On the Inverse of Parabolic Systems of Partial Differential Equations of General Form in an Infinite Space–Time Cylinder[1]

Thomas Krainer[2] and Bert–Wolfgang Schulze

Institut für Mathematik
Universität Potsdam
Postfach 60 15 53
D–14415 Potsdam
Germany

krainer@math.uni-potsdam.de
schulze@math.uni-potsdam.de

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Abstract

We consider general parabolic systems of equations on the infinite time interval in case of the underlying spatial configuration is a closed manifold. The solvability of equations is studied both with respect to time and spatial variables in exponentially weighted anisotropic Sobolev spaces, and existence and maximal regularity statements for parabolic equations are proved. Moreover, we analyze the long-time behaviour of solutions in terms of complete asymptotic expansions.

These results are deduced from a pseudodifferential calculus that we construct explicitly. This algebra of operators is specifically designed to contain both the classical systems of parabolic equations of general form and their inverses, parabolicity being reflected purely on symbolic level. To this end, we assign $t = \infty$ the meaning of an anisotropic conical point, and prove that this interpretation is consistent with the natural setting in the analysis of parabolic PDE. Hence, major parts of this work consist of the construction of an appropriate anisotropic cone calculus of so-called Volterra operators.

In particular, which is the most important aspect, we obtain the complete characterization of the microlocal and the global kernel structure of the inverse of parabolic systems in an infinite space–time cylinder. Moreover, we obtain perturbation results for parabolic equations from the investigation of the ideal structure of the calculus.

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Contents

Chapter 6

Volterra cone calculus

6.1 Green operators

6.1.1 Remark. Throughout this chapter we again employ the notations from Notation 3.1.1 with the corresponding data fixed on the manifold X and the vector bundles E and F.

6.1.2 Definition. a) Let $\Theta = (\theta, 0]$ with $-\infty \le \theta < 0$, and let $P \in \mathrm{As}\big((\gamma,\Theta), C^\infty(X,F)\big)$ and $Q \in \mathrm{As}\big((-\gamma,\Theta), C^\infty(X,E)\big)$ be asymptotic types. Then an operator

$$G \in \mathcal{L}\big(\operatorname*{ind-lim}_{s,t,\delta\in\mathbb{R}} \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta, \operatorname*{proj-lim}_{s,t,\delta\in\mathbb{R}} \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta\big)$$

is called a *Green operator* with respect to the asymptotic types P and Q, if G and its formal adjoint G^* with respect to the $r^{-\frac{n}{2}}L^2$-inner product induce continuous operators

$$G : \operatorname*{ind-lim}_{s,t,\delta\in\mathbb{R}} \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta \longrightarrow \mathcal{S}^\gamma_P(X^\wedge,F),$$
$$G^* : \operatorname*{ind-lim}_{s,t,\delta\in\mathbb{R}} \mathcal{K}^{(s,t),-\gamma;\ell}(X^\wedge,F)_\delta \longrightarrow \mathcal{S}^{-\gamma}_Q(X^\wedge,E).$$

The space of all Green operators is denoted by $C_G(X^\wedge,(\gamma,\Theta);E,F)$. If indication of the concrete asymptotic types is necessary we emphasize this by writing $C_G(X^\wedge,(\gamma,\Theta);E,F)_{P,Q}$.

b) A Green operator $G \in C_G(X^\wedge,(\gamma,\Theta);E,F)$ is called a *Volterra Green operator* provided that one of the following equivalent conditions is fulfilled:

- G restricts to continuous operators

$$G : \mathcal{H}_0^{(s,t),\gamma;\ell}((0,r_0]\times X,E) \longrightarrow \mathcal{H}_0^{(s,t),\gamma;\ell}((0,r_0]\times X,F)$$

for every (some) $s,t \in \mathbb{R}$ and every $r_0 \in \mathbb{R}_+$.
- For every $r_0 \in \mathbb{R}_+$ we have $(Gu)(r) \equiv 0$ for $r > r_0$ for all $u \in C_0^\infty(\mathbb{R}_+, C^\infty(X,E))$ such that $u(r) \equiv 0$ for $r > r_0$.
- For $u \in L^{2,\gamma-\frac{n}{2}}(\mathbb{R}_+, L^2(X,E))$ and $v \in L^{2,-\gamma-\frac{n}{2}}(\mathbb{R}_+, L^2(X,F))$ such that $\mathrm{supp}(u) < \mathrm{supp}(v)$ we have $\langle Gu, v\rangle_{r^{-\frac{n}{2}}L^2} = 0$.

The space of all Volterra Green operators is denoted by $C_{G,V}(X^\wedge,(\gamma,\Theta);E,F)$, respectively $C_{G,V}(X^\wedge,(\gamma,\Theta);E,F)_{P,Q}$ for the space of Volterra Green operators with respect to the asymptotic types P and Q.

6.1.3 Remark. From Definition 6.1.2 we conclude that the class of (Volterra) Green operators is independent of the particular anisotropy $\ell \in \mathbb{N}$. Moreover, it forms an operator algebra, i. e., if H is another vector bundle then the composition induces a well-defined mapping

$$C_{G(,V)}(X^\wedge,(\gamma,\Theta);F,H)\times C_{G(,V)}(X^\wedge,(\gamma,\Theta);E,F) \longrightarrow C_{G(,V)}(X^\wedge,(\gamma,\Theta);E,H).$$

The class of Green operators is closed with respect to taking formal adjoints with respect to the $r^{-\frac{n}{2}}L^2$-inner product, i. e., the mapping

$$^* : C_G(X^\wedge,(\gamma,\Theta);E,F) \longrightarrow C_G(X^\wedge,(-\gamma,\Theta);F,E)$$

is well-defined.

6.1.4 Proposition. *An operator $G : C_0^\infty(\mathbb{R}_+, C^\infty(X,E)) \longrightarrow \mathcal{D}'(\mathbb{R}_+,\mathcal{D}'(X,F))$ belongs to $C_G(X^\wedge,(\gamma,\Theta);E,F)_{P,Q}$ if and only if G can be represented both as*

$$G(u) = \sum_{j=1}^\infty \lambda_j \langle u, x_j\rangle_{r^{-\frac{n}{2}}L^2}\, s_j,$$
$$G(u) = \sum_{j=1}^\infty \tilde{\lambda}_j \langle u, \tilde{s}_j\rangle_{r^{-\frac{n}{2}}L^2}\, \tilde{x}_j,$$

for $u \in C_0^\infty(\mathbb{R}_+, C^\infty(X,E))$, where $(\lambda_j),(\tilde{\lambda}_j) \in \ell^1$, and $(x_j) \in \mathcal{K}^{\infty,-\gamma}(X^\wedge,E)_\infty$, $(s_j) \in \mathcal{S}_P^\gamma(X^\wedge,F)$, $(\tilde{x}_j) \in \mathcal{K}^{\infty,\gamma}(X^\wedge,F)_\infty$, $(\tilde{s}_j) \in \mathcal{S}_Q^{-\gamma}(X^\wedge,E)$ are sequences tending to zero in the corresponding spaces. Here we denote

$$\mathcal{K}^{\infty,\gamma}(X^\wedge,F)_\infty = \bigcap_{s,t,\delta\in\mathbb{R}} \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,F)_\delta,$$

and analogously $\mathcal{K}^{\infty,-\gamma}(X^\wedge,E)_\infty$.

In other words: $G \in C_G(X^\wedge,(\gamma,\Theta);E,F)_{P,Q}$ *if and only if*

$$G \in \big(\mathcal{K}^{\infty,-\gamma}(X^\wedge,E)_\infty \widehat{\otimes}_\pi \mathcal{S}^\gamma_P(X^\wedge,F)\big) \cap \big(\mathcal{S}^{-\gamma}_Q(X^\wedge,E)\widehat{\otimes}_\pi \mathcal{K}^{\infty,\gamma}(X^\wedge,F)_\infty\big).$$

In particular,

$$\begin{aligned} C_G(X^\wedge,(\gamma,\Theta);E,F) &\hookrightarrow \mathcal{S}^{-\gamma}(X^\wedge,E)\widehat{\otimes}_\pi \mathcal{S}^\gamma(X^\wedge,F) \\ &\hookrightarrow \ell^1(\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta, \mathcal{K}^{(s',t'),\gamma;\ell}(X^\wedge,F)_{\delta'}) \end{aligned}$$

for every $s,s',t,t',\delta,\delta' \in \mathbb{R}$.

Proof. By Theorem 4.3.4

$$\begin{aligned} &\{\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_{\frac{n}{2}+\delta}, r^{-\frac{n}{2}}L^2(X^\wedge,E), \mathcal{K}^{(-s,-t),-\gamma;\ell}(X^\wedge,E)_{\frac{n}{2}-\delta}\} \quad \text{and} \\ &\{\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,F)_{\frac{n}{2}+\delta}, r^{-\frac{n}{2}}L^2(X^\wedge,F), \mathcal{K}^{(-s,-t),-\gamma;\ell}(X^\wedge,F)_{\frac{n}{2}-\delta}\} \end{aligned}$$

are Hilbert triples for all $s,t,\delta \in \mathbb{R}$. Moreover, $\mathcal{S}^\gamma_P(X^\wedge,F)$ and $\mathcal{S}^{-\gamma}_Q(X^\wedge,E)$ are nuclear Fréchet spaces which are continuously embedded in the cone Sobolev spaces by Proposition 4.3.9. Let $G \in C_G(X^\wedge,(\gamma,\Theta);E,F)_{P,Q}$. From Proposition 1.3.9 we conclude that G belongs to

$$\begin{aligned} &\bigcap\Big(\big(\mathcal{K}^{(s,t),-\gamma;\ell}(X^\wedge,E)_\delta \widehat{\otimes}_\pi \mathcal{S}^\gamma_P(X^\wedge,F)\big) \cap \big(\mathcal{S}^{-\gamma}_Q(X^\wedge,E)\widehat{\otimes}_\pi \mathcal{K}^{(s',t'),\gamma;\ell}(X^\wedge,F)_{\delta'}\big)\Big) \\ &= \big(\mathcal{K}^{\infty,-\gamma}(X^\wedge,E)_\infty \widehat{\otimes}_\pi \mathcal{S}^\gamma_P(X^\wedge,F)\big) \cap \big(\mathcal{S}^{-\gamma}_Q(X^\wedge,E)\widehat{\otimes}_\pi \mathcal{K}^{\infty,\gamma}(X^\wedge,F)_\infty\big), \end{aligned}$$

where the intersection is taken over all $s,t,s',t',\delta,\delta' \in \mathbb{R}$. The converse is immediate. We have

$$\begin{aligned} &\big(\mathcal{K}^{\infty,-\gamma}(X^\wedge,E)_\infty \widehat{\otimes}_\pi \mathcal{S}^\gamma_P(X^\wedge,F)\big) \cap \big(\mathcal{S}^{-\gamma}_Q(X^\wedge,E)\widehat{\otimes}_\pi \mathcal{K}^{\infty,\gamma}(X^\wedge,F)_\infty\big) \hookrightarrow \\ &\mathcal{S}^{-\gamma}(X^\wedge,E)\widehat{\otimes}_\pi \mathcal{S}^\gamma(X^\wedge,F) \hookrightarrow \ell^1(\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta, \mathcal{K}^{(s',t'),\gamma;\ell}(X^\wedge,F)_{\delta'}) \end{aligned}$$

for all $s,s',t,t',\delta,\delta' \in \mathbb{R}$. This finishes the proof of the proposition. □

6.1.5 Proposition. *Let* $G \in C_G(X^\wedge,(\gamma,\Theta);E,F)$ *and* $\omega,\tilde\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ *be cut-off functions near* $r=0$. *Then we have* $(1-\omega)G(1-\tilde\omega) = op_r(g_\infty)$ *with* $g_\infty \in S^{-\infty}(\mathbb{R}, L^{-\infty}(X;\mathbb{R};E,F))$, *and* $\omega G\tilde\omega = op_M^{\gamma-\frac{n}{2}}(g_0)$ *with* $g_0 \in C_B^\infty(\mathbb{R}_+, L^{-\infty}(X;\Gamma_{\frac{n+1}{2}-\gamma};E,F))$ *such that* $\lim_{r\to 0} g_0(r) = 0$.

For $G \in C_{G,V}(X^\wedge,(\gamma,\Theta);E,F)$ *we even have* $g_\infty \in S^{-\infty}(\mathbb{R}, L_V^{-\infty}(X;\mathbb{H};E,F))$ *and* $g_0 \in C_B^\infty(\mathbb{R}_+, L_V^{-\infty}(X;\mathbb{H}_{\frac{n+1}{2}-\gamma};E,F))$.

Proof. From Proposition 6.1.4 we conclude that

$$\begin{aligned} (1-\omega)G(1-\tilde\omega) &\in \mathcal{S}(\mathbb{R}\times X,E)\widehat{\otimes}_\pi \mathcal{S}(\mathbb{R}\times X,F), \\ \omega G\tilde\omega &\in \mathcal{T}_{-\gamma-\frac{n}{2}}(X^\wedge,E)\widehat{\otimes}_\pi \mathcal{T}_{\gamma-\frac{n}{2}}(X^\wedge,F). \end{aligned}$$

Consider the operator $G_\infty = (1-\omega)G(1-\tilde{\omega})$: We may write

$$\big(G_\infty u\big)(r) = \int_{\mathbb{R}} k_\infty(r,r')u(r')\,dr'$$

for $u \in C_0^\infty(\mathbb{R}_+, C^\infty(X,E))$ with a kernel $k_\infty \in \mathcal{S}(\mathbb{R}\times\mathbb{R}, L^{-\infty}(X;E,F))$. Let $\chi \in \mathcal{S}(\mathbb{R})$ such that $\int_{\mathbb{R}} \chi(\tau)\,đ\tau = 1$, and set $a(r,r',\tau) := e^{-i(r-r')\tau}k(r,r')\chi(\tau)$. Then $a \in S^{-\infty,-\infty}(\mathbb{R}\times\mathbb{R}, L^{-\infty}(X;\mathbb{R};E,F))$ is a double-symbol in the Fourier calculus with global weight conditions from Section 5.4, and we have $G_\infty = \mathrm{op}_r(a)$. Thus we obtain g_∞ as the left-symbol a_L according to Theorem 5.4.3, i. e.,

$$g_\infty(r,\tau) = \iint e^{-ir'\eta}a(r,r+r',\tau+\eta)\,dr'\,đ\eta = \int_{\mathbb{R}} e^{-ir'\tau}k_\infty(r,r-r')\,dr'.$$

If $G \in C_{G,V}(X^\wedge,(\gamma,\Theta);E,F)$ we have $k_\infty(r,r-r') \equiv 0$ for $r' > 0$, and consequently g_∞ extends as an analytic rapidly decreasing function to the upper half-plane in view of the Paley–Wiener theorem (see also Section 1.1), i. e., $g_\infty \in S^{-\infty}(\mathbb{R}, L_V^{-\infty}(X;\mathbb{H};E,F))$ as desired.

Now consider the operator $G_0 = \omega G\tilde{\omega}$: We may write

$$\big(G_0 u\big)(r) = \int_{\mathbb{R}_+} k_0(r,r')u(r')\,\frac{dr'}{r'}$$

for $u \in C_0^\infty(\mathbb{R}_+, C^\infty(X,E))$ with a kernel

$$k_0 \in \mathcal{T}_{\gamma-\frac{n}{2}}(\mathbb{R}_+)\widehat{\otimes}_\pi\mathcal{T}_{-\gamma+\frac{n}{2}+1}(\mathbb{R}_+)\widehat{\otimes}_\pi L^{-\infty}(X;E,F).$$

Let $\chi \in \mathcal{S}(\Gamma_{\frac{n+1}{2}-\gamma})$ such that $\frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \chi(z)\,dz = 1$. Then $G_0 = \mathrm{op}_M^{\gamma-\frac{n}{2}}(b)$ with the Mellin double-symbol $b(r,r',z) := \left(\frac{r}{r'}\right)^z k_0(r,r')\chi(z)$. From Theorem 5.3.2 we obtain $G_0 = \mathrm{op}_M^{\gamma-\frac{n}{2}}(g_0)$ with

$$\begin{aligned} g_0(r,z) &= \int_{\mathbb{R}}\int_{\mathbb{R}_+} s^{i\eta}b(r,sr,z+i\eta)\,\frac{ds}{s}\,đ\eta \\ &= \int_{\mathbb{R}_+} s^z k_0(r,rs^{-1})\,\frac{ds}{s}. \end{aligned}$$

We have $\lim_{r\to 0} g_0(r) = 0$ since $\langle\log(r)\rangle g_0(r)$ is bounded as $r \to 0$.

If $G \in C_{G,V}(X^\wedge,(\gamma,\Theta);E,F)$ we conclude that $k_0(r,rs^{-1}) \equiv 0$ for $s > 1$, and consequently $g_0 \in C_B^\infty(\mathbb{R}_+, L_V^{-\infty}(X;\mathbb{H}_{\frac{n+1}{2}-\gamma};E,F))$ in view of the Paley–Wiener theorem (see Section 1.1). $\square$

6.1.6 Theorem. *a) Let $G \in C_G(X^\wedge,(\gamma,\Theta);E)$ such that $1+G$ is invertible in $\mathcal{L}(\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta)$ for some $s,t,\delta \in \mathbb{R}$. Then $1+G$ is invertible in $\mathcal{L}(\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta)$ for all $s,t,\delta \in \mathbb{R}$, and the inverse is given as $\big(1+G\big)^{-1} = 1+G_1$ with a Green operator $G_1 \in C_G(X^\wedge,(\gamma,\Theta);E)$.*

b) Let $G \in C_{G,V}(X^\wedge,(\gamma,\Theta);E)$. Then $1+G$ is invertible in $\mathcal{L}(\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta)$ for all $s,t,\delta \in \mathbb{R}$, and we have $\big(1+G\big)^{-1} = 1+G_1$ with a Volterra Green operator $G_1 \in C_{G,V}(X^\wedge,(\gamma,\Theta);E)$.

Proof. For the proof of a) note that we may write

$$\big(1+G\big)^{-1} = 1 - G + G\big(1+G\big)^{-1}G.$$

The operator $G_1 := -G + G\big(1+G\big)^{-1}G$ fulfills the conditions in Definition 6.1.2, and consequently belongs to $C_G(X^\wedge,(\gamma,\Theta);E)$. Clearly, $1+G_1$ inverts $1+G$ in $\mathcal{L}(\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta)$ for all $s,t,\delta \in \mathbb{R}$.

Let us now prove b). We first consider the weight $\gamma = \frac{n}{2}$. Since we have

$$(1-G)(1+G) = (1+G)(1-G) = 1 - G^2,$$

where $G^2 \in C_{G,V}(X^\wedge,(\gamma,\Theta);E)$, we just have to prove the assertion for the operator $1-G^2$. We may write $G^2 = G(G^*)^*$. Thus Proposition 1.3.9 gives

$$G^2 \in \mathcal{S}_Q^{-\frac{n}{2}}(X^\wedge,E)\widehat{\otimes}_\pi \mathcal{S}_P^{\frac{n}{2}}(X^\wedge,E)$$

with suitable asymptotic types P and Q, i. e., G^2 has a representation

$$G^2(u) = \sum_{j=1}^\infty \lambda_j \langle u, \tilde{s}_j\rangle_{r^{-\frac{n}{2}}L^2}\, s_j$$

for $u \in C_0^\infty(\mathbb{R}_+, C^\infty(X,E))$ with $(\lambda_j) \in \ell^1$ and sequences $(\tilde{s}_j) \in \mathcal{S}_Q^{-\frac{n}{2}}(X^\wedge,E)$ and $(s_j) \in \mathcal{S}_P^{\frac{n}{2}}(X^\wedge,E)$ tending to zero. In particular, we have

$$\big(G^2u\big)(r) = \int_{\mathbb{R}_+} k(r,r')u(r')\,dr'$$

for $u \in L^2(\mathbb{R}_+, L^2(X,E)) = \mathcal{K}^{(0,0),\frac{n}{2};\ell}(X^\wedge,E)_0$ with a Volterra integral kernel $k \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{L}(L^2(X,E)))$ that satisfies

$$\sup\{g(r)g(r')\|k(r,r')\|_{\mathcal{L}(L^2(X,E))};\ r,r' \in \mathbb{R}_+\} < \infty.$$

Here $g \in C(\mathbb{R}_+)$ is a function of the form

$$g(r) = \omega(r)r^\varepsilon + (1-\omega(r))r$$

with a cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ and a sufficiently small $0 < \varepsilon < \frac{1}{2}$, and thus we have

$$\int\limits_{\mathbb{R}_+} \frac{1}{g(r)^2}\,dr < \infty.$$

Consequently, we may apply Theorem 1.3.6 to the Volterra integral operator $G^2 \in \mathcal{L}(L^2(\mathbb{R}_+, L^2(X,E)))$ and conclude that G^2 is quasinilpotent. Moreover, we may write $\big(1 - G^2\big)^{-1} = 1 - G_1$ with a Volterra integral operator G_1. By a) $G_1 \in C_{G,V}(X^\wedge, (\frac{n}{2}, \Theta); E)$ is a Volterra Green operator, and we have

$$\big(1 - G^2\big)^{-1} = 1 - G_1 \in \mathcal{L}(\mathcal{K}^{(s,t),\frac{n}{2};\ell}(X^\wedge, E)_\delta)$$

for all $s, t, \delta \in \mathbb{R}$. This finishes the proof for the weight $\gamma = \frac{n}{2}$.

Next consider the case of general weights $\gamma \in \mathbb{R}$. We may write

$$1 + G = r^{\gamma - \frac{n}{2}}\big(1 + \tilde{G}\big)r^{-(\gamma - \frac{n}{2})} \in \mathcal{L}(\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge, E)_\delta)$$

for $s, t, \delta \in \mathbb{R}$, where $\tilde{G} := r^{-(\gamma - \frac{n}{2})}Gr^{\gamma - \frac{n}{2}}$ is a Volterra Green operator in $C_{G,V}(X^\wedge, (\frac{n}{2}, \Theta); E)$. From the first part of the proof we conclude that $1 + \tilde{G}$ is invertible, and we have $\big(1 + \tilde{G}\big)^{-1} = 1 + \tilde{G}_1$ with a Volterra Green operator $\tilde{G}_1 \in C_{G,V}(X^\wedge, (\frac{n}{2}, \Theta); E)$. Thus also $1 + G$ is invertible with inverse $\big(1 + G\big)^{-1} = 1 + G_1$, where $G_1 := r^{\gamma - \frac{n}{2}}\tilde{G}_1 r^{-(\gamma - \frac{n}{2})}$ is a Volterra Green operator in the space $C_{G,V}(X^\wedge, (\gamma, \Theta); E)$. This completes the proof of the theorem. □

6.1.7 Corollary. *Let $G \in C_{G,V}(X^\wedge, (\gamma, \Theta); E)$. Then $1 + G$ restricts to an isomorphism*

$$1 + G : \mathcal{H}_0^{(s,t),\gamma;\ell}((0, r_0] \times X, E) \longrightarrow \mathcal{H}_0^{(s,t),\gamma;\ell}((0, r_0] \times X, E)$$

for all $s, t \in \mathbb{R}$ and every $r_0 \in \mathbb{R}_+$, and we have $\big(1 + G\big)^{-1} = 1 + G_1$ with $G_1 \in C_{G,V}(X^\wedge, (\gamma, \Theta); E)$.

6.2 The algebra of conormal operators

Operators that generate asymptotics

6.2.1 Definition. Let $\gamma_1, \gamma_2 \in \mathbb{R}$, and let $\Theta = (\theta, 0]$ with $-\infty \le \theta < 0$. We define spaces of *(Volterra) operators that generate asymptotics* as follows:

a) An operator G belongs to $C_G^{\mu,\varrho;\ell}(X^\wedge, (\gamma_1, \gamma_2, \Theta); E, F)$ for $\mu, \varrho \in \mathbb{R}$, if the following conditions are fulfilled:

- G and the formal adjoint G^* with respect to the $r^{-\frac{n}{2}}L^2$-inner product are well-defined as continuous operators

$$G : \mathcal{K}^{(s,t),\gamma_1;\ell}(X^\wedge, E)_\delta \longrightarrow \mathcal{K}^{(s-\mu,t),\gamma_2;\ell}(X^\wedge, F)_{\delta-\varrho},$$
$$G^* : \mathcal{K}^{(s,t),-\gamma_2;\ell}(X^\wedge, F)_\delta \longrightarrow \mathcal{K}^{(s-\mu,t),-\gamma_1;\ell}(X^\wedge, F)_{\delta-\varrho}$$

for all $s, t, \delta \in \mathbb{R}$.

- There exist asymptotic types $P \in \mathrm{As}\big((\gamma_2, \Theta), C^\infty(X, F)\big)$ and $Q \in \mathrm{As}\big((-\gamma_1, \Theta), C^\infty(X, E)\big)$ such that

$$G : \begin{cases} \mathcal{K}^{(s,t),\gamma_1;\ell}(X^\wedge, E)_\delta \longrightarrow \mathcal{K}_P^{(s-\mu,t),\gamma_2;\ell}(X^\wedge, F)_{\delta-\varrho}, \\ \mathcal{S}^{\gamma_1}(X^\wedge, E) \longrightarrow \mathcal{S}_P^{\gamma_2}(X^\wedge, F), \end{cases}$$

$$G^* : \begin{cases} \mathcal{K}^{(s,t),-\gamma_2;\ell}(X^\wedge, F)_\delta \longrightarrow \mathcal{K}_Q^{(s-\mu,t),-\gamma_1;\ell}(X^\wedge, E)_{\delta-\varrho}, \\ \mathcal{S}^{-\gamma_2}(X^\wedge, F) \longrightarrow \mathcal{S}_Q^{-\gamma_1}(X^\wedge, E), \end{cases}$$

for all $s, t, \delta \in \mathbb{R}$.

b) Let $C_{G,V}^{\mu,\varrho;\ell}(X^\wedge, (\gamma_1, \gamma_2, \Theta); E, F)$ denote the subspace of all operators $G \in C_G^{\mu,\varrho;\ell}(X^\wedge, (\gamma_1, \gamma_2, \Theta); E, F)$ such that one of the following equivalent conditions is fulfilled:

- G restricts to continuous operators

$$G : \mathcal{H}_0^{(s,t),\gamma_1;\ell}((0, r_0] \times X, E) \longrightarrow \mathcal{H}_0^{(s-\mu,t),\gamma_2;\ell}((0, r_0] \times X, F)$$

for every (some) $s, t \in \mathbb{R}$ and every $r_0 \in \mathbb{R}_+$.

- For every $r_0 \in \mathbb{R}_+$ we have $(Gu)(r) \equiv 0$ for $r > r_0$ for all $u \in C_0^\infty(\mathbb{R}_+, C^\infty(X, E))$ such that $u(r) \equiv 0$ for $r > r_0$.

6.2.2 Remark. a) Definition 6.2.1 implies that the spaces of (Volterra) operators that generate asymptotics form (bi-)graded operator algebras, i. e., if H is another vector bundle then the composition induces a well-defined mapping

$$C_{G(,V)}^{\mu',\varrho';\ell}(X^\wedge, (\gamma_2, \gamma_3, \Theta); F, H) \times C_{G(,V)}^{\mu,\varrho;\ell}(X^\wedge, (\gamma_1, \gamma_2, \Theta); E, F) \longrightarrow C_{G(,V)}^{\mu+\mu',\varrho+\varrho';\ell}(X^\wedge, (\gamma_1, \gamma_3, \Theta); E, H).$$

Moreover, taking formal adjoints with respect to the $r^{-\frac{n}{2}}L^2$-inner product induces a well-defined mapping

$$* : C_G^{\mu,\varrho;\ell}(X^\wedge, (\gamma_1, \gamma_2, \Theta); E, F) \longrightarrow C_G^{\mu,\varrho;\ell}(X^\wedge, (-\gamma_2, -\gamma_1, \Theta); F, E),$$

i. e., the operators that generate asymptotics form a (bi-)graded $*$-algebra.

b) For $\mu, \varrho \in \mathbb{R}$ we denote

$$C^{\mu;\ell}_{G(,V)}(X^\wedge,(\gamma_1,\gamma_2,\Theta);E,F) := \bigcap_{\varrho'\in\mathbb{R}} C^{\mu,\varrho';\ell}_{G(,V)}(X^\wedge,(\gamma_1,\gamma_2,\Theta);E,F),$$
$$C^{\varrho}_{G(,V)}(X^\wedge,(\gamma_1,\gamma_2,\Theta);E,F) := \bigcap_{\mu'\in\mathbb{R}} C^{\mu',\varrho;\ell}_{G(,V)}(X^\wedge,(\gamma_1,\gamma_2,\Theta);E,F),$$
$$C_{G(,V)}(X^\wedge,(\gamma_1,\gamma_2,\Theta);E,F) := \bigcap_{\mu',\varrho'\in\mathbb{R}} C^{\mu',\varrho';\ell}_{G(,V)}(X^\wedge,(\gamma_1,\gamma_2,\Theta);E,F).$$

The elements of the latter of these spaces are also called *(Volterra) Green operators* associated with the (double) weight datum $(\gamma_1,\gamma_2,\Theta)$. Indeed, if $\gamma_1 = \gamma_2$ we have

$$C_{G(,V)}(X^\wedge,(\gamma_1,\gamma_2,\Theta);E,F) = C_{G(,V)}(X^\wedge,(\gamma_1,\Theta);E,F)$$

according to Definition 6.1.2.

c) The (Volterra) Green operators form a two-sided ideal in the algebra of (Volterra) operators that generate asymptotics.

d) If $\gamma_1 = \gamma_2$ we simplify the notations by substituting (γ_1,Θ) for $(\gamma_1,\gamma_2,\Theta)$.

Calculus of conormal symbols

6.2.3 Definition. a) Let $(\gamma,(-N,0])$ be a weight datum, $N \in \mathbb{N}$. For $\mu \in \mathbb{R}$ define the space of (classical) *conormal symbols* with respect to the weight datum $(\gamma,(-N,0])$ as

$$\begin{aligned}\Sigma^{\mu;\ell}_{M(cl)}(X,(\gamma,(-N,0]);E,F) := \{&(h_0,\ldots,h_{N-1});\\ &h_j \in M^{\mu;\ell}_{P_j(cl)}(X;E,F),\ \pi_{\mathbb{C}}P_0\cap\Gamma_{\frac{n+1}{2}-\gamma} = \emptyset\}.\end{aligned} \tag{6.2.i}$$

The subspace of (classical) *Volterra conormal symbols* with respect to the weight datum $(\gamma,(-N,0])$ is defined as

$$\begin{aligned}\Sigma^{\mu;\ell}_{M,V(cl)}(X,(\gamma,(-N,0]);E,F) := \{&(h_0,\ldots,h_{N-1});\\ &h_j \in M^{\mu;\ell}_{V,P_j(cl)}(X;\mathbb{H}_{\frac{n+1}{2}-\gamma+j};E,F)\}.\end{aligned} \tag{6.2.ii}$$

We define the spaces of order $-\infty$ as

$$\Sigma_M(X,(\gamma,(-N,0]);E,F) := \bigcap_{\mu\in\mathbb{R}} \Sigma^{\mu;\ell}_M(X,(\gamma,(-N,0]);E,F),$$
$$\Sigma_{M,V}(X,(\gamma,(-N,0]);E,F) := \bigcap_{\mu\in\mathbb{R}} \Sigma^{\mu;\ell}_{M,V}(X,(\gamma,(-N,0]);E,F).$$

These spaces do not depend on the anisotropy $\ell \in \mathbb{N}$, and they consist of all N-tuples of meromorphic (Volterra) Mellin symbols of order $-\infty$ with the same conditions on the Mellin asymptotic types as above.

b) Let G be another vector bundle over X. We define the *Mellin translation product*

$$\begin{aligned} \# : \Sigma_M^{\mu;\ell}(X,(\gamma,(-N,0]);F,G) \times \Sigma_M^{\mu';\ell}(X,(\gamma,(-N,0]);E,F) \\ \longrightarrow \Sigma_M^{\mu+\mu';\ell}(X,(\gamma,(-N,0]);E,G), \\ (g_0,\dots,g_{N-1})\#(h_0,\dots,h_{N-1}) := (\tilde{h}_0,\dots,\tilde{h}_{N-1}), \\ \tilde{h}_k := \sum_{p+q=k} (T_{-q}g_p)(h_q), \end{aligned} \tag{6.2.iii}$$

where T denotes the translation operator for functions in the complex plane, i. e., $\big((T_{-q}g_p)(h_q)\big)(z) = g_p(z-q)h_q(z)$.

c) We define a $*$-operation via

$$\begin{aligned} * : \Sigma_M^{\mu;\ell}(X,(\gamma,(-N,0]);E,F) \longrightarrow \Sigma_M^{\mu;\ell}(X,(-\gamma,(-N,0]);F,E), \\ (h_0,\dots,h_{N-1})^* := (\tilde{h}_0,\dots,\tilde{h}_{N-1}), \\ \tilde{h}_k(z) := \big(h_k(n+1-k-\overline{z})\big)^{(*)}, \end{aligned} \tag{6.2.iv}$$

where $^{(*)}$ denotes the formal adjoint with respect to the L^2-inner product on the manifold.

6.2.4 Theorem. *a) The spaces of (classical) conormal symbols form a graded $*$-algebra with componentwise linear operations, the Mellin translation product (6.2.iii), and the $*$-operation (6.2.iv).*

The conormal symbols of order $-\infty$ form a symmetric two-sided ideal.

More precisely, this means the following: Let $E, F, G, H \in \mathrm{Vect}(X)$ be complex vector bundles with corresponding data fixed according to Notation 3.1.1.

i) $\Sigma_{M(cl)}^{\mu;\ell}(X,(\gamma,(-N,0]);E,F)$ is a vector space with componentwise addition and scalar multiplication.

ii) The Mellin translation product induces an associative product, i. e. it is well-defined as a bilinear mapping

$$\begin{aligned} \# : \Sigma_{M(cl)}^{\mu;\ell}(X,(\gamma,(-N,0]);F,G) \times \Sigma_{M(cl)}^{\mu';\ell}(X,(\gamma,(-N,0]);E,F) \\ \longrightarrow \Sigma_{M(cl)}^{\mu+\mu';\ell}(X,(\gamma,(-N,0]);E,G), \end{aligned}$$

and we have $(a\#b)\#c = a\#(b\#c) \in \Sigma_{M(cl)}^{\mu_1+\mu_2+\mu_3;\ell}(X,(\gamma,(-N,0]);E,H)$ for $a \in \Sigma_{M(cl)}^{\mu_3;\ell}(X,(\gamma,(-N,0]);G,H)$, $b \in \Sigma_{M(cl)}^{\mu_2;\ell}(X,(\gamma,(-N,0]);F,G)$, and $c \in \Sigma_{M(cl)}^{\mu_1;\ell}(X,(\gamma,(-N,0]);E,F)$.

iii) *The $*$-operation is well-defined as an antilinear mapping*

$$* : \Sigma_M^{\mu;\ell}(X,(\gamma,(-N,0]);E,F) \longrightarrow \Sigma_M^{\mu;\ell}(X,(-\gamma,(-N,0]);F,E),$$

and we have $(a\#b)^* = b^*\#a^*$, $(a^*)^* = a$ *for conormal symbols* $a \in \Sigma_{M(cl)}^{\mu;\ell}(X,(\gamma,(-N,0]);F,G)$ *and* $b \in \Sigma_{M(cl)}^{\mu';\ell}(X,(\gamma,(-N,0]);E,F)$.

b) *The spaces of (classical) Volterra conormal symbols form a graded subalgebra, i. e., they share the properties i) and ii) listed in a) with* $\Sigma_{M(cl)}$ *replaced by* $\Sigma_{M,V(cl)}$. *Note that they are not closed with respect to the $*$-operation.*

The Volterra conormal symbols of order $-\infty$ form a two-sided ideal.

Proof. These assertions follow via simple algebraic calculations from Theorem 5.1.8 and Theorem 5.2.4. □

6.2.5 Definition. a) $a = (h_0,\dots,h_{N-1}) \in \Sigma_{M(cl)}^{\mu;\ell}(X,(\gamma,(-N,0]);E,F)$ is called *elliptic* if

- h_0 is elliptic as an element of $M_{P_0(cl)}^{\mu;\ell}(X;E,F)$ in the sense of Definition 5.1.12,
- there exists $s_0 \in \mathbb{R}$ such that $h_0(z) : H^{s_0}(X,E) \longrightarrow H^{s_0-\mu}(X,F)$ is bijective for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$.

b) An element $a = (h_0,\dots,h_{N-1}) \in \Sigma_{M,V(cl)}^{\mu;\ell}(X,(\gamma,(-N,0]);E,F)$ is called *parabolic* if

- h_0 is parabolic as an element of $M_{V,P_0(cl)}^{\mu;\ell}(X;\mathbb{H}_{\frac{n+1}{2}-\gamma};E,F)$ in the sense of Definition 5.2.7,
- there exists $s_0 \in \mathbb{R}$ such that $h_0(z) : H^{s_0}(X,E) \longrightarrow H^{s_0-\mu}(X,F)$ is bijective for all $z \in \mathbb{H}_{\frac{n+1}{2}-\gamma}$.

6.2.6 Notation. Let E be any vector bundle over X. For the moment, we prefer to denote the unit with respect to the Mellin translation product as

$$\mathbb{1} := (1,0,\dots,0) \in \Sigma_{M,V(cl)}^{0;\ell}(X,(\gamma,(-N,0]);E).$$

6.2.7 Theorem. *a) Let* $a \in \Sigma_{M(cl)}^{\mu;\ell}(X,(\gamma,(-N,0]);E,F)$. *Then the following are equivalent:*

i) *a is elliptic in the sense of Definition 6.2.5.*

ii) *a is invertible within the algebra of conormal symbols, i. e. there exists* $b \in \Sigma_{M(cl)}^{-\mu;\ell}(X,(\gamma,(-N,0]);F,E)$ *such that*

$$a\#b = \mathbb{1} \in \Sigma_{M(cl)}^{0;\ell}(X,(\gamma,(-N,0]);F),$$
$$b\#a = \mathbb{1} \in \Sigma_{M(cl)}^{0;\ell}(X,(\gamma,(-N,0]);E).$$

b) Let $a \in \Sigma^{\mu;\ell}_{M,V(cl)}(X,(\gamma,(-N,0]);E,F)$*. Then the following are equivalent:*

i) a is parabolic in the sense of Definition 6.2.5.

ii) a is invertible within the algebra of Volterra conormal symbols, i. e. there exists $b \in \Sigma^{-\mu;\ell}_{M,V(cl)}(X,(\gamma,(-N,0]);F,E)$ *such that*

$$a\#b = \mathtt{1} \in \Sigma^{0;\ell}_{M,V(cl)}(X,(\gamma,(-N,0]);F),$$
$$b\#a = \mathtt{1} \in \Sigma^{0;\ell}_{M,V(cl)}(X,(\gamma,(-N,0]);E).$$

Proof. By Theorem 5.1.14 and Theorem 5.2.8 the conditions ii) in a) and b) are sufficient for the ellipticity (parabolicity) of a in view of the definition of the Mellin translation product.

It remains to show the necessity. Let $a = (h_0,\dots,h_{N-1})$. We define the components of $b = (g_0,\dots,g_{N-1})$ by induction as follows:

By Theorem 5.1.14 and Theorem 5.2.8 a is elliptic, respectively parabolic, if and only if there exists $g_0 \in M^{-\mu;\ell}_{Q_0(cl)}(X;F,E)$, $\pi_{\mathbb{C}}Q_0 \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset$, respectively $g_0 \in M^{-\mu;\ell}_{V,Q_0(cl)}(X;\mathbb{H}_{\frac{n+1}{2}-\gamma};F,E)$, such that $h_0 g_0 \equiv 1$ and $g_0 h_0 \equiv 1$. Assume we have already constructed $g_0,\dots,g_{k-1}$ for some $k < N$. Define

$$g_k := -(T_{-k}g_0)\sum_{\substack{p+q=k\\ q<k}}(T_{-q}h_p)g_q \in \begin{cases} M^{-\mu;\ell}_{Q_k(cl)}(X;F,E) \\ M^{-\mu;\ell}_{V,Q_k(cl)}(X;\mathbb{H}_{\frac{n+1}{2}-\gamma+k};F,E), \end{cases}$$

which is well-defined in view of Theorem 5.1.8 and Theorem 5.2.4. By construction we at once have $a\#b = \mathtt{1}$, and a short calculation reveals $b\#a = \mathtt{1}$. This finishes the proof of the theorem. □

6.2.8 Corollary. *Let* $a \in \Sigma_{M(,V)}(X,(\gamma,(-N,0]);E)$*, where* $a = (h_0,\dots,h_{N-1})$*. Then the following are equivalent:*

a) $\mathtt{1}+a$ *is elliptic (parabolic).*

b) There exists $s_0 \in \mathbb{R}$ *such that* $1 + h_0(z) \in \mathcal{L}(H^{s_0}(X,E))$ *is bijective for all* $z \in \Gamma_{\frac{n+1}{2}-\gamma}$*, respectively* $z \in \mathbb{H}_{\frac{n+1}{2}-\gamma}$*.*

c) $\mathtt{1}+a$ *is invertible within* $\Sigma^{0;\ell}_{M(,V)}(X,(\gamma,(-N,0]);E)$ *with respect to the Mellin translation product, and the inverse is given as* $(\mathtt{1}+a)^{-1} = \mathtt{1}+b$ *with* $b \in \Sigma_{M(,V)}(X,(\gamma,(-N,0]);E)$*.*

Proof. Due to Theorem 6.2.7 we just have to prove that the inverse of $\mathtt{1}+a$ in c) is of the asserted form. But this follows from the identity

$$(\mathtt{1}+a)^{-1} = \mathtt{1} - a + a\#(\mathtt{1}+a)^{-1}a.$$

Note that $b := -a + a\#(\mathtt{1}+a)^{-1}a \in \Sigma_{M(,V)}(X,(\gamma,(-N,0]);E)$ since we handle with a two-sided ideal. □

The operator calculus

6.2.9 Definition. For $\mu, \varrho \in \mathbb{R}$ the set $C^{\mu,\varrho;\ell}_{M+G(cl)}(X^\wedge, (\gamma, (-N, 0]); E, F)$ of (classical) *conormal operators* with respect to the weight datum $(\gamma, (-N, 0])$, $N \in \mathbb{N}$, consists of all operators $A : \mathcal{S}^\gamma(X^\wedge, E) \longrightarrow \mathcal{S}^\gamma(X^\wedge, F)$ of the form

$$A = \sum_{j=0}^{N-1} \omega_j r^j \mathrm{op}_M^{\gamma_j}(h_j)\tilde{\omega}_j + G \tag{6.2.v}$$

with an operator $G \in C^{\mu,\varrho;\ell}_G(X^\wedge, (\gamma, (-N, 0]); E, F)$ and

i) cut-off functions $\omega_j, \tilde{\omega}_j \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$,

ii) $\gamma - \frac{n}{2} - j \leq \gamma_j \leq \gamma - \frac{n}{2}$,

iii) meromorphic Mellin symbols $h_j \in M^{\mu;\ell}_{P_j(cl)}(X; E, F)$ such that $\pi_{\mathbb{C}} P_j \cap \Gamma_{\frac{1}{2}-\gamma_j} = \emptyset$.

The subset $C^{\mu,\varrho;\ell}_{M+G,V(cl)}(X^\wedge, (\gamma, (-N, 0]); E, F)$ of (classical) *Volterra conormal operators* with respect to the weight datum $(\gamma, (-N, 0])$ consists of all those operators A having a representation as in (6.2.v) with $G \in C^{\mu,\varrho;\ell}_{G,V}(X^\wedge, (\gamma, (-N, 0]); E, F)$, and $h_j \in M^{\mu;\ell}_{V,P_j(cl)}(X; \mathbb{H}_{\frac{1}{2}-\gamma_j}; E, F)$.

Note that the meromorphic Mellin symbols $(h_0, \ldots, h_{N-1})$ of a conormal operator $A \in C^{\mu,\varrho;\ell}_{M+G(,V(cl))}(X^\wedge, (\gamma, (-N, 0]); E, F)$ give rise to an element in $\Sigma^{\mu;\ell}_{M(,V(cl))}(X, (\gamma, (-N, 0]); E, F)$.

6.2.10 Remark. The elements of $C^{\mu,\varrho;\ell}_{M+G(cl)}(X^\wedge, (\gamma, (-N, 0]); E, F)$ are indeed well-defined as continuous operators $\mathcal{S}^\gamma(X^\wedge, E) \longrightarrow \mathcal{S}^\gamma(X^\wedge, F)$. In fact, every summand is continuous in

$$\omega_j r^j \mathrm{op}_M^{\gamma_j}(h_j)\tilde{\omega}_j : \mathcal{S}^{\gamma_j + \frac{n}{2}}(X^\wedge, E) \longrightarrow \mathcal{S}^{\gamma_j + \frac{n}{2} + j}(X^\wedge, F),$$

and we have $\mathcal{S}^\gamma(X^\wedge, E) \hookrightarrow \mathcal{S}^{\gamma_j + \frac{n}{2}}(X^\wedge, E)$ and $\mathcal{S}^{\gamma_j + \frac{n}{2} + j}(X^\wedge, F) \hookrightarrow \mathcal{S}^\gamma(X^\wedge, F)$.

Moreover, an operator $A \in C^{\mu,\varrho;\ell}_{M+G,V(cl)}(X^\wedge, (\gamma, (-N, 0]); E, F)$ restricts to a continuous operator

$$A : \mathcal{T}_{\gamma-\frac{n}{2},0}((0, r_0), C^\infty(X, E)) \longrightarrow \mathcal{T}_{\gamma-\frac{n}{2},0}((0, r_0), C^\infty(X, F))$$

for every $r_0 \in \mathbb{R}_+$. This follows from Proposition 5.3.5 and Definition 6.2.1.

6.2.11 Proposition. *Let* $h \in M_R^{\mu;\ell}(X;E,F)$*, and let* $\pi_{\mathbb{C}}R \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset$*. Then* $op_M^{\gamma-\frac{n}{2}}(h)$ *extends by continuity to an operator*

$$op_M^{\gamma-\frac{n}{2}}(h) : \mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge,E) \longrightarrow \mathcal{H}^{(s-\mu,t),\gamma;\ell}(X^\wedge,F)$$

for every $s,t \in \mathbb{R}$*.*

Let $\Theta = (-\theta, 0]$*, where* $-\infty \le \theta < 0$*. For every asymptotic type* $P \in As((\gamma,\Theta), C^\infty(X,E))$ *there exists an asymptotic type* $Q \in As((\gamma,\Theta), C^\infty(X,F))$ *such that* $op_M^{\gamma-\frac{n}{2}}(h)$ *restricts to continuous operators*

$$op_M^{\gamma-\frac{n}{2}}(h) : \begin{cases} \mathcal{H}_P^{(s,t),\gamma;\ell}(X^\wedge,E) \longrightarrow \mathcal{H}_Q^{(s-\mu,t),\gamma;\ell}(X^\wedge,F), \\ \mathcal{T}_{\gamma-\frac{n}{2},P}(X^\wedge,E) \longrightarrow \mathcal{T}_{\gamma-\frac{n}{2},Q}(X^\wedge,F). \end{cases}$$

Proof. The first assertion follows from Theorem 5.3.6. Due to Corollary 5.1.11 we have $M_R^{\mu;\ell}(X;E,F) = M_O^{\mu;\ell}(X;E,F) + M_R^{-\infty}(X;E,F)$. Consequently, the proof of the second assertion reduces to consider the cases $h \in M_O^{\mu;\ell}(X;E,F)$ and $h \in M_R^{-\infty}(X;E,F)$.

The characterization of the Mellin image of $\mathcal{T}_{\gamma-\frac{n}{2},P}(X^\wedge,E)$ from Theorem 4.2.16 shows that h acts as a multiplier in the spaces

$$h : \mathcal{M}_{\gamma-\frac{n}{2}}\big(\mathcal{T}_{\gamma-\frac{n}{2},P}(X^\wedge,E)\big) \longrightarrow \mathcal{M}_{\gamma-\frac{n}{2}}\big(\mathcal{T}_{\gamma-\frac{n}{2},Q}(X^\wedge,F)\big)$$

with a certain asymptotic type Q both in the cases $h \in M_O^{\mu;\ell}(X;E,F)$ and $h \in M_R^{-\infty}(X;E,F)$, i. e.

$$\mathrm{op}_M^{\gamma-\frac{n}{2}}(h) : \mathcal{T}_{\gamma-\frac{n}{2},P}(X^\wedge,E) \longrightarrow \mathcal{T}_{\gamma-\frac{n}{2},Q}(X^\wedge,F)$$

as asserted.

By Theorem 4.2.16 we have $\mathcal{H}_P^{(s,t),\gamma;\ell}(X^\wedge,E) = \mathcal{H}_\Theta^{(s,t),\gamma;\ell}(X^\wedge,E) + \mathcal{T}_{\gamma-\frac{n}{2},P}(X^\wedge,E)$. Thus the remaining proof reduces to consider the case of the empty asymptotic type, i. e., $P = \Theta$. Let $h \in M_O^{\mu;\ell}(X;E,F)$, and let

$$R^s(\tau) \in L^{s;\ell}(X;\mathbb{R};E),$$
$$\tilde{R}^{s-\mu}(\tau) \in L^{s-\mu;\ell}(X;\mathbb{R};F)$$

be parameter-dependent reductions of orders from Theorem 3.1.12. Then we have

$$\tilde{R}^{s-\mu}(\tau)h(\beta+i\tau) = \underbrace{\Big(\tilde{R}^{s-\mu}(\tau)h(\beta+i\tau)R^{-s}(\tau)\Big)}_{\in C^\infty(\mathbb{R}_\beta, S^0(\mathbb{R}_\tau; H^t(X,E), H^t(X,F)))} R^s(\tau),$$

and thus h acts as a multiplier in the spaces

$$h : \mathcal{M}_{\gamma-\frac{n}{2}}\big(\mathcal{H}_\Theta^{(s,t),\gamma;\ell}(X^\wedge,E)\big) \longrightarrow \mathcal{M}_{\gamma-\frac{n}{2}}\big(\mathcal{H}_\Theta^{(s-\mu,t),\gamma;\ell}(X^\wedge,F)\big)$$

due to Theorem 4.2.16.

Now let $h \in M_R^{-\infty}(X;E,F)$, and let $\chi \in C^\infty(\mathbb{C})$ be an arbitrary $\pi_{\mathbb{C}}R$-excision function. Then $\chi(\beta+i\tau)h(\beta+i\tau) \in C^\infty(\mathbb{R}_\beta, \mathcal{S}(\mathbb{R}_\tau, L^{-\infty}(X;E,F)))$. Hence Theorem 4.2.16 implies that h acts as a multiplier in the spaces

$$h : \mathcal{M}_{\gamma-\frac{n}{2}}\big(\mathcal{H}_\Theta^{(s,t),\gamma;\ell}(X^\wedge,E)\big) \longrightarrow \mathcal{M}_{\gamma-\frac{n}{2}}\big(\mathcal{H}_Q^{\infty,\gamma}(X^\wedge,F)\big)$$

with a certain asymptotic type Q such that $\pi_{\mathbb{C}}Q = \pi_{\mathbb{C}}R$. Summing up, we have shown that

$$\mathrm{op}_M^{\gamma-\frac{n}{2}}(h) : \mathcal{H}_\Theta^{(s,t),\gamma;\ell}(X^\wedge,E) \longrightarrow \mathcal{H}_Q^{(s-\mu,t),\gamma;\ell}(X^\wedge,F)$$

both in the cases $h \in M_O^{\mu;\ell}(X;E,F)$ and $h \in M_R^{-\infty}(X;E,F)$. This finishes the proof of the proposition. □

6.2.12 Theorem. *Let* $A \in C_{M+G(cl)}^{\mu,\varrho;\ell}(X^\wedge,(\gamma,(-N,0]);E,F)$.

a) *A extends by continuity to an operator*

$$A : \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta \longrightarrow \mathcal{K}^{(s-\mu,t),\gamma;\ell}(X^\wedge,F)_{\delta-\varrho}$$

for all $s,t,\delta \in \mathbb{R}$.
Moreover, for every asymptotic type $P \in As((\gamma,(-N,0]),C^\infty(X,E))$ *there exists an asymptotic type* $Q \in As((\gamma,(-N,0]),C^\infty(X,F))$ *such that A restricts to continuous operators*

$$A : \begin{cases} \mathcal{K}_P^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta \longrightarrow \mathcal{K}_Q^{(s-\mu,t),\gamma;\ell}(X^\wedge,F)_{\delta-\varrho} \\ \mathcal{S}_P^\gamma(X^\wedge,E) \longrightarrow \mathcal{S}_Q^\gamma(X^\wedge,F) \end{cases}$$

for all $s,t,\delta \in \mathbb{R}$.
Let $A \in C_{M+G,V}^{\mu,\varrho;\ell}(X^\wedge,(\gamma,(-N,0]);E,F)$. *Then A restricts for every* $r_0 \in \mathbb{R}_+$ *to continuous operators*

$$A : \mathcal{H}_0^{(s,t),\gamma;\ell}((0,r_0]\times X,E) \longrightarrow \mathcal{H}_0^{(s-\mu,t),\gamma;\ell}((0,r_0]\times X,F)$$

for all $s,t \in \mathbb{R}$.

b) *The formal adjoint* A^* *with respect to the* $r^{-\frac{n}{2}}L^2$*-inner product belongs to* $C_{M+G(cl)}^{\mu,\varrho;\ell}(X^\wedge,(-\gamma,(-N,0]);F,E)$. *More precisely, let*

$$A = \sum_{j=0}^{N-1} \omega_j r^j \mathrm{op}_M^{\gamma_j}(h_j)\tilde{\omega}_j + G$$

be a representation of A from (6.2.v). Then

$$A^* = \sum_{j=0}^{N-1} \overline{\tilde{\omega}_j} r^j \, op_M^{-\gamma_j - n - j}(\tilde{h}_j)\overline{\omega_j} + G^*,$$
$$\tilde{h}_j(z) := h_j(n+1-j-\overline{z})^{(*)} \in M^{\mu;\ell}_{Q_j(cl)}(X;F,E),$$

is a representation of A^* in the sense of (6.2.v), where $^{(*)}$ denotes the formal adjoint with respect to the L^2-inner product on the manifold.

Proof. From Theorem 5.3.6 we obtain that every summand

$$\omega_j r^j \mathrm{op}_M^{\gamma_j}(h_j)\tilde{\omega}_j : \mathcal{K}^{(s,t),\gamma_j+\frac{n}{2};\ell}(X^\wedge,E)_\delta \longrightarrow \mathcal{K}^{(s-\mu,t),\gamma_j+\frac{n}{2}+j;\ell}(X^\wedge,F)_\infty$$

in the representation of A from (6.2.v) is continuous for all $s,t,\delta \in \mathbb{R}$, and we have

$$\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta \hookrightarrow \mathcal{K}^{(s,t),\gamma_j+\frac{n}{2};\ell}(X^\wedge,E)_\delta,$$
$$\mathcal{K}^{(s-\mu,t),\gamma_j+\frac{n}{2}+j;\ell}(X^\wedge,F)_\infty \hookrightarrow \mathcal{K}^{(s-\mu,t),\gamma;\ell}(X^\wedge,F)_\infty.$$

This proves the first assertion in a). The continuity of A in the subspaces with asymptotics follows from Proposition 6.2.11.
If $A \in C^{\mu,\varrho;\ell}_{M+G,V}(X^\wedge,(\gamma,(-N,0]);E,F)$, then

$$A : \mathcal{H}_0^{(s,t),\gamma;\ell}((0,r_0]\times X,E) \longrightarrow \mathcal{H}_0^{(s-\mu,t),\gamma;\ell}((0,r_0]\times X,F)$$

is continuous due to Proposition 5.3.5 and Theorem 5.3.6, respectively.

Let us now prove b). Using Theorem 5.3.3 and Proposition 2.6.4 we may write

$$\begin{aligned}\left(\omega_j r^j \mathrm{op}_M^{\gamma_j}(h_j)\tilde{\omega}_j\right)^* &= \overline{\tilde{\omega}_j}\mathrm{op}_M^{-\gamma_j-n}(h_j(n+1-\overline{z})^{(*)})r^j\overline{\omega_j} \\ &= \overline{\tilde{\omega}_j} r^j \mathrm{op}_M^{-\gamma_j-n-j}(\tilde{h}_j)\overline{\omega_j}.\end{aligned}$$

This finishes the proof of the theorem. ☐

6.2.13 Lemma. *Let $h \in M^{\mu;\ell}_P(X;E,F)$, and let $\omega,\tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ be cut-off functions. Moreover, let $\gamma_1,\gamma_2 \in \mathbb{R}$, $\gamma_1 < \gamma_2$, such that $\pi_{\mathbb{C}}P \cap \Gamma_{\frac{1}{2}-\gamma_1} = \emptyset$ and $\pi_{\mathbb{C}}P \cap \Gamma_{\frac{1}{2}-\gamma_2} = \emptyset$.*

Then the operator

$$\omega op_M^{\gamma_1}(h)\tilde{\omega} - \omega op_M^{\gamma_2}(h)\tilde{\omega} \in C_G(X^\wedge,(\gamma_2+\frac{n}{2},\gamma_1+\frac{n}{2},(-\infty,0]);E,F),$$

and it is finite-dimensional. If $\pi_{\mathbb{C}}P \cap \Gamma_{(\frac{1}{2}-\gamma_2,\frac{1}{2}-\gamma_1)} = \emptyset$ then the operator is identically zero.

Proof. Using the residue theorem we may write

$$\big(\mathrm{op}_M^{\gamma_1}(h)u\big)(r)-\big(\mathrm{op}_M^{\gamma_2}(h)u\big)(r)=\sum_{p\in\pi_{\mathbb{C}}P\cap\Gamma_I}\mathrm{res}_p\big(r^{-z}h(z)(\mathcal{M}u)(z)\big)$$

for $u\in C_0^\infty(\mathbb{R}_+,C^\infty(X,E))$, where $I:=(\frac{1}{2}-\gamma_2,\frac{1}{2}-\gamma_1)$. Let $(p,m,L)\in P$ such that $p\in\pi_{\mathbb{C}}P\cap\Gamma_I$, and set

$$U_p:=\Big\{\sum_{k=0}^{m}c_{p,k}r^{-p}\log^k(r);\ c_{p,k}\in\langle L(C^\infty(X,E))\rangle\subseteq C^\infty(X,F)\Big\}.$$

Then we have $\mathrm{res}_p\big(r^{-z}h(z)(\mathcal{M}u)(z)\big)\in U_p$, i. e.

$$\big(\omega\mathrm{op}_M^{\gamma_1}(h)\tilde\omega-\omega\mathrm{op}_M^{\gamma_2}(h)\tilde\omega\big)\big(C_0^\infty(\mathbb{R}_+,C^\infty(X,E))\big)\subseteq\omega\sum_{p\in\pi_{\mathbb{C}}P\cap\Gamma_I}U_p=\mathcal{E}_Q(X^\wedge,F)$$

with the induced asymptotic type $Q\in\mathrm{As}\big((\gamma_1+\frac{n}{2},(-\infty,0]),C^\infty(X,F)\big)$. Since $C_0^\infty(\mathbb{R}_+,C^\infty(X,E))$ is dense in $\mathcal{K}^{(s,t),\gamma_2+\frac{n}{2};\ell}(X^\wedge,E)_\delta$ for all $s,t,\delta\in\mathbb{R}$, and $\mathcal{E}_Q(X^\wedge,F)$ is finite-dimensional, we conclude

$$\omega\mathrm{op}_M^{\gamma_1}(h)\tilde\omega-\omega\mathrm{op}_M^{\gamma_2}(h)\tilde\omega:\mathcal{K}^{(s,t),\gamma_2+\frac{n}{2};\ell}(X^\wedge,E)_\delta\longrightarrow\mathcal{E}_Q(X^\wedge,F).$$

Theorem 5.3.3 implies

$$\big(\omega\mathrm{op}_M^{\gamma_1}(h)\tilde\omega-\omega\mathrm{op}_M^{\gamma_2}(h)\tilde\omega\big)^*=\overline{\tilde\omega}\mathrm{op}_M^{-\gamma_1-n}(\tilde h)\overline{\omega}-\overline{\tilde\omega}\mathrm{op}_M^{-\gamma_2-n}(\tilde h)\overline{\omega}$$

with $\tilde h(z):=h(n+1-\overline{z})^{(*)}$, where $^{(*)}$ denotes the formal adjoint with respect to the L^2-inner product on the manifold, and thus we obtain with the same reasoning as above

$$\big(\omega\mathrm{op}_M^{\gamma_1}(h)\tilde\omega-\omega\mathrm{op}_M^{\gamma_2}(h)\tilde\omega\big)^*:\mathcal{K}^{(s,t),-\gamma_1-\frac{n}{2};\ell}(X^\wedge,F)_\delta\longrightarrow\mathcal{E}_{\tilde Q}(X^\wedge,E)$$

for all $s,t,\delta\in\mathbb{R}$ with an asymptotic type $\tilde Q\in\mathrm{As}\big((-\gamma_2-\frac{n}{2},(-\infty,0]),C^\infty(X,E)\big)$. □

6.2.14 Remark. In the notation from Lemma 6.2.13 assume furthermore that $\gamma-\frac{n}{2}-j\le\gamma_1,\gamma_2\le\gamma-\frac{n}{2}$ for some $\gamma\in\mathbb{R}$ and $j\in\mathbb{N}_0$. Then we conclude that

$$\omega r^j\mathrm{op}_M^{\gamma_1}(h)\tilde\omega-\omega r^j\mathrm{op}_M^{\gamma_2}(h)\tilde\omega\in C_G(X^\wedge,(\gamma,(-\infty,0]);E,F),$$

and it is finite-dimensional. If $\pi_{\mathbb{C}}P\cap\Gamma_{(\frac{1}{2}-\gamma_2,\frac{1}{2}-\gamma_1)}=\emptyset$, then the operator is identically zero.

6.2.15 Lemma. *Let $\omega,\tilde\omega\in C_0^\infty(\overline{\mathbb{R}}_+)$ be cut-off functions, and let $\varphi,\psi\in C_0^\infty(\mathbb{R}_+)$. Moreover, let $\gamma-\frac{n}{2}-j\le\gamma_j\le\gamma-\frac{n}{2}$, $j\in\mathbb{N}_0$, and $h\in M_P^{\mu;\ell}(X;E,F)$ such that $\pi_{\mathbb{C}}P\cap\Gamma_{\frac{1}{2}-\gamma_j}=\emptyset$, respectively $h\in M_{V,P}^{\mu;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma_j};E,F)$. Then the following holds:*

a) $\omega r^j op_M^{\gamma_j}(h)\varphi,\ \psi r^j op_M^{\gamma_j}(h)\omega,\ \psi r^j op_M^{\gamma_j}(h)\varphi \in C^{\mu;\ell}_{G(,V)}(X^\wedge,(\gamma,(-\infty,0]);E,F)$.

b) If $j>0$ *then* $\omega r^j op_M^{\gamma_j}(h)\tilde\omega \in C^{\mu;\ell}_{G(,V)}(X^\wedge,(\gamma,(-j,0]);E,F)$.

Proof. Let $\hat\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ be a cut-off function such that $\hat\omega\varphi \equiv \varphi$, and write

$$\omega r^j \mathrm{op}_M^{\gamma_j}(h)\varphi = \big(\omega r^j \mathrm{op}_M^{\gamma_j}(h)\hat\omega\big)\varphi.$$

For every $s,t,\delta \in \mathbb{R}$ we have

$$\varphi : \begin{cases} \mathcal{S}^\gamma(X^\wedge,E) \longrightarrow \mathcal{S}^{\gamma_j+\frac{n}{2}}_{(-\infty,0]}(X^\wedge,E), \\ \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta \longrightarrow \mathcal{K}^{(s,t),\gamma_j+\frac{n}{2};\ell}_{(-\infty,0]}(X^\wedge,E)_\infty, \end{cases}$$

$$\omega r^j \mathrm{op}_M^{\gamma_j}(h)\hat\omega : \begin{cases} \mathcal{S}^{\gamma_j+\frac{n}{2}}_{(-\infty,0]}(X^\wedge,E) \longrightarrow \mathcal{S}^\gamma_Q(X^\wedge,F), \\ \mathcal{K}^{(s,t),\gamma_j+\frac{n}{2};\ell}_{(-\infty,0]}(X^\wedge,E)_\infty \longrightarrow \mathcal{K}^{(s-\mu,t),\gamma;\ell}_Q(X^\wedge,F)_\infty \end{cases}$$

with a certain asymptotic type $Q \in \mathrm{As}\big((\gamma,(-\infty,0]),C^\infty(X,F)\big)$ due to Proposition 6.2.11, i. e.

$$\omega r^j \mathrm{op}_M^{\gamma_j}(h)\varphi : \begin{cases} \mathcal{S}^\gamma(X^\wedge,E) \longrightarrow \mathcal{S}^\gamma_Q(X^\wedge,F), \\ \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta \longrightarrow \mathcal{K}^{(s-\mu,t),\gamma;\ell}_Q(X^\wedge,F)_\infty. \end{cases}$$

Moreover, we have

$$\psi r^j \mathrm{op}_M^{\gamma_j}(h)\omega,\ \psi r^j \mathrm{op}_M^{\gamma_j}(h)\varphi : \begin{cases} \mathcal{S}^\gamma(X^\wedge,E) \longrightarrow \mathcal{S}^\gamma_{(-\infty,0]}(X^\wedge,F), \\ \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta \longrightarrow \mathcal{K}^{(s-\mu,t),\gamma;\ell}_{(-\infty,0]}(X^\wedge,F)_\infty. \end{cases}$$

Theorem 6.2.12 implies

$$\begin{aligned} \big(\omega r^j \mathrm{op}_M^{\gamma_j}(h)\varphi\big)^* &= \overline{\varphi} r^j \mathrm{op}_M^{-\gamma_j-n-j}(\tilde h)\overline{\omega}, \\ \big(\psi r^j \mathrm{op}_M^{\gamma_j}(h)\omega\big)^* &= \overline{\omega} r^j \mathrm{op}_M^{-\gamma_j-n-j}(\tilde h)\overline{\psi}, \\ \big(\psi r^j \mathrm{op}_M^{\gamma_j}(h)\varphi\big)^* &= \overline{\varphi} r^j \mathrm{op}_M^{-\gamma_j-n-j}(\tilde h)\overline{\psi}, \end{aligned}$$

with $\tilde h(z) = h(n+1-j-\overline{z})^{(*)}$, and from the already proven result we finally obtain assertion a). Note that if $h \in M^{\mu;\ell}_{V,P}(X;\mathbb{H}_{\frac12-\gamma_j};E,F)$ then $\omega r^j \mathrm{op}_M^{\gamma_j}(h)\varphi$, $\psi r^j \mathrm{op}_M^{\gamma_j}(h)\omega$ and $\psi r^j \mathrm{op}_M^{\gamma_j}(h)\varphi$ are Volterra operators that generate asymptotics since they fulfill the defining mapping property in Definition 6.2.1, which follows from Proposition 5.3.5.

Let us now prove b). Due to Lemma 6.2.13 we may write

$$\omega r^j \mathrm{op}_M^{\gamma_j}(h)\tilde\omega = \omega r^j \mathrm{op}_M^{\gamma-\frac{n}{2}-\varepsilon}(h)\tilde\omega + G$$

with a Green operator $G \in C_G(X^\wedge,(\gamma,(-\infty,0]);E,F)$ which is independent of $\varepsilon>0$, provided that $\varepsilon>0$ is sufficiently small. Consequently, the operator $\omega r^j \mathrm{op}_M^{\gamma_j}(h)\tilde{\omega}-G$ is continuous in the spaces

$$\mathcal{S}^\gamma(X^\wedge,E)\longrightarrow \bigcap_{\varepsilon>0}\mathcal{S}^{\gamma+j-\varepsilon}(X^\wedge,F)=\mathcal{S}^\gamma_{(-j,0]}(X^\wedge,F),$$
$$\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta\longrightarrow \bigcap_{\varepsilon>0}\mathcal{K}^{(s-\mu,t),\gamma+j-\varepsilon;\ell}(X^\wedge,F)_\infty=\mathcal{K}^{(s-\mu,t),\gamma;\ell}_{(-j,0]}(X^\wedge,F)_\infty,$$

which shows that there is an asymptotic type $Q\in \mathrm{As}\big((\gamma,(-j,0]),C^\infty(X,F)\big)$ such that

$$\omega r^j \mathrm{op}_M^{\gamma_j}(h)\tilde{\omega}:\begin{cases}\mathcal{S}^\gamma(X^\wedge,E)\longrightarrow \mathcal{S}^\gamma_Q(X^\wedge,F),\\ \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta\longrightarrow \mathcal{K}^{(s-\mu,t),\gamma;\ell}_Q(X^\wedge,F)_\infty\end{cases}$$

for all $s,t,\delta\in\mathbb{R}$. From Theorem 6.2.12 we conclude that the same arguments apply to the formal adjoint operator. If $h\in M^{\mu;\ell}_{V,P}(X;\mathbb{H}_{\frac12-\gamma_j};E,F)$ then $\omega r^j\mathrm{op}_M^{\gamma_j}(h)\tilde{\omega}$ is a Volterra operator that generates asymptotics since it fulfills the defining mapping property in Definition 6.2.1. □

6.2.16 Lemma. *Let $\omega_j,\tilde{\omega}_j,\hat{\omega}_j,\breve{\omega}_j\in C_0^\infty(\overline{\mathbb{R}}_+)$ be cut-off functions. Moreover, let $\gamma_j,\tilde{\gamma}_j\in\mathbb{R}$ such that $\gamma-\frac{n}{2}-j\le\gamma_j,\tilde{\gamma}_j\le\gamma-\frac{n}{2}$, and let $h_j\in M^{\mu;\ell}_{P_j}(X;E,F)$ with $\pi_{\mathbb{C}}P_j\cap\Gamma_{\frac12-\gamma_j}=\pi_{\mathbb{C}}P_j\cap\Gamma_{\frac12-\tilde{\gamma}_j}=\emptyset$. Then*

$$\sum_{j=0}^{N-1}\omega_j r^j op_M^{\gamma_j}(h_j)\tilde{\omega}_j-\sum_{j=0}^{N-1}\hat{\omega}_j r^j op_M^{\tilde{\gamma}_j}(h_j)\breve{\omega}_j\in C_G^{\mu;\ell}(X^\wedge,(\gamma,(-\infty,0]);E,F).$$

If $h_j\in M^{\mu;\ell}_{V,P_j}(X;\mathbb{H}_{\frac12-\gamma_j};E,F)\cap M^{\mu;\ell}_{V,P_j}(X;\mathbb{H}_{\frac12-\tilde{\gamma}_j};E,F)$ then

$$\sum_{j=0}^{N-1}\omega_j r^j op_M^{\gamma_j}(h_j)\tilde{\omega}_j-\sum_{j=0}^{N-1}\hat{\omega}_j r^j op_M^{\tilde{\gamma}_j}(h_j)\breve{\omega}_j\in C_{G,V}^{\mu;\ell}(X^\wedge,(\gamma,(-\infty,0]);E,F).$$

Proof. We have $\omega_j=\hat{\omega}_j+\psi_j$ and $\tilde{\omega}_j=\breve{\omega}_j+\varphi_j$ with $\psi_j,\varphi_j\in C_0^\infty(\mathbb{R}_+)$. Thus the assertion follows from Lemma 6.2.13 and Lemma 6.2.15. □

6.2.17 Lemma. *Let H be another vector bundle over X, and let $\omega,\tilde{\omega},\hat{\omega},\breve{\omega}\in C_0^\infty(\overline{\mathbb{R}}_+)$ be cut-off functions. Moreover, let $\gamma_1,\gamma_2\in\mathbb{R}$ such that $\gamma-\frac{n}{2}-k\le\gamma_1\le\gamma-\frac{n}{2}$ and $\gamma-\frac{n}{2}-j\le\gamma_2\le\gamma-\frac{n}{2}$, and let $g\in M^{\mu;\ell}_{P(cl)}(X;F,H)$, $h\in M^{\mu';\ell}_{Q(cl)}(X;E,F)$ with $\pi_{\mathbb{C}}P\cap\Gamma_{\frac12-\gamma_1}=\emptyset$, $\pi_{\mathbb{C}}Q\cap\Gamma_{\frac12-\gamma_2}=\emptyset$. Then*

$$\big(\omega r^k op_M^{\gamma_1}(g)\tilde{\omega}\big)\big(\hat{\omega}r^j op_M^{\gamma_2}(h)\breve{\omega}\big)=\omega r^{k+j}op_M^{\tilde{\gamma}}((T_{-j}g)h)\breve{\omega}+G$$

with an operator $G \in C_G^{\mu+\mu';\ell}(X^\wedge, (\gamma, (-\infty, 0]); E, H)$*, and* $\gamma - \frac{n}{2} - k - j \le \tilde{\gamma} \le \gamma - \frac{n}{2}$*. Here* T *denotes the translation operator for functions in the complex plane, i. e.*

$$\big((T_{-j}g)h\big)(z) = g(z-j)h(z) \in M_{R(cl)}^{\mu+\mu';\ell}(X; E, H).$$

If even $g \in M_{V,P(cl)}^{\mu;\ell}(X; \mathbb{H}_{\frac{1}{2}-\gamma_1}; F, H)$ *and* $h \in M_{V,Q(cl)}^{\mu';\ell}(X; \mathbb{H}_{\frac{1}{2}-\gamma_2}; E, F)$ *we may also choose* $\tilde{\gamma} \in \mathbb{R}$ *such that* $(T_{-j}g)h \in M_{V,R(cl)}^{\mu+\mu';\ell}(X; \mathbb{H}_{\frac{1}{2}-\tilde{\gamma}}; E, H)$*, and* $G \in C_{G,V}^{\mu+\mu';\ell}(X^\wedge, (\gamma, (-\infty, 0]); E, H)$*.*

Proof. We may write

$$\big(\omega r^k \mathrm{op}_M^{\gamma_1}(g)\tilde{\omega}\big)\big(\hat{\omega} r^j \mathrm{op}_M^{\gamma_2}(h)\breve{\omega}\big) = r^{k+j}\big(\omega \mathrm{op}_M^{\gamma_1 - j}(T_{-j}g)\tilde{\omega}\big)\big(\hat{\omega}\mathrm{op}_M^{\gamma_2}(h)\breve{\omega}\big).$$

Choose $\gamma_1 - j \le \tilde{\gamma} \le \gamma_2$ such that no singularity of $T_{-j}g$ and h lies on the weight line $\Gamma_{\frac{1}{2}-\tilde{\gamma}}$. From Lemma 6.2.13 we conclude

$$\begin{aligned}\big(\omega \mathrm{op}_M^{\gamma_1 - j}(T_{-j}g)\tilde{\omega}\big) &= \big(\omega \mathrm{op}_M^{\tilde{\gamma}}(T_{-j}g)\tilde{\omega}\big) + G_1,\\ \big(\hat{\omega}\mathrm{op}_M^{\gamma_2}(h)\breve{\omega}\big) &= \big(\hat{\omega}\mathrm{op}_M^{\tilde{\gamma}}(h)\breve{\omega}\big) + G_2,\end{aligned}$$

with Green operators

$$\begin{aligned}G_1 &\in C_G(X^\wedge, (\tilde{\gamma} + \frac{n}{2}, \gamma_1 - j + \frac{n}{2}, (-\infty, 0]); F, H),\\ G_2 &\in C_G(X^\wedge, (\gamma_2 + \frac{n}{2}, \tilde{\gamma} + \frac{n}{2}, (-\infty, 0]); E, F).\end{aligned}$$

Using Proposition 6.2.11 we obtain the following:

- $G_1 G_2 \in C_G(X^\wedge, (\gamma_2 + \frac{n}{2}, \gamma_1 - j + \frac{n}{2}, (-\infty, 0]); E, H)$,
- $\big(\omega \mathrm{op}_M^{\tilde{\gamma}}(T_{-j}g)\tilde{\omega}\big) G_2 \in C_G(X^\wedge, (\gamma_2 + \frac{n}{2}, \tilde{\gamma} + \frac{n}{2}, (-\infty, 0]); E, H)$,
- $G_1\big(\hat{\omega}\mathrm{op}_M^{\tilde{\gamma}}(h)\breve{\omega}\big) \in C_G(X^\wedge, (\tilde{\gamma} + \frac{n}{2}, \gamma_1 - j + \frac{n}{2}, (-\infty, 0]); E, H)$,

and consequently

$$r^{k+j}\big(\omega \mathrm{op}_M^{\gamma_1 - j}(T_{-j}g)\tilde{\omega}\big)\big(\hat{\omega}\mathrm{op}_M^{\gamma_2}(h)\breve{\omega}\big) \equiv r^{k+j}\big(\omega \mathrm{op}_M^{\tilde{\gamma}}(T_{-j}g)\tilde{\omega}\big)\big(\hat{\omega}\mathrm{op}_M^{\tilde{\gamma}}(h)\breve{\omega}\big)$$

modulo $C_G(X^\wedge, (\gamma, (-\infty, 0]); E, H)$. We may write

$$\begin{aligned}\big(\omega r^{k+j}\mathrm{op}_M^{\tilde{\gamma}}(T_{-j}g)\tilde{\omega}\big)\big(\hat{\omega}\mathrm{op}_M^{\tilde{\gamma}}(h)\breve{\omega}\big) &= \big(\omega r^{k+j}\mathrm{op}_M^{\tilde{\gamma}}((T_{-j}g)h)\breve{\omega}\big)\\ &\quad - \underbrace{\big(\omega r^{k+j}\mathrm{op}_M^{\tilde{\gamma}}(T_{-j}g)(1 - \tilde{\omega}\hat{\omega})\mathrm{op}_M^{\tilde{\gamma}}(h)\breve{\omega}\big)}_{=:\tilde{G}}.\end{aligned}$$

According to Proposition 6.2.11 we have for $s,t,\delta \in \mathbb{R}$

$$\tilde{G}:\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta \hookrightarrow \mathcal{K}^{(s,t),\tilde{\gamma}+\frac{n}{2};\ell}(X^\wedge,E)_\delta \underset{(\mathrm{op}_M^{\tilde{\gamma}}(h)\check{\omega})}{\longrightarrow} \mathcal{H}^{(s-\mu',t),\tilde{\gamma}+\frac{n}{2};\ell}(X^\wedge,F)$$
$$\underset{(1-\tilde{\omega}\hat{\omega})}{\longrightarrow} \mathcal{H}^{(s-\mu',t),\tilde{\gamma}+\frac{n}{2};\ell}_{(-\infty,0]}(X^\wedge,F) \underset{(\mathrm{op}_M^{\tilde{\gamma}}(T_{-j}g))}{\longrightarrow} \mathcal{H}_Q^{(s-\mu-\mu',t),\tilde{\gamma}+\frac{n}{2};\ell}(X^\wedge,H)$$
$$\underset{(\omega r^{k+j})}{\longrightarrow} \mathcal{K}_{\tilde{Q}}^{(s-\mu-\mu',t),\gamma;\ell}(X^\wedge,H)_\infty$$

with certain asymptotic types $Q,\tilde{Q}$.

Analogously, we obtain $\tilde{G}:\mathcal{S}^\gamma(X^\wedge,E)\longrightarrow \mathcal{S}^\gamma_{\tilde{Q}}(X^\wedge,H)$, and the same arguments also apply to the formal adjoint operator $\tilde{G}^*$, i. e., we have $\tilde{G}\in C_G^{\mu+\mu';\ell}(X^\wedge,(\gamma,(-\infty,0]);E,H)$. This implies the first assertion.

In case of Volterra operators we first observe that due to Lemma 6.2.13 we may choose the weight $\tilde{\gamma}:=\gamma-\frac{n}{2}-k-j$, which produces a Green operator as error term, i. e.

$$G:=\big(\omega r^k\mathrm{op}_M^{\gamma_1}(g)\tilde{\omega}\big)\big(\hat{\omega}r^j\mathrm{op}_M^{\gamma_2}(h)\check{\omega}\big)-\omega r^{k+j}\mathrm{op}_M^{\gamma-\frac{n}{2}-k-j}((T_{-j}g)h)\check{\omega}$$
$$\in C_G^{\mu+\mu';\ell}(X^\wedge,(\gamma,(-\infty,0]);E,H).$$

We have $(T_{-j}g)h\in M^{\mu+\mu';\ell}_{V,R(cl)}(X;\mathbb{H}_{\frac{n+1}{2}-\gamma+k+j};E,H)$, and consequently G fulfills the defining mapping property for Volterra operators that generate asymptotics in Definition 6.2.1, which follows from Proposition 5.3.5. This proves the lemma. □

6.2.18 Proposition. *Let $\omega_j,\tilde{\omega}_j\in C_0^\infty(\overline{\mathbb{R}}_+)$ be cut-off functions near $r=0$. Moreover, let $h_j\in M^{\mu;\ell}_{P_j}(X;E,F)$ such that $\pi_{\mathbb{C}}P_j\cap\Gamma_{\frac{1}{2}-\gamma_j}=\emptyset$ with $\gamma-\frac{n}{2}-j\le\gamma_j\le\gamma-\frac{n}{2}$. Assume that*

$$\sum_{j=0}^{N-1}\omega_j r^j op_M^{\gamma_j}(h_j)\tilde{\omega}_j:\mathcal{S}^\gamma(X^\wedge,E)\longrightarrow\mathcal{S}^\gamma_Q(X^\wedge,F)$$

with an asymptotic type $Q\in As\big((\gamma,(\theta,0]),C^\infty(X,F)\big)$, where $-\infty\le\theta<-(N-1)$. Then $h_j\equiv 0$ for $j=0,\dots,N-1$.

Proof. The proof follows by induction over $N\in\mathbb{N}$: Let $N=1$. From Proposition 6.2.11 we conclude

$$(1-\omega_0)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0):\mathcal{T}_{\gamma-\frac{n}{2}}(X^\wedge,E)\longrightarrow\mathcal{T}_{\gamma-\frac{n}{2},(-\infty,0]}(X^\wedge,F),$$
$$\omega_0\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0)(1-\tilde{\omega}_0):\mathcal{T}_{\gamma-\frac{n}{2}}(X^\wedge,E)\longrightarrow\mathcal{T}_{\gamma-\frac{n}{2},\tilde{Q}}(X^\wedge,F)$$

with an asymptotic type $\tilde{Q} \in \mathrm{As}\big((\gamma, (-\infty, 0]), C^\infty(X, F)\big)$. Using the assumption we obtain

$$\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0) : \mathcal{T}_{\gamma-\frac{n}{2}}(X^\wedge, E) \longrightarrow \mathcal{T}_{\gamma-\frac{n}{2},R}(X^\wedge, F)$$

with an asymptotic type $R \in \mathrm{As}\big((\gamma, (\theta, 0]), C^\infty(X, F)\big)$, and by possibly passing to a smaller weight interval $(\theta, 0]$ we may assume that $R = (\theta, 0]$ is the empty asymptotic type. Consequently, h_0 acts as a multiplier in the spaces

$$h_0 : \mathcal{S}(\Gamma_{\frac{n+1}{2}-\gamma}, C^\infty(X, E)) \longrightarrow \mathcal{M}_{\gamma-\frac{n}{2}}\big(\mathcal{T}_{\gamma-\frac{n}{2},(\theta,0]}(X^\wedge, F)\big).$$

Let $\varphi \in C_0^\infty(\Gamma_{\frac{n+1}{2}-\gamma})$ such that $\varphi \equiv 0$ for $|\mathrm{Im}(z)| > 2$, and $\varphi \equiv 1$ for $|\mathrm{Im}(z)| \leq 1$. Hence $\varphi(z)(h_0(z)u) \equiv 0$ for $z \in \Gamma_{\frac{n+1}{2}-\gamma}$ such that $|\mathrm{Im}(z)| > 2$, for all $u \in C^\infty(X, E)$. From Theorem 4.2.16 and uniqueness of analytic continuation we obtain $\varphi(z)(h_0(z)u) \equiv 0$ for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$ and all $u \in C^\infty(X, E)$, and thus $h_0(z) = 0$ for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$ such that $|\mathrm{Im}(z)| \leq 1$. By the meromorphy of h_0 we conclude $h_0 \equiv 0$ everywhere on $\mathbb{C}$. This finishes the proof in the case $N = 1$.

Assume we have already proven the proposition for some $N \in \mathbb{N}$, and

$$\sum_{j=0}^{N} \omega_j r^j \mathrm{op}_M^{\gamma_j}(h_j)\tilde{\omega}_j : \mathcal{S}^\gamma(X^\wedge, E) \longrightarrow \mathcal{S}_Q^\gamma(X^\wedge, F)$$

with an asymptotic type $Q \in \mathrm{As}\big((\gamma, (\theta, 0]), C^\infty(X, F)\big)$, where $-\infty \leq \theta < -N$. By Lemma 6.2.15 the operator $\omega_N r^N \mathrm{op}_M^{\gamma_N}(h_N)\tilde{\omega}_N \in C_G^{\mu;\ell}(X^\wedge, (\gamma, (-N, 0]); E, F)$, which shows that

$$\sum_{j=0}^{N-1} \omega_j r^j \mathrm{op}_M^{\gamma_j}(h_j)\tilde{\omega}_j : \mathcal{S}^\gamma(X^\wedge, E) \longrightarrow \mathcal{S}_{\tilde{Q}}^\gamma(X^\wedge, F)$$

with an asymptotic type $\tilde{Q} \in \mathrm{As}\big((\gamma, (-N, 0]), C^\infty(X, F)\big)$.

Hence $h_j \equiv 0$ for $j = 0, \ldots, N-1$ by induction, i. e.

$$\omega_N r^N \mathrm{op}_M^{\gamma_N}(h_N)\tilde{\omega}_N : \mathcal{S}^\gamma(X^\wedge, E) \longrightarrow \mathcal{S}_Q^\gamma(X^\wedge, F),$$

and consequently

$$\omega_N \mathrm{op}_M^{\gamma_N}(h_N)\tilde{\omega}_N : \mathcal{S}^\gamma(X^\wedge, E) \longrightarrow \mathcal{S}_R^{\gamma-N}(X^\wedge, F)$$

with an asymptotic type $R \in \mathrm{As}\big((\gamma - N, (\theta, 0]), C^\infty(X, F)\big)$.

Choose $\gamma - \frac{n}{2} \leq \tilde{\gamma} < \gamma - \frac{n}{2} - N - \theta$ such that $\pi_{\mathbb{C}} P_N \cap \Gamma_{\frac{1}{2}-\tilde{\gamma}} = \emptyset$. Due to Lemma 6.2.13 we may write

$$\omega_N \mathrm{op}_M^{\tilde{\gamma}}(h_N)\tilde{\omega}_N = \omega_N \mathrm{op}_M^{\gamma_N}(h_N)\tilde{\omega}_N + G$$

with a Green operator $G \in C_G(X^\wedge, (\tilde{\gamma} + \frac{n}{2}, \gamma_N + \frac{n}{2}, (-\infty, 0]); E, F)$. This shows

$$\omega_N \mathrm{op}_M^{\tilde{\gamma}}(h_N)\tilde{\omega}_N : \mathcal{S}^{\tilde{\gamma}+\frac{n}{2}}(X^\wedge, E) \longrightarrow \mathcal{S}_{R'}^{\gamma-N}(X^\wedge, F) \cap \mathcal{S}^{\tilde{\gamma}+\frac{n}{2}}(X^\wedge, F)$$

with an asymptotic type $R' \in \mathrm{As}\big((\gamma - N, (\theta, 0]), C^\infty(X, F)\big)$, while

$$\mathcal{S}_{R'}^{\gamma-N}(X^\wedge, F) \cap \mathcal{S}^{\tilde{\gamma}+\frac{n}{2}}(X^\wedge, F) \hookrightarrow \mathcal{S}_{(\tilde{\theta},0]}^{\tilde{\gamma}+\frac{n}{2}}(X^\wedge, F)$$

with some $-\infty \le \tilde{\theta} < 0$. Hence the first part of the proof implies $h_N \equiv 0$, and by induction the proposition is proved. □

6.2.19 Remark. Let $A \in C_{M+G(cl)}^{\mu,\varrho;\ell}(X^\wedge, (\gamma, (-N, 0]); E, F)$. From Lemma 6.2.16 we obtain that in the representation (6.2.v) any change of the cut-off functions $\omega_j, \tilde{\omega}_j$ as well as of the weights $\gamma - \frac{n}{2} - j \le \gamma_j \le \gamma - \frac{n}{2}$ results in an error in $C_G^{\mu;\ell}(X^\wedge, (\gamma, (-\infty, 0]); E, F)$ only.

Consequently, the following simpler representation of conormal operators is valid:

An operator $A : \mathcal{S}^\gamma(X^\wedge, E) \longrightarrow \mathcal{S}^\gamma(X^\wedge, F)$ is a conormal operator in the space $C_{M+G(cl)}^{\mu,\varrho;\ell}(X^\wedge, (\gamma, (-N, 0]); E, F)$ if and only if

$$A = \sum_{j=0}^{N-1} \omega r^j \mathrm{op}_M^{\gamma_j}(h_j)\omega + G \tag{6.2.vi}$$

with an operator $G \in C_G^{\mu,\varrho;\ell}(X^\wedge, (\gamma, (-N, 0]); E, F)$, and

i) a cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$,

ii) $\gamma - \frac{n}{2} - j \le \gamma_j \le \gamma - \frac{n}{2}$,

iii) meromorphic Mellin symbols $h_j \in M_{P_j(cl)}^{\mu;\ell}(X; E, F)$ such that $\pi_{\mathbb{C}} P_j \cap \Gamma_{\frac{1}{2}-\gamma_j} = \emptyset$.

Moreover, $A : \mathcal{S}^\gamma(X^\wedge, E) \longrightarrow \mathcal{S}^\gamma(X^\wedge, F)$ is a Volterra conormal operator in the space $C_{M+G,V(cl)}^{\mu,\varrho;\ell}(X^\wedge, (\gamma, (-N, 0]); E, F)$ if and only if

$$A = \sum_{j=0}^{N-1} \omega r^j \mathrm{op}_M^{\gamma-\frac{n}{2}-j}(h_j)\omega + G \tag{6.2.vii}$$

with a Volterra operator $G \in C_{G,V}^{\mu,\varrho;\ell}(X^\wedge, (\gamma, (-N, 0]); E, F)$, and

i) a cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$,

ii) meromorphic Volterra Mellin symbols $h_j \in M^{\mu;\ell}_{V,P_j(cl)}(X;\mathbb{H}_{\frac{n+1}{2}-\gamma+j};E,F)$.

6.2.20 Definition. We define the *conormal symbol mapping*

$$\sigma_M : C^{\mu,\varrho;\ell}_{M+G(,V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F) \longrightarrow \Sigma^{\mu;\ell}_{M(,V(cl))}(X,(\gamma,(-N,0]);E,F)$$

as follows:

Let $A \in C^{\mu,\varrho;\ell}_{M+G(,V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F)$, and let

$$A = \sum_{j=0}^{N-1} \omega_j r^j \operatorname{op}^{\gamma_j}_M(h_j)\tilde{\omega}_j + G$$

be a representation according to (6.2.v). Then $\sigma_M(A)$ is defined as

$$\sigma_M(A) := (\sigma^0_M(A),\dots,\sigma^{-(N-1)}_M(A)) := (h_0,\dots,h_{N-1}). \tag{6.2.viii}$$

The component $\sigma^{-k}_M(A)$ is called the *conormal symbol of order* $-k$ of the operator A. The conormal symbol $\sigma^0_M(A)$ of order 0 is also called the conormal symbol simply.

6.2.21 Theorem. *a)* $C^{\mu,\varrho;\ell}_{M+G(,V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F)$ *is a linear space.*

b) The conormal symbol mapping is well-defined, and provides a linear surjection

$$\sigma_M : C^{\mu,\varrho;\ell}_{M+G(,V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F) \to \Sigma^{\mu;\ell}_{M(,V(cl))}(X,(\gamma,(-N,0]);E,F)$$

with kernel

$$\ker(\sigma_M) = C^{\mu,\varrho;\ell}_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F).$$

c) The quotient spaces

$$\begin{aligned} &Quot^{\mu;\ell}_{M+G(,V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F) := \\ &\quad C^{\mu,\varrho;\ell}_{M+G(,V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F)/C^{\mu,\varrho;\ell}_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F)\end{aligned}$$

do not depend on $\varrho \in \mathbb{R}$, *and for* $\mu' \geq \mu$ *the embedding*

$$\begin{aligned} &Quot^{\mu;\ell}_{M+G(,V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F) \hookrightarrow \\ &\qquad Quot^{\mu';\ell}_{M+G(,V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F)\end{aligned}$$

is well-defined.

d) *Taking the formal adjoint* $*$ *with respect to the* $r^{-\frac{n}{2}}L^2$*-inner product induces antilinear mappings*

$$C^{\mu,\varrho;\ell}_{M+G(cl)}(X^\wedge,(\gamma,(-N,0]);E,F) \longrightarrow C^{\mu,\varrho;\ell}_{M+G(cl)}(X^\wedge,(-\gamma,(-N,0]);F,E),$$
$$\mathrm{Quot}^{\mu;\ell}_{M+G(cl)}(X^\wedge,(\gamma,(-N,0]);E,F) \longrightarrow \mathrm{Quot}^{\mu;\ell}_{M+G(cl)}(X^\wedge,(-\gamma,(-N,0]);F,E).$$

For $A \in C^{\mu,\varrho;\ell}_{M+G(cl)}(X^\wedge,(\gamma,(-N,0]);E,F)$ *we have* $\sigma_M(A^*) = \big(\sigma_M(A)\big)^*$ *with the* $*$*-operation (6.2.iv).*

e) *Let* H *be another vector bundle over* X. *The composition as operators on* $\mathcal{S}^\gamma(X^\wedge,E)$ *is well-defined in the spaces*

$$C^{\mu,\varrho;\ell}_{M+G(,V(cl))}(X^\wedge,(\gamma,(-N,0]);F,H)\times C^{\mu',\varrho';\ell}_{M+G(,V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F) \longrightarrow C^{\mu+\mu',\varrho+\varrho';\ell}_{M+G(,V(cl))}(X^\wedge,(\gamma,(-N,0]);E,H).$$

For

$$A \in C^{\mu,\varrho;\ell}_{M+G(,V(cl))}(X^\wedge,(\gamma,(-N,0]);F,H),$$
$$B \in C^{\mu',\varrho';\ell}_{M+G(,V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F)$$

we have $\sigma_M(AB) = \sigma_M(A)\#\sigma_M(B)$ *with the Mellin translation product (6.2.iii).*

In particular,

$$C^{\mu,\varrho;\ell}_{G(,V)}(X^\wedge,(\gamma,(-N,0]);F,H)\times C^{\mu',\varrho';\ell}_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F) \longrightarrow C^{\mu+\mu',\varrho+\varrho';\ell}_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,H),$$
$$C^{\mu,\varrho;\ell}_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);F,H)\times C^{\mu',\varrho';\ell}_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F) \longrightarrow C^{\mu+\mu',\varrho+\varrho';\ell}_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,H),$$

and the composition is well-defined on the quotient spaces.

Proof. Let $A,B \in C^{\mu,\varrho;\ell}_{M+G(,V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F)$. According to (6.2.vi), (6.2.vii) we may write

$$A = \sum_{j=0}^{N-1} \omega r^j \mathrm{op}_M^{\gamma_j}(h_j)\omega + G, \quad B = \sum_{j=0}^{N-1} \omega r^j \mathrm{op}_M^{\gamma_j}(\tilde{h}_j)\omega + \tilde{G},$$

and thus

$$\lambda_1 A + \lambda_2 B = \sum_{j=0}^{N-1} \omega r^j \mathrm{op}_M^{\gamma_j}(\lambda_1 h_j + \lambda_2\tilde{h}_j)\omega + (\lambda_1 G + \lambda_2 \tilde{G})$$

for $\lambda_1, \lambda_2 \in \mathbb{C}$. This proves a). b) follows from a), Lemma 6.2.16 and Proposition 6.2.18, and c) is a consequence of b). d) is subject to Theorem 6.2.12. It remains to prove e). Note first that by Theorem 6.2.12 the composition is well-defined in the spaces

$$\begin{aligned} C^{\mu,\varrho;\ell}_{G(,V)}(X^\wedge,(\gamma,(-N,0]);F,H)\times C^{\mu',\varrho';\ell}_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F) \\ \longrightarrow C^{\mu+\mu',\varrho+\varrho';\ell}_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,H), \\ C^{\mu,\varrho;\ell}_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);F,H)\times C^{\mu',\varrho';\ell}_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F) \\ \longrightarrow C^{\mu+\mu',\varrho+\varrho';\ell}_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,H), \end{aligned}$$

and consequently the complete assertion e) follows from Lemma 6.2.15 and Lemma 6.2.17. □

6.2.22 Remark. By Theorem 6.2.21 we have the following:

The conormal operators $\{C^{\mu,\varrho;\ell}_{M+G(cl)}(X^\wedge,(\gamma,(-N,0]);E,F)\}$ form a (bi-)graded $*$-algebra, and the conormal symbol mapping induces a $*$-homomorphism of graded algebras onto the algebra of conormal symbols. The kernel of this homomorphism is the (bi-)graded symmetric ideal $\{C^{\mu,\varrho;\ell}_{G}(X^\wedge,(\gamma,(-N,0]);E,F)\}$ of operators that generate asymptotics.

The Volterra conormal operators $\{C^{\mu,\varrho;\ell}_{M+G,V(cl)}(X^\wedge,(\gamma,(-N,0]);E,F)\}$ are a (bi-)-graded subalgebra, and the conormal symbol mapping restricts to a homomorphism of graded algebras onto the algebra of Volterra conormal symbols. The kernel of the restriction is the (bi-)graded ideal $\{C^{\mu,\varrho;\ell}_{G,V}(X^\wedge,(\gamma,(-N,0]);E,F)\}$ of Volterra operators that generate asymptotics.

Smoothing Mellin and Green operators

6.2.23 Definition. We define the space of *smoothing (Volterra) Mellin and Green operators* with respect to the weight datum $(\gamma,(-N,0])$, $N \in \mathbb{N}$, as

$$C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F) := \bigcap_{\mu,\varrho\in\mathbb{R}} C^{\mu,\varrho;\ell}_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F).$$

Consequently, $A : \mathcal{S}^\gamma(X^\wedge,E) \longrightarrow \mathcal{S}^\gamma(X^\wedge,F)$ is a smoothing Mellin and Green operator in $C_{M+G}(X^\wedge,(\gamma,(-N,0]);E,F)$ if and only if

$$A = \sum_{j=0}^{N-1} \omega r^j \mathrm{op}_M^{\gamma_j}(h_j)\omega + G \tag{6.2.ix}$$

with a Green operator $G \in C_G(X^\wedge,(\gamma,(-N,0]);E,F)$, and

i) a cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$,

ii) $\gamma - \frac{n}{2} - j \le \gamma_j \le \gamma - \frac{n}{2}$,

iii) meromorphic Mellin symbols $h_j \in M_{P_j}^{-\infty}(X; E, F)$ such that $\pi_{\mathbb{C}} P_j \cap \Gamma_{\frac{1}{2}-\gamma_j} = \emptyset$.

Moreover, $A : \mathcal{S}^\gamma(X^\wedge, E) \longrightarrow \mathcal{S}^\gamma(X^\wedge, F)$ is a smoothing Volterra Mellin and Green operator in $C_{M+G,V}(X^\wedge, (\gamma, (-N, 0]); E, F)$ if and only if

$$A = \sum_{j=0}^{N-1} \omega r^j \mathrm{op}_M^{\gamma - \frac{n}{2} - j}(h_j)\omega + G \tag{6.2.x}$$

with a Volterra Green operator $G \in C_{G,V}(X^\wedge, (\gamma, (-N, 0]); E, F)$, and

i) a cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$,

ii) meromorphic Volterra Mellin symbols $h_j \in M_{V,P_j}^{-\infty}(X; \mathbb{H}_{\frac{n+1}{2}-\gamma+j}; E, F)$.

6.2.24 Remark. According to Theorem 6.2.21 the smoothing (Volterra) Mellin and Green operators form an ideal in the algebra of (Volterra) conormal operators, i. e., the composition as operators on $\mathcal{S}^\gamma(X^\wedge, E)$ is well-defined in the spaces

$$\begin{aligned} &C_{M+G(,V)}^{\mu,\varrho;\ell}(X^\wedge, (\gamma, (-N, 0]); F, H) \times C_{M+G(,V)}(X^\wedge, (\gamma, (-N, 0]); E, F) \\ &\qquad\qquad \longrightarrow C_{M+G(,V)}(X^\wedge, (\gamma, (-N, 0]); E, H), \\ &C_{M+G(,V)}(X^\wedge, (\gamma, (-N, 0]); F, H) \times C_{M+G(,V)}^{\mu,\varrho;\ell}(X^\wedge, (\gamma, (-N, 0]); E, F) \\ &\qquad\qquad \longrightarrow C_{M+G(,V)}(X^\wedge, (\gamma, (-N, 0]); E, H) \end{aligned}$$

for vector bundles $E, F, H \in \mathrm{Vect}(X)$.

6.2.25 Definition. a) Let $A \in C_{M+G}(X^\wedge, (\gamma, (-N, 0]); E)$. Then the operator $1 + A$ is called *elliptic* if there exists $s_0 \in \mathbb{R}$ such that the operator family $1 + \sigma_M^0(A)(z) : H^{s_0}(X, E) \longrightarrow H^{s_0}(X, E)$ is bijective for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$.

b) Let $A \in C_{M+G,V}(X^\wedge, (\gamma, (-N, 0]); E)$. The operator $1 + A$ is called *parabolic* if there exists $s_0 \in \mathbb{R}$ such that $1 + \sigma_M^0(A)(z) : H^{s_0}(X, E) \longrightarrow H^{s_0}(X, E)$ is bijective for all $z \in \mathbb{H}_{\frac{n+1}{2}-\gamma}$.

6.2.26 Remark. The identity belongs to $C_{M+G,V\,cl}^{0,0;\ell}(X^\wedge, (\gamma, (-N, 0]); E)$ with conormal symbol given as $\sigma_M(1) = \imath$:

With a cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ write

$$1 = \omega 1 \omega + \big(\omega 1 (1 - \omega) + (1 - \omega) 1\big),$$

where $\big(\omega 1(1-\omega)+(1-\omega)1\big) \in C_{G,V}^{0,0;\ell}(X^\wedge,(\gamma,(-\infty,0]);E)$.

Let $A \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E)$. Then we see that $1+A$ is elliptic (parabolic) in the sense of Definition 6.2.25 if and only if $\sigma_M(1+A) = \imath + \sigma_M(A)$ is elliptic (parabolic) in the sense of Definition 6.2.5.

6.2.27 Theorem. *a) Let $A \in C_{M+G}(X^\wedge,(\gamma,(-N,0]);E)$. Then the following are equivalent:*

i) $1+A$ is elliptic in the sense of Definition 6.2.25.

ii) There exists $B \in C_{M+G}(X^\wedge,(\gamma,(-N,0]);E)$ such that $(1+A)(1+B) = 1+G_1$ and $(1+B)(1+A) = 1+G_2$ with Green operators $G_1, G_2 \in C_G(X^\wedge,(\gamma,(-N,0]);E)$.

b) Let $A \in C_{M+G,V}(X^\wedge,(\gamma,(-N,0]);E)$. Then the following are equivalent:

i) $1+A$ is parabolic in the sense of Defintion 6.2.25.

ii) There exists $B \in C_{M+G,V}(X^\wedge,(\gamma,(-N,0]);E)$ such that $(1+A)(1+B) = 1$ and $(1+B)(1+A) = 1$, i. e., $1+A$ is invertible with inverse $(1+A)^{-1} = 1+B$.

Proof. According to Corollary 6.2.8 the operator $1+A$ is elliptic (parabolic) if and only if there exists a (Volterra) conormal symbol $b := (g_0,\dots,g_{N-1}) \in \Sigma_{M(,V)}(X^\wedge,(\gamma,(-N,0]);E)$ such that $\imath + \sigma_M(A)$ is invertible with respect to the Mellin translation product with inverse $\imath + b$. Hence we conclude from Theorem 6.2.21 that the conditions ii) in a) and b) are sufficient for the ellipticity (parabolicity) of the operator $1+A$.

Now assume that $1+A$ is elliptic (parabolic). With $(g_0,\dots,g_{N-1})$ we associate an operator $C \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E)$ via

$$C = \sum_{j=0}^{N-1} \omega r^j \operatorname{op}_M^{\gamma_j}(g_j)\omega$$

in the sense of (6.2.ix) or (6.2.x), respectively. Theorem 6.2.21 implies $(1+A)(1+C) = 1+G_1$ and $(1+C)(1+A) = 1+G_2$ with (Volterra) Green operators $G_1, G_2 \in C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E)$. Hence the proof of a) is finished with $B := C$.

In case of b) the operators G_1 and G_2 are Volterra Green operators. Hence, by Theorem 6.1.6, $1+G_1$ and $1+G_2$ are invertible with inverses $(1+G_1)^{-1} = 1+\tilde{G}_1$ and $(1+G_2)^{-1} = 1+\tilde{G}_2$, where $\tilde{G}_1, \tilde{G}_2 \in C_{G,V}(X^\wedge,(\gamma,(-N,0]);E)$. Consequently, $1+A$ is invertible with inverse $(1+A)^{-1} = 1+B$, where

$$B := \tilde{G}_1 + C + C\tilde{G}_1 = \tilde{G}_2 + C + \tilde{G}_2 C \in C_{M+G,V}(X^\wedge,(\gamma,(-N,0]);E).$$

This finishes the proof of the theorem. □

6.3 The algebra of Volterra cone operators

6.3.1 Notation. Let Y be a topological space. For functions $\varphi, \psi : Y \longrightarrow \mathbb{C}$ we write $\varphi \prec \psi$ if $\psi \equiv 1$ in a neighbourhood of $\operatorname{supp}(\varphi)$.

6.3.2 Definition. Let $(\gamma, (-N, 0])$ be a weight datum, $N \in \mathbb{N}$, and let $\mu, \varrho \in \mathbb{R}$.

a) We define the space $C^{\mu,\varrho;\ell}_{(cl)}(X^\wedge, (\gamma, (-N, 0]); E, F)$ of (classical) *cone pseudodifferential operators* (of order (μ, ϱ)) associated with the weight datum $(\gamma, (-N, 0])$ as follows:

$A : \mathcal{S}^\gamma(X^\wedge, E) \longrightarrow \mathcal{S}^\gamma(X^\wedge, F)$ belongs to $C^{\mu,\varrho;\ell}_{(cl)}(X^\wedge, (\gamma, (-N, 0]); E, F)$ if and only if

- for all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ we have
$$\omega A \tilde{\omega} = \operatorname{op}_M^{\gamma - \frac{n}{2}}(h) + A_{M+G} \tag{6.3.i}$$
with some $h \in C_B^\infty(\overline{\mathbb{R}}_+, M^{\mu;\ell}_{O(cl)}(X; E, F))$, and a smoothing Mellin and Green operator $A_{M+G} \in C_{M+G}(X^\wedge, (\gamma, (-N, 0]); E, F)$,
- for all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ we may write
$$(1 - \omega) A (1 - \tilde{\omega}) = \operatorname{op}_r(a) \tag{6.3.ii}$$
with some $a \in S^\varrho(\mathbb{R}, L^{\mu;\ell}_{(cl)}(X; \mathbb{R}; E, F))$,
- for all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ such that $\omega \prec \tilde{\omega}$ we have
$$\omega A (1 - \tilde{\omega}),\ (1 - \tilde{\omega}) A \omega \in C_G(X^\wedge, (\gamma, (-N, 0]); E, F). \tag{6.3.iii}$$

b) The subspace $C^{\mu,\varrho;\ell}_{V(cl)}(X^\wedge, (\gamma, (-N, 0]); E, F)$ of (classical) *Volterra cone pseudodifferential operators* (of order (μ, ϱ)) associated with the weight datum $(\gamma, (-N, 0])$ is defined as follows:

$A : \mathcal{S}^\gamma(X^\wedge, E) \longrightarrow \mathcal{S}^\gamma(X^\wedge, F)$ belongs to $C^{\mu,\varrho;\ell}_{V(cl)}(X^\wedge, (\gamma, (-N, 0]); E, F)$ if and only if

- for all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ we have
$$\omega A \tilde{\omega} = \operatorname{op}_M^{\gamma - \frac{n}{2}}(h) + A_{M+G} \tag{6.3.iv}$$
with some $h \in C_B^\infty(\overline{\mathbb{R}}_+, M^{\mu;\ell}_{V,O(cl)}(X; E, F))$, and a smoothing Volterra Mellin and Green operator $A_{M+G} \in C_{M+G,V}(X^\wedge, (\gamma, (-N, 0]); E, F)$,

- for all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ we may write

$$(1-\omega)A(1-\tilde{\omega}) = \mathrm{op}_r(a) \tag{6.3.v}$$

with some $a \in S^\varrho(\mathbb{R}, L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}; E, F))$,

- for all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ such that $\omega \prec \tilde{\omega}$ we have

$$\omega A(1-\tilde{\omega}),\ (1-\tilde{\omega})A\omega \in C_{G,V}(X^\wedge, (\gamma, (-N, 0]); E, F). \tag{6.3.vi}$$

6.3.3 Theorem. *An operator* $A : C_0^\infty(X^\wedge, E) \longrightarrow C^\infty(X^\wedge, F)$ *belongs to* $C^{\mu,\varrho;\ell}_{(V(cl))}(X^\wedge, (\gamma, (-N, 0]); E, F)$ *if and only if for some (all) cut-off functions* $\omega_3 \prec \omega_1 \prec \omega_2$ *we may write*

$$A = \omega_1 op_M^{\gamma-\frac{n}{2}}(h)\omega_2 + (1-\omega_1)op_r(a)(1-\omega_3) + A_{M+G}, \tag{6.3.vii}$$

where

$$\begin{aligned} A_{M+G} &\in C_{M+G(,V)}(X^\wedge, (\gamma, (-N, 0]); E, F), \\ h &\in C^\infty(\overline{\mathbb{R}}_+, M^{\mu;\ell}_{(V,)O(cl)}(X; E, F)), \\ a &\in \begin{cases} S^\varrho(\mathbb{R}, L^{\mu;\ell}_{(cl)}(X; \mathbb{R}; E, F)) \\ S^\varrho(\mathbb{R}, L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}; E, F)). \end{cases} \end{aligned}$$

Proof. Let $A \in C^{\mu,\varrho;\ell}_{(V(cl))}(X^\wedge, (\gamma, (-N, 0]); E, F)$, and let $\omega_3 \prec \omega_1 \prec \omega_2$ be arbitrary cut-off functions near $r = 0$. Moreover, let $\tilde{\omega}, \hat{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ be cut-off functions such that $\tilde{\omega} \prec \omega_j \prec \hat{\omega}$ for $j = 1, 2, 3$. We write

$$A = \omega_1\big(\hat{\omega}A\hat{\omega}\big)\omega_2 + (1-\omega_1)\big((1-\tilde{\omega})A(1-\tilde{\omega})\big)(1-\omega_3) + \big(\omega_1 A(1-\omega_2) + (1-\omega_1)A\omega_3\big),$$

and consequently A is of the form (6.3.vii) by Definition 6.3.2.

For the proof of the converse note that it suffices to treat each term in the representation (6.3.vii) separately:

Step 1: $A = \omega_1 \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\omega_2 \in C^{\mu,\varrho;\ell}_{(V(cl))}(X^\wedge, (\gamma, (-N, 0]); E, F)$:

Let $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ be arbitrary cut-off functions near $r = 0$. We have

$$\begin{aligned} \omega A\tilde{\omega} &= \mathrm{op}_M^{\gamma-\frac{n}{2}}(\omega(r)\omega_1(r)h(r,z)\omega_2(r')\tilde{\omega}(r')), \\ \omega(r)\omega_1(r)h(r,z)\omega_2(r')\tilde{\omega}(r') &\in C_B^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, M^{\mu;\ell}_{(V,)O(cl)}(X; E, F)), \end{aligned}$$

and thus we conclude from Theorem 5.3.2 that $\omega A\tilde{\omega} = \mathrm{op}_M^{\gamma-\frac{n}{2}}(g)$ with the left-symbol $g \in C_B^\infty(\overline{\mathbb{R}}_+, M^{\mu;\ell}_{(V,)O(cl)}(X; E, F))$ associated with the double-symbol

$\omega(r)\omega_1(r)h(r,z)\omega_2(r')\tilde{\omega}(r')$. This proves that $\omega A\tilde{\omega}$ is of the form (6.3.i) or (6.3.iv), respectively.

Next consider

$$\begin{aligned}(1-\omega)A(1-\tilde{\omega}) &= \big((1-\omega)\omega_1\big)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\big(\omega_2(1-\tilde{\omega})\big)\\ &= \psi_1\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\psi_2\end{aligned}$$

with $\psi_1,\psi_2 \in C_0^\infty(\mathbb{R}_+)$. According to Theorem 5.2.10 and Theorem 2.6.18 we may write

$$\mathrm{op}_M^{\gamma-\frac{n}{2}}(h) = \mathrm{op}_r(a) + \mathrm{op}_M^{\gamma-\frac{n}{2}}((1-\varphi)(\frac{r'}{r})h)$$

with a function $\varphi \in C_0^\infty(\mathbb{R}_+)$ such that $\varphi \equiv 1$ near $r=1$, and $a(r,\tau) := \tilde{Q}_{\gamma-\frac{n}{2}}(\varphi,h)(r,r\tau)$, where $\tilde{Q}_{\gamma-\frac{n}{2}}$ is the inverse Mellin quantization with respect to the weight $\gamma-\frac{n}{2}$ (cf. Definition 2.6.15). Moreover, Theorem 5.2.10 implies

$$\psi_1(r)a(r,\tau)\psi_2(r') \in \begin{cases} S^{-\infty,-\infty}(\mathbb{R}\times\mathbb{R}, L^{\mu;\ell}_{(cl)}(X;\mathbb{R};E,F))\\ S^{-\infty,-\infty}(\mathbb{R}\times\mathbb{R}, L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F)),\end{cases}$$

and from Theorem 5.4.3 we conclude that $\psi_1\mathrm{op}_r(a)\psi_2 = \mathrm{op}_r(\tilde{a})$ with

$$\tilde{a} \in \begin{cases} S^{-\infty}(\mathbb{R}, L^{\mu;\ell}_{(cl)}(X;\mathbb{R};E,F))\\ S^{-\infty}(\mathbb{R}, L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F)).\end{cases}$$

Next observe that

$$\begin{aligned}&\psi_1\mathrm{op}_M^{\gamma-\frac{n}{2}}((1-\varphi)(\frac{r'}{r})h)\psi_2 = \mathrm{op}_M^{\gamma-\frac{n}{2}}(\psi_1(r)(1-\varphi)(\frac{r'}{r})\psi_2(r')h),\\ &\psi_1(r)(1-\varphi)(\frac{r'}{r})\psi_2(r')h(r,z) \in C_B^\infty(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+, M^{\mu;\ell}_{(V,)O(cl)}(X;E,F)),\\ &\psi_1(r)(1-\varphi)(\frac{r'}{r})\psi_2(r')h(r,z) \equiv 0 \text{ for } |\frac{r}{r'}-1| < \varepsilon\end{aligned}$$

with a sufficiently small $\varepsilon > 0$. Proposition 5.3.4 implies

$$\psi_1\mathrm{op}_M^{\gamma-\frac{n}{2}}((1-\varphi)(\frac{r'}{r})h)\psi_2 = \mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h})$$

with $\tilde{h} \in C_B^\infty(\overline{\mathbb{R}}_+, M^{-\infty}_{(V,)O}(X;E,F))$. Let $\tilde{\psi} \in C_0^\infty(\mathbb{R}_+)$ such that $\psi_1,\psi_2 \prec \tilde{\psi}$. Then we see by construction that

$$\mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h}) = \tilde{\psi}\mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h})\tilde{\psi},$$

and thus Theorem 5.3.6, Theorem 5.3.3 and Proposition 5.3.5 (in case of Volterra operators) imply

$$\mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h}) \in C_{G(,V)}(X^\wedge,(\gamma,(-\infty,0]);E,F).$$

So far we have proved that

$$(1-\omega)A(1-\tilde{\omega}) = \mathrm{op}_r(\tilde{a}) + G$$

with $\tilde{a}$ as above, and $G \in C_{G(,V)}(X^\wedge, (\gamma, (-\infty, 0]); E, F)$ such that $G = \tilde{\psi} G \tilde{\psi}$. From Proposition 6.1.5 we obtain $G = \mathrm{op}_r(\tilde{b})$ with

$$\tilde{b} \in \begin{cases} S^{-\infty}(\mathbb{R}, L^{-\infty}(X; \mathbb{R}; E, F)) \\ S^{-\infty}(\mathbb{R}, L_V^{-\infty}(X; \mathbb{H}; E, F)), \end{cases}$$

and consequently $(1-\omega)A(1-\tilde{\omega}) = \mathrm{op}_r(c)$ with

$$c = \tilde{a} + \tilde{b} \in \begin{cases} S^{-\infty}(\mathbb{R}, L^{\mu;\ell}_{(cl)}(X; \mathbb{R}; E, F)) \\ S^{-\infty}(\mathbb{R}, L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}; E, F)). \end{cases}$$

This shows that $(1-\omega)A(1-\tilde{\omega})$ is of the form (6.3.ii) or (6.3.v), respectively.

Next assume that $\omega \prec \tilde{\omega}$. Then we have

$$\begin{aligned} \omega A(1-\tilde{\omega}) &= \big(\omega\omega_1\big)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\big(\omega_2(1-\tilde{\omega})\big) \\ &= \hat{\omega}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\psi, \\ (1-\tilde{\omega})A\omega &= \big((1-\tilde{\omega})\omega_1\big)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\big(\omega_2\omega\big) \\ &= \varphi\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\breve{\omega} \end{aligned}$$

with cut-off functions $\hat{\omega}, \breve{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ and $\varphi, \psi \in C_0^\infty(\mathbb{R}_+)$, such that $\hat{\omega}\psi \equiv 0$ as well as $\breve{\omega}\varphi \equiv 0$. From Proposition 5.3.4 we conclude that

$$\begin{aligned} \hat{\omega}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\psi &= \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_1) \\ & h_1 \in C_B^\infty(\overline{\mathbb{R}}_+, M^{-\infty}_{(V,)O}(X; E, F)), \\ \varphi\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\breve{\omega} &= \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_2) \\ & h_2 \in C_B^\infty(\overline{\mathbb{R}}_+, M^{-\infty}_{(V,)O}(X; E, F)). \end{aligned}$$

Let $\tilde{\psi} \in C_0^\infty(\mathbb{R}_+)$ and $\eta \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\psi, \varphi \prec \tilde{\psi}$ as well as $\hat{\omega}, \breve{\omega} \prec \eta$. By construction we have

$$\begin{aligned} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_1) &= \eta\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_1)\tilde{\psi}, \\ \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_2) &= \tilde{\psi}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_2)\eta. \end{aligned}$$

From Proposition 2.6.4 we conclude $\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_1) = \mathrm{op}_M^{\gamma'-\frac{n}{2}}(h_1)$ as operators on $C_0^\infty(X^\wedge, E)$, for all $\gamma' \in \mathbb{R}$. Hence we obtain from Theorem 5.3.6

$$\begin{aligned} \eta\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_1)\tilde{\psi} &: \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge, E)_\delta \longrightarrow \mathcal{S}^\gamma_{(-\infty,0]}(X^\wedge, F), \\ \tilde{\psi}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_2)\eta &: \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge, E)_\delta \longrightarrow \mathcal{S}^\gamma_{(-\infty,0]}(X^\wedge, F) \end{aligned}$$

for all $s, t, \delta \in \mathbb{R}$. Using Theorem 5.3.3 we obtain with the same reasoning

$$\big(\eta \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_1)\tilde{\psi}\big)^* = \overline{\tilde{\psi}} \mathrm{op}_M^{-\gamma-\frac{n}{2}}(h_1^*)\overline{\eta} : \mathcal{K}^{(s,t),-\gamma;\ell}(X^\wedge, F)_\delta \longrightarrow \mathcal{S}^{-\gamma}_{(-\infty,0]}(X^\wedge, E),$$
$$\big(\tilde{\psi} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_2)\eta\big)^* = \overline{\eta} \mathrm{op}_M^{-\gamma-\frac{n}{2}}(h_2^*)\overline{\tilde{\psi}} : \mathcal{K}^{(s,t),-\gamma;\ell}(X^\wedge, F)_\delta \longrightarrow \mathcal{S}^{-\gamma}_{(-\infty,0]}(X^\wedge, E)$$

for all $s, t, \delta \in \mathbb{R}$. Summing up, using Proposition 5.3.5 in case of Volterra operators, we have shown

$$\omega A(1-\tilde{\omega}),\ (1-\tilde{\omega})A\omega \in C_{G(,V)}(X^\wedge, (\gamma, (-\infty, 0]); E, F).$$

This finishes the proof of Step 1.

Step 2: $A = (1-\omega_1)\mathrm{op}_r(a)(1-\omega_3) \in C^{\mu,\varrho;\ell}_{(V(cl))}(X^\wedge, (\gamma, (-N, 0]); E, F)$:

Let $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ be arbitrary cut-off functions near $r = 0$. We have

$$(1-\omega)A(1-\tilde{\omega}) = \mathrm{op}_r\Big(\big((1-\omega(r))(1-\omega_1(r))\big)a(r,\tau)\big((1-\omega_3(r'))(1-\tilde{\omega}(r'))\big)\Big),$$
$$\big((1-\omega(r))(1-\omega_1(r))\big)a(r,\tau)\big((1-\omega_3(r'))(1-\tilde{\omega}(r'))\big)$$
$$\in \begin{cases} S^{\varrho,0}(\mathbb{R}\times\mathbb{R}, L^{\mu;\ell}_{(cl)}(X;\mathbb{R};E,F)) \\ S^{\varrho,0}(\mathbb{R}\times\mathbb{R}, L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F)). \end{cases}$$

Consequently, $(1-\omega)A(1-\tilde{\omega}) = \mathrm{op}_r(g)$, where

$$g \in \begin{cases} S^{\varrho}(\mathbb{R}, L^{\mu;\ell}_{(cl)}(X;\mathbb{R};E,F)) \\ S^{\varrho}(\mathbb{R}, L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F)) \end{cases}$$

is the left-symbol associated with the double-symbol

$$\big((1-\omega(r))(1-\omega_1(r))\big)a(r,\tau)\big((1-\omega_3(r'))(1-\tilde{\omega}(r'))\big)$$

according to Theorem 5.4.3. This shows that $(1-\omega)A(1-\tilde{\omega})$ is of the form (6.3.ii) or (6.3.v), respectively.

Next consider the operator

$$\omega A\tilde{\omega} = \big(\omega(1-\omega_1)\big)\mathrm{op}_r(a)\big((1-\omega_3)\tilde{\omega}\big) = \psi_1 \mathrm{op}_r(a)\psi_2$$

with $\psi_1, \psi_2 \in C_0^\infty(\mathbb{R}_+)$. According to Theorem 5.2.10 and Theorem 2.6.18 we may write

$$\mathrm{op}_r(a) = \mathrm{op}_M^{\gamma-\frac{n}{2}}(h) + \mathrm{op}_r\big((1-\varphi)(\frac{r'}{r})a\big),$$

where $\varphi \in C_0^\infty(\mathbb{R}_+)$ such that $\varphi \equiv 1$ near $r = 1$, and $h(r,z) := Q(\varphi, \tilde{a})(r,z)$ with $\tilde{a}(r,\tau) := a(r, r^{-1}\tau)$. Here Q denotes the Mellin quantization, see Definition 2.6.15. From Theorem 5.2.10 we obtain

$$\psi_1(r)h(r,z)\psi_2(r') \in C_B^\infty(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+, M^{\mu;\ell}_{(V,)O(cl)}(X;E,F)),$$

and consequently

$$\psi_1 \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\psi_2 = \mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h})$$

with the left-symbol $\tilde{h} \in C_B^\infty(\overline{\mathbb{R}}_+, M^{\mu;\ell}_{(V,)O(cl)}(X;E,F))$ associated with the double-symbol $\psi_1(r)h(r,z)\psi_2(r')$ according to Theorem 5.3.2. Moreover, we have

$$\psi_1(r)(1-\varphi)(\frac{r'}{r})a(r,\tau)\psi_2(r') \in \begin{cases} S^{-\infty,-\infty}(\mathbb{R}\times\mathbb{R}, L^{\mu;\ell}_{(cl)}(X;\mathbb{R};E,F)) \\ S^{-\infty,-\infty}(\mathbb{R}\times\mathbb{R}, L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F)), \end{cases}$$

$$\psi_1(r)(1-\varphi)(\frac{r'}{r})a(r,\tau)\psi_2(r') \equiv 0 \text{ for } |r-r'|<\varepsilon$$

with some sufficiently small $\varepsilon > 0$. Hence Proposition 5.4.5 implies

$$\psi_1 \mathrm{op}_r\big((1-\varphi)(\frac{r'}{r})a\big)\psi_2 = \mathrm{op}_r(c)$$

with a symbol

$$c \in \begin{cases} S^{-\infty}(\mathbb{R}, L^{-\infty}(X;\mathbb{R};E,F)) \\ S^{-\infty}(\mathbb{R}, L_V^{-\infty}(X;\mathbb{H};E,F)). \end{cases}$$

Let $\tilde{\psi} \in C_0^\infty(\mathbb{R}_+)$ such that $\psi_1,\psi_2 \prec \tilde{\psi}$. Then we have $\tilde{\psi}\mathrm{op}_r(c)\tilde{\psi} = \mathrm{op}_r(c)$ by construction, and from Theorem 5.4.7, Theorem 5.4.4 and Proposition 5.4.6 (in case of Volterra operators) we conclude

$$\tilde{\psi}\mathrm{op}_r(c)\tilde{\psi} \in C_{G(,V)}(X^\wedge,(\gamma,(-\infty,0]);E,F).$$

Summing up, we have shown that $\omega A\tilde{\omega}$ is of the form (6.3.i) or (6.3.iv), respectively.

Next assume that $\omega \prec \tilde{\omega}$. Then we have

$$\begin{aligned} \omega A(1-\tilde{\omega}) &= \big(\omega(1-\omega_1)\big)\mathrm{op}_r(a)\big((1-\omega_3)(1-\tilde{\omega})\big) \\ &= \varphi\mathrm{op}_r(a)(1-\hat{\omega}), \\ (1-\tilde{\omega})A\omega &= \big((1-\tilde{\omega})(1-\omega_1)\big)\mathrm{op}_r(a)\big((1-\omega_3)\omega\big) \\ &= (1-\breve{\omega})\mathrm{op}_r(a)\psi, \end{aligned}$$

where $\varphi,\psi \in C_0^\infty(\mathbb{R}_+)$, and $\hat{\omega},\breve{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ are cut-off functions satisfying $\varphi \prec \hat{\omega}$ and $\psi \prec \breve{\omega}$. From Proposition 5.4.5 we conclude

$$\begin{aligned} &\varphi\mathrm{op}_r(a)(1-\hat{\omega}) = \mathrm{op}_r(a_1) \\ &\qquad a_1 \in \begin{cases} S^{-\infty}(\mathbb{R}, L^{-\infty}(X;\mathbb{R};E,F)) \\ S^{-\infty}(\mathbb{R}, L_V^{-\infty}(X;\mathbb{H};E,F)), \end{cases} \\ &(1-\breve{\omega})\mathrm{op}_r(a)\psi = \mathrm{op}_r(a_2) \\ &\qquad a_2 \in \begin{cases} S^{-\infty}(\mathbb{R}, L^{-\infty}(X;\mathbb{R};E,F)) \\ S^{-\infty}(\mathbb{R}, L_V^{-\infty}(X;\mathbb{H};E,F)). \end{cases} \end{aligned}$$

Choose $\eta \in C_0^\infty(\overline{\mathbb{R}}_+)$ and $\tilde{\psi} \in C_0^\infty(\mathbb{R}_+)$ such that $\varphi, \psi \prec \tilde{\psi}$, and $\eta \prec \hat{\omega}, \breve{\omega}$. Then we have

$$\tilde{\psi}\mathrm{op}_r(a_1)(1-\eta) = \mathrm{op}_r(a_1),$$
$$(1-\eta)\mathrm{op}_r(a_2)\tilde{\psi} = \mathrm{op}_r(a_2)$$

by construction. Theorem 5.4.7, Theorem 5.4.4 and Proposition 5.4.6 (in case of Volterra operators) imply

$$\tilde{\psi}\mathrm{op}_r(a_1)(1-\eta),\ (1-\eta)\mathrm{op}_r(a_2)\tilde{\psi} \in C_{G(,V)}(X^\wedge,(\gamma,(-\infty,0]);E,F).$$

Summing up, we conclude

$$\omega A(1-\tilde{\omega}),\ (1-\tilde{\omega})A\omega \in C_{G(,V)}(X^\wedge,(\gamma,(-\infty,0]);E,F),$$

and the proof of Step 2 is finished.

Step 3: $C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F) \subseteq C^{\mu,\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F)$:

Let $A \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F)$, and let $\omega,\tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ be arbitrary cut-off functions. Then we have $\omega A\tilde{\omega} \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F)$ by definition of the smoothing (Volterra) Mellin and Green operators. Moreover, $\omega A(1-\tilde{\omega})$, $(1-\omega)A\tilde{\omega} \in C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F)$ by Lemma 6.2.15. Let $\hat{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ be a cut-off function such that $\hat{\omega} \prec \omega$ and $\hat{\omega} \prec \tilde{\omega}$. Note that $G := (1-\omega)A(1-\tilde{\omega}) \in C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F)$ due to Lemma 6.2.15, and from the choice of $\hat{\omega}$ we conclude $G = (1-\hat{\omega})G(1-\hat{\omega})$. Hence $G = \mathrm{op}_r(g)$ with

$$g \in \begin{cases} S^{-\infty}(\mathbb{R}, L^{-\infty}(X;\mathbb{R};E,F)) \\ S^{-\infty}(\mathbb{R}, L_V^{-\infty}(X;\mathbb{H};E,F)) \end{cases}$$

due to Proposition 6.1.5, and the proof of Step 3 is complete. □

6.3.4 Theorem. *The following inclusions are valid:*

$$C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F) \subseteq C^{\mu,\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F),$$
$$C^{\mu,\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F) \subseteq C^{\mu,\varrho;\ell}_{M+G(,V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F).$$

Let $A \in C^{\mu,\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F)$, *and let*

$$A = \omega_1 op_M^{\gamma-\frac{n}{2}}(h)\omega_2 + (1-\omega_1)op_r(a)(1-\omega_3) + A_{M+G}$$

be any representation of A *according to (6.3.vii). Then the conormal symbols of* A *are given as*

$$\sigma_M^{-k}(A)(z) = \frac{1}{k!}(\partial_r^k h)(0,z) + \sigma_M^{-k}(A_{M+G})(z)$$

for $k = 0,\dots,N-1$.

Proof. The inclusion

$$C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F) \subseteq C^{\mu,\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F)$$

is subject to Theorem 6.3.3.

Assume that $A \in C^{\mu,\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F)$ is given as

$$A = \omega_1 \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\omega_2 + (1-\omega_1)\mathrm{op}_r(a)(1-\omega_3) + A_{M+G}$$

in the sense of (6.3.vii). For short, we set $h_k := \frac{1}{k!}(\partial_r^k h)(0,z)$ for $k = 0,\dots,N-1$. Then Taylor's formula implies

$$A = \Big(\sum_{k=0}^{N-1} \omega_1 r^k \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_k)\omega_2 + A_{M+G}\Big) + \Big(\omega_1 r^N \mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h})\omega_2 + (1-\omega_1)\mathrm{op}_r(a)(1-\omega_3)\Big)$$

with a function $\tilde{h} \in C_B^\infty(\overline{\mathbb{R}}_+, M^{\mu;\ell}_{(V,)O(cl)}(X;E,F))$. Due to Theorem 5.4.7, Theorem 5.4.4 and Proposition 5.4.6 (in case of Volterra operators) we have

$$(1-\omega_1)\mathrm{op}_r(a)(1-\omega_3) \in C^{\mu,\varrho;\ell}_{G(,V)}(X^\wedge,(\gamma,(-\infty,0]);E,F).$$

Moreover, we conclude from Theorem 5.3.6

$$\omega_1 r^N \mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h})\omega_2 : \begin{cases} \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta \longrightarrow \mathcal{K}^{(s-\mu,t),\gamma+N;\ell}(X^\wedge,F)_\infty \\ \mathcal{S}^\gamma(X^\wedge,E) \longrightarrow \mathcal{S}^{\gamma+N}(X^\wedge,F), \end{cases}$$

and by Theorem 5.3.3 and Proposition 2.6.4 we have

$$\big(\omega_1 r^N \mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h})\omega_2\big)^* = \overline{\omega_2} r^N \mathrm{op}_M^{-\gamma-\frac{n}{2}}(T_{-N}\tilde{h}^*)\overline{\omega_1},$$

and consequently

$$\big(\omega_1 r^N \mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h})\omega_2\big)^* : \begin{cases} \mathcal{K}^{(s,t),-\gamma;\ell}(X^\wedge,E)_\delta \longrightarrow \mathcal{K}^{(s-\mu,t),-\gamma+N;\ell}(X^\wedge,F)_\infty \\ \mathcal{S}^{-\gamma}(X^\wedge,E) \longrightarrow \mathcal{S}^{-\gamma+N}(X^\wedge,F). \end{cases}$$

Using Proposition 5.3.5 (in case of Volterra operators) we thus obtain

$$\omega_1 r^N \mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h})\omega_2 \in C^{\mu,\varrho;\ell}_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F).$$

This shows that

$$A \equiv \sum_{k=0}^{N-1} \omega_1 r^k \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_k)\omega_2 + A_{M+G}$$

modulo $C^{\mu,\varrho;\ell}_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F)$, and from Theorem 6.2.21 we obtain

$$A \in C^{\mu,\varrho;\ell}_{M+G(,V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F),$$
$$\sigma_M^{-k}(A) = h_k + \sigma_M^{-k}(A_{M+G})$$

for $k = 0,\dots,N-1$ as asserted. ☐

6.3.5 Corollary. *Let $A \in C^{\mu,\varrho;\ell}(X^\wedge,(\gamma,(-N,0]);E,F)$. Then A extends by continuity to an operator*

$$A : \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta \longrightarrow \mathcal{K}^{(s-\mu,t),\gamma;\ell}(X^\wedge,F)_{\delta-\varrho}$$

for all $s,t,\delta \in \mathbb{R}$.

Moreover, for every asymptotic type $P \in As((\gamma,(-N,0]),C^\infty(X,E))$ there exists an asymptotic type $Q \in As((\gamma,(-N,0]),C^\infty(X,F))$ such that A restricts to continuous operators

$$A : \begin{cases} \mathcal{K}_P^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta \longrightarrow \mathcal{K}_Q^{(s-\mu,t),\gamma;\ell}(X^\wedge,F)_{\delta-\varrho} \\ \mathcal{S}_P^\gamma(X^\wedge,E) \longrightarrow \mathcal{S}_Q^\gamma(X^\wedge,F) \end{cases}$$

for all $s,t,\delta \in \mathbb{R}$.

Let $A \in C_V^{\mu,\varrho;\ell}(X^\wedge,(\gamma,(-N,0]);E,F)$. Then A restricts for every $r_0 \in \mathbb{R}_+$ to continuous operators

$$A : \begin{cases} \mathcal{T}_{\gamma-\frac{n}{2},0}((0,r_0),C^\infty(X,E)) \longrightarrow \mathcal{T}_{\gamma-\frac{n}{2},0}((0,r_0),C^\infty(X,F)) \\ \mathcal{H}_0^{(s,t),\gamma;\ell}((0,r_0]\times X,E) \longrightarrow \mathcal{H}_0^{(s-\mu,t),\gamma;\ell}((0,r_0]\times X,F) \end{cases}$$

for all $s,t \in \mathbb{R}$.

Proof. This follows from Theorem 6.3.4 and Theorem 6.2.12. □

6.3.6 Corollary. *For vector bundles $E,F,H \in Vect(X)$ the composition as operators on $\mathcal{S}^\gamma(X^\wedge,E)$ is well-defined in the spaces*

$$\begin{aligned} C_{(V)}^{\mu,\varrho;\ell}(X^\wedge,(\gamma,(-N,0]);F,H)&\times C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F) \\ &\longrightarrow C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,H), \\ C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);F,H)&\times C_{(V)}^{\mu,\varrho;\ell}(X^\wedge,(\gamma,(-N,0]);E,F) \\ &\longrightarrow C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,H), \end{aligned}$$

as well as

$$\begin{aligned} C_{(V)}^{\mu,\varrho;\ell}(X^\wedge,(\gamma,(-N,0]);F,H)&\times C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F) \\ &\longrightarrow C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,H), \\ C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);F,H)&\times C_{(V)}^{\mu,\varrho;\ell}(X^\wedge,(\gamma,(-N,0]);E,F) \\ &\longrightarrow C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,H). \end{aligned}$$

Proof. This follows from Theorem 6.3.4 and Theorem 6.2.21. □

The symbolic structure

6.3.7 Theorem. *Let $A \in C^{\mu,\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F)$, and let $0 < T_2 < T_1 < \infty$. There exist*

$$h \in C^\infty_B(\overline{\mathbb{R}}_+, M^{\mu;\ell}_{(V,)O(cl)}(X;E,F)),$$
$$a \in \begin{cases} S^\varrho(\mathbb{R}, L^{\mu;\ell}_{(cl)}(X;\mathbb{R};E,F)) \\ S^\varrho(\mathbb{R}, L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F)) \end{cases}$$

with the following properties:

For all cut-off functions $\omega,\tilde{\omega} \in C^\infty_0(\overline{\mathbb{R}}_+)$ near $r=0$ such that $\chi_{[0,T_2]} \prec \omega,\tilde{\omega} \prec \chi_{[0,T_1]}$ we have

$$\begin{gathered} \omega A\tilde{\omega} - \omega op_M^{\gamma-\frac{n}{2}}(h)\tilde{\omega} \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F), \\ (1-\omega)A(1-\tilde{\omega}) - (1-\omega)op_r(a)(1-\tilde{\omega}) \in C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F). \end{gathered} \tag{6.3.viii}$$

In particular, if

$$\chi_{[0,T_2]} \prec \omega_3 \prec \omega_1 \prec \omega_2 \prec \chi_{[0,T_1]} \tag{6.3.ix}$$

are cut-off functions, we have

$$A = \omega_1 op_M^{\gamma-\frac{n}{2}}(h)\omega_2 + (1-\omega_1)op_r(a)(1-\omega_3) + A_{M+G} \tag{6.3.x}$$

with $A_{M+G} \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F)$.

Proof. Let $\hat{\omega},\breve{\omega} \in C^\infty_0(\overline{\mathbb{R}}_+)$ be cut-off functions such that $\chi_{[0,T_1]} \prec \hat{\omega}$ and $\breve{\omega} \prec \chi_{[0,T_2]}$. According to (6.3.i) and (6.3.iv), respectively, we have

$$\begin{aligned} \hat{\omega}A\hat{\omega} &= \mathrm{op}_M^{\gamma-\frac{n}{2}}(h) + \tilde{A}, \\ h &\in C^\infty_B(\overline{\mathbb{R}}_+, M^{\mu;\ell}_{(V,)O(cl)}(X;E,F)), \\ \tilde{A} &\in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F). \end{aligned}$$

Moreover, according to (6.3.ii) and (6.3.v), respectively, we have

$$(1-\breve{\omega})A(1-\breve{\omega}) = \mathrm{op}_r(a),$$
$$a \in \begin{cases} S^\varrho(\mathbb{R}, L^{\mu;\ell}_{(cl)}(X;\mathbb{R};E,F)) \\ S^\varrho(\mathbb{R}, L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F)). \end{cases}$$

Hence we conclude

$$\omega A\tilde{\omega} = \omega\big(\hat{\omega}A\hat{\omega}\big)\tilde{\omega} = \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\tilde{\omega} + \omega\tilde{A}\tilde{\omega},$$
$$\omega\tilde{A}\tilde{\omega} \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F),$$
$$(1-\omega)A(1-\tilde{\omega}) = (1-\omega)\big((1-\breve{\omega})A(1-\breve{\omega})\big)(1-\tilde{\omega}) = (1-\omega)\mathrm{op}_r(a)(1-\tilde{\omega}),$$

i. e., the tuple (h,a) fulfills (6.3.viii).

Next let $\omega_1,\omega_2,\omega_3 \in C_0^\infty(\overline{\mathbb{R}}_+)$ be cut-off functions that satisfy (6.3.ix). We may write

$$A = \omega_1 A\omega_2 + (1-\omega_1)A(1-\omega_3) + \big(\omega_1 A(1-\omega_2) + (1-\omega_1)A\omega_3\big),$$

where

$$\big(\omega_1 A(1-\omega_2) + (1-\omega_1)A\omega_3\big) \in C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F)$$

according to (6.3.iii) and (6.3.vi), respectively. Consequently, A is of the form (6.3.x). This finishes the proof of the theorem. □

6.3.8 Notation. We refer to any system of cut-off functions $\{\omega_1,\omega_2,\omega_3\}$ satisfying (6.3.ix) as subordinated to the covering $\{[0,T_1),(T_2,\infty)\}$ of $\overline{\mathbb{R}}_+$.

6.3.9 Definition. Let $A \in C^{\mu,\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F)$. We associate with A the following triple of symbols:

- **Complete interior symbol:**

  Let $0 < T_2 < T_1 < \infty$. Any tuple

  $$(h,a) \in \begin{cases} C_B^\infty(\overline{\mathbb{R}}_+, M^{\mu;\ell}_{O(cl)}(X;E,F)) \times S^\varrho(\mathbb{R}, L^{\mu;\ell}_{(cl)}(X;\mathbb{R};E,F)) \\ C_B^\infty(\overline{\mathbb{R}}_+, M^{\mu;\ell}_{V,O(cl)}(X;E,F)) \times S^\varrho(\mathbb{R}, L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F)) \end{cases}$$

  that satisfies (6.3.viii) in Theorem 6.3.7 is called a *complete interior symbol* of the operator A, subordinated to the covering $\{[0,T_1),(T_2,\infty)\}$ of $\overline{\mathbb{R}}_+$. For short, we write

  $$\sigma^{\mu,\varrho;\ell}_{\psi,c}(A) := (h,a).$$

  Note that the relation $A \mapsto \sigma^{\mu,\varrho;\ell}_{\psi,c}(A)$ is non-canonical.

- **Conormal symbol:**

  According to Theorem 6.3.4 and Theorem 6.2.21 we associate with A the tuple

  $$\sigma_M(A) = (\sigma_M^0(A),\ldots,\sigma_M^{-(N-1)}(A)) \in \Sigma^{\mu;\ell}_{M(,V(cl))}(X,(\gamma,(-N,0]);E,F)$$

  of conormal symbols.

- **Exit symbol:**
  Let $T > 0$. According to Theorem 6.3.7 there exists a symbol

$$a \in \begin{cases} S^{\varrho}(\mathbb{R}, L^{\mu;\ell}_{(cl)}(X;\mathbb{R};E,F)) \\ S^{\varrho}(\mathbb{R}, L^{\mu;\ell}_{V\,(cl)}(X;\mathbb{H};E,F)) \end{cases}$$

  such that

$$(1-\omega)A(1-\tilde{\omega}) - (1-\omega)\mathrm{op}_r(a)(1-\tilde{\omega}) \in C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F)$$

  for all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ with $\chi_{[0,T]} \prec \omega, \tilde{\omega}$. Any symbol a that satisfies these conditions is called an *exit symbol* of the operator A. The exit symbol is regarded as an operator family

$$\sigma_e^{\mu,\varrho;\ell}(A)(r,\tau) := a(r,\tau) : H^s(X,E) \longrightarrow H^{s-\mu}(X,F)$$

  for $\tau \in \mathbb{R}$, respectively $\tau \in \mathbb{H}$, and $r \gg r_0$ sufficiently large.
  Note that the relation $A \mapsto \sigma_e^{\mu,\varrho;\ell}(A)$ is non-canonical.

Let $A \in C^{\mu,\varrho;\ell}_{(V)cl}(X^\wedge,(\gamma,(-N,0]);E,F)$ be a classical (Volterra) cone operator.

- **Homogeneous principal symbol:**
  The (anisotropic) *homogeneous principal symbol* of A is well-defined as a function

$$\sigma_\psi^{\mu;\ell}(A) \in \begin{cases} C^\infty(\mathbb{R}_+, S^{(\mu;\ell)}((T^*X\times\mathbb{R}) \setminus 0, \mathrm{Hom}(\pi^*E,\pi^*F))) \\ C^\infty(\mathbb{R}_+, S_V^{(\mu;\ell)}((T^*X\times\mathbb{H}) \setminus 0, \mathrm{Hom}(\pi^*E,\pi^*F))), \end{cases}$$

  and it has the following properties:

$$\sigma_\psi^{\mu;\ell}(A)(r,\xi_x,r^{-1}\tau) \in \begin{cases} C^\infty(\overline{\mathbb{R}}_+, S^{(\mu;\ell)}((T^*X\times\mathbb{R}) \setminus 0, \mathrm{Hom}(\pi^*E,\pi^*F))) \\ C^\infty(\overline{\mathbb{R}}_+, S_V^{(\mu;\ell)}((T^*X\times\mathbb{H}) \setminus 0, \mathrm{Hom}(\pi^*E,\pi^*F))), \end{cases}$$

$$(1-\omega)\sigma_\psi^{\mu;\ell}(A) \in \begin{cases} S^{\varrho}(\mathbb{R}, S^{(\mu;\ell)}((T^*X\times\mathbb{R}) \setminus 0, \mathrm{Hom}(\pi^*E,\pi^*F))) \\ S^{\varrho}(\mathbb{R}, S_V^{(\mu;\ell)}((T^*X\times\mathbb{H}) \setminus 0, \mathrm{Hom}(\pi^*E,\pi^*F))) \end{cases}$$

  for every cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$. More precisely, let

$$A = \omega_1 \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\omega_2 + (1-\omega_1)\mathrm{op}_r(a)(1-\omega_3) + A_{M+G}$$

  be any representation of A according to (6.3.vii). Then

$$\begin{aligned} \sigma_\psi^{\mu;\ell}(A)(r,\xi_x,\tau) = {} & \omega_1(r)\sigma_\psi^{\mu;\ell}(h)(r,\xi_x,\frac{n+1}{2}-\gamma-i(r\tau)) \\ & + (1-\omega_1(r))\sigma_\psi^{\mu;\ell}(a)(r,\xi_x,\tau) \end{aligned} \tag{6.3.xi}$$

  for $r \in \mathbb{R}_+$ and $(\xi_x,\tau) \in \big(T^*X\times\mathbb{R}\big) \setminus 0$, respectively $(\xi_x,\tau) \in \big(T^*X\times\mathbb{H}\big) \setminus 0$. This relation follows from the results concerning the (inverse) Mellin quantization in Theorem 5.2.10 and Theorem 2.6.18; note in particular the asymptotic expansion formula (2.6.ix) in Theorem 2.6.16.

Compositions and adjoints

6.3.10 Theorem. *Let H be another vector bundle over X, and let*

$$A \in C^{\mu,\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);F,H),$$
$$B \in C^{\mu',\varrho';\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F).$$

Then we have

$$AB \in C^{\mu+\mu',\varrho+\varrho';\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);E,H)$$

for the composition as operators on $\mathcal{S}^\gamma(X^\wedge,E)$.

Let $0 < T_2 < T_1 < \infty$, and let $\sigma^{\mu,\varrho;\ell}_{\psi,c}(A) = (h_1,a_1)$ and $\sigma^{\mu',\varrho';\ell}_{\psi,c}(B) = (h_2,a_2)$ be complete interior symbols of A and B subordinated to the covering $\{[0,T_1),(T_2,\infty)\}$ of $\overline{\mathbb{R}}_+$. Then

$$\sigma^{\mu+\mu',\varrho+\varrho';\ell}_{\psi,c}(AB) = (h_1\# h_2, a_1\# a_2)$$

is a complete interior symbol of the composition AB, subordinated to the covering $\{[0,T_1),(T_2,\infty)\}$ of $\overline{\mathbb{R}}_+$. The involved Leibniz-products are according to Theorem 5.3.3 and Theorem 5.4.4, respectively.

The following relations hold for the exit symbol and the conormal symbols:

$$\sigma^{\mu+\mu',\varrho+\varrho';\ell}_e(AB) = \sigma^{\mu,\varrho;\ell}_e(A)\#\sigma^{\mu',\varrho';\ell}_e(B),$$
$$\sigma^{-k}_M(AB) = \sum_{p+q=k} T_{-q}\sigma^{-p}_M(A)\sigma^{-q}_M(B)$$

for $k = 0,\ldots,N-1$. The Leibniz-product of the exit symbols is according to Theorem 5.4.4.

If A and B are classical (Volterra) cone operators, then the (anisotropic) homogeneous principal symbol of the composition is given as

$$\sigma^{\mu+\mu';\ell}_\psi(AB) = \sigma^{\mu;\ell}_\psi(A)\sigma^{\mu';\ell}_\psi(B).$$

Proof. Let $\omega,\tilde\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ be cut-off functions near $r=0$.

Step 1: Consider the operator $\omega AB\tilde\omega$:

Choose cut-off functions $\hat\omega,\breve\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\omega,\tilde\omega \prec \hat\omega \prec \breve\omega$. We may write

$$\omega AB\tilde\omega = \big(\omega A\hat\omega\big)\big(\breve\omega B\tilde\omega\big) + \omega A(1-\hat\omega)B\tilde\omega.$$

Now $\omega A(1-\hat\omega) \in C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);F,H)$ according to Definition 6.3.2, and from Corollary 6.3.6 we conclude

$$\omega AB\tilde\omega \equiv \big(\omega A\hat\omega\big)\big(\breve\omega B\tilde\omega\big) \mod C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,H).$$

We have

$$\left.\begin{aligned} \omega A\hat{\omega} &\equiv \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_1)\hat{\omega} \quad \bmod \ C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);F,H), \\ h_1 &\in C_B^\infty(\overline{\mathbb{R}}_+, M^{\mu;\ell}_{(V,)O(cl)}(X;F,H)), \\ \check{\omega} B\tilde{\omega} &\equiv \check{\omega} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_2)\tilde{\omega} \quad \bmod \ C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F), \\ h_2 &\in C_B^\infty(\overline{\mathbb{R}}_+, M^{\mu';\ell}_{(V,)O(cl)}(X;E,F)). \end{aligned}\right\} \tag{1}$$

To see this choose a cut-off function $\omega' \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\check{\omega} \prec \omega'$. From (6.3.i), (6.3.iv) we obtain

$$\begin{aligned} \omega' A\omega' &= \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_1) + \tilde{A}, \\ &\tilde{A} \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);F,H), \\ \omega' B\omega' &= \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_2) + \tilde{B}, \\ &\tilde{B} \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F), \end{aligned}$$

and hence multiplying from the left and from the right with the involved cut-off functions $\omega, \hat{\omega}$ and $\check{\omega}, \tilde{\omega}$, respectively, yields (1). Theorem 6.3.3 and Corollary 6.3.6 imply

$$\big(\omega A\hat{\omega}\big)\big(\check{\omega} B\tilde{\omega}\big) \equiv \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_1)\hat{\omega} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_2)\tilde{\omega}$$

modulo $C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,H)$. From Theorem 5.3.3 and Proposition 5.3.4 we conclude

$$\begin{aligned} \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_1)\hat{\omega} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_2)\tilde{\omega} &= \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_1 \# h_2)\tilde{\omega} - \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_1)(1-\hat{\omega})\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_2)\tilde{\omega}, \\ \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_1)(1-\hat{\omega})\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_2)\tilde{\omega} &= \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h})\tilde{\omega}, \\ \tilde{h} &\in C_B^\infty(\overline{\mathbb{R}}_+, M^{-\infty}_{(V,)O}(X;E,H)). \end{aligned}$$

Carrying out a Taylor expansion we may write

$$\begin{aligned} \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h})\tilde{\omega} &= \sum_{j=0}^{N-1} \omega r^j \mathrm{op}_M^{\gamma-\frac{n}{2}}\Big(\frac{1}{j!}(\partial_r^j \tilde{h})(0,z)\Big)\tilde{\omega} + \omega r^N \mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h}_N)\tilde{\omega}, \\ \frac{1}{j!}(\partial_r^j \tilde{h})(0,z) &\in M^{-\infty}_{(V,)O}(X;E,H), \ j=0,\dots,N-1, \\ \tilde{h}_N &\in C_B^\infty(\overline{\mathbb{R}}_+, M^{-\infty}_{(V,)O}(X;E,H)), \end{aligned}$$

and from Theorem 5.3.6, Theorem 5.3.3, Proposition 2.6.4 and Proposition 5.3.5 (in case of Volterra operators) we obtain

$$\begin{aligned} \omega r^N \mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h}_N)\tilde{\omega} &\in C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,H), \text{ i. e.} \\ \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h})\tilde{\omega} &\in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,H). \end{aligned}$$

Summing up, we have shown

$$\omega AB\tilde{\omega} \equiv \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_1 \# h_2)\tilde{\omega}$$

modulo $C_{M+G(,V)}(X^\wedge, (\gamma, (-N, 0]); E, H)$, and by Theorem 5.3.2 $\omega AB\tilde{\omega}$ is of the form (6.3.i) or (6.3.iv), respectively.

If $0 < T_2 < T_1 < \infty$ and $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ are cut-off functions with $\chi_{[0,T_2]} \prec \omega, \tilde{\omega} \prec \chi_{[0,T_1]}$ we choose the cut-off functions $\hat{\omega}, \breve{\omega}$ at the beginning of the proof of Step 1 with $\omega, \tilde{\omega} \prec \hat{\omega} \prec \breve{\omega} \prec \chi_{[0,T_1]}$. Hence we see that in (1) we may choose the holomorphic Mellin symbols h_1, h_2 as the Mellin components of the complete interior symbols of A and B subordinated to the covering $\{[0, T_1), (T_2, \infty)\}$ of $\overline{\mathbb{R}}_+$. Consequently, the Leibniz-product $h_1 \# h_2$ serves as the Mellin component of a complete interior symbol of the composition AB, subordinated to the covering $\{[0, T_1), (T_2, \infty)\}$ of $\overline{\mathbb{R}}_+$.

Step 2: Consider $(1-\omega)AB(1-\tilde{\omega})$:

Let $\hat{\omega}, \breve{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ be cut-off functions with $\hat{\omega} \prec \breve{\omega} \prec \omega, \tilde{\omega}$. We may write

$$(1-\omega)AB(1-\tilde{\omega}) = \big((1-\omega)A(1-\hat{\omega})\big)\big((1-\breve{\omega})B(1-\tilde{\omega})\big) + (1-\omega)A\breve{\omega}B(1-\tilde{\omega}),$$

where $(1-\omega)A\breve{\omega} \in C_{G(,V)}(X^\wedge, (\gamma, (-N, 0]); F, H)$ by Definition 6.3.2, and consequently

$$(1-\omega)AB(1-\tilde{\omega}) \equiv \big((1-\omega)A(1-\hat{\omega})\big)\big((1-\breve{\omega})B(1-\tilde{\omega})\big) \quad \mathrm{mod}\ C_{G(,V)}(X^\wedge, (\gamma, (-N, 0]); E, H)$$

due to Corollary 6.3.6. We have

$$\left.\begin{aligned} (1-\omega)A(1-\hat{\omega}) &\equiv (1-\omega)\mathrm{op}_r(a_1)(1-\hat{\omega}) \\ &\qquad \mathrm{mod}\ C_{G(,V)}(X^\wedge, (\gamma, (-N, 0]); F, H), \\ a_1 &\in \begin{cases} S^\varrho(\mathbb{R}, L^{\mu;\ell}_{(cl)}(X; \mathbb{R}; F, H) \\ S^\varrho(\mathbb{R}; L^{\mu;\ell}_{V\,(cl)}(X; \mathbb{H}; F, H), \end{cases} \\ (1-\breve{\omega})B(1-\tilde{\omega}) &\equiv (1-\breve{\omega})\mathrm{op}_r(a_2)(1-\tilde{\omega}) \\ &\qquad \mathrm{mod}\ C_{G(,V)}(X^\wedge, (\gamma, (-N, 0]); E, F), \\ a_2 &\in \begin{cases} S^{\varrho'}(\mathbb{R}, L^{\mu';\ell}_{(cl)}(X; \mathbb{R}; E, F) \\ S^{\varrho'}(\mathbb{R}; L^{\mu';\ell}_{V\,(cl)}(X; \mathbb{H}; E, F). \end{cases} \end{aligned}\right\} \qquad (2)$$

Consequently, we obtain from Theorem 6.3.3, Corollary 6.3.6 and Theorem 5.4.4

$$(1-\omega)AB(1-\tilde{\omega}) \equiv (1-\omega)\mathrm{op}_r(a_1 \# a_2)(1-\tilde{\omega}) + (1-\omega)\mathrm{op}_r(a_1)\breve{\omega}\mathrm{op}_r(a_2)(1-\tilde{\omega})$$

modulo $C_{G(,V)}(X^\wedge, (\gamma, (-N, 0]); E, H)$, and by Theorem 5.4.3 and Proposition 5.4.5

$$(1-\omega)\mathrm{op}_r(a_1)\breve{\omega}\mathrm{op}_r(a_2)(1-\tilde{\omega}) = (1-\omega)\mathrm{op}_r(c)(1-\tilde{\omega}),$$
$$c \in \begin{cases} S^{-\infty}(\mathbb{R}, L^{-\infty}(X; \mathbb{R}; E, H) \\ S^{-\infty}(\mathbb{R}; L_V^{-\infty}(X; \mathbb{H}; E, H). \end{cases}$$

Theorem 5.4.7, Theorem 5.4.4 and Proposition 5.4.6 (in case of Volterra operators) imply

$$(1-\omega)\mathrm{op}_r(c)(1-\tilde{\omega}) \in C_{G(,V)}(X^\wedge, (\gamma, (-\infty, 0]); E, H).$$

Summing up, we have shown

$$(1-\omega)AB(1-\tilde{\omega}) \equiv (1-\omega)\mathrm{op}_r(a_1 \# a_2)(1-\tilde{\omega})$$

modulo $C_{G(,V)}(X^\wedge, (\gamma, (-N, 0]); E, H)$, and by Proposition 6.1.5 and Theorem 5.4.3 $(1-\omega)AB(1-\tilde{\omega})$ is of the form (6.3.ii) or (6.3.v), respectively.

Let $0 < T_2 < T_1 < \infty$, and let $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ be cut-off functions with $\chi_{[0,T_2]} \prec \omega, \tilde{\omega} \prec \chi_{[0,T_1]}$. At the beginning of the proof of Step 2 choose suitable cut-off functions $\hat{\omega}, \breve{\omega}$ such that $\chi_{[0,T_2]} \prec \hat{\omega} \prec \breve{\omega} \prec \omega, \tilde{\omega}$. Hence in (2) we may choose the symbols a_1, a_2 as the Fourier components of the complete interior symbols of A and B, subordinated to the covering $\{[0, T_1), (T_2, \infty)\}$ of $\overline{\mathbb{R}}_+$. Consequently, the Leibniz-product $a_1 \# a_2$ is a possible choice of the Fourier component of a complete interior symbol of the composition AB, subordinated to the covering $\{[0, T_1), (T_2, \infty)\}$ of $\overline{\mathbb{R}}_+$. Moreover, we see from the proof that the Leibniz-product of the exit symbols of A and B is an exit symbol of the composition AB.

Step 3: Assume $\omega \prec \tilde{\omega}$, and consider the operators $\omega AB(1-\tilde{\omega})$ and $(1-\tilde{\omega})AB\omega$: Let $\hat{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ be a cut-off function near $r = 0$ such that $\omega \prec \hat{\omega} \prec \tilde{\omega}$. We may write

$$\omega AB(1-\tilde{\omega}) = \big(\omega A(1-\hat{\omega})\big)B(1-\tilde{\omega}) + \omega A\big(\hat{\omega}B(1-\tilde{\omega})\big),$$
$$(1-\tilde{\omega})AB\omega = \big((1-\tilde{\omega})A\hat{\omega}\big)B\omega + (1-\tilde{\omega})A\big((1-\hat{\omega})B\omega\big).$$

Due to (6.3.iii) and (6.3.vi), respectively, we have

$$\omega A(1-\hat{\omega}),\ (1-\tilde{\omega})A\hat{\omega} \in C_{G(,V)}(X^\wedge, (\gamma, (-N, 0]); F, H),$$
$$\hat{\omega}B(1-\tilde{\omega}),\ (1-\hat{\omega})B\omega \in C_{G(,V)}(X^\wedge, (\gamma, (-N, 0]); E, F),$$

and from Corollary 6.3.6 we obtain

$$\omega AB(1-\tilde{\omega}),\ (1-\tilde{\omega})AB\omega \in C_{G(,V)}(X^\wedge, (\gamma, (-N, 0]); E, H).$$

Conclusion: From Step 1 – 3 we see that the composition AB belongs to $C_{(V(cl))}^{\mu+\mu', \varrho+\varrho'; \ell}(X^\wedge, (\gamma, (-N, 0]); E, H)$ in view of Definition 6.3.2. Moreover, the formula for the complete interior symbol of AB subordinated to a given covering

$\{[0,T_1),(T_2,\infty)\}$ of $\overline{\mathbb{R}}_+$, as well as the relationship for the exit symbol, are proved in Step 1 and Step 2. The identities for the conormal symbols of AB are subject to Theorem 6.2.21, see also Theorem 6.3.4.

For classical operators the homogeneous principal symbol of the composition is given as the product of the homogeneous principal symbols. This follows from the relationship for the complete interior symbol of AB and equation (6.3.xi), keeping in mind the asymptotic expansion formulae for the Leibniz-products. $\square$

6.3.11 Theorem. *Let $A \in C^{\mu,\varrho;\ell}_{(cl)}(X^\wedge,(\gamma,(-N,0]);E,F)$. Then the formal adjoint A^* with respect to the $r^{-\frac{n}{2}}L^2$-inner product is a cone pseudodifferential operator in $C^{\mu,\varrho;\ell}_{(cl)}(X^\wedge,(-\gamma,(-N,0]);F,E)$.*

Let $0<T_2<T_1<\infty$, and let $\sigma^{\mu,\varrho;\ell}_{\psi,c}(A)=(h,a)$ be a complete interior symbol of A subordinated to the covering $\{[0,T_1),(T_2,\infty)\}$ of $\overline{\mathbb{R}}_+$. Then

$$\sigma^{\mu,\varrho;\ell}_{\psi,c}(A^*)=(h^*,a^{(*),\frac{n}{2}})$$

is a complete interior symbol of A^* subordinated to the covering $\{[0,T_1),(T_2,\infty)\}$ of $\overline{\mathbb{R}}_+$. The adjoint symbols are according to Theorem 5.3.3 and Theorem 5.4.4, respectively.

The following relations hold for the exit symbol and the conormal symbols:

$$\begin{aligned}\sigma^{\mu,\varrho;\ell}_e(A^*)&=\sigma^{\mu,\varrho;\ell}_e(A)^{(*),\frac{n}{2}},\\ \sigma^{-k}_M(A^*)(z)&=(\sigma^{-k}_M(A)(n+1-k-\overline{z}))^{(*)}\end{aligned}$$

for $k=0,\dots,N-1$, where $^{(*)}$ denotes the formal adjoint with respect to the L^2-inner product on the manifold X in the formula for the conormal symbols, and the adjoint exit symbol is according to Theorem 5.4.4.

If A is a classical cone operator, then the (anisotropic) homogeneous principal symbol of A^* is given as

$$\sigma^{\mu;\ell}_\psi(A^*)=\sigma^{\mu;\ell}_\psi(A)^*.$$

Proof. Let $\omega,\tilde{\omega}\in C_0^\infty(\overline{\mathbb{R}}_+)$ be cut-off functions near $r=0$. According to (6.3.i) we may write

$$\begin{aligned}\overline{\tilde{\omega}}A\overline{\omega}&=\overline{\tilde{\omega}}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\overline{\omega}+A_{M+G},\\ A_{M+G}&\in C_{M+G}(X^\wedge,(\gamma,(-N,0]);E,F),\\ h&\in C_B^\infty(\overline{\mathbb{R}}_+,M^{\mu;\ell}_{O(cl)}(X;E,F)).\end{aligned}$$

Hence

$$\omega A^*\tilde{\omega}=\omega\mathrm{op}_M^{-\gamma-\frac{n}{2}}(h^*)\tilde{\omega}+A^*_{M+G}$$

with the adjoint symbol h^* from Theorem 5.3.3. Note that due to Theorem 6.2.21 we have $A^*_{M+G} \in C_{M+G}(X^\wedge, (-\gamma, (-N, 0]); F, E)$. Consequently, $\omega A^* \tilde{\omega}$ is of the form (6.3.i) by Theorem 5.3.2.

In view of (6.3.ii) we have

$$\begin{aligned}(1-\overline{\tilde{\omega}})A(1-\overline{\omega}) &= (1-\overline{\tilde{\omega}})\mathrm{op}_r(a)(1-\overline{\omega}) + G,\\ G &\in C_G(X^\wedge, (\gamma, (-N, 0]); E, F),\\ a &\in S^\varrho(\mathbb{R}, L^{\mu;\ell}_{(cl)}(X; \mathbb{R}; E, F)),\end{aligned}$$

and thus

$$(1-\omega)A^*(1-\tilde{\omega}) = (1-\omega)\mathrm{op}_r(a^{(*),\frac{n}{2}})(1-\tilde{\omega}) + G^*,$$

where $G^* \in C_G(X^\wedge, (-\gamma, (-N, 0]); F, E)$, and $a^{(*),\frac{n}{2}}$ is the adjoint symbol to a from Theorem 5.4.4. Proposition 6.1.5 and Theorem 5.4.3 imply that $(1-\omega)A^*(1-\tilde{\omega})$ is of the form (6.3.ii).

Next assume that $\omega \prec \tilde{\omega}$. Then also $\overline{\omega} \prec \overline{\tilde{\omega}}$, and thus

$$(1-\overline{\tilde{\omega}})A\overline{\omega},\ \overline{\omega}A(1-\overline{\tilde{\omega}}) \in C_G(X^\wedge, (\gamma, (-N, 0]); E, F)$$

by (6.3.iii). Consequently,

$$\omega A^*(1-\tilde{\omega}),\ (1-\tilde{\omega})A^*\omega \in C_G(X^\wedge, (-\gamma, (-N, 0]); F, E).$$

From Definition 6.3.2 we obtain that $A^* \in C^{\mu,\varrho;\ell}_{(cl)}(X^\wedge, (-\gamma, (-N, 0]); F, E)$ as desired. Moreover, the formulae for the complete interior symbol of A^* subordinated to a covering $\{[0, T_1), (T_2, \infty)\}$ of $\overline{\mathbb{R}}_+$, as well as for the exit symbol, follow immediately from the proof. In the classical case, the identity $\sigma^{\mu;\ell}_\psi(A^*) = \sigma^{\mu;\ell}_\psi(A)^*$ for the homogeneous principal symbol is a consequence of (6.3.xi), the relation for the complete interior symbol, and the asymptotic expansions of h^* in terms of h, and of $a^{(*),\frac{n}{2}}$ in terms of a, from Theorem 5.3.3 and Theorem 5.4.4, respectively. The assertion concerning the conormal symbols follows from Theorem 6.3.4 and Theorem 6.2.21. □

6.4 Ellipticity and Parabolicity

6.4.1 Definition. Let $A \in C^{\mu,\varrho;\ell}(X^\wedge, (\gamma, (-N, 0]); E, F)$.

a) **Interior ellipticity:**

A is called *elliptic in the interior*, respectively *interior elliptic*, if for each $T > 0$ the following conditions are fulfilled:

- For all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\chi_{[0,T]} \prec \omega, \tilde{\omega}$, and all representations

$$\begin{aligned} \omega A \tilde{\omega} &= \mathrm{op}_M^{\gamma-\frac{n}{2}}(h) + A_{M+G}, \\ A_{M+G} &\in C_{M+G}(X^\wedge, (\gamma, (-N,0]); E, F), \\ h &\in C_B^\infty(\overline{\mathbb{R}}_+, M_O^{\mu;\ell}(X; E, F)) \end{aligned}$$

according to (6.3.i), we require that h is elliptic on the interval $[0,T]$ in the sense of Definition 5.3.8.

- For all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\omega, \tilde{\omega} \prec \chi_{[0,T]}$ write

$$\begin{aligned} (1-\omega)A(1-\tilde{\omega}) &= \mathrm{op}_r(a), \\ a &\in S^\varrho(\mathbb{R}, L^{\mu;\ell}(X; \mathbb{R}; E, F)) \end{aligned}$$

according to (6.3.ii). We require that a is interior elliptic on the interval $[T, \infty)$ in the sense of Definition 5.4.9.

b) **Conormal ellipticity:**

A is called *conormal elliptic* if there exists $s_0 \in \mathbb{R}$ such that the conormal symbol

$$\sigma_M^0(A)(z) : H^{s_0}(X, E) \longrightarrow H^{s_0-\mu}(X, F)$$

is an isomorphism for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$.

c) **Exit ellipticity:**

A is called *exit elliptic* if the following is fulfilled: For all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r=0$ write

$$\begin{aligned} (1-\omega)A(1-\tilde{\omega}) &= \mathrm{op}_r(a), \\ a &\in S^\varrho(\mathbb{R}, L^{\mu;\ell}(X; \mathbb{R}; E, F)) \end{aligned}$$

according to (6.3.ii). We require that there exists $s_0 \in \mathbb{R}$ and $r_0 \in \mathbb{R}_+$, such that for $r > r_0$ and all $\tau \in \mathbb{R}$

$$a(r,\tau) : H^{s_0}(X, E) \longrightarrow H^{s_0-\mu}(X, F)$$

is an isomorphism, and we have for some $M \in \mathbb{R}$

$$\|a(r,\tau)^{-1}\|_{\mathcal{L}(H^{s_0-\mu}, H^{s_0})} \langle \tau \rangle^M \langle r \rangle^\varrho = O(1),$$

uniformly for $\tau \in \mathbb{R}$ and $r \to \infty$.

A is called *elliptic* if A is interior elliptic, conormal elliptic, and exit elliptic.

6.4.2 Definition. Let $A \in C_V^{\mu,\varrho;\ell}(X^\wedge, (\gamma, (-N,0]); E, F)$.

a) **Interior parabolicity:**

A is called *parabolic in the interior*, respectively *interior parabolic*, if for each $T > 0$ the following is fulfilled:

- For all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\chi_{[0,T]} \prec \omega, \tilde{\omega}$, and all representations

$$\begin{aligned} \omega A \tilde{\omega} &= \mathrm{op}_M^{\gamma - \frac{n}{2}}(h) + A_{M+G}, \\ A_{M+G} &\in C_{M+G,V}(X^\wedge, (\gamma, (-N, 0]); E, F), \\ h &\in C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O}^{\mu;\ell}(X; E, F)) \end{aligned}$$

  according to (6.3.iv), we require that h is parabolic on the interval $[0, T]$ in the sense of Definition 5.3.8.

- For all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\omega, \tilde{\omega} \prec \chi_{[0,T]}$ write

$$\begin{aligned} (1-\omega) A (1-\tilde{\omega}) &= \mathrm{op}_r(a), \\ a &\in S^\varrho(\mathbb{R}, L_V^{\mu;\ell}(X; \mathbb{H}; E, F)) \end{aligned}$$

  according to (6.3.v). We require that a is interior parabolic on the interval $[T, \infty)$ in the sense of Definition 5.4.9.

b) **Conormal parabolicity:**

A is called *conormal parabolic* if there exists $s_0 \in \mathbb{R}$ such that the conormal symbol

$$\sigma_M^0(A)(z) : H^{s_0}(X, E) \longrightarrow H^{s_0 - \mu}(X, F)$$

is an isomorphism for all $z \in \mathbb{H}_{\frac{n+1}{2} - \gamma}$.

c) **Exit parabolicity:**

A is called *exit parabolic* if the following is fulfilled: For all cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ write

$$\begin{aligned} (1-\omega) A (1-\tilde{\omega}) &= \mathrm{op}_r(a), \\ a &\in S^\varrho(\mathbb{R}, L_V^{\mu;\ell}(X; \mathbb{H}; E, F)) \end{aligned}$$

according to (6.3.v). We require that there exists $s_0 \in \mathbb{R}$ and $r_0 \in \mathbb{R}_+$, such that for $r > r_0$ and all $z \in \mathbb{H}$

$$a(r, z) : H^{s_0}(X, E) \longrightarrow H^{s_0 - \mu}(X, F)$$

is an isomorphism, and we have for some $M \in \mathbb{R}$

$$\|a(r, z)^{-1}\|_{\mathcal{L}(H^{s_0 - \mu}, H^{s_0})} \langle z \rangle^M \langle r \rangle^\varrho = O(1),$$

uniformly for $z \in \mathbb{H}$ and $r \to \infty$.

A is called *parabolic* if A is interior parabolic, conormal parabolic, and exit parabolic.

6.4.3 Notation. For $\varrho \in \mathbb{R}$ we denote

$$C_{(V)}^{-\infty,\varrho}(X^\wedge,(\gamma,(-N,0]);E,F) := \bigcap_{\mu\in\mathbb{R}} C_{(V)}^{\mu,\varrho;\ell}(X^\wedge,(\gamma,(-N,0]);E,F).$$

Consequently, $A \in C_{(V)}^{-\infty,\varrho}(X^\wedge,(\gamma,(-N,0]);E,F)$ if and only if the following holds:

- For all cut-off functions $\omega,\tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ we have

$$\begin{gathered} \omega A\tilde{\omega} \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F), \\ (1-\omega)A(1-\tilde{\omega}) = \mathrm{op}_r(a), \\ a \in \begin{cases} S^\varrho(\mathbb{R},L^{-\infty}(X;\mathbb{R};E,F) \\ S^\varrho(\mathbb{R},L_V^{-\infty}(X;\mathbb{H};E,F). \end{cases} \end{gathered}$$

- For all cut-off functions $\omega \prec \tilde{\omega}$

$$\omega A(1-\tilde{\omega}),\ (1-\tilde{\omega})A\omega \in C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F).$$

Moreover, let $C_{(V)}^{-\infty,\varrho}(X^\wedge,(\gamma,(-N,0]);E,F)_0$ denote the subspace of all those operators A with $\sigma_M^{-k}(A)=0$ for $k=0,\dots,N-1$.

6.4.4 Theorem. *Let $A \in C_{(V(cl))}^{\mu,\varrho;\ell}(X^\wedge,(\gamma,(-N,0]);E,F)$, and let $0<T_2<T_1<\infty$. Moreover, let $\sigma_{\psi,c}^{\mu,\varrho;\ell}(A)=(h,a)$ be a complete interior symbol of A, subordinated to the covering $\{[0,T_1),(T_2,\infty)\}$ of $\overline{\mathbb{R}}_+$.*

The following are equivalent:

a) *A is interior elliptic (parabolic).*

b) *There exist $T_2<\tilde{T}_2<\tilde{T}_1<T_1<\infty$ such that h is elliptic (parabolic) on the interval $[0,\tilde{T}_1]$ in the sense of Definition 5.3.8, and a is interior elliptic (parabolic) on the interval $[\tilde{T}_2,\infty)$ in the sense of Definition 5.4.9.*

c) *There exists $P \in C_{(V(cl))}^{-\mu,-\varrho;\ell}(X^\wedge,(\gamma,(-N,0]);F,E)$ such that*

$$\begin{aligned} AP-1 &\in C_{(V)}^{-\infty,0}(X^\wedge,(\gamma,(-N,0]);F), \\ PA-1 &\in C_{(V)}^{-\infty,0}(X^\wedge,(\gamma,(-N,0]);E). \end{aligned}$$

Moreover, if $A \in C^{\mu,\varrho;\ell}_{(V)\,cl}(X^\wedge, (\gamma, (-N, 0]); E, F)$, *then* A *is interior elliptic (parabolic) if and only if the homogeneous principal symbol* $\sigma^{\mu;\ell}_{\psi}(A)$ *satisfies the following:*

$\sigma^{\mu;\ell}_{\psi}(A)(r, \xi_x, r^{-1}\tau)$ *is invertible in* $Hom(\pi^* E, \pi^* F)$ *for all* $r \in \overline{\mathbb{R}}_+$ *and* $0 \neq (\xi_x, \tau) \in T^*X \times \mathbb{R}$ *(respectively* $0 \neq (\xi_x, \tau) \in T^*X \times \mathbb{H}$*), and for the inverse we have*

$$\|\sigma^{\mu;\ell}_{\psi}(A)(r, \xi_x, \tau)^{-1}\| \langle r\rangle^{\varrho} = O(1),$$

uniformly for $\big(|\xi_x|^{2\ell}_x + |\tau|^2\big) = 1$ *and* $r \to \infty$.

Proof. *c)* $\Rightarrow$ *a):* Let (h_1, a_1) and (h_2, a_2) be any choices of complete interior symbols of the operators A and P, respectively, subordinated to the same covering $\{[0, T_1'), (T_2', \infty)\}$ of $\overline{\mathbb{R}}_+$. Then for every choice of cut-off functions $\omega \prec \tilde{\omega} \prec \chi_{[0,T_1']}$ we have

$$\omega\big(\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_1)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_2) - 1\big)\tilde{\omega} \in C_{M+G(,V)}(X^\wedge, (\gamma, (-N, 0]); F),$$
$$\omega\big(\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_2)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_1) - 1\big)\tilde{\omega} \in C_{M+G(,V)}(X^\wedge, (\gamma, (-N, 0]); E).$$

From Lemma 6.2.15, Proposition 6.1.5 and Theorem 5.3.11 (respectively Theorem 5.3.10) we conclude that h_1 is elliptic (parabolic) on every interval $(0, T]$ such that $T < T_1'$, where h_1 is regarded as an element of $C^\infty_B(\mathbb{R}_+, L^{\mu;\ell}_{(cl)}(X; \Gamma_{\frac{n+1}{2}-\gamma}; E, F))$, respectively $C^\infty_B(\mathbb{R}_+, L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}_{\frac{n+1}{2}-\gamma}; E, F))$. Moreover, Theorem 6.3.10 implies that the conormal symbol $\sigma^0_M(A)$ is elliptic (parabolic) as a meromorphic (Volterra) Mellin symbol in the sense of Definition 5.1.12 or Definition 5.2.7, respectively. Since $h_1(0, z) \equiv \sigma^0_M(A)$ modulo meromorphic (Volterra) Mellin symbols of order $-\infty$ we finally obtain that h_1 is elliptic (parabolic) on the interval $[0, T]$ as an element of $C^\infty_B(\overline{\mathbb{R}}_+, M^{\mu;\ell}_{(V,)O(cl)}(X; E, F))$. Similarly, we conclude that a_1 is interior elliptic (parabolic) on the interval $[T, \infty)$, for every $T_2' < T$. Hence, using Theorem 5.3.2 and Theorem 5.4.3, the interior ellipticity (parabolicity) of the operator A follows.

a) $\Rightarrow$ *b):* Choose arbitrary $T_2 < \tilde{T}_2 < \tilde{T}_1 < T_1 < \infty$. Let $\omega, \tilde{\omega} \in C^\infty_0(\overline{\mathbb{R}}_+)$ such that $\chi_{[0,\tilde{T}_1]} \prec \omega \prec \tilde{\omega} \prec \chi_{[0,T_1]}$. We may write

$$\omega A \tilde{\omega} = \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\tilde{\omega} + A_{M+G},$$
$$A_{M+G} \in C_{M+G(,V)}(X^\wedge, (\gamma, (-N, 0]); E, F).$$

Let $h' \in C^\infty_B(\overline{\mathbb{R}}_+, M^{\mu;\ell}_{(V,)O(cl)}(X; E, F))$ be the left-symbol that is associated with the double-symbol $\omega(r)h(r, z)\tilde{\omega}(r')$ according to Theorem 5.3.2. Then we have $\omega A \tilde{\omega} = \mathrm{op}_M^{\gamma-\frac{n}{2}}(h') + A_{M+G}$, and by Definition 6.4.1, respectively Definition 6.4.2, the symbol h' is elliptic (parabolic) on the interval $[0, \tilde{T}_1]$. For $h' - \omega h \in$

$C_B^\infty(\overline{\mathbb{R}}_+, M_{(V,)O(cl)}^{\mu-1;\ell}(X;E,F))$, we conclude that h is elliptic (parabolic) on the interval $[0,\tilde{T}_1]$. Next let $\omega,\tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\chi_{[0,T_2]} \prec \tilde{\omega} \prec \omega \prec \chi_{[0,\tilde{T}_2]}$. Then

$$\begin{aligned}(1-\omega)A(1-\tilde{\omega}) &= (1-\omega)\mathrm{op}_r(a)(1-\tilde{\omega}) + G,\\ G &\in C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F).\end{aligned}$$

Let a' be the left-symbol associated with the double-symbol $(1-\omega(r))a(r,\tau)(1-\tilde{\omega}(r'))$, see Theorem 5.4.3. Due to Proposition 6.1.5 write $G = \mathrm{op}_r(g)$, and thus

$$\begin{aligned}(1-\omega)A(1-\tilde{\omega}) &= \mathrm{op}_r(a'+g),\\ a'+g &\in \begin{cases} S^\varrho(\mathbb{R}, L_{(cl)}^{\mu;\ell}(X;\mathbb{R};E,F)),\\ S^\varrho(\mathbb{R}, L_{V(cl)}^{\mu;\ell}(X;\mathbb{H};E,F)).\end{cases}\end{aligned}$$

By Definition 6.4.1, respectively Definition 6.4.2, $a'+g$ is interior elliptic (parabolic) on the interval $[\tilde{T}_2,\infty)$. For

$$(a'+g)-(1-\omega)a \in \begin{cases} S^{\varrho-1}(\mathbb{R}, L_{(cl)}^{\mu-1;\ell}(X;\mathbb{R};E,F)),\\ S^{\varrho-1}(\mathbb{R}, L_{V(cl)}^{\mu-1;\ell}(X;\mathbb{H};E,F)),\end{cases}$$

we obtain that a is interior elliptic (parabolic) on $[\tilde{T}_2,\infty)$.

b) ⇒ c): Due to Theorem 5.3.11 (respectively Theorem 5.3.10) and Theorem 5.4.11 there exist

$$\begin{aligned}\tilde{h} &\in C_B^\infty(\overline{\mathbb{R}}_+, M_{(V,)O(cl)}^{-\mu;\ell}(X;F,E)),\\ \tilde{a} &\in \begin{cases} S^{-\varrho}(\mathbb{R}, L_{(cl)}^{-\mu;\ell}(X;\mathbb{R};F,E)),\\ S^{-\varrho}(\mathbb{R}, L_{V(cl)}^{-\mu;\ell}(X;\mathbb{H};F,E)),\end{cases}\end{aligned}$$

such that for all cut-off functions $\omega,\tilde{\omega} \prec \chi_{[0,\tilde{T}_1]}$ and $\chi_{[0,\tilde{T}_2]} \prec \hat{\omega},\breve{\omega}$ we have

$$\left.\begin{aligned}\omega\big(\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h})-1\big)\tilde{\omega} &\in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);F),\\ \omega\big(\mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h})\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)-1\big)\tilde{\omega} &\in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E),\\ (1-\hat{\omega})\big(\mathrm{op}_r(a)\mathrm{op}_r(\tilde{a})-1\big)(1-\breve{\omega}) &= (1-\hat{\omega})\mathrm{op}_r(r_R)(1-\breve{\omega}),\\ r_R &\in \begin{cases} S^0(\mathbb{R}, L^{-\infty}(X;\mathbb{R};F,F)),\\ S^0(\mathbb{R}, L_V^{-\infty}(X;\mathbb{H};F,F)),\end{cases}\\ (1-\hat{\omega})\big(\mathrm{op}_r(\tilde{a})\mathrm{op}_r(a)-1\big)(1-\breve{\omega}) &= (1-\hat{\omega})\mathrm{op}_r(r_L)(1-\breve{\omega}),\\ r_L &\in \begin{cases} S^0(\mathbb{R}, L^{-\infty}(X;\mathbb{R};E,E)),\\ S^0(\mathbb{R}, L_V^{-\infty}(X;\mathbb{H};E,E)).\end{cases}\end{aligned}\right\} \qquad (1)$$

Notice that for all $\varphi,\psi \in C_0^\infty(T_2,T_1)$ we have $\varphi\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\psi \equiv \varphi\mathrm{op}_r(a)\psi$ modulo $C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E,F)$. Now we conclude that also $\tilde{\varphi}\mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h})\tilde{\psi} \equiv$

$\tilde{\varphi}\mathrm{op}_r(\tilde{a})\tilde{\psi}$ modulo $C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);F,E)$, for all $\tilde{\varphi},\tilde{\psi}\in C_0^\infty(\tilde{T}_2,\tilde{T}_1)$. Let $\{\omega_1,\omega_2,\omega_3\}$ be cut-off functions subordinated to the covering $\{[0,\tilde{T}_1),(\tilde{T}_2,\infty)\}$ of $\overline{\mathbb{R}}_+$, and define

$$P := \omega_1\mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h})\omega_2 + (1-\omega_1)\mathrm{op}_r(\tilde{a})(1-\omega_3).$$

According to Theorem 6.3.3 we have $P\in C^{-\mu,-\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);F,E)$. Moreover, $\sigma_{\psi,c}^{-\mu,-\varrho;\ell}(P)=(\tilde{h},\tilde{a})$ is a complete interior symbol of P subordinated to the covering $\{[0,\tilde{T}_1),(\tilde{T}_2,\infty)\}$ of $\overline{\mathbb{R}}_+$. Theorem 6.3.10 and (1) imply

$$\left.\begin{aligned}
\omega_1(AP-1)\omega_2 &\equiv \omega_1\big(\mathrm{op}_M^{\gamma-\frac{n}{2}}(h\#\tilde{h})-1\big)\omega_2\equiv 0\\
&\qquad \mathrm{mod}\ \ C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);F),\\
\omega_1(PA-1)\omega_2 &\equiv \omega_1\big(\mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{h}\#h)-1\big)\omega_2\equiv 0\\
&\qquad \mathrm{mod}\ \ C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E),\\
(1-\omega_1)(AP-1)(1-\omega_3) &\equiv (1-\omega_1)\big(\mathrm{op}_r(a\#\tilde{a})-1\big)(1-\omega_3)\\
&\equiv (1-\omega_1)\mathrm{op}_r(r_R)(1-\omega_3)\\
&\qquad \mathrm{mod}\ \ C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);F),\\
(1-\omega_1)(PA-1)(1-\omega_3) &\equiv (1-\omega_1)\big(\mathrm{op}_r(\tilde{a}\#a)-1\big)(1-\omega_3)\\
&\equiv (1-\omega_1)\mathrm{op}_r(r_L)(1-\omega_3)\\
&\qquad \mathrm{mod}\ \ C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E),
\end{aligned}\right\} \tag{2}$$

i. e., P fulfills the conditions in c).

Interior ellipticity (parabolicity) for classical operators: According to (6.3.xi) we have

$$\begin{aligned}
\sigma_\psi^{\mu;\ell}(h)(r,\xi_x,\frac{n+1}{2}-\gamma-i\tau) &= \sigma_\psi^{\mu;\ell}(A)(r,\xi_x,r^{-1}\tau) \text{ on } [0,\tilde{T}_1],\\
\sigma_\psi^{\mu;\ell}(a)(r,\xi_x,\tau) &= \sigma_\psi^{\mu;\ell}(A)(r,\xi_x,\tau) \text{ on } [\tilde{T}_2,\infty),
\end{aligned}$$

for $(\xi_x,\tau)\in\big(T^*X\times\mathbb{R}\big)\setminus 0$, respectively $(\xi_x,\tau)\in\big(T^*X\times\mathbb{H}\big)\setminus 0$. Consequently, the asserted equivalence for classical (cone) operators follows from b) and Definition 5.3.8, as well as Definition 5.4.9. □

6.4.5 Theorem. *Let* $A\in C^{\mu,\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F)$. *The following are equivalent:*

a) A is interior and exit elliptic (parabolic).

b) A is interior elliptic (parabolic), and for some (every) exit symbol $\sigma_e^{\mu,\varrho;\ell}(A)$ *we have:*

There exists $s_0 \in \mathbb{R}$ such that for $r \gg r_0$ sufficiently large and all $\tau \in \mathbb{R}$ (respectively $\tau \in \mathbb{H}$)

$$\sigma_e^{\mu,\varrho;\ell}(A)(r,\tau) : H^{s_0}(X,E) \longrightarrow H^{s_0-\mu}(X,F)$$

is an isomorphism, and we have for some $M \in \mathbb{R}$

$$\|\sigma_e^{\mu,\varrho;\ell}(A)(r,\tau)^{-1}\|_{\mathcal{L}(H^{s_0-\mu},H^{s_0})}\langle\tau\rangle^M\langle r\rangle^{\varrho} = O(1),$$

uniformly for $\tau \in \mathbb{R}$ (respectively $\tau \in \mathbb{H}$), and $r \to \infty$.

c) *There exists $P \in C^{-\mu,-\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);F,E)$ such that*

$$AP - 1 \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);F),$$
$$PA - 1 \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E).$$

Moreover, the following are equivalent:

i) *A is interior and conormal elliptic (parabolic).*

ii) *There exists $P \in C^{-\mu,-\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);F,E)$ such that*

$$AP - 1 \in C^{-\infty,0}_{(V)}(X^\wedge,(\gamma,(-N,0]);F)_0,$$
$$PA - 1 \in C^{-\infty,0}_{(V)}(X^\wedge,(\gamma,(-N,0]);E)_0.$$

Proof. We first prove the equivalence a) – c): Note that if e_1 and e_2 are exit symbols for the operator A, then there exist cut-off functions $\omega \prec \tilde{\omega}$ such that $(1-\tilde{\omega})e_1(1-\omega) \equiv (1-\tilde{\omega})e_2(1-\omega)$ modulo terms of order $-\infty$, both in μ and ϱ. Hence the condition in b) does not depend on the choice of the exit symbol. In particular, we may consider the Fourier component a of any complete interior symbol (h,a) as an exit symbol that satisfies b).

Let $0 < T_2 < T_1 < \infty$, and let (h,a) be a complete interior symbol of A, subordinated to the covering $\{[0,T_1),(T_2,\infty)\}$ of $\overline{\mathbb{R}}_+$. From Theorem 6.4.4 we obtain that a) and b) are equivalent to the following: There exist $T_2 < \tilde{T}_2 < \tilde{T}_1 < T_1 < \infty$ such that h is elliptic (parabolic) on the interval $[0,\tilde{T}_1]$ in the sense of Definition 5.3.8, and a is elliptic (parabolic) on the interval $[\tilde{T}_2,\infty)$ in the sense of Definition 5.4.9.

Starting from this condition we construct the operator P in c) analogously to the proof of b) $\Rightarrow$ c) in Theorem 6.4.4. Note that in (1) we now obtain

$$r_R \in \begin{cases} S^{-\infty}(\mathbb{R},L^{-\infty}(X;\mathbb{R};F,F)) \\ S^{-\infty}(\mathbb{R},L_V^{-\infty}(X;\mathbb{H};F,F)), \end{cases} \qquad r_L \in \begin{cases} S^{-\infty}(\mathbb{R},L^{-\infty}(X;\mathbb{R};E,E)) \\ S^{-\infty}(\mathbb{R},L_V^{-\infty}(X;\mathbb{H};E,E)), \end{cases}$$

and consequently in (2)

$$(1-\omega_1)(AP-1)(1-\omega_3) \in C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);F),$$
$$(1-\omega_1)(PA-1)(1-\omega_3) \in C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E).$$

This shows that P satisfies the conditions in c).

c) $\Rightarrow$ a) is obvious due to Theorem 6.4.4.

Let us now prove the equivalence i) $\Leftrightarrow$ ii): ii) $\Rightarrow$ i) follows from Theorem 6.4.4 and Theorem 6.3.10. Now assume that i) holds. Choose $P' \in C^{-\mu,-\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);F,E)$ that satisfies c) in Theorem 6.4.4. Hence we obtain from the conormal ellipticity (parabolicity), using Theorem 5.1.14 and Theorem 5.2.8:

$$g := \sigma^0_M(A)^{-1} - \sigma^0_M(P') \in \begin{cases} M^{-\infty}_Q(X;F,E), & \pi_{\mathbb{C}}Q \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset, \\ M^{-\infty}_{V,Q}(X;\mathbb{H}_{\frac{n+1}{2}-\gamma};F,E). \end{cases}$$

With a cut-off function $\omega \in C^\infty_0(\overline{\mathbb{R}}_+)$ define

$$\tilde{P} := P' + \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(g)\omega \in C^{-\mu,-\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);F,E).$$

Then we have $A\tilde{P} = 1 + C_1$, and $\tilde{P}A = 1 + C_2$, where the remainders

$$C_1 \in C^{-\infty,0}_{(V)}(X^\wedge,(\gamma,(-N,0]);F),$$
$$C_2 \in C^{-\infty,0}_{(V)}(X^\wedge,(\gamma,(-N,0]);E)$$

are such that $\sigma^0_M(C_1) = 0$, as well as $\sigma^0_M(C_2) = 0$. Consider the operator C_1: We may write

$$C_1 = \tilde{C}_1 + \hat{C}_1,$$
$$\tilde{C}_1 \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);F),$$
$$\hat{C}_1 \in C^{-\infty,0}_{(V)}(X^\wedge,(\gamma,(-N,0]);F)_0.$$

From Theorem 6.2.27 we conclude that there exists a smoothing (Volterra) Mellin and Green operator $\tilde{D}_1 \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);F)$ such that

$$(1+\tilde{C}_1)(1+\tilde{D}_1) - 1 \in C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);F).$$

Note that $C^{-\infty,0}_{(V)}(X^\wedge,(\gamma,(-N,0]);F)_0$ is a two-sided ideal in the (Volterra) cone algebra. Hence we obtain $AP_R - 1 \in C^{-\infty,0}_{(V)}(X^\wedge,(\gamma,(-N,0]);F)_0$ with $P_R := \tilde{P}(1+\tilde{D}_1)$. Analogously, we construct $P_L \in C^{-\mu,-\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);F,E)$ with $P_LA - 1 \in C^{-\infty,0}_{(V)}(X^\wedge,(\gamma,(-N,0]);E)_0$. But both P_L and P_R differ only by an element in $C^{-\infty,0}_{(V)}(X^\wedge,(\gamma,(-N,0]);F,E)_0$, and thus condition ii) is fulfilled with either $P := P_L$ or $P := P_R$. This finishes the proof of the theorem. $\square$

6.4.6 Theorem. *Let $A \in C^{\mu,\varrho;\ell}_{(cl)}(X^\wedge,(\gamma,(-N,0]);E,F)$. The following are equivalent:*

a) A is elliptic.

b) There exists $P \in C^{-\mu,-\varrho;\ell}_{(cl)}(X^\wedge,(\gamma,(-N,0]);F,E)$ such that $AP = 1 + G_1$ and $PA = 1 + G_2$ with Green operators $G_1 \in C_G(X^\wedge,(\gamma,(-N,0]);F)$ and $G_2 \in C_G(X^\wedge,(\gamma,(-N,0]);E)$.

Moreover, for $A \in C^{\mu,\varrho;\ell}_{V(cl)}(X^\wedge,(\gamma,(-N,0]);E,F)$ the following are equivalent:

i) A is parabolic.

ii) There exists $P \in C^{-\mu,-\varrho;\ell}_{V(cl)}(X^\wedge,(\gamma,(-N,0]);F,E)$ such that $AP = 1+G_1$ and $PA = 1 + G_2$ with Volterra Green operators $G_1 \in C_{G,V}(X^\wedge,(\gamma,(-N,0]);F)$ and $G_2 \in C_{G,V}(X^\wedge,(\gamma,(-N,0]);E)$.

iii) There exists $P \in C^{-\mu,-\varrho;\ell}_{V(cl)}(X^\wedge,(\gamma,(-N,0]);F,E)$ such that $AP = 1$ and $PA = 1$, i. e., A is invertible in the Volterra cone algebra with $A^{-1} = P$.

Proof. We simultaneously prove the equivalences a) $\Leftrightarrow$ b) and i) $\Leftrightarrow$ ii).
Clearly, the conditions in ii) and b) are sufficient for i) and a) by Theorem 6.4.4 and Theorem 6.4.5. Now assume that A is elliptic (parabolic). Using the interior and exit ellipticity (parabolicity), we conclude from Theorem 6.4.5 that there exists $P' \in C^{-\mu,-\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);F,E)$ such that $AP' - 1 \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);F)$ and $P'A - 1 \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E)$. Using the conormal ellipticity (parabolicity) of A we see that

$$g := \sigma^0_M(A)^{-1} - \sigma^0_M(P') \in \begin{cases} M^{-\infty}_Q(X;F,E), & \pi_{\mathbb{C}}Q \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset, \\ M^{-\infty}_{V,Q}(X;\mathbb{H}_{\frac{n+1}{2}-\gamma};F,E). \end{cases}$$

With a cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ define

$$\tilde{P} := P' + \omega \mathrm{op}_M^{\gamma-\frac{n}{2}}(g)\omega \in C^{-\mu,-\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);F,E).$$

Then

$$A\tilde{P} - 1 \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);F),$$
$$\tilde{P}A - 1 \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E).$$

Moreover, we have $\sigma^0_M(A\tilde{P}) = 1$, as well as $\sigma^0_M(\tilde{P}A) = 1$, in view of Theorem 6.3.10. From Theorem 6.2.27 we obtain the existence of

$$D_1 \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);E), \quad D_2 \in C_{M+G(,V)}(X^\wedge,(\gamma,(-N,0]);F)$$

such that $\big((1+D_1)\tilde{P}\big)A-1 \in C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);E)$ and $A\big(\tilde{P}(1+D_2)\big)-1 \in C_{G(,V)}(X^\wedge,(\gamma,(-N,0]);F)$, and thus either $P:=\tilde{P}(1+D_2)$ or $P:=(1+D_1)\tilde{P}$ fulfills the conditions in ii) and b).

It remains to prove that iii) is equivalent to ii), but this follows from Theorem 6.1.6. □

6.4.7 Definition. Let $A \in C^{\mu,\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);E,F)$ be elliptic, respectively parabolic. Then any operator $P \in C^{-\mu,-\varrho;\ell}_{(V(cl))}(X^\wedge,(\gamma,(-N,0]);F,E)$ that satisfies b) or ii) in Theorem 6.4.6 is called a *(Volterra) parametrix* of the operator A.

6.4.8 Corollary. *Let $A \in C^{\mu,\varrho;\ell}(X^\wedge,(\gamma,(-N,0]);E,F)$ be elliptic. Then*

$$A:\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta \longrightarrow \mathcal{K}^{(s-\mu,t),\gamma;\ell}(X^\wedge,F)_{\delta-\varrho}$$

is a Fredholm operator for all $s,t,\delta \in \mathbb{R}$.

Proof. This follows from Theorem 6.4.6, and from the fact that Green operators induce nuclear, in particular compact, operators in the cone Sobolev spaces by Proposition 6.1.4. □

6.4.9 Corollary. *a) Let $A \in C^{\mu,\varrho;\ell}_V(X^\wedge,(\gamma,(-N,0]);E,F)$ be parabolic. Then*

$$A:\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta \longrightarrow \mathcal{K}^{(s-\mu,t),\gamma;\ell}(X^\wedge,F)_{\delta-\varrho}$$

is bijective for all $s,t,\delta \in \mathbb{R}$, i. e., the equation $Au=f$ with $f \in \mathcal{K}^{(s-\mu,t),\gamma;\ell}(X^\wedge,F)_{\delta-\varrho}$ is uniquely solvable with solution $u \in \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta$.

Moreover, if f has asymptotics of "length" N, i. e., if $f \in \mathcal{K}^{(s-\mu,t),\gamma;\ell}_Q(X^\wedge,F)_{\delta-\varrho}$ with some asymptotic type $Q \in As\big((\gamma,(-N,0]),C^\infty(X,F)\big)$, then the solution u belongs to the space $\mathcal{K}^{(s,t),\gamma;\ell}_{\tilde{Q}}(X^\wedge,E)$ with some asymptotic type $\tilde{Q} \in As\big((\gamma,(-N,0]),C^\infty(X,E)\big)$.

b) If $A \in C^{\mu,\varrho;\ell}_{V(cl)}(X^\wedge,(\gamma,(-N,0]);E,F)$ is just interior and conormal parabolic we still have the following:

For every $r_0 \in \mathbb{R}_+$ the operator

$$A:\mathcal{H}^{(s,t),\gamma;\ell}_0((0,r_0]\times X,E) \longrightarrow \mathcal{H}^{(s-\mu,t),\gamma;\ell}_0((0,r_0]\times X,F)$$

is bijective for all $s,t \in \mathbb{R}$.

More precisely, there exists $P \in C^{-\mu,-\varrho;\ell}_{V(cl)}(X^\wedge,(\gamma,(-N,0]);F,E)$ such that

$$\Big(A|_{\mathcal{H}^{(s,t),\gamma;\ell}_0((0,r_0]\times X,E)}\Big)^{-1} = P|_{\mathcal{H}^{(s-\mu,t),\gamma;\ell}_0((0,r_0]\times X,F)}$$

for all $s, t \in \mathbb{R}$.

In particular, the equation $Au = f$ *with* $f \in \mathcal{H}_0^{(s-\mu,t),\gamma;\ell}((0,r_0]\times X, F)$ *is uniquely solvable with solution* $u \in \mathcal{H}_0^{(s,t),\gamma;\ell}((0,r_0]\times X, E)$. *Moreover, if* f *has asymptotics of "length"* N, *then so does* u *(in the sense of a)).*

Proof. a) follows immediately from Theorem 6.4.6. Let us prove b). Choose $P' \in C_{V(cl)}^{-\mu,-\varrho;\ell}(X^\wedge, (\gamma, (-N, 0]); F, E)$ satisfying condition ii) in Theorem 6.4.5. Moreover, let $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ be cut-off functions such that $\chi_{[0,r_0]} \prec \omega \prec \tilde{\omega}$. Then we have $\omega A P' \tilde{\omega} = \omega + G_1$ and $\omega P' A \tilde{\omega} = \omega + G_2$ with Volterra Green operators $G_1 \in C_{G,V}(X^\wedge, (\gamma, (-N, 0]); F)$ and $G_2 \in C_{G,V}(X^\wedge, (\gamma, (-N, 0]); E)$. By Theorem 6.1.6 we have $(1 + G_1)^{-1} = 1 + \tilde{G}_1$ and $(1 + G_2)^{-1} = 1 + \tilde{G}_2$ with Volterra Green operators $\tilde{G}_1$, $\tilde{G}_2$. Hence we see

$$\omega A\big(P'\tilde{\omega}(1 + \tilde{G}_1)\big) = 1 - (1 - \omega)(1 + \tilde{G}_1),$$
$$\big((1 + \tilde{G}_2)\omega P'\big)A\tilde{\omega} = 1 - (1 + \tilde{G}_2)(1 - \omega).$$

Restricting to the $\mathcal{H}_0^{(s,t),\gamma;\ell}((0,r_0]\times X, \cdot)$-spaces reveals that the so-obtained operator $A|_{\mathcal{H}_0^{(s,t),\gamma;\ell}((0,r_0]\times X,E)}$ is indeed invertible from the left and from the right with operators in $C_{V(cl)}^{-\mu,-\varrho;\ell}(X^\wedge, (\gamma, (-N, 0]); F, E)$, and consequently any left- or right-inverse gives rise to an operator P which satisfies the assertion in b). $\square$

Parabolic reductions of orders

6.4.10 Theorem. *Let* $\ell \in 2\mathbb{N}$. *For* $s, \delta \in \mathbb{R}$ *there exist*

$$R^{s,\delta} \in C_{V\,cl}^{s,\delta;\ell}(X^\wedge, (\gamma, (-N, 0]); E)$$

such that $R^{s,\delta}R^{-s,-\delta} = 1$, *i. e., there exist parabolic reductions of orders within the algebra of classical Volterra cone operators.*

Proof. For $s = \delta = 0$ define $R^{0,0} := 1$. Now assume that $s > 0$. Since $\ell \in 2\mathbb{N}$ is even, the function

$$\big((\xi_x, \zeta) \mapsto \big(|\xi_x|_x^\ell + \zeta\big)^{\frac{s}{\ell}} \cdot \mathrm{id}_{\pi^* E_{(\xi_x,\zeta)}}\big) \in S_V^{(s;\ell)}((T^*X \times \mathbb{H}_0) \setminus 0, \mathrm{Hom}(\pi^* E))$$

is well-defined, and $\big(|\xi_x|_x^\ell + \zeta\big)^{\frac{s}{\ell}} \neq 0$ for $(\xi_x, \zeta) \in (T^*X \times \mathbb{H}_0) \setminus 0$.

According to Theorem 3.2.16 there exists

$$h_0 \in L_{V\,cl}^{s;\ell}(X; \mathbb{H}_0; E),$$
$$\sigma_\psi^{s;\ell}(h_0)(\xi_x, \zeta) = \big(|\xi_x|_x^\ell + \zeta\big)^{\frac{s}{\ell}} \cdot \mathrm{id}_{\pi^* E_{(\xi_x,\zeta)}}.$$

Let $\psi \in C_0^\infty(\mathbb{R}_+)$ such that $\psi \equiv 1$ near $r = 1$. With the Mellin kernel cut-off operator define $h_1 := H_{\frac{1}{2}}(\psi)h_0 \in M^{s;\ell}_{V,O\,cl}(X;E)$. Due to Theorem 5.2.5 we have $h_1 - h_0 \in L_V^{-\infty}(X;\mathbb{H}_0;E)$, and thus h_1 is parabolic as an element of $M^{s;\ell}_{V,O\,cl}(X;E)$ in the sense of Definition 5.2.7. Using Theorem 5.2.8 we obtain that for some $\beta \in \mathbb{R}$ the holomorphic Volterra Mellin symbol $h := T_\beta h_1 \in M^{s;\ell}_{V,O\,cl}(X;E)$ is parabolic, and additionally $h(z) : H^{s_0}(X,E) \longrightarrow H^{s_0-s}(X,E)$ is an isomorphism for all $z \in \mathbb{H}_{\frac{n+1}{2}-\gamma}$ and all $s_0 \in \mathbb{R}$.

From Theorem 3.2.16 and Theorem 3.2.19 we see that there exists

$$\begin{aligned} &b \in L^{s;\ell}_{V\,cl}(X;\mathbb{H};E), \\ &\sigma_\psi^{s;\ell}(b)(\xi_x,\tau) = \big(|\xi_x|_x^\ell - i\tau\big)^{\frac{s}{\ell}} \cdot \mathrm{id}_{\pi^* E_{(\xi_x,\tau)}}, \end{aligned}$$

and b is invertible with inverse $b^{-1} \in L^{-s;\ell}_{V\,cl}(X;\mathbb{H};E)$.

Let $\omega_3 \prec \omega_1 \prec \omega_2$ be cut-off functions near $r = 0$. Moreover, let $\varphi \in C^\infty(\overline{\mathbb{R}}_+)$ be an everywhere positive function with $\omega_2 \prec \varphi$, and $r\varphi(r) \equiv 1$ for $r \gg r_0$ sufficiently large. Define

$$R^{s,0} := \omega_1 \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\omega_2 + (1-\omega_1)\mathrm{op}_r(a)(1-\omega_3),$$

where $a(r,\tau) := (1-\omega_3(r))b(r\varphi(r)\tau) \in S^0(\mathbb{R}, L^{s;\ell}_{V\,cl}(X;\mathbb{H};E))$. By Theorem 6.3.3 we have $R^{s,0} \in C^{s,0;\ell}_{V\,cl}(X^\wedge,(\gamma,(-N,0]);E)$, and from the construction we see

$$\begin{aligned} \sigma_\psi^{s;\ell}(R^{s,0})(r,\xi_x,\tau) &= \big(|\xi_x|_x^\ell - i(r\varphi(r)\tau)\big)^{\frac{s}{\ell}} \cdot \mathrm{id}_{\pi^* E_{(\xi_x,\tau)}}, \\ \sigma_M^0(R^{s,0}) &= h, \\ \sigma_e^{s,0;\ell}(R^{s,0})(r,\tau) &= b(\tau). \end{aligned}$$

Hence $R^{s,0}$ is parabolic, and by Theorem 6.4.6 there exists

$$R^{-s,0} := \big(R^{s,0}\big)^{-1} \in C^{-s,0;\ell}_{V\,cl}(X^\wedge,(\gamma,(-N,0]);E).$$

This completes the proof in the case $\delta = 0$.

With a cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ we define for all $s \in \mathbb{R}$ and $\delta > 0$

$$R^{s,\delta} := \big(\omega(r) + (1-\omega(r))\langle r\rangle^\delta\big)R^{s,0} \in C^{s,\delta;\ell}_{V\,cl}(X^\wedge,(\gamma,(-N,0]);E).$$

Consequently, $R^{s,\delta}$ is parabolic, and by Theorem 6.4.6 there exists

$$R^{-s,-\delta} := \big(R^{s,\delta}\big)^{-1} \in C^{-s,-\delta;\ell}_{V\,cl}(X^\wedge,(\gamma,(-N,0]);E).$$

This finishes the proof of the theorem. □

Chapter 7

Remarks on the classical theory of parabolic PDE

We want to conclude the present exposition with some remarks about the classical theory of parabolic partial differential equations; more precisely, we want to give an idea of how it fits into the framework of our Volterra cone calculus. In particular, the intention of this chapter is to offer the reader some guide to the functional analytic structures of the previous chapters. To this end, we shall discuss parabolicity, solvability, and regularity for a generalized heat operator.

A generalization of the scalar heat equation

Consider the following equation:

$$\left.\begin{aligned} \Big(\frac{\partial}{\partial t} - A_t\Big)u(t,x) &= f(t,x) \\ u|_{t=t_0} &= 0 \end{aligned}\right\} \tag{1}$$

- A_t is a smooth family of scalar differential operators on the closed manifold X of order ℓ.

- We impose the following condition on the stabilization of coefficients for $t \to \infty$: $A_{-\log(r)}$ is assumed to be C^∞ up to $r = 0$. In particular, A_t extends continuously up to $t = \infty$, and we find a differential operator A_∞ of order ℓ there.

Notice that the most natural classical equations fulfill the stabilization condition imposed on the coefficients. Among these, in particular, there are the autonomous equations, i. e., the coefficients do not depend on time at all, and, moreover, equations that do not depend on time for $t \gg t_0$ sufficiently large.

On the classical notion of parabolicity

Classically, the notion of parabolicity (more precisely: one notion of parabolicity) for equation (1) is strong ellipticity of the family of differential operators A_t, i. e.,

$$\operatorname{Re}\sigma_\psi^\ell(A_t) < 0 \text{ on } T^*X \setminus 0, \text{ for all } t \in [t_0, \infty]. \tag{2}$$

Writing the operator $\partial_t - A_t$ from (1) in local coordinates as

$$\partial_t - \sum_{|\alpha|=0}^{\ell} a_\alpha(t,x) D_x^\alpha, \tag{3}$$

the parabolicity condition (2) reads

$$\operatorname{Re}\Big(\sum_{|\alpha|=\ell} a_\alpha(t,x)\xi^\alpha\Big) < 0 \tag{4}$$

for all x and $\xi \neq 0$, and all $t \in [t_0, \infty]$.

From the local representation (3) the anisotropic structure of the operator $\partial_t - A_t$ is evident. More precisely, it is an operator of order ℓ with the same anisotropy ℓ, and this is precisely the "gap of orders" of the spatial derivatives and the time derivative. Locally, the anisotropic leading component of the symbol is given as

$$\sigma_\psi^{\ell;\ell}\Big(\partial_t - \sum_{|\alpha|=0}^{\ell} a_\alpha(t,x) D_x^\alpha\Big)(x,t,\xi,\zeta) = i\zeta - \sum_{|\alpha|=\ell} a_\alpha(t,x)\xi^\alpha$$

according to (3), and we have the anisotropic homogeneity

$$\sigma_\psi^{\ell;\ell}\Big(\partial_t - \sum_{|\alpha|=0}^{\ell} a_\alpha(t,x) D_x^\alpha\Big)(x,t,\varrho\xi,\varrho^\ell\zeta) = \varrho^\ell \sigma_\psi^{\ell;\ell}\Big(\partial_t - \sum_{|\alpha|=0}^{\ell} a_\alpha(t,x) D_x^\alpha\Big)(x,t,\xi,\zeta)$$

for $\varrho > 0$.

Now it is easy to see that the local parabolicity condition (4) is equivalent to the following:

$$\sigma_\psi^{\ell;\ell}\Big(\partial_t - \sum_{|\alpha|=0}^{\ell} a_\alpha(t,x) D_x^\alpha\Big)(x,t,\xi,\zeta) \neq 0 \tag{5}$$

for all x and $t \in [t_0, \infty]$, and all $0 \neq (\xi, \zeta) \in \mathbb{R}^n \times \mathbb{H}_-$. Hence we have a condition on invertibility of the anisotropic principal symbol with the time covariable ζ extended to the lower half-plane $\mathbb{H}_- = \{\mathrm{Im}(\zeta) \leq 0\} \subseteq \mathbb{C}$ ("parabolicity in the sense of Petrovskij").

The anisotropic homogeneous principal symbol is invariant under changes of coordinates, and so is the notion of parabolicity (5), i. e., we require

$$\sigma_\psi^{\ell;\ell}(\partial_t - A_t)(t, \xi_x, \zeta) \neq 0$$

for all $t \in [t_0, \infty]$, and $(\xi_x, \zeta) \in \big(T^*X \times \mathbb{H}_-\big) \setminus 0$.

Change of variables and totally characteristic structure

As we already pointed out in the introduction, we shall consider equation (1) not in its original form, but carry out the change of variables $r = e^{-t}$ to end up with the equation

$$\left.\begin{aligned} \Big((-r\partial_r) - B_r\Big)u(r,x) &= f(r,x) \\ u|_{r=r_0} &= 0, \end{aligned}\right\} \tag{6}$$

where $B_r := A_{-\log(r)}$, which is now considered on the transformed time interval $(0, r_0]$ with $r_0 := e^{-t_0}$. Notice, in particular, that the stabilization condition on the coefficients now reads that B_r is required to be smooth up to $r = 0$, and thus equation (6) can be regarded as a totally characteristic equation.

Passing to local coordinates as before reveals that the anisotropic leading component of the symbol of the operator $(-r\partial_r) - B_r$ locally is given as

$$\sigma_\psi^{\ell;\ell}\Big((-r\partial_r) - \sum_{|\alpha|=0}^{\ell} b_\alpha(r,x) D_x^\alpha\Big)(x, r, \xi, \zeta) = -ir\zeta - \sum_{|\alpha|=\ell} b_\alpha(r,x)\xi^\alpha.$$

In particular, we find the typical degenerate structure, and the parabolicity condition (5) is equivalent to

$$\sigma_\psi^{\ell;\ell}\Big((-r\partial_r) - \sum_{|\alpha|=0}^{\ell} b_\alpha(r,x) D_x^\alpha\Big)(x, r, \xi, r^{-1}\zeta) \neq 0 \tag{7}$$

for all x, and all $0 \neq (\xi, \zeta) \in \mathbb{R}^n \times \mathbb{H}$, and all $r \in \overline{\mathbb{R}}_+$. In this condition the upper half-plane $\mathbb{H}$ is involved instead of the lower half-plane $\mathbb{H}_-$, and the degeneracy requires to consider the above "coupled" expression, which is extended up to the origin $r = 0$. The global situation is analogous, i. e., we assume

$$\sigma_\psi^{\ell;\ell}\big((-r\partial_r) - B_r\big)(r, \xi_x, r^{-1}\zeta) \neq 0 \tag{8}$$

for all $r \in [0, r_0]$, and $(\xi_x, \zeta) \in \big(T^*X \times \mathbb{H}\big) \setminus 0$.

Fourier and Mellin representations

We may represent the operator $(-r\partial_r) - B_r$ in the following two ways as a pseudodifferential operator with operator-valued symbol:

1) Fourier representation:

$$(-r\partial_r) - B_r = \mathcal{F}^{-1}\Big(-ir\zeta - B_r\Big)\mathcal{F}$$

   with the (degenerate) symbol $a(r,\zeta) := -ir\zeta - B_r$.

2) Mellin representation:

$$(-r\partial_r) - B_r = \mathcal{M}^{-1}\Big(\zeta - B_r\Big)\mathcal{M}$$

   with the Mellin symbol $h(r,\zeta) := \zeta - B_r$.

Notice, in particular, that both symbols $a(r,\zeta)$ and $h(r,\zeta)$ can be regarded as families parametrized by the "time variable" r taking values in (anisotropic) parameter-dependent operators on the manifold X (with the anisotropic parameter ζ). The parabolicity condition (8) in terms of the operator-valued symbols a and h is given as follows:

1) The parameter-dependent homogeneous principal symbol of $a(r,\zeta)$ satisfies

$$\sigma_\psi^{\ell;\ell}(a(r,r^{-1}\zeta))(\xi_x,\zeta) \neq 0$$

   for all $(\xi_x,\zeta) \in \big(T^*X \times \mathbb{H}\big) \setminus 0$, and all $r \in \overline{\mathbb{R}}_+$.

2) The parameter-dependent homogeneous principal symbol of $h(r,\zeta)$ satisfies

$$\sigma_\psi^{\ell;\ell}(h(r,\zeta))(\xi_x,\zeta) \neq 0$$

   for all $(\xi_x,\zeta) \in \big(T^*X \times \mathbb{H}_0\big) \setminus 0$, and all $r \in \overline{\mathbb{R}}_+$, where $\mathbb{H}_0 = \{\mathrm{Re}(\zeta) \geq 0\} \subseteq \mathbb{C}$ is the right half-plane.

The Volterra cone calculus makes use of both representations of the operator $(-r\partial_r) - B_r$. More precisely, the Mellin representation is used close to $r = 0$, which corresponds to $t \to \infty$ in the original coordinates, while the Fourier representation is used away from $r = 0$.

Necessary basics of parameter-dependent operators are given in Chapter 3, and the discussion of both the Mellin and Fourier calculus is performed separately in Chapter 5. The comments given above for the operator $(-r\partial_r) - B_r$ might be of help to get to a better understanding of their particular structure.

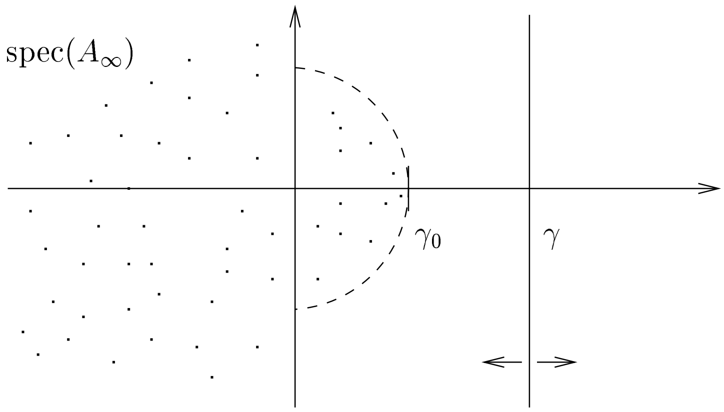

Figure 7.1: The spectrum of A_∞

Solvability and regularity

The following result on solvability and regularity of equation (1) is valid under the classical parabolicity condition:

There exists $\gamma_0 \in \mathbb{R}$ such that for $\gamma > \gamma_0$ we have: Given

$$f \in e^{\gamma t}\big(L^2([t_0,\infty)\times X)\big),$$

equation (1) has a unique solution

$$u \in e^{\gamma t}\big(L^2([t_0,\infty), H^\ell(X)) \cap H^1_0([t_0,\infty), L^2(X))\big).$$

More precisely, γ_0 is given as the spectral bound

$$\gamma_0 := \sup\{\mathrm{Re}(\lambda);\ \lambda \in \mathrm{spec}(A_\infty)\}$$

as is visualized in the figure.

Consequently, under the parabolicity assumption, the question whether equation (1) is solvable for all right hand sides f in some L^2-space with fixed exponential weight $\gamma \in \mathbb{R}$ depends on whether γ is larger than the spectral bound of the operator A_∞. In other words, it is possible if the resolvent $(\lambda - A_\infty)^{-1}$ exists for all $\mathrm{Re}(\lambda) \geq \gamma$.

Let us discuss this in more detail, where we freely interchange between both the original equation (1) and its transformed representation (6):

In the Mellin representation of the operator $(-r\partial_r) - B_r$ above we met the Mellin symbol $h(r,\zeta) = \zeta - B_r$, and thus the so-called conormal symbol is given as

$$\sigma_M^0((-r\partial_r) - B_r)(\lambda) = h(0,\lambda) = \lambda - A_\infty.$$

As a consequence of the (classical) parabolicity assumption, the conormal symbol is a parabolic meromorphic Volterra Mellin symbol as discussed in Section 5.2, and consequently is invertible as such. In particular, the resolvent $(\lambda - A_\infty)^{-1}$ exists for $\mathrm{Re}(\lambda)$ sufficiently large, and it is a meromorphic Volterra Mellin symbol of order $-\ell$ (as a function of $\lambda \in \mathbb{C}$). Notice also that this gives an explanation of the figure.

Consequently, equation (1) has a unique solution u for all $f \in e^{\gamma t}\big(L^2([t_0,\infty)\times X)\big)$ for some fixed $\gamma \in \mathbb{R}$ if and only if

- the parabolicity condition (8) is fulfilled, i. e., the anisotropic homogeneous principal symbol is invertible up to $t = \infty$,
- the conormal symbol
$$\sigma_M^0((-r\partial_r) - B_r)(\lambda) = \lambda - A_\infty : H^{s_0}(X) \longrightarrow H^{s_0-\ell}(X)$$
is invertible for some $s_0 \in \mathbb{R}$ and all $\mathrm{Re}(\lambda) \geq \gamma$.

Under these conditions, we obtain additionally the above-mentioned maximal regularity assertion, i. e., for the solution u we gain one derivative in time (according to the presence of one time derivative in equation (1)), and ℓ derivatives in space (according to the order ℓ of the operators A_t).

Thus we have seen that the (dominant) symbolic structure of the Volterra cone calculus, i. e., the principal symbol and the conormal symbol, both occur in the discussion of the simple equation (1), and the invertibility of both is required to decide about the solvability and regularity in the natural anisotropic Sobolev spaces with an exponential weight, as it is the case for general Volterra cone operators, too.

We shall not pursue the discussion of the asymptotic behaviour of solutions of equation (1) here and refer to the main text, in particular, to Chapter 4 for Sobolev spaces with asymptotics, and to Chapter 6 for the operator calculus, where equation (1) is a special case as we have seen.

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