Elliptic Theory on Manifolds with Nonisolated Singularities

I. The Index of Families of Cone-Degenerate Operators

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Abstract

We study the index problem for families of elliptic operators on manifolds with conical singularities. The relative index theorem concerning changes of the weight line is obtained. An index theorem for families whose conormal symbols satisfy some symmetry conditions is derived.

Keywords: elliptic family, conormal symbol, relative index, index formulas, symmetry conditions.

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Introduction

This is the first paper in the series of papers dealing with elliptic theory on manifolds with nonisolated singularities. The hierarchy of singularity types is known to play an important role in the theory of elliptic operators on manifolds with singularities. The simplest singularities are isolated singularities such as cones and cusps. Then the simplest nonisolated singularities follow, namely, edge type singularities. In the hierarchic approach, one naturally has to deal with families of singularities of simpler types when analyzing operators related to singularities of subsequent types. For example, for elliptic operators on manifolds with edges there is a naturally defined notion of edge symbol, which is a family of elliptic cone-degenerate operators parametrized by the cosphere bundle of the edge. Studying the index of this family (which is an element of the K-group of the parameter space) is necessary if one wishes to know whether a given wedge-degenerate operator admits elliptic wedge boundary and coboundary conditions or find specific wedge (co)boundary conditions. Thus, even in this simple example we see how important the study of the families index for degenerate operators is. In this paper, we study the index of general families of elliptic operators on manifolds with conical singularities. The relative index theorem concerning changes of the weight line for families of elliptic operators on manifolds with conical singularities is obtained. A gluing theorem is proved for the index of two families whose conormal symbols satisfy certain matching conditions. From this theorem, we derive a series of results for the index of families whose conormal symbols satisfy some additional conditions (like symmetry conditions). In subsequent papers, these results will be applied to the study of statements of boundary value problems on manifolds with edge type singularities. Our results are essentially based on the general relative index theorem for abstract elliptic families proved in [5].

1 Elliptic families on manifolds with conical singularities

As a rule, we shall freely use the definition and notation adopted in the theory of (pseudo)differential operators on manifolds with conical singularities. For these definitions and notation, we refer the reader to Schulze's book [6] as well as the papers [7, 3, 5], where issues closely related to the topic of this preprint were studied.

1.1 Bundles of manifolds with conical singularities

Let X be a smooth compact manifold, which will be called the parameter space. Next, let $\mathcal{M} \stackrel{\pi}{\longrightarrow} X$ be a locally trivial bundle over X with fiber $\mathcal{M}_x$ a compact manifold M with conical singularities. This means that there exists an open cover of X such that the bundle is trivial over each element U of the cover, i.e., $\mathcal{M}_U \simeq U \times M$ (with the natural projection to the first factor), and moreover, the transition functions

$$\varphi_{U\widetilde{U}}(x) : M \longrightarrow M, \quad x \in U \cap \widetilde{U}$$

defined on intersections $U \cap \widetilde{U}$ of the cover elements are isomorphisms of manifolds with conical singularities and smoothly depend on $x \in U \cap \widetilde{U}$. For notational convenience, we assume that there is a unique conical singular point α on the manifold M. By the definition of a manifold with conical singularities, some neighborhood $V \subset M$ of the singular point is homeomorphic (smoothly outside α) to a cone:

$$V \simeq \check{K}_\Omega \equiv \{\Omega \times [0,1)\}/\{\Omega \times \{0\}\}, \tag{1.1}$$

where Ω is a smooth compact manifold without boundary called the base of the cone; this homeomorphism takes the point α to the cone vertex $r = 0$ (where $r \in [0,1)$ is the variable along the cone generatrix). We assume that the homeomorphisms (1.1) have been chosen in a neighborhood V_x of the singular point α_x in each fiber $\mathcal{M}_x \simeq M$ of the bundle $\mathcal{M}$ and that they depend on x smoothly, so that we have the bundle[1]

$$\check{\mathcal{K}} \xrightarrow{\pi} X, \quad \check{\mathcal{K}}_x \simeq \check{K}_\Omega \tag{1.2}$$

of cones over the manifold X. Under these assumptions, the following is true.

1. In the neighborhood $\mathcal{V} = \sqcup_{x\in X} V_x$ of the set $\{\alpha_x\}_{x\in X}$ of singular points on $\mathcal{M}$ there is a well-defined function $r : \mathcal{V} \longrightarrow \mathbb{R}$.

2. There is a well-defined bundle

$$\widetilde{\Omega} \xrightarrow{\pi} X, \quad \widetilde{\Omega}_x \simeq \Omega,$$

of cone bases and the associated bundle

$$\mathcal{K} \xrightarrow{\pi} X, \quad \mathcal{K}_x \simeq K_\Omega \equiv \{\Omega \times [0,\infty)\}/\{\Omega \times \{0\}\}$$

of infinite cones over the parameter space X.

Under these assumptions, $\mathcal{M}$ will be called a family of manifolds with conical singularities.

[1]The projections for various bundles over X will be denoted by the same letter π. This will not lead to a misunderstanding.

1.2 Elliptic families

Now let $\gamma \in \mathbb{R}$ be a given number and $\widehat{\mathcal{D}} = \{\widehat{D}_x\}$ a smooth family (with parameter $x \in X$) of elliptic pseudodifferential operators

$$\widehat{D}_x : H^{s,\gamma}(\mathcal{M}_x) \longrightarrow H^{s,\gamma}(\mathcal{M}_x) \tag{1.3}$$

in weighted Sobolev spaces[2] $H^{s,\gamma}(\mathcal{M}_x)$. (Without loss of generality, we consider only zero-order operators.) Recall that this means that the following conditions hold for each $x \in X$ (e.g., see [6, 7]).

1. The principal symbol $D_x \equiv \sigma(\widehat{D}_x)$ is invertible everywhere outside the zero section of the compressed cotangent bundle $T^*\mathcal{M}_x$ (the *formal ellipticity*).
2. The conormal symbol $\mathbf{D}_x(p) \equiv \sigma_c(\widehat{D}_x)$, which is a zero-order Agranovich–Vishik [1] elliptic pseudodifferential operator with parameter p in the Sobolev spaces $H^s(\widetilde{\Omega}_x)$, is defined on the weight line
$$\mathcal{L}_\gamma = \{\operatorname{Im} p = \gamma\}$$
(this is needed for the operator (1.3) to be well defined and bounded) and is invertible for all $p \in \mathcal{L}_\gamma$ (in combination with the formal ellipticity condition, this is known as the *ellipticity condition.*)

By a well-known theorem of the theory of pseudodifferential operators on manifolds with conical singularities, the operators (1.3) are Fredholm (e.g., see [6]), so that we have the Fredholm family (1.3) with parameter space X, for which the index

$$\operatorname{ind} \mathcal{D} \in K(X) \tag{1.4}$$

is well-defined as an element of the K-group of the space X (see [2]). The index depends only on the principal and conormal symbols and is independent of the specific choice of a family with given principal and conormal symbols.

We study the following problems.

1. How does the index (1.4) change under variations of the weight exponent γ? The corresponding assertions have been dubbed as relative index theorems in the literature; we adopt the same terminology. This question is considered in Section 2.
2. How to compute the index (1.4), and what does the index formula look like? Under some additional assumptions (like symmetry conditions) this question is studied in Section 3.

[2] In fact, the operator $\widehat{D}_x$ acts in weighted Sobolev spaces of sections of some *bundles* on $\mathcal{M}_x$. To avoid complicated notation, we omit these bundles in the notation of Sobolev spaces.

2 Changes of the weight line and the relative index formula

In this section, we answer the following question.

Let $\mathcal{D} = \{D_x\}_{x\in X}$ and $\mathbf{D} = \{\mathbf{D}_x(p)\}_{x\in X}$ be given smooth families of principal and conormal symbols on a family $\mathcal{M}$ of manifolds with conical singularities. Suppose also that the conormal symbols are defined and holomorphic in some strip $a < \operatorname{Im} p < b$ and the compatibility condition

$$\sigma(\mathbf{D}_x(p)) = D_x \Big|_{\partial T^*\mathcal{M}_x}, \quad x \in X, \tag{2.1}$$

is valid. Then for each γ in the above-mentioned strip there exists a smooth family of pseudodifferential operators in the spaces $H^{s,\gamma}(\mathcal{M}_x)$ with given principal and conormal symbols. We denote such a family by $\widehat{\mathcal{D}}_\gamma = \{\widehat{D}_{x,\gamma}\}$. Suppose that the ellipticity conditions hold for some γ_1, γ_2 in this strip, so that the families $\widehat{\mathcal{D}}_{\gamma_1}$ and $\widehat{\mathcal{D}}_{\gamma_2}$ are Fredholm. The problem is, *How to compute the difference*

$$\Delta_{\gamma_1,\gamma_2} = \operatorname{ind} \widehat{\mathcal{D}}_{\gamma_1} - \operatorname{ind} \widehat{\mathcal{D}}_{\gamma_2} \in K(X) \tag{2.2}$$

of the indices of these families?

The answer is well known for the case in which X is a point. Namely, the difference is equal to the sum of multiplicities p_j of singular points of the operator family $\mathbf{D}^{-1}(p)$ in the strip between the weight lines $\mathcal{L}_{\gamma_1}$ and $\mathcal{L}_{\gamma_2}$, the difference being taken with the plus sign for $\gamma_1 < \gamma_2$ and the minus sign for $\gamma_1 > \gamma_2$. In the general case, we use the surgery of elliptic families introduced in [5] to reduce our problem to the computation of the index of a special operator family on infinite cones. For the case in which the parameter space is a point, $X = \{\mathrm{pt}\}$, the index of such operators was computed in [7]. Generalizing the computation carried out there, we obtain the desired formula for our case.

2.1 Weighted Sobolev spaces as bottleneck spaces

Let us perform necessary auxiliary constructions.

On the infinite cone K_Ω, we define the Hilbert space $H^{s,\gamma_1,\gamma_2}(K_\Omega)$ (see [7]) as follows. The cone K_Ω can be viewed as a compact manifold with two conical singular points (a "spindle"): one point corresponds to $r = 0$, and the other to $r = \infty$ (the radial variable r' in a neighborhood of the second point is related to r by the change of variables $r' = 1/r$). Then $H^{s,\gamma_1,\gamma_2}(K_\Omega)$ is the weighted Sobolev space of order s with weight exponents γ_1 at $r = 0$ and $(-\gamma_2)$ at $r' = 0$. This choice of notation is related to the fact that in this case

$$H^{s,\gamma,\gamma}(K_\Omega) = H^{s,\gamma}(K_\Omega),$$

where the right-hand side is the "standard" weighted Sobolev space on the infinite cone with the norm

$$\|u\|^2_{s,\gamma} = \int\limits_{K_\Omega} \left| \left(1 - \left(r\frac{\partial}{\partial r} \right)^2 - \triangle_\Omega \right)^{s/2} [r^{-\gamma} u] \right|^2 \frac{dr}{r}\, d\omega.$$

For $\gamma_1 \le \gamma \le \gamma_2$ one has the continuous embedding $H^{s,\gamma}(K_\Omega) \subset H^{s,\gamma_1,\gamma_2}(K_\Omega)$, and for $\gamma_1 \ge \gamma \ge \gamma_2$ the opposite embedding holds by duality.

Let $\mathbf{A} \equiv \mathbf{A}(p)$ be a given conormal symbol on Ω. Suppose that it is defined on the weight line $\mathcal{L}_\gamma$. Then we define an operator

$$\widehat{\mathbf{A}}_\gamma \equiv \mathbf{A}\left(ir\frac{\partial}{\partial r} \right) : C_0^\infty(\overset{\circ}{K}_\Omega) \longrightarrow C_0^\infty(\overset{\circ}{K}_\Omega), \tag{2.3}$$

where $\overset{\circ}{K}_\Omega = \Omega \times (0,\infty)$ is the cone with vertex deleted, by the formula

$$\widehat{\mathbf{A}}_\gamma = \mathfrak{M}_\gamma^{-1} \circ \mathbf{A}(p) \circ \mathfrak{M}_\gamma, \tag{2.4}$$

where

$$[\mathfrak{M}_\gamma] u(p) = \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty r^{ip} u(r)\, \frac{dr}{r} \tag{2.5}$$

is the Mellin transform with respect to the variable r with weight line $\mathcal{L}_\gamma$ and

$$[\mathfrak{M}_\gamma^{-1} v](r) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathcal{L}_\gamma} r^{-ip} v(p)\, dp \tag{2.6}$$

is the inverse transform. The operator $\widehat{\mathbf{A}}_\gamma$ extends by continuity to a bounded operator (denoted by the same letter)

$$\widehat{\mathbf{A}}_\gamma : H^{s,\gamma}(K_\Omega) \longrightarrow H^{s,\gamma}(K_\Omega) \tag{2.7}$$

(recall that we consider only operators, and, accordingly, conormal symbols, of order zero). Next, suppose that the conormal symbol $\mathbf{A}(p)$ is holomorphic in the strip $a < \operatorname{Im} p < b$. By the Cauchy theorem, the operator (2.3) is independent of $\gamma \in (a,b)$ and extends to be a continuous operator

$$\widehat{\mathbf{A}}_{\gamma_1,\gamma_2} : H^{s,\gamma_1,\gamma_2}(K_\Omega) \longrightarrow H^{s,\gamma_1,\gamma_2}(K_\Omega) \tag{2.8}$$

for $a < \gamma_1 \le \gamma_2 < b$. To prove the last assertion, note that on compactly supported functions the operator $\widehat{\mathbf{A}}_\gamma$ can be represented in the form

$$\widehat{\mathbf{A}}_\gamma = \widehat{\mathbf{A}}_\gamma \varphi_1 + \widehat{\mathbf{A}}_\gamma \varphi_2 \equiv \widehat{\mathbf{A}}_{\gamma_1} \varphi_1 + \widehat{\mathbf{A}}_{\gamma_2} \varphi_2,$$

where $\varphi_1(r), \varphi_2(r)$ is a smooth partition of unity on $\mathbb{R}_+$, such that $\varphi_1(r) = 0$ for $r > 1$ and $\varphi_2(r) = 0$ for $r < 1/2$. (From now on, until the end of the paper, we assume that this partition of unity has been chosen and fixed.) Both terms in this representation are continuous in the desired spaces as compositions of continuous operators:

$$\widehat{\mathbf{A}}_{\gamma_1}\varphi_1 : H^{s,\gamma_1,\gamma_2}(K_\Omega) \xrightarrow{\varphi_1} H^{s,\gamma_1}(K_\Omega) \xrightarrow{\widehat{\mathbf{A}}_{\gamma_1}} H^{s,\gamma_1}(K_\Omega) \subset H^{s,\gamma_1,\gamma_2}(K_\Omega),$$
$$\widehat{\mathbf{A}}_{\gamma_2}\varphi_2 : H^{s,\gamma_1,\gamma_2}(K_\Omega) \xrightarrow{\varphi_2} H^{s,\gamma_2}(K_\Omega) \xrightarrow{\widehat{\mathbf{A}}_{\gamma_2}} H^{s,\gamma_2}(K_\Omega) \subset H^{s,\gamma_1,\gamma_2}(K_\Omega).$$

This implies the continuity (2.8).

Let us equip the spaces $H^{s,\gamma}(M)$ and $H^{s,\gamma_1,\gamma_2}(K_\Omega)$ with the structure of bottleneck spaces [4] in the following standard way. Consider a smooth function $\chi(r)$ defined on the interval $r \in [1/2, 1]$ and possessing the following properties:

1. χ is a monotone increasing map of the interval $[1/2, 1]$ onto the interval $[-1, 1]$;
2. $\chi^{(k)}(1/2) = \chi^{(k)}(1) = 0$, $k = 1, 2, \ldots$.

This function can be treated as a function on M or K_Ω defined for $1/2 \le r \le 1$ (this is possible by virtue of our definitions) and then extended to the entire M (respectively, K_Ω) by the value -1 for $r < 1/2$ and by 1 on the remaining set. The continuation (which will be denoted by the same letter) is an infinitely differentiable function. Then we can define the action of $\mathfrak{A} = C^\infty([-1, 1])$ on $H^{s,\gamma}(M)$ (respectively, $H^{s,\gamma_1,\gamma_2}(K_\Omega)$) by the formula

$$\varphi(u) \stackrel{\text{def}}{=} (\varphi \circ \chi) \cdot u, \tag{2.9}$$

so that these spaces become bottleneck spaces [3]. Moreover, it follows from our assumptions about $\mathcal{M}$ that this bottleneck space structure in each $H^{s,\gamma}(\mathcal{M}_x)$ (respectively, $H^{s,\gamma_1,\gamma_2}(\mathcal{K}_x)$) gives families of bottleneck spaces over X in the sense of Definition 2.1 in [5].

2.2 The relative index theorem

Now let us analyze the relative index (2.2). We assume that $\gamma_1 < \gamma_2$. (Otherwise, we just exchange the families.) Along with the original elliptic families $\widehat{\mathcal{D}}_{\gamma_1}$ and $\widehat{\mathcal{D}}_{\gamma_2}$, we also consider the elliptic families

$$\widehat{\mathbf{D}}_{\gamma_1} = \{\widehat{\mathbf{D}}_{x,\gamma_1}\}, \quad \widehat{\mathbf{D}}_{\gamma_2} = \{\widehat{\mathbf{D}}_{x,\gamma_2}\}, \quad \widehat{\mathbf{D}}_{\gamma_1,\gamma_2} = \{\widehat{\mathbf{D}}_{x,\gamma_1,\gamma_2}\} \tag{2.10}$$

on the infinite cone K_Ω with parameter space X. All above-mentioned families are abstract elliptic families in families of bottleneck spaces (more precisely, can be embedded in such families if we multiply the Schwartz kernels by cutoff functions whose supports shrink to the diagonal; for details, see Subsection 2.3 in [5] and also [3]) in the sense of Definition 2.5

in [5]. We have the following commutative diagram of surgeries of abstract elliptic families over X in the sense of Subsection 2.4 in [5]:

$$
\begin{array}{ccc}
\widehat{\mathcal{D}}_{\gamma_1} \oplus \widehat{\mathbf{D}}_{\gamma_2} & \overset{1}{\longleftrightarrow} & \widehat{\mathbf{D}}_{\gamma_1,\gamma_2} \oplus \widehat{\mathbf{D}}_{\gamma_2} \\
{\scriptstyle -1}\updownarrow & & \updownarrow{\scriptstyle -1} \\
\widehat{\mathcal{D}}_{\gamma_2} \oplus \widehat{\mathbf{D}}_{\gamma_1,\gamma_2} & \underset{1}{\longleftrightarrow} & \widehat{\mathbf{D}}_{\gamma_2} \oplus \widehat{\mathbf{D}}_{\gamma_1,\gamma_2}.
\end{array}
\tag{2.11}
$$

The surgeries occurring in this diagram can be described geometrically as follows. (See Fig. 1, where the parts of manifolds corresponding to bottlenecks in the corresponding function spaces are dashed.)

First of all, the surgeries are performed “fiberwise,” that is, separately (but continuously in x) for each parameter value $x \in X$. The fiber for each entry of the left column in the diagram is the disjoint union of the manifold M and the cone K_Ω, and the fiber for each entry of the right column is the disjoint union of two copies of the cone K_Ω. The vertical surgeries consist in cutting away the conical “caps” $\{\chi = -1\}$ of two components of the disjoint union and pasting them back interchanged; the horizontal surgeries amount to cutting away the interior part $\{\chi = 1\}$ of the manifold M and replacing it by the corresponding part of the infinite cone. By applying Theorem 3.1 in [5], we obtain

$$
\operatorname{ind} \widehat{\mathcal{D}}_{\gamma_1} \oplus \widehat{\mathbf{D}}_{\gamma_2} - \operatorname{ind} \widehat{\mathcal{D}}_{\gamma_2} \oplus \widehat{\mathbf{D}}_{\gamma_1,\gamma_2} = \operatorname{ind} \widehat{\mathbf{D}}_{\gamma_1,\gamma_2} \oplus \widehat{\mathbf{D}}_{\gamma_2} - \operatorname{ind} \widehat{\mathbf{D}}_{\gamma_2} \oplus \widehat{\mathbf{D}}_{\gamma_1,\gamma_2} = 0,
$$

since the surgery in the second column is just the interchanging of summands in the direct sum. Since $\widehat{\mathbf{D}}_{\gamma_2}$ is a family of invertible operators, it follows that the desired relative index has the form

$$
\operatorname{ind} \widehat{\mathcal{D}}_{\gamma_1} - \operatorname{ind} \widehat{\mathcal{D}}_{\gamma_2} = \operatorname{ind} \widehat{\mathbf{D}}_{\gamma_1,\gamma_2}. \tag{2.12}
$$

Thus we have reduced the computation of the relative index (2.2) to the computation of the index of a family of operators with coefficients of the radial variable r in special weighted Sobolev spaces on the infinite cone. This problem was solved in [7] for the case of a single operator. For convenience, let us state the desired assertion from [7] as a theorem. We recall that $\gamma_1 < \gamma_2$ by assumption. Then for $x \in X$ the following assertion holds.

Theorem 2.1 (cf. [7]). *The operator $\widehat{\mathbf{D}}_{x,\gamma_1,\gamma_2}$ is an epimorphism. The projection on its kernel is given (on compactly supported functions) by the formula*

$$
\widehat{\mathbf{P}}_{x,\gamma_1,\gamma_2} = \bigl(\widehat{\mathbf{D}}^{-1}_{x,\gamma_1} - \widehat{\mathbf{D}}^{-1}_{x,\gamma_2}\bigr)[\varphi_2, \widehat{\mathbf{D}}_{x,\gamma}], \tag{2.13}
$$

where the choice of $\gamma \in [a, b]$ is irrelevant.

Remark 2.2. 1. In formula (2.13), the function $\varphi_2 = \varphi_2(r)$ is a element of the previously fixed partition of unity on $\mathbb{R}_+$. Needless to say, the projection depends on the choice of the partition of unity in general. (That is why we have fixed the choice.)

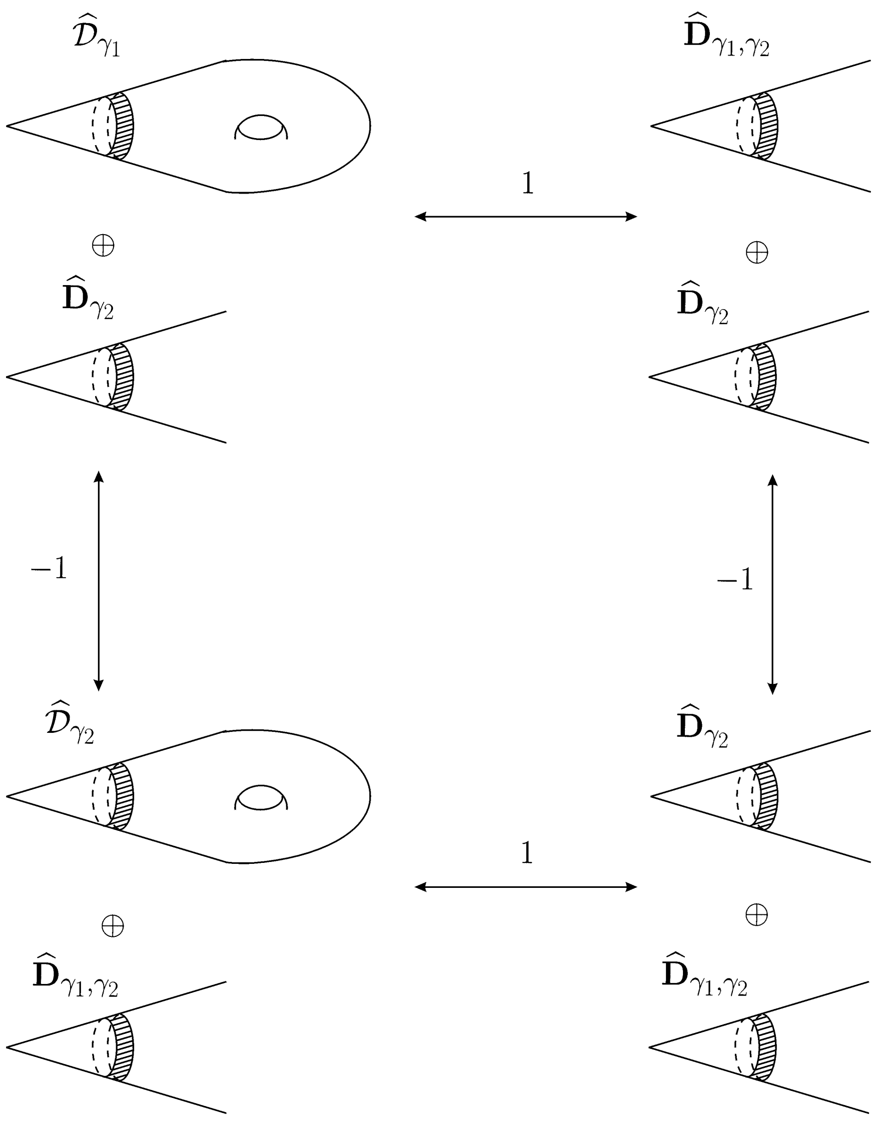

Figure 1. The surgery diagram for the relative index theorem

2. The difference of the operators in parentheses on the right-hand side in (2.13) need not be zero, since the family $\mathbf{D}_x^{-1}(p)$ may have poles between the weight lines $\mathcal{L}_{\gamma_1}$ and $\mathcal{L}_{\gamma_2}$. If there are no such poles, then the projection is zero and the operator $\widehat{\mathbf{D}}_{x,\gamma_1,\gamma_2}$ is an isomorphism.

Now let us compute the index of the family $\widehat{\mathbf{D}}_{\gamma_1,\gamma_2}$. Since the operators $\widehat{\mathbf{D}}_{x,\gamma_1,\gamma_2}$ are epimorphisms (that is, have trivial cokernels), it follows that their kernels form a finite-dimensional vector bundle over X. Thus, the following assertion holds.

Theorem 2.3. *For $\gamma_1 < \gamma_2$ one has*

$$\operatorname{ind} \widehat{\mathbf{D}}_{\gamma_1,\gamma_2} = [\operatorname{Im} \widehat{\mathbf{P}}_{\gamma_1,\gamma_2}] \in K(X). \tag{2.14}$$

Combining this with (2.12), we finally arrive at the following theorem.

Theorem 2.4. *The relative index* (2.2) *is given by the formula*

$$\operatorname{ind} \widehat{\mathcal{D}}_{\gamma_1} - \operatorname{ind} \widehat{\mathcal{D}}_{\gamma_2} = \begin{cases} [\operatorname{Im} \widehat{\mathbf{P}}_{\gamma_1,\gamma_2}] \in K(X), & \gamma_1 \le \gamma_2, \\ -[\operatorname{Im} \widehat{\mathbf{P}}_{\gamma_2,\gamma_1}] \in K(X), & \gamma_1 > \gamma_2, \end{cases} \tag{2.15}$$

where $\widehat{\mathbf{P}}_{\gamma_1,\gamma_2} = \{\widehat{\mathbf{P}}_{x,\gamma_1,\gamma_2}\}_{x \in X}$ *is the projection family* (2.13).

Indeed, the first line in (2.15) readily follows from (2.12) and (2.14), and the transition to the case $\gamma_1 > \gamma_2$ is obvious.

3 The sum of indices formula for operator families on manifolds with conical singularities

In the subsequent two sections we shall prove several theorems concerning the index of operator families on manifolds with conical singularities under additional conditions imposed on the conormal symbol. But first of all, in this section we shall prove a general theorem that permits one in a sense to "glue" together two operator families of this sort, thus obtaining a family of operators on a closed manifold without singularities.

3.1 Conditions on operator families

Let

$$\mathcal{M}_i \xrightarrow{\pi} X, \quad i = 1, 2, \tag{3.1}$$

be two locally trivial bundles of compact manifolds with conical singularities over X of the type considered in Section 1; the fibers of these bundles will be denoted by M_i, $i = 1, 2$ and the corresponding bases of the cones by Ω_i, $i = 1, 2$. We assume that Ω_1 and Ω_2 are in fact diffeomorphic: $\Omega_1 = \Omega_2 = \Omega$. Next, let

$$\widetilde{\Omega}_i \xrightarrow{\pi} X, \quad i = 1, 2, \tag{3.2}$$

be the corresponding bundles of cone bases. Suppose that we are given an isomorphism

$$\begin{array}{ccc} \widetilde{\Omega}_1 & \xrightarrow{h} & \widetilde{\Omega}_2 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & = & X \end{array} \tag{3.3}$$

of these latter bundles over the identity morphism of X. Thus, for each $x \in X$ there is an isomorphism $h_x : \Omega_{1x} \longrightarrow \Omega_{2x}$, which continuously depends on the point x.

Finally, consider two families $\widehat{\mathcal{D}}_i = \{\widehat{D}_{ix}\}_{x\in X}$ of elliptic pseudodifferential operators

$$\widehat{D}_{ix} : H^{s,\gamma}(\mathcal{M}_{ix}, E_{ix}) \longrightarrow H^{s,\gamma}(\mathcal{M}_{ix}, F_{ix}), \quad i = 1, 2, \tag{3.4}$$

on the respective families of manifolds with conical singularities. Here $\mathcal{E}_i = \{E_{ix}\}$, $\mathcal{F}_i = \{F_{ix}\}$ are some families of vector bundles over the corresponding manifolds with conical singularities. We assume that the following conditions are satisfied.

1. The conormal symbols

$$\mathbf{D}_{ix}(p) : H^s(\Omega_{ix}, E_{ix}) \longrightarrow H^s(\Omega_{ix}, F_{ix}) \tag{3.5}$$

   (here we for brevity do not distinguish between, say, E_{ix} and $E_{ix}\big|_{\Omega_x}$ in the notation) are holomorphic in some strip $|\operatorname{Im} p| < R$ including the weight line $\mathcal{L}_\gamma$.

2. These conormal symbols satisfy the *matching condition*[3]

$$\mathbf{D}_{1x}(p) = \varkappa_x h_x^* \mathbf{D}_{2x}(-p)(h_x^*)^{-1}\vartheta_x, \tag{3.6}$$

   where

$$\vartheta_x : E_{1x} \longrightarrow h_x^* E_{2x}, \quad \varkappa_x : h_x^* F_{2x} \longrightarrow F_{1x} \tag{3.7}$$

   are some bundle isomorphisms.

3.2 The index summation theorem

The matching condition (3.6) implies a similar matching condition for the principal symbols of the operator families $\widehat{\mathcal{D}}_1$ and $\widehat{\mathcal{D}}_2$:

$$\sigma(\widehat{D}_{1x})\big|_{\partial T^* M_{1x}\setminus\{0\}} = \varkappa_x h_x^* (\beta\sigma(\widehat{D}_{2x})\big|_{\partial T^* M_{2x}\setminus\{0\}})(h_x^*)^{-1}\vartheta_x, \tag{3.8}$$

where we denote the lifts of h_x, ϑ, and $\varkappa$ to the cotangent bundles by the same letters and β is the change $p \longrightarrow -p$ of the conormal variable in the symbol. Condition (3.8) permits us to perform the following operation. Assuming that the coefficients of our operators are independent of r for small r (which can always be achieved by a homotopy), we can cut away the small conical caps $\{r \le \varepsilon\}$ from both M_{1x} and M_{2x} for each x and glue the resulting manifolds with boundary together using the isomorphism

$$h_x : \Omega_{1x} \equiv \partial M_{1x} \longrightarrow \Omega_{2x} \equiv \partial M_{2x} \tag{3.9}$$

[3] The case in which the matching condition involves some reflection $p \longmapsto p_0 - p$ rather than $p \longmapsto -p$ can be reduced to this one by a standard shift in the complex p-plane, which is represented in the weighted Sobolev spaces by multiplications by powers of r; cf. [7, 3].

of their boundaries. Let us denote the manifold obtained as the result of gluing by $M_{1x} \cup_{h_x} M_{2x}$ and the corresponding family of compact smooth manifolds with parameter space X by $\mathcal{M}_1 \cup_h \mathcal{M}_2 = \{M_{1x} \cup_{h_x} M_{2x}\}$. Now we can concatenate the bundles E_{1x} and E_{2x} using the morphism ϑ_x and likewise the bundles F_{1x} and F_{2x} using the morphism $\varkappa_x$. Condition (3.8) ensures that the principal symbols of $\widehat{D}_{ix}$, $i = 1, 2$ continuously glue together to form some elliptic symbol over $\mathcal{M}_1 \cup_h \mathcal{M}_2$; the corresponding elliptic operator (defined modulo compact operators) will be denoted by $\widehat{D}_{1x} \Box \widehat{D}_{2x}$. As a result, we have the elliptic family

$$\widehat{\mathcal{D}}_1 \Box \widehat{\mathcal{D}}_2 = \{\widehat{D}_{1x} \Box \widehat{D}_{2x}\}_{x \in X}. \tag{3.10}$$

Now note that owing to the matching condition (3.6), the conormal symbols of both operator families are invertible not only on the weight line $\mathcal{L}_\gamma$, but also on $\mathcal{L}_{-\gamma}$. It follows that not only the families $\widehat{\mathcal{D}}_1 \equiv \widehat{\mathcal{D}}_{1,\gamma}$ and $\widehat{\mathcal{D}}_2 \equiv \widehat{\mathcal{D}}_{2,\gamma}$, but also the families $\widehat{\mathcal{D}}_1 \equiv \widehat{\mathcal{D}}_{1,-\gamma}$ and $\widehat{\mathcal{D}}_2 \equiv \widehat{\mathcal{D}}_{2,-\gamma}$ obtained by the transition to the opposite weight line, are elliptic families with parameter space X. Consider the surgery shown in Fig. 2.

The corresponding commutative diagram of surgeries has the form

$$\begin{array}{ccc} \widehat{\mathcal{D}}_{1,\gamma} \oplus \widehat{\mathcal{D}}_{2,\gamma} & \stackrel{1}{\longleftrightarrow} & \widehat{\mathbf{D}}_\gamma \oplus \widehat{\mathbf{D}}_{-\gamma} \\ {\scriptstyle -1}\updownarrow & & \updownarrow{\scriptstyle -1} \\ \widehat{\mathcal{D}}_1 \Box \widehat{\mathcal{D}}_2 & \underset{1}{\longleftrightarrow} & \widehat{\mathbf{D}}_\gamma . \end{array} \tag{3.11}$$

The families in the top left corner are the original families $\widehat{\mathcal{D}}_1$ and $\widehat{\mathcal{D}}_2$ on $\mathcal{M}_1$ and $\mathcal{M}_2$, acting in weighted Sobolev spaces with the weight exponents γ and $-\gamma$, respectively. In the top right corner we have two operators on the families of infinite cones (the notation is the same as in the preceding commutative diagram (2.11)) in the spaces $H^{s,\gamma}$ and $H^{s,-\gamma}$, respectively. (In fact, these operators pass into each other under the change of variables $r \longmapsto 1/r$.) The notation of the operators in the bottom row is now perhaps self-explanatory. The "vertical" surgeries consist in deleting two conical caps on the left and replacing them by a "tube" joining the two resulting parts. The "horizontal" surgeries consist in deleting the interior part of the manifold and replacing it by the corresponding infinite part of the cone. For convenience, weight exponents are shown next to each of the conical points.

Let us have a closer look at this diagram. Note that all operators occurring in the right column are actually invertible, which shows that the relative index of the vertical surgery in this column is zero. By Theorem 3.1 in [5], the same assertion holds for the first column, and accordingly we obtain the following theorem.

Theorem 3.1. *Under the above assumptions, the following identity holds:*

$$\operatorname{ind} \widehat{\mathcal{D}}_{1,\gamma} + \operatorname{ind} \widehat{\mathcal{D}}_{2,-\gamma} = \operatorname{ind} \widehat{\mathcal{D}}_1 \Box \widehat{\mathcal{D}}_2 \in K(X). \tag{3.12}$$

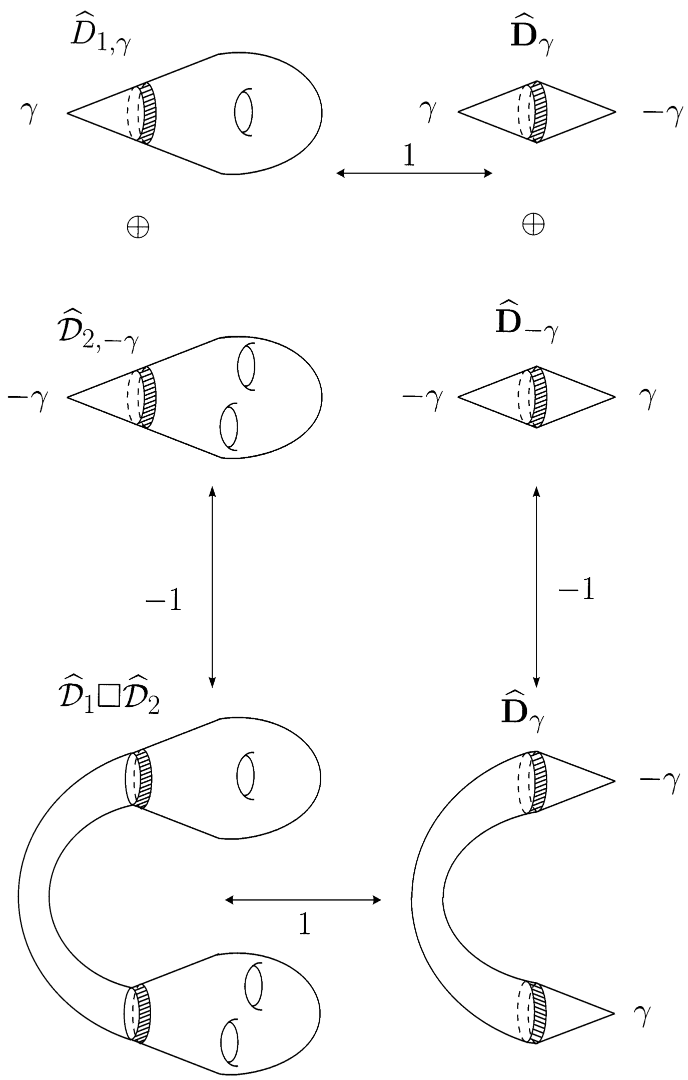

Figure 2. The surgery diagram for the gluing of two families

Observe that under the symmetry condition (3.6) the contributions of conical points to the index of the two summands cancel out, and we can just glue the two families into a single family of elliptic operators on a closed compact manifold without singularities.

4 The index of operator families satisfying symmetry conditions

Now we shall use Theorem 3.1 to obtain various assertions concerning the index of cone-degenerate operator families under certain additional conditions. First, let us consider the case of a single elliptic cone-degenerate operator family $\widehat{\mathcal{D}}_\gamma$ on $\mathcal{M}$ satisfying the following conditions.

1. The conormal symbol $\mathbf{D}_x(p)$ is holomorphic in the strip $|\operatorname{Im} p| < R$, where $\gamma \in (-R, R)$, for every $x \in X$.

2. The following *symmetry condition* holds:

   $$\mathbf{D}_x(p) = \varkappa_x h_x^* \mathbf{D}_{g(x)}(-p)(h_x^*)^{-1}\vartheta_x, \tag{4.1}$$

   where (in contrast to (3.6)) we have the following commutative diagram of maps:

   $$\begin{array}{ccc} \widetilde{\Omega} & \xrightarrow{\ h\ } & \widetilde{\Omega} \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\ g\ ]{} & X. \end{array} \tag{4.2}$$

   Here $g : X \longrightarrow X$ is some smooth mapping (not assumed to be a diffeomorphism) and $h : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}$ is a smooth mapping such that each

   $$h_x = h\Big|_{\Omega_x} : \Omega_x \longrightarrow \Omega_{g(x)}$$

   is a diffeomorphism. Just as before, $\varkappa_x$ and ϑ_x are appropriate vector bundle isomorphisms.

Now we can obviously apply Theorem 3.1 to the operator families $\widehat{\mathcal{D}}_\gamma$ and $g^*\widehat{\mathcal{D}}_{-\gamma}$. Combining this with the relative index theorem (2.15), we arrive at the following lemma.

Lemma 4.1. *Under the above-mentioned assumptions, one has*

$$\operatorname{ind} \widehat{\mathcal{D}}_\gamma + g^* \operatorname{ind} \widehat{\mathcal{D}}_\gamma = \operatorname{ind} \widehat{\mathcal{D}} \Box g^* \widehat{\mathcal{D}} + g^* \mathcal{R}_\gamma, \tag{4.3}$$

where $\mathcal{R}_\gamma$ depends only on the conormal symbol and is given by the expression (2.15) *for the case in which $\gamma_1 = -\gamma$ and $\gamma_2 = \gamma$.*

Remark 4.2. In formula (4.3), g^* stands for the map induced by g in the K-theory of X:

$$g^* : K(X) \longrightarrow K(X).$$

We can find the index of $\widehat{\mathcal{D}}_\gamma$ from the formula obtained in this Lemma provided that the operator $1 + g^*$ is invertible in K-theory. This, of course is not always the case. But still there are some important situations in which one can actually prove the invertibility of $1 + g^*$ (possibly after passing to an appropriate quotient group of $K(X)$) and thus obtain useful corollaries from Lemma 4.1.

4.1 The case in which g is homotopic to the identity map

In this case, the map $g^* : K(X) \longrightarrow K(X)$ is the identity map, and Eq. (4.3) becomes

$$2 \operatorname{ind} \widehat{\mathcal{D}}_\gamma = \operatorname{ind} \widehat{\mathcal{D}} \Box g^* \widehat{\mathcal{D}} + \mathcal{R}_\gamma. \tag{4.4}$$

In general one cannot divide by 2 in K-theory; this is possible only modulo 2-torsion. Thus we arrive at the following theorem.

Theorem 4.3. *Suppose that the operator family $\widehat{\mathcal{D}}$ satisfies the assumptions given in the beginning of this section and g is homotopic to the identity map. Then*

$$\operatorname{ind} \widehat{\mathcal{D}}_\gamma = \frac{1}{2} \left\{ \operatorname{ind} \widehat{\mathcal{D}} \Box g^* \widehat{\mathcal{D}} + \mathcal{R}_\gamma \right\}, \tag{4.5}$$

where the identity holds in the quotient of $K(X)$ by the 2-torsion subgroup.

4.2 The case in which g is homotopic to a constant map

In this case the image of g^* in $K(X)$ is equal to $\mathbb{Z}$, the subgroup of classes of trivial vector bundles over X. Factoring out this subgroup, we arrive at the following answer in this case.

Theorem 4.4. *Suppose that the operator family $\widehat{\mathcal{D}}$ satisfies the assumptions given in the beginning of this section and g is homotopic to a constant map $g : X \longrightarrow \{x_0\} \in X$. Then*

$$\operatorname{ind} \widehat{\mathcal{D}}_\gamma = \operatorname{ind} \widehat{\mathcal{D}} \Box g^* \widehat{\mathcal{D}} \in K(X)/\mathbb{Z}, \tag{4.6}$$

where $\mathbb{Z}$ corresponds to trivial vector bundles.

4.3 The case in which g is of finite order

Let $g^N = \mathrm{id}$. We additionally assume that $g, \ldots, g^{N-1}$ have no fixed points. Then $p : X \longrightarrow X/G$, where G is the cyclic group generated by g, is an n-sheeted covering and there is a well-defined direct image map

$$\begin{aligned} &p_! : K(X) \longrightarrow K(X/G), \\ &\quad (p_! E)_y = \oplus_{x \in p^{-1} y} E_x, \quad E \in \operatorname{Vect}(X), \quad y \in X/G. \end{aligned}$$

Moreover, the diagram

$$\begin{array}{ccc} K(X) & \xrightarrow{g^*} & K(X) \\ {\scriptstyle p_!}\downarrow & & \downarrow{\scriptstyle p_!} \\ K(X/G) & \xrightarrow[\mathrm{id}]{} & K(X/G) \end{array}$$

commutes. Once we apply the map $p_!$ to both sides of Eq. (4.3), g^* turns into the identity map, and factoring out 2-torsion we arrive at the following theorem.

Theorem 4.5. *Suppose that the operator family $\widehat{\mathcal{D}}$ satisfies the assumptions given in the beginning of this section and g is the generator of a finite cyclic group G of diffeomorphisms acting freely on X. Then*

$$\operatorname{ind}\widehat{\mathcal{D}}_\gamma = \frac{1}{2}\left\{\operatorname{ind}\widehat{\mathcal{D}}\Box g^*\widehat{\mathcal{D}} + \mathcal{R}_\gamma\right\} \in K(X/G)/\{\text{2-torsion subgroup}\}. \tag{4.7}$$

5 The index of operator families over a fibered parameter space

In this section we consider the case in which the parameter space X is mapped into another parameter space Y by some map $g : X \longrightarrow Y$ (in particular, one can imagine a fiber bundle X with base Y), there are two bundles

$$\mathcal{M}_1 \xrightarrow{\pi} X, \quad \mathcal{M}_2 \xrightarrow{\pi} Y \tag{5.1}$$

of manifolds with conical singularities over X and Y, respectively, and we deal with two families $\widehat{\mathcal{D}}_{1,\gamma}$ and $\widehat{\mathcal{D}}_{2,\gamma}$ of elliptic cone-degenerate operators on $\mathcal{M}_1$ and $\mathcal{M}_2$. Next, suppose that there is a commutative diagram of mappings

$$\begin{array}{ccc} \widetilde{\Omega}_1 & \xrightarrow{h} & \widetilde{\Omega}_2 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow[g]{} & Y \end{array} \tag{5.2}$$

between the corresponding bundles of cone bases such that

$$h_x = h\Big|_{\Omega_{1x}} : \Omega_{1x} \longrightarrow \Omega_{2g(x)}$$

is a diffeomorphism for each $x \in X$. Let the conormal symbols of our operator families satisfy the symmetry condition

$$\mathbf{D}_{1x}(p) = \varkappa_x h_x^* \mathbf{D}_{2g(x)}(-p)(h_x^*)^{-1}\vartheta_x, \tag{5.3}$$

where $\varkappa_x$ and ϑ_x are appropriate vector bundle isomorphisms. Then we can apply the gluing Theorem 3.1 to the operator families $\widehat{\mathcal{D}}_{1,\gamma}$ and $g^*\widehat{\mathcal{D}}_{2,-\gamma}$ and obtain

$$\operatorname{ind}\widehat{\mathcal{D}}_{1,\gamma} + g^* \operatorname{ind}\widehat{\mathcal{D}}_{2,-\gamma} = \operatorname{ind}\widehat{\mathcal{D}}_1 \Box \operatorname{ind}\widehat{\mathcal{D}}_2. \tag{5.4}$$

By passing to the quotient group $K(X)/g^*K(Y)$, we arrive at the following theorem.

Theorem 5.1. *Under the above assumptions, the following index formula is valid:*

$$\operatorname{ind}\widehat{\mathcal{D}}_{1,\gamma} = \operatorname{ind}\widehat{\mathcal{D}}_1 \Box \operatorname{ind}\widehat{\mathcal{D}}_2 \in K(X)/g^*K(Y). \tag{5.5}$$

Remark 5.2. Even though the operator family on the left-hand side depends on γ, the expression for the index on the right-hand side is independent of γ. This precisely means that by varying γ we obtain a relative index that is the lift of some class from $K^*(Y)$. This is not surprising in view of the fact that the conormal family, owing to the symmetry condition (5.3), depends effectively on the parameter space Y (more precisely, is the lift to X of some family on Y).

Remark 5.3. A special case of this situation arises when one deals with the edge symbols of wedge-degenerate operators. The edge symbol $\sigma_\wedge(A)$, where A is a wedge-degenerate operator, is a family of cone-degenerate operators parametrized by the cosphere bundle $X = S^*Y$, where Y is the edge, while the conormal symbol of the edge symbol does not depend on the covariables and is a family of conormal symbols with parameter space Y. If we find an elliptic cone-degenerate operator family with parameter space Y that, together with the edge symbol, satisfies the symmetry condition (5.3), then we shall be able to write out the element $\operatorname{ind} \sigma_\wedge(A) \in K(S^*Y)/K(Y)$, which is exactly the obstruction to the existence of elliptic boundary and coboundary wedge conditions for A.

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