VISCOSITY SOLUTIONS OF FULLY NONLINEAR PARABOLIC SYSTEMS*

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Abstract. In this paper, we discuss the viscosity solutions of the weakly cou-

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pled systems of fully nonlinear second order degenerate parabolic equations and their Cauchy-Dirichlet problem. We prove the existence, uniqueness and continuity of viscosity solution by combining Perron's method with the technique of coupled solutions. The results here generalize those in [2] and [3].

Keywords. Viscosity solutions, systems of partial differential equations, fully nonlinear degenerate parabolic equations, Perron's method, coupled solution

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1 Introduction.

In this paper, we discuss the following initial-boundary value problem of weakly coupled systems of fully nonlinear second order degenerate parabolic partial differential equations.

$$\begin{cases} F_i(t,x;u,\partial_t u_i, D_x u_i, D_x^2 u_i) = 0, & \text{for } (t,x) \in Q, \\ u_i(t,x) = 0, & \text{for } (t,x) \in \Gamma, \quad i = 1,\cdots,m. \\ u_i(0,x) = u_0(x), & \text{for } x \in \Omega, \end{cases} \tag{1}$$

Here $u = (u_1, \cdots, u_m) : Q \to \mathbf{R}^m$ represents the unknown function, $F = (F_1, \cdots, F_m) : Q \times \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{S}(n) \to \mathbf{R}^m$ is a given function which is locally bounded, where $Q = (0,T) \times \Omega$, $\Omega \subset \mathbf{R}^n$ is a bounded open subset, $\mathbf{S}(n)$ denotes the set of real symmetric $n \times n$ matrices equipped the usual order, i.e., $X \leq Y$ for $X, Y \in \mathbf{S}(n)$ if $\xi^t X \xi \leq \xi^t Y \xi$ for every $\xi \in \mathbf{R}^n$, $\partial_t u_i = \frac{\partial u_i}{\partial t}$, $D_x u_i = (\frac{\partial u}{\partial x_k})_{1\leq k\leq n}$ and $D_x^2 u_i = (\frac{\partial^2 u}{\partial x_k \partial x_l})_{1\leq k,l\leq n}$ denote the gradient and Hessian matrix of the real function u_i with respect to x.

To extend the technique of viscosity solutions to general equations, e.g., random parabolic equations or systems, we will use the following notations:

$$u^*(x) = \lim_{r\to 0} \sup \{ u(y) |\ |x-y| < r \}$$

$$u_*(x) = \lim_{r\to 0} \inf \{u(y)| \ |x-y| < r\}$$

which are called the upper and lower semicontinuous envelope of u, respectively. $F_i^*(t,x;u,a,p,X)$ and $F_{i*}(t,x;u,a,p,X)$ are defined similarly.

A nonlinear operator F is called to be *degenerate parabolic (parabolic* in short) if

$$F_i^*(t,x;u;a,p,X) \le F_{i*}(t,x;u;a,p,Y), \qquad i=1,\cdots,m, \text{ whenever } X \ge Y, \tag{2}$$

and

$$F_i^*(t,x;u;b,p,X) \le F_{i*}(t,x;u;a,p,X), \qquad i=1,\cdots,m, \text{ whenever } b \le a, \tag{3}$$

where $a,b \in \mathbf{R}$, $u \in \mathbf{R}^m$, $x,p \in \mathbf{R}^n$, $X,Y \in \mathbf{S}(n)$. F is called *weakly coupled* if every F_i is independent of the derivatives of the coupled variables u_j, $j \ne i$. Such an operator F as defined above is an extension of heat operator.

The system (1) involves many examples including the systems of ordinary differential equation of the first order, the linear and quasi-linear parabolic and degenerate parabolic partial differential equations (which modeling the reaction-diffusion phenomena in chemistry and biology) and, especially, fully nonlinear partial differential equations (such as Monge-Ampere equation in geometry, and Bellman and Isaacs equations arising from optimal control).

For reader's convenience, we recall first the notation of viscosity solutions. We consider a single fully nonlinear parabolic equation

$$f(t,x;\partial_t u, u, D_x u, D_x^2 u) = 0 \tag{4}$$

$u(t,x)$ is called a classic sub-solution of (4) if $u \in C^2(Q_T)$ and satisfies

$$f(t,x;\partial_t u, u, D_x u, D_x^2 u) \le 0$$

If $\phi(t,x) \in C^2(Q_T)$ and $u-\phi$ takes a local maximum at (t,x), we know that

$$\partial_t \phi(t,x) = \partial_t u(t,x), \quad D_x\phi(t,x) = D_x u(t,x), \quad D_x^2\phi(t,x) \ge D_x^2 u(t,x).$$

By (2) and (3),

$$f(t,x;u,\partial_t\phi,D_x\phi,D_x^2\phi) \le f(t,x;u,\partial_t u,D_x u,D_x^2 u) \le 0.$$

The extremes of this inequality do not depend on the derivatives of u and so we may consider defining a nondifferebtial function u to be (some kind of generalized) subsolution of $F=0$ if

$$f(\hat{t},\hat{x};u(\hat{t},\hat{x}),\partial_t\phi(\hat{t},\hat{x}),D_x\phi(\hat{t},\hat{x}),D_x^2\phi(\hat{t},\hat{x})) \le 0 \tag{5}$$

whenever ϕ is C^2 and $(\hat{t},\hat{x})$ is a local maximum of $u-\phi$. The fact of that $(\hat{t},\hat{x})$ is a local maximum of $u-\phi$ can be expressed as

$$\begin{aligned} u(t,x) \le u(\hat{t},\hat{x}) + a(t-\hat{t}) + \langle p, x-\hat{x}\rangle + \tfrac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle \\ +o(|t-\hat{t}| + |x-\hat{x}|^2), \quad \text{as } (t,x)\to(t,x) \end{aligned} \tag{6}$$

where $(a,p,X)=(\partial_t u, D_x u, D_x^2 u)$. Because the inequality (5) depends only on $(a,p,X)=(\partial_t u, D_x u, D_x^2 u)$, we define "superjet" of u at the point $(\hat{t},\hat{x})$ as

$$\begin{aligned} \mathcal{P}_Q^{2,+}u(\hat{t},\hat{x}) = \{(a,p,X)\in \mathbf{R}\times\mathbf{R}^n\times\mathbf{S}(n) \,|\, u(t,x) \le u(\hat{t},\hat{x}) + a(t-\hat{t}) \\ +\langle p, x-\hat{x}\rangle + \tfrac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle \\ +o(|t-\hat{t}) + |x-\hat{x}|^2), \quad \text{as } (t,x)\to(t,x)\Big\}. \end{aligned} \tag{7}$$

And then, we set the "closure of the superjet"

$$\begin{aligned} \bar{\mathcal{P}}_Q^{2,+}u(\hat{t},\hat{x}) = \{(a,p,X) \,|\, \exists (a_n,p_n,X_n)\in \mathcal{P}_Q^{2,+}u(t_n,x_n) \text{ and} \\ (t_n,x_n,a_n,p_n,X_n)\to(\hat{t},\hat{x},a,p,X)\Big\} \end{aligned}$$

which will be needed in the following sections.

Switching the inequality sign in (7), we arrive at the definition of the "subjet" $\mathcal{P}_Q^{2,-}u(\hat{t},\hat{x})$. Its closure $\bar{\mathcal{P}}_Q^{2,-}u(\hat{t},\hat{x})$ follows similarly.

Definition. *Let $f = 0$ be parabolic. A viscosity sub-solution (or super-solution) of $f = 0$ is a function u such that*

$$f(t, x; u, a, p, X) \leq 0 \text{ for all } (t, x), \text{ and } (a, p, X) \in \bar{\mathcal{P}}_Q^{2,+} u(t, x) \tag{8}$$

$$\Big(\text{ or } f(t, x; u, b, q, Y) \geq 0 \text{ for all } (t, x), \text{ and } (b, q, Y) \in \bar{\mathcal{P}}_Q^{2,-} u(t, x) \Big) . \tag{9}$$

u is a viscosity solution of $f = 0$ if it is both viscosity sub-solution and a viscosity super-solution of $f = 0$.

After being introduced by M.G. Crandall and P.L. Lions [1], the concept of viscosity solution has become a powerful tool to study the fully nonlinear partial differential equations. However, when we wish to study systems, ever those with very special coupled structure, there are only a few results. In fact, the concept of viscosity solution is based on the comparison, it only applies to the scalar degenerate elliptic equations (including the parabolic equations) rather than hyperbolic ones and systems. For the scalar parabolic equation

$$f(t, x; u; \partial_t u, D_x u, D_x^2 u) = 0,$$

because the image of the function $f : Q \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{S} \to \mathbf{R}$ is in a total-ordered space, we have comparison. For the system of parabolic equations,

$$F_i(t, x; u_1, \cdots, u_m; \partial_t u, D_x u_i, D_x^2 u_i) = 0, \qquad i = 1, \cdots, m$$

the image of vector function $F = (F_1, \cdots, F_m)$ is in a partial-ordered space in which trichotomy law is invalid. So it is more difficult to deal with the systems. In order to use the maximum principle method, quasi-decreasing assumption has been introduced in [2],[3], [4] and [5].

In [6], the existence and uniqueness of the viscosity solution have been proved by combining Perron's method with the technique of coupled solutions for Dirichlet problem of degenerate elliptic systems which is quasi-increasing or mixed-quasi-monotone. In fact, the technique of coupled solutions is to introduce a new order in $\mathbf{R}^m$ corresponding to the quasi-monotonicity of F in which the system is "quasi-decreasing" and the comparison holds.

In this paper, we are concerned with the degenerate parabolic systems consisting of m equations. We establish the comparison under weaker conditions than those for degenerate elliptic systems in [6]. We prove the existence and uniqueness of the viscosity solution of problem (1) by conbining Perron's method with the technique of coupled solutions. And we show that the viscosity solution of (1) is continuous even though F are not continuous.

The paper consists of 5 sections. The section 2 devotes to the quasi-monotone parabolic systems. The concepts of the coupled viscosity sub- and super-solution, coupled viscosity solutions and viscosity solution for the weakly coupled quasi-monotone degenerate parabolic systems are introduced first. The definition of the coupled viscosity sub- and super-solutions is not ordinary, which is the key of the technique of coupled solutions. In such a way, the comparison holds true and Perron's method could be applied to general quasi-monotone systems to prove the existence of coupled viscosity solutions, and then the existence, uniqueness and continuity of viscosity solution follow. In the section 3, the same result is proved for the non-quasi-monotone systems by making use of the concept of viscosity sub- and super-solutions in strong sense (c.f. [2],[3]) and fixed point theorem. The section 4 is devoted to existence of the coupled viscosity sub- and super-solutions and the coupled viscosity sub- and super-solutions in strong sense. In the last section, some examples are presented.

2 Quasi-monotone systems

The system of functions F is said to be *quasi-monotone* if every F_i is monotone with respect to every coupled variable $u_j,\ j \neq i$.

A function $U(t,x) = (U_1, \cdots, U_m)$ is called a viscosity sub-solution (or super-solution) of (1) if it is locally bounded in Q and satisfies

$$
\begin{aligned}
&F_{i*}(t,x;U^*;a_i,p_i,X_i) \leq 0,\\
&(\text{ or } F_i^*(t,x;U_*;a_i,p_i,X_i) \geq 0)\\
&\forall (t,x) \in Q,\ (a_i,p_i,X_i) \in \mathcal{P}_Q^{2,+}U_i^*(t,x)(\text{ or } \mathcal{P}_Q^{2,-}U_{i*}(t,x)),\\
&\left.\begin{array}{l} U_i^*(t,x) \leq (\text{ or } \geq)0,\ \text{for } (t,x) \in \Gamma, \\ U_i^*(0,x) \leq (\text{ or } \geq) u_0(x) \text{ for } x \in \Omega, \end{array}\right. \quad i = 1, \cdots, m,
\end{aligned}
\tag{10}
$$

where $\mathcal{P}_Q^{2,+}U_i^*(t,x)$ and $\mathcal{P}_Q^{2,-}U_{i*}(t,x)$ denote the second order superjet and subjet of $U_i(t,x)$, respectively(c.f. (7)).

It is well known that the comparison does not hold for the systems in such a notion of viscosity sub- and super-solutions. We apply the technique of coupled solutions, i.e., to define the sub- and super-solutions in a special way (which is called a coupled sub- and super-solution), to solve this difficulty.

To give the definition of coupled sub- and super-solutions, we first put that $A = \{1, \cdots, m\}$ and $A_i \subset A \backslash \{i\}$ is called the decreasing index set of F_i, i.e., F_i is decreasing with respect to the coupled variable u_j as $j \in A_i$ and increasing with respect to u_k as $k \notin A_i$. Then, we define

$$
W^i(u,v;A_i) \triangleq \{W_1^i, \cdots, W_{i-1}^i, W_{i+1}^i, \cdots, W_m^i\} : \mathbf{R}^m \times \mathbf{R}^m \times 2^A \to \mathbf{R}^{m-1}
$$

in which

$$W_j^i(u,v;A_i) \triangleq \begin{cases} u_j, & \text{as } j \in A_i, \\ v_j, & \text{as } j \notin A_i. \end{cases} \tag{11}$$

Definition 1. *Suppose that F is degenerate parabolic, locally bounded and quasi-monotone. (U,V), here $U(t,x) = (U_1,\cdots,U_m)$ and $V(t,x) = (V_1,\cdots,V_m)$, is called* a coupled viscosity sub- and super-solution of (1) *if it is locally bounded in Q and stisfy*

$$\begin{gathered} F_{i*}(t,x;W^i(U^*,V_*;A_i),U_i^*;a_i,p_i,X_i) \le 0, \\ \forall (t,x) \in Q,\ (a_i,p_i,X_i) \in \mathcal{P}_Q^{2,+}U_i^*(t,x), \\ F_i^*(t,x;W^i(V_*,U^*;A_i),V_{i*};b_i,q_i,Y_i) \ge 0, \\ \forall (t,x) \in Q,\ (b_i,q_1,Y_1) \in \mathcal{P}_Q^{2,-}V_{i*}(t,x), \end{gathered} \tag{12}$$

$$\begin{gathered} U_i^*(t,x) \le 0 \le V_{i*}(t,x),\ \text{for } (t,x) \in \Gamma, \\ U_i^*(0,x) \le u_0(x) \le V_{i*}(0,x),\ \text{for } x \in \Omega, \\ i = 1,\cdots,m. \end{gathered} \tag{13}$$

Definition 2. *$(U(t,x),V(t,x))$ is called a* coupled viscosity solutions *of (1) if (U,V) and (V,U) are both coupled viscosity sub and super-solutions of (1).*

Definition 3. *$u(t,x)$ is called* a viscosity solution of (1) *if (u,u) is a coupled viscosity solution of (1).*

With the concept of the coupled viscosity sub- and super-solution, the following theorem could be proved by Perron's method.

Theorem 1. *Suppose that F is parabolic, locally bounded and quasi-monotone. If (U,V) is a coupled viscosity sub- and super-solution of (1), and $U(t,x) \le V(t,x)$, for* *all $(t,x)\in Q$,then the problem (1) has a coupled viscosity solution (u,v) satisfying*

$$U^*\le v\le u\le V_* \text{ on } \bar{Q}.$$

If the comparison holds in addition, then the problem (1) has unique viscosity solution $u(t,x)\in C(Q)$ satisfying $U^*\le u\le V_*$ on $\bar{Q}$.

To establish the comparison demands the following conditions.

(A1) There exist constants $\beta_i>0$ such that

$$F_{i*}(t,x;W^i,r;b,p,X)-F_i^*(t,x;W^i,r;a,p,X)\ge\beta_i(b-a), \tag{A1}$$

for $a,b\in\mathbf{R}$, $b\ge a$, $(t,x;W_i,r;p,X)\in\bar{Q}\times\mathbf{R}^m\times\mathbf{R}^n\times\mathbf{S}(n)$, $\quad i=1,\cdots,m$.

(A2) There exist constants $\gamma_i>0$ such that

$$F_{i*}(t,x;W^i,r;a,p,X)-F_i^*(t,x;W^i,s;a,p,X)\ge-\gamma_i(r-s), \tag{A2}$$

for $r,s\in\mathbf{R}$, $r\ge s$, $(t,x;W_i;a,p,X)\in\bar{Q}\times\mathbf{R}^{m-1}\times\mathbf{R}\times\mathbf{R}^n\times\mathbf{S}(n)$, $\quad i=1,\cdots,m$.

(A3) There are continuous functions $\omega_i:(0,\infty)\to(0,\infty)$ that satisfy $\omega_i(0^+)=0$ such that, for each fixed $t\in\mathbf{R}$,

$$F_i^*(t,y;W^i,r;a,p,Y)-F_{i*}(t,x;W^i,r;a,p,X)\le\omega_i(\alpha|x-y|^2+|x-y|), \tag{A3}$$

whenever $x,y\in\Omega$, $r,a,\alpha\in\mathbf{R}$, $p=\alpha(x-y)$, $W_i\in\mathbf{R}^{m-1}$ and $X,Y\in\mathbf{S}(n)$ satisfying

$$-3\alpha\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}\le\begin{pmatrix} X & 0\\ 0 & -Y\end{pmatrix}\le 3\alpha\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}.$$

(A4) There are constants $L_{ij},i,j=1,\cdots,m,(L_{ii}=0)$ such that

$$|F_{i*}(t,x;W^i,r;a,p,X)-F_i^*(t,x;V^i,r;a,p,X)|\le\sum_{j\ne i}L_{ij}|W_j-V_j|, \tag{A4}$$

for any $W,V\in\mathbf{R}^m$, $\ (t,x;r;a,p,X)\in\bar{Q}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}^n\times\mathbf{S}(n)$.

Remark 1. In fact, instead of (A2), it might be supposed as well that, for an arbitrary constant λ large enough,

$$F_{i*}(t,x;W^i,r;a,p,X) - F_i^*(t,x;W^i,s;a,p,X) \geq \lambda(r-s), \qquad (A2')$$

for $r,s \in \mathbf{R}$, $r \geq s$, $(t,x;W_i;a,p,X) \in \bar{Q} \times \mathbf{R}^{m-1} \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{S}(n)$, $\quad i = 1,\cdots,m$.

If not, taking $\tilde{\lambda} > (\lambda + \max_{i\in A}\gamma_i)/\max_{i\in A}\beta_i$ and putting $u(t,x) = \tilde{u}(t,x)e^{\tilde{\lambda}t}$, equations in (1) are translated into

$$G_i(t,x;\tilde{u};\partial_t\tilde{u}_i, D\tilde{u}_i, D^2\tilde{u}_i) = 0, \quad t>0, x\in\Omega, \quad i=1,\cdots,m,$$

where

$$G_i(t,x;\tilde{u};\partial_t\tilde{u}_i, D\tilde{u}_i, D^2\tilde{u}_i) = e^{-\tilde{\lambda}t}F_i(t,x;e^{\tilde{\lambda}t}\tilde{u};e^{\tilde{\lambda}t}(\tilde{\lambda}\tilde{u}_i + \partial_t\tilde{u}_i), e^{\tilde{\lambda}t}D\tilde{u}_i, e^{\tilde{\lambda}t}D^2\tilde{u}_i).$$

It can be shown that G satisfies the inequality (A2') if F satisfies (A1) and (A2), and if F satisfies (A1) and (A3), so does G(G satisfies (A1) with the same constants for F and (A3) with probable different continuous functions $\omega_i, i = 1,\cdots,m$).

Theorem 2. *Suppose that F is parabolic, locally bounded and quasi-monotone and satisfies* **(A1)-(A4)**. *If (U,V) is a coupled viscosity sub- and super-solution of (1), then $U^* \leq V_*$ on $\bar{Q}$.*

The following Proposition 3 for parabolic superjet and subjet is an extension of Proposition 4.3 in [7] for elliptic superjet and subjet, which is useful for proving the above theorems.

Proposition 3. *Let $Q \subset (0,T)\times\mathbf{R}^n$ be locally compact, the scalar function $v : Q \to \mathbf{R}$ be upper(lower)-semi-continuous in Q, $(t_0,z) \in Q$, $(a,p,X) \in \mathcal{P}_Q^{2,+}v(t_0,z)$ (or $\mathcal{P}_Q^{2,-}v(t_0,z)$). Suppose also that $u^{(k)}$ is a sequence of upper(lower)- semi-continuous functions on Q such that*

- *(i) there exists* $(t^{(k)}, x^{(k)}) \in Q$ *such that*

$$(t^{(k)}, x^{(k)}, u^{(k)}(t^{(k)}, x^{(k)})) \to (t_0, z, v(t_0, z));$$

- *(ii) if* $(s^{(k)}, y^{(k)}) \in Q$ *and* $(s^{(k)}, y^{(k)}) \to (t, x) \in Q$, *then*

$$\limsup_{k\to\infty} u^{(k)}(s^{(k)}, y^{(k)}) \le v(t, x),$$

$$(\liminf_{k\to\infty} u^{(k)}(s^{(k)}, y^{(k)}) \ge v(t, x)).$$

Then, there exist

$$(\hat{t}^{(k)}, \hat{x}^{(k)}) \in Q, \ (a^{(k)}, p^{(k)}, X^{(k)}) \in \mathcal{P}_Q^{2,+} u^{(k)}(\hat{t}^{(k)}, \hat{x}^{(k)}) \ (\mathcal{P}_Q^{2,-} u^{(k)}(\hat{t}^{(k)}, \hat{x}^{(k)}))$$

such that

$$(\hat{t}^{(k)}, \hat{x}^{(k)}, u^{(k)}(\hat{t}^{(k)}, \hat{x}^{(k)}), a^{(k)}, p^{(k)}, X^{(k)}) \to (t_0, z, v(t_0, z), a, p, X).$$

Proof. It might be set as well that $z = 0$. Let v be a scalar upper-semi-continuous function and $(a, p, X) \in \mathcal{P}_Q^{2,+} v(t_0, z)$. From the definition of (a, p, X), we know that for any $d > 0$, there is a $\gamma > 0$ such that the set $N_\gamma = \{(t, x) \in Q : |x| \le \gamma, \ |t - t_0| \le \gamma\}$ is compact and

$$v(t, x) \le v(t_0, 0) + a(t - t_0) + \langle p, x\rangle + \frac{1}{2}\langle Xx, x\rangle + d(|t - t_0| + |x|^2), \ (t, x) \in N_\gamma. \quad (14)$$

By the supposition (i), there is a function sequence $\{u^{(k)}(t, x)\}$ and a point sequence $\{(t^{(k)}, x^{(k)})\}$ such that

$$(t^{(k)}, x^{(k)}, u^{(k)}(t^{(k)}, x^{(k)})) \to (t_0, 0, v(t_0, 0)).$$

Let $(\hat{t}^{(k)}, \hat{x}^{(k)})$ be the maximum point of the function

$$u^{(k)}(t, x) - (a(t - t_0) + \langle p, x\rangle + \frac{1}{2}\langle Xx, x\rangle + 2d(|t - t_0| + |x|^2)) \text{ in } N_\gamma,$$

hence

$$u^{(k)}(t,x) \le u^{(k)}(\hat{t}^{(k)},\hat{x}^{(k)}) + a(t-\hat{t}^{(k)}) + \langle p, x-\hat{x}^{(k)}\rangle + \frac{1}{2}\langle Xx,x\rangle - \frac{1}{2}\langle X\hat{x}^{(k)},\hat{x}^{(k)}\rangle + 2d(|t-t_0| - |\hat{t}^{(k)}-t_0| + |x|^2 - |\hat{x}^{(k)}|^2). \quad (15)$$

Because we are looking for $(a^{(k)},p^{(k)},X^{(k)}) \in \mathcal{P}_Q^{2,+}u^{(k)}(\hat{t}^{(k)},\hat{x}^{(k)})$ which is only concerned with a small neighborhood of the point $(\hat{t}^{(k)},\hat{x}^{(k)})$, we may as well suppose that $t-t_0$ and $\hat{t}^{(k)}-t_0$ have the same sign, then

$$|t-t_0| - |\hat{t}^{(k)}-t_0| = \delta(t-\hat{t}^{(k)}),$$

where $\delta = \mathrm{sgn}(\hat{t}^{(k)}-t_0)$. And on account of the fact

$$\langle Xx,x\rangle - \langle X\hat{x}^{(k)},\hat{x}^{(k)}\rangle = \langle X(x-\hat{x}^{(k)}), x-\hat{x}^{(k)}\rangle + 2\langle X\hat{x}^{(k)}, x-\hat{x}^{(k)}\rangle,$$

$$|x|^2 - |\hat{x}^{(k)}|^2 = \langle x-\hat{x}^{(k)}, x-\hat{x}^{(k)}\rangle + 2\langle \hat{x}^{(k)}, x-\hat{x}^{(k)}\rangle,$$

we get

$$u^{(k)}(t,x) \le u^{(k)}(\hat{t}^{(k)},\hat{x}^{(k)}) + (a+2d\delta)(t-\hat{t}^{(k)}) + \langle p+(X+4dI)\hat{x}^{(k)}, x-\hat{x}^{(k)}\rangle + \frac{1}{2}\langle (X+4dI)(x-\hat{x}^{(k)}), x-\hat{x}^{(k)}\rangle,$$

that is,

$$(a^{(k)},p^{(k)},X^{(k)}) = (a+2d\delta, p+(X+4dI)\hat{x}^{(k)}, X+4dI) \in \mathcal{P}_Q^{2,+}u^{(k)}(\hat{t}^{(k)},\hat{x}^{(k)}).$$

Because the set N_γ is compact, there is a point $(\tau,\xi) \in N_\gamma$ such that (passing to a subsequence if necessary) $(\hat{t}^{(k)},\hat{x}^{(k)}) \to (\tau,\xi)$ as $k\to\infty$. Putting $(t,x) = (t^{(k)},x^{(k)})$ in (15) and taking the limit superior as $k\to\infty$, we find that

$$v(t_0,0) \le \limsup_{k\to\infty} u^{(k)}(\hat{t}^{(k)},\hat{x}^{(k)}) - a(\tau-t_0) - \langle p,\xi\rangle - \frac{1}{2}\langle X\xi,\xi\rangle - 2d(|\tau-t_0|+|\xi|^2). \quad (16)$$

By the supposition (ii), $\limsup_{k\to\infty} u^{(k)}(t^{(k)},x^{(k)}) \le v(\tau,\xi)$ while (14) implies

$$v(\tau,\xi) - a(\tau-t_0) - \langle p,\xi\rangle - \frac{1}{2}\langle X\xi,\xi\rangle - 2d(|\tau-t_0) + |\xi|^2) \le v(t_0,0) - d(|\tau-t_0|+|\xi|^2).$$

We conclude from (16) that

$$v(t_0,0) \le v(t_0,0) - d(|\tau - t_0| + |\xi|^2).$$

This inequality implies $\tau = t_0,\ \xi = 0$, so $(\hat{t}^{(k)}, \hat{x}^{(k)}) \to (t_0, 0)$ (without passing to a subsequence). At the same time, by the arbitrariness of d, we see that

$$(a^{(k)}, p^{(k)}, X^{(k)}) = (a + 2d\delta, p + (X + 4dI)\hat{x}^{(k)}, X + 4dI) \to (a, p, X), \text{ as } k \to \infty,$$

thus, we conclude that

$$(\hat{t}^{(k)}, \hat{x}^{(k)}, u^{(k)}(\hat{t}^{(k)}, \hat{x}^{(k)}), a^{(k)}, p^{(k)}, X^{(k)}) \to (t_0, 0, v(t_0,0), a, p, X), \text{ as } k \to \infty.$$

The proof is completed.

Let F be degenerate parabolic, locally bounded and quasi-monotone. Suppose that (U, V) is a coupled viscosity sub- and super-solution of (1) satisfying $U(t,x) \le V(t,x)$ for all $(t,x) \in Q$, and we define that

$$S \times P = (S_1 \times \cdots \times S_m) \times (P_1 \times \cdots \times P_m) = \{(u(t,x), v(t,x))$$
$$|(u,v) \text{ is a coupled viscosity sub- and super-solution of (1)}$$
$$\text{satisfying } U \le u, v \le V.\},$$

and $\bar{u} = (\bar{u}_1, \cdots, \bar{u}_m),\ \bar{v} = (\bar{v}_1, \cdots, \bar{v}_m)$, where

$$\bar{u}_i(t,x) = \sup\{u_i(t,x) | u \in S\},$$
$$\bar{v}_i(t,x) = \inf\{v_i(t,x) | v \in P\},$$
$$i = 1, \cdots, m.$$

Then, we have the following lemmas.

Lemma 4. $(\bar{u}, \bar{v}) \in S \times P$.

Proof. S and P are nonempty because $U \in S$ and $V \in P$, and

$$\bar{u}_i^*(t,x) < +\infty, \bar{v}_{i*}(t,x) > -\infty, \forall : (t,x) \in \bar{Q}, i = 1, \cdots, m,$$

because that Q is compact and U and V are upper- and lower-semi-continuous on Q, respectively.

For an arbitrary $i \in A$, suppose that $(s,z) \in Q$, $(a_i, p_i, X_i) \in \mathcal{P}_Q^{2,+}\bar{u}_i^*(s,z)$. By the definition of $\bar{u}^*$, there is a sequence $\{u_i^{(k)}\} \subset S_i$ (we may as well suppose that all components of which are upper semi-continuous) and a sequence of points $\{(t^{(k)}, x^{(k)})\} \subset Q$ such that $(t^{(k)}, x^{(k)}, u_i^{(k)}) \to (s, z, \bar{u}_i^*)$. On virtue of upper semi-continuity of $\bar{u}_i^*$, we see that

$$\limsup_{k\to\infty} u_i^{(k)}(\tau^{(k)}, \xi^{(k)}) \le \bar{u}_i^*(s,z),$$

for every sequence $(\tau^{(k)}, \xi^{(k)}, u_i^{(k)}(\tau^{(k)}, \xi^{(k)}))$. Thus, the conditions of Proposition 3 hold, hence there are $(\hat{t}^{(k)}, \hat{x}^{(k)}) \in Q$, $(a_i^{(k)}, p_i^{(k)}, X_i^{(k)}) \in \mathcal{P}_Q^{2,+}u_i^{(k)}(\hat{t}^{(k)}, \hat{x}^{(k)})$ such that

$$(\hat{t}^{(k)}, \hat{x}^{(k)}, u_i^{(k)}(\hat{t}^{(k)}, \hat{x}^{(k)}), a_i^{(k)}, p_i^{(k)}, X_i^{(k)}) \to (s, z, \bar{u}_i^*(s,z), a_i, p_i, X_i).$$

From the definitions of $u_i^{(k)}$, there are $w_1^{(k)}, \cdots, w_{i-1}^{(k)}, w_{i+1}^{(k)}, \cdots, w_m^{(k)}$ (which might be as well supposed to be continuous for convenience) such that $U_j \le w_j^{(k)} \le V_j$ and $w_j^{(k)} \in S_j$ as $j \in A_i$ or $w_j^{(k)} \in P_j$ as $j \notin A_i$. Furthermore, we know from the definition of $\bar{u}_j$ and $\bar{v}_j$ that $w_j^{(k)} \le \bar{u}_j$ as $j \in A_i$ and $w_j^{(k)} \ge \bar{v}_j$ as $j \notin A_i$, so

$$w_j = \lim_{k\to\infty} w_j^{(k)} \begin{cases} \le \bar{u}_i, \text{ as } j \in A_i, \\ \ge \bar{v}_i, \text{ as } j \notin A_i. \end{cases}$$

On virtue of quasi-monotonicity and lower-semi-continuity of F_{i*},

$$\begin{aligned} & F_{i*}(s,z; W_i(\bar{u}^*, \bar{v}_*, A_i), \bar{u}_i^*; a_i, p_i, X_i) \\ \le\ & F_{i*}(s,z; w_1, \cdots, w_{j-1}, w_{j+1}, \cdots, w_m, \bar{u}_i^*; a_i, p_i, X_i) \\ \le\ & \lim_{k\to\infty} F_{i*}(s,z; w_1^{(k)}, \cdots, w_{j-1}^{(k)}, w_{j+1}^{(k)}, \cdots, w_m^{(k)}, u_i^{(k)}; a_i^{(k)}, p_i^{(k)}, X_i^{(k)}) \\ \le\ & 0. \end{aligned}$$

Similarly, for $(s,z) \in Q$, $(b_i, q_i, Y_i) \in \mathcal{P}_Q^{2,-}\bar{v}_{i*}(s,z)$, from definition, there is a sequence $\{v_i^{(k)}\} \subset P_i$ (all components of which might be as well supposed to be lower-semi-continuous) and a sequence of points $\{(t^{(k)}, x^{(k)})\} \subset \Omega$ such that $(t^{(k)}, x^{(k)}, v_i^{(k)}) \to (s, z, \bar{v}_{i*})$. From its lower-semi-continuity, we see that

$$\liminf_{k\to\infty} v_i^{(k)}(\tau^{(k)}, \xi^{(k)}) \geq \bar{v}_{i*}(s,z),$$

for every sequence $(\tau^{(k)}, \xi^{(k)}, v_i^{(k)}(\tau^{(k)}, \xi^{(k)}))$. Thus, the conditions of Proposition 3 hold again, hence there are $(\hat{t}^{(k)}, \hat{x}^{(k)}) \in Q$, $(b_i^{(k)}, q_i^{(k)}, Y_i^{(k)}) \in \mathcal{P}_Q^{2,-}v_i^{(k)}(\hat{t}^{(k)}, \hat{x}^{(k)})$ such that

$$(\hat{t}^{(k)}, \hat{x}^{(k)}, v_i^{(k)}(\hat{t}^{(k)}, \hat{x}^{(k)}), b_i^{(k)}, q_i^{(k)}, Y_i^{(k)}) \to (s, z, \bar{v}_{1*}(s,z), b_i, q_i, Y_i).$$

From the definitions of $v_i^{(k)}$, there are $w_1^{(k)}, \cdots, w_{i-1}^{(k)}, w_{i+1}^{(k)}, \cdots, w_m^{(k)}$(which might be as well supposed to be continuous again) such that $U_j \leq w_j^{(k)} \leq V_j$ and $w_j^{(k)} \in P_j$ as $j \in A_i$ or $w_j^{(k)} \in S_j$ as $j \notin A_i$. Furthermore, we see that $w_j^{(k)} \geq \bar{v}_j$ as $j \in A_i$ and $w_j^{(k)} \leq \bar{u}_j$ as $j \notin A_i$, so

$$w_j = \lim_{k\to\infty} w_j^{(k)} \begin{cases} \geq \bar{v}_i, \text{ as } j \in A_i, \\ \leq \bar{u}_i, \text{ as } j \notin A_i. \end{cases}$$

On account of quasi-monotonicity and upper-semi-continuity of F_i^*,

$$\begin{aligned} & F_i^*(s,z; W_i(\bar{v}_*, \bar{u}^*, A_i), \bar{v}_i; b_i, q_i, Y_i) \\ \leq\ & F_i^*(s,z; w_1, \cdots, w_{j-1}, w_{j+1}, \cdots, w_m, \bar{v}_{i*}; b_i, q_i, Y_i) \\ \leq\ & \lim\nolimits_{k\to\infty} F_i^*(s,z; w_1^{(k)}, \cdots, w_{j-1}^{(k)}, w_{j+1}^{(k)}, \cdots, w_m^{(k)}, v_i^{(k)}; b_i^{(k)}, q_i^{(k)}, Y_i^{(k)}) \\ \leq\ & 0. \end{aligned}$$

Taking account of the arbitrariness of index i, we have already proved that $(\bar{u}, \bar{v})$ is a coupled viscosity sub- and super-solutions with $U \leq \bar{u}, \bar{v} \leq V$ on $\bar{Q}$.

Lemma 5. *Let F be degenerate parabolic, locally bounded and quasi-monotone. Assume that the sets S and P are both nonempty. If $\{u,v\} \in S \times P$ while $\{v,u\} \notin S \times P$, then there is another couple of sub- and super-solution $\{\hat{U},\hat{V}\}$ such that $\hat{U} \geq u$, $\hat{V} \leq v$ and $\{\hat{U},\hat{V}\} \neq \{u,v\}$.*

Proof. If $\{u,v\} \in S \times P$ while u_{i*} is not a viscosity super-solution of (1), i.e., there is a point $(t_0,x_0) \in Q$,(we might as well suppose $x_0 = 0 \in \Omega$ for convenience) such that

$$F_i^*(t_0,0;W_i(u,v,A_i),u_{i*};b_i,q_i,Y_i) < 0, \text{ for some } (b_i,q_i,Y_i) \in \mathcal{P}_Q^{2,-}u_{i*}(t_0,0).$$

Similar to the proof of Lemma 4.4 in [7], by semi-continuity of F_{i*},

$$\tilde{U}_i = u_{i*}(t_0,0) + \delta + b_i(t-t_0) + \langle q, x0 \rangle + \frac{1}{2}\langle Yx,x \rangle - \gamma|x|^2$$

is a classical solution of $F_{i*} \leq 0$ in $B_\gamma = \{x||x| < \gamma\}$ for all small $r,\delta,\gamma > 0$. It is clear that

$$\hat{U}_i(t,x) = \begin{cases} \max\{u_i(t,x), \tilde{U}_i(t,x) & \text{if } |t-t_0| + |x|^2 < r^2, \\ u_i(t,x) & \text{otherwise }, \end{cases}$$

is a viscosity sub-solution of $F_i = 0$. And let $\hat{U}_j = u_j^*, j \neq i$ and $\hat{V} = v$, then we have that $\hat{U} \geq u, \hat{V} \leq v$ and $\{\hat{U},\hat{V}\} \neq \{u,v\}$. Due to the fact that $F_j, j \neq i$ are quasi-monotone,

$$\begin{gathered} F_{j*}(t,x;W_j(\hat{U},\hat{V},A_j),\hat{U}_j^*;a_j,p_j,X_j) \\ \leq F_{j*}(t,x;W_j(u,v,A_j),u_j^*;a_j,p_j,X_j) \leq 0 \\ \forall\,(t,x) \in Q,\ (a_j,p_j,X_j) \in \mathcal{P}_Q^{2,+}\hat{U}_j^*(t,x), \\ F_j^*(t,x;W_j(\hat{V},\hat{U},A_j),\hat{V}_{j*};b_j,q_j,Y_j) \\ \geq F_j^*(t,x;W_j(v,u,A_j),v_{j*};b_j,q_j,Y_j) \geq 0 \\ \forall\,(t,x) \in Q,\ (b_j,q_j,Y_j) \in \mathcal{P}_Q^{2,-}\hat{V}_{j*}(t,x), \\ j = 1,2,\cdots,i-1,i+1,\cdots,m. \end{gathered}$$

Thus, $(\hat{U}, \hat{V})$ is another couple of viscosity sub- and super-solution of (1), the conclusion of lemma holds.

If $(u, v) \in S \times P$ while v_i^* is not a viscosity sub-solution of (1), i.e.,

$$F_{i*}(t_0, 0; W_i(v, u, A_i), v_i^*; a_i, p_i, X_i) > 0, \text{ for some } (a_i, p_i, X_i) \in \mathcal{P}_Q^{2,+} v_i^*(t_0, 0).$$

Let

$$\hat{V}_i(t, x) = \begin{cases} \min\{v_i(t, x), \tilde{V}_i(t, x) & \text{if } t = t_0, |x| < r, \\ v_i(t, x) & \text{otherwise }, \end{cases}$$

where

$$\tilde{V}_i = v_i^*(t_0, 0) - \delta + a_i(t - t_0) + \langle p_i x, x\rangle + \frac{1}{2}\langle X_i x, x\rangle + \gamma|x|^2,$$

$\hat{V}_j = v_{j*}, j \neq i$ and $\hat{U} = u$, then we can prove similarly that $(\hat{U}, \hat{V})$ is another coupled viscosity sub- and super-solution of (1) with $\hat{U} \geq u, \hat{V} \leq v$ and $(\hat{U}, \hat{V}) \neq (u, v)$. The proof is complete.

Proof of Theorem 1. From Lemma 4, we know that $(\bar{u}, \bar{v})$ is a coupled viscosity sub- and super-solution of (1). If $(\bar{v}, \bar{u})$ is not a coupled sub- and super-solution, according to Lemma 5, there is another one $(\hat{U}, \hat{V}) \in S \times P$ such that $\bar{u}^* \leq \hat{U}, \bar{v}_* \geq \hat{V}$ and $(\bar{u}^*, \bar{v}_*) \neq (\hat{U}, \hat{V})$. This is a contradiction to the definition of $(\bar{u}, \bar{v})$. Hence, $(\bar{v}, \bar{u})$ is a coupled viscosity sub- and super-solution of (1), too.

Knowing that $(\bar{u}, \bar{v})$ is a coupled viscosity solution and the comparison holds, we see that

$$\bar{u} \leq \bar{u}^* \leq \bar{v}_* \leq \bar{v},$$

thanks to $(\bar{u}, \bar{v}) \in S \times P$. We also see that

$$\bar{v} \leq \bar{v}^* \leq \bar{u}_* \leq \bar{u}$$

because at the same time $(\bar{v}, \bar{u}) \in S \times P$. Thus, we conclude that $\bar{u} \equiv \bar{v}$ and $\bar{u} \in C(Q)$. The uniqueness can be proved in the same way.

Proof of Theorem 2. First of all, let $\hat{U}_i = U_i - \epsilon/(T-t),\ \hat{V}_i = V_i + \epsilon/(T-t),\ i = 1, \cdots, m$. Then

$$\mathcal{P}_Q^{2,-}\hat{U}_i^*(t,x) = \Big\{(\hat{a}, p, X) \ \Big|\ \hat{a} = a - \frac{\epsilon}{(T-t)^2},\ (a,p,X) \in \mathcal{P}_Q^{2,-}U_i^*(t,x)\Big\}$$

$$\mathcal{P}_Q^{2,+}\hat{V}_{i*}(t,x) = \Big\{(\hat{b}, q, Y) \ \Big|\ \hat{b} = b + \frac{\epsilon}{(T-t)^2},\ (b,q,Y) \in \mathcal{P}_Q^{2,+}V_{i*}(t,x)\Big\}.$$

Due to $(A1), (A4)$ and $(A2')$ with the constant $\lambda > \sum_{j\neq i} L_{ij}$, we have that

$$\begin{aligned}
& F_{i*}(t,x; W^i(U^*, V_*, A_i), U_i^*; a, p, X) \\
& -F_{i*}(t,x; W^i(\hat{U}^*, \hat{V}_*, A_i), \hat{U}_i^*; \hat{a}, p, X) \\
\geq\ & \beta_i(a - \hat{a}) + \lambda(U_i^* - \hat{U}_i^*) - \sum_{j\neq i} L_{ij}|W_j^i(U^*, V_*, A_i) - W^i(\hat{U}^*, \hat{V}_*, A_i)| \\
\geq\ & \frac{\epsilon\beta_i}{(T-t)^2} + (\lambda - \sum_{j\neq i} L_{ij})\frac{\epsilon}{(T-t)} \\
\geq\ & \frac{\epsilon\beta_i}{(T-t)^2} \\
& \quad (t,x) \in Q,\ (a,p,X) \in \mathcal{P}_Q^{2,-}U_i^*(t,x)
\end{aligned}$$

$$\begin{aligned}
& F_i^*(t,x; W^i(\hat{V}_*, \hat{U}^*, A_i), \hat{V}_{i*}; \hat{b}, q, Y) \\
& -F_i^*(t,x; W^i(V_*, U^*, A_i), V_{i*}; b, q, Y) \\
\geq\ & \beta_i(\hat{b} - b) + \lambda(\hat{V}_{i*} - V_{i*}) - \sum_{j\neq} L_{ij}|W_j^i(V^*, U_*, A_i) - W^i(\hat{V}^*, \hat{U}_*, A_i)| \\
\geq\ & \frac{\epsilon\beta_i}{(T-t)^2} + (\lambda - \sum_{j\neq i} L_{ij})\frac{\epsilon}{(T-t)} \\
\geq\ & \frac{\epsilon\beta_i}{(T-t)^2} \\
& \quad (t,x) \in Q,\ (b,q,Y) \in \mathcal{P}_Q^{2,+}V_{i*}(t,x)
\end{aligned}$$

Thus, we observe that $(\hat{U}, \hat{V})$ is also a coupled viscosity sub- and super-solution of (1) and satisfies the following inequalities

$$\begin{aligned}
F_{i*}(t,x; W^i(\hat{U}^*, \hat{V}_*, A_i), \hat{U}_i^*; \partial_t\hat{U}_i, D\hat{U}_i, D^2\hat{U}_i) \leq\ & -\frac{\epsilon\beta_i}{(T-t)^2} \\
F_i^*(t,x; W^i(\hat{V}_*, \hat{U}^*, A_i), \hat{V}_{i*}; \partial_t\hat{V}_i, D\hat{V}_i, D^2\hat{V}_i) \geq\ & \frac{\epsilon\beta_i}{(T-t)^2}
\end{aligned}$$

Since $U \le V$ follows from $\hat{U} \le \hat{V}$ in the limit $\epsilon \downarrow 0$, it will simply suffice to prove the comparison under the additional assumptions

$$\begin{cases} (i) F_{i*}(t,x; W^i(U^*, V_*, A_i), U_i^*; \partial_t U_i, DU_i, D^2 U_i) \le -\frac{\epsilon \beta_i}{T^2} < 0 \\ (ii) F_i^*(t,x; W^i(V_*, U^*, A_i), V_{i*}; \partial_t V_i, DV_i, D^2 V_i) \ge \frac{\epsilon \beta_i}{T^2} > 0 \\ (iii) \begin{cases} \lim_{t \uparrow T} U_i(t,x) = -\infty \\ \lim_{t \uparrow T} V_i(t,x) = +\infty \end{cases} \quad \text{uniformly on } \bar{\Omega}. \end{cases} \tag{17}$$

Now, we are proving the comparison by contradiction under the assumptions (A2') and (17).

We may suppose as well that there is $i \in \{1, \cdots, m\}$ and $(t_0, x_0) \in Q$ such that

$$\delta \triangleq U_i^*(t_0, x_0) - V_{i*}(t_0, x_0) = \max_{j \in A} \sup_{(t,x) \in \bar{Q}} \{U_j^*(t,x) - V_{j*}(t,x)\} > 0.$$

Consider the function

$$M_i(t,x,y) \triangleq U_i^*(t,x) - V_{i*}(t,y) - \frac{\alpha}{2}|x-y|^2.$$

Let $(t_\alpha, x_\alpha, y_\alpha)$ be the local maximum point of $M_i(t,x,y)$ for a fixed α. (This maximum, denoted by M_α, can be obtained in a neighborhood N of the point (t,x) and is positive and bounded as α is large enough since $M_i(t,x,y) \in \mathrm{USC}([0,T] \times \Omega \times \Omega)$.) It can be seen that

$$U_i^*(t_\alpha, x_\alpha) - V_{i*}(t_\alpha, y_\alpha) \ge M_\alpha \ge \delta_i > 0. \tag{18}$$

From the initial and boundary value conditions, we conclude that $t_\alpha \neq 0$ and $x_\alpha, y_\alpha \notin \partial\Omega$ as α is large enough. From (iii) in (17), we also see that $t_\alpha \neq T$. Thus we may employ the Theorem 8.3 in [7] to learn that there are $a, p = \alpha(x_\alpha - y_\alpha), X, Y \in \mathbf{S}(n)$ such that

$$(a,p,X) \in \bar{\mathcal{P}}_Q^{2,+} U_i^*(t_\alpha, x_\alpha), (a,p,Y) \in \bar{\mathcal{P}}_Q^{2,-} V_{i*}(t_\alpha, y_\alpha). \tag{19}$$

From the inequality (A2'), we have that

$$\begin{aligned}
0 &< \lambda\delta < \lambda M_\alpha < \lambda(U_i^*(t_\alpha, x_\alpha) - V_{i*}(t_\alpha, y_\alpha)) \\
&\leq F_{i*}(t_\alpha, x_\alpha; W^i(U^*(t_\alpha, x_\alpha), V_*(t_\alpha, x_\alpha), A_i), U_i^*(t_\alpha, x_\alpha); a, p, X) \\
&\quad - F_i^*(t_\alpha, x_\alpha; W^i(U^*(t_\alpha, x_\alpha), V_*(t_\alpha, x_\alpha), A_i), V_{i*}(t_\alpha, y_\alpha); a, p, X) \\
&\leq F_{i*}(t_\alpha, x_\alpha; W^i(U^*(t_\alpha, x_\alpha), V_*(t_\alpha, x_\alpha), A_i), U_i^*(t_\alpha, x_\alpha); a, p, X) \\
&\quad - F_i^*(t_\alpha, y_\alpha; W^i(V_*(t_\alpha, y_\alpha), U^*(t_\alpha, y_\alpha), A_i), V_{i*}(t_\alpha, y_\alpha); a, p, Y)) \qquad (20) \\
&\quad + F_i^*(t_\alpha, y_\alpha; W^i(V_*(t_\alpha, y_\alpha), U^*(t_\alpha, y_\alpha), A_i), V_{i*}(t_\alpha, y_\alpha); a, p, Y) \\
&\quad - F_i^*(t_\alpha, y_\alpha; W^i(U^*(t_\alpha, x_\alpha), V_*(t_\alpha, x_\alpha), A_i), V_{i*}(t_\alpha, y_\alpha); a, p, Y) \qquad (21) \\
&\quad + F_i^*(t_\alpha, y_\alpha; W^i(U^*(t_\alpha, x_\alpha), V_*(t_\alpha, x_\alpha), A_i), V_{i*}(t_\alpha, y_\alpha); a, p, Y) \\
&\quad - F_i^*(t_\alpha, x_\alpha; W^i(U^*(t_\alpha, x_\alpha), V_*(t_\alpha, x_\alpha), A_i), V_{i*}(t_\alpha, y_\alpha); a, p, X) \qquad (22)
\end{aligned}$$

Because that (U, V) is a coupled viscosity sub- and super-solution of (1), (20) < 0. It follows from the Lemma 3.1 in [7] that

$$\lim_{\alpha \to +\infty} \alpha |x_\alpha - y_\alpha|^2 = 0$$

and

$$\lim_{\alpha \to +\infty} M_i(t_\alpha, x_\alpha, y_\alpha) = \sup_{(t,x) \in Q} \{U_i^*(t, x) - V_{i*}(t, x)\} \leq M_\alpha.$$

Due to $(A3)$ and $(A4)$,

$$(22) \leq \omega_i(\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha|) \to 0 \text{ as } \alpha \to \infty, \qquad (23)$$

$$(21) \leq \sum_{j \neq i} L_{ij} M_\alpha \text{ as } \alpha \to \infty, \qquad (24)$$

Hence, it leads to the inequality

$$\lambda M_\alpha \leq \sum_{j \neq i} L_{ij} M_\alpha.$$

This is a contradiction because $\lambda > \sum_{j \neq i} L_{ij}$. The comparison is proved.

3 The non-quasi-monotone systems

For the non-quasi-monotone systems, we follow [2] and [3] to introduce the notion of the coupled viscosity sub- and super-solution in strong sense. To do so, U and V are said to satisfy the condition **(C)** if

(C) there is a continuous function $\eta(t,x)$ such that

$$U^*(t,x) \le \eta(t,x) \le V_*(t,x) \quad \text{if } U^*(t,x) \le V_*(t,x)$$

$$V_*(t,x) \le \eta(t,x) \le U^*(t,x) \quad \text{if } V_*(t,x) \le U^*(t,x)$$

Definition 4. *Let F be degenerate parabolic and locally bounded. Suppose that (U,V) is locally bounded and satisfies the condition* **(C)**. *(U,V) is called* a coupled viscosity sub- and super-solution in strong sense to (1) *if*

$$\begin{aligned} &F_{i*}(t,x;\phi^i,U_i^*;a_i,p_i,X_i) \le 0,\\ &\forall (t,x)\in Q,\ (a_i,p_i,X_i)\in \mathcal{P}_Q^{2,+}U_i^*(t,x),\\ &F_i^*(t,x;\phi^i,V_{i*};b_i,q_i,Y_i) \ge 0,\\ &\forall (t,x)\in Q,\ (b_i,q_1,Y_1)\in \mathcal{P}_Q^{2,-}V_{i*}(t,x), \end{aligned} \tag{25}$$

for any continuous function $\phi(t,x)$ between U^* and V_*, and (13), where

$$\phi^i(t,x) = (\phi_1,\cdots,\phi_{i-1},\phi_{i+1},\cdots,\phi_m).$$

Theorem 6. *Suppose that F is parabolic and locally bounded and* **(A1)-(A4)** *hold. If (U,V) is a coupled viscosity sub- and super-solution in strong sense to (1), then*

- *$U^* \le V_*$ on $\bar{Q}$;*
- *there is a unique viscosity solution $u(t,x)\in C(Q)$ to the problem (1) satisfying $U^* \le u \le V_*$ on $\bar{Q}$.*

Proof Prove it by the fixed-point theorem. For a continuous function ϕ between U and V, consider the following system:

$$\begin{cases} F_i(t,x;\phi^i,u_i;\partial_t u_i, D_x u_i, D_x^2 u_i) = 0, & \text{for } (t,x) \in Q, \\ u_i(t,x) = 0, & \text{for } (t,x) \in \Gamma, \\ u_i(0,x) = u_0(x), & \text{for } x \in \Omega, \\ \qquad i = 1,\cdots,m. \end{cases} \tag{26}$$

This system consists of m separated equations without coupling, and the components of U and V are viscosity sub- and super-solutions of these m separated equations, respectively. The comparative result follows the comparison theorem of viscosity solution for scalar equation(see [7]). And from the existence theorem of viscosity solution for scalar equation, there is a viscosity solution u for (26), which depends on the given function ϕ and satisfies $U^* \leq u \leq V_*$. Thus, an operator $A : \phi \mapsto u$ is defined. A fixed point of the operator A is obviously a viscosity solution of (1). Let $(C(Q))^m$ denote the space consisting of the continuous vector functions on Q with the norm

$$||u|| = \max_j \sup_{(t,x)\in Q} |u(t,x)|$$

and

$$\mathcal{B} = \{u \in (C(Q))^m | U^* \leq u \leq V_*\},$$

then $\mathcal{B}$ is a bounded close subset of the Banach space $(C(Q))^m$ and $A(\mathcal{B}) \subset \mathcal{B}$. Now, similar to the proof of Theorem 2, we are proving that there is a fixed point in $\mathcal{B}$ for the operator A under the assumptions (A2') and (17).

Suppose $u = A\phi$ and $v = A\eta$, then u and v are continuous and $u^* = u_* = u$ and so does v. If

$$\sup_{(t,x)\in Q} \{u_i(t,x) - v_i(t,x)\} > 0,$$

consider again the function

$$M_i(t,x,y) \triangleq u_i(t,x) - v_i(t,y) - \frac{\alpha}{2}|x-y|^2,\ 1,\cdots,m.$$

$(t_\alpha, x_\alpha, y_\alpha)$ denotes again the local maximum point of $M_i(t,x,y)$ for a fixed α and $M_i(\alpha)$ denotes this maximum. Hence

$$u_i(t_\alpha, x_\alpha) - v_i(t_\alpha, y_\alpha) \geq M_i(\alpha) > 0. \tag{27}$$

From the initial-boundary value conditions and (iii) in (17), it can be seen that $t_\alpha \neq 0, T$ and $x_\alpha, y_\alpha \notin \partial\Omega$ if α is large enough. By Theorem 8.3 in [7], there are $a, p = \alpha(x_\alpha - y_\alpha), X, Y \in \mathbf{S}(n)$ such that

$$(a,p,X) \in \bar{\mathcal{P}}_Q^{2,+} u_i(t_\alpha, x_\alpha), \quad (a,p,Y) \in \bar{\mathcal{P}}_Q^{2,-} v_i(t_\alpha, y_\alpha).$$

From the inequality (A2'), it can be seen that

$$\begin{aligned}
0 &< \lambda M_i(\alpha) < \lambda(u_i(t_\alpha,x_\alpha) - v_i(t_\alpha,y_\alpha)) \\
&\leq F_{i*}(t_\alpha, x_\alpha; \phi^i(t_\alpha,x_\alpha), u_i(t_\alpha,x_\alpha); a,p,X) \\
&\quad - F_i^*(t_\alpha, x_\alpha; \phi^i(t_\alpha,x_\alpha), v_i(t_\alpha,y_\alpha); a,p,X) \\
&\leq F_{i*}(t_\alpha, x_\alpha; \phi^i(t_\alpha,x_\alpha), u_i(t_\alpha,x_\alpha); a,p,X) \\
&\quad - F_i^*(t_\alpha, y_\alpha; \eta^i(t_\alpha,y_\alpha), v_i(t_\alpha,y_\alpha); a,p,Y) && (28) \\
&\quad + F_i^*(t_\alpha, y_\alpha; \eta^i(t_\alpha,y_\alpha), v_i(t_\alpha,y_\alpha); a,p,Y) \\
&\quad - F_i^*(t_\alpha, y_\alpha; \phi^i(t_\alpha,x_\alpha), v_i(t_\alpha,y_\alpha); a,p,Y) && (29) \\
&\quad + F_i^*(t_\alpha, y_\alpha; \phi^i(t_\alpha,x_\alpha), v_i(t_\alpha,y_\alpha); a,p,Y) \\
&\quad - F_i^*(t_\alpha, x_\alpha; \phi^i(t_\alpha,x_\alpha), v_i(t_\alpha,y_\alpha); a,p,X) && (30)
\end{aligned}$$

Because that $u = A\phi$ and $v = A\eta$, (28) < 0. Due to (A3),

$$(30) \leq \omega_i(\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha|), \tag{31}$$

which goes to 0 as $\alpha \to +\infty$. From (A4),

$$(29) \leq \sum_{j \neq i} L_{ij} |\phi_j(t_\alpha, x_\alpha) - \eta_j(t_\alpha, y_\alpha)|.$$

Then, as $\alpha \to +\infty$,

$$\lambda \sup_{(t,x)\in Q} \{u_i(t,x) - v_i(t,x)\} \leq \sum_{j\neq i} L_{ij} \sup_{(t,x)\in Q} |\phi_j(t,x) - \eta_j(t,x)|. \tag{32}$$

Similarly, if $\sup_{(t,x)\in Q}\{v_i(t,x) - u_i(t,x)\} > 0$, it can be shown that

$$\lambda \sup_{(t,x)\in Q} \{v_i(t,x) - u_i(t,x)\} \leq \sum_{j\neq i} L_{ij} \sup_{(t,x)\in Q} |\eta_j(t,x) - \phi_j(t,x)|. \tag{33}$$

Thus,

$$\lambda ||u - v|| \leq \sum_{i,j=1}^{m} L_{ij} ||\phi - \eta||. \tag{34}$$

Because λ could be taken large enough such that $\lambda > \sum_{i,j=1}^{m} L_{ij}$ according to Remark 1, it has been proved that A is a contraction. By Banach's fixed-point theorem, there is a unique continuous vector function $u \in \mathcal{B}$ such that $u = Au$ which does be the unique viscosity solution of (1).

Remark 2. It can been seen from (20)-(22) and (28)-(30) that some inequalities in **(A1)-(A4)** can be replaced by the following weaker ones:

(A1')

$$F_{i*}(t,x;W^i,r;b,p,X) - F_{i*}(t,x;W^i,r;a,p,X) \geq \beta_i(b-a), \tag{$A1'$}$$

$$F_i^*(t,x;W^i,r;b,p,X) - F_i^*(t,x;W^i,r;a,p,X) \geq \beta_i(b-a), \tag{$A1''$}$$

(A2')

$$F_{i*}(t,x;W^i,r;a,p,X) - F_{i*}(t,x;W^i,s;a,p,X) \geq -\gamma_i(r-s), \tag{$A2'$}$$

$$F_i^*(t,x;W^i,r;a,p,X) - F_i^*(t,x;W^i,s;a,p,X) \geq -\gamma_i(r-s), \tag{$A2''$}$$

(A3')

$$F_{i*}(t,y;W^i,r;a,p,Y) - F_{i*}(t,x;W^i,r;a,p,X) \leq \omega_i(\alpha|x-y|^2 + |x-y|), \tag{$A3'$}$$

$$F_i^*(t,y;W^i,r;a,p,Y) - F_i^*(t,x;W^i,r;a,p,X) \le \omega_i(\alpha|x-y|^2 + |x-y|), \qquad (A3'')$$

(A4')

$$|F_{i*}(t,x;W^i,r;a,p,X) - F_{i*}(t,x;V^i,r;a,p,X)| \le \sum_{j\ne i} L_{ij}|W_j - V_j|, \qquad (A4')$$

$$|F_i^*(t,x;W^i,r;a,p,X) - F_i^*(t,x;V^i,r;a,p,X)| \le \sum_{j\ne i} L_{ij}|W_j - V_j|, \qquad (A4'')$$

For instance, the comparison does still hold if $F_i, i = 1,\cdots,m$ satisfy (A1),(A2), (A3') and (A4') or (A1),(A2), (A3") and (A4") or (A1'),(A2'), (A3) and (A4') etc..

4 On the coupled viscosity sub- and super-solutions

Considering the coupled viscosity sub- and super-solutions in strong sense to (1), we have the following theorem.

Theorem 7. *Suppose that F is degenerate parabolic and locally bounded and satisfies* **(A1)**,**(A2)** *and* **(A4)**. *Let U and V be viscosity sub- and super-solution of (1), respectively, satisfying the condition* **(C)** *and*

$$U_*(t,x) > -\infty, \qquad V^*(t,x) < +\infty,$$

$$d = \max_{1\le i\le m} \sup_{(t,x)\in Q} |V_i(t,x) - U_i(t,x)| < +\infty.$$

Suppose also that

$$\hat{U}(t,x) = U(t,x) - h(t), \quad \hat{V}(t,x) = V(t,x) + h(t)$$

where $h(t) \in (C^1[0,T])^m$ is a solution for the system of ordinary differential equations

$$\begin{cases} \beta_i h_i'(t) - \gamma_i h_i(t) - \sum_{j\ne i} L_{ij}(h_j(t) + d) = 0 \\ h_i(0) = d \end{cases} \quad i = 1,\cdots,m \qquad (35)$$

Then, $(\hat{U}, \hat{V})$ is a coupled viscosity sub- and super-solution in strong sense to (1).

Proof From the theory of ordinary differential equations(see [22] and [23]), there exists a solution of (35) $h(t) \in (C^1[0,T])^m$ and all its components are positive.

Let $\hat{a}_i = a_i - h_i'(t)$ and $\hat{b}_i = b_i + h_i'(t)$. We observe that

$$\mathcal{P}_Q^{2,+}\hat{U}_i^*(t,x) = \Big\{(\hat{a}_i, p_i, X_i)\Big|(a_i, p_i, X_i) \in \mathcal{P}_Q^{2,+}U_i^*(t,x)\Big\}$$

$$\mathcal{P}_Q^{2,-}\hat{V}_{i*}(t,x) = \Big\{(\hat{b}_i, q_i, Y_i)\Big|(b_i, q_i, Y_i) \in \mathcal{P}_Q^{2,-}V_{i*}(t,x)\Big\}.$$

For any continuous function ϕ between $\hat{U}$ and $\hat{V}$, we observe that

$$a_i - \hat{a}_i = \hat{b}_i - b_i = h_i'(t)$$

$$|\phi_i(t,x) - U_i^*(t,x)| \le h_i(t) + d, \; |\phi_i(t,x) - V_{i*}(t,x)| \le h_i(t) + d, \quad \text{on } \bar{Q}$$

Then, by **(A1)**,**(A2)** and **(A4)**,

$$F_{i*}(t,x;U^*;a,p,X) - F_{i*}(t,x;\phi^i,\hat{U}_i^*;\hat{a},p,X)$$

$$\ge \beta_i h_i'(t) - \gamma_i h_i(t) - \textstyle\sum_{j\neq i} L_{ij}\Big(h_j(t) + d\Big) \ge 0,$$

$$\forall (t,x) \in Q, \quad (a,p,X) \in \mathcal{P}_Q^{2,+}U_i^*(t,x),$$

and

$$F_i^*(t,x;\phi^i,\hat{V}_{i*};\hat{b},q,Y) - F_i^*(t,x;V_*;b,q,Y)$$

$$\ge \beta_i h_i'(t) - \gamma_i h_i(t) - \textstyle\sum_{j\neq i} L_{ij}\Big(h_j(t) + d\Big) \ge 0,$$

$$\forall (t,x) \in Q, \quad (b,q,Y) \in \mathcal{P}_Q^{2,-}V_{i*}(t,x).$$

Due to the fact that U and V are viscosity sub- and super-solutions to (1), respectively, we have that

$$F_{i*}(t,x;\phi^i,\hat{U}_i^*;\hat{a},p,X) \le 0$$

$$F_i^*(t,x;\phi^i,\hat{V}_{i*};\hat{b},q,Y) \ge 0,$$

$$\forall (t,x) \in Q,$$

$$(\hat{a},p,X) \in \mathcal{P}_Q^{2,+}\hat{U}_i^*(t,x) \quad (\hat{b},q,Y) \in \mathcal{P}_Q^{2,-}\hat{V}_{i*}(t,x).$$

Meanwhile, (2) holds obviously for $\hat{U}$ and $\hat{V}$. Thus, the theorem is proved.

The following theorem on the coupled viscosity sub- and super-solution can be shown similarly.

Theorem 8. *Suppose that F is degenerate parabolic, locally bounded and quasi-monotone. and* **(A1)**,**(A2)** *and* **(A4)** *hold. Let U and V be a viscosity sub- and a super-solution of (1), respectively, and*

$$U_*(t,x) > -\infty, \qquad V^*(t,x) < +\infty,$$

$$d = \max_{1\le i\le m} \sup_{(t,x)\in Q} |V_i(t,x) - U_i(t,x)| < +\infty.$$

Suppose also that

$$\hat{U}(t,x) = U(t,x) - h(t), \quad \hat{V}(t,x) = V(t,x) + h(t)$$

where $h(t) \in (C^1[0,T])^m$ is a solution of (35). Then, $(\hat{U},\hat{V})$ is a coupled viscosity sub- and super-solution (1).

5 Examples

Example 1. (weakly coupled parabolic systems)

$$\begin{cases} F_i(t,x;u_i,\partial_t u_i, Du_i, D^2u_i) + \sum_{j=1}^m c_{ij}(t,x)u_j = 0, & (t,x)\in Q \\ u_i(t,x) = 0 & (t,x)\in (0,T)\times\partial\Omega \\ u_i(0,x) = \phi_i(x) & x\in\Omega \\ \qquad i = 1,\cdots,m \end{cases} \tag{36}$$

where $F_i (i = 1, \cdots, m)$ are degenerate parabolic. Here $c_{ij}(t,x)$ only are required to be continuous and bounded or not to change their signs. However, they are required in [3],[15] to satisfy

$$\sum_{j=1}^{m} c_{ij} \geq 0 \text{ or } c_{ij} \leq 0 \ (j \neq i) \text{ in } Q$$

It is more stronger.

Suppose that $\phi_i(x) \geq 0$ and are continuous on $\bar{\Omega}$ and

$$F_i(t,x;0,0,0,0) = 0, \text{ and } F_i(t,x,M_i,0,0,0) > 0$$

where $M_i \geq \sup_{x \in \Omega} \phi_i(x)$, then $U = (0, \cdots, 0)$ and $V = (M_1, \cdots, M_m)$ are constant viscosity sub- and super-solutions of (36), respectively. There is a unique viscosity solution for (36), by Theorems 1,2 and 8 if $c_{ij}(t,x)$ do not change their signs, i.e., (36) is quasi-monotone; or by Theorems 6 and 7 if $c_{ij}(t,x)$ are continuous and bounded.

Example 2. (system of Bellman equations)

$$\begin{cases} \sup_{\alpha \in \mathcal{A}} \{\mathcal{L}_i^\alpha u_i + \sum_{j=1}^m c_{ij}^\alpha u_j - f_i^\alpha(t,x)\} = 0, & (t,x) \in Q \\ u_i(t,x) = 0 & (t,x) \in (0,T) \times \partial\Omega \\ u_i(0,x) = \phi_i(x) & x \in \Omega \\ \qquad i = 1, \cdots, m \end{cases} \tag{37}$$

where $\{\mathcal{L}_i^\alpha\}$ is a family of linear parabolic operators with bounded measurable coefficients

$$\mathcal{L}_i^\alpha u = \frac{\partial u_i}{\partial t} - \sum_{k,l=1}^{n} a_{kl}^\alpha \frac{\partial^2 u_i}{\partial x_k \partial x_l} + \sum_{k=1}^{n} b_k^\alpha \frac{\partial u_i}{\partial x_k}$$

in which (a_{kl}^α) have eigenvalues in $[\lambda, \Lambda](0 < \lambda \leq \Lambda)$ for each $(t,x) \in Q$ and $\alpha \in \mathcal{A}$, $c_{ij}^\alpha(t,x)$ and $f_i^\alpha(t,x)$ are continuous and bounded.

If there are a viscosity sub-solution $U(t,x)$ and a viscosity super-solution $V(t,x)$ for (37)(which could be attained according to the specific conditions, e.g., the following paragraph), then there is unique viscosity solution by Theorems 6 and 7, or by Theorems 1, 2 and 8 as all $c_{ij}^{\alpha}(t,x)$ $(\alpha \in \mathcal{A})$ have the same unchanging sign for every $i, j = 1, \cdots, m$.

If all $c_{ij}^{\alpha}(t,x)$ and $f_i^{\alpha}(t,x)$, $(i, j = 1, \cdots, m,\ \alpha \in \mathcal{A})$ are non-negative on Q, there is a λ large enough such that

$$\underline{c}_{ii} + \lambda - \sum_{j \neq i} \overline{c}_{ij} - \overline{f}_i \geq 0,$$

where

$$\underline{c}_{ij} = \inf_{\alpha \in \mathcal{A}} \inf_{(t,x) \in Q} c_{ij}^{\alpha}(t,x), \quad \overline{c}_{ij} = \sup_{\alpha \in \mathcal{A}} \sup_{(t,x) \in Q} c_{ij}^{\alpha}(t,x), \quad \overline{f}_i = \sup_{\alpha \in \mathcal{A}} \sup_{(t,x) \in Q} f_i^{\alpha}(t,x),$$

because $c_{ij}^{\alpha}(t,x)$ and $f_i^{\alpha}(t,x)$, $(i, j = 1, \cdots, m,\ \alpha \in \mathcal{A}$ are bounded. Let

$$U = (-e^{\lambda t}, \cdots, -e^{\lambda t}) \quad V = (e^{\lambda t}, \cdots, e^{\lambda t})$$

It is observed that (U, V) is a coupled viscosity sub- and super-solution of (37).

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