify relation

$$(\pi_+ - \pi_-)i^* \tau = 2^{N+1} Id.$$

□

4.2 Antisymmetric symbols

4.2.1 The difference construction

We consider a bundle $V = V_+ \oplus V_- \in \mathrm{Vect}(B)$ with involution α as in the preceding subsection: $\alpha|_{V_\pm} = \pm Id$.

Definition 4.5. An isomorphism $\sigma : \pi^*E \longrightarrow \pi^*E$, where $\pi : SV \to B$ is the natural projection and $E \in \mathrm{Vect}(B)$ is said to be *antisymmetric* with respect to the involution α if

$$\alpha^*\sigma = \sigma^{-1}. \tag{4.2}$$

The group of stable homotopy classes of odd pairs (E, σ) modulo pairs lifted from the base B will be denoted by $L^{odd}_{\mathbb{Z}_2}(V)$.

Remark 4.6. Using Remark 3.5, one can readily show that the condition $\alpha^*\sigma = \sigma^*$ results in an isomorphic group.

Let us define the difference construction

$$\chi : L^{odd}_{\mathbb{Z}_2}(V) \longrightarrow K_{\mathbb{Z}_2}(S(V \oplus \mathbb{R}_-)),$$

where S is the spherization of a real vector bundle, for odd symbols. (The action on the spheres is induced by the action on the vector bundle $V \oplus \mathbb{R}_-$.)

The difference construction can be described as follows. We treat the space $S(V \oplus \mathbb{R})$ as the gluing of two copies of the unit ball bundle $B_\pm V \overset{\pi_\pm}{\to} B$ in V along the common boundary $\partial(B_\pm V) = SV \subset S(V \oplus \mathbb{R})$:

$$S(V \oplus \mathbb{R}) = B_+V \bigcup_{SV} B_-V.$$

Then each isomorphism σ determines a vector bundle on $S(V \oplus \mathbb{R})$ as the clutching of two copies of the lift $\pi^*_\pm E$ of E to $B_\pm V$ via σ. The resulting vector bundle on $S(V \oplus \mathbb{R})$ is $\mathbb{Z}_2$-equivariant. (More precisely, the antisymmetry condition (4.2) coincides with the compatibility conditions for the actions of $\mathbb{Z}_2$ in the restriction of $\pi^*_\pm E$ to the boundary $\partial(B_\pm V) = SV$).

Lemma 4.7. *The difference construction is an isomorphism*

$$L^{odd}_{\mathbb{Z}_2}(V) \overset{\chi}{\simeq} K_{\mathbb{Z}_2}(S(V \oplus \mathbb{R}_-))/\pi_5^* K_{\mathbb{Z}_2}(B), \quad \pi_5 : S(V \oplus \mathbb{R}_-) \longrightarrow B.$$

Proof. Let us construct the inverse mapping. Let $E \in \mathrm{Vect}_{\mathbb{Z}_2}(S(V \oplus \mathbb{R}_-))$. The restriction $E|_{t=-1}$ will be denoted by $E_0 \in \mathrm{Vect}(B)$. (Here t is the coordinate on $\mathbb{R}_-$.) We take a bundle isomorphism

$$\varphi : E|_{B_+V} \longrightarrow \pi^*_+ E_0$$

that restricts to the identity mapping at $t=-1$. Then to E we assign the odd symbol defined as the transition function

$$(\alpha^*\varphi)(\alpha^*)\varphi^{-1}:\pi^*E_0\longrightarrow\pi^*E_0$$

between the restrictions of the bundles to the two hemispheres. One can readily show that this map extends to a group homomorphism, which is the inverse of χ. □

4.2.2 The exact sequence of the pair

Consider the pair $S(V\oplus\mathbb{R}_-)\subset B(V\oplus\mathbb{R}_-)$ of the bundles of unit spheres and balls. The exact sequence of the pair in $\mathbb{Z}_2$-equivariant K-theory has the form

$$\longrightarrow K_{\mathbb{Z}_2}(B)\xrightarrow{\pi_5^*}K_{\mathbb{Z}_2}(S(V\oplus\mathbb{R}_-))\longrightarrow K^1_{\mathbb{Z}_2}(V\oplus\mathbb{R}_-)\xrightarrow{b}K^1_{\mathbb{Z}_2}(B)\to\cdots$$

By Proposition 4.3, under condition

$$\dim V_-\equiv 0(\mod 2)$$

one has the isomorphism

$$K^1_{\mathbb{Z}_2}(V\oplus\mathbb{R}_-)\otimes\mathbb{Z}\left[\frac{1}{2}\right]\xrightarrow{\Phi}K^1(V_+)\otimes\mathbb{Z}\left[\frac{1}{2}\right],\quad \Phi=(\pi_+-\pi_-)i^*,\quad i:V_+\subset V\oplus\mathbb{R}_-,$$

modulo 2-torsion, and the mapping b in terms of this isomorphism is induced by the restriction.

4.2.3 Computation of the equivariant K-group

We assume that the bundle V_+ has a nonzero section and the following parity condition is satisfied (cf. [37]):

$$\dim V_-\equiv 0\mod 2.$$

The existence of a nonzero section implies that π_5^* is a monomorphism. It follows that the homomorphism b is zero and one has the isomorphism

$$(K_{\mathbb{Z}_2}(S(V\oplus\mathbb{R}_-))/K_{\mathbb{Z}_2}(B))\otimes\mathbb{Z}\left[\frac{1}{2}\right]\simeq K^1(V_+)\otimes\mathbb{Z}\left[\frac{1}{2}\right]$$

Under these assumptions, we obtain the following assertion.

Proposition 4.8. *If* $\dim V_-\equiv 0\mod 2$*, then*

$$L^{odd}_{\mathbb{Z}_2}(V)\otimes\mathbb{Z}\left[\frac{1}{2}\right]\simeq K^1(V_+)\otimes\mathbb{Z}\left[\frac{1}{2}\right].$$

5 Index formulas

In § 3, we have computed the *main* term in the index formula for edge problems. Here we give a number of examples of computation of *correction* terms under various symmetry conditions and hence obtain the corresponding index formulas on manifolds with edges.

5.1 Symmetry with respect to the conormal variable

We shall consider elliptic edge problems whose principal symbols $\sigma(A)$ satisfy condition (3.2), where the involution in the fibers of $\partial T^*\mathcal{M} = T^*\partial M \oplus \mathbb{R}$ is given by the formula

$$\alpha(v,p) = (v,-p), \tag{5.1}$$

that is, merely *changes the sign of the conormal variable* p. Let $\mathbf{A}$ be an elliptic edge problem of this sort. Without loss of generality (using homotopies that affect neither the boundary symbol $\sigma_\partial(A)$, nor the edge symbol $\sigma_\wedge(\mathbf{A})$, nor the principal symbol $\sigma(A)$ far from the edge), we can assume that the principal symbol is independent of the variable r in a collar neighborhood U of the edge. Let us cut U away from $\mathcal{M}$. The manifold $\mathcal{M}\setminus U$ is diffeomorphic to M, and we glue together two copies of this manifold along the common boundary, thus obtaining the double $2M$ of the manifold M. This gluing naturally extends to the cotangent bundles: the point (v,p) on one copy of T^*M over a given point of ∂M is identified with the point $(v,-p)$ over the same point of the boundary on the other copy (the minus sign occurs since the directions of the r-axis on the two copies of M are opposite), and the gluing results in the cotangent bundle of $T^*(2M)$. The principal symbol $\sigma(A)$ can be extended to $T_0^*(2M)$ by virtue of the symmetry with respect to the conormal variable (and the assumption that it does not depend on the radial variable over U); the bundles $2E$ and $2F$ where the extended symbol acts are obtained from two copies of the respective bundles E and F by clutching along the common boundary ∂M of two copies of M by the automorphisms ε_E and ε_F, respectively. We denote the resulting symbol by

$$2\sigma(A) : \pi_{2M}^*(2E) \longrightarrow \pi_{2M}^*(2F).$$

The corresponding elliptic pseudodifferential operator

$$2A \stackrel{\text{def}}{=} \widehat{2\sigma(A)}$$

on the closed manifold $2M$ is defined modulo compact operators.

Next, the closure $\overline{U}$ is a manifold with edge X and with boundary $\partial U \equiv \overline{U}\setminus U \simeq \partial M$. By gluing two copies of this manifold along the common boundary ∂M, we obtain the double $2U$, which is a manifold with edge, the edge being the disjoint union of two copies of X. The manifold $2U$ can also be represented as the bundle of suspensions with base Ω over the manifold X. (Recall that the suspension with base Ω is defined as the manifold with two conical points of the form

$$S\Omega = \{\{\Omega\times[0,2]\}/\{\Omega\times\{0\}\}\}/\{\Omega\times\{2\}\}. \tag{5.2}$$

The radial variable r on $2U$ can be assumed to vary on the interval $[0,2]$, so that the gluing is along $r=1$.) By the same argument as above, the principal symbol defined on T^*U admits the gluing and determines an elliptic principal symbol

$$\sigma_{2U}(A): \pi_{2U}^*(2E_{2U}) \longrightarrow \pi_{2U}^*(2F_{2U})$$

on $T_0^*(2U)$. The latter depends only on the boundary symbol $\sigma_\partial(A)$ (since the principal symbol is independent of the radial variable in U) and on the involutions ε_E and ε_F. On each of the components of the edge, we have the elliptic edge symbol $\sigma_\wedge(\mathbf{A})$ compatible with $\sigma_{2U}(A)$. Thus, the principal symbol $\sigma_{2U}(A)$ is uniquely determined by the edge symbol together with the involutions ε_E and ε_F. These data specify (modulo compact operators) an elliptic problem on $2U$, which will be denoted by $\widehat{2\sigma_\wedge(\mathbf{A})}$.

Now we can state the index theorem for this case. Note that it was earlier stated by G. Rozenblum in his talk at Potsdam University in October 1999.

Theorem 5.1. *Suppose that the principal symbol of an elliptic edge problem* $\mathbf{A}$ *satisfies the symmetry condition with respect to the conormal variable*:

$$\sigma_\partial(A)(v,-p) = \varepsilon_F \sigma_\partial(A)(v,p)\varepsilon_E.$$

Then the index formula

$$\operatorname{ind}\mathbf{A} = \frac{1}{2}\left\{\operatorname{ind} 2A + \operatorname{ind}\widehat{2\sigma_\wedge(\mathbf{A})}\right\} \tag{5.3}$$

is valid, where $2A$ *and* $\widehat{2\sigma_\wedge(\mathbf{A})}$ *are the above-defined operators on the smooth compact manifold* $2M$ *and on the suspension bundle* $2U$, *depending on the principal symbol and the edge symbol, respectively, of problem* $\mathbf{A}$.

Proof. The proof is based on surgery and the locality principle for the relative index. It reproduces, *mutatis mutandis*, the proof in [24] of the index theorem for operators with symmetry conditions on manifolds with conical singularities. □

Remark 5.2. The expression given by Theorem 5.1 for the contribution of the principal symbol seems to differ from the homotopy invariant constructed in § 3. However, this is only an apparent difference. Indeed, by applying item 1 of Lemma 3.9, we obtain

$$\operatorname{ind} 2A = \operatorname{ind}\mathcal{H}_{\sigma(A)}, \tag{5.4}$$

where $\mathcal{H}_{\sigma(A)}$ is the Hirzebruch operator (3.6) on $D(T^*2M)$ corresponding to the symbol $\sigma(A)$ and the involution $\alpha: p \mapsto -p$.

Remark 5.3. Theorem 5.1 provides the desired index splitting: the index of the problem $\mathbf{A}$ is half the sum of indices of two operators, one of which is determined by the principal symbol of the problem and the second by the edge symbol. The first term is a homotopy invariant of the principal symbol and can be represented by the Atiyah–Singer formula [6]. The second term is an invariant of the edge symbol and can also be represented by an appropriate formula. Namely, the following assertion holds.

Proposition 5.4. *The operator* $\widehat{2\sigma_\wedge(\mathbf{A})}$ *can be represented modulo compact operators as a pseudodifferential operator on* X *with elliptic operator-valued symbol* $F(x,\xi)$ *with compact fiber variation in Hilbert function spaces on the fibers of the suspension bundle with base* Ω *over* X*:*

$$\widehat{2\sigma_\wedge(\mathbf{A})} = F\left(x, -i\frac{\partial}{\partial x}\right).$$

The index of this operator is the direct image under the mapping of X *into a point of the difference construction of this symbol in* $K(T^*X)$ *(see* [4]*):*

$$\operatorname{ind} \widehat{2\sigma_\wedge(\mathbf{A})} = p_![F(x,\xi)], \quad p : X \longrightarrow \{pt\}.$$

Proof. This is a special case of Proposition 1.2. □

5.2 Symmetry with respect to the edge covariables

Suppose that there is a decomposition

$$\partial T^*\mathcal{M} \simeq \pi^*(T^*X) \oplus ({}^vT^*\partial M \oplus \mathbf{1}) \in \operatorname{Vect}(\partial M) \tag{5.5}$$

of the restriction of the compressed cotangent bundle of $\mathcal{M}$ to the boundary ∂M into "horizontal" and "vertical" components. Next, let $\partial T^*\mathcal{M}$ be equipped with the involution described in this decomposition by the formula

$$\alpha(\xi, q, p) = (\alpha\xi, q, p), \tag{5.6}$$

where $\alpha : T^*X \longrightarrow T^*X$ on the right-hand side is an orientation-reversing involution (denoted by the same letter) in the cotangent bundle of the edge X.

Remark 5.5. The involution $\xi \longrightarrow -\xi$ can serve as an example provided that X is odd-dimensional.

We consider elliptic edge problems $\mathbf{A}$ whose principal symbol satisfies the symmetry condition (3.2).

Let $\mathbf{A}$ be a problem of this sort. We set

$$\sigma_\wedge^+(\mathbf{A})(x,\xi) = \varepsilon_F \sigma_\wedge(\mathbf{A})(x, \alpha\xi)\varepsilon_E, \quad (x,\xi) \in T_0^*X. \tag{5.7}$$

By the symmetry condition, the edge symbols $\sigma_\wedge^+(\mathbf{A})(x,\xi)$ and $\sigma_\wedge(\mathbf{A})(x,\xi)$ have the same principal symbol $\sigma_\partial(A)$. Their ratio

$$\mathbf{B}(x,\xi) = \left(\sigma_\wedge^+(\mathbf{A})(x,\xi)\right)^{-1} \sigma_\wedge(\mathbf{A})(x,\xi) \tag{5.8}$$

has a unit principal symbol and, by Proposition A.31, defines a Fredholm operator $\mathbf{B}(x, -i\partial/\partial x)$ in the space $\mathcal{W}(W) \oplus L^2(X)$ on the infinite-dimensional wedge W associated with $\mathcal{M}$.

Theorem 5.6. *Let* $\mathbf{A}$ *be a Fredholm edge problem whose principal symbol satisfies the symmetry condition* (3.2) *with respect to the orientation-reversing involution* (5.6)*. Then the following index formula holds from the problem* $\mathbf{A}$:

$$\operatorname{ind}\mathbf{A} = \frac{1}{2}\operatorname{ind}\mathcal{H}_{\sigma(A)} + \frac{1}{2}\operatorname{ind}\mathbf{B}\left(x, -i\frac{\partial}{\partial x}\right), \tag{5.9}$$

where $\mathcal{H}_{\sigma(A)}$ *is the Hirzebruch operator* (3.6) *and the symbol* $\mathbf{B}(x,\xi)$ *is given by formula* (5.8)*.*

Remark 5.7. The contribution of the edge symbol can be expressed by Theorem 1.1 as the direct image of the index $\operatorname{ind}\mathbf{B}(x,\xi) \in K(T^*X)$ of the Fredholm family $\mathbf{B}(x,\xi)$, invertible outside a compact set, under the mapping of X into a point:

$$\operatorname{ind}\mathbf{B}\left(x, -i\frac{\partial}{\partial x}\right) = p_!\operatorname{ind}\mathbf{B}(x,\xi), \quad p : X \longrightarrow \{pt\}.$$

Proof of Theorem 5.6. First, let us describe the scheme of the proof. We transpose the second term on the right-hand side in (5.9) to the left-hand side, so that the desired assertion acquires the form

$$\operatorname{ind}\mathbf{A} - \frac{1}{2}\operatorname{ind}\mathbf{B}\left(x, -i\frac{\partial}{\partial x}\right) = \frac{1}{2}\operatorname{ind}\mathcal{H}_{\sigma(A)}. \tag{5.10}$$

We denote the left-hand side by

$$\varphi(\sigma(A), \sigma_\wedge(\mathbf{A})) = \operatorname{ind}\mathbf{A} - \frac{1}{2}\operatorname{ind}\mathbf{B}\left(x, -i\frac{\partial}{\partial x}\right). \tag{5.11}$$

The proof contains several steps.

1. We prove that $\varphi(\sigma(A), \sigma_\wedge(\mathbf{A}))$ is actually independent of $\sigma_\wedge(\mathbf{A})$ and is a homotopy invariant of the principal symbol $\sigma(A)$ in the class $\widetilde{\Sigma}_\alpha$:

   $$\varphi(\sigma(A), \sigma_\wedge(\mathbf{A})) = \operatorname{inv}(a).$$

   This is done in Lemma 5.9 with the use of the relative index formula provided by Proposition 1.3 and the auxiliary Lemma 5.8. Thus, both sides of (5.10) are homotopy invariants of the principal symbol.

2. We show that an arbitrary symbol in Σ_α is stably rationally homotopic to a symbol symmetric with respect to the inversion $p \mapsto -p$ of the conormal variable (Lemma 5.10).

3. Since both sides of (5.10) are homotopy invariant, it suffices to verify the formula for symbols that are also symmetric with respect to the involution $p \mapsto -p$. We do this in Lemma 5.11.

Let us now proceed to the proof itself.

1. *The homotopy invariance of* (5.11). First, we prove an auxiliary assertion.

Lemma 5.8. *Let $a \in \widetilde{\Sigma}_\alpha$ be an elliptic symbol. If we deform a in the class $\widetilde{\Sigma}_\alpha$, then for sufficiently small intervals of the deformation parameter there exists an invertible edge symbol $\mathbf{a}_\wedge$ compatible with a and continuously depending on the deformation parameter.*

Proof. By Proposition A.39 there exists a *Fredholm* edge symbol $a_\wedge$ compatible with a. Since the obstruction to the existence of edge problems for a is zero, one can add (co)boundary conditions such that the complete matrix Green symbol $\mathbf{a}_\wedge$ will be invertible. Next, if we deform a and deform $a_\wedge$ compatibly, the other entries of the Green matrix symbol being fixed, then $\mathbf{a}_\wedge$ remains invertible for small values of the deformation parameter. □

In what follows, by $\mathbf{A}(\mathbf{c}_\wedge)$ we denote the edge elliptic problem with principal symbol $\sigma(A)$ and (compatible) edge symbol $\mathbf{c}_\wedge(x,\xi)$. For brevity, we denote $\sigma_\wedge(\mathbf{A})$ by $\mathbf{a}_\wedge$. Thus, $\mathbf{A} = \mathbf{A}(\mathbf{a}_\wedge)$.

Lemma 5.9. *The quantity $\varphi(\sigma(A), \sigma_\wedge(\mathbf{A}))$ (5.11) can be expressed by the formula*

$$\varphi(\sigma(A), \sigma_\wedge(\mathbf{A})) = \frac{1}{2}\operatorname{ind}\bigl(\mathbf{A}(\mathbf{a}_\wedge) \oplus \mathbf{A}(\mathbf{a}_\wedge^+)\bigr). \tag{5.12}$$

It depends only on the symbol a (i.e., is independent of the choice of an elliptic problem for this symbol) and is its stable homotopy invariant.

Proof. By the relative index theorem 1.3, we have

$$\operatorname{ind}\mathbf{A}(\mathbf{a}_\wedge) - \operatorname{ind}\mathbf{A}(\mathbf{a}_\wedge^+) = \operatorname{ind}\mathbf{B}\left(x, -i\frac{\partial}{\partial x}\right).$$

Now

$$\begin{aligned}\frac{1}{2}\operatorname{ind}\bigl(\mathbf{A}(\mathbf{a}_\wedge) \oplus \mathbf{A}(\mathbf{a}_\wedge^+)\bigr) &= \operatorname{ind}\mathbf{A}(\mathbf{a}_\wedge) - \frac{1}{2}\bigl(\operatorname{ind}\mathbf{A}(\mathbf{a}_\wedge) - \operatorname{ind}\mathbf{A}(\mathbf{a}_\wedge^+)\bigr) \\ &= \operatorname{ind}\mathbf{A}(\mathbf{a}_\wedge) - \frac{1}{2}\operatorname{ind}\mathbf{B}\left(x, -i\frac{\partial}{\partial x}\right) = \varphi(\sigma(A), \sigma_\wedge(\mathbf{A})),\end{aligned}$$

which proves (5.12).

To prove that the right-hand side of (5.12) is independent of $\mathbf{a}_\wedge$, note that if we replace $\mathbf{a}_\wedge$ by another elliptic edge symbol $\mathbf{c}_\wedge$ compatible with $\sigma(A)$, then the right-hand side of (5.12) is changed according to the relative index formula by

$$p_!\bigl(\operatorname{ind}(\mathbf{c}_\wedge\mathbf{a}_\wedge^{-1} \oplus \mathbf{c}_\wedge^+(\mathbf{a}_\wedge^+)^{-1})\bigr), \quad p : X \longrightarrow \{pt\}.$$

But the family $\mathbf{c}_\wedge\mathbf{a}_\wedge^{-1} \oplus \mathbf{c}_\wedge^+(\mathbf{a}_\wedge^+)^{-1}$ is invariant with respect to the involution $\xi \mapsto \alpha\xi$; since the involution reverses the orientation, this family defines a 2-torsion element in $K(T^*X)$, whose index is zero. Now the homotopy invariance of $\varphi(\sigma(A), \sigma_\wedge(\mathbf{A}))$ is also clear, since for small deformations of the principal symbol one can continuously deform the compatible elliptic edge symbol $\mathbf{a}_\wedge$. The proof of the lemma is complete. □

In view of the result of this lemma, we write

$$\operatorname{inv}(\sigma(A)) \stackrel{\text{def}}{=} \varphi(\sigma(A), \sigma_\wedge(\mathbf{A}))$$

in what follows.

2. The principal symbol is homotopic to a symbol symmetric with respect to the conormal variable. Now that the homotopy invariance of both sides of the desired relation has been proved, we deform the principal symbol to the simplest form, for which the proof will be essentially elementary.

Lemma 5.10. *For an arbitrary elliptic boundary symbol $a \in \Sigma_{\alpha\partial}$, there exist positive integers N and N' such that the direct sum $2^N a \oplus 1_{N'}$ is homotopic in $\Sigma_{\alpha\partial}$ to a symbol symmetric with respect to the inversion $p \mapsto -p$ of the conormal variable, where by $1_{N'}$ we denote the unit symbol in the trivial bundle of dimension N' equipped with an arbitrary involution.*

Proof. The proof is based on the classification of symmetric boundary symbols obtained in § 4.1. In this case, the classification results in the K-group $K_{\mathbb{Z}_2}(V_+ \oplus V_- \oplus \mathbb{R}_+)$, where V_+ and V_- are the positive and negative subspaces generated by the involution in the cotangent bundle $T^*\partial M$). By the results of § 4.1, this group is isomorphic to $K(V_+ \oplus \mathbb{R})$ modulo 2-torsion.

It turns out that the group $K(V_+ \oplus \mathbb{R})$ is generated modulo 2-torsion by endomorphisms $\sigma(v_+, p)$ invertible outside the zero section and satisfying the condition

$$\sigma(v_+, -p) = U\sigma^*(v_+, p)U, \tag{5.13}$$

where U is a unitary involution.

Indeed, for an arbitrary symbol $\sigma(v_+, p)$ we set

$$\sigma_0(v_+, p) = \begin{pmatrix} \sigma(v_+, p) & 0 \\ 0 & \sigma^*(v_+, -p) \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.14}$$

The resulting symbol satisfies condition (5.13). One can readily show that the mapping $\sigma \mapsto \sigma_0$ induces an isomorphism in K-theory modulo 2-torsion. Indeed, the passage to adjoint symbols, as well as the involution $p \to -p$, acts as $-\operatorname{Id}$ on the K-groups. Thus, the composition of these two mappings is the identity mapping of the K-group. Finally, the 2-torsion arises from the doubling of the symbol in (5.14).

Gathering the results, we obtain the following set of symbols generating the group $K_{\mathbb{Z}_2}(V_+ \oplus V_- \oplus \mathbb{R}_+)$:

$$c\#\sigma(v_+, v_-, p) = \begin{pmatrix} c(v_-) \otimes 1 & 1 \otimes \sigma^*(v_+, p) \\ 1 \otimes \sigma(v_+, p) & -c(v_-) \otimes 1 \end{pmatrix}, \tag{5.15}$$

and the relation $\sigma(v_+,-p)=U\sigma^*(v_+,p)U$ holds.

These generating elements satisfy the symmetry condition with respect to the involution $\alpha_0: p\mapsto -p$. More precisely,

$$(c\#\sigma)(v_+,v_-,-p)=A((c\#\sigma)(v_+,v_-,p))A,\quad A=\begin{pmatrix}0 & -U\\ U & 0\end{pmatrix}.$$

The proof is straightforward; as the involutions one can take $\varepsilon_E=iA$ and $\varepsilon_F=-iA$.

It follows that for some positive integer N one has

$$2^N[a]=[c\#\sigma]\in K_{\mathbb{Z}_2}(\partial T^*\mathcal{M}),$$

where the symbol $c\#\sigma$ has the form (5.15) and is symmetric. In the symbol $c\#\sigma$, one should smooth (see [6]) the second component $\sigma=\sigma(v_+,p)$, which is originally not a smooth function of the variables (v_+,v_-,p). With regard to properties of K-groups, we see that there exists a homotopy between these symbols. The proof of the lemma is complete. □

3. The proof of formula (5.9) ***for symbols symmetric with respect to the conormal variable.*** It remains to show that the following assertion holds.

Lemma 5.11. *The identity*

$$\operatorname{inv}(a)=\frac{1}{2}\operatorname{ind}\mathcal{H}_a, \tag{5.16}$$

where $\mathcal{H}_a$ is the Hirzebruch operator (3.6)*, holds for symbols $a\in\widetilde{\Sigma}_\alpha$ symmetric with respect to the conormal variable.*

Using the doubling construction (i.e., passing to the problem $\mathbf{A}(\mathbf{a}_\wedge)\oplus\mathbf{A}(\mathbf{a}_\wedge^+)$), we can assume that a is equipped with a compatible edge symbol $\mathbf{a}_\wedge$ satisfying the symmetry condition

$$\mathcal{G}_\wedge(x,\xi)=\mathcal{G}_\wedge(x,\alpha\xi).$$

For the case of the principal symbol symmetric with respect to the conormal variable, an index splitting formula was established in the preceding subsection. It differs from the one two be proved in two respects:

1. The Hirzebruch operator is defined on $D(T^*(2M))$ rather than $D(T^*\mathcal{M}\underset{\alpha}{\cup}T^*\mathcal{M})$.

2. The contribution of the edge symbol is expressed not as the index of an operator on an infinite wedge but as the index of the operator $\widehat{2\sigma_\wedge(\mathbf{A})}$ on the total space of the suspension bundle.

However, the indices of the two Hirzebruch operators coincide by item 2 in Lemma 3.9. Next, by virtue of our assumption on the symmetry of the edge symbol, both the operator on the infinite wedge and the operator on the suspension bundle, whose indices are to be compared, can be represented as pseudodifferential operators on X with elliptic operator-valued symbols (see Proposition 1.2) symmetric with respect to the involution $\xi \mapsto \alpha\xi$. (Indeed, the symmetry of the operator on the infinite wedge is clear from the formula determining this symbol, and the symmetry of the symbol of the operator on the suspension bundle can be observed from the construction of this symbol, described in the proof of Proposition 1.2, with regard to the fact that both the edge symbol and a_∂ are symmetric.) Since the involution is orientation-reversing, it follows that these symbols generate torsion elements in the K-group $K(T^*X)$ with compact supports. Applying propositions 1.1 and 1.2, we see that both indices are zero (and, in particular, coincide). The proof of Theorem 5.6 is complete. $\square$

5.3 Antisymmetry with respect to the edge covariables

In this subsection, we prove an index theorem for the case of an orientation-preserving involution (the antisymmetric case). Many of the constructions are parallel to the symmetric case, considered in the preceding subsection. Hence we focus our attention on the differences between these two cases.

Let again α be an involution of the form (5.6). Now we assume that it is orientation-preserving.

Remark 5.12. The involution $\xi \longrightarrow -\xi$ can serve as an example provided that X is even-dimensional.

We consider elliptic edge problems $\mathbf{A}$ whose principal symbol satisfies the antisymmetry condition (3.3).

Let $\mathbf{A}$ be a problem of this sort. We set

$$\sigma_\wedge^+(\mathbf{A})(x,\xi) = \varepsilon\sigma_\wedge(\mathbf{A})(x,\alpha\xi)^{-1}\varepsilon, \quad (x,\xi) \in T_0^*X. \tag{5.17}$$

By the antisymmetry condition, the edge symbols $\sigma_\wedge^+(\mathbf{A})(x,\xi)$ and $\sigma_\wedge(\mathbf{A})(x,\xi)$ have the same principal symbol $\sigma_\partial(A)$. Their ratio

$$\mathbf{B}(x,\xi) = \left(\sigma_\wedge^+(\mathbf{A})(x,\xi)\right)^{-1}\sigma_\wedge(\mathbf{A})(x,\xi) \tag{5.18}$$

has a unit principal symbol and hence defines a Fredholm operator $\mathbf{B}(x,-i\partial/\partial x)$ in the space $\mathcal{W}(W) \oplus L^2(X)$ on the infinite-dimensional wedge W associated with $\mathcal{M}$.

Theorem 5.13. *Let* $\mathbf{A}$ *be a Fredholm edge problem whose principal symbol satisfies the antisymmetry condition* (3.3) *with respect to the orientation-preserving involution* (5.6). *Then the following index formula holds from the problem* $\mathbf{A}$:

$$\operatorname{ind}\mathbf{A} = \frac{1}{2}\operatorname{ind}\mathcal{H}_{\sigma(A)} + \frac{1}{2}\operatorname{ind}\mathbf{B}\left(x,-i\frac{\partial}{\partial x}\right), \tag{5.19}$$

where $\mathcal{H}_{\sigma(A)}$ *is the Hirzebruch operator* (3.6) *and the symbol* $\mathbf{B}(x,\xi)$ *is given by formula* (5.18).

Remark 5.14. The contribution of the edge symbol can be expressed by Theorem 1.1 as the direct image of the index $\operatorname{ind}\mathbf{B}(x,\xi) \in K(T^*X)$ of the Fredholm family $\mathbf{B}(x,\xi)$, invertible outside a compact set, under the mapping of X into a point:

$$\operatorname{ind}\mathbf{B}\left(x,-i\frac{\partial}{\partial x}\right) = p_! \operatorname{ind}\mathbf{B}(x,\xi), \quad p : X \longrightarrow \{pt\}.$$

Proof of Theorem 5.13. The proof is very close to that of Theorem 5.6. Before proceeding to details, we briefly outline the scheme of the proof. We transpose the second term on the right-hand side in (5.19) into the left-hand side, so that the right-hand side will contain only the homotopy invariant of the principal symbol equal to half the index of the Hirzebruch operator. The proof contains the following steps.

1. We prove that the left-hand side

$$\varphi(\sigma(A),\sigma_\wedge(A)) = \operatorname{ind}\mathbf{A} - \frac{1}{2}\operatorname{ind}\mathbf{B}\left(x,-i\frac{\partial}{\partial x}\right)$$

   of the equation to be proved is actually independent of $\sigma_\wedge(A)$ and is a homotopy invariant of the principal symbol$\sigma(A)$ in the class $\widetilde{\Sigma}_\alpha$ (Lemma 5.15).

2. We show that an arbitrary symbol in Σ_α is stably rationally homotopic to a symbol symmetric with respect to the inversion $p \mapsto -p$ of the conormal variable (Proposition 5.17 and Corollary 5.18).

3. Now it suffices to verify the formula for symbols that are also symmetric with respect to the involution $p \mapsto -p$. This is done in Lemma 5.19.

We now proceed to the detailed proof.

1. The homotopy invariance of $\varphi(\sigma(A),\sigma_\wedge(\mathbf{A}))$**.** Just as in the preceding subsection, by $\mathbf{A}(\mathbf{c}_\wedge)$ we denote the elliptic morphism with principal symbol $\sigma(A)$ and (compatible) edge symbol $\mathbf{c}_\wedge$. By $\mathbf{c}_\wedge^+$ we denote the elliptic edge symbol (cf. (5.17))

$$\mathbf{c}_\wedge^+(x,\xi) = \mathbf{c}_\wedge(x,\alpha\xi)^{-1}.$$

It follows from the antisymmetry condition that the problems $\mathbf{A}(\mathbf{c}_\wedge)$ and $\mathbf{A}(\mathbf{c}_\wedge^+)$ are Fredholm or non-Fredholm simultaneously.

Lemma 5.15. *The quantity* $\varphi(\sigma(A),\sigma_\wedge(\mathbf{A}))$ *can be expressed by the formula*

$$\varphi(\sigma(A),\sigma_\wedge(\mathbf{A})) = \frac{1}{2}\operatorname{ind}\big(\mathbf{A}(\mathbf{a}_\wedge)\oplus\mathbf{A}(\mathbf{a}_\wedge^+)\big). \tag{5.20}$$

It depends only on the symbol $\sigma(A)$ (*i.e., is independent of the choice of an elliptic problem for this symbol*) *and is its stable homotopy invariant.*

Proof. By the relative index theorem 1.3, we have

$$\operatorname{ind}\mathbf{A}(\mathbf{a}_\wedge) - \operatorname{ind}\mathbf{A}(\mathbf{a}_\wedge^+) = \operatorname{ind}\mathbf{B}\left(x, -i\frac{\partial}{\partial x}\right).$$

Now

$$\begin{aligned}\frac{1}{2}\operatorname{ind}\bigl(\mathbf{A}(\mathbf{a}_\wedge)\oplus\mathbf{A}(\mathbf{a}_\wedge^+)\bigr) &= \operatorname{ind}\mathbf{A}(\mathbf{a}_\wedge) - \frac{1}{2}\bigl(\operatorname{ind}\mathbf{A}(\mathbf{a}_\wedge) - \operatorname{ind}\mathbf{A}(\mathbf{a}_\wedge^+)\bigr)\\ &= \operatorname{ind}\mathbf{A}(\mathbf{a}_\wedge) - \frac{1}{2}\operatorname{ind}\mathbf{B}\left(x, -i\frac{\partial}{\partial x}\right) = \varphi(\sigma(A), \sigma_\wedge(\mathbf{A})),\end{aligned}$$

which proves (5.20).

To prove that the right-hand side of (5.20) is independent of $\mathbf{a}_\wedge$, note that if we replace $\mathbf{a}_\wedge$ by another elliptic edge symbol $\mathbf{c}_\wedge$ compatible with $\sigma(A)$, then the right-hand side of (5.20) is changed according to the relative index formula by

$$p_!\bigl(\operatorname{ind}(\mathbf{c}_\wedge\mathbf{a}_\wedge^{-1}\oplus\mathbf{c}_\wedge^+(\mathbf{a}_\wedge^+)^{-1})\bigr), \quad p : X \longrightarrow \{pt\}.$$

But a straightforward verification shows that the family $\mathbf{c}_\wedge\mathbf{a}_\wedge^{-1}\oplus\mathbf{c}_\wedge^+(\mathbf{a}_\wedge^+)^{-1}$ is this time antisymmetric with respect to the involution $\xi \mapsto \alpha\xi$; since the involution preserves the orientation, this family defines a 2-torsion element in $K(T^*X)$, whose index is zero. Now the homotopy invariance of $\varphi(\sigma(A), \sigma_\wedge(\mathbf{A}))$ is also clear, since for small deformations of the principal symbol one can continuously deform the compatible elliptic edge symbol $\mathbf{a}_\wedge$. The proof of the lemma is complete.. □

In view of the result of this lemma, we again write

$$\operatorname{inv}(\sigma(A)) \stackrel{\text{def}}{=} \varphi(\sigma(A), \sigma_\wedge(\mathbf{A}))$$

in what follows.

2. The principal symbol is homotopic to a symbol symmetric with respect to the conormal variable. Now that the homotopy invariance of both sides of the desired relation has been proved, we deform the principal symbol to the simplest form, for which the proof will be essentially elementary.

In this case, we deal with the group $L^{odd}_{\mathbb{Z}_2}(V_+\oplus V_-\oplus\mathbb{R}_+)$, where V_+ and V_- are the positive and negative subspaces generated by the involution in the cotangent bundle $T^*\partial M$. By the results of § 4.2, this group is isomorphic to $K^1(V_+\oplus\mathbb{R})$ modulo 2-torsion.

Lemma 5.16. *The group $K^1(V_+\oplus\mathbb{R})$ is generated modulo 2-torsion by Hermitian endomorphisms $\sigma(v_+, p)$ invertible outside the zero section and satisfying the condition*

$$\sigma(v_+, -p) = -U\sigma(v_+, p)U,$$

where U is a unitary involution.

Proof. For an arbitrary symbol $\sigma(v_+,p)$, we set

$$\sigma_0(v_+,p)=\begin{pmatrix}\sigma(v_+,p) & 0\\ 0 & -\sigma(v_+,-p)\end{pmatrix},\quad U=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

The resulting symbol satisfies the assertion of the lemma. The mapping $\sigma\mapsto\sigma_0$ induces an isomorphism in K-theory modulo 2-torsion. □

Gathering the results of Propositions 4.4 and 4.8 and Lemma 5.16, we obtain the following set of symbols generating the group $L^{odd}_{\mathbb{Z}_2}(V_+\oplus V_-\oplus\mathbb{R}_+)$:

$$\sigma\#c(v_+,v_-,p)=\begin{pmatrix}\sigma(v_+,p)\otimes 1 & 1\otimes c(v_-)\\ -1\otimes c(v_-) & \sigma(v_+,p)\otimes 1\end{pmatrix},\quad \chi([\sigma\#c])\in K_{\mathbb{Z}_2}(S(V_+\oplus V_-\oplus\mathbb{R}_+\oplus\mathbb{R}_-)),\tag{5.21}$$

and the relation $\sigma(v_+,-p)=-U\sigma(v_+,p)U$ holds.

Proposition 5.17. *These generating elements satisfy the symmetry condition with respect to the involution $\alpha_0: p\mapsto -p$. More precisely,*

$$(\sigma\#c)(v_+,v_-,-p)=-A((\sigma\#c)(v_+,v_-,p))A,\quad A=\begin{pmatrix}0&U\\U&0\end{pmatrix}.$$

Proof. It suffices to multiply the matrices and use the fact that U and c commute. □

Corollary 5.18. *If* $\dim V_-\equiv 0 \mod 2$, *then for an arbitrary odd symbol σ on the bundle $V_+\oplus V_-\oplus\mathbb{R}_+$ there exist positive integers N and N' such that the direct sum $2^N\sigma\oplus 1_{N'}$ is homotopic to a symbol symmetry with respect to the involution $p\mapsto -p$.*

To prove the corollary, it suffices to note that the component $\sigma(v_+,p)$ in the external tensor product (5.21) can be smoothed as in [6]. Now the existence of the desired homotopy follows from Proposition 4.8, Lemma 5.16, and Remark 3.5.

3. The proof of the formula for symbols symmetric with respect to the conormal variable. To complete the proof of Theorem 5.13, it remains to show that the following assertion holds.

Lemma 5.19. *For symbols $a\in\Sigma_a$ symmetric with respect to the conormal variable, the identity*

$$\operatorname{inv}(a)=\frac{1}{2}\operatorname{ind}\mathcal{H}_a\tag{5.22}$$

holds, where $\mathcal{H}_a$ is the Hirzebruch operator (3.6).

Proof. Using the doubling construction (i.e., passing to the problem $\mathbf{A}(\mathbf{a}_\wedge)\oplus\mathbf{A}(\mathbf{a}_\wedge^+)$), we can assume that a is equipped with a compatible edge symbol $\mathbf{a}_\wedge$ satisfying the anti-symmetry condition

$$\mathbf{a}_\wedge(x,\xi)=\mathbf{a}_\wedge(x,\alpha\xi)^{-1}.$$

For the case of the principal symbol symmetric with respect to the conormal variable, an index splitting formula was established in § 5.1. It gives the expansion

$$\operatorname{inv}(a) = \operatorname{ind} A(a, \mathbf{a}_\wedge) = \frac{1}{2}\left(\operatorname{ind} 2A + \operatorname{ind} \widehat{2\mathbf{a}_\wedge}\right).$$

By Lemma 3.9, the index of the operator $2A$ on the double of the manifold is equal to the index of the Hirzebruch operator $\mathcal{H}_a$. It remains to show that $\operatorname{ind} \widehat{2\mathbf{a}_\wedge} = 0$.

Arguing as in the symmetric case, we see that the operator-valued symbol $2\mathbf{a}_\wedge$ is antisymmetric modulo compact operator families decaying as $|\xi| \to \infty$:

$$\alpha^*(2\mathbf{a}_\wedge)(2\mathbf{a}_\wedge) = 1 + K(x,\xi), \quad \text{where } \|K\| \to 0 \text{ as } |\xi| \to \infty.$$

The corresponding element of the K-group lies in the 2-torsion subgroup by virtue of the antisymmetry condition. Hence the index of the corresponding operator is zero.

The proof of Lemma 5.19 as well as Theorem 5.13 is complete. □

6 Examples

6.1 Operators on a manifold with conical singularities

As the first example, we consider the trivial case in which the edge degenerates into a point, i.e., the manifold $\mathcal{M}$ in question is a manifold with conical singularity. Let A be an elliptic operator on $\mathcal{M}$. Its edge symbol, which can be computed as the limit as $\lambda \to \infty$ of the family (A.39), is just the conormal symbol $\sigma_c(A)$ (since the factor $e^{ix\xi}$ in the definition of the one-parameter group (A.38) disappears for $\dim X = 0$). Suppose that the principal symbol of A is symmetric with respect to the conormal variable (that is, with respect to the involution $p \to -p$):

$$\sigma_\partial(A)(\omega, q, p) = \varepsilon_F \sigma_\partial(A)(\omega, q, -p)\varepsilon_E.$$

By Theorem 5.1, for the index of A we obtain the formula

$$\operatorname{ind} A = \frac{1}{2}\operatorname{ind} 2A + \frac{1}{2}\operatorname{ind} B, \tag{6.1}$$

where $2A$ is the operator on the double $2M$ with principal symbol $2\sigma(A)$ obtained from $\sigma(A)$ by gluing (which is possible by the symmetry condition) and B is an operator in spaces of sections of infinite-dimensional bundle over the point X), i.e., an operator in function spaces on the suspension over the base of the cone; the principal symbol of this operator is independent of r, and the conormal symbol at either conical point coincides with the conormal symbol of the original operator.

The index of the latter operator is equal to the spectral flow of the linear homotopy

$$\operatorname{ind} B = \operatorname{sf}\{t\sigma_c(A)(p) + (1-t)\varepsilon_F \sigma_c(A)(-p)\varepsilon_E\}, \quad t \in [0,1],$$

and can be expressed as the sum of multiplicities of the poles of the family $[\sigma_c(A)(p)]^{-1}$ in some horizontal strip on the complex plane provided that the stronger symmetry condition (involving not only the principal symbol, but also the conormal symbol)

$$\sigma_c(A)(p_0 - p) = \varepsilon_F \sigma_c(A)(p_0 + p)\varepsilon_E$$

with respect to some point p_0 of the complex plane is satisfied.

Thus, we recover the well-known index theorem for manifolds with conical singularities, originally obtained in [42] and then generalized in numerous papers (e.g., see [24] and bibliography therein).

6.2 The Euler operator

Consider the Euler operator on a compact manifold M with boundary X, where the boundary is treated as an edge (i.e., the model cone is just $\overline{\mathbb{R}}_+$ with base a single point). If the metric in a collar neighborhood U of the boundary has the direct product form $d\rho_M^2 = dr^2 + d\rho_X^2$, then the Euler operator χ_M can be represented in this neighborhood as the external tensor product

$$\chi_M = \chi_X \#(-id/dr) = r^{-1}\chi_X \#(ird/dr). \tag{6.2}$$

Its symbol satisfies the symmetry condition with respect to the involution $p \longrightarrow -p$ (see [42]), and the obstruction (A.44) for this operator (coinciding with the Atiyah–Bott obstruction, since $X = \partial M$ and $\pi : \partial M \longrightarrow X$ is the identity mapping) is zero. Thus, we can apply Theorem 5.1 and remark 5.2 and arrive at the following assertion.

Proposition 6.1. *The contribution of the principal symbol to the index of the Euler operator χ_M is equal to half the Euler characteristic of the double $2M$ of the manifold M:*

$$f(\sigma(\chi_M)) = \frac{1}{2}\chi(2M).$$

To descend to more specific computations and write out a complete index formula, we consider the special case in which M is two-dimension. Accordingly, the edge ∂M is the circle $\mathbb{S}^1$ (or a disjoint union of finitely many circles, which can be treated in a similar way). A collar neighborhood of the edge in the manifold is isomorphic to a collar neighborhood of the edge in the two-dimensional model wedge $\mathbb{S}^1 \times \overline{\mathbb{R}}_+$ with coordinates $x \in \mathbb{S}^1 = \mathbb{R} \mod 2\pi$ and $r \in \overline{\mathbb{R}}_+$. We assume that in a neighborhood of the boundary the metric is the direct product metric $ds^2 = dr^2 + dx^2$. We consider the Euler operator (acting from the space of odd forms to the space of even forms) in the spaces

$$\chi \equiv d + \delta : \mathcal{W}^{1,1}(M, \Lambda^1) \longrightarrow \mathcal{W}^{0,0}(M, \Lambda^0 \oplus \Lambda^2).$$

It is given by the formula[5] $\chi = d - *d*$, since the dimension of the manifold is even. For the explicit computation in a neighborhood of the boundary, we note that the Hodge operator is given by the formulas

$$*1 = dx \wedge dr, \quad *dx = dr, \quad *dr = -dx, \quad *dx \wedge dr = 1,$$

so that for an arbitrary 1-form $\omega = adx + bdr$ we have

$$\begin{aligned}\chi\omega &= \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial r}\right) dx \wedge dr - *d(adr - bdx) \\ &= \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial r}\right) dx \wedge dr - * \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial r}\right) dx \wedge dr \\ &= \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial r}\right) dx \wedge dr - \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial r}\right).\end{aligned}$$

Thus, the Euler operator is represented by the matrix

$$\chi = \begin{pmatrix} \frac{\partial}{\partial x} & -\frac{\partial}{\partial r} \\ -\frac{\partial}{\partial r} & -\frac{\partial}{\partial x} \end{pmatrix} = ir^{-1} \begin{pmatrix} -ir\frac{\partial}{\partial x} & ir\frac{\partial}{\partial r} \\ ir\frac{\partial}{\partial r} & ir\frac{\partial}{\partial x} \end{pmatrix}.$$

Its edge symbol has the form[6]

$$\begin{aligned}\sigma_\wedge(\chi)(x,\xi) = ir^{-1} \begin{pmatrix} r\xi & ir\frac{\partial}{\partial r} \\ ir\frac{\partial}{\partial r} & -r\xi \end{pmatrix} : \mathcal{K}^{1,1}(\mathbb{R}_+) \equiv \psi(r) H^{1,1/2}(\mathbb{R}_+) + [1-\psi(r)] H^1(\mathbb{R}_+) \\ \longrightarrow \mathcal{K}^{0,0}(\mathbb{R}_+) \equiv L^2(\mathbb{R}_+) \equiv H^{0,-1/2}(\mathbb{R}_+), \quad (6.3)\end{aligned}$$

where $\psi(r)$ is an R-function (an infinitely differentiable function such that $\psi(r) = 1$ for $r < 1$ and $\psi(r) = 0$ for $r > 2$) and the bundles (in this case, $\mathbb{C}^2$) are omitted in the notation of spaces.

The edge symbol is Fredholm, since the conormal symbol

$$\sigma_c(\sigma_\wedge(\chi))(p) = \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$$

is invertible on the weight line $\mathcal{L}_{1/2} = \{\operatorname{Im} p = 1/2\}$. To compute the index of the edge symbol, we note that the conormal symbol is symmetry with respect to the involution $p \mapsto -p$ and use the standard theorem on the index of operators on manifolds with conical singularities for this case. By this theorem, the index of the operator (6.3) is

[5] where d is the exterior differential, δ is the adjoint operator, and $*$ is the Hodge star operator (e.g., see [29]).

[6] The apparent controversy in the exponents is due to a difference of 1/2 between the weight exponents accepted here and in the book [39]. The exponents in $\mathcal{W}^{s,\gamma}$ and $\mathcal{K}^{s,\gamma}$ are used according to the convention in [39], while for the cone spaces $H^{s,\gamma}$ we use the convention in [42].

half the difference between the index of an operator on the double of $\mathbb{R}_+$ and the sum of multiplicities of the singular points of the conormal symbol in the strip between the weight lines $\mathcal{L}_{1/2}$ and $\mathcal{L}_{-1/2}$. The operator on the double has the form

$$\begin{pmatrix} \xi & i\frac{\partial}{\partial r} \\ i\frac{\partial}{\partial r} & -\xi \end{pmatrix} : H^1(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

and is obviously invertible (recall that $\xi \neq 0$), so that its index is zero. The half-sum of multiplicities is equal to 1; thus, the index of the edge symbol is independent of ξ and x and is equal to -1, more precisely, $-[\mathbb{C}] \in K(\mathbb{S}^1)$. Let us compute the kernel and cokernel of the operator (6.3). This is easy, since we deal with an operator with constant coefficients on the half-line:

$$\sigma_\wedge(\chi)(x,\xi) = i \begin{pmatrix} \xi & i\frac{\partial}{\partial r} \\ r\frac{\partial}{\partial r} & -\xi \end{pmatrix}. \tag{6.4}$$

An arbitrary function annihilated by the operator (6.4) has the form

$$\varphi(r) = C_1 e^{\xi r} \begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 e^{-\xi r} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \tag{6.5}$$

where C_1 and C_2 are arbitrary constants. None of these functions lies in $\mathcal{K}^{1,1}(\mathbb{R}_+)$, since the elements of this space must be square integrable at infinity and vanish at zero. The cokernel of the edge symbol is equal to the kernel of the adjoint operator

$$\sigma_\wedge(\chi)(x,\xi)^* = -i \begin{pmatrix} \xi & i\frac{\partial}{\partial r} \\ r\frac{\partial}{\partial r} & -\xi \end{pmatrix} : L^2(\mathbb{R}_+) \equiv \mathcal{K}^{0,0}(\mathbb{R}_+) \longrightarrow \mathcal{K}^{-1,-1}(\mathbb{R}_+). \tag{6.6}$$

It is one-dimensional and consists of functions of the form

$$\varphi(r) = Ce^{-|\xi| r} \begin{pmatrix} 1 \\ -i \operatorname{sign} \xi \end{pmatrix}. \tag{6.7}$$

Thus, to make the edge symbol invertible, we must equip it with a single coboundary condition, that is, extend it to a matrix of the form

$$\big(\sigma_\wedge(\chi)(x,\xi), P(x,\xi)\big) : \mathcal{K}^{1,1}(\mathbb{R}_+) \oplus \mathbb{C} \longrightarrow \mathcal{K}^{0,0}(\mathbb{R}_+). \tag{6.8}$$

The operator $P(x,\xi)$ can be specified as follows. Consider an arbitrary function $\varphi_0(r) \in C_0^\infty(\mathbb{R}_+)$ (vanishing at zero) such that

$$\int_0^\infty \varphi_0(r) e^{-r}\, dr \neq 0$$

and set

$$P(x,\xi)(\lambda) = r^{-1}\varphi_0(|\xi|r)\begin{pmatrix}1\\0\end{pmatrix}.$$

One can readily see that the resulting edge symbol is an isomorphism and is first-order twisted homogeneous in ξ. Thus, for the Euler operator on M we obtain a Fredholm problem of the form

$$\chi u + \psi(r)r^{-1}\varphi_0(|-ir\partial/\partial x|)v\begin{pmatrix}1\\0\end{pmatrix} = f \in \mathcal{W}^{0,0}(M,\Lambda^{ev}), \quad u \in \mathcal{W}^{1,1}(M,\Lambda^{odd}), v \in H^1(X), \tag{6.9}$$

where $\psi(r)$ is an arbitrary R-function (on the choice of which the problem is independent modulo infinitely smoothing operators). The edge symbol of this problem is symmetric in ξ, and the circle is odd-dimensional. Applying the theorem in § 3.3, we arrive at the following assertion.

Proposition 6.2. *The index of the edge problem* (6.9) *is equal to half the Euler characteristic of the double* $2M$*.*

Likewise, if we consider the Euler operator in the spaces

$$\chi : \mathcal{W}^{s,\gamma}(M,\Lambda^{odd}) \longrightarrow \mathcal{W}^{s-1,\gamma-1}(M,\Lambda^{ev}) \tag{6.10}$$

for $\gamma < 1/2$, then we obtain for it a Fredholm edge problem with edge condition, say, of the form

$$\begin{cases}\chi u = f \in \mathcal{W}^{s-1,\gamma-1}(M,\Lambda^{ev}),\\ \int_0^\infty \varphi_0(|-ir\partial/\partial x|)u_1(x,r)\frac{dr}{r} = g \in H^{s-1}(X),\end{cases} \quad u \in \mathcal{W}^{s,\gamma}(M,\Lambda^{odd}) \tag{6.11}$$

(here u_1 is the first component of the vector function u). For the index of this problem, we have the following assertion.

Proposition 6.3. *The index of problem* (6.11) *is equal to half the Euler characteristic of the double* $2M$*.*

For $\gamma = 1/2$, the edge symbol is not Fredholm, and there are no elliptic edge problems.

Appendix 1. Pseudodifferential operators in sections of infinite-dimensional bundles

The technique of pseudodifferential operators in spaces of sections of infinite-dimensional bundles (or pseudodifferential operators with operator-valued symbols) plays an important role in elliptic theory on manifolds with singularities and is substantially used in this paper in the proof of main theorems. Such pseudodifferential operators were studied by many authors (in particular, see [16] or [23], where further references can be found). As applied to elliptic theory, where the remainders in composition formulas for pseudodifferential operators must be compact operators in order to provide informative results, an appropriate technique was developed in [15]. In this appendix, following that article with slight modifications, we briefly recall the main definitions and facts concerning pseudodifferential operators in sections of infinite-dimensional bundles in a form convenient to us. The main theorem on the index of elliptic pseudodifferential operators with operator-valued symbols was proved in [15] in the special case of symbols homogeneous for large values of the momentum variables and only announced in the general case of arbitrary elliptic symbols (which is of main interest to us) with the indication that the proof can be based on the ideas developed by Hörmander for the corresponding class of symbols in finite-dimensional bundles. This has given rise to some doubts (e.g., see [31] and [9]). In this connection, we also give a more or less detailed proof of this theorem with a reduction to the above-mentioned special case.

A.1 Symbols with compact fiber variation

We consider pseudodifferential operators on a smooth compact closed manifold X in spaces of sections of Hilbert bundles over X. By the Kuiper theorem (e.g., see [4]), any Hilbert bundle is trivial, and so in the abstract theory one can assume that these bundles have the form $X \times H \longrightarrow X$, where H is a Hilbert space. By $\mathcal{B}(H)$ we denote the Banach algebra of bounded operators in H and $\mathcal{K}(H) \subset \mathcal{B}(H)$ is the closed ideal of compact operators. The canonical coordinates on T^*X will be denoted by (x,ξ). For convenience, we choose and fix a smooth norm $|\xi|$ in the fibers of T^*X.

Definition A.1. By $S^0_{CV}(T^*X)$ we denote the space of infinitely differentiable (in the uniform operator topology) functions

$$f : T^*X \longrightarrow \mathcal{B}(H),$$

satisfying the estimates

$$\left\| \frac{\partial^{\alpha+\beta} f(x,\xi)}{\partial x^\alpha \partial \xi^\beta} \right\|_{\mathcal{B}(H)} \le C_{\alpha\beta}(1+|\xi|)^{-|\beta|}, \quad |\alpha|+|\beta| = 0,1,2,\dots, \tag{A.1}$$

in any canonical coordinate system (x,ξ) on T^*M and possessing the *compact fiber variation* property

$$f(x,\xi) - f(x,\widetilde{\xi}) \in \mathcal{K}(H) \quad \text{for } \xi,\widetilde{\xi} \in T^*_x(X). \tag{A.2}$$

By $S^0_{CV}(T^*_0X)$ we denote the space of functions

$$f : T^*_0X \longrightarrow \mathcal{B}(H),$$

satisfying the estimates (A.1) for $|\xi| > \varepsilon$ for any $\varepsilon > 0$ with constants $C_{\alpha\beta}$ depending on ε and possessing the property (A.2) for $\xi, \widetilde{\xi} \neq 0$.

The elements of the spaces $S^0_{CV}(T^*X)$ and $S^0_{CV}(T^*_0X)$ will be referred to as *symbols* (of degree 0) *with compact fiber variation* on T^*X (respectively, on T^*_0X).

Remark A.2. Under the estimates for the derivatives, condition (A.2) can be replaced by the equivalent condition

$$\frac{\partial^{\alpha+\beta} f(x,\xi)}{\partial x^\alpha \partial \xi^\beta} \in \mathcal{K}(H) \quad \text{for } |\beta| \geq 1. \tag{A.3}$$

(See Lemma 2.4 in [15].)

Both spaces $S^0_{CV}(T^*X)$ and $S^0_{CV}(T^*_0X)$ are obviously algebras (with respect to fiber-wise multiplication). The algebra $S^0_{CV}(T^*X)$ contains the ideal $J^{-1}_K(T^*X)$ of compact-valued symbols

$$f : T^*X \longrightarrow \mathcal{K}(H),$$

satisfying the estimates

$$\left\| \frac{\partial^{\alpha+\beta} f(x,\xi)}{\partial x^\alpha \partial \xi^\beta} \right\|_{\mathcal{B}(H)} \leq C_{\alpha\beta}(1+|\xi|)^{-1-|\beta|}, \quad |\alpha|+|\beta| = 0,1,2,\ldots \tag{A.4}$$

in any canonical coordinate system (x,ξ).

Lemma A.3. *For each symbol $f \in S^0_{CV}(T^*_0X)$, there exists a symbol $\widetilde{f} \in S^0_{CV}(T^*X)$ such that $f(x,\xi) = \widetilde{f}(x,\xi)$ for sufficiently large $|\xi|$. If $\widetilde{\widetilde{f}} \in S^0_{CV}(T^*X)$ is another symbol with the same property, then the difference $\widetilde{f} - \widetilde{\widetilde{f}}$ is compact-valued and compactly supported.*

Proof. (a) The existence of $\widetilde{f}$. On the unit spheres

$$\mathbb{S}^{n-1}(x) = \{|\xi| = 1\} \subset T^*_xX,$$

we define an arbitrary smooth measure $d\mu(\xi)$ with the property

$$\int_{\mathbb{S}^{n-1}(x)} d\mu(\xi) = 1$$

and set

$$\widetilde{f}(x,\xi) = \chi(|\xi|) f(x,\xi) + \big(1 - \chi(|\xi|)\big) \int_{\mathbb{S}^{n-1}(x)} f(x,\xi')\, d\mu(\xi'), \tag{A.5}$$

where $\chi(t) \in C^\infty(\mathbb{R}_+)$ is an excision function equal to zero in a neighborhood of zero and unity in a neighborhood of infinity. (For $\xi = 0$, we set $f(x,\xi)$ equal to zero on the right-hand side.) Then, obviously, $f(x,\xi) = \widetilde{f}(x,\xi)$ for sufficiently large $|\xi|$. Moreover, $f(x,\xi)$

is infinitely differentiable and satisfies the estimates (A.1). Let us verify the compact fiber variation property (it suffices to do this for $\widetilde{\xi}=0$):

$$\widetilde{f}(x,\xi)-\widetilde{f}(x,0)=\chi(|\xi|)\int\limits_{\mathbb{S}^{n-1}(x)}(f(x,\xi)-f(x,\xi'))\,d\mu(\xi')\in\mathcal{K}(H)$$

for any $\xi\in T_x^*X$, since for $\xi=0$ the right-hand side is zero, while for $\xi\neq 0$ the integrand is compact (and depends on ξ' continuously).

(b) Uniqueness of the symbol $\widetilde{f}$ modulo compactly supported compact remainders. If $\widetilde{f}$ and $\widetilde{\widetilde{f}}$ are two symbol with the above-mentioned properties, then they both coincide with f for sufficiently large ξ, so that the difference is compactly supported. Since both symbols have compact fiber variation, we obtain

$$\begin{aligned}\widetilde{f}(x,\xi)-\widetilde{\widetilde{f}}(x,\xi)&=[\widetilde{f}(x,\xi)-\widetilde{f}(x,\xi_0)]-[\widetilde{\widetilde{f}}(x,\xi)-\widetilde{f}(x,\xi_0)]\\&=[\widetilde{f}(x,\xi)-\widetilde{f}(x,\xi_0)]-[\widetilde{\widetilde{f}}(x,\xi)-\widetilde{\widetilde{f}}(x,\xi_0)]\in\mathcal{K}(H);\end{aligned}$$

one should only take a sufficiently large $\xi_0\in T_x^*X$ such that $\widetilde{f}(x,\xi_0)=\widetilde{\widetilde{f}}(x,\xi_0)$.

The proof of the lemma is complete. □

A symbol $\widetilde{f}$ whose existence is established in Lemma A.3 will be called the *tightening* of f. The main role in our exposition is played by symbols in $S^0_{CV}(T_0^*X)$, since such symbols (undefined for $\xi=0$) naturally occur in edge problems.

A.2 The algebra of pseudodifferential operators

A.2.1 Definition of PDO

For the case of operator-valued symbols, pseudodifferential operators are defined in a standard way (e.g., see [12]). Namely, we use the following construction. Let

$$f(x,\xi):H\longrightarrow H,\quad (x,\xi)\in T^*X$$

be an operator-valued symbol on X. We cover X by coordinate neighborhoods U_j, $j=1,\dots,N$, and consider a smooth partition of unity

$$1=\sum_{j=1}^{N}\chi_j(x)^2$$

subordinate to the cover. We define a pseudodifferential operator with symbol $f(x,\xi)$ by the formula

$$f\left(x,-i\frac{\partial}{\partial x}\right)\overset{\text{def}}{=}\sum_{j=1}^{N}(\chi_j f)\left(x,-i\frac{\partial}{\partial x}\right)\circ\chi_j(x),\tag{A.6}$$

where the jth term in the sum is defined in the local coordinates of the chart U_j as the composition of the operator of multiplication by the function $\chi_j(x)$ (which localizes the function to which the pseudodifferential operator must be applied into the chart U_j) and a pseudodifferential operator with symbol $\chi_j(x)f(x,\xi)$ in $\mathbb{R}^n$ (here the symbol is expressed via the canonical coordinates in T^*U_j, denoted by the same letters (x,ξ)), defined with the help of the Fourier transform:

$$(\chi_j f)\left(x, -i\frac{\partial}{\partial x}\right)u(x) = \left(\frac{i}{2\pi}\right)^{n/2} \int e^{ix\xi}\chi_j(x)f(x,\xi)\widetilde{u}(\xi)\,d\xi, \quad u \in C_0^\infty(\mathbb{R}^n_x), \tag{A.7}$$

where $\widetilde{u}(\xi)$ is the Fourier transform of $u(x)$.

Proposition A.4. *Let $f \in S^0_{CV}(T^*X)$. Formulas* (A.6) *and* (A.7) *define a continuous operator*

$$f\left(x, -i\frac{\partial}{\partial x}\right) : L^2(X,H) \longrightarrow L^2(X,H), \tag{A.8}$$

which is modulo compact operators independent of the choice of the atlas U_j and the subordinate partition of unity.

This proposition permits us to give the following definition.

Definition A.5. Let $f \in S^0_{CV}(T^*X)$. A *pseudodifferential operator with symbol f* is the operator (A.8) defined modulo compact operators by formulas (A.6)–(A.7).

Let $f \in S^0_{CV}(T^*_0X)$. A *pseudodifferential operator with symbol f* is the operator

$$\widetilde{f}\left(x, -i\frac{\partial}{\partial x}\right) : L^2(X,H) \longrightarrow L^2(X,H),$$

where $\widetilde{f}(x,\xi) \in S^0_{CV}(T^*X)$ is an arbitrary tightening of $f(x,\xi)$.

By Lemma A.3, the symbol $\widetilde{f}$ exists and is uniquely determined modulo compact-valued compactly supported symbols, which, in particular, belong to the ideal $J_K^{-1}(T^*X)$, so that the corresponding pseudodifferential operators are compact by Proposition 2.1 in [15]. Thus, a pseudodifferential operator with symbol $f \in S^0_{CV}(T^*_0X)$ is well defined modulo compact operators. By abuse of notation, we denote this operator by

$$f\left(x, -i\frac{\partial}{\partial x}\right).$$

A.2.2 The composition theorem

Pseudodifferential operators with operator-valued symbols thus defined form an algebra. More precisely, the following assertion holds.

Proposition A.6. *Let $p, q \in S^0_{CV}(T^*_0X)$, and let P and Q be pseudodifferential operators with symbols p and q, respectively. Then PQ is a pseudodifferential operator with symbol pq.*

This theorem is essentially about the principal symbol (since we have defined pseudodifferential operators modulo compact operators). Needless to say, having chosen the ambiguous elements in the construction of pseudodifferential operators in some way (the atlas, the partition of unity, the tightening, etc.), one can obtain more precise composition formulas, which in local coordinates have the standard form (e.g., see [13]). We use such formulas below in the proof of the index theorem.

A.3 Ellipticity and the Fredholm property

The composition formula modulo compact operators contained in Proposition A.6 permits one to give a natural definition of ellipticity and prove the finiteness theorem.

Definition A.7. A symbol $p \in S^0_{CV}(T^*_0X)$ is said to be *elliptic* if there exists an $R > 0$ such that $p(x, \xi)$ is invertible for $|\xi| > R$ and the inverse satisfies the estimate

$$\left\|p(x,\xi)^{-1}\right\| \leq C, \quad |\xi| > R, \tag{A.9}$$

for some constant C.

Proposition A.8. *A symbol $p \in S^0_{CV}(T^*_0X)$ is elliptic if and only if there exists a symbol $q \in S^0_{CV}(T^*X)$ such that the symbols $q\widetilde{p} - 1$ and $\widetilde{p}q - 1$ are compactly supported and compact-valued for any tightening $\widetilde{p} \in S^0_{CV}(T^*X)$ of p. In particular, if p is elliptic, then $\widetilde{p}$ is Fredholm for all $(x, \xi) \in T^*X$ (and p is Fredholm for all $(x, \xi) \in T^*_0X$).*

Proof. We define $q(x, \xi)$ for $|\xi| > R$ by the formula

$$q(x, \xi) = p(x, \xi)^{-1}.$$

Outside the balls of radius R, this symbol satisfies the estimates (A.1) and has a compact fiber variation. We extend it into the interior of the balls using the construction in the proof of Lemma A.3 with an excision function $\chi(t)$ vanishing for $t \leq R$. The resulting symbol lies in $S^0_{CV}(T^*X)$ by construction; we denote it again by $q(x, \xi)$. Now consider the symbol $q\widetilde{p} - 1$. By construction, $q(x, \xi)\widetilde{p}(x, \xi) - 1 = 0$ for sufficiently large $|\xi|$. Now for arbitrary ξ we have

$$\begin{aligned} q(x,\xi)\widetilde{p}(x,\xi) - 1 &= q(x,\xi)\widetilde{p}(x,\xi) - q(x,\xi_0)\widetilde{p}(x,\xi_0) + q(x,\xi_0)\widetilde{p}(x,\xi_0) - 1 \\ &= q(x,\xi)\widetilde{p}(x,\xi) - q(x,\xi_0)\widetilde{p}(x,\xi_0), \end{aligned}$$

provided that $|\xi_0|$ is sufficiently large. The last expression is compact, and so we finally obtain,

$$q(x, \xi)\widetilde{p}(x, \xi) = 1 + K(x, \xi),$$

where $K(x,\xi)$ is a compact-valued function equal to zero for large $|\xi|$. The symbol $\widetilde{p}q-1$ can be treated in a similar manner. Thus, $\widetilde{p}(x,\xi)$ has a two-sided almost inverse and hence is Fredholm. The proof of the proposition is complete. □

Theorem A.9 (finiteness theorem). *If $p \in S^0_{CV}(T^*_0X)$ is an elliptic symbol, then the operator $P = p(x, -i\partial/\partial x)$ is Fredholm.*

Proof. Indeed, the operator $Q = q(x, -i\partial/\partial x)$ with the symbol $q(x,\xi)$ constructed in Proposition A.8 is the two-sided almost inverse for P. □

A.4 The index theorem

Let $p \in S^0_{CV}(T^*_0X)$ be an elliptic symbol. Consider an arbitrary tightening $\widetilde{p}$. Since the family $\widetilde{p}(x,\xi)$ is everywhere Fredholm and is invertible outside a compact set, it has a well-defined index $\operatorname{ind}\widetilde{p} \in K(T^*X)$ (the K-group with compact supports). Since distinct tightenings differ by compactly supported compact-valued families, it follows that the index is independent of the choice of tightening, that is, is defined by the symbol p itself, and we denote it by $\operatorname{ind} p$.

Theorem A.10. *Let $p \in S^0_{CV}(T^*_0X)$ be an elliptic symbol. Then the index of the corresponding elliptic operator can be obtained as the direct image in K-theory:*

$$\operatorname{ind} p\left(x, -i\frac{\partial}{\partial x}\right) = \pi_! \operatorname{ind} p, \tag{A.10}$$

where $\pi : X \longrightarrow \{pt\}$ is the mapping of X into a point.

Proof. Without loss of generality, one can assume from the very beginning that $p \in S^0_{CV}(T^*X)$ (i.e., the tightening has already been done). The proof was given in [15] for the case of symbols p satisfying the condition

$$p(x,t\xi) = p(x,\xi) \quad \text{for } \xi \geq R,\ t > 1. \tag{A.11}$$

We use Hörmander's method [12, Theorem 19.2.3] to reduce the case of general elliptic symbols to the one considered in [15]. Hörmander's theorem gives a reduction for the case of symbols acting in finite-dimensional bundles, and our main task is to verify that all constructions still work in the infinite-dimensional case.

The reduction is based on the following assertion, which we give here for the special case that we need.

Proposition A.11 ([12, Theorem 19.1.10]). *Let $T_\varepsilon, S_\varepsilon \in \mathcal{B}(H)$, $\varepsilon \in [0,\varepsilon_0]$, be two strongly continuous operator families such that the families $S_\varepsilon T_\varepsilon - 1$ and $T_\varepsilon S_\varepsilon - 1$ are uniformly compact. Then $\operatorname{ind} S_\varepsilon = -\operatorname{ind} T_\varepsilon$ is independent of ε.*

(An operator family $Q_\varepsilon \in \mathcal{B}(H)$ is said to be *uniformly compact* if the union of images of the unit ball under the operators in the family is precompact.)

Let $q \in S^0_{CV}(T^*X)$ be a symbol such that $pq = qp = 1$ for $|\xi| > R$.

Let $\psi(t)$, $t \geq 0$, be an infinitely differentiable function such that

$$\psi(t) = \begin{cases} 1 & \text{for } t < 1, \\ 1/t & \text{for } t > 2. \end{cases} \tag{A.12}$$

Consider the symbols

$$p_\varepsilon(x,\xi) = p(x, \xi\psi(\varepsilon\xi)), \quad q_\varepsilon(x,\xi) = q(x, \xi\psi(\varepsilon\xi)). \tag{A.13}$$

They have the following properties (the proof coincides word for word with the one given in [12, Theorem 19.2.3] for the finite-dimensional case):

1. $p_0 = p$ and $q_0 = q$;
2. p_ε and q_ε are uniformly bounded in $S^0_{CV}(T^*X)$ for $\varepsilon \in [0,1]$;
3. for $\varepsilon > 0$, the symbols p_ε and q_ε satisfy condition (A.11), and for sufficiently small ε they are elliptic;
4. for sufficiently small $\varepsilon > 0$, the compactly supported compact-valued symbols

$$r_1 = p_\varepsilon q_\varepsilon - 1, \quad r_2 = q_\varepsilon p_\varepsilon - 1 \tag{A.14}$$

   are independent of ε.

We define operators

$$P_\varepsilon = p_\varepsilon\left(x, -i\frac{\partial}{\partial x}\right), \quad Q_\varepsilon = q_\varepsilon\left(x, -i\frac{\partial}{\partial x}\right) \tag{A.15}$$

by formulas (A.6)–(A.8) with coordinate neighborhoods and partition of unity independent of ε. We claim that (for sufficiently small ε)

(a) $\operatorname{ind} p_\varepsilon \in K(T^*X)$ is independent of ε;

(b) the families P_ε and Q_ε satisfy the assumptions of Proposition A.11.

This implies our assertion, since for $\varepsilon > 0$ the symbol p_ε satisfies condition (A.11) and $\operatorname{ind} P_\varepsilon = \pi_! \operatorname{ind} p_\varepsilon$ by Luke's result, while the passage to the limit as $\varepsilon \to 0$ is possible by Proposition A.11. Thus, it suffices to prove (a) and (b).

The first assertion is obvious, since for sufficiently small ε the symbol p_ε varies with ε only in the exterior of a sufficiently large ball $\{|\xi| > R \simeq 1/\varepsilon\}$ and remains invertible there.

To prove the second assertion, note that the following is true.

1) The families P_ε and Q_ε are strongly continuous, since they are uniformly bounded and each term occurring in the definition of these families via the sum over coordinate charts on X is strongly continuous on the set of functions with compactly supported Fourier transform.

2) The operators $P_\varepsilon Q_\varepsilon - 1$ and $Q_\varepsilon P_\varepsilon - 1$ are compact and depend on ε continuously; indeed, the second assertion follows from the fact that their complete symbols in local coordinate systems together with derivatives are uniformly continuous in ε on compact subsets of ξ, are uniformly bounded, and decay as $\xi \to \infty$, which implies the uniform continuity in ε for all ξ.

It follows from 2) that the families $P_\varepsilon Q_\varepsilon - 1$ and $Q_\varepsilon P_\varepsilon - 1$ are uniformly compact, which completes the proof. $\square$

Appendix 2. Pseudodifferential operators on manifolds with edges

In this appendix, we describe the calculus of pseudodifferential operators on manifolds with edges. As was already noted in the introduction, the constructions given below are specializations of the general scheme of the construction of the algebra of pseudodifferential operators on manifolds with fibered boundary and hence differs from the constructions given in [8] concerning manifolds with edges. In particular, we nowhere use the analyticity properties of symbols with respect to the conormal variable. The structure of the appendix is as follows. In § A.5, we give the definitions of manifolds and function spaces. This material is standard (it can be found, say, in [8]), and so all proofs are omitted. In § A.6 we describe the calculus itself, namely, define symbols and operators and give their main properties, including the composition theorem. Our algebra is slightly different from the algebra of edge pseudodifferential operators described in [8] (one of the differences is the larger supply of Green operators), and so some of the results are provided with brief proofs or hints. In § A.6, the definition of ellipticity is given and the finiteness (Fredholm property) theorem is presented. The proof is again standard and hence omitted.

A.5 Manifolds and function spaces

First, we give the main definitions of manifolds and spaces that we deal with.

A.5.1 Manifolds with edges

Let

$$\pi : Y \xrightarrow{\Omega} X \tag{A.16}$$

be a locally trivial bundle over a smooth compact manifold X with fiber a smooth compact manifold Ω, and let M be a smooth compact manifold with boundary $\partial M = Y$. In a collar neighborhood U of the boundary, we choose and fix a trivialization

$$U \simeq Y \times [0, 1).$$

The coordinate on the interval $[0, 1)$ ("the distance from the boundary") will be denoted by r, and local coordinates (sometimes, points) on X and Ω will be denoted by x and ω, respectively. The dimension of X will be denoted by n, and the dimension of Ω by k, so that $\dim M = n + k + 1$.

A *manifold with edges* is the space obtained from M by shrinking each fiber of π into a point, i.e., by identifying all boundary points lying in the same fiber of π. The manifold M is called the *stretched manifold* of $\mathcal{M}$. The image of U under the factorization $M \longrightarrow \mathcal{M}$ will again be denoted by U.

There is a natural diffeomorphism

$$\mathcal{M} \setminus X \equiv \overset{\circ}{\mathcal{M}} \simeq \overset{\circ}{M} \equiv M \setminus Y.$$

An invariant definition of the *compressed cotangent bundle* of $\mathcal{M}$ can be found in [19]. This is a smooth manifold with boundary $T^*\mathcal{M}$; it is isomorphic to T^*M, and there is a diffeomorphism of the interiors $T^*\mathcal{M} \setminus \partial T^*\mathcal{M} \equiv T^*\overset{\circ}{\mathcal{M}}$ and $T^*M \setminus \partial T^*M \equiv T^*\overset{\circ}{M}$ of these manifolds, specified in U by the formulas

$$\begin{gathered} T^*\mathcal{M} \setminus \partial T^*\mathcal{M} \longrightarrow T^*M \setminus \partial T^*M, \\ (x, \omega, r, \eta, q, p) \longmapsto (x, \omega, r, \xi, q, \zeta), \\ \eta = \xi r, \quad p = -\zeta r. \end{gathered} \tag{A.17}$$

Here (η, q, p) are the momenta dual to $(x, \omega, t = -\ln r)$ in the fibers of $T^*\mathcal{M}$, and (ξ, q, ζ) are the momenta dual to the same variables in the fibers of the usual cotangent bundle T^*M. Note that the conormal variable p can be interpreted as a usual momentum (dual to the variable $t = -\ln r$). The change of variables $t = -\ln r$, or, equivalently, $r = e^{-t}$ takes U to the half-infinite cylinder $Y \times (0, \infty)_t$, on which $t \to \infty$ corresponds to approaching the edge. The isomorphism (A.17) can be extended to the entire $T^*\mathcal{M} \setminus \partial T^*\mathcal{M}$ in a trivial way: the function r is extended to be equal to 1 on $\mathcal{M} \setminus U$ (and is then smoothed near the interior boundary of $\overline{U}$), so that the isomorphism proves to be identical outside U. In what follows, we often use the "mixed" set (ξ, ω, p) of momentum variables; this will not result in a confusion.

Along with $\mathcal{M}$, we consider the model wedge $W_\pi \equiv W$ obtained from the semiinfinite cylinder $Y \times \mathbb{R}_+$ by the same shrinking of the fibers of π over $Y \times \{0\}$:

$$W = (Y \times \mathbb{R}_+)/\sim, \quad \text{where } (y, r) \sim (y', r') \Leftrightarrow r = r' = 0 \text{ and } \pi(y) = \pi(y').$$

The neighborhood U of the edge in $\mathcal{M}$ will be identified with the corresponding neighborhood in W, which will be denoted by the same letter. Thus, functions on $\mathcal{M}$ supported in U can be viewed as functions on W and vice versa. For the localization in U, we shall use cutoff functions ψ supported in U such that ψ depends only on r ($\psi = \psi(r)$), belongs to the space $C_0^\infty\bigl([0,1)\bigr)$, and is identically equal to unity in a neighborhood of zero. For brevity, such functions will be called *R-functions*. Next, we sometimes need localization along X in the same neighborhood. We then use cutoff functions of the form $\psi(r)\phi(x)$, where $\psi(r)$ is an R-function and $\phi(x)$ is a smooth function on X (supported, say, in a coordinate neighborhood on X). Similar cutoff functions $\phi(\omega)$ will be used on Ω.

The model wedge bears the natural bundle structure

$$\pi_W : W \longrightarrow X \tag{A.18}$$

over X; the fiber is the model cone

$$K \equiv K_\Omega = \{\Omega \times [0,\infty)\}/\{\Omega \times \{0\}\}.$$

By $\overset{\circ}{K} = \Omega \times (0,\infty)$ we denote the interior of K.

A.5.2 Edge Sobolev spaces

These spaces can be obtained by applying the construction [39], [8] of abstract edge spaces to the spaces $\mathcal{K}^{s,\gamma}(K)$ of functions on the model cone.

Definition A.12. By $\mathcal{K}^{s,\gamma}(K)$, where $s,\gamma \in \mathbb{R}$, we denote the subspace of the space $\mathcal{D}(\overset{\circ}{K})$ of distributions formed by elements u such that for each R-function $\psi(r)$ and each smooth function $\phi(\omega)$ supported in a coordinate neighborhood on Ω one has the inclusions

$$\begin{aligned} \bigl[r^{(\gamma-k/2)}\psi\phi u\bigr](e^{-t},\omega) &\in H^s(\mathbb{R}^{k+1}_{t,\omega}), \\ \bigl[(1-\psi)u\bigr](r,\omega/r) &\in H^s(\mathbb{R}^{k+1}_{r,\omega}), \end{aligned}$$

where $H^s(\mathbb{R}^{k+1})$ is the usual Sobolev space on the $(k+1)$-dimensional real vector space and in the first inclusion the standard change of variables $r = e^{-t}$ has been made. The norm in $\mathcal{K}^{s,\gamma}(K)$ is introduced in the usual way on the basis of a partition of unity $\{\phi_j(\omega)\}$ subordinate to a cover of Ω by coordinate neighborhoods.

The special case of this space for $s = \gamma = 0$ is [8, p. 213]

$$\mathcal{K}^{0,0}(K) \equiv \mathcal{K} = r^{-k/2}L^2(\Omega \times \mathbb{R}_+),$$

where the coordinate on $\mathbb{R}_+$ is r and the space L^2 is considered with respect to the measure $dr\, d\omega$ (here $d\omega$ is a volume form on Ω). This equation in particular means an equivalence of norms, and we always use the norm of the space on the right-hand side in this special case.

In each of the spaces $\mathcal{K}^{s,\gamma}(K)$, we have a well-defined one-parameter group

$$\begin{gathered}\varkappa_\lambda : \mathcal{K}^{s,\gamma}(K) \longrightarrow \mathcal{K}^{s,\gamma}(K), \lambda \in \mathbb{R}_+\\ \varkappa_\lambda u(r,\omega) = \lambda^{(k+1)/2} u(\lambda r, \omega)\end{gathered} \tag{A.19}$$

of bounded operators (the group law is multiplicative, i.e., $\varkappa_\lambda \varkappa_\mu = \varkappa_{\lambda\mu}$). One can readily see that this group is *unitary* in $\mathcal{K}^{0,0}(K)$, since

$$\int \lambda^{k+1} |u|^2(\lambda r, \omega) r^k \, dr \, d\omega = \int |u|^2(r,\omega) r^k \, dr \, d\omega.$$

For $\xi \in \mathbb{R}^n$, we take a smooth function $[\xi]$ such that $[\xi]$ is strictly positive for all ξ and

$$[\xi] = |\xi| \equiv \sqrt{\xi^2} \text{ for } |\xi| \geq 1.$$

Now we can define the space $\mathcal{W}^{s,\gamma}(W)$.

Definition A.13. The space $\mathcal{W}^{s,\gamma}(W)$ is the completion of the space $C_0^\infty(\overset{\circ}{W})$ of compactly supported smooth functions on $\overset{\circ}{W}$ with respect to the norm defined on functions $u \in C_0^\infty(\overset{\circ}{W})$ with the projection of support on X contained in some coordinate neighborhood by the formula

$$\|u\|^2_{s,\gamma} = \int [\xi]^{2s} \left\| \varkappa^{-1}_{[\xi]} \widetilde{u}(\xi) \right\|^2_{s,\gamma} d\xi, \tag{A.20}$$

where $\widetilde{u} \equiv \widetilde{u}(\xi)$ is the Fourier transform of $u(x,\omega,r)$ with respect to x and the norm of the integrand on the right-hand side is taken in $\mathcal{K}^{s,\gamma}(K)$.

The special case of this space for $s = \gamma = 0$ is [8, p. 297]

$$\mathcal{W}^{0,0}(W) = r^{-k/2} L^2(W) \equiv L^2(X, \mathcal{K});$$

Here the coordinate on $\mathbb{R}_+$ is r and $L^2(W) = L^2(Y \times \mathbb{R}_+, dr\, dv)$, where dv is a volume form on Y (in local computations, one can assume that $dv = d\omega\, dx$).

The group $\varkappa_\lambda$ is strongly continuous in all these edge spaces and is unitary in $\mathcal{W}^{0,0}(W)$.

Now we can define edge spaces on the manifold $\mathcal{M}$ itself.

Definition A.14. The space $\mathcal{W}^{s,\gamma}(\mathcal{M})$ is the completion of $C_0^\infty(\overset{\circ}{\mathcal{M}})$ with respect to the norm

$$\|u\|^2_{\mathcal{W}^{s,\gamma}(\mathcal{M})} = \|\psi u\|^2_{s,\gamma} + \|(1-\psi)u\|^2_s, \tag{A.21}$$

where ψ is an arbitrary R-function, the first term is the norm in the edge space on the infinite wedge, and the second term is the norm in the standard Sobolev space on the smooth part of $\mathcal{M}$.

The space $\mathcal{W}^{0,0}(\mathcal{M})$ can be represented in the form

$$\mathcal{W}^{0,0}(\mathcal{M}) = \widetilde{r}^{-k/2} L^2(\mathcal{M}),$$

where $\widetilde{r}$ is a smooth nonvanishing real function on $\overset{\circ}{\mathcal{M}}$ equal to r in $\overset{\circ}{U}$ and to a positive constant outside a larger neighborhood of the edge.

In what follows, we write $\mathcal{W}(\mathcal{M})$ and $\mathcal{W}(W)$ instead of $\mathcal{W}^{0,0}(\mathcal{M})$ and $\mathcal{W}^{0,0}(W)$, respectively. If the argument is clear, we simply write $\mathcal{W}$.

The definition of edge spaces of sections of vector bundles are obtained from the above definitions by standard modifications, which are omitted here altogether. Such spaces will be denoted by $\mathcal{W}(\mathcal{M}, E)$, $\mathcal{W}(W, E)$, or even $\mathcal{W}(E)$, where E is the corresponding bundle. The letter E will also be often omitted. We only note in passing that vector bundles on $\mathcal{M}$ are defined just as vector bundles on M.

A.5.3 The structure of a bottleneck space

Our proof of index theorems uses, among other tools, the abstract relative index locality principle introduced in [24]. To apply this principle, one must have the structure of a bottleneck space in the sense of [24] on the spaces where the elliptic operators in question act. In our case, these spaces are $\mathcal{W}(\mathcal{M})$ and $\mathcal{W}(W)$. They can be equipped with the structure of bottleneck spaces as follows. Consider a smooth function $\chi(r)$ such that

$$\chi(r) = \begin{cases} -1, & 0 \le r \le 1/2, \\ \text{increases monotonically from } -1 \text{ to } 1, & 1/2 \le r \le 1, \\ 1, & r \ge 1. \end{cases}$$

It can be interpreted as a smooth function on the entire $\mathcal{M}$ (or $\mathcal{W}$) if we define it to be equal to 1 identically outside U. We define an action of the algebra $C^\infty([-1,1])$ on W by setting, in accordance with the general recipe in [24],

$$fu \overset{\text{def}}{=} (f \circ \chi) \cdot u, \quad f \in C^\infty([-1,1]), \quad u \in W,$$

where the functions on the right-hand side are multiplied pointwise. Geometrically, the bottleneck is the subset of $\mathcal{M}$ (or W) specified by the inequalities $1/2 \le r \le 1$.

A.6 Pseudodifferential morphisms

Here we introduce the algebra of pseudodifferential morphisms used in this preprint. First of all, we explain why it suffices to deal with zero-order operators when studying index theory.

A.6.1 Order reduction

The theory of (pseudo)differential operators on manifolds with edges deals with elliptic edge problems of arbitrary order m given by operators of the form [8, p. 320, Definition 1; p. 326, Definition 15; p. 327, Theorem 16]

$$\mathbf{A} : \mathcal{W}^{s,\gamma}(\mathcal{M}, E_1) \oplus H^s(X, J_1) \longrightarrow \mathcal{W}^{s-m,\gamma-m}(\mathcal{M}, E_2) \oplus H^{s-m}(X, J_2). \tag{A.22}$$

Here E_1 and E_2 are bundles over $\mathcal{M}$ and J_1 and J_2 are bundles over X. Such problems can always be reduced to problems specified by *zero-order operators acting in spaces of the form* $\mathcal{W}(\mathcal{M}, E_j) \oplus L^2(X, J_j)$. This does not restrict generality in index theory. Indeed, the index of the Fredholm operator (A.22) is independent of s (see [8, p. 329, Remark 19]). Hence we can assume without loss of generality that $s = \gamma$. To simplify the exposition, we assume that the bundle $\partial M \to X$ is trivial (the direct product case). Next, on the manifold $\mathcal{M}$ we consider the Laplace operator $\triangle$ corresponding in a neighborhood of the edge to the edge-degenerate metric

$$dm^2 = dr^2 + r^2 dm_\Omega^2 + dm_X^2,$$

where dm_Ω^2 and dm_X^2 are some metrics on the edge and the base of the cone, respectively. This operator has the form

$$\triangle = r^{-2}\left[\left(r\frac{\partial}{\partial r}\right)^2 + \triangle_\Omega + r^2\triangle_X\right]$$

in U, where $\triangle_\Omega$ and $\triangle_X$ are the Laplace operators on Ω and X, respectively. The operator

$$Q = c\widetilde{r}^{-2} - \triangle, \tag{A.23}$$

where c is a positive constant, is elliptic and, moreover, invertible in the spaces

$$Q : \mathcal{W}^{s,\gamma}(\mathcal{M}) \longrightarrow \mathcal{W}^{s-2,\gamma-2}(\mathcal{M})$$

for $|\gamma| < R$ provided that $c > c(R)$, and for the order reduction in problem (A.22) it suffices to multiply $\mathbf{A}$ on the right and the left by the operators

$$\begin{pmatrix} Q & 0 \\ 0 & 1-\triangle_X \end{pmatrix}^{(s-m)/2} \quad \text{and} \quad \begin{pmatrix} Q & 0 \\ 0 & 1-\triangle_X \end{pmatrix}^{-s/2},$$

respectively. The new operator

$$\widetilde{\mathbf{A}} = \begin{pmatrix} Q & 0 \\ 0 & 1-\triangle_X \end{pmatrix}^{-s/2} \mathbf{A} \begin{pmatrix} Q & 0 \\ 0 & 1-\triangle_X \end{pmatrix}^{(s-m)/2}$$

defines an elliptic problem with the same index in the spaces

$$\widetilde{\mathbf{A}} : \mathcal{W}(\mathcal{M}, E_1) \oplus L^2(X, J_1) \longrightarrow \mathcal{W}(\mathcal{M}, E_2) \oplus L^2(X, J_2).$$

Note that the order reduction procedure does not affect the interior principal symbol on the unit cospheres, while the edge symbol is multiplied by an invertible family symmetric with respect to the involution $\xi \longrightarrow -\xi$. Hence, applying the index theorems proved in the present paper to the reduced problem, one can readily restate the results in terms of the original problem.

A.6.2 The general form of morphisms

We shall construct the calculus of pseudodifferential morphisms on a manifold M with edges modulo compact operators. To avoid unnecessary wordiness, we do not explicitly mention this each and every time. (For example, the statement that Green operators form an ideal in the algebra of pseudodifferential morphisms should be understood in the sense that the product of a pseudodifferential operator by a Green operator is again a Green operator plus a compact operator. This will not result in a misunderstanding.) Pseudodifferential morphisms have the form of matrix operators

$$\mathbf{A} = \begin{pmatrix} A+G & C \\ B & D \end{pmatrix} : \begin{array}{c} \mathcal{W}(\mathcal{M}, E_1) \\ \oplus \\ L^2(X, J_1) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}(\mathcal{M}, E_2) \\ \oplus \\ L^2(X, J_2) \end{array}, \tag{A.24}$$

where E_1 and E_2 are bundles over $\mathcal{M}$ and J_1 and J_2 are bundles over X. Here A is an *edge-degenerate pseudodifferential operator* on $\mathcal{M}$ and

$$\begin{pmatrix} G & C \\ B & D \end{pmatrix} \stackrel{\text{def}}{=} \mathbf{G}$$

is a *Green matrix*, or a *matrix Green operator*. The structure of these components will be described in detail in what follows. For brevity, we sometimes write $\mathbf{A} = A + \mathbf{G}$ omitting the matrix factor $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ on A.

A.6.3 Matrix Green operators

First, we describe the matrix Green operators $\mathbf{G}$. These operators act in direct sums of the form

$$\mathcal{W}(\mathcal{M}) \oplus L^2(X) \equiv \mathcal{W}(\mathcal{M}, E) \oplus L^2(X, J)$$

and, as we shall see, form a subalgebra (and even an ideal) in the algebra of pseudodifferential morphisms. Modulo compact operators, they are determined by their edge symbols, which we shall now describe.

Green symbols.

Definition A.15. A *matrix Green symbol* is a family

$$\mathbf{g}_\wedge(x,\xi) = \begin{pmatrix} g_\wedge(x,\xi) & c_\wedge(x,\xi) \\ b_\wedge(x,\xi) & d_\wedge(x,\xi) \end{pmatrix} : \mathcal{K}(K_x, E_1) \oplus J_1 \longrightarrow \mathcal{K}(K_x, E_2) \oplus J_2, \quad (x,\xi) \in T_0^*X$$

(where $E_{1,2}$ are finite-dimensional vector bundles over W and $J_{1,2}$ are finite-dimensional vector bundles over X), parameterized by T_0^*X, of continuous operators on the cones K_x (the fibers of the bundle (A.18)) smoothly depending on $(x,\xi)\in T_0^*X$ and possessing the following properties.

1. The family $\mathbf{g}_\wedge(x,\xi)$ satisfies the *twisted homogeneity condition*

$$\mathbf{g}_\wedge(x,\lambda\xi)=\varkappa_\lambda\mathbf{g}_\wedge(x,\xi)\varkappa_\lambda^{-1},\quad \lambda>0, \tag{A.25}$$

   where the group $\varkappa_\lambda$ acts on the second component of the direct sum $\mathcal{K}(K_x,E_j)\oplus J_j$ as the identity operator for all values of λ.[7]

2. The operator $\mathbf{g}_\wedge(x,\xi)$ is compact for all $(x,\xi)\in T_0^*X$.

Remark A.16. All entries of $\mathbf{g}_\wedge(x,\xi)$ except for the upper left are finite rank operators, and so the compactness condition is nontrivial only for the upper left entry.

The set of matrix Green symbols will be denoted by $S_G^{(0)}(T_0^*X)$. Obviously, the product of matrix Green symbols (if it is well defined, i.e., if the spaces fit) is again a matrix Green symbol.

Proposition A.17. *The embedding $S_G^{(0)}(T_0^*X)\subset S_{CV}^0(T_0^*X)$ holds. In other words, a matrix Green symbol is a symbol with compact fiber variation on T_0^*X.*

Proof. The property of compact fiber variation follows from the compactness of the matrix Green symbol itself, and so we only need to verify the estimates (A.1) on sets of the form $|\xi|>\varepsilon>0$. We differentiate (A.25) α times with respect to x and β with respect to ξ and set $\lambda=|\xi|^{-1}$; then we obtain

$$|\xi|^{-|\beta|}\mathbf{g}_\wedge^{\alpha,\beta}(x,\xi/|\xi|)=\varkappa_{|\xi|^{-1}}\mathbf{g}_\wedge^{\alpha,\beta}(x,\xi)\varkappa_{|\xi|}.$$

The desired estimate now follows, since the group $\varkappa_\lambda$ is unitary. $\square$

Quantization. Since $\mathbf{g}_\wedge(x,\xi)$ is a symbol with compact fiber variation, we can use Definition A.5 and obtain the pseudodifferential operator

$$\widehat{\mathbf{g}_\wedge}\equiv\mathbf{g}_\wedge\left(x,-i\frac{\partial}{\partial x}\right):\begin{matrix}\mathcal{W}(W,E_1)\\ \oplus\\ L^2(X,J_1)\end{matrix}\longrightarrow\begin{matrix}\mathcal{W}(W,E_2)\\ \oplus\\ L^2(X,J_2)\end{matrix} \tag{A.26}$$

[7] Thus, instead of $\varkappa_\lambda$ we should write the matrix

$$\begin{pmatrix}\varkappa_\lambda & 0\\ 0 & 1\end{pmatrix},$$

which has not been done to avoid clumsy notation. This will not result in a misunderstanding.

in spaces of sections of vector bundles on the infinite wedge W and on X, which will be called a *matrix Green operator with symbol* $\mathbf{g}_\wedge(x,\xi)$.

Applying Proposition A.6, we see that the product of two matrix Green operators is again (modulo compact operators) a matrix Green operator, and the product of operators corresponds to the product of their symbols.

However, we wish to interpret a matrix Green operator as an operator in spaces of sections of vector bundles on $\mathcal{M}$ and X. To this end, we use the following assertion, which shows that modulo compact operators a matrix Green operator is concentrated in an arbitrarily small neighborhood of the edge. To state the assertion most concisely, for elements $v \in L^2(X,J)$ we define multiplication by functions $\psi(r)$ of the variable r by setting $\psi v = \psi(0)v$. (In particular, if $\psi(r)$ is an R-function, then $\psi v = v$.)

Proposition A.18. *Let $\psi(r)$ be an arbitrary R-function. Then the operators*

$$(1-\psi(r))\mathbf{g}_\wedge\left(x,-i\frac{\partial}{\partial x}\right), \quad \mathbf{g}_\wedge\left(x,-i\frac{\partial}{\partial x}\right)(1-\psi(r))$$

are compact.

Proof. Consider the first of these two operators. (For the second operator, one passes to the adjoint operator

$$\left[\mathbf{g}_\wedge\left(x,-i\frac{\partial}{\partial x}\right)(1-\psi(r))\right]^* = (1-\psi(r))\mathbf{g}^*_\wedge\left(x,-i\frac{\partial}{\partial x}\right) + \text{a compact operator,}$$

thus reducing the proof to the case of the first operator.) We have

$$(1-\psi(r))\mathbf{g}_\wedge\left(x,-i\frac{\partial}{\partial x}\right) = f\left(x,-i\frac{\partial}{\partial x}\right),$$

we have

$$f(x,\xi) = (1-\psi(r))\mathbf{g}_\wedge(x,\xi).$$

Let us estimate the norm of the symbol $f(x,\xi)$ as $|\xi|\to\infty$. To this end, we use twisted homogeneity and write

$$f(x,\xi) = (1-\psi(r))\varkappa_\lambda \mathbf{g}_\wedge(x,\xi')\varkappa_\lambda^{-1} = \varkappa_\lambda(1-\psi(r/\lambda))\mathbf{g}_\wedge(x,\xi')\varkappa_\lambda^{-1},$$

where $\lambda = |\xi|$ and $\xi' = \xi/|\xi|$. Since the group $\varkappa_\lambda$ is unitary, we see that

$$\|f(x,\xi)\| = \|(1-\psi(r/\lambda))\mathbf{g}_\wedge(x,\xi')\|.$$

We are interested in an estimate as $\lambda\to\infty$ uniformly with respect to $(x,\xi')\in S^*X$; hence we treat $\mathbf{g}_\wedge(x,\xi')$ as an operator in the spaces

$$\mathbf{g}_\wedge : H_1 \longrightarrow C(S^*X, H_2), \tag{A.27}$$

where

$$H_j = \mathcal{K}(W,E_j)\oplus J_j, \quad j=1,2.$$

Lemma A.19. *The operator* (A.27) *is compact.*

Proof. Let us prove that the image of the unit ball in H_1 under the mapping $\mathbf{g}_\wedge$ is precompact in $C(S^*X, H_2)$. Let $u_j = u_j(\alpha)$, $\alpha = (x, \xi) \in S^*X$, be a sequence of functions from the image of the unit ball. We must show that it contains a convergent subsequence. One has the inequality

$$\|u_j(\alpha) - u_j(\alpha')\| \le C \operatorname{dist}(\alpha, \alpha'),$$

where the constant C is determined by the maximum of the first derivatives of $\mathbf{g}_\wedge(\alpha)$. Let $\{\alpha_s\}$ be a sequence everywhere dense in S^*X. Since $\mathbf{g}_\wedge(\alpha)$ is compact for each given α, it follows that the sequence $u_j(\alpha_s)$ contains a convergent subsequence for each given s. Using the diagonal process, we choose a subsequence of u_j convergent at all points α_s. We denote this subsequence again by u_j. It is a Cauchy sequence in $C(S^*X, H_2)$ and hence converges. Indeed, for any $\varepsilon > 0$ in $C(S^*X, H_2)$ there exists a finite ε-net that is a subset of $\{\alpha_s\}$. We choose N large enough that at all points α_s of this ε-net the function $u_j(\alpha_s)$, $j \ge N$, differs from the limit value at most by ε. Now if $j, j' > N$ and $\alpha \in S^*X$ is an arbitrary point, then

$$\left\|u_j(\alpha) - u_j'(\alpha)\right\| \le \|u_j(\alpha) - u_j(\alpha_s)\| + \left\|u_j'(\alpha) - u_j'(\alpha_s)\right\| + \left\|u_j(\alpha_s) - u_j'(\alpha_s)\right\| \le (2+2C)\varepsilon,$$

where α_s is the point of the ε-net closest to α. The proof of the lemma is complete. $\square$

The operator family

$$1 - \psi(r/\lambda) : C(S^*X, H_2) \longrightarrow C(S^*X, H_2)$$

strongly converges to zero as $\lambda \to \infty$. (Indeed, this family is uniformly bounded and strongly converges to zero on the dense subset of functions with support compact with respect to the variable r.) Hence the product $(1 - \psi(r/\lambda))\mathbf{g}_\wedge(x, \xi')$ tends to zero in the operator norm (the first factor strongly tends to zero and the second factor is compact). Thus, we obtain an estimate of the form

$$\|f(x, \xi)\| \le \varphi_{00}(|\xi|),$$

where $\phi_{00}(\lambda)$ is a monotone decaying function such that $\varphi_{00}(\lambda) \xrightarrow{\lambda\to\infty} 0$.

The derivatives

$$f^{(\alpha,\beta)}(x, \xi) \equiv \frac{\partial^{\alpha+\beta} f(x, \xi)}{\partial x^\alpha \partial \xi^\beta} = (1 - \psi(r))\mathbf{g}_\wedge^{(\alpha,\beta)}(x, \xi)$$

can be estimated with the use of twisted homogeneity in a similar way:

$$\left\|f^{(\alpha,\beta)}(x, \xi)\right\| \le |\xi|^{-|\beta|}\varphi_{\alpha\beta}(|\xi|), \quad \varphi_{\alpha\beta}(\lambda) \xrightarrow{\lambda\to\infty} 0. \tag{A.28}$$

Let $\chi(\xi)$ be a smooth function identically equal to 1 for $|\xi| < 1$ and to zero for $|\xi| > 2$. We set

$$f_\varepsilon(x, \xi) = f(x, \xi)\chi(\varepsilon\xi), \quad \varepsilon > 0.$$

The symbol $f_\varepsilon(x,\xi)$ is compactly supported and compact-valued, so that the operator

$$\widehat{f_\varepsilon} \equiv f_\varepsilon\left(x, -i\frac{\partial}{\partial x}\right)$$

is compact for every $\varepsilon > 0$. On the other hand, the difference $g_\varepsilon = f - f_\varepsilon$ satisfies the estimates

$$\left\| g_\varepsilon^{(\alpha,\beta)}(x,\xi) \right\| \le C_{\alpha\beta} |\xi|^{-|\beta|} \max_{\alpha' \le \alpha, \beta' \le \beta} \varphi_{\alpha'\beta'}(\varepsilon^{-1}). \tag{A.29}$$

For some N depending only on the dimension of X, the L^2-norm of a pseudodifferential operator on X can be estimated via the norms of the derivatives of its symbol of order $\le N$. Hence it follows from the estimates (A.29) that $\widehat{g_\varepsilon} \to 0$ in the operator norm as $\varepsilon \to 0$ and hence the operator $\widehat{f}$ is compact as the uniform limit of the compact operators $\widehat{f_\varepsilon}$.

The proof of the proposition is complete. □

Now let $\mathbf{g}_\wedge(x,\xi)$ be a matrix Green symbol. By the proposition we have just proved, the product

$$\mathbf{G} = \psi \mathbf{g}_\wedge\left(x, -i\frac{\partial}{\partial x}\right)\psi_1, \tag{A.30}$$

where ψ and ψ_1 are arbitrary R-functions, is independent of the choice of these functions modulo compact operators. Since the product is localized in the collar neighborhood U, it can be interpreted as an operator in spaces of sections of bundles over $\mathcal{M}$ and X.

Definition A.20. The *matrix Green operator* corresponding to a Green symbol $\mathbf{g}_\wedge(x,\xi) \in S_G^{(0)}(T_0^*X)$ is the operator

$$\mathbf{G} : \begin{matrix} \mathcal{W}(\mathcal{M}, E_1) \\ \oplus \\ L^2(X, J_1) \end{matrix} \longrightarrow \begin{matrix} \mathcal{W}(\mathcal{M}, E_2) \\ \oplus \\ L^2(X, J_2) \end{matrix}$$

given by formula (A.30).

Omitting the cutoff R-functions, we also denote this operator by $\widehat{\mathbf{g}_\wedge}$ or $\mathbf{g}_\wedge\left(x, -i\frac{\partial}{\partial x}\right)$. In more detailed notation, a matrix Green operator has the form

$$\widehat{\mathbf{g}_\wedge} = \begin{pmatrix} \widehat{g_\wedge} & \widehat{c_\wedge} \\ \widehat{b_\wedge} & \widehat{d_\wedge} \end{pmatrix}.$$

The operator $G = \widehat{g_\wedge}$ will be called simply a *Green operator*, and its symbol will be called a *Green symbol*. The operators $B = b_\wedge$ and $C = c_\wedge$ will be called the *boundary* and *coboundary operators*, respectively (cf. [45]). The operator $D = d_\wedge$ is a classical zero-order pseudodifferential operator on the edge X.

A.6.4 Pseudodifferential operators

Let us now describe the class of operators A in the upper left corner of the matrix of a pseudodifferential morphism. These operators are naturally called *pseudodifferential operators*. It will be convenient for us to describe three operator algebras, each of which is contained in the preceding (and hence inherits some nice properties of the latter). The last algebra is the one we deal with in our index theorems. This way we clarify which elements in the definition of the algebra of pseudodifferential operators are responsible for which properties of the algebra.

By $C^\infty(\mathcal{M})$ we shall denote the class of smooth functions on $\mathring{\mathcal{M}}$ that depend only on the variable x in some neighborhood of X contained in U (and hence extend naturally to the entire $\mathcal{M}$ by continuity). In particular, R-functions belong to $C^\infty(\mathcal{M})$. We also use the notation $S^0(T^*X)$ (respectively, $S^0(T_0^*X)$) for spaces of operator-valued symbols that satisfy the same estimates as those in the definition of $S^0_{CV}(T^*X)$ (respectively, $S^0_{CV}(T_0^*X)$) but do not necessarily have compact fiber variation.

By $\chi(|\xi|)$ we denote a cutoff function on T^*X that is compactly supported and is equal to unity in a neighborhood of $\xi = 0$.

The widest algebra.

Definition A.21. By $P(M)$ we denote the set of continuous operators

$$A : \mathcal{W}(\mathcal{M}) \longrightarrow \mathcal{W}(\mathcal{M})$$

with the following properties.

1. The commutator $[A, \varphi]$ is compact for any $\varphi \in C^\infty(\mathcal{M})$.
2. If $\varphi, \psi \in C^\infty(\mathcal{M})$ are supported away from the edge, then $\varphi A \psi$ is, modulo compact operators, a classical pseudodifferential operator of order zero in the interior of $\mathcal{M}$.
3. If $\varphi, \psi \in C^\infty(\mathcal{M})$ are supported in U, then $\varphi A \psi$ is, modulo compact operators, a pseudodifferential operator with symbol $A(x, \xi) \in S^0_{CV}(T_0^*X)$ on the infinite wedge W. (Here we use the identification of neighborhoods U of the edge in W and M.)

It is standard routine to show that $P(\mathcal{M})$ is an algebra. Each element $A \in P(M)$ has a well-defined principal symbol $\sigma(A)$, which is a function on the interior of the compressed cotangent bundle $T^*\mathcal{M}$. This function may well have singularities on the boundary $\partial T^*\mathcal{M}$. We also note that if the principal symbol of an operator $A \in P(\mathcal{M})$ is zero and for some R-functions φ, ψ the operator $\varphi A \psi$ is compact, then A itself is compact. However, this is probably all we can say about the compactness of operators in the class $P(\mathcal{M})$, because in general these operators have no edge symbols. Our next step is to introduce a narrower algebra, in which one can already speak of edge symbols and write out a precise compactness criterion.

The middle algebra. Now let us introduce a subset $PS(\mathcal{M}) \subset P(\mathcal{M})$ by imposing certain restrictions on the symbols of pseudodifferential operators arising in item 3 of Definition A.21. By $\mathcal{O}^{(0)}(T_0^*X)$ we denote the class of infinitely differentiable bounded families

$$a_\wedge(x,\xi) : \mathcal{K}(K_x) \longrightarrow \mathcal{K}(K_x), \quad (x,\xi) \in T_0^*X,$$

of operators on the cones K_x smoothly depending on $(x,\xi) \in T_0^*X$ and possessing the following properties.

1. The family $a_\wedge(x,\xi)$ is *twisted homogeneous*:

   $$a_\wedge(x,\lambda\xi) = \varkappa_\lambda a_\wedge(x,\xi)\varkappa_\lambda^{-1}, \quad \lambda > 0.$$

   (Hence it lies in $S^0_{CV}(T_0^*X)$; cf. Proposition A.17.)

2. If ψ is an R-function, then $\psi a_\wedge$ and $a_\wedge\psi$ lie in $S^0_{CV}(T_0^*X)$.

3. If ψ is an R-function, then the commutator $[\psi, a_\wedge]$ is compact-valued, and moreover

   $$[\psi, a_\wedge]\chi(\varepsilon|\xi|) \to [\psi, a_\wedge] \quad \text{in } S^0_{CV}(T_0^*X) \text{ as } \varepsilon \to 0.$$

One can readily see that $\mathcal{O}^{(0)}(T_0^*X)$ is an algebra. Only property 3) might cause some difficulty, but it is also in fact easy: one has

$$[\psi, a_\wedge b_\wedge] = [\psi, a_\wedge]b_\wedge + a_\wedge[\psi, b_\wedge],$$

and it remains to note that the product of two factors in $S^0(T_0^*X)$ necessarily lies in $S^0_{CV}(T_0^*X)$ provided that one of the factors is compact; the desired convergence also follows readily.

Now we introduce the space $\widetilde{S}^0_{CV}(T_0^*X) \subset S^0_{CV}(T_0^*X)$ of symbols $a(x,\xi)$ with the following two properties.

1. If ψ is an R-function, then the commutator $[\psi, a]$ is compact-valued, and moreover

   $$[\psi, a]\chi(\varepsilon|\xi|) \to [\psi, a] \quad \text{in } S^0_{CV}(T_0^*X) \text{ as } \varepsilon \to 0.$$

2. Set

   $$a_\lambda(x,\xi) = \varkappa_\lambda^{-1} a(x,\lambda\xi)\varkappa_\lambda.$$

   Then there exists a symbol $a_\wedge \in \mathcal{O}^{(0)}(T_0^*X)$ such that for each R-function $\psi(r)$ and each $\delta > 0$ one has the estimates

   $$\left\| \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial \xi^\beta}(a_\lambda - a_\wedge)\psi(r) \right\| \le \delta,$$

   $$\left\| \psi(r) \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial \xi^\beta}(a_\lambda - a_\wedge) \right\| \le \delta|\xi|^{-|\beta|}$$

   whenever $\lambda > R$ and $\xi > R_1$, where R and R_1 depend on δ (and also on ψ, α, and β), and R_1 may depend also on λ.

The space $\widetilde{S}^0_{CV}(T^*X)$ is just the space of tightenings of symbols $a \in \widetilde{S}^0_{CV}(T^*_0X)$.

Proposition A.22. 1. *For each $a \in \widetilde{S}^0_{CV}(T^*_0X)$ there exists a unique $a_\wedge$ with the above-mentioned property.*

2. *If ψ is an R-function, then $\psi \in \widetilde{S}^0_{CV}(T^*X)$ and $\psi_\wedge = 1$.*
3. *The space $\widetilde{S}^0_{CV}(T^*_0X)$ (as well as $\widetilde{S}^0_{CV}(T^*X)$) is an algebra.*
4. *The mapping $a \mapsto a_\wedge$ is an algebra homomorphism:*

$$(ab)_\wedge = a_\wedge b_\wedge.$$

Sketch of proof. Item 2 is obvious. Taking it into account, we see that it suffices to prove item 1 for the case in which the function ψ is fixed. It makes little difference whether ψ is on the right or on the left in the product, since we can always commute it from one side to another using the properties concerning commutators. Hence we assume that it is, say, on the right. Assuming that there are two elements $a_\wedge$ and $\widetilde{a_\wedge}$ for the same a, we subtract two estimates and obtain the similar estimates for the difference $(a_\wedge(x,\xi) - \widetilde{a_\wedge}(x,\xi))\psi(r)$, which is however independent of λ. Namely (we take $\alpha = \beta = 0$)

$$\|(a_\wedge(x,\xi) - \widetilde{a_\wedge}(x,\xi))\psi(r)\| \le \delta$$

for $|\xi| > R_1(\delta)$. Using the twisted homogeneity and the fact that $\varkappa_\mu$ is unitary, we obtain

$$\|(a_\wedge(x,\xi) - \widetilde{a_\wedge}(x,\xi))\psi(r/\mu)\| \le \delta$$

for a given ξ and for $\mu > R_1(\delta)|\xi|^{-1}$. But this is only possible if $a_\wedge(x,\xi) = \widetilde{a_\wedge}(x,\xi)$, since $\psi(r/\mu)$ strongly converges to unity as $\mu \to \infty$.

Now we prove items 3 and 4. Let $c = ab$ and accordingly $c_\wedge = a_\wedge b_\wedge$. We have

$$(c_\lambda - c_\wedge)\psi = a_\lambda(b_\lambda - b_\wedge)\psi + (a_\lambda - a_\wedge)\psi b_\wedge \psi_0 + (a_\lambda - a_\wedge)[b_\wedge, \psi]\psi_0,$$

where ψ_0 is an R-function such that $\psi\psi_0 = \psi$, whence the desired estimates follow with regard to the fact that a_λ is uniformly bounded in $S^0_{CV}(T^*_0X)$ and the above-mentioned properties of commutators. The case in which ψ is on the left can be considered in a similar way. ☐

Definition A.23. By $PS(\mathcal{M})$ we denote the set of operators $A \in P(\mathcal{M})$ such that the following property holds.

> If $\varphi, \psi \in C^\infty(\mathcal{M})$ are R-functions, then $\varphi A \psi$ is, modulo compact operators, a pseudodifferential operator with symbol $A(x,\xi) \in \widetilde{S}^0_{CV}(T^*_0X)$ on the infinite wedge W.

It follows from the preceding results that to each $A \in PS(\mathcal{M})$ there corresponds a uniquely determined symbol $a_\wedge$ constructed in the above-mentioned way via the symbol $A(x,\xi)$ of the operator $\varphi A \psi$, where ψ and φ are arbitrary R-functions. Indeed, to observe the uniqueness, one has only to note that if $A(x,\xi) \in \widetilde{S}^0_{CV}(T^*_0X)$ is the symbol of a compact operator, then it necessarily decays for large ξ, thus giving rise to a zero symbol $a_\wedge$.

Definition A.24. The symbol $a_\wedge$ corresponding to an operator $A \in PS(\mathcal{M})$ via the procedure described above is called the *edge symbol* of A and is denoted by $\sigma_\wedge(A)$.

Proposition A.25. *The set $PS(\mathcal{M})$ is an algebra. The mappings $A \mapsto \sigma(A)$ and $A \mapsto \sigma_\wedge(A)$ are algebra homomorphisms.*

The proof is obvious. □

The operators in the middle algebra $PS(\mathcal{M})$ possess not only principal symbols but also edge symbols. It turns out that these two symbols already permit one to find out whether a given operator is compact.

Theorem A.26. *An operator $A \in PS(\mathcal{M})$ is compact if and only if $\sigma(A) = 0$ and $\sigma_\wedge(A) = 0$.*

Sketch of proof. The necessity is obvious. Indeed, A is compact if and only if so are $\psi A \psi$ and $(1-\psi)A(1-\psi)$ for arbitrary R-functions ψ. We have already noted that the edge symbol of the first operator must be zero if the operator itself is compact. The second operator is a usual pseudodifferential operator, so one can apply standard results to the desired effect.

Let us prove sufficiency. If both symbols are zero, then the operator $(1-\psi)A(1-\psi)$ is compact by virtue of well-known results for pseudodifferential operators, and it remains to prove the compactness of the operator $B = \psi A \psi$. Let φ be an R-function. Then

$$B(1-\varphi(\lambda r)) = (1-\varphi_1(\lambda r))B(1-\varphi(\lambda r)) + [B, (1-\varphi_1(\lambda r))](1-\varphi(\lambda r))$$

is compact (the first term is a usual pseudodifferential operator with zero principal symbol and the second term contains a commutator), and so we shall arrive the desired assertion once we prove that $B\varphi(\lambda r)$ can be represented as the sum of a compact operator and an operator whose norm tends to zero as $\lambda \to \infty$. The operator $B\varphi(\lambda r)$ has the operator-valued symbol

$$c(x,\xi,\lambda) = b(x,\xi)\varphi(\lambda r) = \varkappa_\lambda \varkappa_\lambda^{-1} b(x,\xi)\varkappa_\lambda \varphi(r)\varkappa_\lambda^{-1} = \varkappa_\lambda [b_\lambda(x,\xi/\lambda)\varphi(r)]\varkappa_\lambda^{-1}.$$

The group $\varkappa_\lambda$ is unitary and does not affect estimates; since $\sigma_\wedge(B) = \sigma_\wedge(A) = 0$, it follows that the symbol in brackets (and hence $c(x,\xi,\lambda)$) admits the estimates

$$\left\| \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial \xi^\beta} c(x,\xi,\lambda) \right\| \le \delta |\xi|^{-|\beta|}$$

for $\lambda > R(\delta)$ and $|\xi| \geq R(\delta)R_1(\delta, \lambda)$. Now

$$
\begin{aligned}
B\varphi(\lambda r)u &= \int e^{ix\xi} c(x,\xi,\lambda)\widetilde{u}(\xi)\, d\xi \\
&= \int e^{ix\xi}\chi(\varepsilon|\xi|)c(x,\xi,\lambda)\widetilde{u}(\xi)\, d\xi + \int e^{ix\xi}(1-\chi(\varepsilon|\xi|))c(x,\xi,\lambda)\widetilde{u}(\xi)\, d\xi \\
&= \int e^{ix\xi}\chi(\varepsilon|\xi|)[c(x,\xi,\lambda) - c_R(x,\lambda)]\widetilde{u}(\xi)\, d\xi + \int e^{ix\xi}\chi(\varepsilon|\xi|)c_R(x,\lambda)\widetilde{u}(\xi)\, d\xi \\
&\qquad + \int e^{ix\xi}(1-\chi(\varepsilon|\xi|))c(x,\xi,\lambda)\widetilde{u}(\xi)\, d\xi = I + II + III,
\end{aligned}
$$

where c_R is the mean value of the symbol over the sphere of radius R. Now the first term is a compact operator by virtue of the compact fiber variation condition, while the second and third terms can be made arbitrarily small in the norm if we take sufficiently large λ and then choose a sufficiently large $R(> R(\delta)R_1(\delta, \lambda))$ and sufficiently small ε. The proof of the theorem is complete. □

The algebra of pseudodifferential operators. Although for middle-class symbols we have the main ingredient of the calculus, namely, the composition formula modulo compact operators (which readily follows from the preceding), we in general have no compatibility condition for the principal and edge symbols and cannot construct an operator with given principal and edge symbols, which prevents us from constructing almost inverses. In particular, one cannot prove the finiteness theorem in this generality.

Thus we descend to a still narrower algebra, which is the main algebra used in our preprint. This algebra will be denoted by $PSD(M)$, and its elements will be called pseudodifferential operators. This algebra will be constructed in the remaining items of this subsection. It is singled out by further conditions imposed on the principal and edge symbols.

Principal symbols. The principal symbols that we shall consider are smooth (up to the boundary) zero-order homogeneous functions on $T_0^*\mathcal{M}$. The space of such symbols will be denoted by $S^{(0)}(T_0^*\mathcal{M})$. In local coordinates near the boundary (in U), each element $a \in S^{(0)}(T_0^*\mathcal{M})$ is represented by a smooth function

$$a(x,\omega,r,\eta,q,p) = a(x,\omega,r,\lambda\eta,\lambda q,\lambda p), \quad \lambda > 0, \tag{A.31}$$

defined for $r \geq 0$ and $\eta^2 + q^2 + p^2 \neq 0$ (the variables x and ω range in the corresponding coordinate neighborhood). For each such function, using the change of variables (A.17), one defines a smooth (also up to the boundary) symbol $\widetilde{a} \in S^{(0)}(T_0^*M)$ by the formula

$$\widetilde{a}(x,\omega,r,\xi,q,\zeta) = a(x,\omega,r,r\xi,q,-r\zeta). \tag{A.32}$$

Edge symbols. Now we describe additional conditions imposed on the edge symbol of pseudodifferential operators.

Recall that edge symbols, as well as Green symbols, act in the space $\mathcal{K} \equiv \mathcal{K}^{0,0}(K)$ of functions on the model cone $K = K_\Omega$. More precisely, these are operator families in the spaces $\mathcal{K}(K_x)$ of functions on the fibers K_x of the bundle $\pi_W : W \longrightarrow X$ (where W is the model wedge with edge X). They are conically degenerate pseudodifferential operators near the vertex of the cone and pseudodifferential operators with a special behavior of the coefficients far from the vertex (as $r \to \infty$), and so, prior to introducing the definitions of the edge symbols themselves, we recall some auxiliary notions.

Definition A.27. A *boundary symbol* is a smooth zero-order homogeneous function

$$a_\partial \equiv a_\partial(x, \omega, \eta, q, p) \in S^{(0)}(\partial T_0^*\mathcal{M}), \tag{A.33}$$

defined on $\partial T_0^*\mathcal{M}$. One says that a boundary symbol is *compatible* with a principal symbol $a \in S^{(0)}(T_0^*\mathcal{M})$ if

$$a \,\big|_{\partial T_0^*\mathcal{M}} = a_\partial. \tag{A.34}$$

Let a_∂ be a given boundary symbol and $\psi(r)$ and R-function. To the product $(1-\psi)a_\partial$ we shall assign a pseudodifferential operator on the cones K_x in the following special way. Consider a finite cover of Ω by coordinate neighborhoods Ω_j and a subordinate partition of unity

$$\sum_j \chi_j^2(\omega) = 1.$$

In $\Omega_j \times \mathbb{R}_+$ we introduce local coordinates $\widetilde{\omega} = \omega r$ and r. Note that the change of variables $(\omega, r) \mapsto (\widetilde{\omega}, r)$ takes $\mathcal{W}(W)|_{\Omega_j \times \mathbb{R}_+}$ into the usual L^2 space with respect to the measure $dr\, d\widetilde{\omega}$.

We consider the symbol

$$f_j(x, \xi, \widetilde{\omega}, r, q, \zeta) = \chi_j(\widetilde{\omega}/r(1 - \psi(r))a_\partial(x, \widetilde{\omega}/r, \xi, q, \zeta) \tag{A.35}$$

depending on the parameters $(x, \xi) \in S^*X$ and define pseudodifferential operators

$$\widehat{f}(x, \xi) \stackrel{\text{def}}{=} \sum_j g_j^{-1} f_j\left(x, \xi, \widetilde{\omega}, r, -i\frac{\partial}{\partial\widetilde{\omega}}, -i\frac{\partial}{\partial r}\right) \circ g_j \chi_j(\omega), \tag{A.36}$$

where g_j is the coordinate mapping in the jth coordinate chart and the pseudodifferential operator in each term of the sum is defined in the standard way in the corresponding local coordinates with the help of the Fourier transform (cf. (A.6)).

The symbols f_j belong to the class $S^{[0],0}(\mathbb{R}^{2n})$ of symbols in $\mathbb{R}_n$ (the physical coordinates are $\widetilde{\omega}, r$) studied, e.g., in [8, § 8.2]. The results obtained there imply that the following assertion holds.

Proposition A.28. *The operator* (A.36) *is bounded in* $\mathcal{K}(K_x)$ *and, modulo compact operators in* $\mathcal{K}(K_x)$*, is independent of the choice of coordinate neighborhoods, local coordinates and the partition of unity.*

Definition A.29. A (*pseudodifferential*) *edge symbol* is a family, parameterized by the cotangent bundle of the edge X with zero section deleted, family

$$a_\wedge(x,\xi) : \mathcal{K}(K_x) \longrightarrow \mathcal{K}(K_x), \quad (x,\xi) \in T_0^*X,$$

of operators on the cones K_x smoothly depending on $(x,\xi) \in T_0^*X$ and possessing the following properties, modulo the addition of a Green family.

1. the family $a_\wedge(x,\xi)$ satisfies the twisted homogeneity condition

$$a_\wedge(x,\lambda\xi) = \varkappa_\lambda a_\wedge(x,\xi)\varkappa_\lambda^{-1}, \quad \lambda > 0.$$

2. In a neighborhood of the point $r = 0$ (the vertex of the cone), the operator $a_\wedge(x,\xi)$ is a conically degenerated zero-order pseudodifferential operator 0 with a principal symbol $a_0(x,\omega,r\xi,q,\zeta)$ and with conormal symbol

$$\sigma_c(a_\wedge)(x,p), \quad p \in \mathcal{L} = \{\operatorname{Im} p = -(k-1)/2\},$$

independent of the parameter ξ. Outside a neighborhood of the vertex, $a_\wedge(x,\xi)$ is a pseudodifferential operator of the form (A.36). More precisely, for $|\xi| = 1$ one has

$$a_\wedge(x,\xi) = \psi_1 A_1 \varphi_1 + (1-\psi_2)A_2(1-\varphi_2),$$

where $\varphi_1, \varphi_2, \psi_1, \psi_2$ are R-functions, A_1 is a cone-degenerate operator, and A_2 is an operator of the form (A.36).

The functions $\sigma(a_\wedge)(x,\omega,\eta,q,p) \stackrel{\text{def}}{=} a_0(x,\omega,\eta,q,p)$ and $\sigma_c(a_\wedge)(x,p)$ are called the *principal* and the *conormal* symbol of the edge symbol, respectively.

The space of pseudodifferential edge symbols will be denoted by $\mathcal{O}_\wedge^0 \equiv \mathcal{O}_\wedge^0(T^*X)$.

Proposition A.30. *The space* $\mathcal{O}_\wedge^0 \equiv \mathcal{O}_\wedge^0(T^*X)$ *is a subalgebra of* $\mathcal{O}^{(0)} \equiv \mathcal{O}^0(T^*X)$*.*

The proof follows from the composition theorems for conically degenerate pseudodifferential operators as well as for pseudodifferential operators with exit behavior [8].

Proposition A.31. *If the principal symbol of an edge symbol* $a_\wedge(x,\xi)$ *is a constant* (*or, more generally, depends on* x *but is independent of* ξ)*, then* $a_\wedge(x,\xi) \in S_{CV}^0(T_0^*X)$*.*

Definition of the algebra $PSD(\mathcal{M})$***.***

Definition A.32. By $PSD(\mathcal{M})$ we denote the subset of $PS(\mathcal{M})$ formed by operators A such that the edge symbol $a_\wedge = \sigma_\wedge(A)$ belongs to $\mathcal{O}^0_\wedge \equiv \mathcal{O}^0_\wedge(T^*X)$, the principal symbol $a = \sigma(A)$ belongs to $S^{(0)}(T^*_0\mathcal{M})$, and the compatibility condition

$$\sigma(a_\wedge)(x, \omega, \eta, q, p) = a(x, \omega, 0, \eta, q, p) \tag{A.37}$$

holds, i.e., the principal symbol of the edge symbol coincides with the restriction of the principal symbol to the edge $\partial T^*\mathcal{M}$.

Theorem A.33. *The set* $PSD(\mathcal{M})$ *is an algebra, and the mappings taking operators to symbols are multiplicative.*

Computation of the symbols of a pseudodifferential operator. To a compatible pair of symbols, one can always assign a pseudodifferential operator in $PSD(\mathcal{M})$ (e.g., see [39], [8], [40]), which is unique modulo compact operators. (The uniqueness holds already in $PS(\mathcal{M})$; we have proved this.) A construction procedure will be given in the next item. Here we indicate a method for reconstructing symbols from a given operator. Let $A \in PSD(\mathcal{M})$. Then for any R-functions $\psi(r)$ and $\psi_1(r)$ the operator $(1-\psi_1)A(1-\psi)$ is a *classical pseudodifferential operator* on $\overset{\circ}{M}$. It has a well-defined principal symbol $T^*_0\overset{\circ}{M}$ (in the usual sense), which can be computed via a procedure due to Hörmander [11] and which is independent of the choice of ψ, ψ_1 outside their supports. This gives a method for computing the principal symbol.

To compute the order symbol of A, consider the operator

$$B = \psi A \psi_1,$$

where ψ and ψ_1 are some R-functions (which do not affect the result.) The operator B can be viewed as an operator on the model cone W, since the support of its Schwartz kernel is contained in $U \times U$.

Now let $\phi, \phi_1 \in C^\infty(X)$ be arbitrary functions supported in some coordinate neighborhood V on X. Consider the operator

$$\widetilde{A} = \phi B \phi_1.$$

In the space $\mathcal{W} = \mathcal{W}(W)$, we consider the following one-parameter group depending on an additional parameter $\xi \in \mathbb{R}^n \setminus \{0\}$:

$$U_\lambda(\xi) = \varkappa_\lambda e^{ix\xi\lambda} : \mathcal{W} \longrightarrow \mathcal{W}. \tag{A.38}$$

(Outside the union of supports of ϕ and ϕ_1, we modify the product $x\xi$ in the exponent by multiplying $x\xi$ by a real-valued cutoff function.)

Then as $\lambda \to \infty$ there exists a limit $\widetilde{A}_\infty(\xi)$ in the strong operator topology of the operator family

$$U_\lambda^{-1}(\xi)\widetilde{A}U_\lambda(\xi) : \mathcal{W} \longrightarrow \mathcal{W}, \tag{A.39}$$

and the limit operator splits into a family of operators

$$\widetilde{A}_\infty(x,\xi) : \mathcal{K} \longrightarrow \mathcal{K}$$

smoothly depending on the parameters (x,ξ), $\xi \neq 0$, in the spaces $\mathcal{K}$ on the model cones that are the fibers of the bundle

$$\pi_X^*(W) \xrightarrow{K} T_0^*X;$$

here $\pi_X : T^*X \longrightarrow X$ is the natural projection. This operator has the form

$$\widetilde{A}_\infty(x,\xi) = \phi(x)\phi_1(x)\sigma_\wedge(A)(x,\xi),$$

so that the procedure described above, which resembles Hörmander's procedure for computing the principal symbol on smooth manifolds as well as the procedure [26, 18, 22] of computing the conormal symbol on manifolds with conical singularities permits one to compute the edge symbol of a pseudodifferential operator $A \in PSD(\mathcal{M})$.

Quantization. Now we describe the construction of a pseudodifferential operator $A \in PSD(\mathcal{M})$ with given compatible principal symbol $a \in \mathcal{O}^0(T^*\mathcal{M})$ an edge symbol $a_\wedge \in \mathcal{O}^0_\wedge(T^*X)$. First, we construct a pseudodifferential operator$B$ such that

$$\sigma_\wedge(B)(x,\xi) = a_\wedge(x,\xi). \tag{A.40}$$

To this end, we take R-functions ψ and ψ_1, consider the operator family

$$b(x,\xi) = \psi a_\wedge(x,\xi)\psi_1 : \mathcal{K} \longrightarrow \mathcal{K},$$

and set

$$B = b\left(x, -i\frac{\partial}{\partial x}\right)$$

(see Appendix 1). This operator can be treated as an operator on $\mathcal{M}$, since its Schwartz kernel is nonzero only in $U \times U$. One can readily see that (A.40) holds. Indeed, to compute the edge symbol, we need not multiply the operator by additional R-functions; instead, we can directly return to the infinite wedge W. Next, we localize along X (we omit the corresponding cutoff functions in the notation) and consider the family

$$B(\lambda,\xi) = U_\lambda^{-1}(\xi)BU_\lambda(\xi) : \mathcal{W} \longrightarrow \mathcal{W}.$$

Note that

$$B(\lambda,\xi) = \varkappa_\lambda^{-1}\psi\varkappa_\lambda \cdot \varkappa_\lambda^{-1}e^{-ix\xi\lambda}a_\wedge\left(x, -i\frac{\partial}{\partial x}\right)e^{ix\xi\lambda}\varkappa_\lambda \cdot \varkappa_\lambda^{-1}\psi\varkappa_\lambda.$$

The first and third factors strongly converge to the operator of multiplication by $\psi(0) = \psi_1(0) = 1$, so it suffices to consider the family in the middle. This family is bounded uniformly in λ, and so it suffices to prove the strong convergence as $\lambda \to \infty$ on the dense subset $C_0^\infty(K) \subset \mathcal{W}$. Let $u \in C_0^\infty(K)$. Then

$$\varkappa_\lambda^{-1} e^{-ix\xi} a_\wedge\left(x, -i\frac{\partial}{\partial x}\right) e^{ix\xi} \varkappa_\lambda u = \left(\frac{1}{2\pi}\right)^{n/2} \int \varkappa_\lambda^{-1} e^{ix\rho} a_\wedge(x, \lambda\xi + \rho) \varkappa_\lambda \widetilde{u}(\rho)\, d\rho,$$

where $\widetilde{u}(\rho)$ is the Fourier transform of ρ with respect to x. We take some ξ with $|\xi| = 1$. For sufficiently large λ, the function $\widetilde{u}(\rho)$, which rapidly decays as $\rho \to \infty$, will be arbitrarily small for $|\lambda\xi + \rho| < 1$. Hence with an error tending to zero as $\lambda \to \infty$ we can use twisted homogeneity and conclude that the last expression is equal to

$$\left(\frac{1}{2\pi}\right)^{n/2} \int e^{ix\rho} a_\wedge(x, \xi + \rho/\lambda) \widetilde{u}(\rho)\, d\rho,$$

which, in turn, converges as $\lambda \to \infty$ to

$$\left(\frac{1}{2\pi}\right)^{n/2} \int e^{ix\rho} a_\wedge(x, \xi) \widetilde{u}(\rho)\, d\rho = a_\wedge(x, \xi) u,$$

as desired. The computation of the principal symbol of B shows that it is equal to

$$\sigma(B) = \psi(r)\psi_1(r)\sigma(a_\wedge)(x, \omega, \eta, q, p) = \psi(r)\psi_1(r) a(x, \omega, 0, \eta, q, p)$$

(the second equation holds since a and $a_\wedge$ are compatible), and so A should be sought in the form

$$A = B + C,$$

where $\sigma_\wedge(C) = 0$ and $\sigma(C) = c \overset{\text{def}}{=} \sigma(A) - \sigma(B)$. In particular, $\sigma(C)$ is identically zero on the boundary $\partial T^*\mathcal{M}$. Thus, we arrive at the following problem: for a given symbol $c \in \mathcal{O}^0(T^*\mathcal{M})$ vanishing on $\partial T^*\mathcal{M}$, construct a zero-order pseudodifferential operator C on $\mathcal{M}$ with principal symbol c and with zero edge symbol. Using a partition of unity, we reduce the solution of this problem to the corresponding problems in coordinate neighborhoods on $\mathcal{M}$. The problem is trivial for the neighborhoods lying away from the edge. Now let us consider a neighborhood adjacent to the edge. Suppose that the symbol $c(x, \omega, r, \eta, q, p)$ defined $\omega \in \mathbb{R}^k$, $x \in \mathbb{R}^n$, and $r \in [0, 1)$ is compactly supported in (ω, x, r), zero-order homogeneous 0 in (η, q, p), and identically zero for $r = 0$. We smooth the symbol in a neighborhood of $r = 0$ preserving the latter property, so that we obtain a classical zero-order symbol (denoted by the same letter) $c \in S^0_{cl}(\mathbb{R}^{k+n} \times [0, 1) \times \mathbb{R}^{n+k+1}$ and vanishing for $r = 0$. Now we define an operator C in the space $\mathcal{W}(W)$ by the formula

$$Cu(x, \omega, e^{-t}) = \left(\frac{1}{2\pi}\right)^{(n+k+1)/2} \int\limits_{\mathbb{R}^{n+k}} dx\, d\omega \int\limits_{\mathcal{L}} dp \\ \times \left\{ e^{i(tp + x\xi + q\omega)} c(x, \omega, r, r\xi, q, p + i(k-1)/2) \widetilde{u}(\xi, \omega, p) \right\}, \quad \text{(A.41)}$$

where $\widetilde{u}$ is the Fourier transform of $u\bigl(x, \omega, e^{-t}\bigr)$ with respect to the variables (x, ω, t). This Fourier transform (owing to the weight $r^{-k/2}$) is well defined as an element of the space $L^2(\mathbb{R}^{n+k}_{\xi,q} \times \mathcal{L}$, where $\mathcal{L} = \{\operatorname{Im} p = -(k-1)/2\}$ is the weight line mentioned earlier, and the integral in (A.41) is taken over the corresponding space, so that the argument $p + i(k-1)/2$ of c is real (the function c is not assumed to be analytic).

One can readily see that the principal symbol of the operator (A.41) is equal to c. Indeed, multiplying it on the right and on the left by functions of the form $1 - \psi(r)$, where $\psi(r)$ is an R-function, in the computation of the principal symbol we can assume that the function u on which the operator acts is compactly supported in r on the open interval $(0, 1)$, so that the Fourier transform of u is holomorphic in the entire complex plane and belongs to the space of rapidly decaying functions on any horizontal line. Using an analytic smoothing of the symbol (see [39]), one can proceed to the integration over the real variable p by Cauchy's theorem. The subsequent argument is standard in the spirit of Hörmander [11] or Kohn–Nirenberg [13]. To see that the edge symbol of C is zero, it suffices to note that $C = r\psi_2(r)C_1$, where C_1 is the pseudodifferential operator (A.41) with principal symbol

$$c_1(x, \omega, r, \eta, q, p) = c(x, \omega, r, \eta, q, p)/r,$$

so that

$$\sigma_\wedge(C)(x, \xi) = \sigma_\wedge(r\psi_2(r))(x, \xi)\sigma_\wedge(C_1)(x, \xi) = 0 \cdot \sigma_\wedge(C_1)(x, \xi) = 0.$$

(Note that the edge symbol of C_1 exists, and so the computation is valid: the strong limit of a product is equal to the product of strong limits provided the latter exist.)

Remark A.34. 1. The above-described construction gives an operator continuously depending on parameters if the symbols continuously depend on the parameters.

2. If the principal symbol is independent of r in U (in elliptic theory this can always be achieved via homotopies), then one can simplify the quantization procedure by setting

$$A = \psi a_\wedge\left(x, -i\frac{\partial}{\partial x}\right)\psi_1 + (1 - \psi)\widehat{a}(1 - \psi_2), \tag{A.42}$$

where the first term is quantized as a pseudodifferential operator with operator-valued symbol on X and the hat in the second term stands for the usual pseudodifferential quantization in the interior of $\mathcal{M}$. Here ψ, ψ_1,and ψ_2 are R-functions satisfying the relations

$$\psi_1\psi = \psi, \quad \psi_2\psi = \psi_2.$$

Thus we have described the class $PSD(\mathcal{M})$ of operators on manifolds with edges for the case of scalar operators. The construction can be directly transferred to the case of operators acting in spaces of sections of vector bundles.

Calculus. Now we state the composition theorem (cf. [8] as well as Theorem 1.3.4 and Remark 1.3.3 in [40]).

Theorem A.35. *Operators of the form*

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G}, \tag{A.43}$$

where $A \in PSD(\mathcal{M})$ and $\mathcal{G}$ is a matrix Green operator with symbol in $\mathcal{O}^0_G(T^*X)$, form an algebra $\mathfrak{A}$, and the product in $\mathfrak{A}$ is determined by the symbol product:

$$\sigma(\mathcal{A}\mathcal{B}) = \sigma(\mathcal{A})\sigma(\mathcal{B}), \quad \sigma_\wedge(\mathcal{A}\mathcal{B})(x,\xi) = \sigma_\wedge(\mathcal{A})(x,\xi)\sigma_\wedge(\mathcal{B})(x,\xi).$$

An operator $\mathcal{A} \in \mathfrak{A}$ is compact if and only if $\sigma(\mathcal{A}) = 0$ and $\sigma_\wedge(\mathcal{A})(x,\xi) \equiv 0$.

Needless to say, here $\sigma(\mathcal{A}) = \sigma(A) \in \mathcal{O}^0(T^*\mathcal{M})$, and

$$\sigma_\wedge(\mathcal{A}) = \begin{pmatrix} \sigma_\wedge(A) & 0 \\ 0 & 0 \end{pmatrix} + \sigma_\wedge(\mathcal{G}) \in \mathcal{O}^0_\wedge(T^*X) + \mathcal{O}^0_G(T^*X).$$

Remark A.36. 1. By our definitions, the operator D in the lower right corner of $\mathcal{A}$ is a pseudodifferential operator in spaces of sections of finite-dimensional bundles on X.

2. The set of matrix Green operators is a module over the algebra of diagonal matrices of pseudodifferential operators.

3. Our definition of Green operators is wider than that in [39] or [8] (where the description involves some asymptotic information) and even in [40] (where an improvement of the weight exponent is required). The only property essential to us is that these operators must be concentrated in an arbitrarily small neighborhood of the edge. This is satisfied automatically under our definitions.

General matrix operators of the form (A.43) will be called *morphisms*, on the analogy with the terminology pertaining to Sobolev problems ([44, 45] etc.).

A.7 Ellipticity and the Fredholm property

The composition theorem in the preceding item implies the following natural definition and theorem. (Cf. [39] or [8]).

Definition A.37. A morphism $\mathcal{A}$ is said to be *elliptic* if its principal symbol $\sigma(\mathcal{A})$ and edge symbol $\sigma_\wedge(\mathcal{A})$ are invertible (on $S^*\mathcal{M}$ and S^*X, respectively).

Theorem A.38. *Each elliptic morphism is Fredholm.*

Elliptic morphisms are also called *elliptic problems.*

An elliptic problem can be stated for an elliptic pseudodifferential operator A with elliptic principal symbol under the following conditions:

1. The conormal symbol $\sigma_c(\sigma_\wedge(A))(x,p)$ is invertible for all $x \in X$ and $p \in \mathcal{L}$. (This guarantees the *Fredholm property* of the edge symbol $\sigma_\wedge(A)(x,\xi)$, since the exit ellipticity conditions [8] are satisfied automatically once the principal symbol is elliptic.)

2. The obstruction to the existence of elliptic edge problems for the operator A computed in [19] is zero.

Under these two conditions, there exists bundles J_1 and J_2 on X and operators B, C, D such that the problem

$$\mathcal{A} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

is elliptic. The corresponding procedure is described in [8, Chapter 9].

We note that the obstruction mentioned in condition 2 has the form[8]

$$\pi_![\sigma_\partial(A)] \in K(T^*X \times \mathbb{R}), \quad \pi_! : K(T^*\partial M \times \mathbb{R}) \longrightarrow K(T^*X \times \mathbb{R}), \tag{A.44}$$

and is determined by the restriction $\sigma_\partial(A)$ of the principal symbol $\sigma(A)$ to the boundary $\partial T^*\mathcal{M} \simeq T^*\partial M \times \mathbb{R}$ of the compressed cotangent bundle $T^*\mathcal{M}$.

Finally, we note that invertible conormal symbols (or, equivalently, Fredholm edge symbols) always exist for an elliptic interior symbol. Indeed, the following proposition holds.

Proposition A.39. *For each elliptic symbol a on a manifold $\mathcal{M}$ with edge X there exists a compatible Fredholm edge symbol $a_\wedge$.*

Proof. Let $a_\wedge$ be an edge symbol compatible with a. Since a is elliptic, we see that the edge symbol is Fredholm if and only if the conormal symbol $\sigma_c(a_\wedge) = \sigma_c(a_\wedge)(x,p)$ is invertible. The conormal symbol is a family of operators elliptic with parameter p on the weight line $\mathcal{L} = \{\operatorname{Im} p = -(k-1)/2\}$; this family smoothly depends on the parameter $x \in X$. Thus, the problem of constructing a Fredholm edge symbol is equivalent to the problem of constructing an invertible conormal symbol compatible with a. It was shown in [27] that the obstruction to the existence of an invertible conormal symbol is the element

$$\operatorname{ind} \sigma_c(a_\wedge) \in K^1(X).$$

The index of families elliptic with parameter was also computed in [27]. Namely, the principal symbol of the family determines an element of the K-group $K^1({}^vT^*\partial M)$, and the index of the family is just the direct image of this element. (The direct image is induced by the projection ${}^vT^*\partial M \to X$, where ${}^vT^*\partial M$ is the bundle of "vertical vectors" under the projection $\pi : M \longrightarrow X$.)

[8] Here we do not touch general edge problems with (co)boundary conditions in *subspaces* of Sobolev spaces on the edge [40], similar to general boundary value problems [41], [21]; in these classes there are no obstructions to the existence of Fredholm problems for a given principal symbol.

On the other hand, the principal symbol of the family $\sigma_c(a_\wedge)$ can be viewed as the restriction to $\{\xi = 0\}$ of a family of elliptic principal symbols with parameters (ξ, p). Namely, the latter family is the restriction a_∂ of the symbol a to the boundary ∂M. The corresponding difference element lies in $K^1(T^*\partial M)$, and there is a commutative diagram

$$\begin{array}{ccc} K^1(T^*\partial M) & \stackrel{i^*}{\longrightarrow} & K^1({}^vT^*\partial M) \\ \mathrm{ind}\downarrow & & \downarrow \mathrm{ind} \\ K^1(T^*X) & \stackrel{j^*}{\longrightarrow} & K^1(X). \end{array}$$

Here $i : {}^vT^*\partial M \to T^*\partial M$ and $j : X \to T^*X$ are natural embeddings. It was proved in [3] that the mapping j^* is zero. Hence it follows from this commutative diagram that the obstruction is zero:

$$\mathrm{ind}\, \sigma_c(a_\wedge) = 0.$$

The proof of the proposition is complete. □

Remark A.40. Note that our elliptic problems are elliptic operators in bottleneck spaces in the sense of [24] with respect to the bottleneck structure in $\mathcal{W}$.

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