SOME REMARKS RELATED TO DE GIORGI'S CONJECTURE

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ABSTRACT. For several classes of functions including the special case $f(u) = u - u^3$, we obtain boundedness and symmetry results for solutions of the problem $-\Delta u = f(u)$ defined on R^n. Our results complement a number of recent results related to a conjecture of De Giorgi.

1. INTRODUCTION

In this paper, we make several observations related to the following conjecture of De Giorgi [dG]: If u is a solution of the scalar Ginzburg-Landau equation

$$(1.1) \qquad \Delta u + u(1 - u^2) = 0 \text{ on } R^n$$

such that $|u| \le 1$ and $\partial_n u > 0$ on R^n, and

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1, \ \forall x' \in R^{n-1},$$

then all level sets of u are hyperplanes, at least for $n \le 8$. Here $\partial_n u$ denotes the partial derivative of u with respect to x_n, the last component of x, and x' denotes the first $n-1$ components of x.

When $n = 2$, this conjecture was completely resolved by Ghoussoub and Gui [GG]. When $n = 3$, it was very recently proved by Ambrosio and Cabre [AC]. Both solutions of the conjecture are based on a Liouville-type theorem due to Berestycki, Caffarelli and Nirenberg [BCN2]. The first partial answer to the De Giorgi conjecture is from the work of 1980 by Modica and Motola [MM]. In 1985, Modica found a pointwise gradient bound for all bounded solutions. This estimate was further generalized by Caffarelli, Garofalo and Segala [CGS] to more general nonlinear partial differential equations which include the p-Laplacian. Under more assumptions on the solutions, for example, if $u(x) = u(x', x_n) \to \pm 1$ as $x_n \to \pm\infty$ holds uniformly for $x' \in R^{n-1}$, the conclusion of this conjecture was confirmed in [BBG], [BHM] and [F] for any $n \ge 2$. The conjecture in its original form, however, remains open for $n > 3$. We refer to [AAC] for a fuller account of the history and progress about this conjecture.

All these previous studies obtain results which are valid not only for the special nonlinearity appearing in (1.1), but also for much more general nonlinearities. In this

paper, we show that for some narrower classes of nonlinearities, but still including the one in (1.1), several further results can be obtained.

Our first observation comes from a simple application of the techniques in [DM], notably the use of boundary blow-up solutions. It implies the following result as a special case.

Theorem 1.1. *Suppose* $u \in C^2(R^n)$ *is a solution of (1.1). Then* $u \in L^\infty(R^n)$. *Moreover,* u *is either a constant (hence,* $u \equiv 1$, *or* $u \equiv 0$, *or* $u \equiv -1$*), or* u *changes sign and satisfies* $|u| < 1$ *on* R^n.

Therefore, the condition that $|u| \leq 1$ in De Giorgi's conjecture is unnecessary. This point has already been observed by Farina [F], but his conclusion does not seem to include those nonlinearities covered by our general result (see Theorem 2.1 below).

Our second observation is motivated by some recent results in [BBG], [BHM] and [F]. One of the main results in these papers says that if u solves (1.1) and satisfies $|u| \leq 1$ and

(1.2) $u(x', x_n) \to \pm 1$ as $x_n \to \pm\infty$ uniformly in $x' \in R^{n-1}$,

then $u(x) = u(x_n)$. Farina [F] further observes that, under the restrictions already put on the nonlinearity, a result in [BCN1] implies that (1.2) is equivalent to

$$x_n u(x', x_n) > 0, \ \forall \, |x_n| > R, \ \forall x' \in R^{n-1},$$

for some $R > 0$. Or equivalently, the zero set of u, denoted by $u^{-1}(0)$, lies between two parallel hyperplanes $\{x_n = -R\}$ and $\{x_n = R\}$. This somehow relates the above result to Theorem 2 in [BBG], which improves a result in [MM] and asserts that if a solution u to (1.1) satisfies $|u| \leq 1$ and $\partial_n u > 0$ and that all its level sets are graphs of Lipschitzian functions of x', then u is a function of one variable.

All these above mentioned results are proved for rather general nonlinearities. Under some further restrictions, we have a result which implies the following.

Theorem 1.2. *Suppose that* $u \in C^2(R^n)$ *solves (1.1) on* R^n. *If* $u^{-1}(0)$ *lies on one side of a hyperplane and touches that hyperplane, i.e., there exists* $\nu \in S^{n-1}$ *and* $x_0 \in u^{-1}(0)$ *such that* $\nu \cdot (x - x_0) \geq 0$ *for all* $x \in u^{-1}(0)$, *then* u *depends on one variable only (in the direction of* ν*).*

Note that, in Theorem 1.2, a condition of the type $\partial_n u > 0$ is not required. Moreover, no regularity on $u^{-1}(0)$ is assumed. In our proof of Theorem 1.2, the oddness of the nonlinearity will be employed.

We end this introduction with the statement of a simple consequence of Theorem 1.2. Assume that M is the set of one-dimensional solutions of (1.1). Then M can be explicitly expressed in the following way. Let $v = v(t)$ be the function $tanh(t/\sqrt{2})$, which is, up to a translation, the unique solution of the problem

$$h'' = h^3 - h, \ h(\pm\infty) = \pm 1.$$

For $a \in S^{n-1}$ and $c \in R$, we let

$$u_{a,c}(x) = tanh((a.x - c)/\sqrt{2})$$

Then

$$M = \{u_{a,c}; a \in S^{n-1}, c \in R\}.$$

Corollary 1.3. *Assume that $u \in C^2(R^n)$ solves (1.1) and the C^0-distance of u to M is less than 1, i.e.,*

$$\inf_{v \in M} \sup_{x \in R^n} |u(x) - v(x)| < 1.$$

Then $u \in M$.

Note that, M is a closed set in the C^0-topology, and if we denote the C^0 distance function by $dist_0$, then, by Theorem 1.1, for any solution u of (1.1), $dist_0(u, M) \leq 2$. In particular, if $u = 0$, then $dist_0(u, M) = 1$, and if $u = 1$ or $u = -1$, then $dist_0(u, M) = 2$.

The rest of this paper is organized as follows. In section 2, we present a general result which implies Theorem 1.1. Theorem 1.2 will be discussed in section 3, again as a special case of a general result.

2. GLOBAL BOUNDEDNESS AND RELATED RESULTS

In this section, we prove a general result which contains Theorem 1.1 as a special case. We consider the problem

$$(2.1) \qquad\qquad -\Delta u = f(u), \quad x \in R^n,$$

where the real valued C^1 function f is assumed to satisfy

$$(2.2) \qquad \begin{cases} f(0) = f(1) = f(-1) = 0, \\ uf(u) > 0 \quad \text{for } 0 < |u| < 1, \ uf(u) < 0 \quad \text{for } |u| > 1, \end{cases}$$

and for some large constant $M > 1$

$$(2.3) \qquad \begin{cases} \underline{\lim}_{u \to 0} \frac{f(u)}{u|u|^{2/n}} \in (0, \infty], \\ uf(u) \leq ug(|u|) < 0 \text{ for } |u| > M, \\ g(u) \text{ is decreasing in } [M, \infty) \text{ and } \int_M^\infty \left[\int_M^u |g(s)|ds \right]^{-1/2} du < \infty. \end{cases}$$

It is easily checked that if $p > 1$, then $f(u) = u(1 - |u|^p)$ satisfies both (2.2) and (2.3).

Theorem 2.1. *Suppose f is C^1 and satisfies (2.2) and (2.3). Let $u \in C^2(R^n)$ be a solution of (2.1). Then the conclusions in Theorem 1.1 hold.*

Proof. From Theorem 5.2 in [DM] and our assumptions on f and the Harnack inequality, we know that any solution u which does not change sign in R^n must be constant, that is, $u \equiv 1$ or $u \equiv -1$ or $u \equiv 0$. Therefore we only need to consider solutions of (2.1) which changes sign in R^n.

Suppose now u is a sign-changing solution of (2.1). We want to show that $|u| < 1$ on R^n. Let us first observe that it suffices to show $|u| \leq 1$ in R^n. Indeed, if $|u(x_0)| = 1$, say $u(x_0) = -1$, then, $w := u + 1$ satisfies

$$-\Delta w = c(x)w, \ w \geq 0, \ w(x_0) = 0,$$

where $c \in L^\infty(R^n)$ is given by $c(x) = [f(u(x)) - f(-1)]/(u(x) + 1)$ when $u(x) \neq -1$, and $c(x) = f'(-1)$ otherwise. Hence it follows from the Harnack inequality that $w \equiv 0$, contradicting our assumption that u changes sign.

To show $|u| \leq 1$ on R^n, we adapt the argument used in the proof of Theorem 5.2 in [DM].

Let $h(u) = -g(u + M)$ for $u \geq 0$ and $h(u) = -g(M)$ for $u < 0$. Then h is positive and due to (2.3), we can use the proof of Theorem 1 of Keller [Ke] to conclude that the problem

$$\Delta u = h(u), \ u|_{\partial B} = \infty$$

has a unique *positive* solution u_∞, where B stands for a ball centered at the origin with *small* radius (the proof in [Ke] actually shows that $\min_{R^n} u_\infty \to \infty$ as the radius of B goes to 0). It follows that $v_\infty = u_\infty + M$ is a positive solution to

$$-\Delta v = g(v), \ v|_{\partial B} = \infty.$$

We claim that $u \leq c := \min_B v_\infty(x)$ on R^n. Otherwise, we can find $x_0 \in R^n$ such that $u(x_0) > c$. Define $v(x) = v_\infty(x - x_0)$. We find that the set $\{x \in B(x_0) : u(x) > v(x)\}$ has a component Ω whose closure lies entirely in the open ball $B(x_0) = \{x : x - x_0 \in B\}$. On Ω, we have $u(x) > v(x) \geq c > M$ and $\Delta u + g(u) \geq 0 = \Delta v + g(v)$. Moreover, $u = v$ on $\partial \Omega$. As $g(u)$ is decreasing for $u > M$, from

$$\Delta(u - v) + c(x)(u - v) \geq 0, \ c(x) = [g(u) - g(v)]/(u - v) \leq 0 \text{ on } \Omega$$

and the maximum principle, we deduce that $u \equiv v$ in Ω. This contradiction shows that we must have $u \leq c$ on R^n.

Applying the above argument to $w = -u$ which satisfies

$$-\Delta w = g(w), \ g(w) = -f(-w),$$

we deduce that $u \geq -c$ on R^n. Therefore we have

$$-c \leq u(x) \leq c, \ \forall x \in R^n.$$

Let u_c and u_{-c} denote the unique solution of

(2.4) $u' = f(u), \ u(0) = u_0,$

with $u_0 = c$ and $u_0 = -c$, respectively. Then it follows from elementary analysis that $u_c(t) \to 1$ and $u_{-c}(t) \to -1$ as $t \to +\infty$. On the other hand, u, u_c and u_{-c} are all bounded solutions of the parabolic problem

$$u_t - \Delta u = f(u).$$

Since $u_c(0) \geq u(x) \geq u_{-c}$ on R^n, by the parabolic maximum principle and the boundedness of u, u_c and u_{-c} ([Fr, Theorem 9, page 43]), we conclude that $u_{-c}(t) \leq u(x) \leq u_c(t)$ for all $t > 0$. Letting $t \to \infty$, we obtain $-1 \leq u(x) \leq 1$, as required. This finishes our proof of Theorem 2.1.   □

Remark 2.2. *As was remarked in [DM], the behavior of the function f near infinity described by (2.3) is necessary for Theorem 2.1 to hold true. One can find functions f which satisfy all the other conditions in (2.2) and (2.3) except the restriction near infinity in (2.3), such that, (2.1) has unbounded entire positive and negative solutions. The interested reader may check [DM] for more details.*

3. Odd nonlinearity and related results

In this section, we prove a generalization of Theorem 1.2.

We first recall two lemmas. The first one is from [BHM] which is a simple consequence of Lemma 2.1 in [BCN1].

Lemma 3.1. *Let f be a Lipschitz continuous function, non-increasing on $[-1, -1+\delta]$ and on $[1-\delta, 1]$ for some $\delta > 0$. Assume that u_1, u_2 are solutions of (2.1) over some open connected set $\Omega \subset R^n$ and $|u_i| \leq 1$ $(i = 1, 2)$. Assume further that $u_2 \geq u_1$ on $\partial\Omega$ and either $u_2 \geq 1 - \delta$ or $u_1 \leq -1 + \delta$ in Ω. Then $u_2 \geq u_1$ in Ω provided that $R^n \setminus \Omega$ contains an infinite open connected cone.*

The second is Lemma 2.4 in [F] which follows from Lemmas 3.2 and 3.3 in [BCN1].

Lemma 3.2. *Let f be a Lipschitz continuous function which is positive over $(0, 1)$, and satisfies $f(1) = 0$, $f(t) \geq \delta_0 t$ on $(0, t_0]$ for some small $\delta_0 > 0$ and $t_0 > 0$. If u is C^2 on the half plane $\Sigma_M := \{x \in R^n : x_n > M\}$ and satisfies*

$$\Delta u + f(u) \leq 0, \ 0 < u \leq 1 \ \ on \ \Sigma_M,$$

then $u(x', x_n) \to 1$ uniformly in $x' \in R^{n-1}$ as $x_n \to +\infty$.

Theorem 3.3. *Suppose f is Lipschitz continuous and satisfies*

$$f(-1) = f(0) = f(1) = 0, \ tf(t) > 0 \ \ when \ 0 < |t| < 1,$$

and for some small positive constants δ_0, t_0 and δ,

$$f(t)/t \geq \delta_0 \ \ when \ 0 < |t| < t_0,$$

$$f \ is \ non\text{-}increasing \ on \ [-1, -1+\delta] \cup [1-\delta, 1].$$

Furthermore, assume that $f(t)$ is odd in t. Then the statement in Theorem 1.2 holds for any solution $u \in C^2(R^n)$ of (2.1) satisfying $|u| \leq 1$.

Proof. Through a rotation and translation, we may assume that the hyperplane is given by $x_n = 0$, $u(0) = 0$ and $u^{-1}(0) \subset \{x : x_n \leq 0\}$. We may assume that $u(x', x_n) > 0$, $\forall x' \in R^{n-1}$, $\forall x_n > 0$; the other possibility that $u(x', x_n) < 0$, $\forall x' \in R^{n-1}$, $\forall x_n > 0$ can be handled analogously.

For $\tau \geq 0$, let us define

$$u_\tau(x', x_n) = -u(x', 2\tau - x_n).$$

Since f is odd, we easily see that

$$-\Delta u_\tau = f(u_\tau).$$

Clearly

$$u|_{\{x_n=\tau\}} \geq 0 \geq u_\tau|_{\{x_n=\tau\}}.$$

We want to show that for every $\tau \geq 0$, $u \geq u_\tau$ on the half space $\{x : x_n \geq \tau\}$.

Since $u(x) > 0$ when $x_n > 0$, it follows from Lemma 3.2 that $u(x', x_n) \to 1$ as $x_n \to +\infty$ uniformly in $x' \in R^{n-1}$. Therefore, for large τ we can apply Lemma 3.1 to

$$\Omega := \{x : x_n > \tau\}$$

to conclude that $u \geq u_\tau$ on Ω. Now define

$$\tau_0 = \inf\{\tau \in [0,\infty) : u(x', x_n) \geq u_\tau(x', x_n), \forall x' \in R^{n-1}, \forall x_n \geq \tau\}.$$

Claim: $\tau_0 = 0$.

Otherwise, $\tau_0 > 0$ and $u(x) \geq u_{\tau_0}(x)$ on the set $\Omega_0 := \{x : x_n \geq \tau_0\}$. Clearly $w := u - u_{\tau_0}$ satisfies

$$-\Delta w = c(x)w, \ w \geq 0, \ \forall x \in \Omega_0,$$

where $c \in L^\infty(\Omega_0)$. Since $u > 0 > u_{\tau_0}$ on $\partial\Omega_0$, by the definition of τ_0, we have two possibilities:

(a) $w(x_0) = 0$ for some $x_0 \in \Omega_0$, or
(b) $w(x) > 0$ in Ω_0 and $w(z_k) \to 0$ for some $z_k \in \overline{\Omega}_0$ with $|z_k| \to \infty$.

If case (a) occurs, then the Harnack inequality forces $w \equiv 0$ on Ω_0, which is impossible as $w > 0$ on $\partial\Omega_0$.

If (b) occurs, we set $u_k(x) = u(x + z_k)$. By standard elliptic estimates, up to extraction of a subsequence, u_k converges in $C^2_{loc}(R^n)$ to a solution u^* of (2.1) as $k \to \infty$. Moreover,

$$v := u^* - u^*_{\tau_0}$$

satisfies $v(0) = 0$ and

$$-\Delta v = c^*(x)v, \ v \geq 0, \ \forall x \in \Omega^*,$$

where $c^* \in L^\infty(\Omega^*)$ and $\Omega^* = \{x : x_n > \tau^*\}$ with $\tau^* \in [-\infty, 0]$ determined by (passing to a subsequence when necessary)

$$\tau^* = -\lim_{k\to\infty} d(z_k, \partial\Omega_0).$$

If $0 \in \Omega^*$ then we obtain from the Harnack inequality that $v \equiv 0$ on Ω^*, i.e.,

(3.1) $$u^*(x', x_n) = -u^*(x', 2\tau_0 - x_n), \forall x' \in R^{n-1}, \forall x_n > \tau^*.$$

Taking $x_n = \tau_0$ we deduce $u^*(x', \tau_0) = 0$. This implies that $\{d(z_k, \partial\Omega_0)\}$ is bounded, for otherwise, due to $u(x', x_n) \to 1$ uniformly in $x' \in R^{n-1}$ as $x_n \to +\infty$, we would have $u^* \equiv 1$. The boundedness of $\{d(z_k, \partial\Omega_0)\}$ and the fact that $u(x', x_n) \to 1$ uniformly in $x' \in R^{n-1}$ as $x_n \to +\infty$ imply $u^*(x', x_n) \to 1$ uniformly in $x' \in R^{n-1}$ as $x_n \to +\infty$. This together with (3.1) implies that $u^*(x', x_n) \to -1$ uniformly in x' as $x_n \to -\infty$. Hence we can use Theorem 1 in [BHM] to conclude that $u^*(x) = u^*(x_n)$ and is increasing in x_n. On the other hand, since $u_k(0) = u(z_k) > 0$, we have $u^*(0) \geq 0$, a contradiction to the monotonicity of $u^*(x_n)$ and $u^*(\tau_0) = 0$.

If $0 \in \partial\Omega^*$, we necessarily have $d(z_k, \partial\Omega_0) \to 0$ and hence $\tau^* = 0, \Omega^* = \{x : x_n > 0\}$. As before, this implies that $u^*(x', x_n) \to 1$ uniformly in x' as $x_n \to +\infty$. Moreover, for any $\eta \geq -\tau_0$, since $u_k(x', \eta) = u((x', \eta) + z_k) \geq 0$, we deduce

$$u^*(x', \eta) \geq 0, \ \forall x' \in R^{n-1}.$$

In particular,

(3.2) $$u^*(0, x_n) \geq 0, \ \forall x_n \geq -\tau_0.$$

As $v(0) = 0$, we have $u^*(0) = -u^*(0, 2\tau_0)$. Therefore we necessarily have $u^*(0) = u^*(0, 2\tau_0) = 0$. In view of (3.2), the function $g(t) := u^*(0, t)$ has a local minimum at $t = 0$ and at $t = 2\tau_0$. Therefore, $g'(0) = g'(2\tau_0) = 0$. This implies that $\partial_n v(0) = 0$. Since v satisfies

$$-\Delta v = c^*(x)v, \ v \geq 0, \ \forall x \in \Omega^*, \ v(0) = 0, \ 0 \in \partial\Omega^*,$$

an application of the maximum principle and the Hopf boundary lemma gives $v \equiv 0$, i.e., $u^*(x', x_n) = -u^*(x', 2\tau_0 - x_n)$ for all $x' \in R^{n-1}$ and all $x_n \geq 0$. We can now argue as in the case that $0 \in \Omega^*$ to conclude that $u^*(x) = u^*(x_n)$ and is increasing in x_n. But this is in contradiction with our earlier observation that $u^*(0) = u^*(2\tau_0)$. This proves our Claim.

From $\tau_0 = 0$ we obtain $u(x', x_n) \geq -u(x', -x_n)$ for all $x' \in R^{n-1}$ and $x_n \geq 0$. Hence $w(x) := u(x', x_n) + u(x', -x_n)$ satisfies

$$-\Delta w = c(x)w, \ w \geq 0 \text{ on } \Omega := \{x : x_n > 0\}, w(0) = 0,$$

where $c \in L^\infty(\Omega)$. By Harnack's inequality and the Hopf boundary lemma, we have either $w \equiv 0$ or $\partial_n w(0) > 0$. A direct calculation yields

$$\partial_n w(0) = \partial_n u(0) - \partial_n u(0) = 0.$$

Hence we must have $w \equiv 0$ on Ω, i.e., $u(x', x_n) = -u(x', -x_n)$ for all $x' \in R^{n-1}$ and all $x_n > 0$. Recall that we have $u(x', x_n) \to 1$ uniformly in $x' \in R^{n-1}$ as $x_n \to +\infty$. The above identity gives $u(x', x_n) \to -1$ uniformly in x' as $x_n \to -\infty$. Therefore we can use Theorem 1 of [BHM] and conclude. The proof of Theorem 4.2 is complete. $\qquad\square$

Remark 3.4. *Clearly, if f satisfies also the conditions in Theorem 2.1, then $|u| \leq 1$ is satisfied.*

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