PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

A FIXED POINT FORMULA IN ONE COMPLEX VARIABLE

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(Communicated by XXX)

ABSTRACT. We show a Lefschetz fixed point formula for holomorphic functions in a bounded domain $\mathcal{D}$ with smooth boundary in the complex plane. To introduce the Lefschetz number for a holomorphic map of $\mathcal{D}$, we make use of the Bergman kernel of this domain. The Lefschetz number is proved to be the sum of usual contributions of fixed points of the map in $\mathcal{D}$ and contributions of boundary fixed points, these latter being different for attracting and repulsing fixed points.

1. INTRODUCTION

Let $\mathcal{D}$ be a bounded domain with smooth boundary in the complex plane $\mathbb{C}$, and f be a holomorphic map of $\mathcal{D}$ which is C^∞ up to the boundary of $\mathcal{D}$. The pull-back operator f^* on differential forms preserves the bidegree and commutes with the Cauchy-Riemann operator $\bar{\partial}$. Hence it induces a homomorphism Hf^* of the cohomology of the complex

$$0 \longrightarrow \mathcal{E}^0(\overline{\mathcal{D}}) \stackrel{\bar{\partial}}{\longrightarrow} \mathcal{E}^1(\overline{\mathcal{D}}) \longrightarrow 0 \tag{1.1}$$

where $\mathcal{E}^q(\overline{\mathcal{D}})$ stands for the space of all $(0,q)$-forms in $\mathcal{D}$ with coefficients smooth up to the boundary, $q = 0, 1$.

The cohomology of (1.1) at step $q = 0$ just amounts to the space $\mathcal{A}(\overline{\mathcal{D}})$ of holomorphic functions in $\mathcal{D}$ which are C^∞ up to the boundary. This space is infinite-dimensional. On the other hand, the cohomology of (1.1) at step $q = 1$ vanishes. It follows that the usual definition of the holomorphic Lefschetz number leads to

$$L(f) = \operatorname{Tr} f^*|_{\mathcal{A}(\overline{\mathcal{D}})},$$

the trace of f^* on $\mathcal{A}(\overline{\mathcal{D}})$. As the space $\mathcal{A}(\overline{\mathcal{D}})$ is of infinite dimension, this trace fails to be defined for all maps f. The problem arises of defining a regularised trace of f^* on the space $\mathcal{A}(\overline{\mathcal{D}})$.

To this end one might invoke any right fundamental solution Φ of the operator $\bar{\partial}$ in $\overline{\mathcal{D}}$. Then the operator $\Pi = 1 - \Phi\bar{\partial}$ is a projection in the space $\mathcal{E}^0(\overline{\mathcal{D}}) = \mathcal{E}(\overline{\mathcal{D}})$ whose range is $\mathcal{A}(\overline{\mathcal{D}})$. The kernel $K_\Pi(\zeta, z)$ of Π is a representation of the Dirac functional $\delta_z(\zeta)$ on the space of holomorphic functions. The regularised trace of f^*

Received by the editors January 1, 2003.
2000 *Mathematics Subject Classification.* Primary 32S50; Secondary 58J20.
Key words and phrases. The Lefschetz number, the Neumann problem, the Bergman kernel.

on $\mathcal{A}(\overline{\mathcal{D}})$ is then defined by

$$\operatorname{Tr} f^*|_{\mathcal{A}(\overline{\mathcal{D}})} = \text{p.v.} \int K_\Pi(z, f(z)), \tag{1.2}$$

the principal value referring to fixed points of f on the boundary of $\mathcal{D}$ if there are any.

This agrees on the one hand with the trace of the pull-back operator f^* on $\mathcal{A}(\overline{\mathcal{D}})$ for constant maps f of $\mathcal{D}$, i.e., $f(z) = w_0 \in \mathcal{D}$. Indeed, in the latter case both sides of (1.2) are equal to 1. On the other hand, equality (1.2) gives readily the most elementary result of the Lefschetz theory. Namely, if f has no fixed points in $\overline{\mathcal{D}}$ then the holomorphic Lefschetz number of f vanishes, for the integral localises to the set of fixed points.

In this paper we take as Φ the right fundamental solution of the Cauchy-Riemann operator $\bar{\partial}$ given by the Neumann problem for the complex (1.1). At step 1 the latter problem actually reduces to the Dirichlet problem for the Laplace operator in $\mathcal{D}$. The kernel $K_\Pi(\zeta, z)$ obtained this way is nothing but the Bergman kernel of the domain $\mathcal{D}$. We obtain:

Theorem 1.1. *Suppose f is a holomorphic map of $\mathcal{D}$ which extends smoothly to the closure of $\mathcal{D}$. If f has only isolated fixed points in $\overline{\mathcal{D}}$ then the holomorphic Lefschetz number of f is*

$$L(f) = \sum_{p \in \operatorname{Fix}(f, \overline{\mathcal{D}})} \mu(p).$$

The local indices $\mu(p)$ are infinitesimal invariants of f at p. If $p \in \operatorname{Fix}(f, \mathcal{D})$ then $\mu(p)$ coincides with that appearing in the case of compact Riemannian surfaces. Namely, $\mu(p)$ is the trace of the meromorphic function $1/(1 - f'(z))$ near $z = p$ with respect to the map $z - f(z)$ at 0, cf. § 6 in Tsikh [12]. The local indices of boundary fixed points are more artful, cf. § 5.

A Lefschetz fixed point formula for closed holomorphic curves was first proved by Eichler in [4]. Atiyah and Bott [1] generalised it to the Dolbeault complex on a compact closed complex manifold. For direct constructions along more classical lines we refer the reader to Patodi [9], Toledo and Tong [11], et al.

For strictly pseudoconvex domains $\mathcal{D}$ in $\mathbb{C}^n$, a holomorphic Lefschetz formula was proved by Donnelly and Fefferman in [3], who worked within the framework of L^2-cohomology of the Bergman metric. Recall that the Bergman metric, whose Kaehler potential is given by the Bergman kernel, is a complete Kaehler metric on $\mathcal{D}$. This actually corresponds to the case of a non-compact closed manifold, and so f was assumed to have no fixed points on $\partial\mathcal{D}$. Our results extend to higher-dimensional situation, too.

Brenner and Shubin [2] showed a fixed point formula for elliptic boundary value problems in Boutet de Monvel's algebra. Their results do not apply to the Cauchy-Riemann system, for the latter admits no boundary value problem satisfying the Lopatinskii condition.

2. The Neumann problem

The Neumann problem for complex (1.1) at extreme step 1 consists of finding, for a given $F \in \mathcal{E}^1(\overline{\mathcal{D}})$, a differential form $u \in \mathcal{E}^1(\overline{\mathcal{D}})$ such that

$$\begin{array}{rcll} \bar{\partial}\bar{\partial}^* u & = & F & \text{in } \mathcal{D}, \\ n(u) & = & 0 & \text{on } \partial\mathcal{D}, \end{array} \tag{2.1}$$

where $\bar{\partial}^*$ is the formal adjoint for $\bar{\partial}$, and $n(u)$ the complex normal part of u on the boundary. Write

$$\begin{aligned} F &= F_1(z)\, d\bar{z}, \\ u &= u_1(z)\, d\bar{z}, \end{aligned}$$

then the problem (2.1) becomes in fact the Dirichlet problem for the function $u_1(z)$, namely

$$\begin{aligned} -(1/2)\,\Delta u_1 &= F_1 \quad \text{in} \quad \mathcal{D}, \\ u_1 &= 0 \quad \text{on} \quad \partial\mathcal{D}. \end{aligned}$$

The latter problem has a unique solution given by

$$u_1(z) = 2\imath \int_{\mathcal{D}} F_1(\zeta)\, G(\zeta, z)\, d\bar{\zeta} \wedge d\zeta$$

for $z \in \mathcal{D}$, where $G(\zeta, z)$ is the Green function of the Dirichlet problem. It has the form

$$G(\zeta, z) = \frac{1}{2\pi} \log|\zeta - z| - h(\zeta, z) \tag{2.2}$$

where $h(\zeta, z)$ is a smooth function defined away from the boundary diagonal in the product $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$. For a fixed $z \in \mathcal{D}$, this function is harmonic in $\zeta \in \mathcal{D}$, continuous in $\zeta \in \overline{\mathcal{D}}$, and satisfies $h(\zeta, z) = (1/2\pi) \log|\zeta - z|$ in $\zeta \in \partial\mathcal{D}$. This forces it to be symmetric in ζ and z, whence $G(\zeta, z) = G(z, \zeta)$. It follows that the Neumann operator at step 1 is

$$NF\,(z) = \int_{\mathcal{D}} F(\zeta) \wedge \Big(2\imath\, G(\zeta, z)\, d\zeta d\bar{z}\Big), \tag{2.3}$$

the integral being over $\zeta \in \mathcal{D}$.

The composition $\Phi = \bar{\partial}^* N$ gives a right fundamental solution of the Cauchy-Riemann operator in $\overline{\mathcal{D}}$. Indeed, (2.1) implies at once that $\bar{\partial}\Phi = 1$ on $\mathcal{E}^1(\overline{\mathcal{D}})$, as desired.

Lemma 2.1. *When regarded as a map in $\mathcal{L}(\mathcal{E}^1(\overline{\mathcal{D}}), \mathcal{E}(\overline{\mathcal{D}}))$, the operator Φ has the Schwartz kernel*

$$K_\Phi(\zeta, z) = \Big(-\frac{1}{2\pi\imath} \frac{1}{\zeta - z} + 2\imath \frac{\partial}{\partial z} h(\zeta, z) \Big) d\zeta.$$

Proof. Since

$$\bar{\partial}^*\,(u_1(z) d\bar{z}) = -\frac{\partial}{\partial z} u_1(z),$$

it follows from (2.3) that

$$\begin{aligned} \Phi F\,(z) &= \int_{\mathcal{D}} F(\zeta) \wedge -\frac{\partial}{\partial z}\Big(2\imath\, G(\zeta, z)\, d\zeta\Big) \\ &= \int_{\mathcal{D}} F(\zeta) \wedge K_\Phi(\zeta, z) \end{aligned}$$

for all $z \in \mathcal{D}$. □

Note that the second term in $K_\Phi(\zeta, z)$ is holomorphic in $z \in \mathcal{D}$ for any fixed $\zeta \in \overline{\mathcal{D}}$.

3. Augmented complex

The operator $\Pi = 1 - \Phi\bar{\partial}$ belongs to $\mathcal{L}(\mathcal{E}(\overline{\mathcal{D}}))$ and extends to an orthogonal projection of $L^2(\mathcal{D})$ onto the subspace of $L^2(\mathcal{D})$ consisting of holomorphic functions of class $L^2(\mathcal{D})$. The Schwartz kernel of Π is just the Bergman kernel of the domain $\mathcal{D}$. We next compute it.

Lemma 3.1. *When regarded as a map in $\mathcal{L}(\mathcal{E}(\overline{\mathcal{D}}))$, the operator Π has the Schwartz kernel*

$$K_{\Pi}(\zeta, z) = \Big(2\imath \frac{\partial^2}{\partial\bar{\zeta}\partial z} h(\zeta, z)\Big) d\bar{\zeta} \wedge d\zeta.$$

Proof. Let $u \in \mathcal{E}(\overline{\mathcal{D}})$. Combining Lemma 2.1, the Cauchy-Pompey theorem and Stokes' integral formula, we obtain

$$\Pi u\,(z) = -\int_{\partial\mathcal{D}} u(\zeta)\, K_{\Phi}(\zeta, z) + \int_{\mathcal{D}} u(\zeta) \Big(2\imath \frac{\partial^2}{\partial\bar{\zeta}\partial z} h(\zeta, z)\Big) d\bar{\zeta} \wedge d\zeta$$

for all $z \in \mathcal{D}$. The proof of Lemma 2.1 actually shows that

$$K_{\Phi}(\zeta, z) = -\frac{\partial}{\partial z}\Big(2\imath\, G(\zeta, z)\, d\zeta\Big)$$

for $(\zeta, z) \in \overline{\mathcal{D}} \times \mathcal{D}$. Since $G(\zeta, z)$ vanishes whenever $(\zeta, z) \in \partial\mathcal{D} \times \mathcal{D}$, so does $K_{\Phi}(\zeta, z)$, too. Hence the boundary integral in the formula for Πu vanishes, which completes the proof. □

The operators $\{\Pi, \Phi\}$ fit together to give a parametrix of the so-called augmented complex

$$0 \longrightarrow \mathcal{A}(\overline{\mathcal{D}}) \stackrel{i}{\longrightarrow} \mathcal{E}(\overline{\mathcal{D}}) \stackrel{\bar{\partial}}{\longrightarrow} \mathcal{E}^1(\overline{\mathcal{D}}) \longrightarrow 0, \tag{3.1}$$

where i stands for the embedding operator. This means they satisfy the fundamental equation

$$\begin{array}{rcll} \Pi i & = & 1 - S_{-1} & \text{on} \quad \mathcal{A}(\overline{\mathcal{D}}), \\ i\Pi + \Phi\bar{\partial} & = & 1 - S_0 & \text{on} \quad \mathcal{E}(\overline{\mathcal{D}}), \\ \bar{\partial}\Phi & = & 1 - S_1 & \text{on} \quad \mathcal{E}^1(\overline{\mathcal{D}}) \end{array} \tag{3.2}$$

up to operators S_{-1}, S_0 and S_1 with smooth kernels on $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$. In fact, the operators S_0 and S_{-1} vanish by the very construction, while S_1 vanishes on the range of $\bar{\partial}$, i.e., $\bar{\partial}\mathcal{E}(\overline{\mathcal{D}})$.

The pull-back operator f^* defines an endomorphism of (3.1), too, inducing a homomorphism of the cohomology of (3.1). Since this cohomology is finite-dimensional, the Lefschetz number of the later homomorphism is well defined. We denote it by $L_p(f)$, the sub "p" indicating to "partial", for the cohomology of (3.1) at steps -1 and 0 is zero.

4. Lefschetz number

By the above, the total holomorphic Lefschetz number is

$$L(f) = \operatorname{Tr} f^*|_{\mathcal{A}(\overline{\mathcal{D}})} + L_p(f), \tag{4.1}$$

the first term on the right-hand side being the regularised trace (1.2). We first evaluate the partial Lefschetz number $L_p(f)$.

Lemma 4.1. *Suppose f is a holomorphic map of $\mathcal{D}$, smooth up to the boundary and having only isolated fixed points in $\overline{\mathcal{D}}$. Then*

$$L_p(f) = -\text{p.v.} \int_{\mathcal{D}} \Delta^*(1 \times f)^* K_{\Pi} - \text{p.v.} \int_{\mathcal{D}} d\varphi(\Phi),$$

where Δ is the diagonal map $\overline{\mathcal{D}} \to \overline{\mathcal{D}} \times \overline{\mathcal{D}}$, and $\varphi(\Phi) = -\Delta^*(1 \times f)^* K_{\Phi}$ a smooth $(1,0)$-form away from the set of fixed points of f in the closure of $\mathcal{D}$.

Proof. Applying f^* to both sides of (3.2) we conclude that the endomorphisms f^* and $f^* S$ of (3.1) are homotopic. Hence it follows that $L_p(f) = L_p(f^* S)$. Since the endomorphism $f^* S$ is smoothing, the alternating sum formula readily yields $L_p(f) = -\operatorname{Tr} f^* S_1$, for $S_{-1} = S_0 = 0$. We now use the fundamental equation (3.2) once again, taking into account that the kernel of the identity operator is supported on the diagonal of $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$. This gives

$$\begin{aligned} L_p(f) &= \lim_{\varepsilon \to 0} \int_{\mathcal{D} \setminus U_\varepsilon} \Delta^*(1 \times f)^* K_{-S_1} \\ &= \lim_{\varepsilon \to 0} \int_{\mathcal{D} \setminus U_\varepsilon} \Delta^*(1 \times f)^* K_{\bar{\partial} \Phi}, \end{aligned}$$

where U_ε is an ε-neighbourhood of the set of fixed points of f in $\overline{\mathcal{D}}$. This establishes the formula because $\Delta^*(1 \times f)^* K_{\bar{\partial} \Phi}$ coincides with $-\Delta^*(1 \times f)^* K_{\Pi} - d\varphi(\Phi)$ away from the set $\operatorname{Fix}(f, \overline{\mathcal{D}})$. □

Lemma 4.1 gives some suggestive evidence to defining the regularised trace of f^* on holomorphic functions by (1.2). Indeed, from (4.1) and the lemma it follows that

$$L(f) = -\text{p.v.} \int_{\mathcal{D}} d\varphi(\Phi), \tag{4.2}$$

the formula looking like that for the case of compact closed manifolds, cf. Theorem 6.2.15 in [10].

5. Local indices

Given a point $p \in \operatorname{Fix}(f, \overline{\mathcal{D}})$, we write $U(p, \varepsilon)$ for the disk with centre p and radius $\varepsilon > 0$ in $\mathbb{C}$. By (4.2) and Stokes' formula, we get

$$L(f) = -\text{p.v.} \int_{\partial \mathcal{D}} \varphi(\Phi) + \sum_{p \in \operatorname{Fix}(f, \overline{\mathcal{D}})} \mu(p) \tag{5.1}$$

where

$$\mu(p) = \lim_{\varepsilon \to 0} \int_{\mathcal{D} \cap \partial U(p, \varepsilon)} \varphi(\Phi) \tag{5.2}$$

is an infinitesimal invariant of f at p. Note that in general $\mu(p)$ is a complex number, and not an integer.

Lemma 5.1. *Assume that $p \in \mathcal{D}$ is an isolated fixed point of f. If $\varepsilon > 0$ is small enough then*

$$\mu(p) = \int_{\partial U(p, \varepsilon)} \frac{1}{2\pi\imath} \frac{dz}{z - f(z)}.$$

Proof. Combining formula (5.2) and Lemma 2.1 we obtain

$$\mu(p) = \lim_{\varepsilon \to 0} \int_{\partial U(p,\varepsilon)} \frac{1}{2\pi\imath} \frac{dz}{z - f(z)} - \lim_{\varepsilon \to 0} \int_{\partial U(p,\varepsilon)} 2\imath\, h'_z(z, f(z))\, dz,$$

$h'_z(\zeta, z)$ meaning the derivative of $h(\zeta, z)$ in z. The differential form in the first integral on the right-hand side is closed in a small punctured neighbourhood of the fixed point p. Hence the integral does not depend on ε, provided that $\varepsilon > 0$ is sufficiently small. On the other hand, the differential form in the second integral on the right-hand side is smooth in a neighbourhood of p, for $h(\zeta, z)$ is smooth in the product $\mathcal{D} \times \mathcal{D}$. It follows that the second limit is equal to zero, which establishes the formula. □

In particular, if $p \in \mathcal{D}$ is a simple fixed point of f, i.e., $f'(p) \neq 1$, then

$$\mu(p) = \frac{1}{1 - f'(p)}, \tag{5.3}$$

as is easy to check by the Cauchy formula. In the general case the integral is evaluated by the residue theory.

For boundary fixed points of f the computation of the local index $\mu(p)$ is much more subtle. We will touch only the case of simple fixed points of f on the boundary.

Brenner and Shubin [2] specified attracting and repulsing simple fixed points of f on the boundary. Each simple point $p \in \mathrm{Fix}(f, \partial\mathcal{D})$ is either attracting or repulsing. The contribution of an attracting point $p \in \mathrm{Fix}(f, \partial\mathcal{D})$ to the Lefschetz number $L(f)$ amounts to that of any interior simple fixed point, cf. (5.3), while the repulsing points do not contribute to the Lefschetz number at all. In the non-elliptic case the specification is more tricky. A boundary fixed point p of f is said to be attracting, if $|f'(p)| < 1$, and repulsing, if $|f'(p)| > 1$. Once again each simple fixed point p of f on the boundary of $\mathcal{D}$ is either attracting or repulsing. Indeed, if $|f'(p)| = 1$ then close to p the map f reduces to a rotation around p, and so it cannot keep $\mathcal{D}$ invariant.

Lemma 5.2. *Assume that $p \in \partial\mathcal{D}$ is a simple fixed point of f. Then*

$$\mu(p) = \begin{cases} \dfrac{1}{2} \dfrac{1}{1 - f'(p)}, & \text{if } p \text{ is attracting;} \\ \dfrac{1}{2} \dfrac{1}{f'(p)} \dfrac{1}{1 - f'(p)}, & \text{if } p \text{ is repulsing.} \end{cases}$$

It is worth pointing out that the contribution of a repulsing fixed point is still smaller by absolute value than the contribution of an attracting fixed point.

Proof. The proof follows from calculations of § 7 and a familiar construction of the Green function for planar domains $\mathcal{D}$. Namely, given a point $z \in \mathcal{D}$, let $C(\zeta, z)$ be a conformal map of $\mathcal{D}$ onto the unit disk with centre at 0 in $\mathbb{C}$, such that $C(z, z) = 0$. Then

$$G(\zeta, z) = \frac{1}{2\pi} \log |C(\zeta, z)|$$

is the Green function of $\mathcal{D}$. □

6. Holomorphic Lefschetz formula

We are now in a position to formulate our fixed point theorem which makes more precise Theorem 1.1 of the Introduction.

Theorem 6.1. *Suppose f is a holomorphic map of $\mathcal{D}$ which extends smoothly to the closure of $\mathcal{D}$. If f has only isolated fixed points in $\overline{\mathcal{D}}$ then the holomorphic Lefschetz number of f is*

$$L(f) = \sum_{p \in \mathrm{Fix}(f, \overline{\mathcal{D}})} \mu(p),$$

the local indices $\mu(p)$ being infinitesimal invariants of f at p given by formula (5.2).

Proof. By (5.1) it suffices to show that the integral

$$-\mathrm{p.v.} \int_{\partial \mathcal{D}} \varphi(\varPhi) = \lim_{\varepsilon \to 0} \int_{\partial \mathcal{D} \setminus \cup_{p \in \mathrm{Fix}(f, \partial \mathcal{D})} U(p,\varepsilon)} K_{\varPhi}(z, f(z))$$

is equal to zero. As has been mentioned in the proof of Lemma 3.1, the kernel $K_{\varPhi}(\zeta, z)$ vanishes for all $(\zeta, z) \in \partial \mathcal{D} \times \mathcal{D}$. Since this kernel is actually C^∞ away from the diagonal in the product $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$, we deduce at once that $K_{\varPhi}(z, f(z))$ vanishes for all z away from the set of fixed points of f on the boundary. This finishes the proof. □

Theorem 6.1 extends obviously to the Dolbeault complex on a strictly pseudoconvex domain in $\mathbb{C}^n$, thus implying the fixed point formula of [3] as a highly special case. But we will not develop this point here.

7. Automorphisms of the unit disk

Let $\mathcal{D} = U$ be the unit disk centered at the origin in the complex plane. Then the Green function is

$$G(\zeta, z) = \frac{1}{2\pi} \log |\zeta - z| - \frac{1}{2\pi} \log |1 - \bar{\zeta} z|,$$

cf. (2.2). An easy computation shows that

$$-K_{\varPhi}(\zeta, z) = \Big(\frac{1}{2\pi\imath} \frac{1}{\zeta - z} - \frac{1}{2\pi\imath} \frac{\bar{\zeta}}{1 - \bar{\zeta} z} \Big) d\zeta$$

for (ζ, z) away from the diagonal in $\overline{U} \times \overline{U}$.

By (5.2), we get

$$\mu(p) = \lim_{\varepsilon \to 0} \int_{U \cap \partial U(p,\varepsilon)} \frac{1}{2\pi\imath} \frac{dz}{z - f(z)} - \lim_{\varepsilon \to 0} \int_{U \cap \partial U(p,\varepsilon)} \frac{1}{2\pi\imath} \frac{\bar{z} dz}{1 - \bar{z} f(z)} \tag{7.1}$$

if $p \in \overline{U}$ is an isolated fixed point of f. We restrict our discussion to the case $p \in \partial U$, for the local index of interior fixed points $p \in U$ is computed in Lemma 5.1.

The first limit on the right-hand side of (7.1) is standard in the theory of the Cauchy integral, unless p fails to be simple. This is

$$\lim_{\varepsilon \to 0} \int_{U \cap \partial U(p,\varepsilon)} \frac{1}{2\pi\imath} \frac{dz}{z - f(z)} = \frac{1}{2} \frac{1}{1 - f'(p)}. \tag{7.2}$$

To evaluate the second limit on the right-hand side of (7.1) we use the Taylor formula to write

$$f(z) = p + f'(p)(z - p) + o(|z - p|)$$

in a small neighbourhood of p. After changing the variables $z - p = \varepsilon\, p\, \zeta$ with $|\zeta| = 1$, we get

$$
\begin{aligned}
-\lim_{\varepsilon\to 0}\int_{U\cap\partial U(p,\varepsilon)} \frac{1}{2\pi\imath}\,\frac{\bar z dz}{1-\bar z f(z)} &= \lim_{\varepsilon\to 0}\int_{\substack{\zeta\in\partial U\\ \Re\zeta<0}} \frac{1}{2\pi\imath}\,\frac{(1+\varepsilon\bar\zeta)\,d\zeta}{f'(p)\zeta+\bar\zeta+\varepsilon f'(p)+o(\varepsilon)}\\
&= \lim_{\varepsilon\to 0}\int_{\substack{\zeta\in\partial U\\ \Re\zeta<0}} \frac{1}{2\pi\imath}\,\frac{(\zeta+\varepsilon)\,d\zeta}{f'(p)\zeta^2+1+\varepsilon f'(p)\zeta+o(\varepsilon)},
\end{aligned}
$$

for the boundary of U is smooth at p. If $|f'(p)| \neq 1$ then we can pass to the limit under the integral sign, thus obtaining

$$
\int_{\substack{\zeta\in\partial U\\ \Re\zeta<0}} \frac{1}{2\pi\imath}\,\frac{\zeta\, d\zeta}{f'(p)\zeta^2+1} = \frac{1}{2}\int_{\zeta\in\partial U} \frac{1}{2\pi\imath}\,\frac{\zeta\, d\zeta}{f'(p)\zeta^2+1},
$$

the last equality being a consequence of the invariance of the differential form under the transformation $\zeta \mapsto -\zeta$.

In the case $|f'(p)| < 1$ the polynomial $f'(p)\zeta^2 + 1$ is different from zero in the closure of U. Hence the latter integral vanishes by the Cauchy theorem.

For $|f'(p)| > 1$ the integral in question is easily evaluated by the residue theorem. It is equal to $1/2f'(p)$. Summarising, we have

$$
(7.3)\qquad -\lim_{\varepsilon\to 0}\int_{U\cap\partial U(p,\varepsilon)} \frac{1}{2\pi\imath}\,\frac{\bar z dz}{1-\bar z f(z)} = \begin{cases} 0, & \text{if } p \text{ is attracting;}\\ \dfrac{1}{2}\,\dfrac{1}{f'(p)}, & \text{if } p \text{ is repulsing.}\end{cases}
$$

Combining the equalities (7.1) and (7.2), (7.3), we arrive at the formula of Lemma 5.2, namely

$$
\mu(p) = \begin{cases} \dfrac{1}{2}\,\dfrac{1}{1-f'(p)}, & \text{if } p \text{ is attracting;}\\ \dfrac{1}{2}\,\dfrac{1}{f'(p)}\,\dfrac{1}{1-f'(p)}, & \text{if } p \text{ is repulsing.}\end{cases}
$$

Example 7.1. Consider the family of linear-fractional automorphisms of the unit disk U, given by

$$
f(z) = \frac{z-a}{1-\bar a z}
$$

where $a \in U$ is different from zero. The map f has two fixed points $p = \pm a/|a|$, both points belonging to the boundary. Since

$$
f'\Big(\pm\frac{a}{|a|}\Big) = \frac{1\pm|a|}{1\mp|a|},
$$

the point $+a/|a|$ is repulsing and the point $-a/|a|$ is attracting. Hence it follows that

$$
\begin{aligned}
\mu\Big(+\frac{a}{|a|}\Big) &= -\frac{1}{4}\,\frac{1-|a|}{1+|a|}\,\frac{1-|a|}{|a|},\\
\mu\Big(-\frac{a}{|a|}\Big) &= \frac{1}{4}\,\frac{1+|a|}{|a|},
\end{aligned}
$$

and so

$$
L(f) = \frac{1}{1+|a|}.
$$

Note that $L(f)$ tends to 1 when $a \to 0$, while single local indices $\mu(\pm a/|a|)$ have no limit values for $a \to 0$.

The author gratefully acknowledges the many helpful suggestions of Professor B. Fedosov during the preparation of the paper.

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