On the Inverse of Parabolic Systems of Partial Differential Equations of General Form in an Infinite Space–Time Cylinder[1]

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Abstract

We consider general parabolic systems of equations on the infinite time interval in case of the underlying spatial configuration is a closed manifold. The solvability of equations is studied both with respect to time and spatial variables in exponentially weighted anisotropic Sobolev spaces, and existence and maximal regularity statements for parabolic equations are proved. Moreover, we analyze the long-time behaviour of solutions in terms of complete asymptotic expansions.

These results are deduced from a pseudodifferential calculus that we construct explicitly. This algebra of operators is specifically designed to contain both the classical systems of parabolic equations of general form and their inverses, parabolicity being reflected purely on symbolic level. To this end, we assign $t = \infty$ the meaning of an anisotropic conical point, and prove that this interpretation is consistent with the natural setting in the analysis of parabolic PDE. Hence, major parts of this work consist of the construction of an appropriate anisotropic cone calculus of so-called Volterra operators.

In particular, which is the most important aspect, we obtain the complete characterization of the microlocal and the global kernel structure of the inverse of parabolic systems in an infinite space–time cylinder. Moreover, we obtain perturbation results for parabolic equations from the investigation of the ideal structure of the calculus.

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Contents

Chapter 3

Parameter-dependent Volterra calculus on a closed manifold

3.1 Anisotropic parameter-dependent operators

3.1.1 Notation. Let X be a closed manifold of dimension $\dim X = n$, and E and F be complex vector bundles over X of dimensions N_- and N_+, respectively. A local chart will be denoted as a tuple (κ, Ω, U) (or simply κ), where $\Omega \subseteq X$ and $U \subseteq \mathbb{R}^n$ are open subsets and $\kappa : \Omega \to U$ is a diffeomorphism. We will throughout assume that the bundles are trivial over Ω. The transition matrices of the fibres (local trivializations of the bundles) are suppressed from the notation.

On X we fix the following data:

- A finite open covering $\{\widehat{\Omega_j};\ j = 1, \ldots, N\}$ of X, where $\widehat{\Omega_j} \subseteq \overline{\widehat{\Omega_j}} \Subset \Omega_j$ with suitable coordinate neighbourhoods $\{(\kappa_j, \Omega_j, U_j);\ j = 1, \ldots, N\}$ such that E and F are trivial over Ω_j.
- A subordinated C^∞-partition of unity $\{\varphi_j\}_{j=1,\ldots,N}$, i. e. $\varphi_j \in C_0^\infty(\widehat{\Omega_j})$ with $0 \leq \varphi_j \leq 1$ and $\sum_{j=1}^{N} \varphi_j \equiv 1$.
- Suitable functions $\psi_j \in C_0^\infty(\widehat{\Omega_j})$ and $\theta_j \in C_0^\infty(\Omega_j)$ with $\psi_j \equiv 1$ in a neighbourhood of $\mathrm{supp}(\varphi_j)$ and $\theta_j \equiv 1$ on $\overline{\widehat{\Omega_j}}$ for $j = 1, \ldots, N$.

- A Riemannian metric on X.
- Hermitean inner products on E and F.

The Riemannian metric and the Hermitean inner products on the bundles are fixed in order to avoid inconveniences what the measure on X and the Hilbert space structure of $L^2(X,E)$ are concerned. Alternatively, we could also fix a positive section in the density bundle over X replacing the Riemannian metric, or consider sections in the $\frac{1}{2}$-density bundle instead of working with functions.

Pull-backs and push-forwards of distributions and operators with respect to a chart κ will be denoted by κ^* and κ_*, respectively. Note once more that local trivializations of the bundles are suppressed from the notation.

3.1.2 Definition. a) We define $L^{-\infty}(X;\mathbb{R}^q;E,F) := \mathcal{S}(\mathbb{R}^q, L^{-\infty}(X;E,F))$, where $L^{-\infty}(X;E,F)$ denotes the space of smoothing pseudodifferential operators on X acting from $C^\infty(X,E)$ to $C^\infty(X,F)$. This space carries a natural Fréchet topology, and it is characterized as

$$\begin{aligned} L^{-\infty}(X;\mathbb{R}^q;E,F) &= \mathcal{S}(\mathbb{R}^q, \bigcap_{s,t\in\mathbb{R}} \mathcal{L}(H^s(X,E),H^t(X,F))) \\ &= \bigcap_{s,t\in\mathbb{R}} \mathcal{S}(\mathbb{R}^q, \mathcal{L}(H^s(X,E),H^t(X,F))) \\ &= S^{-\infty}(\mathbb{R}^q; \underset{s\in\mathbb{R}}{\text{ind-lim}}\, H^s(X,E), \underset{t\in\mathbb{R}}{\text{proj-lim}}\, H^t(X,F)). \end{aligned}$$

In other words, the elements $A(\lambda) \in L^{-\infty}(X;\mathbb{R}^q;E,F)$ are precisely those operators having integral kernels $k(x,y,\lambda) \in \mathcal{S}(\mathbb{R}^q_\lambda, C^\infty(X_x \times X_y, F \boxtimes E^*))$.

b) The space $L^{\mu;\ell}_{(cl)}(X;\mathbb{R}^q;E,F)$ consists of all families of operators $A(\lambda) : C^\infty(X,E) \to C^\infty(X,F)$ with the following properties:

i) For all functions $\varphi,\psi \in C^\infty(X)$ that are supported in the same coordinate neighbourhood (κ,Ω,U) the push-forward $\kappa_*(\varphi A(\lambda)\psi)$ belongs to $L^{\mu;\ell}_{\text{comp}\,(cl)}(U;\mathbb{R}^q;\mathbb{C}^{N_-},\mathbb{C}^{N_+})$.
Note that $\varphi A(\lambda)\psi$ acts as an operator from $C_0^\infty(\Omega,E)$ to $C_0^\infty(\Omega,F)$, and consequently $\kappa_*(\varphi A(\lambda)\psi) : C_0^\infty(U,\mathbb{C}^{N_-}) \to C_0^\infty(U,\mathbb{C}^{N_+})$. The condition is, that this operator should belong to $L^{\mu;\ell}_{\text{comp}\,(cl)}(U;\mathbb{R}^q;\mathbb{C}^{N_-},\mathbb{C}^{N_+}) \hookrightarrow \mathcal{L}(C_0^\infty(U,\mathbb{C}^{N_-}), C_0^\infty(U,\mathbb{C}^{N_+}))$.

ii) For all $\varphi,\psi \in C^\infty(X)$ with disjoint support the operator $\varphi A(\lambda)\psi$ belongs to $L^{-\infty}(X;\mathbb{R}^q;E,F)$.

We endow the space $L^{\mu;\ell}_{(cl)}(X;\mathbb{R}^q;E,F)$ with the projective topology with respect to the mappings $A(\lambda) \longmapsto$

$$\begin{cases} \kappa_*(\varphi A(\lambda)\psi) \in L^{\mu;\ell}_{\text{comp}\,(cl)}(U;\mathbb{R}^q;\mathbb{C}^{N_-},\mathbb{C}^{N_+}) & \text{for } \varphi,\psi \text{ supported in } (\kappa,\Omega,U) \\ \varphi A(\lambda)\psi \in L^{-\infty}(X;\mathbb{R}^q;E,F) & \text{for } \operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset. \end{cases}$$

3.1.3 Theorem. *a) Let* (κ, Ω, U) *be a chart. Then the pull-back* $\kappa^*(A(\lambda))$ *of every compactly supported operator family* $A(\lambda) \in L^{\mu;\ell}_{comp\,(cl)}(U; \mathbb{R}^q; \mathbb{C}^{N_-}, \mathbb{C}^{N_+})$ *belongs to* $L^{\mu;\ell}_{(cl)}(X; \mathbb{R}^q; E, F)$. *Moreover, this induces a continuous linear mapping*

$$L^{\mu;\ell}_{comp\,(cl)}(U; \mathbb{R}^q; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \ni A(\lambda) \longmapsto \kappa^*(A(\lambda)) \in L^{\mu;\ell}_{(cl)}(X; \mathbb{R}^q; E, F).$$

Note: $A(\lambda)$ *acts as a continuous operator* $C_0^\infty(U, \mathbb{C}^{N_-}) \to C_0^\infty(U, \mathbb{C}^{N_+})$, *and there exists a function* $\varphi \in C_0^\infty(U)$ *with* $\varphi A(\lambda)\varphi = A(\lambda)$. *Therefore, the pull-back* $\kappa^*(A(\lambda)) = \kappa^*(\varphi)\kappa^*(A(\lambda))\kappa^*(\varphi)$ *is defined as a continuous operator* $C_0^\infty(\Omega, E) \to C_0^\infty(\Omega, F)$, *which extends by means of the latter identity to an operator* $C^\infty(X, E) \to C^\infty(X, F)$. *The assertion now is, that* $\kappa^*(A(\lambda))$ *belongs to* $L^{\mu;\ell}_{(cl)}(X; \mathbb{R}^q; E, F)$ *in this sense.*

b) Let (κ, Ω, U) *be a chart,* $\varphi, \psi \in C_0^\infty(\Omega)$, *and* $A(\lambda) \in L^{\mu;\ell}_{(cl)}(\mathbb{R}^n; \mathbb{R}^q; \mathbb{C}^{N_-}, \mathbb{C}^{N_+})$. *Then the operator* $\varphi\kappa^*(A(\lambda))\psi$ *belongs to* $L^{\mu;\ell}_{(cl)}(X; \mathbb{R}^q; E, F)$, *and the mapping*

$$L^{\mu;\ell}_{(cl)}(\mathbb{R}^n; \mathbb{R}^q; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \ni A(\lambda) \longmapsto \varphi\kappa^*(A(\lambda))\psi \in L^{\mu;\ell}_{(cl)}(X; \mathbb{R}^q; E, F)$$

is continuous. Note that for short we write $\kappa^*(A(\lambda))$ *for* $\kappa^*(A(\lambda)|_{C_0^\infty(U, \mathbb{C}^{N_-})})$.

c) Let $A(\lambda) \in L^{\mu;\ell}_{(cl)}(X; \mathbb{R}^q; E, F)$. *We may write*

$$A(\lambda) = \sum_{j=1}^N \varphi_j A(\lambda)\psi_j + \underbrace{\Big(\sum_{j=1}^N \varphi_j A(\lambda)(1 - \psi_j)\Big)}_{=:K(\lambda) \in L^{-\infty}(X; \mathbb{R}^q; E, F)} = \sum_{j=1}^N \varphi_j(\theta_j A(\lambda)\theta_j)\psi_j + K(\lambda).$$

The mapping

$$\begin{aligned} L^{\mu;\ell}_{(cl)}(X; \mathbb{R}^q; E, F) \ni A(\lambda) &\longmapsto (\kappa_{1,*}(\theta_1 A(\lambda)\theta_1), \dots, \kappa_{N,*}(\theta_N A(\lambda)\theta_N); K(\lambda)) \\ &\in \mathop{\times}_{j=1}^N L^{\mu;\ell}_{comp\,(cl)}(U_j; \mathbb{R}^q; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \times L^{-\infty}(X; \mathbb{R}^q; E, F) \end{aligned} \tag{3.1.i}$$

is continuous. Conversely, the mapping

$$\begin{aligned} \mathop{\times}_{j=1}^N L^{\mu;\ell}_{(cl)}(\mathbb{R}^n; \mathbb{R}^q; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \times L^{-\infty}(X; \mathbb{R}^q; E, F) \ni (A_1(\lambda), \dots, A_N(\lambda); K(\lambda)) \\ \longmapsto \sum_{j=1}^N \varphi_j \kappa_j^*(A_j(\lambda))\psi_j + K(\lambda) \in L^{\mu;\ell}_{(cl)}(X; \mathbb{R}^q; E, F) \end{aligned} \tag{3.1.ii}$$

is continuous, and (3.1.i) is right-inverse to (3.1.ii).

d) $L^{\mu;\ell}_{(cl)}(X;\mathbb{R}^q;E,F)$ *is a Fréchet space, and the embedding* $L^{\mu;\ell}_{(cl)}(X;\mathbb{R}^q;E,F) \hookrightarrow C^\infty(\mathbb{R}^q, L^\mu(X;E,F))$ *is continuous.*

For $\beta \in \mathbb{N}_0^q$ *the operator* $\partial_\lambda^\beta : L^{\mu;\ell}_{(cl)}(X;\mathbb{R}^q;E,F) \to L^{\mu-\ell|\beta|;\ell}_{(cl)}(X;\mathbb{R}^q;E,F)$ *is continuous.*

Proof. a) follows from Theorem 2.2.15 and Remark 2.2.5. Note that the transition between different trivializations of the vector bundles results in "conjugation" with the transition matrices in the local representations for the operator. To prove b), choose a function $\theta \in C_0^\infty(\Omega)$ with $\theta\psi = \psi$ and $\theta\varphi = \varphi$. Then we have $\varphi\kappa^*(A(\lambda))\psi = \varphi\kappa^*(\kappa_*\theta A(\lambda)\kappa_*\theta)\psi$. For the mapping

$$L^{\mu;\ell}_{(cl)}(\mathbb{R}^n;\mathbb{R}^q;\mathbb{C}^{N_-},\mathbb{C}^{N_+}) \ni A(\lambda) \mapsto \kappa_*\theta A(\lambda)\kappa_*\theta \in L^{\mu;\ell}_{\mathrm{comp}\,(cl)}(U;\mathbb{R}^q;\mathbb{C}^{N_-},\mathbb{C}^{N_+})$$

is continuous, we obtain b) from a). c) follows immediately from a) and b), while d) is a consequence of c). □

3.1.4 Remark. The mapping (3.1.ii) gives rise to a continuous mapping op_x

$$\mathop{\times}_{j=1}^{N} S^{\mu;\ell}_{(cl)}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^q;\mathbb{C}^{N_-},\mathbb{C}^{N_+}) \times L^{-\infty}(X;\mathbb{R}^q;E,F) \ni (a_1,\dots,a_N;K(\lambda))$$
$$\longmapsto \sum_{j=1}^N \varphi_j\kappa_j^*(\mathrm{op}_x(a_j)(\lambda))\psi_j + K(\lambda) \in L^{\mu;\ell}_{(cl)}(X;\mathbb{R}^q;E,F), \tag{3.1.iii}$$

the so called *operator convention* for parameter-dependent pseudodifferential operators, while (3.1.i) induces a continuous mapping

$$L^{\mu;\ell}_{(cl)}(X;\mathbb{R}^q;E,F) \ni A(\lambda) \longmapsto ((a_1,\dots,a_N);K(\lambda))$$
$$\in \mathop{\times}_{j=1}^{N} S^{\mu;\ell}_{(cl)}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^q;\mathbb{C}^{N_-},\mathbb{C}^{N_+}) \times L^{-\infty}(X;\mathbb{R}^q;E,F), \tag{3.1.iv}$$

which is right-inverse to (3.1.iii).

We call a tuple $(a_1,\dots,a_N) \in \mathop{\times}_{j=1}^{N} S^{\mu;\ell}_{(cl)}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^q;\mathbb{C}^{N_-},\mathbb{C}^{N_+})$ a *complete symbol* for the operator $A(\lambda) \in L^{\mu;\ell}_{(cl)}(X;\mathbb{R}^q;E,F)$, if the following conditions are fulfilled:

- For any $\varphi,\psi \in C_0^\infty(\widehat{\Omega_j})$ it holds $\kappa_{j,*}(\varphi A(\lambda)\psi) = (\kappa_{j,*}\varphi)\mathrm{op}_x(a_j)(\lambda)(\kappa_{j,*}\psi)$ modulo $L^{-\infty}(\mathbb{R}^n;\mathbb{R}^q;\mathbb{C}^{N_-},\mathbb{C}^{N_+})$.
- For any choice of the partition of unity $\{\hat\varphi_j\}_j$ and functions $\{\hat\psi_j\}_j$ in Notation 3.1.1 we have $A(\lambda) - \sum_{j=1}^N \hat\varphi_j\kappa_j^*(\mathrm{op}_x(a_j)(\lambda))\hat\psi_j \in L^{-\infty}(X;\mathbb{R}^q;E,F)$.

The tuple $(a_1, \dots, a_N)$ obtained from (3.1.iv) yields a complete symbol for the operator $A(\lambda)$. We will refer to the mapping $A(\lambda) \mapsto (a_1, \dots, a_N)$ also as the *symbol mapping.*

In the classical case, equation (2.2.x) in Theorem 2.2.15 shows, that with $A(\lambda) \in L^{\mu;\ell}_{cl}(X; \mathbb{R}^q; E, F)$ we can associate uniquely the *principal symbol*

$$\sigma^{\mu;\ell}_{\psi}(A) \in C^{\infty}((T^*X \times \mathbb{R}^q) \setminus 0, \mathrm{Hom}(\pi^*E, \pi^*F)) \tag{3.1.v}$$

which is anisotropic homogeneous of degree μ in the fibres of $(T^*X \times \mathbb{R}^q) \setminus 0$. Here π^* denotes the pull-back with respect to the projection $\pi : (T^*X \times \mathbb{R}^q) \setminus 0 \to X$. The mapping $A(\lambda) \mapsto \sigma^{\mu;\ell}_{\psi}(A)$ is continuous. The following sequence is topologically exact and splits:

$$\begin{aligned} 0 \longrightarrow & L^{\mu-1;\ell}_{cl}(X; \mathbb{R}^q; E, F) \xrightarrow{\imath} L^{\mu;\ell}_{cl}(X; \mathbb{R}^q; E, F) \xrightarrow{\sigma^{\mu;\ell}_{\psi}} \\ & S^{(\mu;\ell)}((T^*X \times \mathbb{R}^q) \setminus 0, \mathrm{Hom}(\pi^*E, \pi^*F)) \longrightarrow 0, \end{aligned} \tag{3.1.vi}$$

where $S^{(\mu;\ell)}((T^*X \times \mathbb{R}^q) \setminus 0, \mathrm{Hom}(\pi^*E, \pi^*F))$ denotes the space of anisotropic homogeneous functions of degree μ as a closed subspace of $C^{\infty}((T^*X \times \mathbb{R}^q) \setminus 0, \mathrm{Hom}(\pi^*E, \pi^*F))$.

3.1.5 Theorem. *Let $A(\lambda) \in L^{\mu;\ell}(X; \mathbb{R}^q; E, F)$. Then $A(\lambda)$ extends by continuity to a family of continuous operators $A(\lambda) : H^s(X, E) \to H^{s-\nu}(X, F)$ for every $s, \nu \in \mathbb{R}$ with $\nu \geq \mu$. The following estimates for the norms are valid:*

$$\|A(\lambda)\|_{\mathcal{L}(H^s(X,E), H^{s-\nu}(X,F))} \leq \begin{cases} C_{s,\nu} \langle\lambda\rangle^{\frac{\mu}{\ell}} & \nu \geq 0 \\ C_{s,\nu} \langle\lambda\rangle^{\frac{\mu-\nu}{\ell}} & \nu \leq 0, \end{cases}$$

where $C_{s,\nu} > 0$ are suitable constants depending on s, ν and $A(\lambda)$, which may be chosen uniformly for $A(\lambda)$ in bounded subsets of $L^{\mu;\ell}(X; \mathbb{R}^q; E, F)$. More precisely, this induces a continuous embedding

$$L^{\mu;\ell}(X; \mathbb{R}^q; E, F) \hookrightarrow \begin{cases} S^{\frac{\mu}{\ell}}(\mathbb{R}^q; H^s(X, E), H^{s-\nu}(X, F)) & \nu \geq 0 \\ S^{\frac{\mu-\nu}{\ell}}(\mathbb{R}^q; H^s(X, E), H^{s-\nu}(X, F)) & \nu \leq 0 \end{cases}$$

into the space of operator-valued symbols in the Sobolev spaces.

Proof. This follows by means of Theorem 3.1.3 from Theorem 2.2.13. □

3.1.6 Theorem. a) *Let G be another vector bundle over X. Then the composition of operators on $C^{\infty}(X, E)$ gives rise to a continuous bilinear mapping*

$$L^{\mu;\ell}_{(cl)}(X; \mathbb{R}^q; G, F) \times L^{\mu';\ell}_{(cl)}(X; \mathbb{R}^q; E, G) \to L^{\mu+\mu';\ell}_{(cl)}(X; \mathbb{R}^q; E, F)$$

for $\mu, \mu' \in \mathbb{R}$. *If* $(a_1, \ldots, a_N)$ *is a complete symbol for* $A(\lambda) \in L^{\mu;\ell}_{(cl)}(X; \mathbb{R}^q; G, F)$, *and* $(b_1, \ldots, b_N)$ *a complete symbol for* $B(\lambda) \in L^{\mu';\ell}_{(cl)}(X; \mathbb{R}^q; E, G)$, *then* $(a_1 \# b_1, \ldots, a_N \# b_N)$ *is a complete symbol for the composition* $A(\lambda)B(\lambda) \in L^{\mu+\mu';\ell}_{(cl)}(X; \mathbb{R}^q; E, F)$.

In particular, the operators of order $-\infty$ *remain invariant with respect to compositions from the left and from the right, i. e., they share the properties of a two-sided ideal in the pseudodifferential operators.*

If $A(\lambda)$ *and* $B(\lambda)$ *are classical, then the following relation holds for the principal symbol of the composition:*

$$\sigma_\psi^{\mu+\mu';\ell}(AB) = \sigma_\psi^{\mu;\ell}(A) \cdot \sigma_\psi^{\mu';\ell}(B).$$

b) Let $A(\lambda) \in L^{\mu;\ell}_{(cl)}(X; \mathbb{R}^q; E, F)$. *Then the formal adjoint operator belongs to* $L^{\mu;\ell}_{(cl)}(X; \mathbb{R}^q; F, E)$. *Moreover, this induces a continuous antilinear mapping* $L^{\mu;\ell}_{(cl)}(X; \mathbb{R}^q; E, F) \ni A(\lambda) \mapsto A(\lambda)^{(*)} \in L^{\mu;\ell}_{(cl)}(X; \mathbb{R}^q; F, E)$.

In the classical case, we have the following relation for the homogeneous principal symbol: $\sigma_\psi^{\mu;\ell}(A^{(*)}) = \sigma_\psi^{\mu;\ell}(A)^*$.

Proof. In the proof we suppress the vector bundles from the notation for better readability.

To prove a), note that by Theorem 3.1.5 and the closed graph theorem we just have to show that the spaces of (classical) parameter-dependent pseudodifferential operators remain invariant under composition, and secondly that the formulae for the symbols are valid.

According to Definition 3.1.2 and Theorem 3.1.5 the space $L^{-\infty}(X; \mathbb{R}^q)$ clearly remains invariant with respect to compositions from the left and from the right with operators in $L^{\mu;\ell}(X; \mathbb{R}^q)$, i. e.,

$$\begin{aligned} L^{\mu;\ell}(X; \mathbb{R}^q) \times L^{-\infty}(X; \mathbb{R}^q) &\to L^{-\infty}(X; \mathbb{R}^q) \\ L^{-\infty}(X; \mathbb{R}^q) \times L^{\mu;\ell}(X; \mathbb{R}^q) &\to L^{-\infty}(X; \mathbb{R}^q). \end{aligned}$$

Let $\varphi, \psi \in C^\infty(X)$ such that $\mathrm{supp}(\varphi) \cap \mathrm{supp}(\psi) = \emptyset$. Choose a function $\eta \in C^\infty(X)$ with $\mathrm{supp}(\eta) \cap \mathrm{supp}(\varphi) = \emptyset$, and $\eta \equiv 1$ in a neighbourhood of $\mathrm{supp}(\psi)$. Hence we obtain

$$\varphi A(\lambda) B(\lambda) \psi = \underbrace{(\varphi A(\lambda) \eta)}_{\in L^{-\infty}(X;\mathbb{R}^q)} B(\lambda)\psi + \varphi A(\lambda) \underbrace{\big((1-\eta) B(\lambda) \psi\big)}_{\in L^{-\infty}(X;\mathbb{R}^q)} \in L^{-\infty}(X; \mathbb{R}^q)$$

as desired.

Now let $\varphi, \psi \in C_0^\infty(\Omega)$ be supported in the same coordinate neighbourhood (κ, Ω, U). Choose a function $\theta \in C_0^\infty(\Omega)$ such that $\theta \equiv 1$ in a neighbourhood of the support of φ and ψ. Then we see

$$\begin{aligned}\varphi A(\lambda)B(\lambda)\psi &= \varphi A(\lambda)\theta^2 B(\lambda)\psi + \varphi A(\lambda)\underbrace{\big((1-\theta^2)B(\lambda)\psi\big)}_{\in L^{-\infty}(X;\mathbb{R}^q)} \\ &= (\varphi A(\lambda)\theta)(\theta B(\lambda)\psi) + R(\lambda)\end{aligned}$$

with $R(\lambda) \in L^{-\infty}(X;\mathbb{R}^q)$. For

$$\kappa_*((\varphi A(\lambda)\theta)(\theta B(\lambda)\psi)) = \kappa_*(\varphi A(\lambda)\theta) \cdot \kappa_*(\theta B(\lambda)\psi) \in L_{(cl)}^{\mu+\mu';\ell}(\mathbb{R}^n;\mathbb{R}^q)$$

according to Definition 3.1.2 and Theorem 2.2.4, we finally obtain the desired assertion about the composition. Moreover, the latter identity also implies the corresponding results about the complete symbol and the homogeneous principal symbol.

To show b) note first that the assertion is immediately clear for operators belonging to $L^{-\infty}(X;\mathbb{R}^q)$. Consequently, we may restrict ourselves to operators that are supported in a coordinate neighbourhood (κ, Ω, U), such that the bundles are trivial over Ω. But in this case we may apply locally Theorem 2.2.4 from which we deduce the desired result. □

3.1.7 Lemma. *There exists a family of operators* $\{H_\theta;\ \theta \in \mathbb{R}_+\}$ *on the space* $\bigcup_{\mu\in\mathbb{R}} L^{\mu;\ell}(X;\mathbb{R}^q;E,F)$ *of parameter-dependent pseudodifferential operators with the following properties:*

$$\begin{aligned}H_\theta : L_{(cl)}^{\mu;\ell}(X;\mathbb{R}^q;E,F) &\longrightarrow L_{(cl)}^{\mu;\ell}(X;\mathbb{R}^q;E,F), \\ I - H_\theta : L_{(cl)}^{\mu;\ell}(X;\mathbb{R}^q;E,F) &\longrightarrow L^{-\infty}(X;\mathbb{R}^q;E,F)\end{aligned}$$

are continuous for each $\mu \in \mathbb{R}$. *Moreover, given a sequence* $(\mu_k) \subseteq \mathbb{R}$ *such that* $\mu_k > \mu_{k+1} \underset{k\to\infty}{\longrightarrow} -\infty$ *and countable systems of bounded sets* $(A_{k_j})_{j\in\mathbb{N}} \subseteq L^{\mu_k;\ell}(X;\mathbb{R}^q;E,F)$, *we may find a sequence* $(c_i) \subseteq \mathbb{R}_+$ *with* $c_i < c_{i+1} \underset{k\to\infty}{\longrightarrow} \infty$ *having the property, that for each* $k \in \mathbb{N}_0$

$$\sum_{i=k}^{\infty} \sup_{a\in A_{i_j}} p(H_{d_i}a) < \infty$$

for all continuous seminorms p *on* $L^{\mu_k;\ell}(X;\mathbb{R}^q;E,F)$ *and every* $j \in \mathbb{N}$, *for all sequences* $(d_i) \subseteq \mathbb{R}_+$ *with* $d_i \geq c_i$. *If the sequence* $(\mu_k)_{k\in\mathbb{N}_0}$ *is given as* $\mu_k := \mu - k$ *and the subsets are bounded in the classical operators, the same assertion holds for all continuous seminorms* p *on* $L_{cl}^{\mu_k;\ell}(X;\mathbb{R}^q;E,F)$.

Proof. In view of the operator convention and the symbol mappings it suffices to construct the operators H_θ on the symbol spaces and on $L^{-\infty}(X;\mathbb{R}^q;E,F)$ with the corresponding properties there (cf. Remark 3.1.4). Let $\chi_1 \in C^\infty(\mathbb{R}^n \times \mathbb{R}^q)$ and $\chi_2 \in C^\infty(\mathbb{R}^q)$ be 0-excision functions. Define for $\theta \in \mathbb{R}_+$ the operators H_θ on $S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;\mathbb{C}^{N_-},\mathbb{C}^{N_+})$ and on $L^{-\infty}(X;\mathbb{R}^q;E,F)$, respectively, via

$$(H_\theta a)(x,\xi,\lambda) := \chi_1\big(\frac{\xi}{\theta},\frac{\lambda}{\theta^\ell}\big)\cdot a(x,\xi,\lambda) \quad \text{for } a \in S^{\mu;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;\mathbb{C}^{N_-},\mathbb{C}^{N_+}),$$

$$(H_\theta A)(\lambda) := \chi_2\big(\frac{\lambda}{\theta}\big)\cdot A(\lambda) \quad \text{for } A(\lambda) \in L^{-\infty}(X;\mathbb{R}^q;E,F).$$

Then we obtain the assertion from Lemma 2.1.7 (see also Remark 2.1.10). □

3.1.8 Definition. Let $(\mu_k) \subseteq \mathbb{R}$ be a sequence of reals such that $\mu_k \underset{k\to\infty}{\longrightarrow} -\infty$ and $\overline{\mu} := \max_{k\in\mathbb{N}} \mu_k$. Moreover, let $A_k(\lambda) \in L^{\mu_k;\ell}(X;\mathbb{R}^q;E,F)$. An operator $A(\lambda) \in L^{\overline{\mu};\ell}(X;\mathbb{R}^q;E,F)$ is called the *asymptotic expansion* of the $A_k(\lambda)$, if for every $R \in \mathbb{R}$ there is a $k_0 \in \mathbb{N}$ such that for $k > k_0$

$$A(\lambda) - \sum_{j=1}^{k} A_j(\lambda) \in L^{R;\ell}(X;\mathbb{R}^q;E,F).$$

The operator $A(\lambda)$ is uniquely determined up to $L^{-\infty}(X;\mathbb{R}^q;E,F)$.

In analogous manner as we proved Theorem 2.1.8 using Lemma 2.1.7, we now obtain from Lemma 3.1.7 corresponding existence results of operators (or operator families) having a prescribed asymptotic expansion.

Ellipticity and parametrices

3.1.9 Definition. Let $A(\lambda) \in L^{\mu;\ell}_{(cl)}(X;\mathbb{R}^q;E,F)$. Then $A(\lambda)$ is called *parameter-dependent elliptic*, if the following condition is fulfilled:

For every compact set $K \Subset (\kappa,\Omega,U)$ contained in a coordinate neighbourhood (with the vector bundles being trivial over Ω), and every $\varphi,\psi \in C_0^\infty(\Omega)$ such that $\varphi,\psi \equiv 1$ on K, the push-forward $\kappa_*(\varphi A(\lambda)\psi) \in L^{\mu;\ell}_{\mathrm{comp}\,(cl)}(U;\mathbb{R}^q;\mathbb{C}^{N_-},\mathbb{C}^{N_+})$ is parameter-dependent elliptic on $\kappa(K) \Subset U$ in the sense of Definition 2.2.8.

In the classical case, the condition of parameter-dependent ellipticity simplifies to the invertibility of $\sigma_\psi^{\mu;\ell}(A)(x,\xi,\lambda)$ for $(x,\xi,\lambda) \in \big(T^*X \times \mathbb{R}^q\big)\setminus 0$, see Remark 2.2.9.

Note that for the existence of parameter-dependent elliptic elements it is necessary that the dimensions of the vector bundles coincide, i. e. $N_- = N_+$.

3.1.10 Theorem. *Let $A(\lambda) \in L^{\mu;\ell}(X;\mathbb{R}^q;E,F)$. The following assertions are equivalent:*

a) $A(\lambda)$ *is parameter-dependent elliptic.*

b) The components of the complete symbol $(a_1, \dots, a_N)$ *of* $A(\lambda)$ *are parameter-dependent elliptic on* $\kappa_j(\mathrm{supp}\psi_j)$ *for* $j = 1, \dots, N$.

c) There exists an operator $P(\lambda) \in L^{-\mu;\ell}(X;\mathbb{R}^q;F,E)$ *such that* $A(\lambda)P(\lambda) - I \in L^{-\varepsilon;\ell}(X;\mathbb{R}^q;F,F)$ *and* $P(\lambda)A(\lambda) - I \in L^{-\varepsilon;\ell}(X;\mathbb{R}^q;E,E)$ *for some* $\varepsilon > 0$.

d) There exists an operator $P(\lambda) \in L^{-\mu;\ell}(X;\mathbb{R}^q;F,E)$ *such that* $A(\lambda)P(\lambda) - I \in L^{-\infty}(X;\mathbb{R}^q;F,F)$ *and* $P(\lambda)A(\lambda) - I \in L^{-\infty}(X;\mathbb{R}^q;E,E)$.

Moreover, if $A(\lambda) \in L^{\mu;\ell}_{cl}(X;\mathbb{R}^q;E,F)$ *is parameter-dependent elliptic, then every operator* $P(\lambda)$ *satisfying d) belongs to* $L^{-\mu;\ell}_{cl}(X;\mathbb{R}^q;F,E)$. *Every* $P(\lambda)$ *satisfying d) is called a* (parameter-dependent) parametrix *of* $A(\lambda)$.

Proof. a) implies b) follows from the definition of parameter-dependent ellipticity. Now assume that b) holds. From Corollary 2.2.11 we obtain the existence of $b_1, \dots, b_N \in S^{-\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q;\mathbb{C}^{N_+},\mathbb{C}^{N_-})$ and suitable functions $\hat{\varphi}_j, \hat{\psi}_j \in C_0^\infty(\mathbb{R}^n)$ such that $\hat{\psi}_j\hat{\varphi}_j = \hat{\varphi}_j$ and $\hat{\varphi}_j \equiv 1$ on $\kappa_j(\mathrm{supp}\psi_j) \Subset \mathbb{R}^n$ with the property that $\hat{\varphi}_j(\mathrm{op}_x(a_j)(\lambda)\mathrm{op}_x(b_j)(\lambda) - 1)\hat{\psi}_j$ belongs to $L^{-\infty}(\mathbb{R}^n;\mathbb{R}^q;\mathbb{C}^{N_+},\mathbb{C}^{N_+})$. Now define $P(\lambda) := \mathrm{op}_x((b_1,\dots,b_N);0)$ with the operator convention (3.1.iii), which yields an operator $P(\lambda) \in L^{-\mu;\ell}_{(cl)}(X;\mathbb{R}^q;F,E)$. Now it is straightforward to check that $A(\lambda)P(\lambda) - I$ belongs to $L^{-1;\ell}_{(cl)}(X;\mathbb{R}^q;F,F)$. Analogously, we obtain a parametrix from the left. But both the left- and the right-parametrix differ only by a term of order -1 which gives c). c) implies d) follows by means of a formal Neumann series argument as, e. g., in the proof of Theorem 2.2.10, where now Theorem 3.1.6 and Definition 3.1.8 enter the argument. This also yields the existence of a classical parametrix if we started with an elliptic classical parameter-dependent operator $A(\lambda)$. d) implies a) is part of Corollary 2.2.11 when passing to local coordinates. □

3.1.11 Theorem. *Let* Δ *be a compact* C^∞*-manifold (not necessarily with empty boundary), and* $\Delta \ni \delta \longmapsto A^\delta(\lambda) \in L^{\mu;\ell}_{(cl)}(X;\mathbb{R}^q;E,F)$ *be a smooth family that is locally uniformly parameter-dependent elliptic. Then the set*

$$K := \{(\delta,\lambda) \in \Delta \times \mathbb{R}^q;\ \left(A^\delta(\lambda)\right)^{-1} \in \mathcal{L}(H^{s-\mu}(X,F), H^s(X,E)) \text{ does not exist}\}$$

is compact in $\Delta \times \mathbb{R}^q$ *and independent of* $s \in \mathbb{R}$. *Moreover, for any given neighbourhood* $U(K) \subseteq \Delta \times \mathbb{R}^q$ *of* K, *there exists a* C^∞*-family* $\Delta \ni \delta \mapsto P^\delta(\lambda) \in L^{-\mu;\ell}_{(cl)}(X;\mathbb{R}^q;F,E)$ *such that* $A^\delta(\lambda)P^\delta(\lambda) - I$ *and* $P^\delta(\lambda)A^\delta(\lambda) - I$ *depend smoothly on* $\delta \in \Delta$ *with values in the operators of order* $-\infty$, *and* $P^\delta(\lambda) = (A^\delta(\lambda))^{-1}$ *for* $(\delta,\lambda) \in \left(\Delta \times \mathbb{R}^q\right) \setminus U(K)$.

Proof. In the proof we omit the vector bundles for better readability. From elliptic regularity we obtain that the set K is indeed independent of $s \in \mathbb{R}$. Consequently, we may fix $s \in \mathbb{R}$ in the sequel. Let $B^\delta(\lambda) \in L^{-\mu;\ell}_{(cl)}(X;\mathbb{R}^q)$ be C^∞ such that $A^\delta(\lambda)B^\delta(\lambda) - I =: R^\delta_R(\lambda)$ and $B^\delta(\lambda)A^\delta(\lambda) - I =: R^\delta_L(\lambda)$ depend smoothly on $\delta \in \Delta$ with values in $L^{-\infty}(X;\mathbb{R}^q)$ (cf. Theorem 3.1.10). From Theorem 3.1.5 (or the defining characterization of $L^{-\infty}(X;\mathbb{R}^q)$ in Definition 3.1.2) we see, that for $|\lambda|$ sufficiently large and all $\delta \in \Delta$ the operators $I + R^\delta_R(\lambda)$ and $I + R^\delta_L(\lambda)$ are invertible in $\mathcal{L}(H^{s-\mu}(X))$ and $\mathcal{L}(H^s(X))$, respectively. Thus it remains to show the closedness of the set $K \subseteq \big(\Delta \times \mathbb{R}^q\big)$. For the set of invertible operators in $\mathcal{L}(H^s(X), H^{s-\mu}(X))$ is open, and since $A^\delta(\lambda)$ may be viewed as a continuous function of $(\delta,\lambda) \in \Delta \times \mathbb{R}^q$ with values in this space, we obtain the closedness and consequently the asserted compactness of $K \subseteq \Delta \times \mathbb{R}^q$.

Now let $U(K) \subseteq \Delta \times \mathbb{R}^q$ be any given neighbourhood of K. Let $\chi \in C^\infty(\Delta \times \mathbb{R}^q)$ such that $\chi \equiv 0$ on K and $\chi \equiv 1$ on $\big(\Delta \times \mathbb{R}^q\big) \setminus U(K)$. Define

$$P^\delta_L(\lambda) := B^\delta(\lambda) - R^\delta_L(\lambda)B^\delta(\lambda) + R^\delta_L(\lambda)\chi(\delta,\lambda)(A^\delta(\lambda))^{-1}R^\delta_R(\lambda),$$
$$P^\delta_R(\lambda) := B^\delta(\lambda) - B^\delta(\lambda)R^\delta_R(\lambda) + R^\delta_L(\lambda)\chi(\delta,\lambda)(A^\delta(\lambda))^{-1}R^\delta_R(\lambda).$$

From Theorem 3.1.5 we obtain that $R^\delta_L(\lambda)\chi(\delta,\lambda)(A^\delta(\lambda))^{-1}R^\delta_R(\lambda)$ is a C^∞-function of $\delta \in \Delta$ taking values in $L^{-\infty}(X;\mathbb{R}^q)$, and thus $P^\delta_L(\lambda)$ and $P^\delta_R(\lambda)$ depend smoothly on $\delta \in \Delta$ with values in $L^{-\mu;\ell}_{(cl)}(X;\mathbb{R}^q)$. Moreover, we have $P^\delta_L(\lambda)A^\delta(\lambda) = I$ as well as $A^\delta(\lambda)P^\delta_R(\lambda) = I$ for $(\delta,\lambda) \in \big(\Delta \times \mathbb{R}^q\big) \setminus U(K)$. Now define $P^\delta(\lambda)$ either as $P^\delta_L(\lambda)$ or as $P^\delta_R(\lambda)$, which concludes the proof. □

3.1.12 Theorem. *There exist parameter-dependent reductions of orders, i. e., there exist operators $R^\mu(\lambda) \in L^{\mu;\ell}(X;\mathbb{R}^q;E,E)$ such that $R^\mu(\lambda)R^{-\mu}(\lambda) = I$ for every $\mu \in \mathbb{R}$.*

Proof. For $\mu = 0$ choose the identity. Now let $\mu > 0$. With the given Riemannian metric on X we define the anisotropic homogeneous function of degree μ on $T^*X \times \mathbb{R}^{q+1} \setminus 0$ with values in $\mathrm{Hom}(\pi^*E)$ via

$$a_{(\mu)}(\xi_x,(\lambda,\lambda_{q+1})) := \big(\langle \xi_x,\xi_x\rangle_x^\ell + |(\lambda,\lambda_{q+1})|^2\big)^{\frac{\mu}{2\ell}} \cdot \mathrm{id}_{\pi^*E_{(\xi_x,(\lambda,\lambda_{q+1}))}}.$$

The associated operator in $L^{\mu;\ell}(X;\mathbb{R}^{q+1};E,E)$ therefore is parameter-dependent elliptic. From Theorem 3.1.11 we now obtain, that if we fix λ_{q+1} with $|\lambda_{q+1}|$ sufficiently large, we obtain invertible operators $R^\mu(\lambda) \in L^{\mu;\ell}(X;\mathbb{R}^q;E,E)$ with inverses $R^{-\mu}(\lambda)$ in $L^{-\mu;\ell}(X;\mathbb{R}^q;E,E)$. This proves the theorem. □

3.1.13 Remark. In analogous manner, we also have the calculus of anisotropic parameter-dependent pseudodifferential operators on closed manifolds X, where the parameter-space $\mathbb{R}^q$ is substituted by a conical set $\Lambda \subseteq \mathbb{R}^q$ that is assumed to be the closure of its interior. This will be employed, in particular, with half-planes in $\mathbb{C} \cong \mathbb{R}^2$.

3.2 Parameter-dependent Volterra operators

3.2.1 Remark. Throughout this section we employ again the notations from Notation 3.1.1 with the same data fixed on X and the vector bundles E and F.

3.2.2 Definition. a) Define

$$L_V^{-\infty}(X;\mathbb{H};E,F) := L^{-\infty}(X;\mathbb{H};E,F) \cap \mathcal{A}(\mathring{\mathbb{H}}, L^{-\infty}(X;E,F)),$$

which is a closed subspace of $L^{-\infty}(X;\mathbb{H};E,F)$. It is characterized as

$$\begin{aligned}
&\mathcal{S}(\mathbb{H}, \bigcap_{s,t\in\mathbb{R}} \mathcal{L}(H^s(X,E), H^t(X,F))) \cap \mathcal{A}(\mathring{\mathbb{H}}, \bigcap_{s,t\in\mathbb{R}} \mathcal{L}(H^s(X,E), H^t(X,F)))\\
&= \bigcap_{s,t\in\mathbb{R}} \mathcal{S}(\mathbb{H}, \mathcal{L}(H^s(X,E), H^t(X,F))) \cap \mathcal{A}(\mathring{\mathbb{H}}, \mathcal{L}(H^s(X,E), H^t(X,F)))\\
&= S_V^{-\infty}(\mathbb{H}; \operatorname*{ind-lim}_{s\in\mathbb{R}} H^s(X,E), \operatorname*{proj-lim}_{t\in\mathbb{R}} H^t(X,F)).
\end{aligned}$$

b) For $\mu \in \mathbb{R}$ define

$$L_{V(cl)}^{\mu;\ell}(X;\mathbb{H};E,F) := L_{(cl)}^{\mu;\ell}(X;\mathbb{H};E,F) \cap \mathcal{A}(\mathring{\mathbb{H}}, L^{\mu}(X;E,F)),$$

with the space of pseudodifferential operators $L^\mu(X;E,F)$ on X acting in sections of the vector bundles E and F. Thus $L_{V(cl)}^{\mu;\ell}(X;\mathbb{H};E,F)$ becomes a closed subspace of $L_{(cl)}^{\mu;\ell}(X;\mathbb{H};E,F)$.

3.2.3 Remark. From the considerations about the calculus of Volterra pseudodifferential operators in Section 2.4 on the one hand as well as Definition 3.1.2 on the other hand we conclude, that the space $L_{V(cl)}^{\mu;\ell}(X;\mathbb{H};E,F)$ consists of all families of operators $A(\lambda) : C^\infty(X,E) \to C^\infty(X,F)$ with the following properties:

i) For all functions $\varphi, \psi \in C^\infty(X)$ that are supported in the same coordinate neighbourhood (κ, Ω, U) the push-forward $\kappa_*(\varphi A(\lambda)\psi)$ belongs to $L_{\mathrm{comp}\,V(cl)}^{\mu;\ell}(U;\mathbb{H};\mathbb{C}^{N_-},\mathbb{C}^{N_+})$.

ii) For all $\varphi, \psi \in C^\infty(X)$ having disjoint support the operator $\varphi A(\lambda)\psi$ is an element of $L_V^{-\infty}(X;\mathbb{H};E,F)$.

The projective topology on the space $L_{V(cl)}^{\mu;\ell}(X;\mathbb{H};E,F)$ with respect to the mappings $A(\lambda) \longmapsto$

$$\begin{cases}
\kappa_*(\varphi A(\lambda)\psi) \in L_{\mathrm{comp}\,V(cl)}^{\mu;\ell}(U;\mathbb{H};\mathbb{C}^{N_-},\mathbb{C}^{N_+}) & \text{for } \varphi,\psi \text{ supported in } (\kappa,\Omega,U)\\
\varphi A(\lambda)\psi \in L_V^{-\infty}(X;\mathbb{H};E,F) & \text{for } \mathrm{supp}\varphi \cap \mathrm{supp}\psi = \emptyset
\end{cases}$$

is exactly the given one from Definition 3.2.2.

Theorem 3.1.3 holds within parameter-dependent Volterra pseudodifferential operators:

a) Let (κ, Ω, U) be a chart and $A(\lambda) \in L^{\mu;\ell}_{\mathrm{comp}\,V(cl)}(U; \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+})$. Then the pull-back $\kappa^*(A(\lambda))$ belongs to $L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}; E, F)$ and provides a continuous linear mapping

$$L^{\mu;\ell}_{\mathrm{comp}\,V(cl)}(U; \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \ni A(\lambda) \longmapsto \kappa^*(A(\lambda)) \in L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}; E, F).$$

b) Let (κ, Ω, U) be a chart and $\varphi, \psi \in C_0^\infty(\Omega)$.
Moreover, let $A(\lambda) \in L^{\mu;\ell}_{V(cl)}(\mathbb{R}^n; \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+})$. Then the operator $\varphi\kappa^*(A(\lambda))\psi$ belongs to $L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}; E, F)$, and the mapping

$$L^{\mu;\ell}_{V(cl)}(\mathbb{R}^n; \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \ni A(\lambda) \longmapsto \varphi\kappa^*(A(\lambda))\psi \in L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}; E, F)$$

is continuous.

c) The restriction of the mapping (3.1.i)

$$L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}; E, F) \longrightarrow \mathop{\times}_{j=1}^{N} L^{\mu;\ell}_{\mathrm{comp}\,V(cl)}(U_j; \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \times L^{-\infty}_V(X; \mathbb{H}; E, F) \tag{3.2.i}$$

to Volterra pseudodifferential operators is well-defined and continuous, and so is the restriction of the mapping (3.1.ii)

$$\mathop{\times}_{j=1}^{N} L^{\mu;\ell}_{V(cl)}(\mathbb{R}^n; \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \times L^{-\infty}_V(X; \mathbb{H}; E, F) \longrightarrow L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}; E, F). \tag{3.2.ii}$$

d) For $\beta \in \mathbb{N}_0$ the complex derivative acts continuously in the spaces $\partial^\beta_\lambda : L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}; E, F) \to L^{\mu-\ell|\beta|;\ell}_{V(cl)}(X; \mathbb{H}; E, F)$.

The restriction of the parameter to the real line induces a continuous embedding

$$L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}; E, F) \hookrightarrow L^{\mu;\ell}_{(cl)}(X; \mathbb{R}; E, F).$$

This follows from Proposition 2.3.2, Remark 2.4.4.

3.2.4 Remark. From (3.2.ii) and (3.2.i) we see that the restriction of the operator convention op_x from (3.1.iii)

$$\mathop{\times}_{j=1}^{N} S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \times L^{-\infty}_V(X; \mathbb{H}; E, F) \longrightarrow L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}; E, F), \tag{3.2.iii}$$

as well as the restriction of the mapping (3.1.iv)

$$L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F) \longrightarrow \mathop{\times}_{j=1}^{N} S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H};\mathbb{C}^{N_-},\mathbb{C}^{N_+}) \times L^{-\infty}_V(X;\mathbb{H};E,F) \tag{3.2.iv}$$

to Volterra symbols respectively operators are well-defined and continuous.
The symbol mapping induced from (3.2.iv) now associates to a given operator $A(\lambda) \in L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F)$ a complete symbol $(a_1,\dots,a_N)$ consisting of (classical) Volterra symbols.

In the classical case, the parameter-dependent anisotropic homogeneous principal symbol $\sigma^{\mu;\ell}_{\psi}(A) \in C^\infty((T^*X \times \mathbb{H}) \setminus 0, \mathrm{Hom}(\pi^*E,\pi^*F))$ of an operator $A(\lambda) \in L^{\mu;\ell}_{V\,cl}(X;\mathbb{H};E,F)$ is analytic in the interior of $\mathbb{H}$. This follows from Proposition 2.3.2.

3.2.5 Theorem. *Let G be another vector bundle over X. Then the composition of operators (cf. Theorem 3.1.6) restricts to a continuous bilinear mapping*

$$L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};G,F) \times L^{\mu';\ell}_{V(cl)}(X;\mathbb{H};E,G) \to L^{\mu+\mu';\ell}_{V(cl)}(X;\mathbb{H};E,F)$$

for $\mu,\mu' \in \mathbb{R}$. In particular, the Volterra operators of order $-\infty$ remain invariant with respect to compositions from the left and from the right, i. e. they share the properties of a two-sided ideal in the Volterra pseudodifferential operators.

Proof. By Theorem 3.1.6 the bilinear mappings

$$\mathcal{A}(\overset{\circ}{\mathbb{H}}, L^{\mu}(X;G,F)) \times \mathcal{A}(\overset{\circ}{\mathbb{H}}, L^{\mu'}(X;E,G)) \longrightarrow \mathcal{A}(\overset{\circ}{\mathbb{H}}, L^{\mu+\mu'}(X;E,F)),$$
$$L^{\mu;\ell}_{(cl)}(X;\mathbb{H};G,F) \times L^{\mu';\ell}_{(cl)}(X;\mathbb{H};E,G) \longrightarrow L^{\mu+\mu';\ell}_{(cl)}(X;\mathbb{H};E,F)$$

are well-defined and continuous. This implies the assertion in view of Definition 3.2.2. □

3.2.6 Theorem. *Let $A(\lambda) \in L^{\mu;\ell}_V(X;\mathbb{H};E,F)$. Then $A(\lambda)$ extends by continuity to a family of continuous operators $A(\lambda) : H^s(X,E) \to H^{s-\nu}(X,F)$ for every $s,\nu \in \mathbb{R}$ with $\nu \geq \mu$. Moreover, this induces a continuous embedding*

$$L^{\mu;\ell}_V(X;\mathbb{H};E,F) \hookrightarrow \begin{cases} S_V^{\frac{\mu}{\ell}}(\mathbb{H};H^s(X,E),H^{s-\nu}(X,F)) & \nu \geq 0 \\ S_V^{\frac{\mu-\nu}{\ell}}(\mathbb{H};H^s(X,E),H^{s-\nu}(X,F)) & \nu \leq 0 \end{cases}$$

into the space of operator-valued Volterra symbols in the Sobolev spaces.

Proof. This follows from Theorem 3.1.5 and Definition 3.2.2. □

Kernel cut-off behaviour and asymptotic expansion

3.2.7 Definition. Let $(\mu_k) \subseteq \mathbb{R}$ be a sequence of reals such that $\mu_k \underset{k\to\infty}{\longrightarrow} -\infty$ and $\overline{\mu} := \max_{k\in\mathbb{N}} \mu_k$. Moreover, let $A_k(\lambda) \in L_V^{\mu_k;\ell}(X;\mathbb{H};E,F)$. An operator $A(\lambda) \in L_V^{\overline{\mu};\ell}(X;\mathbb{H};E,F)$ is called the *asymptotic expansion* of the $A_k(\lambda)$, if for every $R \in \mathbb{R}$ there is a $k_0 \in \mathbb{N}$ such that for $k > k_0$

$$A(\lambda) - \sum_{j=1}^{k} A_j(\lambda) \in L_V^{R;\ell}(X;\mathbb{H};E,F).$$

The operator $A(\lambda)$ is uniquely determined up to $L_V^{-\infty}(X;\mathbb{H};E,F)$. For short, we write $A(\lambda) \underset{V}{\sim} \sum_{j=1}^{\infty} A_j(\lambda)$.

Note that the distinction between the notion of asymptotic expansion from Definition 3.1.8 is that we require the remainders to have the Volterra property (cf. Definition 2.3.3). In order to obtain existence results of Volterra operators having a prescribed asymptotic expansion we need to carry over the considerations concerning the kernel cut-off operator (cf. Definition 2.3.5, Theorem 2.3.6, and Corollary 2.3.7) and deduce an analogue of Proposition 2.3.8.

3.2.8 Remark. Let $\mu \in \mathbb{R}$ and $\mu_+ := \max\{0,\mu\}$. From Theorem 3.1.5 and Theorem 3.2.6 we see that for every $s \in \mathbb{R}$ we have continuous embeddings

$$L^{\mu;\ell}(X;\mathbb{R};E,F) \hookrightarrow S^{\frac{\mu}{\ell}}(\mathbb{R};H^s(X,E),H^{s-\mu_+}(X,F)),$$
$$L_V^{\mu;\ell}(X;\mathbb{H};E,F) \hookrightarrow S_V^{\frac{\mu}{\ell}}(\mathbb{H};H^s(X,E),H^{s-\mu_+}(X,F)).$$

By Theorem 2.3.6 the kernel cut-off operator H (see Definition 2.3.5) acts as a bilinear and continuous map in the spaces

$$C_b^\infty(\mathbb{R})\times S^{\frac{\mu}{\ell}}(\mathbb{R};H^s(X,E),H^{s-\mu_+}(X,F)) \longrightarrow S^{\frac{\mu}{\ell}}(\mathbb{R};H^s(X,E),H^{s-\mu_+}(X,F)),$$
$$C_b^\infty(\mathbb{R})\times S_V^{\frac{\mu}{\ell}}(\mathbb{H};H^s(X,E),H^{s-\mu_+}(X,F)) \longrightarrow S_V^{\frac{\mu}{\ell}}(\mathbb{H};H^s(X,E),H^{s-\mu_+}(X,F)).$$

3.2.9 Theorem. *The kernel cut-off operator H (cf. Remark 3.2.8) restricts to continuous bilinear mappings*

$$H : \begin{cases} C_b^\infty(\mathbb{R})\times L_{(cl)}^{\mu;\ell}(X;\mathbb{R};E,F) \longrightarrow L_{(cl)}^{\mu;\ell}(X;\mathbb{R};E,F) \\ C_b^\infty(\mathbb{R})\times L_{V(cl)}^{\mu;\ell}(X;\mathbb{H};E,F) \longrightarrow L_{V(cl)}^{\mu;\ell}(X;\mathbb{H};E,F). \end{cases}$$

Moreover, the following asymptotic expansion (in the sense of Definitions 3.1.8, 3.2.7) holds for $\big(H(\varphi)A\big)(\lambda)$ in terms of φ and $A(\lambda)$:

$$\big(H(\varphi)A\big)(\lambda) \underset{(V)}{\sim} \sum_{k=0}^{\infty} \Big(\frac{(-1)^k}{k!} D_t^k\varphi(0)\Big)\cdot\big(\partial_\lambda^k A\big)(\lambda)$$

where ∂_λ denotes the complex derivative with respect to $\lambda \in \mathbb{H}$ in case of Volterra operators.

Proof. For the proof of the first assertion we simply have to check that H maps the corresponding spaces into each other as asserted. The (separate) continuity then follows from the closed graph theorem.

Employing (3.1.iv) and (3.2.iv) we obtain continuous linear mappings

$$L^{\mu;\ell}_{(cl)}(X;\mathbb{R};E,F) \longrightarrow \mathop{\times}_{j=1}^{N} S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \times L^{-\infty}(X;\mathbb{R};E,F),$$
$$L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F) \longrightarrow \mathop{\times}_{j=1}^{N} S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \times L^{-\infty}_{V}(X;\mathbb{H};E,F).$$

From Theorem 2.3.6 we conclude that the kernel cut-off operator acts bilinear and continuous in each of the factors on the right-hand sides, i. e.,

$$C^\infty_b(\mathbb{R}) \times S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \longrightarrow S^{\mu;\ell}_{(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+})$$
$$C^\infty_b(\mathbb{R}) \times S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \longrightarrow S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}),$$

as well as

$$C^\infty_b(\mathbb{R}) \times L^{-\infty}(X;\mathbb{R};E,F) \longrightarrow L^{-\infty}(X;\mathbb{R};E,F)$$
$$C^\infty_b(\mathbb{R}) \times L^{-\infty}_{V}(X;\mathbb{H};E,F) \longrightarrow L^{-\infty}_{V}(X;\mathbb{H};E,F),$$

keeping in mind the characterizations of $L^{-\infty}(X;\mathbb{R};E,F)$ and $L^{-\infty}_V(X;\mathbb{H};E,F)$ as operator-valued symbols from Definition 3.1.2 and Definition 3.2.2. Moreover, the asymptotic expansion (2.3.ii) holds in the factors corresponding to the $\mathcal{L}(\mathbb{C}^{N_-}, \mathbb{C}^{N_+})$-valued symbols.

Now we see that we find the kernel cut-off operator (restricted to parameter-dependent pseudodifferential operators as in the assertion of the theorem) as composition of the mappings (3.1.iv) resp. (3.2.iv), the "local" kernel cut-off operators in the factors as discussed above, and the operator convention (3.1.iii) resp. (3.2.iii). Moreover, the "local" expansions give also the second assertion of the theorem concerning the asymptotic expansion. □

3.2.10 Corollary. *Let $\varphi \in C^\infty_0(\mathbb{R})$ such that $\varphi \equiv 1$ near $t = 0$. Then the operator $I - H(\varphi)$ acts continuous in the spaces*

$$I - H(\varphi) : \begin{cases} L^{\mu;\ell}(X;\mathbb{R};E,F) \longrightarrow L^{-\infty}(X;\mathbb{R};E,F) \\ L^{\mu;\ell}_V(X;\mathbb{H};E,F) \longrightarrow L^{-\infty}_V(X;\mathbb{H};E,F). \end{cases}$$

3.2.11 Proposition. *Let $(\mu_k) \subseteq \mathbb{R}$ such that $\mu_k \geq \mu_{k+1} \underset{k\to\infty}{\longrightarrow} -\infty$. Furthermore, for each $k \in \mathbb{N}$ let $(A_{k_j})_{j\in\mathbb{N}} \subseteq L_V^{\mu_k;\ell}(X;\mathbb{H};E,F)$ be a countable system of bounded sets. Let $\varphi \in C_0^\infty(\mathbb{R})$, and for $c \in [1,\infty)$ let $\varphi_c \in C_0^\infty(\mathbb{R})$ be defined as $\varphi_c(t) := \varphi(ct)$. Then there is a sequence $(c_i) \subseteq [1,\infty)$ with $c_i < c_{i+1} \underset{i\to\infty}{\longrightarrow} \infty$ such that for each $k \in \mathbb{N}$*

$$\sum_{i=k}^{\infty} \sup_{A(\lambda)\in A_{i_j}} p\big((H(\varphi_{d_i})A)(\lambda)\big) < \infty$$

for all continuous seminorms p on $L_V^{\mu_k;\ell}(X;\mathbb{H};E,F)$ and every $j \in \mathbb{N}$, and for all sequences $(d_i) \subseteq \mathbb{R}_+$ with $d_i \geq c_i$.

Proof. Employing (3.2.iv) and the operator convention (3.2.iii) as well as Theorem 3.2.9 reduces the proof to the case of Volterra symbol spaces $S_V^{\mu_k;\ell}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{H};\mathbb{C}^{N_-},\mathbb{C}^{N_+})$ and $L_V^{-\infty}(X;\mathbb{H};E,F)$. But for these the assertion follows at once from Proposition 2.3.8. □

3.2.12 Theorem. *Let $(\mu_k) \subseteq \mathbb{R}$ such that $\mu_k \underset{k\to\infty}{\longrightarrow} -\infty$ and $\overline{\mu} := \max\limits_{k\in\mathbb{N}_0} \mu_k$. Moreover, let $A_k(\lambda) \in L_V^{\mu_k;\ell}(X;\mathbb{H};E,F)$. Then there exists $A(\lambda) \in L_V^{\overline{\mu};\ell}(X;\mathbb{H};E,F)$ such that $A(\lambda) \underset{V}{\sim} \sum\limits_{k=0}^{\infty} A_k(\lambda)$. The asymptotic sum $A(\lambda)$ is uniquely determined modulo $L_V^{-\infty}(X;\mathbb{H};E,F)$.*

If the sequence $(\mu_k)_{k\in\mathbb{N}_0}$ is given as $\mu_k = \overline{\mu} - k$ and $A_k(\lambda) \in L_{V\,cl}^{\overline{\mu}-k;\ell}(X;\mathbb{H};E,F)$, then also $A(\lambda) \in L_{V\,cl}^{\overline{\mu};\ell}(X;\mathbb{H};E,F)$.

Proof. This follows analogously to the proof of Theorem 2.3.9, but now Proposition 3.2.11 enters the argument replacing Proposition 2.3.8 which was used there. □

The translation operator in Volterra pseudodifferential operators

3.2.13 Remark. Let $\mu \in \mathbb{R}$ and $\mu_+ := \max\{0,\mu\}$. According to Proposition 2.3.11 the translation operator $T_{i\tau}$ for $\tau \geq 0$ (cf. Definition 2.3.10) acts as a linear continuous operator in the spaces

$$T_{i\tau} : S_V^{\frac{\mu}{\ell}}(\mathbb{H};H^s(X,E),H^{s-\mu_+}(X,F)) \longrightarrow S_V^{\frac{\mu}{\ell}}(\mathbb{H};H^s(X,E),H^{s-\mu_+}(X,F))$$

for every $s \in \mathbb{R}$.

3.2.14 Proposition. *For every $\tau \geq 0$ the translation operator $T_{i\tau}$ (cf. Remarks 3.2.13, 3.2.8) restricts to a linear and continuous operator in the spaces*

$$T_{i\tau} : L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F) \longrightarrow L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F).$$

Moreover, $\big(T_{i\tau}A\big)(\lambda)$ has the following asymptotic expansion in terms of τ and $A(\lambda)$ (in the sense of Definition 3.2.7):

$$\big(T_{i\tau}A\big)(\lambda) \underset{V}{\sim} \sum_{k=0}^{\infty} \frac{(i\tau)^k}{k!} \cdot \big(\partial_\lambda^k A\big)(\lambda).$$

In particular, the operator $I - T_{i\tau}$ is continuous in the spaces

$$I - T_{i\tau} : L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F) \longrightarrow L^{\mu-\ell;\ell}_{V(cl)}(X;\mathbb{H};E,F).$$

Proof. This follows from Proposition 2.3.11 when passing via (3.2.iv) to "local" symbols and remainders in $L_V^{-\infty}(X;\mathbb{H};E,F)$. □

3.2.15 Notation. For every $\mu \in \mathbb{R}$ let $S_V^{(\mu;\ell)}((T^*X \times \mathbb{H}) \setminus 0, \mathrm{Hom}(\pi^*E, \pi^*F))$ denote the space of anisotropic homogeneous functions of degree μ that are analytic in the interior of $\mathbb{H}$. This is a closed subspace of $C^\infty((T^*X \times \mathbb{H}) \setminus 0, \mathrm{Hom}(\pi^*E, \pi^*F))$.

3.2.16 Theorem. *The restriction of the principal symbol sequence to Volterra operators is topologically exact and splits:*

$$0 \longrightarrow L^{\mu-1;\ell}_{V\,cl}(X;\mathbb{H};E,F) \stackrel{\imath}{\longrightarrow} L^{\mu;\ell}_{V\,cl}(X;\mathbb{H};E,F) \stackrel{\sigma_\psi^{\mu;\ell}}{\longrightarrow}$$
$$S_V^{(\mu;\ell)}((T^*X \times \mathbb{H}) \setminus 0, Hom(\pi^*E, \pi^*F)) \longrightarrow 0.$$

Proof. Every element in $S_V^{(\mu;\ell)}((T^*X \times \mathbb{H}) \setminus 0, \mathrm{Hom}(\pi^*E, \pi^*F))$ can be represented by a vector of local representatives corresponding to the given covering of X by coordinate neighbourhoods from Notation 3.1.1 (i. e. they satisfy the transition conditions of the bundles involved over the intersections). For every $j = 1, \ldots, N$, the representative over Ω_j may be viewed as a C^∞-function on Ω_j taking values in the space $S_V^{(\mu;\ell)}\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}\big)$ (cf. Notation 2.3.12). Thus, if we multiply this function by θ_j and pass from Ω_j to U_j via κ_j for every $j = 1, \ldots, N$, we get an N-tuple of compactly supported smooth functions on $\mathbb{R}^n$ taking values in $S_V^{(\mu;\ell)}\big((\mathbb{R}^n \times \mathbb{H}) \setminus \{0\}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}\big)$. To each of these we now apply the translation operator $T_{i\tau}$ for some $\tau > 0$. By Theorem 2.3.13 we so obtained an N-tuple of classical $\mathcal{L}(\mathbb{C}^{N_-}, \mathbb{C}^{N_+})$-valued Volterra symbols of order μ. Now we associate to this tuple an operator in $L^{\mu;\ell}_{V\,cl}(X;\mathbb{H};E,F)$ via (3.2.iii). Summing up, we constructed a continuous linear right-inverse to the principal symbol mapping

$$\sigma_\psi^{\mu;\ell} : L^{\mu;\ell}_{V\,cl}(X;\mathbb{H};E,F) \to S_V^{(\mu;\ell)}((T^*X \times \mathbb{H}) \setminus 0, \mathrm{Hom}(\pi^*E, \pi^*F))$$

which shows the assertion of the theorem. □

Parabolicity for Volterra operators on manifolds

3.2.17 Definition. Let $A(\lambda) \in L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F)$. Then $A(\lambda)$ is called *parabolic*, if $A(\lambda)$ is parameter-dependent elliptic as an element of $L^{\mu;\ell}_{(cl)}(X;\mathbb{H};E,F)$.

3.2.18 Theorem. *Let $A(\lambda) \in L^{\mu;\ell}_{V}(X;\mathbb{H};E,F)$. The following assertions are equivalent:*

a) $A(\lambda)$ is parabolic.

b) The components of the complete symbol $(a_1,\dots,a_N)$ of $A(\lambda)$ are parabolic on $\kappa_j(\mathrm{supp}\psi_j)$ for $j=1,\dots,N$.

c) There exists an operator $P(\lambda) \in L^{-\mu;\ell}_{V}(X;\mathbb{H};F,E)$ such that $A(\lambda)P(\lambda)-I \in L^{-\varepsilon;\ell}_{V}(X;\mathbb{H};F,F)$ and $P(\lambda)A(\lambda)-I \in L^{-\varepsilon;\ell}_{V}(X;\mathbb{H};E,E)$ for some $\varepsilon>0$.

d) There exists an operator $P(\lambda) \in L^{-\mu;\ell}_{V}(X;\mathbb{H};F,E)$ such that $A(\lambda)P(\lambda)-I \in L^{-\infty}_{V}(X;\mathbb{H};F,F)$ and $P(\lambda)A(\lambda)-I \in L^{-\infty}_{V}(X;\mathbb{H};E,E)$.

Moreover, if $A(\lambda) \in L^{\mu;\ell}_{V\,cl}(X;\mathbb{H};E,F)$ is parabolic, then every $P(\lambda)$ satisfying d) belongs to $L^{-\mu;\ell}_{V\,cl}(X;\mathbb{H};F,E)$. Every $P(\lambda)$ satisfying d) is called a (parameter-dependent) Volterra parametrix *of $A(\lambda)$.*

Proof. From Theorem 3.1.10 and (3.2.iv) and the definition of parabolicity as parameter-dependent ellipticity we conclude that we only have to prove that b) implies c), and c) implies d).

Now assume that b) holds. From Corollary 2.4.14 we obtain the existence of $b_1,\dots,b_N \in S^{-\mu;\ell}_{V(cl)}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{H};\mathbb{C}^{N_+},\mathbb{C}^{N_-})$ and suitable functions $\hat{\varphi}_j,\hat{\psi}_j \in C_0^\infty(\mathbb{R}^n)$ such that $\hat{\psi}_j\hat{\varphi}_j=\hat{\varphi}_j$ and $\hat{\varphi}_j\equiv 1$ on $\kappa_j(\mathrm{supp}\psi_j) \Subset \mathbb{R}^n$ with the property that $\hat{\varphi}_j(\mathrm{op}_x(a_j)(\lambda)\mathrm{op}_x(b_j)(\lambda)-1)\hat{\psi}_j$ belongs to $L^{-\infty}_{V}(\mathbb{R}^n;\mathbb{H};\mathbb{C}^{N_+},\mathbb{C}^{N_+})$. Now define $P(\lambda):=\mathrm{op}_x((b_1,\dots,b_N);0)$ with the operator convention (3.2.iii), which yields an operator $P(\lambda)\in L^{-\mu;\ell}_{V(cl)}(X;\mathbb{H};F,E)$. Now we see that $A(\lambda)P(\lambda)-I$ belongs to $L^{-1;\ell}_{V(cl)}(X;\mathbb{H};F,F)$. Analogously, we obtain a (rough) Volterra parametrix from the left. But both the left- and the right-parametrix differ only by a term of order -1 which gives c).

c) implies d) follows analogously to the proof of Theorem 2.2.10 by means of a formal Neumann series argument, where Theorem 3.2.5 and Theorem 3.2.12 enter the argument for carrying out the compositions and asymptotic expansions. □

3.2.19 Theorem. *Let* $A(\lambda) \in L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F)$ *be parabolic. Then the set*

$$K := \{\lambda \in \mathbb{H};\ \big(A(\lambda)\big)^{-1} \in \mathcal{L}(H^{s-\mu}(X,F), H^s(X,E)) \text{ does not exist}\}$$

is compact in $\mathbb{H}$ *and independent of* $s \in \mathbb{R}$. *Moreover, let* $\tau \geq 0$ *such that* $\sup_{\lambda\in K}|\lambda| < \tau$. *Then the operator* $(T_{i\tau}A)(\lambda) \in L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F)$ *is invertible with inverse* $\big((T_{i\tau}A)(\lambda)\big)^{-1} \in L^{-\mu;\ell}_{V(cl)}(X;\mathbb{H};F,E)$.

Proof. The first assertion follows from Theorem 3.1.11. In view of Proposition 3.2.14 this also implies that the operator $(T_{i\tau}A)(\lambda) \in L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F)$ is invertible with inverse $\big((T_{i\tau}A)(\lambda)\big)^{-1} \in L^{-\mu;\ell}_{(cl)}(X;\mathbb{H};F,E)$. In particular, the operator family $(T_{i\tau}A)(\lambda) \in \mathcal{A}(\overset{\circ}{\mathbb{H}}, L^{\mu}(X;E,F))$ is invertible with $\big((T_{i\tau}A)(\lambda)\big)^{-1} \in C^\infty(\overset{\circ}{\mathbb{H}}, L^{-\mu}(X;F,E))$. From the resolvent identity we conclude that for $\lambda_0, \lambda_1 \in \overset{\circ}{\mathbb{H}}$ with $\lambda_0 \neq \lambda_1$ the difference quotient $\frac{\left((T_{i\tau}A)(\lambda_0)\right)^{-1} - \left((T_{i\tau}A)(\lambda_1)\right)^{-1}}{\lambda_0-\lambda_1}$ is given as

$$-\big((T_{i\tau}A)(\lambda_0)\big)^{-1}\cdot\Big(\frac{(T_{i\tau}A)(\lambda_0)-(T_{i\tau}A)(\lambda_1)}{\lambda_0-\lambda_1}\Big)\cdot\big((T_{i\tau}A)(\lambda_1)\big)^{-1},$$

which implies the analyticity of $\big((T_{i\tau}A)(\lambda)\big)^{-1}$ in the interior of $\mathbb{H}$. $\square$

Chapter 4

Weighted Sobolev spaces

4.1 Anisotropic Sobolev spaces on the infinite cylinder

4.1.1 Remark. In this chapter we again employ the notations from Notation 3.1.1 with the corresponding data fixed on X and the vector bundle E.

The material in this section is standard in the isotropic case, i. e. $\ell = 1$ and $t = 0$ (in the notation of Definition 4.1.2 below). There are many variants of anisotropic Sobolev spaces discussed in the literature, e. g. [2], [27], [49]. Therefore, we restrict ourselves to give the basic definitions and results in that form as they are needed in this work.

4.1.2 Definition. For $s, t \in \mathbb{R}$ define the Sobolev space $H^{(s,t);\ell}(\mathbb{R} \times X, E)$ as the space of all $u \in \bigcup_{s' \in \mathbb{R}} \mathcal{S}'(\mathbb{R}, H^{s'}(X, E))$ such that $\mathcal{F}u$ is a regular distribution and

$$\|u\|_{H^{(s,t);\ell}(\mathbb{R}\times X,E)} := \Big(\int_{\mathbb{R}} \|R^s(\tau)(\mathcal{F}_{r\to\tau}u)(\tau)\|^2_{H^t(X,E)}\, d\tau \Big)^{\frac{1}{2}} < \infty. \tag{4.1.i}$$

Here $R^s(\tau) \in L^{s;\ell}(X; \mathbb{R}; E, E)$ is a parameter-dependent reduction of orders from Theorem 3.1.12.

4.1.3 Remark. a) The space $H^{(s,t);\ell}(\mathbb{R}\times X, E)$ is well-defined, i. e. other choices of the reduction of orders give rise to equivalent norms (see Theorem 3.1.5, and Theorem 3.1.6).

b) For $s = t = 0$ we have $H^{(0,0);\ell}(\mathbb{R} \times X, E) = L^2(\mathbb{R} \times X, E) = L^2(\mathbb{R}, L^2(X, E))$, see also Proposition 4.1.7.

c) The local space $H^{(s,0);\ell}(\mathbb{R}^{n+1},\mathbb{C}^{N_-})$ consists of all $u \in \mathcal{S}'(\mathbb{R}^{n+1},\mathbb{C}^{N_-})$ such that $\mathcal{F}u$ is a regular distribution and

$$\|u\|_{H^{(s,0);\ell}(\mathbb{R}^{n+1},\mathbb{C}^{N_-})} := \Big(\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^n} \langle\xi,\tau\rangle_\ell^{2s}\, \|(\mathcal{F}u)(\xi,\tau)\|^2_{\mathbb{C}^{N_-}}\, d\xi\, d\tau\Big)^{\frac{1}{2}} < \infty.$$

Using the covering of X by coordinate neighbourhoods as well as the subordinated partition of unity from Notation 3.1.1 we see that the space $H^{(s,0);\ell}(\mathbb{R}\times X,E)$ consists precisely of those distributions that locally belong to $H^{(s,0);\ell}(\mathbb{R}^{n+1},\mathbb{C}^{N_-})$.

4.1.4 Definition. For $s,t,\delta \in \mathbb{R}$ define

$$H^{(s,t);\ell}(\mathbb{R}\times X,E)_\delta := \langle r\rangle^{-\delta} H^{(s,t);\ell}(\mathbb{R}\times X,E) \tag{4.1.ii}$$

with the induced norm.

4.1.5 Theorem. *a) $H^{(s,t);\ell}(\mathbb{R}\times X,E)_\delta$ is a Hilbert space with respect to the scalar product*

$$\langle u,v\rangle = \int\limits_{\mathbb{R}} \langle R^s(\tau)\big(\mathcal{F}_{r\to\tau}\langle r\rangle^\delta u(r)\big)(\tau), R^s(\tau)\big(\mathcal{F}_{r\to\tau}\langle r\rangle^\delta v(r)\big)(\tau)\rangle_{H^t(X,E)}\, d\tau.$$

b) The embedding $\mathcal{S}(\mathbb{R}\times X,E) = \mathcal{S}(\mathbb{R},C^\infty(X,E)) \hookrightarrow H^{(s,t);\ell}(\mathbb{R}\times X,E)_\delta$ is continuous and dense.

c) The operator of multiplication with a function $\varphi \in C_b^\infty(\mathbb{R})$ induces a continuous operator in $\mathcal{L}(H^{(s,t);\ell}(\mathbb{R}\times X,E)_\delta)$, and the mapping $C_b^\infty(\mathbb{R}) \ni \varphi \mapsto M_\varphi \in \mathcal{L}(H^{(s,t);\ell}(\mathbb{R}\times X,E)_\delta)$ is continuous.

d) For $s'-s \le \min\{0,t-t'\}$ and $\delta \ge \delta'$ the embedding $H^{(s,t);\ell}(\mathbb{R}\times X,E)_\delta \hookrightarrow H^{(s',t');\ell}(\mathbb{R}\times X,E)_{\delta'}$ is well-defined and continuous. Moreover, it is compact if $s'-s < \min\{0,t-t'\}$ and $\delta > \delta'$; Hilbert-Schmidt if $s'-s+\frac{n+\ell}{2} < \min\{0,t-t'\}$ and $\delta-\delta' > \frac{1}{2}$.

e) For $k \in \mathbb{N}_0$ let $C_b^{k;\ell}(\mathbb{R}\times X,E)$ denote the Banach space of all sections $u : \mathbb{R}\times X \to E$ such that for $|\alpha|_\ell \le k$ there exists $\partial^\alpha_{(x,t)}u$ as a bounded continuous function with respect to any choice of local coordinates and trivializations of the vector bundle, endowed with the topology of uniform convergence of all derivatives up to (anisotropic) order k.

Sobolev embedding theorem: *Let $k \in \mathbb{N}_0$. Then for $s > k+\frac{n+\ell}{2}$ the embedding $H^{(s,0);\ell}(\mathbb{R}\times X,E)_\delta \hookrightarrow \langle t\rangle^{-\delta} C_b^{k;\ell}(\mathbb{R}\times X,E)$ is well-defined and continuous.*

In particular, we have $\mathcal{S}(\mathbb{R}\times X,E) = \bigcap\limits_{s,\delta\in\mathbb{R}} H^{(s,0);\ell}(\mathbb{R}\times X,E)_\delta$, which holds topologically with the projective limit topology on the right-hand side.

f) For every $\delta_0 \in \mathbb{R}$ the $\langle r\rangle^{-\delta_0}L^2(\mathbb{R}\times X,E)$-inner product extends to a non-degenerate sesquilinear pairing

$$\langle\cdot,\cdot\rangle_{\delta_0} : H^{(s,t);\ell}(\mathbb{R}\times X,E)_{\delta_0+\delta} \times H^{(-s,-t);\ell}(\mathbb{R}\times X,E)_{\delta_0-\delta} \longrightarrow \mathbb{C}$$

which induces an identification of the dual

$$H^{(s,t);\ell}(\mathbb{R}\times X,E)'_{\delta_0+\delta} \cong H^{(-s,-t);\ell}(\mathbb{R}\times X,E)_{\delta_0-\delta}.$$

In particular, this provides a topological (antilinear) isomorphism $\mathcal{S}(\mathbb{R}\times X,E)' \cong \bigcup_{s,\delta\in\mathbb{R}} H^{(s,0);\ell}(\mathbb{R}\times X,E)_\delta$ with the inductive limit topology on the right-hand side.

4.1.6 Remark. Let Y be a Hausdorff-topological vector space. Moreover, let F and G be Fréchet spaces which are continuously embedded in Y. Then the *non-direct sum* of the spaces F and G is defined as

$$F+G := \{y=f+g\in Y;\ f\in F,\ g\in G\},$$

endowed with the following topology: For every continuous seminorm $\|\cdot\|_F$ on F and every continuous seminorm $\|\cdot\|_G$ on G define the seminorm $\|\cdot\|_{F+G}$ on $F+G$ as $\|y\|_{F+G} := \inf\{\|f\|_F + \|g\|_G;\ y=f+g\}$.

Consider the addition $+ : F\oplus G \longrightarrow F+G$ which provides a linear surjection. The kernel is given as $\Delta = \{(f,-f);\ f\in F\cap G\subseteq Y\}$, and is a closed subspace of $F\oplus G$. The induced mapping on the quotient space $(F\oplus G)/\Delta \cong F+G$ is a topological isomorphism.

In particular, $F+G$ is a Fréchet space, and for Hilbert spaces F and G also $F+G$ is a Hilbert space (more precisely a hilbertizable space), and we have $F+G\cong\Delta^\perp\subseteq F\oplus G$.

4.1.7 Proposition. *For $s,t,\delta\in\mathbb{R}$ the following identities hold algebraically and topologically:*

$$H^{(s,t);\ell}(\mathbb{R}\times X,E)_\delta = \begin{cases} \big(\langle r\rangle^{-\delta}L^2(\mathbb{R},H^{s+t}(X,E))\big)\cap\big(\langle r\rangle^{-\delta}H^{\frac{s}{\ell}}(\mathbb{R},H^t(X,E))\big) \\ \hfill \text{for } s\geq 0, \\ \big(\langle r\rangle^{-\delta}L^2(\mathbb{R},H^{s+t}(X,E))\big)+\big(\langle r\rangle^{-\delta}H^{\frac{s}{\ell}}(\mathbb{R},H^t(X,E))\big) \\ \hfill \text{for } s\leq 0. \end{cases} \tag{4.1.iii}$$

Proof. Without loss of generality assume $\delta=0$. First we consider the case $s\geq 0$. Let $u\in H^{(s,t);\ell}(\mathbb{R}\times X,E)$ and $R^s(\tau)\in L^{s;\ell}(X;\mathbb{R};E,E)$ be a parameter-dependent reduction of orders from Theorem 3.1.12. From Theorem 3.1.5 we conclude that

$$R^{-s}(\tau)\in S^0(\mathbb{R};H^t(X,E),H^{s+t}(X,E))\cap S^{-\frac{s}{\ell}}(\mathbb{R};H^t(X,E),H^t(X,E)).$$

Thus we have

$$\begin{aligned}\|\mathcal{F}(u)(\tau)\|_{H^{s+t}(X,E)} &= \|R^{-s}(\tau)\big(R^s(\tau)\mathcal{F}(u)(\tau)\big)\|_{H^{s+t}(X,E)}\\ &\le C\|R^s(\tau)\mathcal{F}(u)(\tau)\|_{H^t(X,E)},\\ \|\mathcal{F}(u)(\tau)\|_{H^t(X,E)} &= \|R^{-s}(\tau)\big(R^s(\tau)\mathcal{F}(u)(\tau)\big)\|_{H^t(X,E)}\\ &\le C\langle\tau\rangle^{-\frac{s}{\ell}}\cdot\|R^s(\tau)\mathcal{F}(u)(\tau)\|_{H^t(X,E)},\end{aligned}$$

and consequently

$$H^{(s,t);\ell}(\mathbb{R}\times X,E)\hookrightarrow L^2(\mathbb{R},H^{s+t}(X,E))\cap H^{\frac{s}{\ell}}(\mathbb{R},H^t(X,E)).$$

Let us show that the embedding is onto. First recall the following elementary inequality for $\alpha,\beta\in\mathbb{C}$ and $p>0$:

$$|\alpha-\beta|^p\le\max\{1,2^{p-1}\}\big(|\alpha|^p+|\beta|^p\big). \tag{1}$$

Let $u\in L^2(\mathbb{R},H^s(X,E))\cap H^{\frac{s}{\ell}}(\mathbb{R},L^2(X,E))$. Passing to local coordinates on X and E we conclude that

$$\Big(\int_{\mathbb{R}}\int_{\mathbb{R}^n}\big(\langle\xi\rangle^{2s}+\langle\tau\rangle^{2\frac{s}{\ell}}\big)\|(\mathcal{F}u)(\xi,\tau)\|^2_{\mathbb{C}^{N_-}}\,d\xi\,d\tau\Big)^{\frac12}<\infty.$$

From (1) we see that

$$\langle\xi,\tau\rangle_\ell^{2s}\le C\big(\langle\xi\rangle^{2s}+\langle\tau\rangle^{2\frac{s}{\ell}}\big)$$

for $(\xi,\tau)\in\mathbb{R}^n\times\mathbb{R}$ with a suitable constant $C>0$, i. e. locally u belongs to the space $H^{(s,0);\ell}(\mathbb{R}^{n+1},\mathbb{C}^{N_-})$. This finishes the proof in the case $s\ge 0$ and $t=0$. Now let $u\in L^2(\mathbb{R},H^{s+t}(X,E))\cap H^{\frac{s}{\ell}}(\mathbb{R},H^t(X,E))$, and let $\Lambda^t\in L^t(X;E,E)$ be a reduction of orders. Then $\mathrm{op}(\Lambda^t)u\in L^2(\mathbb{R},H^s(X,E))\cap H^{\frac{s}{\ell}}(\mathbb{R},L^2(X,E))$, i. e. $\mathrm{op}(\Lambda^t)u\in H^{(s,0);\ell}(\mathbb{R}\times X,E)$. Following Seeley's construction we can arrange that the reductions of orders $R^s(\tau)$ and Λ^t are commuting, e. g. choose $\Lambda^t=\big(C-\Delta\big)^{\frac{t}{2}}$ and $R^s(\tau)=\big(C+\big(-\Delta\big)^\ell+\tau^2\big)^{\frac{s}{2\ell}}$ with a suitable connection Laplacean Δ and a sufficiently large constant $C>0$. Then we obtain that $u\in H^{(s,t);\ell}(\mathbb{R}\times X,E)$ from Definition 4.1.2.

The case $s\le 0$ follows by duality:
Due to Theorem 4.1.5 the space $H^{(s,t);\ell}(\mathbb{R}\times X,E)$ equals the dual space of $H^{(-s,-t);\ell}(\mathbb{R}\times X,E)$ with respect to the sesquilinear pairing induced by the $L^2(\mathbb{R},L^2(X,E))$-inner product. Moreover, we have $L^2(\mathbb{R},H^{s+t}(X,E))\cong L^2(\mathbb{R},H^{-s-t}(X,E))'$ and $H^{\frac{s}{\ell}}(\mathbb{R},H^t(X,E))\cong H^{-\frac{s}{\ell}}(\mathbb{R},H^{-t}(X,E))'$, while the space $\mathcal{S}(\mathbb{R}\times X,E)$ is dense both in $L^2(\mathbb{R},H^{-s-t}(X,E))$ and $H^{-\frac{s}{\ell}}(\mathbb{R},H^{-t}(X,E))$. Thus we obtain the assertion from the already proven result for the space $H^{(-s,-t);\ell}(\mathbb{R}\times X,E)$. □

4.1.8 Definition. Let $\emptyset\neq U\subseteq\mathbb{R}$ be an open set.

a) Let $H_0^{(s,t);\ell}(\overline{U} \times X, E)_\delta$ be the subspace of all $u \in H^{(s,t);\ell}(\mathbb{R} \times X, E)_\delta$ such that $\operatorname{supp} u \subseteq \overline{U}$.

b) Let $H_{\mathrm{loc}}^{(s,t);\ell}(U \times X, E)$ denote the space of all $u \in \bigcup_{s' \in \mathbb{R}} \mathcal{D}'(U, H^{s'}(X, E))$ such that for all $\varphi \in C_0^\infty(U)$ the distribution φu belongs to $H^{(s,t);\ell}(\mathbb{R} \times X, E)$, endowed with the projective topology with respect to the mappings $u \mapsto \varphi u \in H^{(s,t);\ell}(\mathbb{R} \times X, E)$ for all $\varphi \in C_0^\infty(U)$.

c) Let $H_{\mathrm{comp}}^{(s,t);\ell}(U \times X, E)$ denote the space of all $u \in H^{(s,t);\ell}(\mathbb{R} \times X, E)$ such that $\operatorname{supp} u \Subset U$ is compact. We equip this space with the inductive topology with respect to the mappings

$$H^{(s,t);\ell}(K \times X, E) \hookrightarrow H_{\mathrm{comp}}^{(s,t);\ell}(U \times X, E)$$

for every compact set $K \Subset U$, where $H^{(s,t);\ell}(K \times X, E)$ is the closed subspace of all u in $H^{(s,t);\ell}(\mathbb{R} \times X, E)$ with $\operatorname{supp} u \subseteq K$. Hence $H_{\mathrm{comp}}^{(s,t);\ell}(U \times X, E)$ is a strict (countable) inductive limit.

4.1.9 Theorem. *Let $\emptyset \neq U \subseteq \mathbb{R}$ be an open set.*

a) *The closure of $C_0^\infty(U, C^\infty(X, E))$ in $H^{(s,t);\ell}(\mathbb{R} \times X, E)_\delta$ is contained in $H_0^{(s,t);\ell}(\overline{U} \times X, E)_\delta$.*

If U is an interval then the closure coincides with the space.

b) *$H_{loc}^{(s,t);\ell}(U \times X, E)$ is a Fréchet space. If $V \subseteq \mathbb{R}$ is another open set and $\chi : U \longrightarrow V$ is a diffeomorphism, then the distributional pull-back χ^* induces topological isomorphisms*

$$\chi^* : \begin{cases} H_{comp}^{(s,t);\ell}(V \times X, E) \longrightarrow H_{comp}^{(s,t);\ell}(U \times X, E), \\ H_{loc}^{(s,t);\ell}(V \times X, E) \longrightarrow H_{loc}^{(s,t);\ell}(U \times X, E). \end{cases}$$

4.2 Anisotropic Mellin Sobolev spaces

4.2.1 Remark. Material on isotropic Mellin Sobolev spaces can be found, e. g., in [13], [59], [60], [61].

4.2.2 Notation. For any set Y we denote $Y^\wedge := \mathbb{R}_+ \times Y$.

4.2.3 Definition. For $s, t, \gamma \in \mathbb{R}$ the Mellin Sobolev space $\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge, E)$ is defined as the space of all $u \in \bigcup_{s' \in \mathbb{R}} \mathcal{T}'_{\gamma - \frac{n}{2}}(\mathbb{R}_+, H^{s'}(X, E))$ such that $\mathcal{M}_{\gamma - \frac{n}{2}} u \in$

$\bigcup_{s'\in\mathbb{R}} \mathcal{S}'(\Gamma_{\frac{n+1}{2}-\gamma}, H^{s'}(X,E))$ is a regular distribution and

$$\|u\|_{\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge,E)} := \Big(\frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{n+1}{2}-\gamma}} \|R^s(z)(\mathcal{M}_{\gamma-\frac{n}{2}}u)(z)\|^2_{H^t(X,E)}\, dz\Big)^{\frac{1}{2}} < \infty. \tag{4.2.i}$$

Here $R^s(z) \in L^{s;\ell}(X;\Gamma_{\frac{n+1}{2}-\gamma};E,E)$ is a parameter-dependent reduction of orders from Theorem 3.1.12.

4.2.4 Remark. The space $\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge,E)$ is well-defined. From relation (1.1.iv) we see that the transformation $S_{\gamma-\frac{n}{2}}$ from (1.1.i) induces a topological isomorphism

$$S_{\gamma-\frac{n}{2}} : \mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge,E) \longrightarrow H^{(s,t);\ell}(\mathbb{R}\times X,E). \tag{4.2.ii}$$

Using (4.2.ii) we consequently obtain many properties of the Mellin Sobolev spaces $\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge,E)$ from Theorem 4.1.5.

4.2.5 Proposition. *a) The relation $r^\delta\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge,E) = \mathcal{H}^{(s,t),\gamma+\delta;\ell}(X^\wedge,E)$ is valid, and we have*

$$\mathcal{H}^{(0,0),0;\ell}(X^\wedge,E) = r^{-\frac{n}{2}}L^2(X^\wedge,E) = r^{-\frac{n}{2}}L^2(\mathbb{R}_+,L^2(X,E)).$$

More precisely, the following identity holds algebraically and topologically:

$$\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge,E) = \begin{cases} L^{2,\gamma-\frac{n}{2}}(\mathbb{R}_+,H^{s+t}(X,E))\cap\mathcal{H}^{\frac{s}{\ell},\gamma-\frac{n}{2}}(\mathbb{R}_+,H^t(X,E)) \\ \qquad \text{for } s\geq 0, \\ L^{2,\gamma-\frac{n}{2}}(\mathbb{R}_+,H^{s+t}(X,E)) + \mathcal{H}^{\frac{s}{\ell},\gamma-\frac{n}{2}}(\mathbb{R}_+,H^t(X,E)) \\ \qquad \text{for } s\leq 0. \end{cases} \tag{4.2.iii}$$

b) The embeddings $H^{(s,t);\ell}_{comp}(X^\wedge,E) \hookrightarrow \mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge,E) \hookrightarrow H^{(s,t);\ell}_{loc}(X^\wedge,E)$ are well-defined and continuous.

Proof. Assertion a) follows from (4.2.ii), Proposition 4.1.7 and Definition 2.5.10. b) is a consequence of (4.2.ii) and Theorem 4.1.9. □

4.2.6 Theorem. *a) $\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge,E)$ is a Hilbert space with respect to the scalar product*

$$\langle u,v\rangle = \frac{1}{2\pi i}\int\limits_{\Gamma_{\frac{n+1}{2}-\gamma}} \langle R^s(z)\big(\mathcal{M}_{\gamma-\frac{n}{2}}u\big)(z), R^s(z)\big(\mathcal{M}_{\gamma-\frac{n}{2}}v\big)(z)\rangle_{H^t(X,E)}\, dz.$$

b) *The embedding* $\mathcal{T}_{\gamma-\frac{n}{2}}(X^\wedge, E) = \mathcal{T}_{\gamma-\frac{n}{2}}(\mathbb{R}_+, C^\infty(X,E)) \hookrightarrow \mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge, E)$ *is continuous and dense.*

c) *The operator of multiplication with a function* $\varphi \in C_B^\infty(\mathbb{R}_+)$ *induces a continuous operator in* $\mathcal{L}(\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge, E))$, *and the mapping* $C_B^\infty(\mathbb{R}_+) \ni \varphi \mapsto M_\varphi \in \mathcal{L}(\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge, E))$ *is continuous.*

d) *For* $s'-s \le \min\{0, t-t'\}$ *the embedding* $\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge, E) \hookrightarrow \mathcal{H}^{(s',t'),\gamma;\ell}(X^\wedge, E)$ *is well-defined and continuous.*

e) *For* $k \in \mathbb{N}_0$ *let* $C_B^{k;\ell}(\mathbb{R}_+ \times X, E)$ *denote the Banach space of all sections* $u : \mathbb{R}_+ \times X \to E$ *such that for* $j\ell + |\alpha| \le k$ *there exists* $(-r\partial_r)^j \partial_x^\alpha u$ *as a bounded continuous function with respect to any choice of local coordinates and trivializations of the vector bundle, endowed with the topology of uniform convergence of all derivatives* $(-r\partial_r)^j \partial_x^\alpha u$ *up to (anisotropic) order* k.

   Sobolev embedding theorem: *Let* $k \in \mathbb{N}_0$. *Then for* $s > k + \frac{n+\ell}{2}$ *the embedding* $\mathcal{H}^{(s,0),\gamma;\ell}(X^\wedge, E) \hookrightarrow r^{-(\frac{n+1}{2}-\gamma)} C_B^{k;\ell}(\mathbb{R}_+ \times X, E)$ *is well-defined and continuous.*

f) *For every* $\gamma_0 \in \mathbb{R}$ *the* $r^{\gamma_0-\frac{n}{2}} L^2(\mathbb{R}_+ \times X, E)$*-inner product extends to a non-degenerate sesquilinear pairing*

$$\langle \cdot, \cdot \rangle_{\gamma_0} : \mathcal{H}^{(s,t),\gamma+\gamma_0;\ell}(X^\wedge, E) \times \mathcal{H}^{(-s,-t),-\gamma+\gamma_0;\ell}(X^\wedge, E) \longrightarrow \mathbb{C}$$

   which induces an (antilinear) identification of the dual

$$\mathcal{H}^{(s,t),\gamma+\gamma_0;\ell}(X^\wedge, E)' \cong \mathcal{H}^{(-s,-t),-\gamma+\gamma_0;\ell}(X^\wedge, E).$$

4.2.7 Definition. A function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\omega \equiv 1$ near $r = 0$ is called a *cut-off function (near $r = 0$).*

4.2.8 Notation. Let Y be a locally convex space and $A \in \mathcal{L}(Y)$. Then we denote the closure of $A(Y)$ in Y by $[A]Y$.

This notation will be employed frequently in case of function spaces Y and multiplication operators A.

4.2.9 Theorem. *Let* $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ *be a cut-off function near* $r = 0$. *Then the embeddings*

$$[\omega]\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge, E) \hookrightarrow [\omega]\mathcal{H}^{(s',t'),\gamma';\ell}(X^\wedge, E),$$
$$[1-\omega]\mathcal{H}^{(s,t),\gamma';\ell}(X^\wedge, E) \hookrightarrow [1-\omega]\mathcal{H}^{(s',t'),\gamma;\ell}(X^\wedge, E),$$

are well-defined and continuous for $s' - s \le \min\{0, t-t'\}$ *and* $\gamma \ge \gamma'$. *Moreover, they are compact if* $s'-s < \min\{0, t-t'\}$ *and* $\gamma > \gamma'$; *Hilbert-Schmidt if* $s' - s + \frac{n+\ell}{2} < \min\{0, t-t'\}$ *and* $\gamma > \gamma'$.

4.2.10 Corollary. *Let $\gamma, \gamma' \in \mathbb{R}$ with $\gamma < \gamma'$. Then for every $s, t \in \mathbb{R}$ we have*

$$\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge, E) \cap \mathcal{H}^{(s,t),\gamma';\ell}(X^\wedge, E) = \bigcap_{\gamma \leq \delta \leq \gamma'} \mathcal{H}^{(s,t),\delta;\ell}(X^\wedge, E).$$

The intersection is taken in $H^{(s,t);\ell}_{loc}(X^\wedge, E)$.

4.2.11 Definition. Let $\emptyset \neq U \subseteq \mathbb{R}_+$ be an open set. Define $\mathcal{H}_0^{(s,t),\gamma;\ell}(\overline{U} \times X, E)$ to be the subspace of all $u \in \mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge, E)$ such that $\mathrm{supp} u \subseteq \overline{U}$.

Note that the closure of U is taken with respect to the topology of $\mathbb{R}_+$.

4.2.12 Proposition. *The closure of $C_0^\infty(U, C^\infty(X, E))$ in $\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge, E)$ is contained in $\mathcal{H}_0^{(s,t),\gamma;\ell}(\overline{U} \times X, E)$ for every open set $\emptyset \neq U \subseteq \mathbb{R}_+$. If U is an interval then the closure coincides with the space.*

Mellin Sobolev spaces with asymptotics

4.2.13 Definition. a) Let $-\infty \leq \theta < 0$ and $\Theta := (\theta, 0]$. For $\gamma \in \mathbb{R}$ the tuple (γ, Θ) is called a *weight datum*.

The strip $\Gamma_{(\frac{n+1}{2}-\gamma+\theta, \frac{n+1}{2}-\gamma)} \subseteq \mathbb{C}$ is called the *weight strip* associated with the weight datum (γ, Θ).

b) An *asymptotic type* associated with the weight datum (γ, Θ) is a finite or countably infinite set

$$P = \{(p_j, m_j, L_j);\ j \in \mathbb{Z}\} \tag{4.2.iv}$$

where the $m_j \in \mathbb{N}_0$ are integers, the L_j are finite-dimensional subspaces of $C^\infty(X, E)$ and the $p_j \in \mathbb{C}$ are complex numbers such that with the “projection” $\pi_{\mathbb{C}} P := \{p_j;\ j \in \mathbb{Z}\}$ of P to $\mathbb{C}$ the following properties are fulfilled:

- $\pi_{\mathbb{C}} P \subseteq \Gamma_{(\frac{n+1}{2}-\gamma+\theta, \frac{n+1}{2}-\gamma)}$.
- $\pi_{\mathbb{C}} P \cap \Gamma_I$ is finite for every subinterval $I \subseteq (\frac{n+1}{2} - \gamma + \theta, \frac{n+1}{2} - \gamma)$ of finite length.

The collection of all asymptotic types associated with the weight datum (γ, Θ) is denoted by $\mathrm{As}\big((\gamma, \Theta), C^\infty(X, E)\big)$.

c) Let (γ, Θ) be a weight datum such that $\theta > -\infty$. For an asymptotic type P associated with (γ, Θ) and an arbitrary but fixed cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ we define

$$\mathcal{E}_P(X^\wedge, E) := \{\omega(r) \sum_{(p,m,L) \in P} \sum_{k=0}^{m} c_{p,k} r^{-p} \log^k(r);\ c_{p,k} \in L\}. \tag{4.2.v}$$

This is a finite-dimensional subspace of $C^\infty(X^\wedge, E)$, and we endow this space with the norm topology.

4.2.14 Definition. Let (γ, Θ) be a weight datum and $P \in \operatorname{As}\big((\gamma,\Theta), C^\infty(X,E)\big)$.

a) Define

$$\mathcal{H}_\Theta^{(s,t),\gamma;\ell}(X^\wedge, E) := \bigcap_{\delta\in\Theta} \mathcal{H}^{(s,t),\gamma-\delta;\ell}(X^\wedge, E), \tag{4.2.vi}$$

$$\mathcal{T}_{\gamma-\frac{n}{2},\Theta}(X^\wedge, E) := \bigcap_{\delta\in\Theta} \mathcal{T}_{\gamma-\frac{n}{2}-\delta}(X^\wedge, E), \tag{4.2.vii}$$

endowed with the projective topology with respect to the mappings

$$\mathcal{H}_\Theta^{(s,t),\gamma;\ell}(X^\wedge, E) \ni u \mapsto u \in \mathcal{H}^{(s,t),\gamma-\delta;\ell}(X^\wedge, E),$$
$$\mathcal{T}_{\gamma-\frac{n}{2},\Theta}(X^\wedge, E) \ni u \mapsto u \in \mathcal{T}_{\gamma-\frac{n}{2}-\delta}(X^\wedge, E),$$

for $\delta \in \Theta$.

Actually, these spaces are Fréchet spaces. In (4.2.vi) and (4.2.vii) we only need to take the intersection over the elements of a sequence $\{\delta_\nu;\ \nu \in \mathbb{N}_0\} \subseteq \Theta$ such that $\delta_0 = 0$ and $\lim_{\nu\to\infty} \delta_\nu = \theta$ to obtain them algebraically and topologically, see also Corollary 4.2.10.

b) Let $\theta > -\infty$. Define

$$\mathcal{H}_P^{(s,t),\gamma;\ell}(X^\wedge, E) := \mathcal{H}_\Theta^{(s,t),\gamma;\ell}(X^\wedge, E) + \mathcal{E}_P(X^\wedge, E), \tag{4.2.viii}$$

$$\mathcal{T}_{\gamma-\frac{n}{2},P}(X^\wedge, E) := \mathcal{T}_{\gamma-\frac{n}{2},\Theta}(X^\wedge, E) + \mathcal{E}_P(X^\wedge, E). \tag{4.2.ix}$$

These spaces are well-defined, i. e. independent of the choice of the particular cut-off function involved in (4.2.v). We equip these spaces with the topology of the direct sum which turns them into Fréchet spaces.

c) In case of $\theta = -\infty$ we define

$$\mathcal{H}_P^{(s,t),\gamma;\ell}(X^\wedge, E) := \bigcap_{\nu\in\mathbb{N}} \mathcal{H}_{P_\nu}^{(s,t),\gamma;\ell}(X^\wedge, E), \tag{4.2.x}$$

$$\mathcal{T}_{\gamma-\frac{n}{2},P}(X^\wedge, E) := \bigcap_{\nu\in\mathbb{N}} \mathcal{T}_{\gamma-\frac{n}{2},P_\nu}(X^\wedge, E), \tag{4.2.xi}$$

where the asymptotic type P_ν associated with the weight datum $(\gamma, (-\nu, 0])$ contains those elements $(p, m, L) \in P$ with $p \in \Gamma_{(\frac{n+1}{2}-\gamma-\nu,\frac{n+1}{2}-\gamma)}$.

These spaces are Fréchet spaces with the projective limit topology induced by the right-hand sides of (4.2.x), (4.2.xi).

4.2.15 Theorem. *Let $\gamma,\gamma' \in \mathbb{R}$ with $\gamma < \gamma'$ and $s,t \in \mathbb{R}$. Then for every $\gamma \le \delta \le \gamma'$ the weighted Mellin transform*

$$\mathcal{M}_{\delta-\frac{n}{2}} : \mathcal{T}'_{\delta-\frac{n}{2}}(\mathbb{R}_+, H^{s+t}(X,E)) \longrightarrow \mathcal{S}'(\Gamma_{\frac{n+1}{2}-\delta}, H^{s+t}(X,E))$$

restricts to a topological isomorphism from the intersection $\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge,E) \cap \mathcal{H}^{(s,t),\gamma';\ell}(X^\wedge,E)$ onto the following space of analytic functions in the strip $\Gamma_{(\frac{n+1}{2}-\gamma',\frac{n+1}{2}-\gamma)}$:

Let $R^s(\tau) \in L^{s;\ell}(X;\mathbb{R};E,E)$ be a parameter-dependent reduction of orders from Theorem 3.1.12. Then $a \in \mathcal{M}_{\delta-\frac{n}{2}}\big(\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge,E) \cap \mathcal{H}^{(s,t),\gamma';\ell}(X^\wedge,E)\big)$ if and only if

- $a \in \mathcal{A}(\Gamma_{(\frac{n+1}{2}-\gamma',\frac{n+1}{2}-\gamma)}, H^{s+t}(X,E))$,
- $\|a\| := \sup\limits_{\gamma<\delta'<\gamma'} \Big(\frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{n+1}{2}-\delta'}} \|R^s(Im(z))a(z)\|^2_{H^t(X,E)}\, dz\Big)^{\frac{1}{2}} < \infty.$

4.2.16 Theorem. *Let (γ,Θ) be a weight datum and $P \in As\big((\gamma,\Theta),C^\infty(X,E)\big)$.*

a) For every $\gamma \le \delta < \gamma - \theta$ the weighted Mellin transform $\mathcal{M}_{\delta-\frac{n}{2}}$ restricts to a topological isomorphism from $\mathcal{H}^{(s,t),\gamma;\ell}_\Theta(X^\wedge,E)$ onto the space of all

- $a \in \mathcal{A}(\Gamma_{(\frac{n+1}{2}-\gamma+\theta,\frac{n+1}{2}-\gamma)}, H^{s+t}(X,E))$,
- $a|_{\Gamma_{(\frac{n+1}{2}-\gamma+\delta',\frac{n+1}{2}-\gamma)}} \in \mathcal{M}_{\gamma-\frac{n}{2}}\big(\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge,E) \cap \mathcal{H}^{(s,t),\gamma-\delta';\ell}(X^\wedge,E)\big)$ for $\delta' \in (\theta,0)$,

endowed with the topology of the projective limit.

b) The weighted Mellin transform $\mathcal{M}_{\gamma-\frac{n}{2}}$ restricts to a topological isomorphism from $\mathcal{T}_{\gamma-\frac{n}{2},P}(X^\wedge,E)$ onto the following space of meromorphic functions:

$a \in \mathcal{M}_{\gamma-\frac{n}{2}}\big(\mathcal{T}_{\gamma-\frac{n}{2},P}(X^\wedge,E)\big)$ if and only if:

- $a \in \mathcal{A}(\Gamma_{(\frac{n+1}{2}-\gamma+\theta,\frac{n+1}{2}-\gamma)} \setminus \pi_{\mathbb{C}}P, C^\infty(X,E))$.
- *For every $(p,m,L) \in P$ we may write in a neighbourhood $U(p) \setminus \{p\}$*

$$a(z) = \sum_{k=0}^{m} \nu_k (z-p)^{-(k+1)} + a_0(z)$$

with $\nu_k \in L$, $k = 0,\dots,m$, and a_0 holomorphic in p taking values in $C^\infty(X,E)$.

- *For every $\pi_{\mathbb{C}}P$-excision function $\chi \in C^\infty(\mathbb{C})$ we have $(\chi{\cdot}a)|_{\Gamma_\delta} \in \mathcal{S}(\Gamma_\delta, C^\infty(X,E))$, uniformly for δ in subintervals $I \subseteq (\frac{n+1}{2}-\gamma+\theta, \frac{n+1}{2}-\gamma)$ of the form $I=[\delta', \frac{n+1}{2}-\gamma)$.*

This space carries a canonical Fréchet topology, namely the convergence of $(\chi{\cdot}a)|_{\Gamma_\delta} \in \mathcal{S}(\Gamma_\delta, C^\infty(X,E))$ for every $\pi_{\mathbb{C}}P$-excision function $\chi \in C^\infty(\mathbb{C})$, uniformly for δ in subintervals I as above.

c) *The following identities hold algebraically and topologically with the topology of the non-direct sum of Fréchet spaces on the right-hand sides:*

$$\mathcal{H}_P^{(s,t),\gamma;\ell}(X^\wedge,E) = \mathcal{H}_\Theta^{(s,t),\gamma;\ell}(X^\wedge,E) + \mathcal{T}_{\gamma-\frac{n}{2},P}(X^\wedge,E),$$

$$\mathcal{M}_{\gamma-\frac{n}{2}}\big(\mathcal{H}_P^{(s,t),\gamma;\ell}(X^\wedge,E)\big) = \mathcal{M}_{\gamma-\frac{n}{2}}\big(\mathcal{H}_\Theta^{(s,t),\gamma;\ell}(X^\wedge,E)\big) + \mathcal{M}_{\gamma-\frac{n}{2}}\big(\mathcal{T}_{\gamma-\frac{n}{2},P}(X^\wedge,E)\big),$$

the latter within meromorphic $H^{s+t}(X,E)$-valued functions.

Proof. a) follows immediately from Theorem 4.2.15. Let us prove b). Note first that for $\gamma' \in \mathbb{R}$ we have $u \in \mathcal{T}_{\gamma'-\frac{n}{2}}(X^\wedge,E)$ if and only if the function $\langle\log(r)\rangle^\varrho u(r)$ belongs to $\mathcal{H}^{(s,0),\gamma';\ell}(X^\wedge,E)$ for every $s,\varrho \in \mathbb{R}$. To see this observe that the transformation $S_{\gamma'-\frac{n}{2}}$ provides a topological isomorphism

$$S_{\gamma'-\frac{n}{2}} : \langle\log(r)\rangle^{-\varrho}\mathcal{H}^{(s,0),\gamma';\ell}(X^\wedge,E) \longrightarrow H^{(s,0);\ell}(\mathbb{R}\times X,E)_\varrho,$$

and we have $\bigcap_{s,\varrho\in\mathbb{R}} H^{(s,0);\ell}(\mathbb{R}\times X,E)_\varrho = \mathcal{S}(\mathbb{R}\times X,E)$ by Theorem 4.1.5. Employing relation (1.1.iii) and again Theorem 4.1.5 we obtain assertion b) from a) in case of the empty asymptotic type, i. e. $P=\Theta$. Let us consider general asymptotic types P. Note that it suffices to prove the assertion for the finite weight interval, i. e. $\theta > -\infty$. Let $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ be a cut-off function near $r=0$, and $v(r) := \omega(r) \sum_{(p,m,L)\in P} \sum_{k=0}^{m} c_{p,k} r^{-p} \log^k(r) \in \mathcal{E}_P(X^\wedge,E)$ with $c_{p,k} \in L$. Then, by the properties of the Mellin transform, we have

$$(\mathcal{M}_{\gamma-\frac{n}{2}}v)(z) = \sum_{(p,m,L)\in P} \sum_{k=0}^{m} c_{p,k}\Big(\frac{d}{dz}\Big)^k \Big(\frac{1}{z-p}\mathcal{M}(-r\partial_r\omega)(z-p)\Big) \tag{1}$$

for $z \in \mathbb{C}$, and consequently the asserted characterizations hold for the functions in $\mathcal{E}_P(X^\wedge,E)$. Summing up, we have proved that the Mellin transform of a function in $\mathcal{T}_{\gamma-\frac{n}{2},P}(X^\wedge,E)$ is meromorphic in the weight strip with the desired properties. Conversely, let a be a meromorphic function in the weight strip with the properties listed in b). Then we see from (1) that there is a function $v \in \mathcal{E}_P(X^\wedge,E)$ such that $a(z)-(\mathcal{M}_{\gamma-\frac{n}{2}}v)(z)$ is holomorphic and satisfies the conditions in b) for the empty asymptotic type, i. e. there is a function $u \in \mathcal{T}_{\gamma-\frac{n}{2},\Theta}(X^\wedge,E)$ such that

$(\mathcal{M}_{\gamma-\frac{n}{2}}u)(z) = a(z) - (\mathcal{M}_{\gamma-\frac{n}{2}}v)(z)$. Thus we have $a = \mathcal{M}_{\gamma-\frac{n}{2}}(u+v)$ which finishes the proof of b).

Clearly, the assertions in c) are equivalent by Mellin transform. For the finite weight interval we have nothing to prove, so let us consider the infinite interval $\Theta = (-\infty, 0]$. For $\mathcal{H}_P^{(s,t),\gamma;\ell}(X^\wedge, E) \supseteq \mathcal{H}_\Theta^{(s,t),\gamma;\ell}(X^\wedge, E) + \mathcal{T}_{\gamma-\frac{n}{2},P}(X^\wedge, E)$ is obvious we only have to check the opposite inclusion. Let $a \in \mathcal{M}_{\gamma-\frac{n}{2}}\big(\mathcal{H}_P^{(s,t),\gamma;\ell}(X^\wedge, E)\big)$ be arbitrary, and let $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ be a cut-off function near $r = 0$. Let $\{q_j;\ j \in \mathbb{N}_0\} \subseteq \pi_{\mathbb{C}}P$ be the pole pattern of the meromorphic function a. For every pole $q_j \in \mathbb{C}$ with $(q_j, m_j, L_j) \in P$ choose functions $c_{q_j,k} \in L_j$ such that $a(z) - \mathcal{M}_{\gamma-\frac{n}{2}}\big(\omega(r)\sum_{k=0}^{m_j} c_{q_j,k} r^{-q_j}\log^k(r)\big)(z)$ is holomorphic in q_j. For $c > 0$ define

$$\psi_{c,q_j}(z) := \mathcal{M}_{\gamma-\frac{n}{2}}\big(\omega(cr)\sum_{k=0}^{m_j} c_{q_j,k} r^{-q_j}\log^k(r)\big)(z).$$

Then also $a(z) - \psi_{c,q_j}(z)$ is holomorphic in q_j by (1). Note that for $c > 0$ we have $\mathcal{M}(-r\partial_r\omega(cr))(z) = c^{-z}\mathcal{M}(-r\partial_r\omega)(z)$, and consequently $\mathcal{M}(-r\partial_r\omega(cr))$ converges to 0 in $\mathcal{S}(\Gamma_\beta)$ as $c \to \infty$, locally uniformly for β in $\mathbb{R}_+$. A Borel argument now shows that there is a sequence $(c_j) \subseteq \mathbb{R}_+$ with $\lim_{j\to\infty} c_j = \infty$ such that the series

$$b(z) := \sum_{j=0}^{\infty} \psi_{c_j,q_j}(z)$$

converges and defines an element $b \in \mathcal{M}_{\gamma-\frac{n}{2}}\big(\mathcal{T}_{\gamma-\frac{n}{2},P}(X^\wedge, E)\big)$ in view of b). Moreover, $a - b$ belongs to $\mathcal{M}_{\gamma-\frac{n}{2}}\big(\mathcal{H}_{(-\infty,0]}^{(s,t),\gamma;\ell}(X^\wedge, E)\big)$ by a) which finishes the proof of the theorem. □

4.3 Cone Sobolev spaces

4.3.1 Remark. In this section we introduce anisotropic Sobolev spaces on $X^\wedge$ which coincide near $r = 0$ with the Mellin Sobolev spaces from Section 4.2, and near $r = \infty$ with the Sobolev spaces from Section 4.1. The construction is analogous to that of the cone Sobolev spaces considered in the analysis on spaces with conical singularities, cf. [13], [59], [60], [61], which motivates the name and the notations involved. Nevertheless, even in the isotropic case, i. e. $\ell = 1$ and $t = 0$, the spaces differ from each other near $r = \infty$: While the "classical" cone Sobolev spaces reflect the conical structure near infinity in polar coordinates, the spaces from this section impose the structure of a cylindrical end.

Our main interest in this part is the analysis of parabolic pseudodifferential operators and the behaviour of solutions on the closed compact manifold X. The space–time configuration for these problems is $[t_0,\infty)\times X$ for some $t_0\in\mathbb{R}$, where $t=\infty$ is treated as a cylindrical end with an exponential weight. In our approach, this configuration is transformed via (1.1.i) to $(0,r_0]\times X$ with $r_0\in\mathbb{R}_+$, and the corresponding function spaces are the (weighted) Mellin Sobolev spaces from Section 4.2. Consequently, for the applications we have in mind, the particular choice of the function space on $X^\wedge$ near $r=\infty$ is irrelevant as far as it is compatible with the Mellin Sobolev space away from infinity.

The analysis of the operators within the cone Sobolev spaces from this section turns out to be quite natural in view of the examples involved in the applications, which motivates the definitions and constructions given below.

4.3.2 Definition. Let $\omega\in C_0^\infty(\overline{\mathbb{R}}_+)$ be a cut-off function near $r=0$.

a) For $\gamma\in\mathbb{R}$ define

$$\mathcal{S}^\gamma(X^\wedge,E):=[\omega]\mathcal{T}_{\gamma-\frac{n}{2}}(X^\wedge,E)+[1-\omega]\mathcal{S}(\mathbb{R}\times X,E). \tag{4.3.i}$$

b) For $s,t,\gamma,\delta\in\mathbb{R}$ define the cone Sobolev space $\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta$ as

$$\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta:=[\omega]\mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge,E)+[1-\omega]H^{(s,t);\ell}(\mathbb{R}\times X,E)_\delta. \tag{4.3.ii}$$

The non-direct sums are carried out in $H^{(s,t);\ell}_{\mathrm{loc}}(X^\wedge,E)$, and the resulting spaces do not depend on the particular choice of the cut-off function.

4.3.3 Notation. For $\gamma,\delta\in\mathbb{R}$ let $k_{\gamma,\delta}\in C^\infty(\mathbb{R}_+)$ be an everywhere positive function with

$$k_{\gamma,\delta}\equiv\begin{cases} r^{\gamma-\frac{n}{2}} & \text{near } r=0\\ r^{-\delta} & \text{near } r=\infty.\end{cases}$$

4.3.4 Theorem. *Let $\omega\in C_0^\infty(\overline{\mathbb{R}}_+)$ be a cut-off function.*

a) $\mathcal{S}^\gamma(X^\wedge,E)$ is a nuclear Fréchet space.

b) $\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta$ is a Hilbert space (more precisely a hilbertizable space).

c) We have $k_{\gamma'+\frac{n}{2},\delta'}\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta=\mathcal{K}^{(s,t),\gamma+\gamma';\ell}(X^\wedge,E)_{\delta+\delta'}$ algebraically and topologically. Moreover, we have $\mathcal{K}^{(0,0),\gamma;\ell}(X^\wedge,E)_\delta=k_{\gamma,\delta}L^2(X^\wedge,E)$ and in particular $\mathcal{K}^{(0,0),0;\ell}(X^\wedge,E)_{\frac{n}{2}}=r^{-\frac{n}{2}}L^2(X^\wedge,E)$ (see also i) below).

d) The embedding $\mathcal{S}^\gamma(X^\wedge,E)\hookrightarrow\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta$ is continuous and dense.

e) *The operator of multiplication with a function* $\varphi \in [\omega]C_B^\infty(\mathbb{R}_+) + [1-\omega]C_b^\infty(\mathbb{R})$ *induces a continuous operator in* $\mathcal{L}(\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta)$, *and the mapping*

$$[\omega]C_B^\infty(\mathbb{R}_+) + [1-\omega]C_b^\infty(\mathbb{R}) \ni \varphi \mapsto M_\varphi \in \mathcal{L}(\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta)$$

is continuous.

f) *For* $s'-s \leq \min\{0,t-t'\}$, $\gamma \geq \gamma'$ *and* $\delta \geq \delta'$ *the embedding* $\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta \hookrightarrow \mathcal{K}^{(s',t'),\gamma';\ell}(X^\wedge,E)_{\delta'}$ *is well-defined and continuous. Moreover, it is compact if* $s'-s < \min\{0,t-t'\}$, $\gamma > \gamma'$ *and* $\delta > \delta'$; *Hilbert-Schmidt if* $s'-s+\frac{n+\ell}{2} < \min\{0,t-t'\}$, $\gamma > \gamma'$ *and* $\delta - \delta' > \frac{1}{2}$.

g) Sobolev embedding theorem: *Let* $k \in \mathbb{N}_0$. *Then for* $s > k + \frac{n+\ell}{2}$ *the embedding*

$$\mathcal{K}^{(s,0),\gamma;\ell}(X^\wedge,E)_\delta \hookrightarrow [\omega]r^{-(\frac{n+1}{2}-\gamma)}C_B^{k;\ell}(\mathbb{R}_+\times X,E) + [1-\omega]\langle r\rangle^{-\delta}C_b^{k;\ell}(\mathbb{R}\times X,E)$$

is well-defined and continuous.

h) *For every* $\delta_0,\gamma_0 \in \mathbb{R}$ *the* $k_{\gamma_0,\delta_0}L^2(X^\wedge,E)$*-inner product extends to a non-degenerate sesquilinear pairing*

$$\langle\cdot,\cdot\rangle_{\gamma_0,\delta_0} : \mathcal{K}^{(s,t),\gamma+\gamma_0;\ell}(X^\wedge,E)_{\delta_0+\delta} \times \mathcal{K}^{(-s,-t),-\gamma+\gamma_0;\ell}(X^\wedge,E)_{\delta_0-\delta} \longrightarrow \mathbb{C}$$

which induces an identification of the dual

$$\mathcal{K}^{(s,t),\gamma+\gamma_0;\ell}(X^\wedge,E)'_{\delta_0+\delta} \cong \mathcal{K}^{(-s,-t),-\gamma+\gamma_0;\ell}(X^\wedge,E)_{\delta_0-\delta}.$$

The $r^{-\frac{n}{2}}L^2(X^\wedge,E)$*-inner product* $\langle\cdot,\cdot\rangle$ *serves as the reference inner product in the scale* $\big(\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta\big)_{s,t,\gamma,\delta\in\mathbb{R}}$.

i) *For* $s,t,\gamma,\delta \in \mathbb{R}$ *the following identities hold algebraically and topologically:*

$$\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge,E)_\delta = \begin{cases} k_{\gamma,\delta}L^2(\mathbb{R},H^{s+t}(X,E)) \cap \mathcal{K}^{\frac{s}{\ell},\gamma-\frac{n}{2}}(\mathbb{R}_+,H^t(X,E))_\delta & \text{for } s \geq 0, \\ k_{\gamma,\delta}L^2(\mathbb{R},H^{s+t}(X,E)) + \mathcal{K}^{\frac{s}{\ell},\gamma-\frac{n}{2}}(\mathbb{R}_+,H^t(X,E))_\delta & \text{for } s \leq 0, \end{cases} \tag{4.3.iii}$$

where

$$\mathcal{K}^{\frac{s}{\ell},\gamma-\frac{n}{2}}(\mathbb{R}_+,H^t(X,E))_\delta := [\omega]\mathcal{H}^{\frac{s}{\ell},\gamma-\frac{n}{2}}(\mathbb{R}_+,H^t(X,E)) + [1-\omega]\langle r\rangle^{-\delta}H^{\frac{s}{\ell}}(\mathbb{R},H^t(X,E)).$$

Proof. a) and b) are consequences of Remark 4.1.6 and the permanence properties of nuclear spaces. c)–e), g) and h) follow from Theorem 4.1.5, Theorem 4.2.6 and Proposition 4.2.5. f) is a consequence of Theorem 4.1.5 and Theorem 4.2.9. i) follows from Proposition 4.1.7 and Proposition 4.2.5. ☐

4.3.5 Remark. Let

$$A : \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge, E)_\delta \longrightarrow \mathcal{K}^{(s-\mu,t),\gamma;\ell}(X^\wedge, F)_{\delta-\varrho}$$

be continuous for all $s, t, \delta \in \mathbb{R}$. Then the *formal adjoint operator* A^* with respect to the $r^{-\frac{n}{2}}L^2$-inner product $\langle \cdot, \cdot \rangle$ is defined by means of the identity $\langle Au, v\rangle = \langle u, A^* v\rangle$. By Theorem 4.3.4 the operator A^* is well-defined as a continuous operator

$$A^* : \mathcal{K}^{(s,t),-\gamma;\ell}(X^\wedge, F)_\delta \longrightarrow \mathcal{K}^{(s-\mu,t),-\gamma;\ell}(X^\wedge, E)_{\delta-\varrho}$$

for all $s, t, \delta \in \mathbb{R}$. In the remaining part we will take formal adjoints of operators on $X^\wedge$ in this sense.

4.3.6 Definition. Let $\emptyset \neq U \subseteq \mathbb{R}_+$ be an open set. Define $\mathcal{K}_0^{(s,t),\gamma;\ell}(\overline{U} \times X, E)_\delta$ to be the subspace of all $u \in \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge, E)_\delta$ such that $\operatorname{supp} u \subseteq \overline{U}$.

The closure of U is taken with respect to the topology of $\mathbb{R}_+$.

4.3.7 Proposition. *The closure of $C_0^\infty(U, C^\infty(X, E))$ in $\mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge, E)_\delta$ is contained in $\mathcal{K}_0^{(s,t),\gamma;\ell}(\overline{U} \times X, E)_\delta$ for every open set $\emptyset \neq U \subseteq \mathbb{R}_+$. If U is an interval then the closure coincides with the space. Moreover, the following identities are valid:*

$$\mathcal{K}_0^{(s,t),\gamma;\ell}((0, r_0] \times X, E)_\delta = \mathcal{H}_0^{(s,t),\gamma;\ell}((0, r_0] \times X, E),$$
$$\mathcal{K}_0^{(s,t),\gamma;\ell}([r_0, \infty) \times X, E)_\delta = H_0^{(s,t);\ell}([r_0, \infty) \times X, E)_\delta$$

for every $r_0 \in \mathbb{R}_+$.

Proof. These assertions follow from Theorem 4.1.9 and Proposition 4.2.12. □

4.3.8 Definition. Let (γ, Θ) be a weight datum and $P \in \mathrm{As}\big((\gamma, \Theta), C^\infty(X, E)\big)$. For an arbitrary but fixed cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ define

$$\mathcal{S}_P^\gamma(X^\wedge, E) := [\omega]\mathcal{T}_{\gamma-\frac{n}{2},P}(X^\wedge, E) + [1-\omega]\mathcal{S}(\mathbb{R} \times X, E),$$
$$\mathcal{K}_P^{(s,t),\gamma;\ell}(X^\wedge, E)_\delta := [\omega]\mathcal{H}_P^{(s,t),\gamma;\ell}(X^\wedge, E) + [1-\omega]H^{(s,t);\ell}(\mathbb{R} \times X, E)_\delta,$$

for $s, t, \delta \in \mathbb{R}$. These spaces are independent of the particular choice of the cut-off function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$, and we endow them with the (Fréchet) topology of the non-direct sum.

4.3.9 Proposition. a) *The embeddings*

$$\mathcal{S}_P^\gamma(X^\wedge, E) \hookrightarrow \mathcal{S}^\gamma(X^\wedge, E) \hookrightarrow \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge, E)_\delta,$$
$$\mathcal{S}_P^\gamma(X^\wedge, E) \hookrightarrow \mathcal{K}_P^{(s,t),\gamma;\ell}(X^\wedge, E)_\delta \hookrightarrow \mathcal{K}^{(s,t),\gamma;\ell}(X^\wedge, E)_\delta$$

are well-defined and continuous for every $s, t, \delta \in \mathbb{R}$.

b) If $\theta > -\infty$ we have direct decompositions

$$\mathcal{S}_P^\gamma(X^\wedge, E) = \mathcal{S}_\Theta^\gamma(X^\wedge, E) + \mathcal{E}_P(X^\wedge, E),$$
$$\mathcal{K}_P^{(s,t),\gamma;\ell}(X^\wedge, E)_\delta = \mathcal{K}_\Theta^{(s,t),\gamma;\ell}(X^\wedge, E)_\delta + \mathcal{E}_P(X^\wedge, E).$$

c) We have

$$\mathcal{S}_P^\gamma(X^\wedge, E) = \bigcap_{s,\delta\in\mathbb{R}} \mathcal{K}_P^{(s,0),\gamma;\ell}(X^\wedge, E)_\delta,$$

and $\mathcal{S}_P^\gamma(X^\wedge, E)$ is a nuclear Fréchet space.

Proof. a) and b) are obvious. We have to prove the representation of $\mathcal{S}_P^\gamma(X^\wedge, E)$ in c) as the intersection over the $\mathcal{K}_P^{(s,0),\gamma;\ell}(X^\wedge, E)_\delta$–spaces in case of the finite weight interval only. From b) we conclude that it suffices to consider the empty asymptotic type. But then the desired identity holds in view of the Sobolev embedding theorem in Theorem 4.3.4. Let us prove the nuclearity of the space $\mathcal{S}_P^\gamma(X^\wedge, E)$. By the permanence properties of nuclear spaces we just have to consider the finite weight interval and the empty asymptotic type. By the closed graph theorem the representation in c) holds topologically with the projective limit topology on the right-hand side. Employing the embedding properties from Theorem 4.3.4 f) we get the asserted nuclearity. □

Chapter 5

Calculi built upon parameter-dependent operators

5.1 Anisotropic meromorphic Mellin symbols

5.1.1 Remark. In this chapter we shall again employ the notations from Notation 3.1.1 with the corresponding data fixed on X and the vector bundles E and F.

5.1.2 Lemma. *Let $\emptyset \neq I \subseteq \mathbb{R}$ be an open interval and $\mu \in \mathbb{R}$. Let*

$$\begin{aligned} \ell^\infty_{loc}\mathcal{A}^{\mu;\ell}_{(cl)} &:= \{a \in \mathcal{A}(\Gamma_I, L^\mu(X;E,F));\ a|_{\Gamma_\beta} \in L^{\mu;\ell}_{(cl)}(X;\Gamma_\beta;E,F) \\ &\qquad\qquad \text{locally uniformly for } \beta \in I\} \\ C^\infty\mathcal{A}^{\mu;\ell}_{(cl)} &:= \{a \in \mathcal{A}(\Gamma_I, L^\mu(X;E,F));\ a \in C^\infty(I_\beta, L^{\mu;\ell}_{(cl)}(X;\Gamma_\beta;E,F))\} \end{aligned}$$

endowed with their natural Fréchet topologies.

Then the embedding $\iota : C^\infty\mathcal{A}^{\mu;\ell}_{(cl)} \hookrightarrow \ell^\infty_{loc}\mathcal{A}^{\mu;\ell}_{(cl)}$ is onto and provides an isomorphism between these spaces. The complex derivative acts linear and continuous in the spaces $\partial_z : \ell^\infty_{loc}\mathcal{A}^{\mu;\ell}_{(cl)} \to \ell^\infty_{loc}\mathcal{A}^{\mu-\ell;\ell}_{(cl)}$.

Moreover, given $a \in \ell^\infty_{loc}\mathcal{A}^{\mu;\ell}_{(cl)}$, we have the following asymptotic expansion for $a|_{\Gamma_{\beta_0}}$ in terms of $a|_{\Gamma_\beta}$ for every $\beta_0, \beta \in I$ which depends smoothly on $(\beta_0,\beta) \in I \times I$:

$$a|_{\Gamma_{\beta_0}} \sim \sum_{k=0}^\infty \frac{(\beta_0-\beta)^k}{k!}\,(\partial_z^k a)|_{\Gamma_\beta}.$$

Proof. Passing via (3.1.iv) to local symbols and global remainders on X reveals that the assertions follow from Proposition 2.6.3. □

5.1.3 Notation. For $p \in \mathbb{C}$ and $k \in \mathbb{N}_0$ let $\psi_{p,k} \in \mathcal{A}(\mathbb{C}\setminus\{p\})$ be an analytic function which is meromorphic in p with a pole of multiplicity $k+1$ such that for every p-excision function $\chi \in C^\infty(\mathbb{C})$ the function $\chi \cdot \psi_{p,k}$ belongs to $C^\infty(\mathbb{R}_\beta, \mathcal{S}(\Gamma_\beta))$.

In view of the properties of the Mellin transform the function

$$\psi_{p,k}(z) := \mathcal{M}_\gamma(\omega(r)r^{-p}\log^k r)(z) = \Big(\frac{d}{dz}\Big)^k\Big(\frac{1}{z-p}\cdot\mathcal{M}(-r\partial_r\omega)(z-p)\Big),$$

where $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ is a cut-off function near $r=0$ and $\gamma < \frac{1}{2} - \mathrm{Re}(p)$, fulfills these conditions.

5.1.4 Definition. A *Mellin asymptotic type* is a finite or countably infinite set

$$P = \{(p_j, m_j, L_j);\ j \in \mathbb{Z}\} \tag{5.1.i}$$

where the $m_j \in \mathbb{N}_0$ are integers, the L_j are finite-dimensional subspaces of $L^{-\infty}(X;E,F)$ consisting of finite-dimensional operators, and the $p_j \in \mathbb{C}$ are complex numbers such that with the "projection" $\pi_{\mathbb{C}}P := \{p_j;\ j \in \mathbb{Z}\}$ of P to $\mathbb{C}$ we have that the set $\pi_{\mathbb{C}}P \cap \Gamma_I$ is finite for every compact interval $I \subseteq \mathbb{R}$. For the empty asymptotic type we use the notation O.

The collection of all Mellin asymptotic types is denoted by $\mathrm{As}\big(L^{-\infty}(X;E,F)\big)$.

5.1.5 Definition. For $\mu \in \mathbb{R}$ and $P \in \mathrm{As}\big(L^{-\infty}(X;E,F)\big)$ we define the space $M^{\mu;\ell}_{P(cl)}(X;E,F)$ of *(anisotropic) meromorphic Mellin symbols* of order μ with asymptotic type P to consist of all functions $a \in \mathcal{A}(\mathbb{C}\setminus\pi_{\mathbb{C}}P, L^\mu(X;E,F))$ with the following properties:

- For every $(p,m,L) \in P$ we may write in a neighbourhood $U(p)\setminus\{p\}$

  $$a(z) = \sum_{k=0}^{m} \nu_k(z-p)^{-(k+1)} + a_0(z)$$

  with $\nu_k \in L$, $k = 0,\dots,m$, and a_0 holomorphic in p taking values in $L^\mu(X;E,F)$.

- For every compact interval $I \subseteq \mathbb{R}$ we have

  $$a(\beta+i\tau) - \sum_{\{(p_j,m_j,L_j);\ \mathrm{Re}(p_j)\in I\}}\ \sum_{k=0}^{m_j}\sigma_{j_k}\psi_{p_j,k} \in L^{\mu;\ell}_{(cl)}(X;\Gamma_\beta;E,F) \tag{5.1.ii}$$

  uniformly for $\beta \in I$ with suitable $\sigma_{j_k} \in L_j$.

Analogously, we define the space $M_P^{-\infty}(X; E, F)$ of meromorphic Mellin symbols of order $-\infty$ with asymptotic type P.

If $P = O$ is the empty asymptotic type the spaces are called holomorphic Mellin symbols.

5.1.6 Remark. The topology on the space $M_{P(cl)}^{\mu;\ell}(X; E, F)$ is determined by the following ingredients:

- The topology of $\mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}} P, L^{\mu}(X; E, F))$.
- Convergence of the Laurent coefficients ν_k in the corresponding coefficient spaces $L_j \subseteq L^{-\infty}(X; E, F)$.
- Uniform convergence of (5.1.ii) for $\beta \in I$ for every compact interval $I \subseteq \mathbb{R}$.

With this topology $M_{P(cl)}^{\mu;\ell}(X; E, F)$ is a Fréchet space. Note that the topology does not depend on the particular choice of the functions $\psi_{p_j,k}$ from Notation 5.1.3 involved in (5.1.ii) and the coefficients σ_{j_k} determined by them in view of the closed graph theorem.

In order $-\infty$ we have an equivalent characterization of $M_P^{-\infty}(X; E, F)$ as the space of all analytic functions $a \in \mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}} P, L^{-\infty}(X; E, F))$ such that

- a is meromorphic in p for every $(p, m, L) \in P$, and in a neighbourhood $U(p) \setminus \{p\}$ we have
$$a(z) = \sum_{k=0}^{m} \nu_k (z - p)^{-(k+1)} + a_0(z)$$
with $\nu_k \in L$, $k = 0, \dots, m$, and a_0 holomorphic in p taking values in $L^{-\infty}(X; E, F)$.
- We have $\chi{\cdot}a \in C^{\infty}(\mathbb{R}_\beta, L^{-\infty}(X; \Gamma_\beta; E, F))$ for every $\pi_{\mathbb{C}} P$-excision function $\chi \in C^{\infty}(\mathbb{C})$.

If we choose a countable collection of $\pi_{\mathbb{C}} P$-excision functions $\{\chi_k\}_k \subseteq C^{\infty}(\mathbb{C})$ which shrink to $\pi_{\mathbb{C}} P$ we find that the Fréchet space structure on $M_P^{-\infty}(X; E, F)$ is determined by the projective topology with respect to the mappings

$$M_P^{-\infty}(X; E, F) \ni a \longmapsto \chi_k{\cdot}a \in C\big(\mathbb{R}_\beta, L^{-\infty}(X; \Gamma_\beta; E, F)\big).$$

Note that Lemma 5.1.2 enters these considerations.

Material on (scalar) isotropic meromorphic Mellin symbols, i. e. $\ell = 1$, can be found, e. g., in [13], [59], [60], [61].

5.1.7 Proposition. *Let $\mu, \mu' \in \mathbb{R}$, $\mu' \leq \mu$. Moreover, let $P \in As\big(L^{-\infty}(X;E,F)\big)$ and $\beta \in \mathbb{R}$ with $\Gamma_\beta \cap \pi_{\mathbb{C}} P = \emptyset$. The following identities hold algebraically and topologically:*

a) $M_P^{\mu;\ell}(X;E,F) \cap L^{\mu';\ell}(X;\Gamma_\beta;E,F) = M_P^{\mu';\ell}(X;E,F)$.

b) $M_P^{\mu;\ell}(X;E,F) \cap L_{cl}^{\mu;\ell}(X;\Gamma_\beta;E,F) = M_{P\,cl}^{\mu;\ell}(X;E,F)$.

For holomorphic Mellin symbols we have:

i) $M_{O(cl)}^{\mu;\ell}(X;E,F) = \mathcal{A}(\mathbb{C}, L^\mu(X;E,F)) \cap C^\infty(\mathbb{R}_\beta, L_{(cl)}^{\mu;\ell}(X;\Gamma_\beta;E,F))$.

ii) The complex derivative acts continuous in the spaces

$$\partial_z : M_{O(cl)}^{\mu;\ell}(X;E,F) \to M_{O(cl)}^{\mu-\ell;\ell}(X;E,F).$$

iii) For $a \in M_{O(cl)}^{\mu;\ell}(X;E,F)$ the following asymptotic expansion holds for $a|_{\Gamma_{\beta_0}}$ in terms of $a|_{\Gamma_\beta}$ which depends smoothly on $(\beta_0,\beta) \in \mathbb{R}\times\mathbb{R}$:

$$a|_{\Gamma_{\beta_0}} \sim \sum_{k=0}^{\infty} \frac{(\beta_0-\beta)^k}{k!}(\partial_z^k a)|_{\Gamma_\beta}.$$

In the classical case we consequently obtain for arbitrary $\beta_0, \beta \in \mathbb{R}$ the following relationship for the parameter-dependent homogeneous principal symbol:

$$\sigma_\psi^{\mu;\ell}(a|_{\Gamma_{\beta_0}}) = \sigma_\psi^{\mu;\ell}(a|_{\Gamma_\beta}).$$

iv) For $s, \nu \in \mathbb{R}$ with $\nu \geq \mu$ we have a continuous embedding

$$M_O^{\mu;\ell}(X;E,F) \hookrightarrow \begin{cases} S_O^{\frac{\mu}{\ell}}(\mathbb{C}; H^s(X,E), H^{s-\nu}(X,F)) & \nu \geq 0 \\ S_O^{\frac{\mu-\nu}{\ell}}(\mathbb{C}; H^s(X,E), H^{s-\nu}(X,F)) & \nu \leq 0. \end{cases}$$

Proof. The assertions i) – iii) are subject to Lemma 5.1.2, while iv) follows from i) and Theorem 3.1.5.

To prove a) let $a \in M_P^{\mu;\ell}(X;E,F) \cap L^{\mu';\ell}(X;\Gamma_\beta;E,F)$. Consider for $N \in \mathbb{N}$ such that $N > |\beta|$ the open interval $I := (-N,N) \subseteq \mathbb{R}$. According to Definition 5.1.5 of meromorphic Mellin symbols we conclude from Lemma 5.1.2 that the expression (5.1.ii) belongs to the space $\ell_{\mathrm{loc}}^\infty \mathcal{A}^{\mu;\ell}$ over the interval I. From the asymptotic expansion result in Lemma 5.1.2 and the assumption that a is of order μ' on the weight line Γ_β we even conclude that (5.1.ii) belongs to the space $\ell_{\mathrm{loc}}^\infty \mathcal{A}^{\mu';\ell}$ over I. But since $N \in \mathbb{N}$ with $N > |\beta|$ was arbitrary we see that $a \in M_P^{\mu';\ell}(X;E,F)$ as asserted. This shows that the identity in a) holds algebraically, but then it holds also topologically in view of the closed graph theorem. The same reasoning as in the proof of a) also yields assertion b). ∎

5.1.8 Theorem. *a) Let $P, Q \in As\big(L^{-\infty}(X;E,F)\big)$. Then (pointwise) addition as $L^{\mu}(X;E,F)$-valued operator functions on $\mathbb{C} \setminus \big(\pi_{\mathbb{C}}P \cup \pi_{\mathbb{C}}Q\big)$ induces a bilinear and continuous mapping*

$$+ : M^{\mu;\ell}_{P(cl)}(X;E,F) \times M^{\mu;\ell}_{Q(cl)}(X;E,F) \longrightarrow M^{\mu;\ell}_{R(cl)}(X;E,F),$$

where $R \in As\big(L^{-\infty}(X;E,F)\big)$ consists (in general) of elements (q,m,L) of the form

$$\begin{cases} (q, \max\{m_1,m_2\}, L_1 + L_2) & \text{if } (q,m_1,L_1) \in P \text{ and } (q,m_2,L_2) \in Q \\ (q,m,L) & \text{if } q \in \pi_{\mathbb{C}}P \triangle \pi_{\mathbb{C}}Q \text{ and } (q,m,L) \in P \cup Q. \end{cases}$$

b) Let G be another vector bundle over X and $a \in M^{\mu;\ell}_{P(cl)}(X;G,F)$, $b \in M^{\mu';\ell}_{Q(cl)}(X;E,G)$. Then the pointwise composition (multiplication) as functions on $\mathbb{C} \setminus \big(\pi_{\mathbb{C}}P \cup \pi_{\mathbb{C}}Q\big)$ gives rise to an element $ab \in M^{\mu+\mu';\ell}_{R(cl)}(X;E,F)$ with a resulting asymptotic type $R \in As\big(L^{-\infty}(X;E,F)\big)$ which consists (in general) of elements (q,m,L) of the form

$$\begin{cases} (q, m_1 + m_2 + 1, L) & \text{if } (q,m_1,L_1) \in P \text{ and } (q,m_2,L_2) \in Q \\ (q,m,L) & \text{if } q \in \pi_{\mathbb{C}}P \triangle \pi_{\mathbb{C}}Q \text{ and } (q,m,\tilde{L}) \in P \cup Q. \end{cases}$$

c) For holomorphic Mellin symbols the multiplication as functions on $\mathbb{C}$ gives rise to a continuous bilinear mapping

$$\cdot : M^{\mu;\ell}_{O(cl)}(X;G,F) \times M^{\mu';\ell}_{O(cl)}(X;E,G) \longrightarrow M^{\mu+\mu';\ell}_{O(cl)}(X;E,F).$$

Proof. These assertions follow from the Definition 5.1.5 of meromorphic Mellin symbols and the properties of anisotropic parameter-dependent pseudodifferential operators on closed compact manifolds as discussed in the Section 3.1 (for the composition note in particular Theorem 3.1.6). □

5.1.9 Remark. Let $\mu \in \mathbb{R}$ and $\mu_+ := \max\{0,\mu\}$. Then the Mellin kernel cut-off operator with respect to the weight $\gamma \in \mathbb{R}$ is bilinear and continuous in the spaces

$$\begin{aligned} H_\gamma : C^\infty_B(\mathbb{R}_+) \times S^{\frac{\mu}{\ell}}(\Gamma_{\frac{1}{2}-\gamma}; &H^s(X,E), H^{s-\mu_+}(X,F)) \\ &\longrightarrow S^{\frac{\mu}{\ell}}(\Gamma_{\frac{1}{2}-\gamma}; H^s(X,E), H^{s-\mu_+}(X,F)) \end{aligned}$$

by Theorem 2.6.13. Analogously to Theorem 3.2.9 and Corollary 3.2.10 we obtain the following theorem for the Mellin kernel cut-off operator from Theorem 2.6.13.

5.1.10 Theorem. *The Mellin kernel cut-off operator with respect to the weight $\gamma \in \mathbb{R}$ restricts to continuous bilinear mappings in the spaces*

$$H_\gamma : \begin{cases} C_B^\infty(\mathbb{R}_+) \times L_{(cl)}^{\mu;\ell}(X;\Gamma_{\frac{1}{2}-\gamma};E,F) \longrightarrow L_{(cl)}^{\mu;\ell}(X;\Gamma_{\frac{1}{2}-\gamma};E,F) \\ C_0^\infty(\mathbb{R}_+) \times L_{(cl)}^{\mu;\ell}(X;\Gamma_{\frac{1}{2}-\gamma};E,F) \longrightarrow M_{O(cl)}^{\mu;\ell}(X;E,F). \end{cases}$$

Given $\varphi \in C_B^\infty(\mathbb{R}_+)$ and $a \in L^{\mu;\ell}(X;\Gamma_{\frac{1}{2}-\gamma};E,F)$ we have the following asymptotic expansion of $H_\gamma(\varphi)a$ in terms of φ and a in the sense of Definition 3.1.8:

$$H_\gamma(\varphi)a \sim \sum_{k=0}^{\infty} \frac{1}{k!}(r\partial_r)^k \varphi(r)|_{r=1} \cdot D_\tau^k a.$$

For $\psi \in C_0^\infty(\mathbb{R}_+)$ such that $\psi \equiv 1$ near $r=1$ the operator $I - H_\gamma(\psi)$ is continuous in the spaces

$$I - H_\gamma(\psi) : L^{\mu;\ell}(X;\Gamma_{\frac{1}{2}-\gamma};E,F) \longrightarrow L^{-\infty}(X;\Gamma_{\frac{1}{2}-\gamma};E,F).$$

5.1.11 Corollary. *For $P \in As\big(L^{-\infty}(X;E,F)\big)$ we have*

$$M_{P(cl)}^{\mu;\ell}(X;E,F) = M_{O(cl)}^{\mu;\ell}(X;E,F) + M_P^{-\infty}(X;E,F)$$

algebraically and topologically with the topology of the non-direct sum of Fréchet spaces on the right hand sides.

Proof. Let $\psi \in C_0^\infty(\mathbb{R}_+)$ such that $\psi \equiv 1$ near $r=1$. Let $\gamma \in \mathbb{R}$ such that $\Gamma_{\frac{1}{2}-\gamma} \cap \pi_{\mathbb{C}} P = \emptyset$. In view of Theorem 5.1.10 and Theorem 5.1.8 we may write for $a \in M_{P(cl)}^{\mu;\ell}(X;E,F)$

$$a = H_\gamma(\psi)a + \big(I - H_\gamma(\psi)\big)a,$$

where $H_\gamma(\psi)a \in M_{O(cl)}^{\mu;\ell}(X;E,F)$, and $\big(I - H_\gamma(\psi)\big)a$ belongs to the space $L^{-\infty}(X;\Gamma_{\frac{1}{2}-\gamma};E,F) \cap M_{P(cl)}^{\mu;\ell}(X;E,F) = M_P^{-\infty}(X;E,F)$ due to Proposition 5.1.7 and Theorem 5.1.8. This provides a topological isomorphism as asserted. □

5.1.12 Definition. Let $P \in \mathrm{As}\big(L^{-\infty}(X;E,F)\big)$. A meromorphic Mellin symbol $a \in M_{P(cl)}^{\mu;\ell}(X;E,F)$ is called *elliptic*, if the restriction $a|_{\Gamma_\beta} \in L_{(cl)}^{\mu;\ell}(X;\Gamma_\beta;E,F)$ to some weight line Γ_β is parameter-dependent elliptic in the sense of Definition 3.1.9, where $\beta \in \mathbb{R}$ is such that $\Gamma_\beta \cap \pi_{\mathbb{C}} P = \emptyset$.

According to Corollary 5.1.11 and Proposition 5.1.7 this is well-defined in the sense that for every $\beta \in \mathbb{R}$ such that $\Gamma_\beta \cap \pi_{\mathbb{C}} P = \emptyset$ the restriction $a|_{\Gamma_\beta} \in L_{(cl)}^{\mu;\ell}(X;\Gamma_\beta;E,F)$ is parameter-dependent elliptic if and only if it is the case for some $\beta \in \mathbb{R}$.

5.1.13 Proposition. *Let $c \in M_P^{-\infty}(X;E,E)$. Then there exists an element $d \in M_Q^{-\infty}(X;E,E)$ such that $\big(1+c\big)^{-1} = 1+d$ as meromorphic operator functions.*

Proof. First observe that the function $1+c \in \mathcal{A}(\mathbb{C}\setminus\pi_{\mathbb{C}}P, L^0(X;E,E))$ is a finitely meromorphic Fredholm family taking values in $L^0(X;E,E) \hookrightarrow \mathcal{L}(L^2(X,E))$. Let $\beta \in \mathbb{R}$ such that $\Gamma_\beta \cap \pi_{\mathbb{C}}P = \emptyset$. Then $\big(1+c\big)|_{\Gamma_\beta} \in L^{0;\ell}(X;\Gamma_\beta;E,E)$ is parameter-dependent elliptic. Thus, by virtue of Theorem 3.1.11, there exists $\big(1+c(z)\big)^{-1} \in L^0(X;E,E)$ for $|\mathrm{Im}(z)|$ sufficiently large on Γ_β. Consequently, we may apply Theorem 1.2.6 on the inversion of finitely meromorphic Fredholm families to $1+c$, i. e., $1+c$ is invertible as a finitely meromorphic Fredholm family (taking values in $L^0(X;E,E)$). The Laurent-coefficients of the principal part of $\big(1+c\big)^{-1}$ at a pole $p \in \mathbb{C}$ are finite-dimensional pseudodifferential operators, and thus they necessarily belong to $L^{-\infty}(X;E,E)$. Moreover, we have $\big(1+c\big)^{-1} = 1-c+c\big(1+c\big)^{-1}c$, which shows that $d := -c+c\big(1+c\big)^{-1}c$ is a meromorphic function on $\mathbb{C}$ taking values in $L^{-\infty}(X;E,E)$.

Let us study the inverse $\big(1+c\big)^{-1}$ in more detail. Let $R>0$ be arbitrary such that $\big(\Gamma_{-R}\cup\Gamma_R\big)\cap\pi_{\mathbb{C}}P = \emptyset$. Let $\chi \in C^\infty(\mathbb{C})$ such that $\chi \equiv 0$ near $\pi_{\mathbb{C}}P \cap \Gamma_{[-R,R]}$ and $\chi \equiv 1$ outside some small neighbourhood U of $\pi_{\mathbb{C}}P\cap\Gamma_{[-R,R]}$. Then in view of Definition 5.1.5 and Remark 5.1.6 the function $\chi{\cdot}c$ depends smoothly on $\beta \in [-R,R]$ taking values in $L^{-\infty}(X;\Gamma_\beta;E,E)$. Now apply Theorem 3.1.11 to $1+\chi{\cdot}c$. This shows at first that for $|\mathrm{Im}(z)|$ sufficiently large in $\Gamma_{[-R,R]}$ we have that $1+\chi(z)c(z)$ is invertible. Moreover, we have $\big(1+\chi(z)c(z)\big)^{-1} = \big(1+c(z)\big)^{-1}$ outside U. But since the neighbourhood U (i. e. the excision function χ) may be chosen arbitrarily small we conclude that only finitely many poles of $\big(1+c\big)^{-1}$ are located in the strip $\Gamma_{[-R,R]}$. Consequently, the pattern of poles together with the data of the Laurent expansions of $\big(1+c\big)^{-1}$ determines a Mellin asymptotic type $Q \in \mathrm{As}\big(L^{-\infty}(X;E,E)\big)$.

Now let $R>0$ be arbitrary such that $\big(\Gamma_{-R}\cup\Gamma_R\big)\cap\big(\pi_{\mathbb{C}}P\cup\pi_{\mathbb{C}}Q\big) = \emptyset$. We have that $1+c(z)$ is invertible for $z \in \Gamma_{[-R,R]}\setminus\big(\pi_{\mathbb{C}}P\cup\pi_{\mathbb{C}}Q\big)$. Let V be some small neighbourhood of $\big(\pi_{\mathbb{C}}P\cup\pi_{\mathbb{C}}Q\big)\cap\Gamma_{[-R,R]}$. Choose $\chi \in C^\infty(\mathbb{C})$ such that $\chi\equiv 0$ near $\big(\pi_{\mathbb{C}}P\cup\pi_{\mathbb{C}}Q\big)\cap\Gamma_{[-R,R]}$ and $\chi \equiv 1$ outside V. For $z \in \Gamma_{[-R,R]}\setminus V$ we may write using Theorem 3.1.11

$$1+d(z) = \big(1+c(z)\big)^{-1} = \big(1+\chi(z)c(z)\big)^{-1} = 1+\tilde{c}(z)$$

where $\tilde{c} \in C^\infty([-R,R], L^{-\infty}(X;\Gamma_\beta;E,E))$. This shows that $d \in M_Q^{-\infty}(X;E,E)$ which finishes the proof of the proposition. □

5.1.14 Theorem. *An element $a \in M_{P(cl)}^{\mu;\ell}(X;E,F)$ is elliptic if and only if there exists $b \in M_{Q(cl)}^{-\mu;\ell}(X;F,E)$ such that $a{\cdot}b \equiv 1$ and $b{\cdot}a \equiv 1$, i. e., a is invertible as a meromorphic operator function with $a^{-1} = b \in M_{Q(cl)}^{-\mu;\ell}(X;F,E)$.*

Proof. Let $a \in M^{\mu;\ell}_{P(cl)}(X;E,F)$ be elliptic and $\gamma \in \mathbb{R}$ such that $\Gamma_{\frac{1}{2}-\gamma} \cap \pi_{\mathbb{C}} P = \emptyset$. According to Theorem 3.1.10 there exists $p \in L^{-\mu;\ell}_{(cl)}(X;\Gamma_{\frac{1}{2}-\gamma};F,E)$ such that $(a|_{\Gamma_{\frac{1}{2}-\gamma}}) \cdot p - 1 \in L^{-\infty}(X;\Gamma_{\frac{1}{2}-\gamma};F,F)$ and $p \cdot (a|_{\Gamma_{\frac{1}{2}-\gamma}}) - 1 \in L^{-\infty}(X;\Gamma_{\frac{1}{2}-\gamma};E,E)$. Let $\psi \in C_0^\infty(\mathbb{R}_+)$ with $\psi \equiv 1$ near $r=1$ and define $\tilde{b} := H_\gamma(\psi)p$. Using Theorem 5.1.10 we get $\tilde{b} \in M^{-\mu;\ell}_{O(cl)}(X;F,E)$ and $\big(p-\tilde{b}\big)|_{\Gamma_{\frac{1}{2}-\gamma}} \in L^{-\infty}(X;\Gamma_{\frac{1}{2}-\gamma};F,E)$. This shows that $\tilde{b}|_{\Gamma_{\frac{1}{2}-\gamma}}$ is a parameter-dependent parametrix of $a|_{\Gamma_{\frac{1}{2}-\gamma}}$. Moreover, from Theorem 5.1.8 and Proposition 5.1.7 we obtain that $a \cdot \tilde{b} = 1 + r_R$ and $\tilde{b} \cdot a = 1 + r_L$ with meromorphic Mellin symbols r_L and r_R of order $-\infty$. Now apply Proposition 5.1.13 to $1+r_L$ and $1+r_R$. Then we conclude from Theorem 5.1.8 that

$$a^{-1} = b = \big(1+r_L\big)^{-1}\tilde{b} = \tilde{b}\big(1+r_R\big)^{-1} \in M^{-\mu;\ell}_{Q(cl)}(X;F,E)$$

as asserted. If conversely a is invertible as a meromorphic Mellin symbol with inverse $b \in M^{-\mu;\ell}_{Q(cl)}(X;F,E)$ we see that $b|_{\Gamma_\beta}$ is a parameter-dependent parametrix of $a|_{\Gamma_\beta}$ for every $\beta \in \mathbb{R}$ such that $\Gamma_\beta \cap \big(\pi_{\mathbb{C}} P \cup \pi_{\mathbb{C}} Q\big) = \emptyset$. Consequently, a is elliptic in the sense of Definition 5.1.12 due to Theorem 3.1.10. □

5.1.15 Theorem. *Let $I \subseteq \mathbb{R}$ be a compact interval. Then there exists for every $\mu \in \mathbb{R}$ an elliptic element $h \in M_O^{\mu;\ell}(X;E,E)$ such that its inverse $h^{-1} \in M_Q^{-\mu;\ell}(X;E,E)$ (cf. Theorem 5.1.14) has no poles in the strip Γ_I, i. e., $\Gamma_I \cap \pi_{\mathbb{C}} Q = \emptyset$.*

Proof. According to Theorem 3.1.12 there exist for $\mu \in \mathbb{R}$ operators $a^\mu \in L^{\mu;\ell}(X;\Gamma_0 \times \mathbb{R}_\lambda;E,E)$ such that $a^\mu a^{-\mu} = 1$. Now let $\mu \in \mathbb{R}$ be given. Let $\varphi \in C_0^\infty(\mathbb{R}_+)$ such that $\varphi \equiv 1$ near $r=1$. Define for $(z,\lambda) \in \Gamma_0 \times \mathbb{R}$

$$a(z,\lambda) := \big(H_{\frac{1}{2}}(\varphi)a^\mu\big)(z,\lambda).$$

From Theorem 5.1.10 and Theorem 2.6.13 we conclude that $a(z,\lambda)$ gives rise to a parameter-dependent family in $M_O^{\mu;\ell}(X;E,E)$ depending on the parameter $\lambda \in \mathbb{R}$. More precisely, we have for $\beta \in \mathbb{R}$

$$a(\cdot,\lambda)|_{\Gamma_\beta} = \big(H_{\frac{1}{2}}(\varphi)a^\mu\big)|_{\Gamma_\beta}(\lambda) = \big(H_{\frac{1}{2}}(r^\beta\varphi)a^\mu\big)|_{\Gamma_0}(\lambda).$$

For the family $\{r^\beta\varphi(r);\ \beta \in I\} \subseteq C_0^\infty(\mathbb{R}_+)$ is bounded we conclude from Theorem 5.1.10 and Theorem 3.1.11 that if we fix $\lambda_0 \in \mathbb{R}$ with $|\lambda_0|$ sufficiently large we can arrange the invertibility of $a(z,\lambda_0) : H^s(X,E) \to H^{s-\mu}(X,E)$ for all $z \in \Gamma_I$. Thus the symbol $h := a(z,\lambda_0) \in M_O^{\mu;\ell}(X;E,E)$ has the desired properties in view of Theorem 5.1.14. □

5.2 Meromorphic Volterra Mellin symbols

5.2.1 Definition. Let $\mathbb{H}_\beta$ be a right half-plane in $\mathbb{C}$. For $\mu \in \mathbb{R}$ and a Mellin asymptotic type $P \in \mathrm{As}\big(L^{-\infty}(X;E,F)\big)$ such that $\pi_{\mathbb{C}}P \cap \mathbb{H}_\beta = \emptyset$ we define the space of *meromorphic Volterra Mellin symbols* of order μ with asymptotic type P as

$$M^{\mu;\ell}_{V,P(cl)}(X;\mathbb{H}_\beta;E,F) := M^{\mu;\ell}_{P}(X;E,F) \cap L^{\mu;\ell}_{V(cl)}(X;\mathbb{H}_\beta;E,F)$$

with the induced Fréchet topology.

Analogously, we define the space $M^{-\infty}_{V,P}(X;\mathbb{H}_\beta;E,F)$ of meromorphic Mellin symbols of order $-\infty$ with asymptotic type P. If $P = O$ is the empty asymptotic type the spaces are called holomorphic Volterra Mellin symbols.

5.2.2 Remark. Recall that the embedding

$$L^{\mu;\ell}_{V(cl)}(X;\mathbb{H}_\beta;E,F) \hookrightarrow L^{\mu;\ell}_{(cl)}(X;\Gamma_\beta;E,F)$$

is well-defined and continuous in view of Proposition 2.3.2 and the considerations in Section 3.2. Using Proposition 5.1.7 we conclude that also the embedding

$$M^{\mu;\ell}_{V,P(cl)}(X;\mathbb{H}_\beta;E,F) \hookrightarrow M^{\mu;\ell}_{P(cl)}(X;E,F)$$

is well-defined and continuous.

Moreover, the spaces of meromorphic Volterra Mellin symbols are independent of the right half-plane $\mathbb{H}_\beta$ as far as $\mathbb{H}_\beta \cap \pi_{\mathbb{C}}P = \emptyset$. This follows from (2.1.i) together with the considerations about the translation operator in parameter-dependent Volterra pseudodifferential operators from Sections 2.3 and 3.2 (see also Proposition 2.6.3).

In particular, holomorphic Volterra Mellin symbols are parameter-dependent Volterra pseudodifferential operators with respect to any right half-plane $\mathbb{H}_\beta \subseteq \mathbb{C}$. Therefore, we suppress the half-plane from the notation when we deal with holomorphic Volterra Mellin symbols.

5.2.3 Proposition. *Let $\mu, \mu' \in \mathbb{R}$, $\mu' \leq \mu$. Moreover, let $P \in \mathrm{As}\big(L^{-\infty}(X;E,F)\big)$ and $\beta \in \mathbb{R}$ with $\mathbb{H}_\beta \cap \pi_{\mathbb{C}}P = \emptyset$. The following identity holds algebraically and topologically:*

$$M^{\mu;\ell}_{P}(X;E,F) \cap L^{\mu';\ell}_{V}(X;\mathbb{H}_\beta;E,F) = M^{\mu';\ell}_{V,P}(X;\mathbb{H}_\beta;E,F).$$

For holomorphic Volterra Mellin symbols we have:

a) The complex derivative acts continuous in the spaces

$$\partial_z : M^{\mu;\ell}_{V,O(cl)}(X;E,F) \to M^{\mu-\ell;\ell}_{V,O(cl)}(X;E,F).$$

b) *For* $a \in M^{\mu;\ell}_{V,O(cl)}(X;E,F)$ *the following asymptotic expansion holds for* $a|_{\mathbb{H}_{\beta_0}}$ *in terms of* $a|_{\mathbb{H}_\beta}$ *which depends smoothly on* $(\beta_0,\beta) \in \mathbb{R} \times \mathbb{R}$*:*

$$a|_{\mathbb{H}_{\beta_0}} \underset{V}{\sim} \sum_{k=0}^{\infty} \frac{(\beta_0-\beta)^k}{k!} (\partial_z^k a)|_{\mathbb{H}_\beta}.$$

In the classical case we thus obtain for arbitrary $\beta_0,\beta \in \mathbb{R}$ *the following relationship for the parameter-dependent homogeneous principal symbol:*

$$\sigma_\psi^{\mu;\ell}(a|_{\mathbb{H}_{\beta_0}}) = \sigma_\psi^{\mu;\ell}(a|_{\mathbb{H}_\beta}).$$

c) *For* $s,\nu \in \mathbb{R}$ *with* $\nu \geq \mu$ *we have a continuous embedding*

$$M^{\mu;\ell}_{V,O}(X;E,F) \hookrightarrow \begin{cases} S^{\frac{\mu}{\ell}}_{V,O}(\mathbb{C};H^s(X,E),H^{s-\nu}(X,F)) & \nu \geq 0 \\ S^{\frac{\mu-\nu}{\ell}}_{V,O}(\mathbb{C};H^s(X,E),H^{s-\nu}(X,F)) & \nu \leq 0. \end{cases}$$

Proof. These assertions follow from Proposition 5.1.7 and Remark 5.2.2. The asymptotic expansion in b) follows as in the proof of Lemma 5.1.2 from Proposition 2.6.3. For c) see also Theorem 3.2.6. □

5.2.4 Theorem. a) *Let* $P,Q \in As\big(L^{-\infty}(X;E,F)\big)$ *such that* $\big(\pi_{\mathbb{C}}P \cup \pi_{\mathbb{C}}Q\big) \cap \mathbb{H}_\beta = \emptyset$*. Then pointwise addition as* $L^\mu(X;E,F)$*-valued operator functions on* $\mathbb{C} \setminus \big(\pi_{\mathbb{C}}P \cup \pi_{\mathbb{C}}Q\big)$ *induces a bilinear and continuous mapping*

$$+ : M^{\mu;\ell}_{V,P(cl)}(X;\mathbb{H}_\beta;E,F) \times M^{\mu;\ell}_{V,Q(cl)}(X;\mathbb{H}_\beta;E,F) \longrightarrow M^{\mu;\ell}_{V,R(cl)}(X;\mathbb{H}_\beta;E,F),$$

where $R \in As\big(L^{-\infty}(X;E,F)\big)$ *is (in general) determined from* P *and* Q *as in Theorem 5.1.8.*

b) *Let* $G \in Vect(X)$ *be another vector bundle, and let* $a \in M^{\mu;\ell}_{V,P(cl)}(X;\mathbb{H}_\beta;G,F)$*, as well as* $b \in M^{\mu';\ell}_{V,Q(cl)}(X;\mathbb{H}_\beta;E,G)$*. Then the pointwise composition (multiplication) as operator functions on* $\mathbb{C} \setminus \big(\pi_{\mathbb{C}}P \cup \pi_{\mathbb{C}}Q\big)$ *gives rise to an element* $ab \in M^{\mu+\mu';\ell}_{V,R(cl)}(X;\mathbb{H}_\beta;E,F)$ *with a resulting asymptotic type* $R \in As\big(L^{-\infty}(X;E,F)\big)$ *which is determined from* P *and* Q *as in Theorem 5.1.8.*

c) *For holomorphic Volterra Mellin symbols the multiplication as functions on* $\mathbb{C}$ *gives rise to a continuous bilinear mapping*

$$\cdot : M^{\mu;\ell}_{V,O(cl)}(X;G,F) \times M^{\mu';\ell}_{V,O(cl)}(X;E,G) \longrightarrow M^{\mu+\mu';\ell}_{V,O(cl)}(X;E,F).$$

Proof. These assertions follow from Theorem 5.1.8 and Section 3.2, see in particular Theorem 3.2.5 what the composition is concerned. □

5.2.5 Theorem. *The Mellin kernel cut-off operator (cf. Remark 5.1.9, Theorem 5.1.10) with respect to the weight $\gamma \in \mathbb{R}$ restricts to continuous bilinear mappings in the spaces*

$$H_\gamma : \begin{cases} C_B^\infty(\mathbb{R}_+) \times L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}_{\frac{1}{2}-\gamma}; E, F) \longrightarrow L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}_{\frac{1}{2}-\gamma}; E, F) \\ C_0^\infty(\mathbb{R}_+) \times L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}_{\frac{1}{2}-\gamma}; E, F) \longrightarrow M^{\mu;\ell}_{V,O(cl)}(X; E, F). \end{cases}$$

Given $\varphi \in C_B^\infty(\mathbb{R}_+)$ and $a \in L_V^{\mu;\ell}(X; \mathbb{H}_{\frac{1}{2}-\gamma}; E, F)$ we have the following asymptotic expansion of $H_\gamma(\varphi)a$ in terms of φ and a in the sense of Definition 3.2.7:

$$H_\gamma(\varphi)a \underset{V}{\sim} \sum_{k=0}^{\infty} \frac{1}{k!}(r\partial_r)^k \varphi(r)|_{r=1} \cdot \partial_z^k a.$$

If $\psi \in C_0^\infty(\mathbb{R}_+)$ such that $\psi \equiv 1$ near $r = 1$ then the operator $I - H_\gamma(\psi)$ is continuous in the spaces

$$I - H_\gamma(\psi) : L_V^{\mu;\ell}(X; \mathbb{H}_{\frac{1}{2}-\gamma}; E, F) \longrightarrow L_V^{-\infty}(X; \mathbb{H}_{\frac{1}{2}-\gamma}; E, F).$$

Proof. This follows as Remark 5.1.9, Theorem 5.1.10 analogously to Theorem 3.2.9 and Corollary 3.2.10 from Theorem 2.6.13. □

5.2.6 Corollary. *For $P \in As\big(L^{-\infty}(X; E, F)\big)$ such that $\pi_{\mathbb{C}} P \cap \mathbb{H}_\beta = \emptyset$ we have*

$$M^{\mu;\ell}_{V,P(cl)}(X; \mathbb{H}_\beta; E, F) = M^{\mu;\ell}_{V,O(cl)}(X; E, F) + M^{-\infty}_{V,P}(X; \mathbb{H}_\beta; E, F)$$

algebraically and topologically with the topology of the non-direct sum of Fréchet spaces on the right hand sides.

Proof. Let $\psi \in C_0^\infty(\mathbb{R}_+)$ such that $\psi \equiv 1$ near $r = 1$. In view of Theorem 5.2.5 and Theorem 5.2.4 we may write for $a \in M^{\mu;\ell}_{V,P(cl)}(X; \mathbb{H}_\beta; E, F)$ analogously to Corollary 5.1.11

$$a = H_{\frac{1}{2}-\beta}(\psi)a + \big(I - H_{\frac{1}{2}-\beta}(\psi)\big)a,$$

where $H_{\frac{1}{2}-\beta}(\psi)a \in M^{\mu;\ell}_{V,O(cl)}(X; E, F)$, and $\big(I - H_{\frac{1}{2}-\beta}(\psi)\big)a$ is an element of

$$L_V^{-\infty}(X; \mathbb{H}_\beta; E, F) \cap M^{\mu;\ell}_{V,P(cl)}(X; \mathbb{H}_\beta; E, F) = M^{-\infty}_{V,P}(X; \mathbb{H}_\beta; E, F)$$

due to Proposition 5.2.3 and Theorem 5.2.4. This provides a topological isomorphism as asserted. □

5.2.7 Definition. Let $P \in \mathrm{As}\big(L^{-\infty}(X; E, F)\big)$ such that $\mathbb{H}_\beta \cap \pi_{\mathbb{C}} P = \emptyset$. An element $a \in M^{\mu;\ell}_{V,P(cl)}(X; \mathbb{H}_\beta; E, F)$ is called *parabolic*, if $a|_{\mathbb{H}_\beta} \in L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}_\beta; E, F)$ is parabolic in the sense of Definition 3.2.17.

According to Corollary 5.2.6 we may write $a = a_0 + r$ with $a_0 \in M^{\mu;\ell}_{V,O(cl)}(X; E, F)$ and $r \in M^{-\infty}_{V,P}(X; \mathbb{H}_\beta; E, F)$. Thus we see that a is parabolic if and only if a_0 is parabolic, i. e., $a|_{\mathbb{H}_\beta}$ is parabolic in the sense of Definition 3.2.17. But the latter condition is independent of the particular choice of the half-plane $\mathbb{H}_\beta$ according to Proposition 5.2.3. In this sense we may speak about parabolicity for meromorphic Volterra Mellin symbols without referring to the particular half-plane involved.

5.2.8 Theorem. *An element $a \in M^{\mu;\ell}_{V,P(cl)}(X; \mathbb{H}_\beta; E, F)$ is parabolic if and only if there exists some $\beta' \geq \beta$ and $b \in M^{-\mu;\ell}_{V,Q(cl)}(X; \mathbb{H}_{\beta'}; F, E)$ such that $a \cdot b \equiv 1$ and $b \cdot a \equiv 1$, i. e., a is invertible as a meromorphic operator function with $a^{-1} = b \in M^{-\mu;\ell}_{V,Q(cl)}(X; \mathbb{H}_{\beta'}; F, E)$. If $a(z) : H^s(X, E) \to H^{s-\mu}(X, F)$ is invertible for some $s \in \mathbb{R}$ for all $z \in \mathbb{H}_\beta$ we may choose $\beta' = \beta$.*

Proof. Let $a \in M^{\mu;\ell}_{V,P(cl)}(X; \mathbb{H}_\beta; E, F)$ be parabolic. Then a is elliptic as an element of $M^{\mu;\ell}_{P(cl)}(X; E, F)$. Consequently we may apply Theorem 5.1.14 which shows that a is invertible as a meromorphic operator function with $a^{-1} = b \in M^{-\mu;\ell}_{Q(cl)}(X; F, E)$. For $a|_{\mathbb{H}_\beta} \in L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}_\beta; E, F)$ is parabolic we may apply Theorem 3.2.19 to $a|_{\mathbb{H}_\beta}$. This shows that b necessarily belongs to the space $M^{-\mu;\ell}_{V,Q(cl)}(X; \mathbb{H}_{\beta'}; F, E)$ with some $\beta' \geq \beta$, where we may choose $\beta' = \beta$ if $a(z)$ is pointwise invertible in the Sobolev spaces for $z \in \mathbb{H}_\beta$. This proves the theorem, for the converse is immediate. □

Mellin quantization

5.2.9 Remark. For later purposes let us note, that the Mellin quantization operator Q and its inverse $\tilde{Q}$, see Definition 2.6.15, are well-behaved within parameter-dependent Volterra operators. The proof is analogous to that of Theorem 3.2.9, and it is based on Theorem 2.6.16 in the abstract framework. Hence we restrict ourselves to state the result.

5.2.10 Theorem. *a) The operator Q from (2.6.vi) restricts to continuous bilinear mappings*

$$Q : \begin{cases} C_0^\infty(\mathbb{R}_+) \times L^{\mu;\ell}_{(cl)}(X; \mathbb{R}; E, F) \longrightarrow M^{\mu;\ell}_{O(cl)}(X; E, F) \\ C_0^\infty(\mathbb{R}_+) \times L^{\mu;\ell}_{V(cl)}(X; \mathbb{H}; E, F) \longrightarrow M^{\mu;\ell}_{V,O(cl)}(X; E, F). \end{cases}$$

Moreover, the asymptotic expansion result (2.6.viii) of $Q(\varphi, a)|_{\Gamma_{\frac{1}{2}-\gamma}}$, respectively $Q(\varphi, a)|_{\mathbb{H}_{\frac{1}{2}-\gamma}}$, in terms of a is valid in the sense of the Definitions 3.1.8 and 3.2.7:

$$Q(\varphi, a)(\frac{1}{2} - \gamma + i\tau) \underset{(V)}{\sim} \varphi(1) a(-\tau) + \sum_{k=1}^{\infty} \sum_{j=0}^{k} c_{k,j}(\varphi, \gamma)(-\tau)^j \left(\partial_\tau^{k+j} a\right)(-\tau)$$

for $\tau \in \mathbb{R}$, *respectively* $\tau \in \mathbb{H}_-$.

b) *The operator* $\tilde{Q}_\gamma$ *from (2.6.vii) restricts to continuous bilinear mappings*

$$\tilde{Q}_\gamma : \begin{cases} C_0^\infty(\mathbb{R}_+) \times L^{\mu;\ell}_{(cl)}(X;\Gamma_{\frac{1}{2}-\gamma};E,F) \longrightarrow M^{\mu;\ell}_{iO(cl)}(X;E,F) \\ C_0^\infty(\mathbb{R}_+) \times L^{\mu;\ell}_{V(cl)}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,F) \longrightarrow M^{\mu;\ell}_{V,iO(cl)}(X;E,F). \end{cases}$$

The spaces in the image are the multiples by the imaginary unit in the parameter of the ordinary spaces of meromorphic Mellin symbols.

The asymptotic expansion (2.6.ix) of $\tilde{Q}_\gamma(\psi,a)|_{\mathbb{R}}$, *respectively* $\tilde{Q}_\gamma(\psi,a)|_{\mathbb{H}}$, *in terms of* a *is valid in the sense of the Definitions 3.1.8 and 3.2.7:*

$$\tilde{Q}_\gamma(\psi,a)(\tau) \underset{(V)}{\sim} \psi(1)a(\frac{1}{2}-\gamma-i\tau)+\sum_{k=1}^{\infty}\sum_{j=0}^{k} d_{k,j}(\psi,\gamma)(-i\tau)^j\big(\partial_\tau^{k+j}a\big)(\frac{1}{2}-\gamma-i\tau)$$

for $\tau \in \mathbb{R}$, *respectively* $\tau \in \mathbb{H}$.

c) *For* $\varphi, \psi \in C_0^\infty(\mathbb{R}_+)$ *such that* $\varphi \equiv 1$ *and* $\psi \equiv 1$ *near* $r = 1$ *we have*

$$Q(\tilde{Q}_\gamma(a)) - a \in \begin{cases} L^{-\infty}(X;\Gamma_{\frac{1}{2}-\gamma};E,F) \\ L_V^{-\infty}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,F), \end{cases}$$

$$\tilde{Q}_\gamma(Q(a)) - a \in \begin{cases} L^{-\infty}(X;\mathbb{R};E,F) \\ L_V^{-\infty}(X;\mathbb{H};E,F). \end{cases}$$

5.3 Elements of the Mellin calculus

5.3.1 Remark. In this section we introduce subcalculi of the Mellin pseudodifferential calculi with operator-valued symbols from Sections 2.5 and 2.6, where the symbols are built upon parameter-dependent pseudodifferential operators on the manifold X. To this end recall from Theorem 3.1.5, Theorem 3.2.6, Proposition 5.1.7 and Proposition 5.2.3 the following embeddings:

$$\begin{aligned} L^{\mu;\ell}(X;\Gamma_\beta;E,F) &\hookrightarrow S^{\frac{\mu}{\ell}}(\Gamma_\beta;H^s(X,E),H^{s-\mu_+}(X,F)), \\ L_V^{\mu;\ell}(X;\mathbb{H}_\beta;E,F) &\hookrightarrow S_V^{\frac{\mu}{\ell}}(\mathbb{H}_\beta;H^s(X,E),H^{s-\mu_+}(X,F)), \\ M_O^{\mu;\ell}(X;E,F) &\hookrightarrow S_O^{\frac{\mu}{\ell}}(\mathbb{C};H^s(X,E),H^{s-\mu_+}(X,F)), \\ M_{V,O}^{\mu;\ell}(X;E,F) &\hookrightarrow S_{V,O}^{\frac{\mu}{\ell}}(\mathbb{C};H^s(X,E),H^{s-\mu_+}(X,F)). \end{aligned}$$

for $s, \mu \in \mathbb{R}$, where $\mu_+ := \max\{0,\mu\}$.

In particular, for every $s \in \mathbb{R}$ we have

$$
\begin{aligned}
&C_B^\infty((\mathbb{R}_+)^q, L^{\mu;\ell}(X;\Gamma_{\frac{1}{2}-\gamma};E,F)) \hookrightarrow \\
&\qquad\qquad M_\gamma S^{\frac{\mu}{\ell}}((\mathbb{R}_+)^q \times \Gamma_{\frac{1}{2}-\gamma}; H^s(X,E), H^{s-\mu_+}(X,F)),\\
&C_B^\infty((\mathbb{R}_+)^q, L_V^{\mu;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,F)) \hookrightarrow \\
&\qquad\qquad M_\gamma S_V^{\frac{\mu}{\ell}}((\mathbb{R}_+)^q \times \mathbb{H}_{\frac{1}{2}-\gamma}; H^s(X,E), H^{s-\mu_+}(X,F)),
\end{aligned}
$$

for $q = 1,2$, see Definition 2.5.1. This shows that for every double-symbol $a \in C_B^\infty(\mathbb{R}_+ \times \mathbb{R}_+, L^{\mu;\ell}(X;\Gamma_{\frac{1}{2}-\gamma};E,F))$ the associated Mellin pseudodifferential operator acts continuously in the spaces

$$
\operatorname{op}_M^\gamma(a) : \mathcal{T}_\gamma(X^\wedge,E) \longrightarrow \mathcal{T}_\gamma(X^\wedge,F),
$$

and left- or right-symbols a are uniquely determined by this action in view of Theorem 2.5.4 and the density of $\mathcal{T}_\gamma(X^\wedge,E)$ in $\mathcal{T}_\gamma(\mathbb{R}_+,H^t(X,E))$ for every $t \in \mathbb{R}$. As turns out, the classes of Mellin pseudodifferential operators based on such symbols remain preserved by the manipulations in the (holomorphic) Mellin calculus from Section 2.5 and Section 2.6.

5.3.2 Theorem. *Consider a double-symbol a belonging to one of the following spaces:*

i) $C_B^\infty(\mathbb{R}_+ \times \mathbb{R}_+, L_{(cl)}^{\mu;\ell}(X;\Gamma_{\frac{1}{2}-\gamma};E,F))$,

ii) $C_B^\infty(\mathbb{R}_+ \times \mathbb{R}_+, L_{V(cl)}^{\mu;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,F))$,

iii) $C_B^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, M_{O(cl)}^{\mu;\ell}(X;E,F))$,

iv) $C_B^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, M_{V,O(cl)}^{\mu;\ell}(X;E,F))$.

Then the corresponding left- and right-symbols a_L and a_R obtained from Theorem 2.5.4 in the cases i) and ii), respectively from Theorem 2.6.7 in the cases iii) and iv), belong to the spaces

i) $C_B^\infty(\mathbb{R}_+, L_{(cl)}^{\mu;\ell}(X;\Gamma_{\frac{1}{2}-\gamma};E,F))$,

ii) $C_B^\infty(\mathbb{R}_+, L_{V(cl)}^{\mu;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,F))$,

iii) $C_B^\infty(\overline{\mathbb{R}}_+, M_{O(cl)}^{\mu;\ell}(X;E,F))$,

iv) $C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O(cl)}^{\mu;\ell}(X;E,F))$.

Moreover, the asymptotic expansions of a_L and a_R in terms of a from Theorem 2.5.4 and 2.6.7 are valid within these smaller classes (see also the Definitions 3.1.8 and 3.2.7).

Proof. From the correspondence (3.1.iv) and (3.1.iii) we conclude that the proof of the theorem reduces to consider local symbols and global remainders of order $-\infty$, since we have explicit oscillatory integral formulas for the left- and right-symbol at hand. Keeping in mind the characterization of the remainders on the manifold as operator-valued symbols (cf. Definition 3.1.2 and 3.2.2) we see that the cases i) and ii) follow from Theorem 2.5.4, while iii) and iv) follow from Theorem 2.6.7. Note that the global asymptotic expansions on the manifold follow from the corresponding asymptotic expansions on the level of local symbols, which are subject to the theorems in the abstract framework. □

5.3.3 Theorem. *a) Let a and b be given, where*

i) $a \in C_B^\infty(\mathbb{R}_+, L_{(cl)}^{\mu;\ell}(X;\Gamma_{\frac{1}{2}-\gamma};F,G))$, $b \in C_B^\infty(\mathbb{R}_+, L_{(cl)}^{\mu';\ell}(X;\Gamma_{\frac{1}{2}-\gamma};E,F))$,

ii) $a \in C_B^\infty(\mathbb{R}_+, L_{V(cl)}^{\mu;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};F,G))$, $b \in C_B^\infty(\mathbb{R}_+, L_{V(cl)}^{\mu';\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,F))$,

iii) $a \in C_B^\infty(\overline{\mathbb{R}}_+, M_{O(cl)}^{\mu;\ell}(X;F,G))$, $b \in C_B^\infty(\overline{\mathbb{R}}_+, M_{O(cl)}^{\mu';\ell}(X;E,F))$,

iv) $a \in C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O(cl)}^{\mu;\ell}(X;F,G))$, $b \in C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O(cl)}^{\mu';\ell}(X;E,F))$.

Then the Leibniz-product $a\#b$ (cf. Theorem 2.5.6 and Theorem 2.6.9) belongs to

i) $C_B^\infty(\mathbb{R}_+, L_{(cl)}^{\mu+\mu';\ell}(X;\Gamma_{\frac{1}{2}-\gamma};E,G))$,

ii) $C_B^\infty(\mathbb{R}_+, L_{V(cl)}^{\mu+\mu';\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,G))$,

iii) $C_B^\infty(\overline{\mathbb{R}}_+, M_{O(cl)}^{\mu+\mu';\ell}(X;E,G))$,

iv) $C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O(cl)}^{\mu+\mu';\ell}(X;E,G))$,

and the asymptotic expansions (2.5.iii), (2.5.iv) and (2.6.i) hold within the smaller classes. The formulas for the conormal symbols of the composition in the cases iii) and iv) are inherited from the abstract framework; see Definition 2.6.10, in particular the defining relation (2.6.iii) and (2.6.iv). In the classical case we conclude that the homogeneous principal symbol of the Leibniz-product is given as the product of the homogeneous principal symbols of a and b.

b) Let

i) $a \in C_B^\infty(\mathbb{R}_+, L_{(cl)}^{\mu;\ell}(X;\Gamma_{\frac{n+1}{2}-\gamma};E,F))$,

ii) $a \in C_B^\infty(\overline{\mathbb{R}}_+, M_{O(cl)}^{\mu;\ell}(X;E,F))$, *and* $A = op_M^{\gamma-\frac{n}{2}}(a)$. *Then the formal adjoint with respect to the* $r^{-\frac{n}{2}}L^2$*-inner product is given as* $A^* = op_M^{-\gamma-\frac{n}{2}}(a^*)$ *with the symbol*

$$a^*(r,z) = \big(a(r', n+1-\overline{z})^{(*)}\big)_L.$$

Here $^{(*)}$ *denotes the formal adjoint with respect to the* L^2*-inner product on the manifold. This shows that*

i) $a^* \in C_B^\infty(\mathbb{R}_+, L_{(cl)}^{\mu;\ell}(X;\Gamma_{\frac{n+1}{2}+\gamma}; F, E))$,

ii) $a^* \in C_B^\infty(\overline{\mathbb{R}}_+, M_{O(cl)}^{\mu;\ell}(X;F,E))$.

Moreover, the following asymptotic expansion of a^* *in terms of* a *is valid:*

$$a^*\big(r, \frac{n+1}{2}+\gamma+i\tau\big) \sim \sum_{k=0}^{\infty} \frac{1}{k!}(-1)^k D_\tau^k(-r\partial_r)^k a\big(r, \frac{n+1}{2}-\gamma+i\tau\big)^{(*)}.$$

In the classical case we thus obtain the following formula for the (anisotropic) homogeneous principal symbol:

$$\sigma_\psi^{\mu;\ell}(A^*) = \sigma_\psi^{\mu;\ell}(A)^*.$$

In case of ii) we have the following relations for the conormal symbols:

$$\sigma_M^{-k}\big(A^*\big)(z) = \sigma_M^{-k}\big(A\big)(n+1-k-\overline{z})^{(*)}$$

for $k \in \mathbb{N}_0$.

5.3.4 Proposition. *Let* a *belong to one of the following spaces*

i) $C_B^\infty(\mathbb{R}_+ \times \mathbb{R}_+, L^{\mu;\ell}(X;\Gamma_{\frac{1}{2}-\gamma}; E, F))$,

ii) $C_B^\infty(\mathbb{R}_+ \times \mathbb{R}_+, L_V^{\mu;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma}; E, F))$,

iii) $C_B^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, M_O^{\mu;\ell}(X;E,F))$,

iv) $C_B^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, M_{V,O}^{\mu;\ell}(X;E,F))$,

and assume that $a(r,r') \equiv 0$ *for* $\left|\frac{r}{r'}-1\right| < \varepsilon$ *for a sufficiently small* $\varepsilon > 0$. *Then* $op_M^\gamma(a) = op_M^\gamma(c)$ *with a symbol* c *in*

i) $C_B^\infty(\mathbb{R}_+, L^{-\infty}(X;\Gamma_{\frac{1}{2}-\gamma}; E, F))$,

ii) $C_B^\infty(\mathbb{R}_+, L_V^{-\infty}(X;\mathbb{H}_{\frac{1}{2}-\gamma}; E, F))$,

iii) $C_B^\infty(\overline{\mathbb{R}}_+, M_O^{-\infty}(X;E,F))$, *iv)* $C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O}^{-\infty}(X; E, F))$.

5.3.5 Proposition. *Let* $a \in C_B^\infty(\mathbb{R}_+, L_V^{\mu;\ell}(X; \mathbb{H}_{\frac{1}{2}-\gamma}; E, F))$. *Then* $op_M^\gamma(a)$ *restricts for every* $r_0 \in \mathbb{R}_+$ *to a continuous operator*

$$op_M^\gamma(a) : \mathcal{T}_{\gamma,0}((0, r_0), C^\infty(X, E)) \longrightarrow \mathcal{T}_{\gamma,0}((0, r_0), C^\infty(X, F)).$$

Proof. This follows from Theorem 2.5.8. □

5.3.6 Theorem. *Let* $a \in C_B^\infty(\mathbb{R}_+, L^{\mu;\ell}(X; \Gamma_{\frac{n+1}{2}-\gamma}; E, F))$. *Then* $op_M^{\gamma-\frac{n}{2}}(a)$ *extends for every* $s, t \in \mathbb{R}$ *to a continuous operator*

$$op_M^{\gamma-\frac{n}{2}}(a) : \mathcal{H}^{(s,t),\gamma;\ell}(X^\wedge, E) \longrightarrow \mathcal{H}^{(s-\mu,t),\gamma;\ell}(X^\wedge, F).$$

If $a \in C_B^\infty(\mathbb{R}_+, L_V^{\mu;\ell}(X; \mathbb{H}_{\frac{n+1}{2}-\gamma}; E, F))$, *then* $op_M^{\gamma-\frac{n}{2}}(a)$ *restricts for every* $r_0 \in \mathbb{R}_+$ *to a continuous operator*

$$op_M^{\gamma-\frac{n}{2}}(a) : \mathcal{H}_0^{(s,t),\gamma;\ell}((0, r_0] \times X, E) \longrightarrow \mathcal{H}_0^{(s-\mu,t),\gamma;\ell}((0, r_0] \times X, F).$$

Proof. Let

$$R^{-s}(z) \in L^{-s;\ell}(X; \Gamma_{\frac{n+1}{2}-\gamma}; E, E),$$
$$\tilde{R}^{s-\mu}(z) \in L^{s-\mu;\ell}(X; \Gamma_{\frac{n+1}{2}-\gamma}; F, F),$$

be parameter-dependent reductions of orders according to Theorem 3.1.12. In view of Definition 4.2.3 the asserted boundedness is equivalent to the continuity of

$$\mathrm{op}_M^{\gamma-\frac{n}{2}}(\tilde{R}^{s-\mu} \# a \# R^{-s}) : \mathcal{H}^{(0,t),\gamma;\ell}(X^\wedge, E) \longrightarrow \mathcal{H}^{(0,t),\gamma;\ell}(X^\wedge, F).$$

Recall that $\mathcal{H}^{(0,t),\gamma;\ell}(X^\wedge, E) = L^{2,\gamma-\frac{n}{2}}(\mathbb{R}_+, H^t(X, E))$, and that $\tilde{R}^{s-\mu} \# a \# R^{-s}$ belongs to

$$C_B^\infty(\mathbb{R}_+, L^{0;\ell}(X; \Gamma_{\frac{n+1}{2}-\gamma}; E, F)) \hookrightarrow \mathcal{M}_{\gamma-\frac{n}{2}} S^0(\mathbb{R}_+ \times \Gamma_{\frac{n+1}{2}-\gamma}; H^t(X, E), H^t(X, F)).$$

Consequently, we obtain the desired boundedness from Theorem 2.5.11. The second assertion follows from Proposition 5.3.5 and Proposition 4.2.12. □

Ellipticity and Parabolicity

5.3.7 Remark. Let

$$a \in \begin{cases} C_B^\infty(\mathbb{R}_+, L_{(cl)}^{\mu;\ell}(X; \Gamma_{\frac{1}{2}-\gamma}; E, F)) \\ C_B^\infty(\mathbb{R}_+, L_{V(cl)}^{\mu;\ell}(X; \mathbb{H}_{\frac{1}{2}-\gamma}; E, F)). \end{cases}$$

According to Remark 3.1.4, 3.2.4 we associate to a a complete symbol $(a_1,\dots,a_N)$ subordinate to the given covering of X from Notation 3.1.1. Thus we have

$$(a_1,\dots,a_N)\in\begin{cases} C_B^\infty(\mathbb{R}_+,\bigtimes_{j=1}^N S^{\mu;\ell}_{(cl)}(\mathbb{R}^n\times\mathbb{R}^n\times\Gamma_{\frac12-\gamma};\mathbb{C}^{N_-},\mathbb{C}^{N_+}))\\ C_B^\infty(\mathbb{R}_+,\bigtimes_{j=1}^N S^{\mu;\ell}_{V(cl)}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{H}_{\frac12-\gamma};\mathbb{C}^{N_-},\mathbb{C}^{N_+})).\end{cases}$$

5.3.8 Definition. a) Let $a\in C_B^\infty(\mathbb{R}_+,L^{\mu;\ell}_{(cl)}(X;\Gamma_{\frac12-\gamma};E,F))$. Let either $I=\mathbb{R}_+$ or $I=(0,r_0]$ with $r_0\in\mathbb{R}_+$. We call a *elliptic* (on I), if for the complete symbol $(a_1,\dots,a_N)$ we have:

For every $j=1,\dots,N$ there exists $R>0$ and a neighbourhood $U(\kappa_j(\mathrm{supp}\psi_j))$ such that for $(\xi,\tau)\in\mathbb{R}^n\times\mathbb{R}$ with $|(\xi,\tau)|_\ell\geq R$ and all $r\in I$ and $x\in U(\kappa_j(\mathrm{supp}\psi_j))$ there exists $a_j(r,x,\xi,\frac12-\gamma+i\tau)^{-1}$ with

$$\|a_j\big(r,x,\xi,\frac12-\gamma+i\tau\big)^{-1}\|=O\big(\langle\xi,\tau\rangle_\ell^{-\mu}\big)$$

as $|(\xi,\tau)|_\ell\to\infty$, uniformly for $r\in I$ and $x\in U(\kappa_j(\mathrm{supp}\psi_j))$.

In the classical case this condition is equivalent to require that the homogeneous principal symbol

$$\sigma_\psi^{\mu;\ell}(a)\in C_B^\infty(\mathbb{R}_+,S^{(\mu;\ell)}((T^*X\times\Gamma_{\frac12-\gamma})\setminus 0,\mathrm{Hom}(\pi^*E,\pi^*F)))$$

is invertible (on I), and for the inverse we have

$$\sup\{\|\sigma_\psi^{\mu;\ell}(a)(r,\xi_x,\tau)^{-1}\|;\ r\in I,\ \big(|\xi_x|_x^{2\ell}+|\tau|^2\big)^{\frac{1}{2\ell}}=1\}<\infty.$$

Note that we identified $\Gamma_{\frac12-\gamma}$ with $\mathbb{R}$ via $\tau=\mathrm{Im}(z)$.

b) Let $a\in C_B^\infty(\overline{\mathbb{R}}_+,M^{\mu;\ell}_{O(cl)}(X;E,F))$. Let either $I=\overline{\mathbb{R}}_+$ or $I=[0,r_0]$ with $r_0\in\mathbb{R}_+$. We call a *elliptic* (on I), if there exists $\gamma\in\mathbb{R}$ such that the conditions in a) hold with the interval I.

c) Let $a\in C_B^\infty(\mathbb{R}_+,L^{\mu;\ell}_{V(cl)}(X;\mathbb{H}_{\frac12-\gamma};E,F))$. Let either $I=\mathbb{R}_+$ or $I=(0,r_0]$ with $r_0\in\mathbb{R}_+$. We call a *parabolic* (on I) if the following condition is fulfilled:

For every $j=1,\dots,N$ there exists $R>0$ and a neighbourhood $U(\kappa_j(\mathrm{supp}\psi_j))$ such that for $(\xi,\zeta)\in\mathbb{R}^n\times\mathbb{H}_0$ with $|(\xi,\zeta)|_\ell\geq R$ and all $r\in I$ and $x\in U(\kappa_j(\mathrm{supp}\psi_j))$ there exists $a_j(r,x,\xi,\frac12-\gamma+\zeta)^{-1}$ with

$$\|a_j\big(r,x,\xi,\frac12-\gamma+\zeta\big)^{-1}\|=O\big(\langle\xi,\zeta\rangle_\ell^{-\mu}\big)$$

as $|(\xi,\zeta)|_\ell\to\infty$, uniformly for $r\in I$ and $x\in U(\kappa_j(\mathrm{supp}\psi_j))$.

In the classical case this condition is equivalent to require that the homogeneous principal symbol

$$\sigma_{\psi}^{\mu;\ell}(a) \in C_B^\infty(\mathbb{R}_+, S_V^{(\mu;\ell)}((T^*X \times \mathbb{H}_{\frac{1}{2}-\gamma}) \setminus 0, \mathrm{Hom}(\pi^*E, \pi^*F)))$$

is invertible (on I), and for the inverse we have

$$\sup\{\|\sigma_{\psi}^{\mu;\ell}(a)(r,\xi_x,\zeta)^{-1}\|;\ r \in I,\ \left(|\xi_x|_x^{2\ell} + |\zeta|^2\right)^{\frac{1}{2\ell}} = 1\} < \infty.$$

Here we identified $\mathbb{H}_{\frac{1}{2}-\gamma}$ with $\mathbb{H}_0$ via translation.

d) Let $a \in C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O(cl)}^{\mu;\ell}(X;E,F))$. Let either $I = \overline{\mathbb{R}}_+$ or $I = [0, r_0]$ with $r_0 \in \mathbb{R}_+$. We call a *parabolic* (on I), if there exists $\gamma \in \mathbb{R}$ such that the conditions in c) hold with the interval I.

If $I = \mathbb{R}_+$ in a), c) or $I = \overline{\mathbb{R}}_+$ in b), d) we say that a is elliptic, respectively parabolic, without refering to the interval.

5.3.9 Lemma. *a) Let $a \in C_B^\infty(\mathbb{R}_+, L_{(cl)}^{\mu;\ell}(X;\Gamma_{\frac{1}{2}-\gamma};E,F))$ and $I = \mathbb{R}_+$ or $I = (0, r_0]$ with $r_0 \in \mathbb{R}_+$. Then the following are equivalent:*

- *a is elliptic (on I).*
- *There exists $b \in C_B^\infty(\mathbb{R}_+, L_{(cl)}^{-\mu;\ell}(X;\Gamma_{\frac{1}{2}-\gamma};F,E))$ such that $ab-1$ and $ba-1$ coincide in a neighbourhood of I with symbols belonging to $C_B^\infty(\mathbb{R}_+, L_{(cl)}^{-1;\ell}(X;\Gamma_{\frac{1}{2}-\gamma};F,F))$ and $C_B^\infty(\mathbb{R}_+, L_{(cl)}^{-1;\ell}(X;\Gamma_{\frac{1}{2}-\gamma};E,E))$, respectively.*

b) Let $a \in C_B^\infty(\overline{\mathbb{R}}_+, M_{O(cl)}^{\mu;\ell}(X;E,F))$ and $I = \overline{\mathbb{R}}_+$ or $I = [0, r_0]$ with $r_0 \in \mathbb{R}_+$. Then the following are equivalent:

- *a is elliptic (on I).*
- *There exists $b \in C_B^\infty(\overline{\mathbb{R}}_+, M_{O(cl)}^{-\mu;\ell}(X;F,E))$ such that $ab-1$ and $ba-1$ coincide in a neighbourhood of I with symbols in $C_B^\infty(\overline{\mathbb{R}}_+, M_{O(cl)}^{-1;\ell}(X;F,F))$ and $C_B^\infty(\overline{\mathbb{R}}_+, M_{O(cl)}^{-1;\ell}(X;E,E))$, respectively.*

c) Let $a \in C_B^\infty(\mathbb{R}_+, L_{V(cl)}^{\mu;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,F))$ and $I = \mathbb{R}_+$ or $I = (0, r_0]$ with $r_0 \in \mathbb{R}_+$. Then the following are equivalent:

- *a is parabolic (on I).*
- *There exists $b \in C_B^\infty(\mathbb{R}_+, L_{V(cl)}^{-\mu;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};F,E))$ such that $ab-1$ and $ba-1$ coincide in a neighbourhood of I with symbols belonging to $C_B^\infty(\mathbb{R}_+, L_{V(cl)}^{-1;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};F,F))$ and $C_B^\infty(\mathbb{R}_+, L_{V(cl)}^{-1;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,E))$, respectively.*

d) Let $a \in C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O(cl)}^{\mu;\ell}(X;E,F))$ *and* $I = \overline{\mathbb{R}}_+$ *or* $I = [0, r_0]$ *with* $r_0 \in \mathbb{R}_+$*. Then the following are equivalent:*

- *a is parabolic (on I).*
- *There exists* $b \in C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O(cl)}^{-\mu;\ell}(X;F,E))$ *such that* $ab-1$ *and* $ba-1$ *coincide in a neighbourhood of* I *with symbols in* $C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O(cl)}^{-1;\ell}(X;F,F))$ *and* $C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O(cl)}^{-1;\ell}(X;E,E))$*, respectively.*

In particular, the notions of ellipticity and parabolicity on an interval I from Definition 5.3.8 are well-defined, i. e. independent of the choice of the data on X and the subordinated complete symbol, as well as independent of the choice of the particular weight line or right half-plane for analytic symbols.

Proof. a) follows from Theorem 3.1.10 and c) follows from Theorem 3.2.18. For the proof of b) and d) note first that the existence of symbols b with the asserted properties is sufficient for the ellipticity or parabolicity of a on the interval I in view of Theorem 3.1.10 and Theorem 3.2.18. Now let $a \in C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O(cl)}^{\mu;\ell}(X;E,F))$ be parabolic on the interval I. From Theorem 3.2.18 we obtain a symbol $\tilde{b} \in C_B^\infty(\overline{\mathbb{R}}_+, L_{V(cl)}^{-\mu;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};F,E))$ such that $a\tilde{b}-1$ and $\tilde{b}a-1$ coincide in a neighbourhood of I with symbols belonging to $C_B^\infty(\overline{\mathbb{R}}_+, L_{V(cl)}^{-1;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};F,F))$ and $C_B^\infty(\overline{\mathbb{R}}_+, L_{V(cl)}^{-1;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,E))$, respectively, for some $\gamma \in \mathbb{R}$. Let $\varphi \in C_0^\infty(\mathbb{R}_+)$ such that $\varphi \equiv 1$ near $r = 1$, and define $b := H_\gamma(\varphi)\tilde{b}$ with the Mellin kernel cut-off operator H_γ. Then we obtain from Theorem 5.2.5 that $b \in C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O(cl)}^{-\mu;\ell}(X;F,E))$, and we have $b-\tilde{b} \in C_B^\infty(\overline{\mathbb{R}}_+, L_{V(cl)}^{-\infty}(X;\mathbb{H}_{\frac{1}{2}-\gamma};F,E))$. Consequently, $ab-1$ and $ba-1$ coincide in a neighbourhood of I with symbols in $C_B^\infty(\overline{\mathbb{R}}_+, L_{V(cl)}^{-1;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};F,F))$ respectively $C_B^\infty(\overline{\mathbb{R}}_+, L_{V(cl)}^{-1;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,E))$, but both $ab-1$ and $ba-1$ are analytic symbols. Thus we obtain from Proposition 5.2.3 the desired assertion which completes the proof of d). The proof of b) is analogous. □

5.3.10 Theorem. *a) Let* $a \in C_B^\infty(\mathbb{R}_+, L_{V(cl)}^{\mu;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,F))$*. The following are equivalent:*

- *a is parabolic.*
- *There exists a symbol* $b \in C_B^\infty(\mathbb{R}_+, L_{V(cl)}^{-\mu;\ell}(X;\mathbb{H}_{\frac{1}{2}-\gamma};F,E))$ *such that*

$$op_M^\gamma(a) op_M^\gamma(b) = 1 + op_M^\gamma(r_R),$$
$$op_M^\gamma(b) op_M^\gamma(a) = 1 + op_M^\gamma(r_L),$$

with remainders

$$r_R \in C_B^\infty(\mathbb{R}_+, L_V^{-\infty}(X;\mathbb{H}_{\frac{1}{2}-\gamma};F,F)),$$
$$r_L \in C_B^\infty(\mathbb{R}_+, L_V^{-\infty}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,E)).$$

Let $I = (0, r_0]$ for some $r_0 \in \mathbb{R}_+$. Then the following are equivalent:

- *a is parabolic on I.*
- *There exists a symbol $b \in C_B^\infty(\mathbb{R}_+, L_{V(cl)}^{-\mu;\ell}(X; \mathbb{H}_{\frac{1}{2}-\gamma}; F, E))$ as well as cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ with $\omega, \tilde{\omega} \equiv 1$ on I, such that*

$$\omega\big(op_M^\gamma(a)op_M^\gamma(b) - 1\big)\tilde{\omega} = op_M^\gamma(r_R),$$
$$\omega\big(op_M^\gamma(b)op_M^\gamma(a) - 1\big)\tilde{\omega} = op_M^\gamma(r_L),$$

with remainders

$$r_R \in C_B^\infty(\mathbb{R}_+, L_V^{-\infty}(X; \mathbb{H}_{\frac{1}{2}-\gamma}; F, F)),$$
$$r_L \in C_B^\infty(\mathbb{R}_+, L_V^{-\infty}(X; \mathbb{H}_{\frac{1}{2}-\gamma}; E, E)).$$

b) Let $a \in C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O(cl)}^{\mu;\ell}(X; E, F))$. The following are equivalent:

- *a is parabolic.*
- *There exists $b \in C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O(cl)}^{-\mu;\ell}(X; F, E))$ such that*

$$op_M^\gamma(a)op_M^\gamma(b) = 1 + op_M^\gamma(r_R),$$
$$op_M^\gamma(b)op_M^\gamma(a) = 1 + op_M^\gamma(r_L),$$

for some (every) $\gamma \in \mathbb{R}$, where

$$r_R \in C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O}^{-\infty}(X; F, F)),$$
$$r_L \in C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O}^{-\infty}(X; E, E)).$$

Let $I = [0, r_0]$ for some $r_0 \in \mathbb{R}_+$. Then the following are equivalent:

- *a is parabolic on I.*
- *There exists $b \in C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O(cl)}^{-\mu;\ell}(X; F, E))$ as well as cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r = 0$ with $\omega, \tilde{\omega} \equiv 1$ on I such that*

$$\omega\big(op_M^\gamma(a)op_M^\gamma(b) - 1\big)\tilde{\omega} = op_M^\gamma(r_R),$$
$$\omega\big(op_M^\gamma(b)op_M^\gamma(a) - 1\big)\tilde{\omega} = op_M^\gamma(r_L),$$

for some (every) $\gamma \in \mathbb{R}$, where

$$r_R \in C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O}^{-\infty}(X; F, F)),$$
$$r_L \in C_B^\infty(\overline{\mathbb{R}}_+, M_{V,O}^{-\infty}(X; E, E)).$$

Proof. In view of Theorem 5.3.2, Theorem 5.3.3 and Lemma 5.3.9 the above conditions in a) and b) are clearly sufficient for the parabolicity of the symbols (on the interval I). Now assume that

$$a \in \begin{cases} C_B^\infty(\mathbb{R}_+, L^{\mu;\ell}_{V(cl)}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,F)) \\ C_B^\infty(\overline{\mathbb{R}}_+, M^{\mu;\ell}_{V,O(cl)}(X;E,F)) \end{cases}$$

is parabolic. From Lemma 5.3.9 and Theorem 5.3.3 we conclude that there exists

$$\tilde{b} \in \begin{cases} C_B^\infty(\mathbb{R}_+, L^{-\mu;\ell}_{V(cl)}(X;\mathbb{H}_{\frac{1}{2}-\gamma};F,E)) \\ C_B^\infty(\overline{\mathbb{R}}_+, M^{-\mu;\ell}_{V,O(cl)}(X;F,E)) \end{cases}$$

such that $a\#\tilde{b} = 1 - \tilde{r}_R$ and $\tilde{b}\#a = 1 - \tilde{r}_L$ with

$$\tilde{r}_R \in \begin{cases} C_B^\infty(\mathbb{R}_+, L^{-1;\ell}_{V(cl)}(X;\mathbb{H}_{\frac{1}{2}-\gamma};F,F)) \\ C_B^\infty(\overline{\mathbb{R}}_+, M^{-1;\ell}_{V,O(cl)}(X;F,F)), \end{cases}$$

$$\tilde{r}_L \in \begin{cases} C_B^\infty(\mathbb{R}_+, L^{-1;\ell}_{V(cl)}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,E)) \\ C_B^\infty(\overline{\mathbb{R}}_+, M^{-1;\ell}_{V,O(cl)}(X;E,E)). \end{cases}$$

Now choose $\hat{r}_R, \hat{r}_L$ such that

$$\left.\begin{matrix} C_B^\infty(\mathbb{R}_+, L^{0;\ell}_{V(cl)}(X;\mathbb{H}_{\frac{1}{2}-\gamma};F,F)) \\ C_B^\infty(\overline{\mathbb{R}}_+, M^{0;\ell}_{V,O(cl)}(X;F,F)) \end{matrix}\right\} \ni \hat{r}_R \underset{V}{\sim} \sum_{j=0}^{\infty} \#^{(j)}\tilde{r}_R,$$

$$\left.\begin{matrix} C_B^\infty(\mathbb{R}_+, L^{0;\ell}_{V(cl)}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,E)) \\ C_B^\infty(\overline{\mathbb{R}}_+, M^{0;\ell}_{V,O(cl)}(X;E,E)) \end{matrix}\right\} \ni \hat{r}_L \underset{V}{\sim} \sum_{j=0}^{\infty} \#^{(j)}\tilde{r}_L.$$

These asymptotic expansions are to be carried out within the corresponding symbol classes. Recall that the terms in the asymptotic expansions are well-defined in the corresponding symbol classes with decreasing orders by Theorem 5.3.3, and that the classes themselves are closed with respect to taking asymptotic sums by Theorem 2.6.14, Theorem 3.2.12, Theorem 5.2.5 and Proposition 5.2.3. Now we see that

$$a\#\big(\tilde{b}\#\hat{r}_R\big) - 1 \in \begin{cases} C_B^\infty(\mathbb{R}_+, L^{-\infty}_{V}(X;\mathbb{H}_{\frac{1}{2}-\gamma};F,F)) \\ C_B^\infty(\overline{\mathbb{R}}_+, M^{-\infty}_{V,O}(X;F,F)), \end{cases}$$

$$\big(\hat{r}_L\#\tilde{b}\big)\#a - 1 \in \begin{cases} C_B^\infty(\mathbb{R}_+, L^{-\infty}_{V}(X;\mathbb{H}_{\frac{1}{2}-\gamma};E,E)) \\ C_B^\infty(\overline{\mathbb{R}}_+, M^{-\infty}_{V,O}(X;E,E)), \end{cases}$$

and consequently the same relations hold with either $b := \tilde{b}\#\hat{r}_R$ or $b := \hat{r}_L\#\tilde{b}$. This completes the proof of the first assertions in a) and b).

Now we consider the case of a finite interval I. Let a be parabolic on I. Choose $\tilde{b}$ satisfying the conditions in Lemma 5.3.9, and let $\tilde{r}_L$ and $\tilde{r}_R$ be (classical holomorphic) Volterra symbols of order -1, such that $a\#\tilde{b} = 1+\tilde{r}_R$ and $\tilde{b}\#a = 1+\tilde{r}_L$ in a neighbourhood of I. Observe that $1+\tilde{r}_R$ and $1+\tilde{r}_L$ are parabolic, and consequently we obtain from the already proven part of the theorem that there exist (classical holomorphic) Volterra symbols $\hat{r}_L$ and $\hat{r}_R$ of order -1, such that $(1+\hat{r}_L)\#(1+\tilde{r}_L)-1$ and $(1+\tilde{r}_R)\#(1+\hat{r}_R)-1$ are of order $-\infty$. Now we see that if we set either $b := (1+\hat{r}_L)\#\tilde{b}$ or $b := \tilde{b}\#(1+\hat{r}_R)$, and choose cut-off functions $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ supported sufficiently close to the interval I, we obtain the second assertions in a) and b). This finishes the proof of the theorem. □

5.3.11 Theorem. *a) Let* $a \in C_B^\infty(\mathbb{R}_+, L^{\mu;\ell}_{(cl)}(X;\Gamma_{\frac{1}{2}-\gamma};E,F))$. *The following are equivalent:*

- *a is elliptic.*
- *There exists a symbol* $b \in C_B^\infty(\mathbb{R}_+, L^{-\mu;\ell}_{(cl)}(X;\Gamma_{\frac{1}{2}-\gamma};F,E))$, *and symbols* $r_R \in C_B^\infty(\mathbb{R}_+, L^{-\infty}(X;\Gamma_{\frac{1}{2}-\gamma};F,F))$, $r_L \in C_B^\infty(\mathbb{R}_+, L^{-\infty}(X;\Gamma_{\frac{1}{2}-\gamma};E,E))$, *such that*

$$op_M^\gamma(a)op_M^\gamma(b) = 1 + op_M^\gamma(r_R),$$
$$op_M^\gamma(b)op_M^\gamma(a) = 1 + op_M^\gamma(r_L).$$

Let $I = (0, r_0]$ *for some* $r_0 \in \mathbb{R}_+$. *Then the following are equivalent:*

- *a is elliptic on I.*
- *There exists a symbol* $b \in C_B^\infty(\mathbb{R}_+, L^{-\mu;\ell}_{(cl)}(X;\Gamma_{\frac{1}{2}-\gamma};F,E))$, *and symbols* $r_R \in C_B^\infty(\mathbb{R}_+, L^{-\infty}(X;\Gamma_{\frac{1}{2}-\gamma};F,F))$, $r_L \in C_B^\infty(\mathbb{R}_+, L^{-\infty}(X;\Gamma_{\frac{1}{2}-\gamma};E,E))$, *as well as cut-off functions* $\omega, \tilde{\omega} \in C_0^\infty(\overline{\mathbb{R}}_+)$ *near* $r=0$ *with* $\omega, \tilde{\omega} \equiv 1$ *on* I, *such that*

$$\omega\big(op_M^\gamma(a)op_M^\gamma(b) - 1\big)\tilde{\omega} = op_M^\gamma(r_R),$$
$$\omega\big(op_M^\gamma(b)op_M^\gamma(a) - 1\big)\tilde{\omega} = op_M^\gamma(r_L).$$

b) Let $a \in C_B^\infty(\overline{\mathbb{R}}_+, M^{\mu;\ell}_{O(cl)}(X;E,F))$. *The following are equivalent:*

- *a is elliptic.*
- *There exists a symbol* $b \in C_B^\infty(\overline{\mathbb{R}}_+, M^{-\mu;\ell}_{O(cl)}(X;F,E))$ *and remainders* $r_R \in C_B^\infty(\overline{\mathbb{R}}_+, M_O^{-\infty}(X;F,F))$, $r_L \in C_B^\infty(\overline{\mathbb{R}}_+, M_O^{-\infty}(X;E,E))$, *such that*

$$op_M^\gamma(a)op_M^\gamma(b) = 1 + op_M^\gamma(r_R),$$
$$op_M^\gamma(b)op_M^\gamma(a) = 1 + op_M^\gamma(r_L),$$

  for some (every) $\gamma \in \mathbb{R}$.

Let $I=[0,r_0]$ for some $r_0\in\mathbb{R}_+$. Then the following are equivalent:

- *a is elliptic on I.*
- *There exists a symbol $b\in C_B^\infty(\overline{\mathbb{R}}_+,M_{O(cl)}^{-\mu;\ell}(X;F,E))$ and remainders $r_R\in C_B^\infty(\overline{\mathbb{R}}_+,M_O^{-\infty}(X;F,F))$, $r_L\in C_B^\infty(\overline{\mathbb{R}}_+,M_O^{-\infty}(X;E,E))$, as well as cut-off functions $\omega,\tilde\omega\in C_0^\infty(\overline{\mathbb{R}}_+)$ near $r=0$ with $\omega,\tilde\omega\equiv 1$ on I, such that*

$$\omega\big(op_M^\gamma(a)op_M^\gamma(b)-1\big)\tilde\omega=op_M^\gamma(r_R),$$
$$\omega\big(op_M^\gamma(b)op_M^\gamma(a)-1\big)\tilde\omega=op_M^\gamma(r_L),$$

for some (every) $\gamma\in\mathbb{R}$.

Proof. The proof is analogous to that of Theorem 5.3.10. □

5.4 Elements of the Fourier calculus with global weights

5.4.1 Remark. Analogously to Section 5.3 we are going to introduce subcalculi of the pseudodifferential calculi with operator-valued symbols satisfying global weight conditions from Section 2.7, where the symbols are built upon parameter-dependent pseudodifferential operators on the manifold X.

Recall that for every $s\in\mathbb{R}$ we have

$$S^{\varrho_1,\varrho_2}(\mathbb{R}^2,L^{\mu;\ell}(X;\mathbb{R};E,F))\hookrightarrow S^{\frac{\mu}{\ell},\varrho_1,\varrho_2}(\mathbb{R}^2\times\mathbb{R};H^s(X,E),H^{s-\mu_+}(X,F)),$$
$$S^{\varrho}(\mathbb{R},L^{\mu;\ell}(X;\mathbb{R};E,F))\hookrightarrow S^{\frac{\mu}{\ell},\varrho}(\mathbb{R}\times\mathbb{R};H^s(X,E),H^{s-\mu_+}(X,F)),$$
$$S^{\varrho_1,\varrho_2}(\mathbb{R}^2,L_V^{\mu;\ell}(X;\mathbb{H};E,F))\hookrightarrow S_V^{\frac{\mu}{\ell},\varrho_1,\varrho_2}(\mathbb{R}^2\times\mathbb{H};H^s(X,E),H^{s-\mu_+}(X,F)),$$
$$S^{\varrho}(\mathbb{R},L_V^{\mu;\ell}(X;\mathbb{H};E,F))\hookrightarrow S_V^{\frac{\mu}{\ell},\varrho}(\mathbb{R}\times\mathbb{H};H^s(X,E),H^{s-\mu_+}(X,F)),$$

where $\mu_+:=\max\{0,\mu\}$, see Definition 2.7.2. Consequently, for every double-symbol $a\in S^{\varrho_1,\varrho_2}(\mathbb{R}^2,L^{\mu;\ell}(X;\mathbb{R};E,F))$ the associated pseudodifferential operator acts continuously in the spaces

$$\mathrm{op}_r(a):\mathcal{S}(\mathbb{R}\times X,E)\longrightarrow\mathcal{S}(\mathbb{R}\times X,F),$$

and left- or right-symbols a are uniquely determined by this action in view of Theorem 2.7.4 and the density of $\mathcal{S}(\mathbb{R}\times X,E)$ in $\mathcal{S}(\mathbb{R},H^t(X,E))$ for every $t\in\mathbb{R}$. The classes of pseudodifferential operators with global weight conditions based on such symbols are invariant with respect to the manipulations in the calculus from Section 2.7. The technique to see this is the same as before. Therefore, we will only state the results what the basic elements of the calculus are concerned, and skip the proofs.

5.4.2 Remark. If explicitly stated, the asymptotic expansions in the sequel are to be regarded as follows:

Let $(\mu_k), (\varrho_k) \subseteq \mathbb{R}$ be sequences such that $\mu_k, \varrho_k \underset{k\to\infty}{\longrightarrow} -\infty$, and $\overline{\mu} := \max_{k\in\mathbb{N}} \mu_k$ as well as $\overline{\varrho} := \max_{k\in\mathbb{N}} \varrho_k$. Moreover, let

$$a_k \in \begin{cases} S^{\varrho_k}(\mathbb{R}, L^{\mu_k;\ell}(X;\mathbb{R};E,F)) \\ S^{\varrho_k}(\mathbb{R}, L_V^{\mu_k;\ell}(X;\mathbb{H};E,F)), \end{cases}$$

$$a \in \begin{cases} S^{\overline{\varrho}}(\mathbb{R}, L^{\overline{\mu};\ell}(X;\mathbb{R};E,F)) \\ S^{\overline{\varrho}}(\mathbb{R}, L_V^{\overline{\mu};\ell}(X;\mathbb{H};E,F)). \end{cases}$$

We write $a \underset{(V)}{\sim} \sum_{j=1}^{\infty} a_j$ if for every $R \in \mathbb{R}$ there is a $k_0 \in \mathbb{N}$ such that for $k > k_0$

$$a - \sum_{j=1}^{k} a_j \in \begin{cases} S^{R}(\mathbb{R}, L^{R;\ell}(X;\mathbb{R};E,F)) \\ S^{R}(\mathbb{R}, L_V^{R;\ell}(X;\mathbb{H};E,F)). \end{cases}$$

5.4.3 Theorem. *Consider a double-symbol*

$$a \in \begin{cases} S^{\varrho_1,\varrho_2}(\mathbb{R}^2, L_{(cl)}^{\mu;\ell}(X;\mathbb{R};E,F)) \\ S^{\varrho_1,\varrho_2}(\mathbb{R}^2, L_{V(cl)}^{\mu;\ell}(X;\mathbb{H};E,F)). \end{cases}$$

Then the corresponding left- and right-symbols a_L and a_R obtained from Theorem 2.7.4 belong to the spaces

$$a_L, a_R \in \begin{cases} S^{\varrho_1+\varrho_2}(\mathbb{R}, L_{(cl)}^{\mu;\ell}(X;\mathbb{R};E,F)) \\ S^{\varrho_1+\varrho_2}(\mathbb{R}, L_{V(cl)}^{\mu;\ell}(X;\mathbb{H};E,F)), \end{cases}$$

and the mappings $a \longmapsto a_L, a_R$ are continuous within

$$\left.\begin{matrix} S^{\varrho_1,\varrho_2}(\mathbb{R}^2, L_{(cl)}^{\mu;\ell}(X;\mathbb{R};E,F)) \\ S^{\varrho_1,\varrho_2}(\mathbb{R}^2, L_{V(cl)}^{\mu;\ell}(X;\mathbb{H};E,F)) \end{matrix}\right\} \longrightarrow \begin{cases} S^{\varrho_1+\varrho_2}(\mathbb{R}, L_{(cl)}^{\mu;\ell}(X;\mathbb{R};E,F)) \\ S^{\varrho_1+\varrho_2}(\mathbb{R}, L_{V(cl)}^{\mu;\ell}(X;\mathbb{H};E,F)). \end{cases}$$

Moreover, the asymptotic expansions of a_L and a_R in terms of a from Theorem 2.7.4 are valid in the sense of Remark 5.4.2.

5.4.4 Theorem. *a) Let*

$$a \in \begin{cases} S^{\varrho}(\mathbb{R}, L_{(cl)}^{\mu;\ell}(X;\mathbb{R};F,G)) \\ S^{\varrho}(\mathbb{R}, L_{V(cl)}^{\mu;\ell}(X;\mathbb{H};F,G)) \end{cases} \quad \text{and} \quad b \in \begin{cases} S^{\varrho'}(\mathbb{R}, L_{(cl)}^{\mu';\ell}(X;\mathbb{R};E,F)) \\ S^{\varrho'}(\mathbb{R}, L_{V(cl)}^{\mu';\ell}(X;\mathbb{H};E,F)). \end{cases}$$

Then we have for the Leibniz-product (cf. Theorem 2.7.5)

$$a\#b \in \begin{cases} S^{\varrho+\varrho'}(\mathbb{R}, L^{\mu+\mu';\ell}_{(cl)}(X;\mathbb{R};E,G)) \\ S^{\varrho+\varrho'}(\mathbb{R}, L^{\mu+\mu';\ell}_{V(cl)}(X;\mathbb{H};E,F)). \end{cases}$$

Moreover, the asymptotic expansion

$$a\#b \underset{(V)}{\sim} \sum_{k=0}^{\infty} \frac{1}{k!}(\partial_\tau^k a)(D_r^k b)$$

holds in the sense of Remark 5.4.2.

b) Let $[\cdot] : \mathbb{R} \to \mathbb{R}_+$ be a smoothed norm function, and $\delta \in \mathbb{R}$. Moreover, let $a \in S^{\varrho}(\mathbb{R}, L^{\mu;\ell}_{(cl)}(X;\mathbb{R};E,F))$. Let $a^{(*),\delta}$ be the formal adjoint symbol with respect to the $[\cdot]^{-\delta}L^2$-inner product (see also Theorem 2.7.9). Then we have

$$a^{(*),\delta} = \big([r]^{-2\delta}\big(a(r',\tau)\big)^{(*)}[r']^{2\delta}\big)_L \in S^{\varrho}(\mathbb{R}, L^{\mu;\ell}_{(cl)}(X;\mathbb{R};F,E)),$$

and we have the asymptotic expansion

$$a^{(*),\delta}(r,\tau) \sim \sum_{k=0}^{\infty}\sum_{p+q=k}\frac{1}{p!q!}\big([r]^{-2\delta}D_r^p[r]^{2\delta}\big)\big(\partial_\tau^k D_r^q(a(r,\tau))^{(*)}\big)$$

in the sense of Remark 5.4.2. Here $^{(*)}$ denotes the formal adjoint with respect to the L^2-inner product on the manifold.

5.4.5 Proposition. *Let*

$$a \in \begin{cases} S^{\varrho_1,\varrho_2}(\mathbb{R}^2, L^{\mu;\ell}(X;\mathbb{R};E,F)) \\ S^{\varrho_1,\varrho_2}(\mathbb{R}^2, L^{\mu;\ell}_V(X;\mathbb{H};E,F)) \end{cases}$$

such that $a(r,r') \equiv 0$ for $|r-r'| < \varepsilon$ for a sufficiently small $\varepsilon > 0$. Then $op_r(a) = op_r(c)$ with a symbol

$$c \in \begin{cases} S^{-\infty}(\mathbb{R}, L^{-\infty}(X;\mathbb{R};E,F)) \\ S^{-\infty}(\mathbb{R}, L^{-\infty}_V(X;\mathbb{H};E,F)). \end{cases}$$

5.4.6 Proposition. *Let $a \in S^{\varrho}(\mathbb{R}, L^{\mu;\ell}_V(X;\mathbb{H};E,F))$. Then $op_r(a)$ restricts for every $r_0 \in \mathbb{R}_+$ to a continuous operator*

$$op_r(a) : \mathcal{S}((-\infty,r_0),C^\infty(X,E)) \longrightarrow \mathcal{S}((-\infty,r_0),C^\infty(X,F)).$$

5.4.7 Theorem. *Let* $a \in S^{\varrho}(\mathbb{R}, L^{\mu;\ell}(X;\mathbb{R};E,F))$*.Then* $op_r(a)$ *extends for every* $s,t,\delta \in \mathbb{R}$ *to a continuous operator*

$$op_r(a) : H^{(s,t);\ell}(\mathbb{R}\times X, E)_{\delta} \longrightarrow H^{(s-\mu,t);\ell}(\mathbb{R}\times X, F)_{\delta-\varrho}.$$

If $a \in S^{\varrho}(\mathbb{R}, L_V^{\mu;\ell}(X;\mathbb{H};E,F))$*, then* $op_r(a)$ *restricts for every* $r_0 \in \mathbb{R}_+$ *to a continuous operator*

$$op_r(a) : H_0^{(s,t);\ell}((-\infty,r_0]\times X, E)_{\delta} \longrightarrow H_0^{(s-\mu,t);\ell}((-\infty,r_0]\times X, F)_{\delta-\varrho}.$$

Ellipticity and Parabolicity

5.4.8 Remark. Let

$$a \in \begin{cases} S^{\varrho}(\mathbb{R}, L_{(cl)}^{\mu;\ell}(X;\mathbb{R};E,F)) \\ S^{\varrho}(\mathbb{R}, L_{V(cl)}^{\mu;\ell}(X;\mathbb{H};E,F)). \end{cases}$$

As in Remark 5.3.7 we associate to a a complete symbol $(a_1,\dots,a_N)$ subordinate to the given covering of X from Notation 3.1.1. Then we have

$$(a_1,\dots,a_N) \in \begin{cases} S^{\varrho}(\mathbb{R}, \mathop{\times}\limits_{j=1}^{N} S_{(cl)}^{\mu;\ell}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R};\mathbb{C}^{N_-},\mathbb{C}^{N_+})) \\ S^{\varrho}(\mathbb{R}, \mathop{\times}\limits_{j=1}^{N} S_{V(cl)}^{\mu;\ell}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{H};\mathbb{C}^{N_-},\mathbb{C}^{N_+})). \end{cases}$$

5.4.9 Definition. a) Let $a \in S^{\varrho}(\mathbb{R}, L_{(cl)}^{\mu;\ell}(X;\mathbb{R};E,F))$. Let either $I=\mathbb{R}$ or $I=[r_0,\infty)$, $I=(-\infty,r_0]$ with $r_0\in\mathbb{R}$. We call *a interior elliptic* (on I), if for the complete symbol $(a_1,\dots,a_N)$ we have:

For every $j=1,\dots,N$ there exists $R>0$ and a neighbourhood $U(\kappa_j(\mathrm{supp}\psi_j))$ such that for $(\xi,\tau)\in\mathbb{R}^n\times\mathbb{R}$ with $|(\xi,\tau)|_{\ell}\geq R$ and all $r\in I$ and $x\in U(\kappa_j(\mathrm{supp}\psi_j))$ there exists $a_j(r,x,\xi,\tau)^{-1}$ with

$$\sup\{\|a_j(r,x,\xi,\tau)^{-1}\|\,\langle\xi,\tau\rangle_{\ell}^{\mu}\,\langle r\rangle^{\varrho};\ |(\xi,\tau)|_{\ell}\geq R,\ r\in I,\ x\in U(\kappa_j(\mathrm{supp}\psi_j))\} < \infty.$$

In the classical case this condition is equivalent to require that the homogeneous principal symbol

$$\sigma_{\psi}^{\mu;\ell}(a) \in S^{\varrho}(\mathbb{R}, S^{(\mu;\ell)}((T^*X\times\mathbb{R})\setminus 0, \mathrm{Hom}(\pi^*E,\pi^*F)))$$

is invertible (on I), and for the inverse we have

$$\sup\{\|\sigma_{\psi}^{\mu;\ell}(a)(r,\xi_x,\tau)^{-1}\|\langle r\rangle^{\varrho};\ r\in I,\ \left(|\xi_x|_x^{2\ell}+|\tau|^2\right)^{\frac{1}{2\ell}}=1\}<\infty.$$

We call a *elliptic* (on I), if a is interior elliptic (on I), and there exists some $s_0 \in \mathbb{R}$ such that

$$a(r,\tau) : H^{s_0}(X,E) \longrightarrow H^{s_0-\mu}(X,F)$$

is invertible for all $\tau \in \mathbb{R}$ and $|r|$ sufficiently large (on I), and

$$\sup\{\|a(r,\tau)^{-1}\|_{\mathcal{L}(H^{s_0-\mu},H^{s_0})}\langle r\rangle^{\varrho}\langle\tau\rangle^{M};\ |r| > \tilde{R},\ \tau \in \mathbb{R}\} < \infty$$

for some $\tilde{R}, M \in \mathbb{R}$.

b) Let $a \in S^{\varrho}(\mathbb{R}, L^{\mu;\ell}_{V\,(cl)}(X;\mathbb{H};E,F))$. Let either $I = \mathbb{R}$ or $I = [r_0,\infty),\ (-\infty,r_0]$ with $r_0 \in \mathbb{R}$. We call a *interior parabolic* (on I), if the following condition is fulfilled:

For every $j = 1,\dots,N$ there exists $R > 0$ and a neighbourhood $U(\kappa_j(\mathrm{supp}\psi_j))$ such that for $(\xi,\zeta) \in \mathbb{R}^n \times \mathbb{H}$ with $|(\xi,\zeta)|_\ell \geq R$ and all $r \in I$ and $x \in U(\kappa_j(\mathrm{supp}\psi_j))$ there exists $a_j(r,x,\xi,\zeta)^{-1}$ with

$$\sup\{\|a_j\big(r,x,\xi,\zeta\big)^{-1}\|\,\langle\xi,\zeta\rangle_\ell^{\mu}\,\langle r\rangle^{\varrho};\ |(\xi,\zeta)|_\ell \geq R,\ r \in I,\ x \in U(\kappa_j(\mathrm{supp}\psi_j))\} < \infty.$$

In the classical case this condition is equivalent to require that the homogeneous principal symbol

$$\sigma_{\psi}^{\mu;\ell}(a) \in S^{\varrho}(\mathbb{R}, S_V^{(\mu;\ell)}((T^*X \times \mathbb{H}) \setminus 0, \mathrm{Hom}(\pi^*E,\pi^*F)))$$

is invertible (on I), and for the inverse we have

$$\sup\{\|\sigma_{\psi}^{\mu;\ell}(a)(r,\xi_x,\zeta)^{-1}\|\langle r\rangle^{\varrho};\ r \in I,\ \big(|\xi_x|_x^{2\ell} + |\zeta|^2\big)^{\frac{1}{2\ell}} = 1\} < \infty.$$

We call a *parabolic* (on I), if a is interior parabolic (on I), and there exists some $s_0 \in \mathbb{R}$ such that

$$a(r,\zeta) : H^{s_0}(X,E) \longrightarrow H^{s_0-\mu}(X,F)$$

is invertible for all $\zeta \in \mathbb{H}$ and $|r|$ sufficiently large (on I), and

$$\sup\{\|a(r,\zeta)^{-1}\|_{\mathcal{L}(H^{s_0-\mu},H^{s_0})}\langle r\rangle^{\varrho}\langle\zeta\rangle^{M};\ |r| > \tilde{R},\ \zeta \in \mathbb{H}\} < \infty$$

for some $\tilde{R}, M \in \mathbb{R}$.

If $I = \mathbb{R}$ in a) or b) we say that a is (interior) elliptic, respectively (interior) parabolic, without refering to the interval.

5.4.10 Lemma. *a) Let $a \in S^{\varrho}(\mathbb{R}, L^{\mu;\ell}_{(cl)}(X;\mathbb{R};E,F))$. Let either $I = \mathbb{R}$ or $I = [r_0,\infty)$, $I = (-\infty,r_0]$ with $r_0 \in \mathbb{R}$. Then the following are equivalent:*

- *a is interior elliptic (on I).*
- *There exists $b \in S^{-\varrho}(\mathbb{R}, L^{-\mu;\ell}_{(cl)}(X;\mathbb{R};F,E))$ such that $ab-1$ and $ba-1$ coincide in a neighbourhood of I with symbols in $S^0(\mathbb{R}, L^{-1;\ell}_{(cl)}(X;\mathbb{R};F,F))$ and $S^0(\mathbb{R}, L^{-1;\ell}_{(cl)}(X;\mathbb{R};E,E))$, respectively.*

Moreover, the following are equivalent:

- *a is elliptic (on I).*
- *There exists $b \in S^{-\varrho}(\mathbb{R}, L^{-\mu;\ell}_{(cl)}(X;\mathbb{R};F,E))$ such that $ab-1$ and $ba-1$ coincide in a neighbourhood of I with symbols in $S^{-1}(\mathbb{R}, L^{-1;\ell}_{(cl)}(X;\mathbb{R};F,F))$ and $S^{-1}(\mathbb{R}, L^{-1;\ell}_{(cl)}(X;\mathbb{R};E,E))$, respectively.*

b) *Let $a \in S^{\varrho}(\mathbb{R}, L^{\mu;\ell}_{V\,(cl)}(X;\mathbb{H};E,F))$. Let either $I=\mathbb{R}$ or $I=[r_0,\infty)$, $(-\infty,r_0]$ with $r_0 \in \mathbb{R}$. Then the following are equivalent:*

- *a is interior parabolic (on I).*
- *There exists $b \in S^{-\varrho}(\mathbb{R}, L^{-\mu;\ell}_{V\,(cl)}(X;\mathbb{H};F,E))$ such that $ab-1$ and $ba-1$ coincide in a neighbourhood of I with symbols in $S^0(\mathbb{R}, L^{-1;\ell}_{V\,(cl)}(X;\mathbb{H};F,F))$ and $S^0(\mathbb{R}, L^{-1;\ell}_{V\,(cl)}(X;\mathbb{H};E,E))$, respectively.*

Moreover, the following are equivalent:

- *a is parabolic (on I).*
- *There exists $b \in S^{-\varrho}(\mathbb{R}, L^{-\mu;\ell}_{V\,(cl)}(X;\mathbb{H};F,E))$ such that $ab-1$ and $ba-1$ coincide in a neighbourhood of I with symbols in $S^{-1}(\mathbb{R}, L^{-1;\ell}_{V\,(cl)}(X;\mathbb{H};F,F))$ and $S^{-1}(\mathbb{R}, L^{-1;\ell}_{V\,(cl)}(X;\mathbb{H};E,E))$, respectively.*

Proof. The first equivalences in a) and b) follow from Theorem 3.1.10 and Theorem 3.2.18. It suffices to prove the necessity of the existence of b with the asserted properties for the ellipticity or parabolicity on the interval I. We will concentrate on b) only, for the proof of a) is analogous.

Let $a \in S^{\varrho}(\mathbb{R}, L^{\mu;\ell}_{V\,(cl)}(X;\mathbb{H};E,F))$ be parabolic on I. From the parabolicity in the interior we obtain together with Theorem 3.2.18 the existence of $\tilde{b} \in S^{-\varrho}(\mathbb{R}, L^{-\mu;\ell}_{V\,(cl)}(X;\mathbb{H};F,E))$ and $r_L \in S^0(\mathbb{R}, L^{-\infty}_{V}(X;\mathbb{H};E,E))$ as well as $r_R \in S^0(\mathbb{R}, L^{-\infty}_{V}(X;\mathbb{H};F,F))$ such that $a\tilde{b} = 1 + r_R$ and $\tilde{b}a = 1 + r_L$ in a neighbourhood of I. With a suitable excision function $\chi \in C^\infty(\mathbb{R})$ we define

$$b_L := \tilde{b} - r_L\tilde{b} + r_L\big(\chi(r)a(r,\zeta)^{-1}\big)r_R,$$
$$b_R := \tilde{b} - \tilde{b}r_R + r_L\big(\chi(r)a(r,\zeta)^{-1}\big)r_R.$$

Note that

$$r_L\big(\chi(r)a(r,\zeta)^{-1}\big)r_R \in S^{-\varrho}(\mathbb{R}, L_V^{-\infty}(X;\mathbb{H};F,E))$$

by Definition 3.2.2 and the parabolicity of a: the function is analytic and rapidly decreasing in $\mathbb{H}$ taking values in the bounded operators acting in the scale of Sobolev spaces on the manifold X; the corresponding estimates in the variable $r \in \mathbb{R}$ are straightforward.

Consequently, we have that $ab_R - 1$ and $b_La - 1$ coincide with symbols belonging to $S^{-\infty}(\mathbb{R}, L_V^{-\infty}(X;\mathbb{H};F,F))$ and $S^{-\infty}(\mathbb{R}, L_V^{-\infty}(X;\mathbb{H};E,E))$, respectively, in a neighbourhood of I. Now we may choose either $b = b_L$ or $b = b_R$ to obtain the desired properties. □

5.4.11 Theorem. *a) Let $a \in S^{\varrho}(\mathbb{R}, L_{(cl)}^{\mu;\ell}(X;\mathbb{R};E,F))$. Let either $I = \mathbb{R}$ or $I = [r_0,\infty)$, $I = (-\infty, r_0]$ with $r_0 \in \mathbb{R}$. Then the following are equivalent:*

- *a is interior elliptic on I.*
- *There exists a symbol $b \in S^{-\varrho}(\mathbb{R}, L_{(cl)}^{-\mu;\ell}(X;\mathbb{R};F,E))$, and elements $r_R \in S^0(\mathbb{R}, L^{-\infty}(X;\mathbb{R};F,F))$, $r_L \in S^0(\mathbb{R}, L^{-\infty}(X;\mathbb{R};E,E))$, as well as functions $\chi, \tilde{\chi} \in C^\infty(\mathbb{R})$ with $\chi, \tilde{\chi} \equiv 1$ on I and $\chi, \tilde{\chi} \equiv 0$ outside a neighbourhood of I, such that*

$$\chi\big(op_r(a)op_r(b) - 1\big)\tilde{\chi} = op_r(r_R),$$
$$\chi\big(op_r(b)op_r(a) - 1\big)\tilde{\chi} = op_r(r_L).$$

Moreover, the following are equivalent:

- *a is elliptic on I.*
- *There exists a symbol $b \in S^{-\varrho}(\mathbb{R}, L_{(cl)}^{-\mu;\ell}(X;\mathbb{R};F,E))$, and elements $r_R \in S^{-\infty}(\mathbb{R}, L^{-\infty}(X;\mathbb{R};F,F))$, $r_L \in S^{-\infty}(\mathbb{R}, L^{-\infty}(X;\mathbb{R};E,E))$, as well as functions $\chi, \tilde{\chi} \in C^\infty(\mathbb{R})$ with $\chi, \tilde{\chi} \equiv 1$ on I and $\chi, \tilde{\chi} \equiv 0$ outside a neighbourhood of I, such that*

$$\chi\big(op_r(a)op_r(b) - 1\big)\tilde{\chi} = op_r(r_R),$$
$$\chi\big(op_r(b)op_r(a) - 1\big)\tilde{\chi} = op_r(r_L).$$

b) Let $a \in S^{\varrho}(\mathbb{R}, L_{V(cl)}^{\mu;\ell}(X;\mathbb{H};E,F))$. Let either $I = \mathbb{R}$ or $I = [r_0,\infty)$, $(-\infty, r_0]$ with $r_0 \in \mathbb{R}$. Then the following are equivalent:

- *a is interior parabolic on I.*
- *There exists a symbol $b \in S^{-\varrho}(\mathbb{R}, L_{V(cl)}^{-\mu;\ell}(X;\mathbb{H};F,E))$, and elements $r_R \in S^0(\mathbb{R}, L_V^{-\infty}(X;\mathbb{H};F,F))$, $r_L \in S^0(\mathbb{R}, L_V^{-\infty}(X;\mathbb{H};E,E))$, as well as*

functions $\chi, \tilde{\chi} \in C^\infty(\mathbb{R})$ *with* $\chi, \tilde{\chi} \equiv 1$ *on* I *and* $\chi, \tilde{\chi} \equiv 0$ *outside a neighbourhood of* I*, such that*

$$\chi\big(op_r(a)op_r(b) - 1\big)\tilde{\chi} = op_r(r_R),$$
$$\chi\big(op_r(b)op_r(a) - 1\big)\tilde{\chi} = op_r(r_L).$$

Moreover, the following are equivalent:

- a *is parabolic on* I.
- *There exists a symbol* $b \in S^{-\varrho}(\mathbb{R}, L^{-\mu;\ell}_{V(cl)}(X;\mathbb{H};F,E))$*, and elements* $r_R \in S^{-\infty}(\mathbb{R}, L^{-\infty}_V(X;\mathbb{H};F,F))$*,* $r_L \in S^{-\infty}(\mathbb{R}, L^{-\infty}_V(X;\mathbb{H};E,E))$*, as well as functions* $\chi, \tilde{\chi} \in C^\infty(\mathbb{R})$ *with* $\chi, \tilde{\chi} \equiv 1$ *on* I *and* $\chi, \tilde{\chi} \equiv 0$ *outside a neighbourhood of* I*, such that*

$$\chi\big(op_r(a)op_r(b) - 1\big)\tilde{\chi} = op_r(r_R),$$
$$\chi\big(op_r(b)op_r(a) - 1\big)\tilde{\chi} = op_r(r_L).$$

Proof. In view of Theorem 5.4.3, Theorem 5.4.4 and Lemma 5.4.10 the above conditions in a) and b) are clearly sufficient for the (interior) ellipticity or parabolicity of the symbols (on the interval I). For the proof of the necessity we restrict ourselves to consider b), and to the case $I = \mathbb{R}$. Note first that the case of interior parabolicity is analogous to Theorem 5.3.10, but now applied with similar arguments to the setting of Fourier operators with global weight conditions.

Now assume that $a \in S^{\varrho}(\mathbb{R}, L^{\mu;\ell}_{V(cl)}(X;\mathbb{H};E,F))$ is parabolic. From Lemma 5.4.10 and Theorem 5.4.4 we conclude that there exists $\tilde{b} \in S^{-\varrho}(\mathbb{R}, L^{-\mu;\ell}_{V(cl)}(X;\mathbb{H};F,E))$ such that $a\#\tilde{b} = 1 - \tilde{r}_R$ and $\tilde{b}\#a = 1 - \tilde{r}_L$ with

$$\tilde{r}_R \in S^{-1}(\mathbb{R}, L^{-1;\ell}_{V(cl)}(X;\mathbb{H};F,F)),$$
$$\tilde{r}_L \in S^{-1}(\mathbb{R}, L^{-1;\ell}_{V(cl)}(X;\mathbb{H};E,E)).$$

Now choose $\hat{r}_R, \hat{r}_L$ such that

$$S^{-1}(\mathbb{R}, L^{-1;\ell}_{V(cl)}(X;\mathbb{H};F,F)) \ni \hat{r}_R \sim \sum_{j=1}^{\infty} \#^{(j)}\tilde{r}_R,$$
$$S^{-1}(\mathbb{R}, L^{-1;\ell}_{V(cl)}(X;\mathbb{H};E,E)) \ni \hat{r}_L \sim \sum_{j=1}^{\infty} \#^{(j)}\tilde{r}_L.$$

These asymptotic expansions are to be understood in the following sense:

$$\hat{r}_R - \sum_{j=1}^{N} \#^{(j)}\tilde{r}_R \in S^{-1-N}(\mathbb{R}, L^{-1;\ell}_{V(cl)}(X;\mathbb{H};F,F)),$$
$$\hat{r}_L - \sum_{j=1}^{N} \#^{(j)}\tilde{r}_L \in S^{-1-N}(\mathbb{R}, L^{-1;\ell}_{V(cl)}(X;\mathbb{H};E,E)).$$

for $N \in \mathbb{N}_0$. Note that the existence of $\hat{r}_R, \hat{r}_L$ with the corresponding asymptotic expansions can be proved analogously to Section 2.1 by employing a Borel-argument with a 0-excision function in the variable $r \in \mathbb{R}$ involved. Now define $\hat{b} := \tilde{b}\#(1+\hat{r}_R)$ or $\hat{b} := (1+\hat{r}_L)\#\tilde{b}$. Then we see that $a\#\hat{b} = 1-\bar{r}_R$ and $\hat{b}\#a = 1-\bar{r}_L$ with $\bar{r}_R \in S^{-\infty}(\mathbb{R}, L^{-1;\ell}_{V\,(cl)}(X;\mathbb{H};F,F))$ and $\bar{r}_L \in S^{-\infty}(\mathbb{R}, L^{-1;\ell}_{V\,(cl)}(X;\mathbb{H};E,E))$. Now choose $\check{r}_R, \check{r}_L$ such that

$$S^{-\infty}(\mathbb{R}, L^{-1;\ell}_{V\,(cl)}(X;\mathbb{H};F,F)) \ni \check{r}_R \underset{V}{\sim} \sum_{j=1}^{\infty} \#^{(j)}\bar{r}_R,$$
$$S^{-\infty}(\mathbb{R}, L^{-1;\ell}_{V\,(cl)}(X;\mathbb{H};E,E)) \ni \check{r}_L \underset{V}{\sim} \sum_{j=1}^{\infty} \#^{(j)}\bar{r}_L,$$

where the asymptotic expansions are to be carried out analogously to Theorem 3.2.12 (with rapidly decreasing behaviour of the extra-parameter $r \in \mathbb{R}$). Now we see that if we define either $b := \hat{b}\#(1+\check{r}_R)$ or $b := (1+\check{r}_L)\#\hat{b}$ we obtain the desired assertion, i. e.

$$a\#b - 1 \in S^{-\infty}(\mathbb{R}, L^{-\infty}_V(X;\mathbb{H};F,F)),$$
$$b\#a - 1 \in S^{-\infty}(\mathbb{R}, L^{-\infty}_V(X;\mathbb{H};E,E)).$$

This finishes the proof of the theorem. □

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